Dinamika kružnog gibanja

$$\vec{a}_c = -\frac{v^2}{r}\hat{r}$$

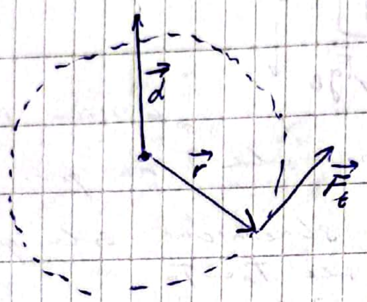
$$\vec{a} = \frac{\vec{F}}{m}$$

centripetalna sila (mora postojati zbog cp. akceleracije).

$$\vec{F}_c = m_c \cdot \vec{a}_c$$

Sada definiramo neke nove veličine:

Moment sile



Zbog djelovanja tangencijalne sile dolazi do tangencijalne akceleracije:

$$\vec{r}\times / \quad \vec{a}_t = \frac{\vec{F}}{m}$$

$\boxed{\vec{a}_t = \vec{\alpha}\times\vec{r}}$ izvedeno u kinematici kružnog gibanja.

$$\vec{r}\times\vec{a}_t = \frac{1}{m}(\vec{r}\times\vec{F})$$

$$\vec{r}\times\vec{a}_t = \vec{r}\times(\vec{\alpha}\times\vec{r}) = \left|\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})\right| =$$

$$= \vec{\alpha}\underbrace{(\vec{r}\cdot\vec{r})}_{|\vec{r}||\vec{r}|\cos 0 = r^2} - \vec{r}\underbrace{(\vec{r}\cdot\vec{\alpha})}_{0} = r^2\vec{\alpha}$$

Pa se vratimo u jednadžbu:

$$r^2\vec{\alpha} = \frac{1}{m}\underbrace{(\vec{r}\times\vec{F}_t)}_{\vec{M}}$$

$\vec{M}$ → Moment sile

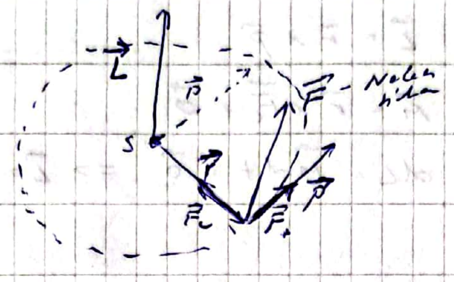
Nadalje: $J \equiv m \cdot r^2$ (moment tromosti)

Pa imamo: $\vec{\alpha} = \frac{\vec{M}}{J}$ (1)

Dobili smo ekvivalent Newtonovog zapisa za kružno gibanje.

Pokušajmo dobiti ekvivalent Newtonove jednadžbe

$$\vec{L} = \vec{r} \times \vec{p} \quad \text{(kutna količina gibanja)}$$



Zanima nas njena promjena:

$$d\vec{L} = d(\vec{r} \times \vec{p}) = (d\vec{r}) \times \vec{p} + \vec{r} \times (d\vec{p}) =$$

$$= \left| d\vec{r} = \vec{v}\,dt \Rightarrow d\vec{r} \parallel \vec{v} \Rightarrow d\vec{r} \parallel \vec{p} \Rightarrow (d\vec{r}) \times \vec{p} = 0 \right|$$

$$= \vec{r} \times (d\vec{p})$$

$$\vec{r} \times \vec{F} = \vec{r} \times (\vec{F}_t + \vec{F}_c) = \underbrace{\vec{r} \times \vec{F}_c}_{\vec{M}} + \underbrace{\vec{r} \times \vec{F}_t}_{0} = \vec{M}$$

Sad iz II. N.Z. $\quad \vec{r} \times / \; d\vec{p} = \vec{F}\,dt$

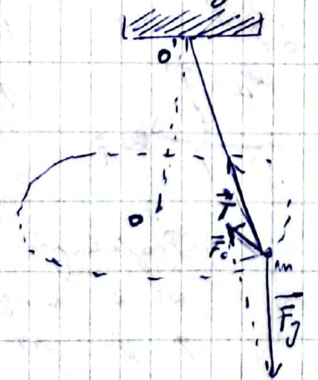
$$\vec{r} \times d\vec{p} = (\vec{r} \times \vec{F})\,dt$$

$$d\vec{L} = \vec{M}\,dt \quad (2)$$

Kada imamo kružno gibanje primjenjujemo ili (1) ili (2)

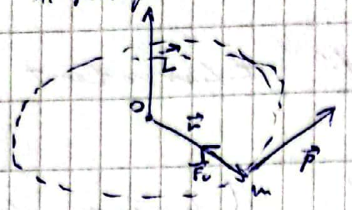
Pokus: Moment sile = $mg \cdot r$

Konusno njihalo:



$$\vec{F}_c = \vec{F}_g + \vec{T}$$

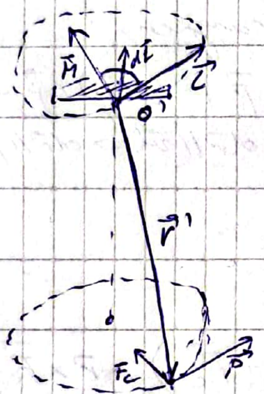
Kada promatramo 1. slučaj



$\vec L = \vec r \times \vec p$

$\vec M = \vec r \times \vec F_c = \vec 0$

$dL = \vec M dt = \vec 0 \Rightarrow \vec L = konst.$



$\vec L \perp \pi(\vec r', \vec p)$

$\vec M = \vec r' \times \vec F_c \neq 0$

$\vec M \perp \vec r' \wedge \vec F_c$

$\vec F_c \perp \vec p$

$\vec L \perp \vec r' \wedge \vec p$

$\vec M \perp \vec L$

$d\vec L \parallel \vec M$

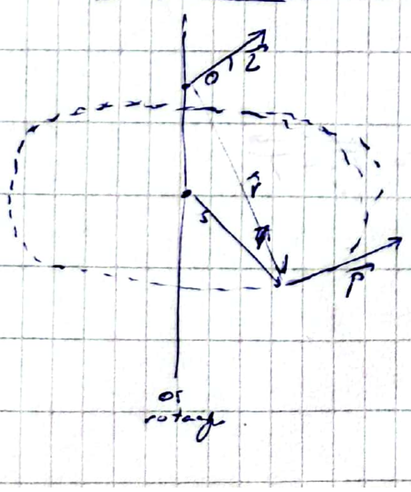
Pa zaključujemo

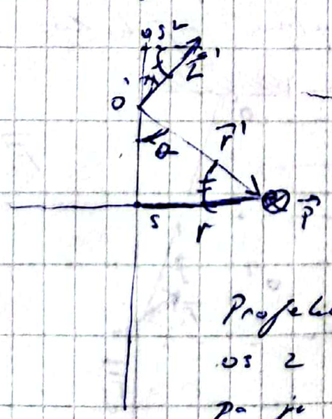
$d\vec L \perp \vec L$

$\Downarrow$

Nema promjene iznosa $\vec L$, ali se on rotira.

Naginjanje osi rotacije





$\vec L = \vec r' \times \vec p$

$|\vec L'| = r' p$ ~~na~~

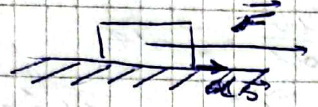
projekcija $\vec L$ na os z:

$\vec L_z = |\vec L'| \cdot \sin\theta = p \cdot r' \sin\theta$

$= p \cdot r$

Projekcija vektora kutne k.g. na os z uvijek je jednaka $p \cdot r$, pa je istaknuta os

— Rad i energija —



$$dW = \vec{F} \cdot d\vec{s}$$

$$dW = F \cdot ds$$

Ako imamo slučaj $\vec{F} \perp d\vec{s} - \parallel \vec{v}$

$$\vec{F} = \frac{d\vec{p}}{dt} \parallel d\vec{v} \rightarrow W = 0$$

Ako je $\vec{F} \parallel d\vec{s}$:

$$dW = ma \cdot v dt$$

$$dW = m \frac{dv}{dt} \cdot v dt$$

$$dW = m v dv \quad / \; v dv = d\left(\frac{1}{2} v^2\right)$$

$$dW = m \left(\frac{1}{2} v^2\right)_2$$

$$W = \int_1^2 dW = \int_1^2 d\left(\frac{1}{2} m v^2\right)$$

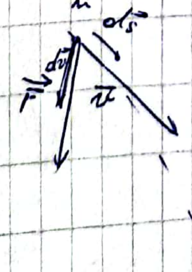
$$W = \frac{1}{2} m v^2 \Big|_{v_1}^{v_2}$$

$$W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

Definiramo: $E_k \equiv \frac{1}{2} m v^2$; Pa imamo:

$$W = \Delta E_k = \begin{cases} > 0 & v_2 > v_1 \\ < & v_2 < v_1 \end{cases}$$

Sada pogledajmo što je sa vektorima:



$$dW = \vec{F}\cdot d\vec{s}$$
$$= m\vec{a}\cdot\vec{v}\,dt$$
$$= m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt$$
$$= m\vec{v}\cdot d\vec{v} = m\cdot d\left(\tfrac{1}{2}v^2\right)$$

Dokaz: $\Big[\; d(v^2) = d(\vec{v}\cdot\vec{v}) = \vec{v}\,d\vec{v} + d\vec{v}\cdot\vec{v} = 2\vec{v}\cdot d\vec{v}$

$d\vec{v} = d\vec{v}_\perp + d\vec{v}_\parallel \implies 2\vec{v}\,d\vec{v} = 2\vec{v}\cdot d\vec{v}_\parallel$

centripetalna sila:

$$\vec{F}_{cp}\perp\vec{v} \implies dW_{cp} = \vec{F}_{cp}\cdot d\vec{s} = 0$$

vektori su kolinearni

Ali općenito imamo

$$\vec{F} = \vec{F}_t + \vec{F}_{cp} \implies dW = \vec{F}\cdot d\vec{s} = \vec{F}_t\cdot d\vec{s} = F_t\cdot ds$$

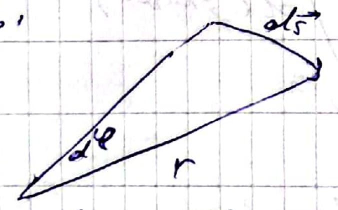
(Prepisano) $\quad dW = F_t\cdot ds \qquad (1)$

Moment sile:

okomiti: $\sin = 1$

$$\vec{M} = \vec{r}\times\vec{F} = \vec{r}\times(\vec{F}_{cp}+\vec{F}_t) = \vec{r}\times\vec{F}_t \implies M = rF_t$$

Znamo:



$$ds = r\,d\varphi$$

To ubacimo u (1)

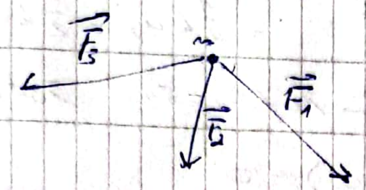
$$dW = F_t\,r\,d\varphi$$
$$dW = \vec{M}\cdot d\vec{\varphi}$$

$dW = \vec{F}\cdot d\vec{s}, \quad dW = \vec{M}\,d\vec{\varphi}$

$W = \Delta E_k$ → Tražimo kružni ekvivalent

$$E_k = \frac{1}{2} m v^2 = / v = \omega \cdot r / = \frac{1}{2} m r^2 \omega^2 = / J = m \cdot r^2 / = \underline{\frac{1}{2} J \omega^2}$$

Princip superpozicije



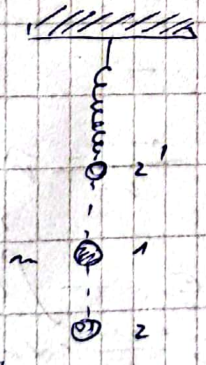
$$\vec F_{uk} = \sum_i \vec F_i$$

$$dW_{uk} = \vec F_{uk} \cdot d\vec s = \sum_i \vec F_i \cdot d\vec s = \sum_i dW_i$$

— Potencijalna energija —

$$W_{12} = \int_1^2 \vec F_{el} \cdot d\vec s = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = -\frac{1}{2} m v_1^2$$

elastična opruga

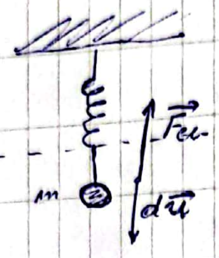


$$W^{1-2} = \int_1^2 \vec F_{el} \cdot d\vec s = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = -\frac{1}{2} m v_1^2$$

$$W = \int_2^1 \vec F_{el} \cdot d\vec s = \frac{1}{2} m v_1^2 - 0$$

Def: $dE_p = -\vec F_{el} \cdot d\vec s$

$$\Delta E_p = E_{p2} - E_{p1} = -\int_1^2 \vec{F_u} \cdot d\vec{s}$$



$$\Delta E_{p2} - \Delta E_{p1} = -\int -k \cdot \vec{u} \cdot d\vec{u}$$
$$= \int_{u_1}^{u_2} k u \, du = \frac{1}{2} k u^2 \Big|_{u_1}^{u_2}$$
$$= \frac{1}{2} k u_2^2 - \frac{1}{2} k u_1^2$$

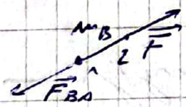
Kad je $u = 0$ → referentna točka

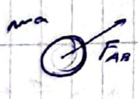
Definiramo $E_p(u=0) = 0$

$$\Downarrow$$

$$E_p = \frac{1}{2} k u^2$$

— Potencijalna energija gravitacijskog polja —

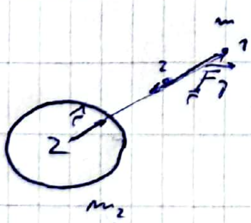




$$W_B = W_{F_{AB}} + W_{\vec F} = \underbrace{\int_1^2 \vec{F}_{AB}\, d\vec{s}}_{-\Delta E_p} + \int_1^2 \vec{F} \cdot d\vec{s} = \Delta E_k$$

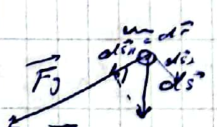
$$\Delta E_k + \Delta E_p = \int \vec{F} \cdot d\vec{s}$$

— Pot. en. sustava Zemlja-tijelo: —



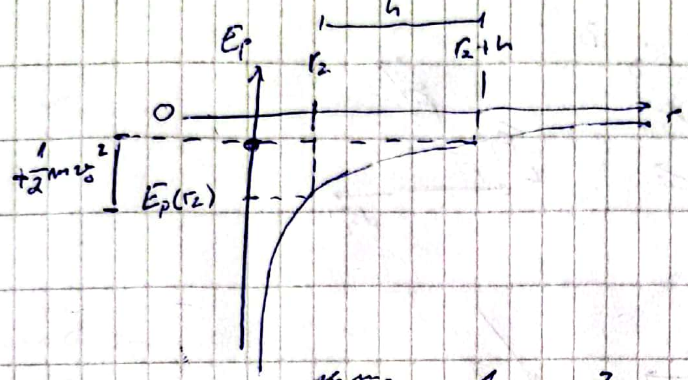
$$\vec{F}_g = -G \frac{m m_2}{r^2} \hat{r}$$

$$\Delta E_p = E_{p2} - E_{p1} = -\int_1^2 \vec{F}_g \cdot d\vec{s}$$
$$= -\int_{r_1}^{r_2} -G \frac{m_2 m}{r^2} 1\, dr = G m_1 m_2 \int_{r_1}^{r_2} \frac{dr}{r^2}$$
$$= -G \cdot m_1 \cdot m_2 \frac{1}{r} \Big|_{r_1}^{r_2}$$
$$\Rightarrow \Delta E_p = G m_1 m_2 \left(-\frac{1}{r_2} + \frac{1}{r_1} \right)$$



$$\vec{F}_g \cdot d\vec{s} = \vec{F}_g \cdot (d\vec{s}_\parallel + d\vec{s}_\perp)$$
$$= \vec{F}_g \cdot d\vec{r}$$

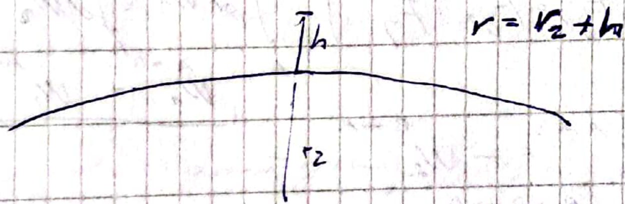
Def: $E_p(r\to\infty) = 0\,J \Rightarrow E_p = -G\frac{m\, m_z}{r}$



Ako: $G\frac{m\, m_z}{r_z} = \frac{1}{2} m v_z^2$

$$v_z = \sqrt{\frac{2Gm_z}{r_z}} \approx \sqrt{\frac{2\cdot 10^{-10}\cdot 10^{24}}{10^{7}}} \approx 10^{4}\ m/s$$

Gravitacijska potencijalna energija u blizini Zemlje:



$$\Delta E = E_p(h) - E_p(0)$$
$$= G m m_z\left(-\frac{1}{r_z+h} + \frac{1}{r_z}\right)$$
$$= \Big|\ (r_z+h)^{-1} = \left[r_z\left(1+\frac{h}{r_z}\right)\right]^{-1} = \frac{1}{r_z}\left(1+\frac{h}{r_z}\right)^{-1} \approx \frac{1}{r_z}\left(1-\frac{h}{r_z}\right)$$

Taylor: $(1\pm x^n) \approx 1 \pm nx$

$$\approx G m m_z\left[-\frac{1}{r_z}\left(1-\frac{h}{r_z}\right) + \frac{1}{r_z}\right] =$$
$$= G\frac{m\, m_z}{r_z^2} h = \underbrace{G\frac{m_z}{r_z^2}}_{=g\ \text{Def.}} m \cdot h = g \cdot m \cdot h$$

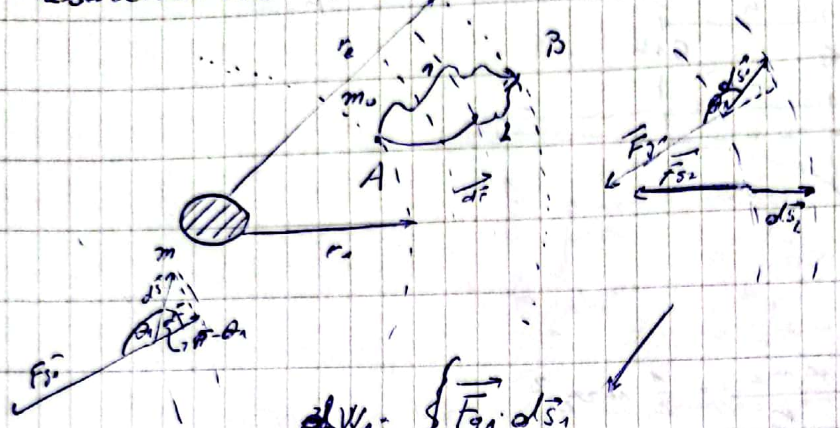
Konačno: $\Delta E_p = E_p(h) - E_p(0) = mgh$

Blizu Zemlje uzmemo referencu da je

E_p na površini Zemlje $= 0$

Općenito uzmemo $E_p(r\to\infty) = 0$

– konzervativne i nekonzervativne sile –



$$dW_1 = \oint \vec{F_{g1}} \cdot d\vec{s_1}$$
$$= -F_{g1}(ds_1 \cos\theta_1) = -F_{g1} dr_1$$

Analogno: →

$$\left.\begin{aligned} dW_2 &= -F_{g2} \cdot dr_2 \\ dr_1 &= dr_2 = dr \\ F_{g1} &= F_{g2} = F_g \end{aligned}\right\} \begin{aligned} &dW_1 = dW_2 \;/\!\int_A^B \\ &\int_A^B dW_1 = \int_A^B dW_2 \\ &\underline{W_1^{A\to B} = W_2^{A\to B}} \end{aligned}$$

$$W_1^{A\to B} = -W_2^{B\to A}$$

$$\underline{W^{A\to B\to A} = 0}$$

Sile koje ispunjavaju ovaj uvjet zovu se konzervativne sile.

$$\oint \vec{F} \cdot d\vec{s} = 0$$

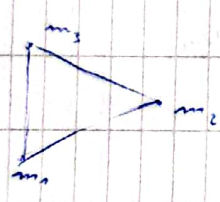
Sile trenja nisu konzervativne sile.

Sve konzervativne sile imaju potencijalnu energiju.

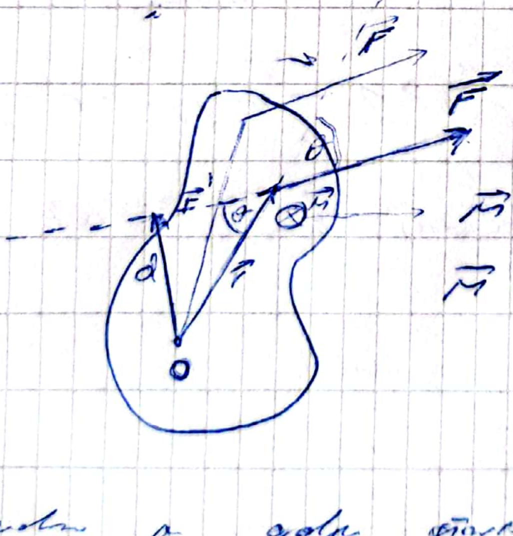
– Mehanika krutog tijela –

Definiramo: $\rho(\vec{r}) = \dfrac{dm}{dV} \Rightarrow m = \int \rho(\vec{r})\, dV$

Ako imamo:



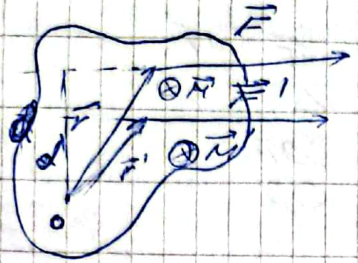
imamo: $m = \sum_i m_i$



$\vec{M} = \vec{r} \times \vec{F} = M = r \cdot F \sin\theta = F \cdot d$

$\vec{M} = \vec{r}' \times \vec{F}$

Pregledom je golo gornje [illegible] dvi [illegible] iste [illegible] momenta.



$|\vec F| = |\vec F'|$

$\vec M_{uk} = \vec M + \vec M'$ → v isti smeri

$= \vec r \times \vec F + \vec r' \times \vec F'$

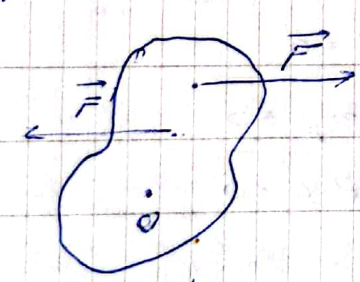
$M_{uk} = M + M'$

$= F \cdot d + F' \cdot d'$

$= F(d+d')$

$M_{uk} = F_{uk} \cdot D \Rightarrow D = \frac{F(d+d')}{F \cdot 2} = \frac{d+d'}{2}$

– Spreg sil –



$D = \frac{d-d'}{2}$

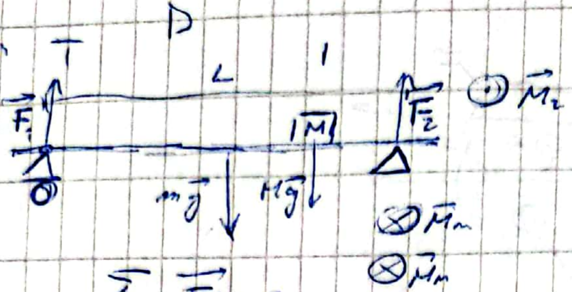
Zaradi sile ... pri O, ... momenti se razlikujejo, pa se ... ...

Pa imamo ... pri ... ravnovesju:

$\sum_i F_i = 0$

$\sum_i M_i = 0$

- model masa



$$\sum_i \vec{F}_i = 0$$

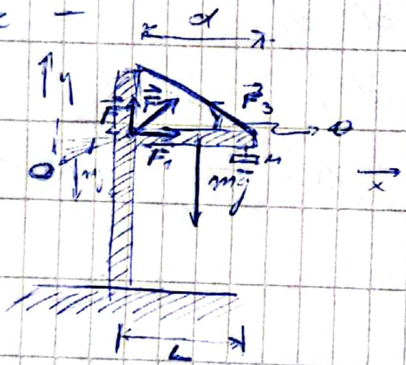
$$\sum_i \vec{M}_i = 0$$

$$\vec{F}_1 + \vec{F}_2 + m\vec{g} + M\vec{g} = 0$$

$$F_1 + F_2 - mg - Mg = 0 \quad (1)$$

$$mg\frac{L}{2} + Mg \cdot D - F_2 \cdot L = 0 \quad (2)$$

- model dirlice -



$$X: \quad F_1 - F_3 \cos\theta = 0$$

$$Y: \quad F_2 + F_3 \sin\theta - mg = 0$$

$$mg \cdot \frac{L}{2} \underset{+Mgd}{\wedge} - F_3 \sin\theta \cdot L = 0$$

– Rotacija nekog krutog tijela –

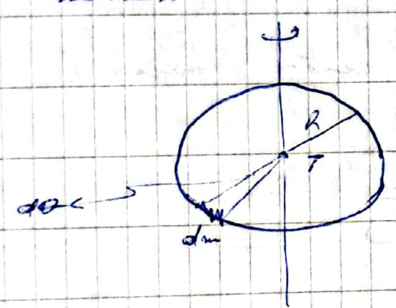
$$L_z = \sum_i L_{zi} = \left| \vec{L}_i = \vec{r}_i \times \vec{p}_i \right| =$$
$$= \sum_i R_i p_i = \sum_i m_i R_i v_i = \left| v_i = \omega R_i \right|$$
$$= \sum_i m_i R_i^2 \omega = \left(\sum_i m_i R_i^2\right)\omega = \underbrace{J\omega}$$

moment tromosti J · središ.

$$E_k = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i R_i^2 \omega^2 =$$
$$= \frac{1}{2}\left(\sum_i m_i R_i^2\right)\omega^2 = \underline{\frac{1}{2} J \omega^2} \rightarrow$$ kinetička energija rotacije krutog tijela.

– Određivanje momenta tromosti nekih tijela –

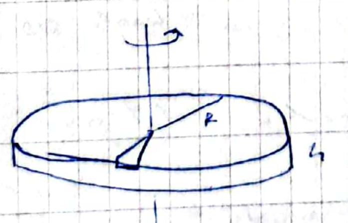
– Prsten –

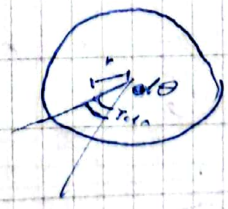


$$dm = \lambda\, dl = \lambda R\, d\theta$$

$$dJ = r^2 dm = |r = R|$$
$$dJ = R^2 dm$$
$$dJ = \lambda R^3 d\theta \quad / \int$$
$$J = \int_0^{2\pi} \lambda R^3 d\theta = \lambda R^3 \cdot \theta \Big|_0^{2\pi} = 2\pi R^3 \lambda$$
$$= (2\pi R \cdot \lambda)\cdot R^2 = mR^2$$

– Disk –





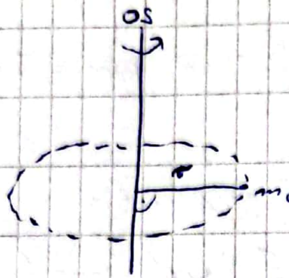
$$dm = \rho \cdot dV$$
$$= \rho \cdot d\theta \cdot dr \cdot h$$
$$dJ = r^2 \cdot \rho r\, dr\, d\theta \cdot h$$

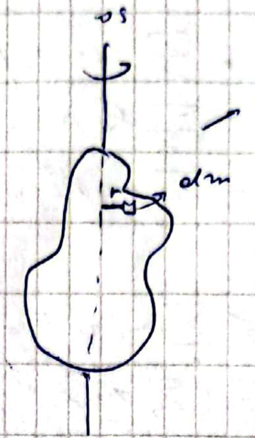
$$J = \int dJ$$
$$= \int_0^{2\pi}\int_0^R h \rho r^3\, dr\, d\theta = \rho \cdot h \int_0^R r^3 dr \int_0^{2\pi} d\theta$$
$$= \frac{1}{4}\rho r^4 h \Big|_0^R \cdot \theta \Big|_0^{2\pi} = \frac{1}{4}\rho R^4 h 2\pi = \dots = \frac{1}{2} m R^2$$

- Rotacija krutog tijela -

Znamo: $I = \sum_{i=1}^{N} m_i r_i^2$

$E_k = \frac{1}{2} m v^2 = \frac{1}{2} I \omega^2$



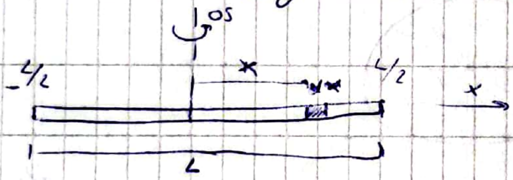


$dI = r^2 dm$

$I = \int r^2 dm$

- Moment tromosti štapa -

Štap se rotira oko svoga centra mase.



definiramo linearnu gustoću mase: $\lambda = \frac{dm}{dx}$

$dI = r^2 dm$

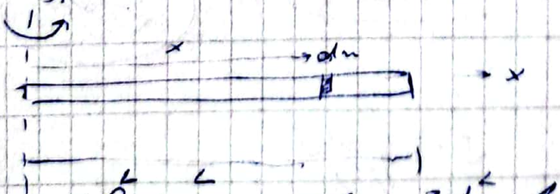
$dI = x^2 dm = x^2 \lambda \cdot dx = \lambda x^2 dx \quad / \int$

$I = \int dI = \int_{-L/2}^{+L/2} \lambda x^2 dx = \lambda \frac{1}{3} x^3 \Big|_{-L/2}^{+L/2}$

$= \frac{1}{3} \lambda \left(\frac{L^3}{8} + \frac{L^3}{8} \right) = \frac{1}{12} \lambda L^3 = \frac{1}{12} (\lambda L) L^2 = \underline{\frac{1}{12} m L^2}$

kada os prolazi kroz težište!

- Ako je os rotacije u rubu štapa.



$I = \int dI = \lambda \int_0^L x^2 dx = \lambda \frac{1}{3} x^3 \Big|_0^L = \frac{1}{3} \lambda L^3 = \underline{\underline{\frac{1}{3} m L^2}}$

Steinerov [illegible] o [illegible]

$J_z' = ?$ [illegible]

$J_z' = \sum_i m_i R_i'^2$

Potrebno [illegible]

$R_i'^2 = x_i'^2 + y_i'^2$

$R_i^2 = x_i^2 + y_i^2$

$d^2 = a^2 + b^2$

$x_i' = x_i - a$

$y_i' = y_i - b$

$J_z' = \sum_i m_i [(x_i - a)^2 + (y_i - b)^2]$

$= \sum_i m_i (x_i^2 - 2a x_i + a^2 + y_i^2 - 2b y_i + b^2)$

$= \sum_i m_i (x_i^2 + y_i^2) + \sum_i m_i (a^2 + b^2) - 2a \underbrace{\sum_i m_i x_i}_{0} - 2b \underbrace{\sum_i m_i y_i}_{0}$

$\vec{r}_{cm} = \dfrac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \vec{0}$

$x_{cm} = \dfrac{\sum_i m_i x_i}{\sum_i m_i} = 0$

$y_{cm} = \dfrac{\sum_i m_i y_i}{\sum_i m_i} = 0$

c.m. je u [illegible]

(1): $J_z' = \sum_i m_i (x_i^2 + y_i^2) + \sum_i m_i (a^2 + b^2) =$

$= \underbrace{\sum_i m_i R_i^2}_{J_z} + \sum_i m_i d^2$

$= J_z + m d^2$ → Steinerov [illegible]

Za kin. e.

$$E_k' = \frac{1}{2} J_z' \omega^2 = \frac{1}{2} (J_z + md^2)\cdot\omega^2$$

$$= \underbrace{\frac{1}{2} J_z \cdot \omega^2}_{} + \frac{1}{2} m(\omega\cdot d)^2$$

$$= E_k + \frac{1}{2} m v^2$$

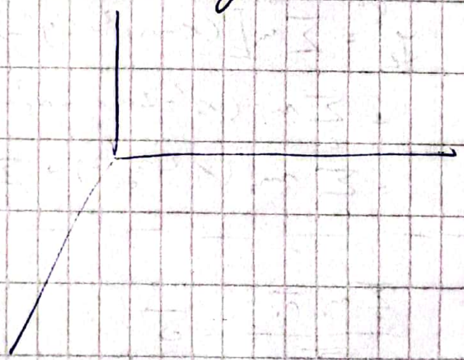
Za k. kol. g.

$$L_z' = J_z'\omega = (J_z + md^2)\cdot\omega$$

$$= J_z\omega + md(d\omega)$$

$$= L_z + m\cdot v_T\cdot d$$

— Glavne osi kretanja tijela —



$$\vec{F}_i = -m_i\omega^2\vec{R}_i$$

$$F_{ix} = -m_i\omega^2 x_i$$

$$F_{iy} = -m_i\omega^2 y_i$$

$$F_{iz} = 0$$

$$\vec{M}_i = \vec{r}_i \times \vec{F}_i$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ F_{ix} & F_{iy} & 0 \end{vmatrix} = + m_i\omega^2 y_i z_i\hat{i} - m_i\omega^2 x_i z_i\hat{j} - (m_i\omega^2 x_i y_i + m_i\omega^2 x_i y_i)\hat{k}$$

$$\vec{M}_{ix} = m_i y_i z_i \omega^2 \qquad M_x = \omega^2\sum_i m_i y_i z_i$$

$$\vec{M}_{iy} = -m_i x_i z_i \omega^2 \qquad M_y = \omega^2\sum_i m_i x_i z_i$$

$$\vec{M}_{iz} = 0 \qquad M_z = 0$$

Općenito

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i\, m_i \vec{v}_i$$

$$\vec{v}_i = \vec{\omega}\times\vec{r}_i = \begin{vmatrix} \omega_x & \omega_y & \omega_z \\ x_i & y_i & z_i \end{vmatrix} =$$

$$= (\omega_y z_i - \omega_z y_i)\hat{i} - (\omega_x z_i - \omega_z x_i)\hat{j} + (\omega_x y_i - \omega_y x_i)\hat{k}$$

$$\vec{L}_i = m_i \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ v_x & v_y & v_z \end{vmatrix} =$$

$$= m_i \left[\omega_x y_i^2 - \omega_y x_i y_i + \omega_x z_i^2 - \omega_z x_i z_i\right] \hat{i} -$$

$$- m_i \left[\omega_x x_i y_i - \omega_y x_i^2 - \omega_y z_i^2 + \omega_z y_i z_i\right] \hat{j} +$$

$$+ m_i \left[-\omega_x x_i z_i + \omega_z x_i^2 - \omega_y y_i z_i + \omega_z y_i^2\right] \hat{k}$$

za 1 jednu česticu.

$$\vec{L}_i = m_i \left[\omega_x (y_i^2 + z_i^2) - \omega_y x_i y_i - \omega_z x_i z_i\right] \hat{i} +$$

$$+ m_i \left[\omega_x y_i x_i + \omega_y (x_i^2 + z_i^2) - \omega_z y_i z_i\right] \hat{j} +$$

$$+ m_i \left[-\omega_x x_i z_i - \omega_y y_i z_i + \omega_z (x_i^2 + y_i^2)\right] \hat{k}$$

$$\vec{L} = \sum_i L_i = \sum_i m_i - '' - \hat{i} +$$

$$+ \sum_i m_i - '' - \hat{j} +$$

$$+ \sum_i m_i - '' - \hat{k}$$

momenti tromosti:

$$J_x = \sum_i m_i (y_i^2 + z_i^2) \qquad P_{xy} = \sum_i m_i x_i y_i$$

$$J_y = \sum_i m_i (x_i^2 + z_i^2) \qquad P_{xz} = \sum_i m_i x_i z_i$$

$$J_z = \sum_i m_i (x_i^2 + y_i^2) \qquad P_{yz} = \sum_i m_i y_i z_i$$

produkti tromosti

$$L_x = J_x \cdot \omega_x - P_{xy} - P_{xz} \omega_z$$

$$L_y = - P_{xy} \omega_x + J_y \omega_y - P_{yz} \omega_z$$

$$L_z = - P_{xz} \omega_x - P_{xz} \omega_y + J_z \omega_z$$

Tenzor tromosti

Matrični zapis vektora

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} J_x & -P_{xy} & -P_{xz} \\ -P_{xy} & J_y & -P_{yz} \\ -P_{xz} & -P_{yz} & J_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Kruto tijelo ima <u>tenzor</u>, a ne skalar tromosti

Pr. $\omega_x = \omega_y = 0$

$\omega_z \neq 0$

$$\left.\begin{aligned} L_x &= -P_{xz}\omega_z \\ L_y &= -P_{yz}\omega_z \\ L_z &= J_z\omega_z \end{aligned}\right\} \frac{d\vec{L}}{dt} \neq 0 \Rightarrow \vec{M} \neq 0$$

nije glavna os.

Činjenica da je tenzor tromosti simetrična matrica je dokaz da svako kruto tijelo ima barem tri glavne osi.

– Eulerove jednadžbe –

Dijagonalizirani tenzor za glavne osi:

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Znamo $\frac{dL}{dt} = \vec{M}$, $\left[\vec{M} = M_a\hat{a} + M_b\hat{b} + M_c\hat{c}\right]$

Pokazali smo da u sustavu koji se rotira vrijedi:

$$\left[\frac{d\vec{L}}{dt}\right]_{S\,\{x,y,z\}} = \vec{\omega} \times \vec{L} + \left[\frac{d\vec{L}}{dt}\right]_{S'\,\{a,b,c\}}$$

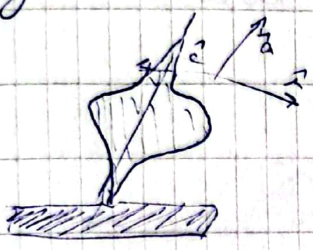
$$\left[\frac{d\vec{L}}{dt}\right]_{S'} = J_a \frac{d\omega_a}{dt}\hat{a} + J_b\frac{d\omega_b}{dt}\hat{b} + J_c\frac{d\omega_c}{dt}\hat{c}$$

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{a} & \hat{b} & \hat{c} \\ \omega_a & \omega_b & \omega_c \\ L_a & L_b & L_c \end{vmatrix} = \dots = (\omega_b L_c - \omega_c L_b)\hat{a} + -(\omega_a L_c - \omega_c L_a)\hat{b} + (\omega_a L_b - \omega_b L_a)\hat{c}$$

To uvrstimo i dobijemo

$$M_a = J_a \frac{d\omega_a}{dt} - (J_b - J_c)\omega_b\omega_c$$
$$M_b = J_b \frac{d\omega_b}{dt} - (J_c - J_a)\omega_a\omega_c$$
$$M_c = J_c \frac{d\omega_c}{dt} - (J_a - J_b)\omega_a\omega_b$$
Eulerove jj.

-Rotacija zvrka u bestežinskom stanju-



Važi: $J_a \neq J_b = J_c$

$$\vec{M} = 0$$

opšti vektor $\vec{\omega} = \omega_a \vec{a} + \omega_b \vec{b} + \omega_c \vec{c}$

Iz Eulerove jednačine.

$$J_a \frac{d\omega_a}{dt} = 0 \quad (J_b = J_c \text{ pa je}, M = 0) \quad (1)$$

$$\Longrightarrow \omega_a = const.$$

$$J_b \frac{d\omega_b}{dt} (J_c - J_a)\omega_a\omega_c = 0 \quad (2)$$

$$J_c \frac{d\omega_c}{dt} (J_a - J_b)\omega_a\omega_b = 0 \quad (3)$$

Sada uzimamo da je $J_c = J_b$ pa delimo (2) i (3) sa J_b. Dobijamo:

$$\frac{d\omega_b}{dt} - \frac{J_b - J_a}{J_b}\omega_a\omega_c = 0$$
$$\frac{d\omega_c}{dt} + \frac{J_b - J_a}{J_b}\omega_a\omega_b = 0$$

Uvedimo veličinu $\Omega = \frac{J_b - J_a}{J_b}\omega_a$

Pa imamo sistem dif. jj.

$$\frac{d\omega_b}{dt} - \Omega\omega_c = 0 \quad \Big/ \frac{d}{dt}$$
$$\frac{d\omega_c}{dt} + \Omega\omega_b = 0 \Rightarrow \frac{d\omega_c}{dt} = -\Omega\omega_b$$

$$\frac{d^2\omega_b}{dt^2} - \Omega\frac{d\omega_c}{dt} = 0$$
$$\frac{d^2\omega_b}{dt} + \Omega^2\omega_b = 0$$

Pravtako sproti: $\omega_b = B\cos(\Omega t)$

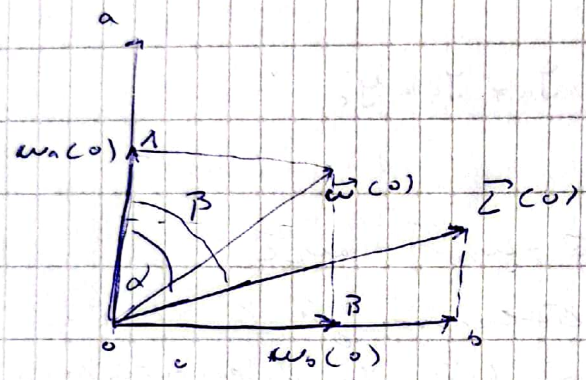
$\dot{\omega}_b = -B\Omega\sin(\Omega t)$

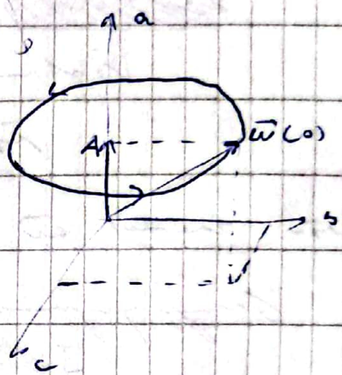
$\ddot{\omega}_b = -B\Omega^2\cos(\Omega t) \rightarrow$ Točno!

Po zaključimo

$\omega_a = konst. = \omega_a(0) = \omega\cos\alpha$

$\left.\begin{aligned} &\omega_b = B\cos(\Omega t) \\ &\omega_c = B\sin(\Omega t) \end{aligned}\right\}$





$$tg\alpha = \frac{\omega_b(0)}{\omega_a(0)} = \frac{B}{A}$$

$$tg\beta = \frac{J_b\,\omega_b(0)}{J_a\,\omega_a(0)} = \frac{J_b}{J_a}\,tg\alpha$$

Obodna hitrost dotične točke dna stožca

$\omega_p = \frac{\sin\alpha}{\sin\beta}\omega$

$\left.\begin{aligned} &\vec{v} = \vec{\Omega}\times\vec{r} \\ &\vec{v} = \vec{\omega}_p \times \vec{r} \end{aligned}\right\} \Rightarrow -\Omega r \sin\alpha = \omega_p \cdot r\sin(\beta-\alpha)$

$\omega_p \rightarrow$ precesija

$\frac{J_b - J_a}{J_b}\omega\cos\alpha\cdot\sin\alpha = \omega_p(\sin\beta\cos\alpha - \cos\beta\sin\alpha)$

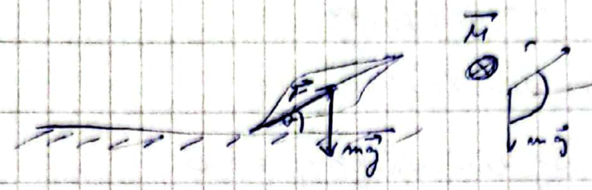
$\frac{\sin\beta}{\cos\beta} = \frac{J_b}{J_a}\frac{\sin\alpha}{\cos\alpha}$

$\frac{J_b - J_a}{J_b}\omega\cos\alpha\sin\alpha = \omega_p\cdot\sin\beta\cos\alpha$

$\cos\beta\sin\alpha = \frac{J_a}{J_b}\sin\beta\cos\alpha$

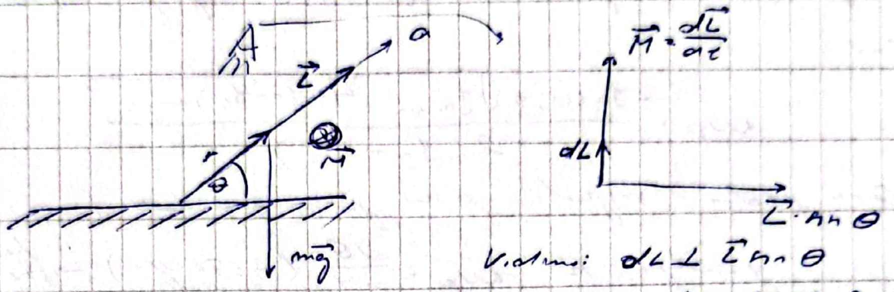
$\omega_p = \omega\cdot\frac{\sin\alpha}{\sin\beta}$

– [illegible] –



→ Also [illegible] [illegible]

Also [illegible] [illegible] [illegible] um a.



Wir haben: $dL \perp \vec L \sin\theta$

[illegible] [illegible] [illegible] [illegible], [illegible]

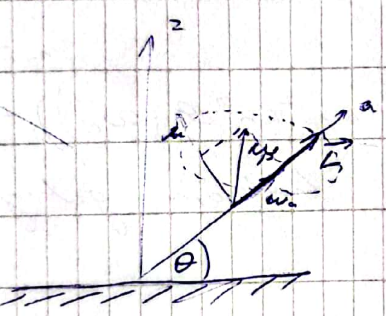
[illegible] [illegible] precessiert $\vec L$ um

vertikalen [illegible].

$\vec M = \vec r \times m\vec g =$

$\to M = r m g \sin(\pi-\theta)$

$= r m g \sin\theta$



$\omega = \omega_a + \omega_p$ → [illegible]

$= \omega_a \cdot \hat a + (\omega_p \cos\theta\, \hat a + \omega_p \sin\theta\, \hat n)$

$= (\omega_a + \omega_p\cos\theta)\hat a + \omega_p \sin\theta\, \hat n$

Zu $\vec L$ [illegible]: $\vec L = J_a \vec\omega_a + J_n \vec\omega_n =$

$= J_a(\omega_a + \omega_p\cos\theta)\hat a + J_n \omega_p \sin\theta\, \hat n$

[illegible] [illegible] [illegible]:

$$\left.\frac{d\vec L}{dt}\right|_s = \vec\omega \times \vec L + \underbrace{\left.\frac{d\vec L}{dt}\right|_{s'}}_{=0}$$

$$\left.\frac{d\vec L}{dt}\right|_s = \vec\omega_p \times \vec L = (\omega_p\cos\theta\,\hat a + \omega_p\sin\theta\,\hat n) \times$$

$$\times \left[J_a(\omega_a + \omega_p\cos\theta)\hat a + J_n \omega_p \sin\theta\,\hat n\right] =$$

$$= \omega_p \sin\theta\left[J_a(\omega_a + \omega_p\cos\theta) - J_n \omega_p\cos\theta\right]\underbrace{(\hat n \times \hat a)}_{\hat\sigma}$$

Izjednačimo dobivene izraze i rezultat $\vec{M}$

$\omega_p \sin\theta [- \ldots -] = mgs\sin\theta$

Dobivamo kvadratnu jednadžbu:

$(J_a - J_s)\cos\theta\, \omega_p^2 + J_a \omega_a \omega_p - mgs = 0$

Pa za zgodnu dostupno (rješenje odabiremo fizikalno smisleno) dobivamo:

$$\omega_p = \frac{-J_a\omega_a + \sqrt{J_a^2\omega_a^2 + 4(J_a - J_s)mgs}}{2(J_a - J_s)}$$

Ova jednadžba vrijedi ako je ω_p konstantan

Ako nije pravilna tj. $\omega_a = \frac{d\theta}{dt}$ (nutacija) $\to$ stalno kinetička energija jer zbroj nije održan

$$E_k = \frac{1}{2} J_a (\omega_a + \omega_p \cos\theta)^2 + \frac{1}{2} J_s (\omega_p^2 \sin^2\theta + \omega_c^2) + mgs\sin\theta$$

(nutacija iznad ω_c^2)

— Zakon očuvanja kutne količine gibanja —

$\vec{F} \times / \quad \vec{F}_{AB} = -\vec{F}_{BA}$

$\vec{M}_{AB} = -\vec{M}_{BA} \quad (1)$

$d\vec{L}_A = \vec{M}_A dt$

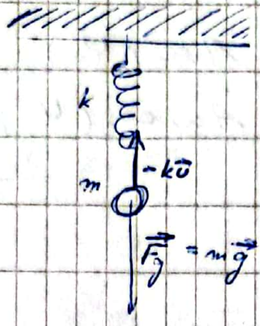
$d\vec{L}_B = \vec{M}_B dt \quad / +$

$dL_A + dL_B = (\vec{M}_A + \vec{M}_B)dt$

$d(\vec{L}_A + \vec{L}_B) = \vec{0} \quad \longrightarrow \quad \vec{L}_A + \vec{L}_B = \text{konst.}$

CamScanner

Harmonijski oscilator



Za ravnotežni položaj vrijedi:

$$ku_0 = mg$$

$$\vec{F}_{uk} = m\vec{g} - k(\vec{u}_0 + \vec{u}) = -k\vec{u}$$

$$\underline{F_{uk} = -ku}$$

$$\rightarrow m \cdot \frac{d^2u}{dt^2} + ku = 0 \;/ :m$$

$$\omega_0^2 \equiv \frac{k}{m}$$

(1) $\quad \dfrac{d^2u}{dt^2} + \omega_0^2 u = 0$ → jednadžba neprigušenog i neprisiljenog harmonijskog oscilatora.

Ovo je obič. lin. homogena diferencijalna jednadžba drugog reda.

Probno rješenje:

$$u(t) = A\sin(\omega t + \varphi_0)$$

$$\dot{u} = A\omega\cos(\omega t + \varphi_0)$$

$$\ddot{u} = -A\omega^2\sin(\omega t + \varphi_0)$$

uvrstimo u (1):

$$-A\omega^2\sin(\omega t + \varphi_0) + \omega_0^2 A\sin(\omega t + \varphi_0) = 0$$

$$A(-\omega^2 + \omega_0^2)\sin(\omega t + \varphi_0) = 0$$

$$u(t) = A\sin(\omega_0 t + \varphi_0)$$ → početni kut

elongacija

Amplituda

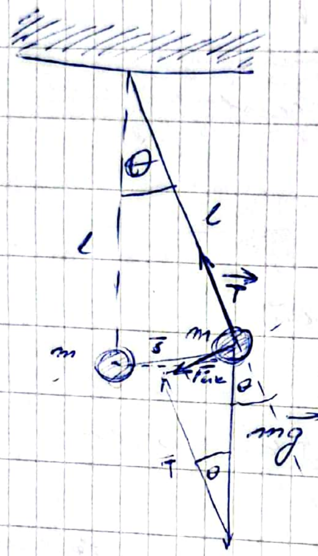
→ faza kretanja t. trenutka

Početni uvjeti

$$x(0) = A\sin(\varphi_0)$$

$$v(0) = A\omega\cos(\varphi_0)$$

Matematičko njihalo



$$F_{tan} = -mg\sin\theta$$

$$m\cdot\frac{d^2s}{dt^2} = -mg\sin\theta$$

Iz geometrije znamo $ds = l\,d\theta$ pa imamo:

$$m\,l\cdot\frac{d^2\theta}{dt^2} + mg\sin\theta = 0 \quad /:ml$$

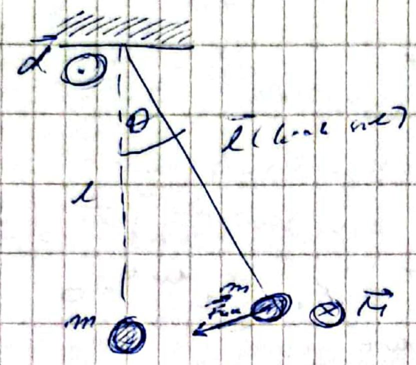
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

Nelinearna diferencijalna jednadžba drugog reda.

Za male kutove $\sin\theta \approx \theta$ pa imamo

$$\frac{d^2\theta}{dt^2} + \omega_0^2\theta = 0 \quad , \quad \omega_0^2 = \frac{g}{l}$$

Sedaj ponovimo [illegible] z [illegible] [illegible]:



$F_{un} = mg\sin\theta$

$\vec{M} = \vec{l} \times \vec{F}_{un}$

$M = l \cdot mg\sin\theta$

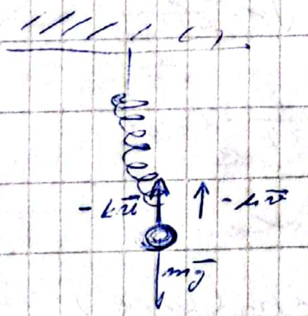
Znano: $I\vec{\alpha} = \vec{M}$ $\vec{\alpha} \parallel \vec{M}$ nasprotne sm.

$I\alpha = -\vec{M}$

$$m \cdot l^2 \frac{d^2\theta}{dt^2} = -mgl\sin\theta \; / : ml^2$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \longrightarrow \text{dobili smo isto enačbo.}$$

Poglejmo dušenje



$\vec{F}_{un} = -k\vec{u} - \mu\vec{v}$

$F_{un} = -ku - \mu v$

$$m\frac{d^2u}{dt^2} = -ku - \mu\frac{du}{dt} \; / : m \qquad \omega_0^2 = \frac{k}{m};\; \frac{\mu}{m} = \frac{1}{\tau}$$

$$\frac{d^2u}{dt^2} + \frac{\mu}{m}\frac{du}{dt} + \frac{k}{m}u = 0 \rightarrow$$

$$\frac{d^2u}{dt^2} + \frac{1}{\tau}\frac{du}{dt} + \omega_0^2 u = 0$$

pretp.: $x(t) = Ce^{\omega t}$

$\dot{x}(t) = C\omega e^{\omega t}$

$\ddot{x}(t) = C\omega^2 e^{\omega t}$

Pa imamo

$$C\omega^2 e^{\omega t} + \frac{1}{\tau} C\omega e^{\omega t} + C\omega_0^2 e^{\omega t} = 0$$

$$C(\omega^2 + \frac{1}{\tau}\omega + \omega_0^2) e^{\omega t} = 0$$

Ima smisla samo kad je zagrada = 0 → kvadratna

$$\omega^2 + \frac{1}{\tau}\omega + \omega_0^2 = 0$$

$$\omega_{1,2} = \frac{-\frac{1}{\tau} \pm \sqrt{(\frac{1}{\tau})^2 - 4\omega_0^2}}{2}$$

Opće rješenje je lin. kombinacija:

$$x(t) = C_1 e^{\omega_1 t} + C_2 e^{\omega_2 t} \quad (2)$$

Također imamo: $\omega_{1,2} = -\frac{1}{2\tau} \pm \sqrt{(\frac{1}{2\tau})^2 - \omega_0^2}$

Slabo gušenje:

$$(\frac{1}{2\tau})^2 < \omega_0^2 \Rightarrow \frac{1}{2\tau} < \omega_0$$

Pa imamo $\sqrt{(\frac{1}{2\tau})^2 - \omega_0^2} = i\omega_0\sqrt{1 - \frac{1}{4\tau^2\omega_0^2}} = i \cdot \omega_g$

Uvrstimo u (2)

$$= C_1 e^{-\frac{t}{2\tau}} \cdot e^{i\omega_g t} + C_2 e^{-\frac{t}{2\tau}} e^{-i\omega_g t}$$

Eulerova formula:

$$e^{\pm i\omega_g t} = \cos(\omega_g t) \pm i\sin(\omega_g t)$$

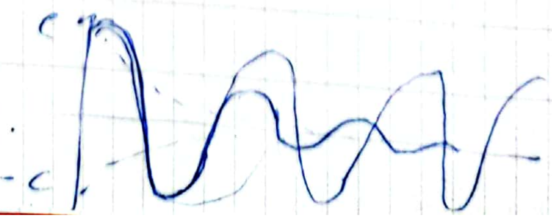
Kada to uvrstimo imamo:

$$x(t) = C_1 e^{-\frac{t}{2\tau}}[\cos(\omega_g t) + i\sin(\omega_g t)] +$$
$$+ C_2 e^{-\frac{t}{2\tau}}[\cos(\omega_g t) - i\sin(\omega_g t)]$$

$$= (C_1 + C_2) e^{-\frac{t}{2\tau}} \cos(\omega_g t) + i(C_1 - C_2) e^{-\frac{t}{2\tau}} \sin(\omega_g t)$$

$$x(t) \in \mathbb{R} \Rightarrow C_1 = C_2 = \frac{C}{2}$$

$$x(t) = C e^{-\frac{t}{2\tau}} \cos(\omega_g t) \quad \Downarrow$$



gubitak energije:

$$E_k = \frac{1}{2} m v^2$$

$$v(t) = \frac{dx}{dt} = C\left[-\frac{1}{2\tau} e^{-\frac{t}{2\tau}} \cos(\omega_g t) - \right.$$
$$\left. - e^{-\frac{t}{2\tau}} \cdot \omega_g \sin(\omega_g t)\right]$$
$$= -C e^{-\frac{t}{2\tau}} \left[\frac{1}{2\tau} \cos(\omega_g t) + \omega_g \sin(\omega_g t)\right]$$

$$v^2 = C^2 e^{-\frac{t}{\tau}} \left[\frac{1}{(2\tau)^2} \cos^2(\omega_g t) + \omega_g^2 \sin^2(\omega_g t) + \right.$$
$$\left. + \frac{1}{\tau} \cos(\omega_g t) \cdot \sin(\omega_g t)\right]$$

Aproksimacija: gušenje je konstantno.

$$\langle E_k \rangle = \frac{1}{T} \int_t^{t+T} E_k \, dt = \frac{m}{2T} C^2 \int e^{-\frac{t}{2\tau}} \cdot [\dots] \, dt$$

$\Rightarrow$ ostaje nam izračunati

$$\int_t^{t+T} \cos^2(\omega_g t') dt' = \frac{1}{2} t' + \frac{1}{4\omega_g} \cdot \sin(2\omega_g t') \Big|_t^{t+T} = \dots = \frac{T}{2}$$

$$\int_t^{t+T} \sin^2(\omega_g t') dt' = \frac{1}{2} t' + \frac{1}{4\omega_g} \cos(2\omega_g t') = \frac{T}{2}$$

$$\int_t^{t+T} \sin(\omega_g t') \cdot \cos(\omega_g t') dt' = \frac{1}{2\omega_g} \sin(2\omega_g t')$$

Dobivamo $\langle E_k \rangle = \frac{C^2 m}{2T} e^{-\frac{t}{2\tau}} \left[\frac{1}{(2\tau)^2} \cdot \frac{T}{2} + \omega_g^2 \cdot \frac{T}{2}\right] =$

$$= \frac{1}{4} C m e^{-\frac{t}{2\tau}} \underbrace{\left(\frac{1}{(2\tau)^2} + \omega_g^2\right)}_{\omega_0^2} =$$

$$= \frac{1}{4} C \omega_0^2 m e^{-t/2\tau} \rightarrow \text{gubi se mehanička energija}$$

Ako je izraz ispod korijena > 0

$$\omega_{1,2} = -\frac{1}{2\tau}\left[1 \pm \sqrt{\frac{1}{\tau^2} - \omega_0^2}\right]$$

$$x(t) = C_1 e^{-|\omega_1| t} + C_2 e^{-|\omega_2| t}$$ — nije oscilatorno gibanje jer nema im. jedinice u eksponentu, pa ne možemo primjeniti Eulerove formule

— Prisilni harmonijski oscilator —

Ako mi uvrstimo silu dif jed poprima oblik:

$$m\frac{d^2x}{dt^2} = -kx - \mu v - F_0\cos(\omega t) \quad /:m$$

$$\frac{d^2x}{dt^2} + \frac{1}{\tau}\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$ nehomogena

Rješenje ove jednadžbe piše se: $x(t) = x_H(t) + x_p(t)$ Prisilno

$$x_H(t) = C e^{-\frac{t}{2\tau}}\cos(\omega t)$$

Ovo gibanje trne, pa za $t\to\infty$ možemo zanemariti homogeni dio x_H. ($t\to\infty$, $x_H(t)\longrightarrow 0$).

frekvencija vanjskog titranja

$$x_p(t) = A\cos(\omega t - \varphi)$$

$$\dot{x} = -A\omega\sin(\omega t - \varphi)$$

$$\ddot{x} = -A\omega^2\cos(\omega t - \varphi)$$

$$\cos(\omega t - \varphi) = \cos(\omega t)\cdot\cos(\varphi) + \sin(\omega t)\sin(\varphi)$$

$$\sin(\omega t - \varphi) = \sin(\omega t)\cos(\varphi) - \cos(\omega t)\sin(\varphi)$$

Uvrštavamo

$$-A\omega^2\cos(\omega t-\varphi) - \frac{1}{\tau}A\omega\sin(\omega t - \varphi) + \omega_0^2 A\cos(\omega t - \varphi) = \frac{F_0}{m}\cos(\omega t) \quad /:A$$

$$(\omega_0^2 - \omega^2)\cos(\omega t - \varphi) - \frac{\omega}{\tau}\sin(\omega t - \varphi) = \frac{F_0}{mA}\cos(\omega t)$$

$$(\omega_0^2-\omega^2)\left[\cos(\omega t)\cos(\varphi) + \sin(\omega t)\sin(\varphi)\right] - \frac{\omega}{\tau}\left[\sin(\omega t)\cos\varphi - \cos(\omega t)\sin\varphi\right] = \frac{F_0}{mA}\cos(\omega t)$$

$$\left[(\omega_0^2-\omega^2)\cos\varphi + \frac{\omega}{\tau}\sin\varphi + \frac{\omega}{\tau}\sin\varphi\right]\cos(\omega t) + \left[(\omega_0^2-\omega^2)\sin\varphi - \frac{\omega}{\tau}\cos\varphi\right]\sin(\omega t) = \frac{F_0}{mA}\cos(\omega t)$$

To se svodi na dvije jednadžbe:

$$(\omega_0^2-\omega^2)\sin\varphi - \frac{\omega}{\tau}\cos\varphi = 0 \quad /:\cos\varphi$$

$$(\omega_0^2-\omega^2)\cos\varphi + \frac{\omega}{\tau}\sin\varphi = \frac{F_0}{mA}$$

$$\operatorname{tg}\varphi = \frac{\omega/\tau}{\omega_0^2-\omega^2}$$

tel. $\sin\varphi = \frac{\operatorname{tg}\varphi}{\sqrt{1+\operatorname{tg}^2\varphi}}$ $\cos\varphi = \frac{1}{\sqrt{1+\operatorname{tg}^2\varphi}}$

$$\frac{\omega_0^2-\omega^2}{\sqrt{1+\operatorname{tg}^2\varphi}} + \frac{\omega/\tau\cdot\operatorname{tg}\varphi}{\sqrt{1+\operatorname{tg}^2\varphi}} = \frac{F_0}{mA} \quad /\sqrt{\dots}$$

$$(\omega_0^2-\omega^2) + \frac{(\omega/\tau)^2}{(\omega_0^2-\omega^2)} = \frac{F_0}{mA}\sqrt{\frac{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}{(\omega_0^2-\omega^2)^2}}$$

$$\frac{(\omega_0^2-\omega^2)^2 + (\omega/\tau)^2}{\omega_0^2-\omega^2} = \frac{F_0}{mA}\,\frac{\sqrt{\dots}}{\omega_0^2-\omega^2}$$

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2}}$$

frekvencija odnosno oscilatora

frekvencija prisilnog titranja

Tražimo kad je amplituda maksimalna. (ekstrem) $f'(x)=0$

$$\frac{\partial}{\partial\omega}\left[(\omega_0^2-\omega^2)^2+(\omega/\tau)^2\right]=0$$

$$2(\omega_0^2-\omega^2)(-2\omega)+2\cdot\omega/\tau\cdot\frac{1}{\tau}=0 \quad /:2\omega$$

$$-2(\omega_0^2-\omega^2)=-\frac{1}{\tau^2}$$

$$\omega^2=\omega_0^2-\frac{1}{2\tau^2}$$

$$\omega_{rez}=\sqrt{\omega_0^2-\frac{1}{2\tau^2}}\approx\omega_0 \text{ za malo gušenje.}$$

Širina rezonancije $[\Delta\omega]$

$$A\left(\omega_0\pm\frac{\Delta\omega}{2}\right)=\frac{1}{2}A(\omega_0)$$

$$A(\omega_0)=\frac{F_0/m}{\omega_0/\tau}=\frac{F_0\tau}{m\omega_0}$$

$$A\left(\omega_0\pm\frac{\Delta\omega}{2}\right)=\frac{F_0/m}{\sqrt{\left[\omega_0^2-\left(\omega_0^2-\frac{\Delta\omega}{2}\right)^2\right]^2+\left(\frac{\omega_0-\Delta\omega/2}{\tau}\right)^2}}=$$

$$=\frac{F_0/m}{\sqrt{\left(\frac{\Delta\omega}{2}\right)\left(2\omega_0-\frac{\Delta\omega}{2}\right)+\left(\frac{\omega_0-\Delta\omega/2}{\tau}\right)^2}}=/\Delta\omega\ll\omega_0/\approx\frac{F_0/m}{\sqrt{(\omega_0\Delta\omega)^2+\frac{\omega_0^2}{\tau^2}}}=\frac{1}{\sqrt{2}}\frac{F_0/m}{\frac{\omega_0}{\tau}}$$

$$\sqrt{(\omega_0\Delta\omega)^2+\frac{\omega_0^2}{\tau^2}}=\sqrt{2}\,\omega_0/\tau \quad /^2$$

$$(\omega_0\Delta\omega)^2+\frac{\omega_0^2}{\tau^2}=2\frac{\omega_0^2}{\tau^2} \quad /:\omega_0^2$$

$$\Delta\omega^2=\frac{1}{\tau^2}\ ;\ \Delta\omega=\frac{1}{\tau}$$

Granični slučajevi $\omega\ll\omega_0-\frac{\Delta\omega}{2}$

I.

$$\omega_0^2-\omega^2\approx\omega_0^2$$

$$\frac{\omega}{\tau}\approx0$$

$$A=\frac{F_0/m}{\sqrt{\omega_0^2}}=\frac{F_0}{m\omega_0}$$

$$\operatorname{tg}\varphi\approx\frac{\omega/\tau}{\omega_0^2}\ll\frac{\omega_0/\tau}{\omega_0^2}=\frac{1}{\omega_0\tau}\to0^+\Rightarrow\operatorname{tg}\varphi\approx0$$

II. $\omega=\omega_0$ (rez.)

$$\operatorname{tg}\varphi\to\infty\Rightarrow\varphi\to\frac{\pi}{2}$$

$$A=\frac{F_0/m}{\omega_0/\tau}$$

$\overline{III}$ $\omega >> \omega_0 - \frac{\Delta\omega}{2}$

$$\omega_0^2 - \omega^2 \approx -\omega^2$$

$$tg\,\varphi \approx \frac{\omega/\tau}{-\omega^2} = -\frac{1}{\omega\tau} = 0^- \Rightarrow tg\,\varphi \to \pi$$

$$A = \frac{F_0/m}{\sqrt{\omega^4 - \frac{\omega^2}{\tau^2}}} \approx \frac{F_0/m}{\omega^2} \to 0$$

$$x(t) = A[\cos(\omega t)\cdot\cos\varphi + \sin(\omega t)\sin\varphi] =$$

$$= \underbrace{A\cos(\omega t)\underbrace{\cos\varphi}}_{} + \underbrace{A\sin(\omega t)\underbrace{\sin\varphi}}_{}$$

disperzijska amplituda $\leftarrow A_d$ $\qquad$ $A_a \rightarrow$ Apsorpcijska amplituda

Pa imamo:

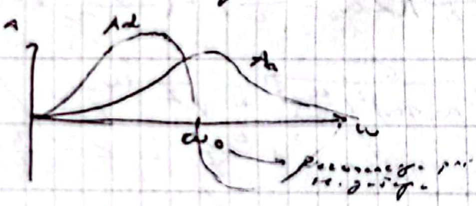
$$A_a = A\sin\varphi$$

$$A_d = A\cos\varphi$$

Izrazimo $tg\,\varphi$ preko sin i cos, iskoristimo jedn. za $tg\,\varphi$ pa dobijemo

$$A_d = \frac{F_0}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + (\omega/\tau)^2}$$

$$A_d = \frac{F_0}{m}\frac{\omega/\tau}{(\omega_0^2-\omega^2)^2 + (\omega/\tau)^2}$$



— Apsorbirana snaga —

Kolika snaga mora biti dovedena oscilatoru tijekom 1 perioda.

$$P = F\cdot v = F(t)\cdot v(t)$$

$$\vec{v}(t') = -A_d\omega\sin(\omega t) + A_a\omega\cos(\omega t)$$

$$F(t') = F_0\cos(\omega t)$$

$$\overline{P} = \frac{1}{T}\int_t^{t+T} F(t')\cdot v(t')\,dt'$$

$$\overline{P} = \frac{1}{T}\int_t^{t+T}[-A_d\omega\sin(\omega t) + A_a\omega\cos(\omega t)]\cdot F_0\cos(\omega t)\,dt'$$

$$= \ldots = \frac{1}{2}A_a\omega F_0$$ — srednja snaga prisilnog oscilatora

- Prisilno titranje: metoda rotirajućih vektora -

$$\frac{d^2u}{dt^2} + \frac{1}{\tau}\frac{du}{dt} + \omega_0^2 u = \frac{F_0}{m}\cos(\omega t)$$

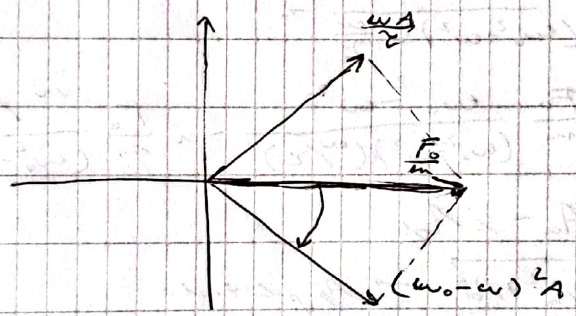
$$u(t) = A\cos(\omega t - \varphi)$$

$$\Downarrow$$

$$-A\omega^2\cos(\omega t - \varphi) + \frac{1}{\tau}A\omega\underbrace{\sin(\omega t - \varphi)}_{-\cos\left[(\omega t - \varphi) + \frac{\pi}{2}\right]} + \omega_0^2 A\cos(\omega t - \varphi) = \frac{F_0}{m}\cos(\omega t)$$

$$\underbrace{(\omega_0^2 - \omega^2)A\cos(\omega t - \varphi) + \frac{1}{\tau}A\omega\cos\left[(\omega t - \varphi) + \frac{\pi}{2}\right] = \frac{F_0}{m}\cos(\omega t)}_{3 \text{ rotirajuća vektora}}$$

u $t = 0$:



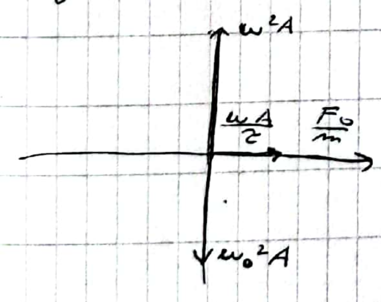
Pitagorin poučak

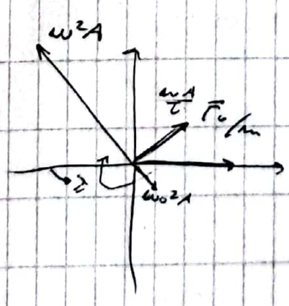
$$\left[(\omega_0^2 - \omega^2)A\right]^2 + \left(\frac{\omega A}{\tau}\right)^2 = \left(\frac{F_0}{m}\right)^2$$

Iz ovoga se dobiva već poznat izraz za A

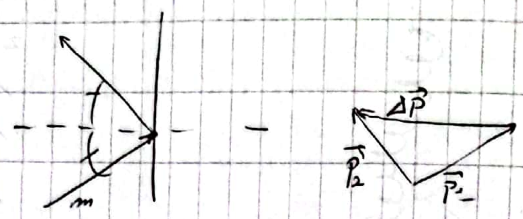
$$\operatorname{tg}\varphi = \frac{\omega A/\tau}{(\omega_0^2 - \omega^2)A}$$

Rezonancija: $\omega_0 = \omega$ | $\omega >> \omega_0$





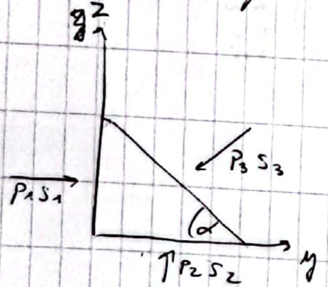
– Mehanika fluida –



$$\frac{\Delta\vec{p}}{\Delta t} = \vec{F}$$

$$p = \frac{\Delta F_\perp}{\Delta S} = \frac{dF_\perp}{\Delta S}$$

Pascalov princip:



$S_1 = S_3 \sin\alpha$

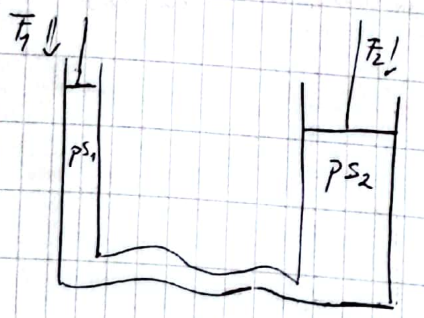
$S_2 = S_3 \cos\alpha$

Mirovanje: $\sum F = 0$

$p_1 S_1 = p_3 S_3 \sin\alpha$

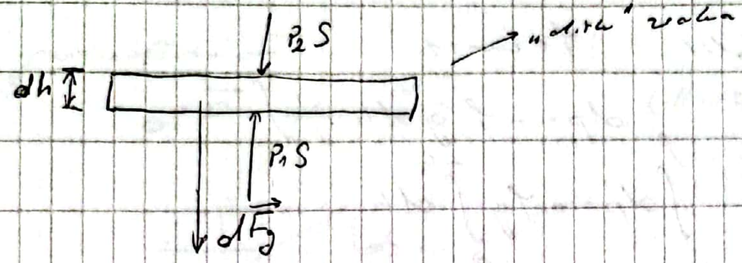
$p_1 = p_3$

Hidraulička dizalica



$\boxed{p_1 S_1 = p_2 S_2}$

Atmosferski tlak:



Uzmemo: $p_2 = p_1 + dp$

$d\vec{F}_g = g \cdot dm = g \cdot \rho \cdot S \cdot dh$

Po uvjetu ravnoteže imamo:

$$dF_g = pS - (p + dp)S$$

$$\rho g S dh = -dp S$$

$$\underline{dp = -\rho g dh} \quad (1)$$

plinovi: Predpostavimo $\frac{p_1}{\rho_1} = \frac{p_2}{\rho_2}$ pomoću

tj. ako su ρ_0 i p_0 na zemlji:

$$\frac{p}{\rho} = \frac{p_0}{\rho_0} \Rightarrow p = \rho \cdot \frac{p_0}{\rho_0} \Rightarrow \underline{\rho = p \cdot \frac{\rho_0}{p_0}}$$

(1): $dp = -\rho g dh = -p \cdot \frac{\rho_0}{p_0} g dh \quad / : p$

$$\frac{dp}{p} = -\frac{\rho_0 g}{p_0} dh \quad / \int$$

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{\rho g}{p_0} \int_0^h dh$$

$$\ln p \Big|_{p_0}^{p} = -\frac{\rho g}{p_0} h$$

$$\ln \frac{p}{p_0} = -\frac{\rho g h}{p_0}$$

$$\underline{p(h) = p_0 e^{-\frac{\rho_0 g}{p_0} \cdot h}}$$

Tlak tekućine:

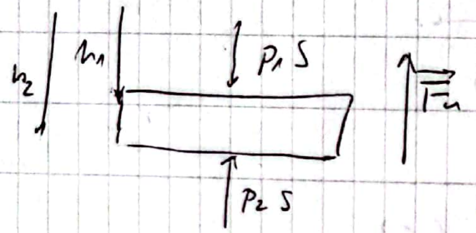
Vrijedi ρ = konst.

$$dp = -\rho g\, dh \quad / \int$$

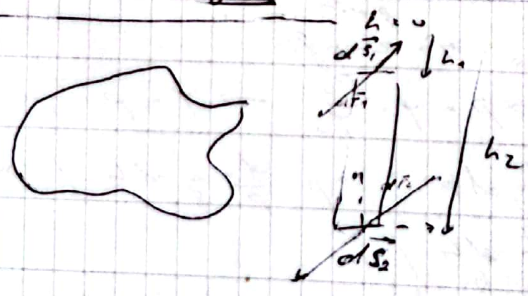
$$\int_{p_0}^{p} dp = -\rho g \int_{0}^{-h} dh$$

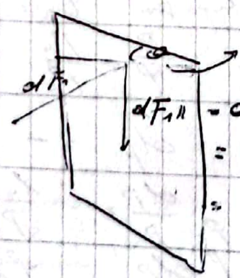
$$p(h) = p_0 + \rho g h$$

Uzgon



$$F_u = p_2 S - p_1 S = (p_0 + \rho g h_2) S - (p_0 + \rho g h_1) S$$
$$= \rho g S (h_2 - h_1) = \rho g V$$





$dS_1 \cdot \cos\theta$

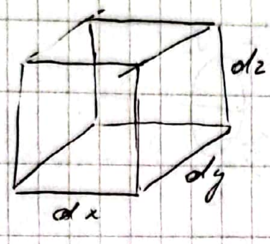
$dF_{1\parallel} = dF_1 \cos\theta$
$= \rho g h_1 \cos\theta \cdot \dfrac{dS_1}{\cos\theta}$

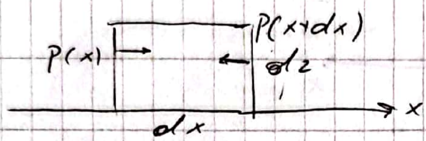
$$dF_1 = p_1\, dS_1 = (p_0 + \rho g h_1)\, dS_1$$
$$dF_2 = (p_0 + \rho g h_2)\, dS_2$$

— Dinamika fluida —

- Eulerova jednadžba —

$$d\vec{F} = dm \cdot \frac{d\vec{v}}{dt} \qquad (II\,NZ)$$

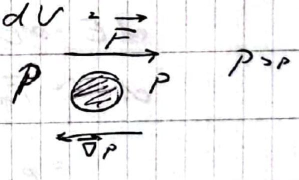




promjena tlaka po x osi

$$dF_x = p(x)\,dy\,dz - p(x+dx)\,dy\,dz$$
$$= \left| p(x+dx) = p(x) + \frac{\partial p}{\partial x}dx \right|$$
$$= -\frac{\partial p}{\partial x}dx\,dy\,dz = -\frac{\partial p}{\partial x}dV$$
$$dF_y = -\frac{\partial p}{\partial y}dV$$
$$dF_z = -\frac{\partial p}{\partial z}dV$$

$$d\vec{F} = dF_x\hat{i} + dF_y\hat{j} + dF_z\hat{k} =$$
$$= \left(-\frac{\partial p}{\partial x}\hat{i} - \frac{\partial p}{\partial y}\hat{j} - \frac{\partial p}{\partial z}\hat{k}\right)dV =$$
$$(1) \qquad \underline{= -\vec{\nabla} p \cdot dV} \Longrightarrow$$



Iz II.NZ. $\frac{d\vec{v}}{dt} = \left| \vec{v} = f(t,x,y,z) \right| = ?$

Brzina ovisi o vremenu i širini cijevi

tj. koliko vode protječe.

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t}dt + \frac{\partial \vec{v}}{\partial x}dx + \frac{\partial \vec{v}}{\partial y}dy + \frac{\partial \vec{v}}{\partial z}dz \;/\; :dt$$
$$= \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}v_x + \frac{\partial \vec{v}}{\partial y}v_y + \frac{\partial \vec{v}}{\partial z}v_z \qquad (2)$$

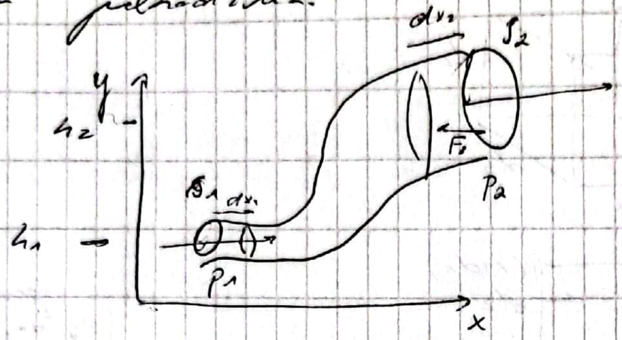
Za sile promatramo samo utjecaj gravitacijske i sile uzgona:

$$d\vec{F} = dF_g + dF_p = \rho g\,dV - \vec{\nabla} p \cdot dV$$

$$\rho g\, dV - \frac{\vec{\nabla} p}{\rho} \cdot dV = \left(\frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x} v_x + \frac{\partial \vec{v}}{\partial y} v_y + \frac{\partial \vec{v}}{\partial z} v_z\right) \cdot \rho \cdot dV \,/ : \rho dV$$

ε ...

Bernoullieva jednadžba:



$dm_1 = dm_2$ → sve što je ušlo mora izaći.

$$S_1 v_1 \cancel{dt \cdot \rho} = S_2 v_2 \, dt \cdot \rho$$

$$\underline{S_1 v_1 = S_2 v_2}$$ → jednadžba kontinuiteta.

---

meh. energija: $dE_1 = \frac{1}{2} \underset{\parallel}{dm_1} v_1^2 + dm_1 g h$

$$\underline{dE_2 = \frac{1}{2} dm_2 v_2^2 + dm_2 g h}$$

$$dE = dE_2 - dE_1 = dW$$

$$dW = F_1 dx_1 - F_2 dx_2$$

$$= p_1 S_1 v_1 dt - p_2 S_2 v_2 dt =$$

$$= \frac{p_1 dm}{\rho} - \frac{p_2 dm}{\rho} = \frac{dm}{\rho}(p_1 - p_2)$$

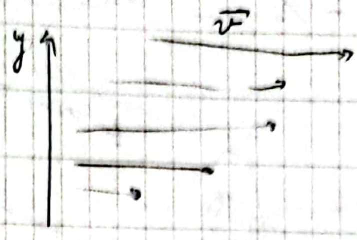
Uvrstimo u zakon očuvanja:

$$\frac{1}{2} dm\, v_2^2 - \frac{1}{2} dm\, v_1^2 + dm\, \rho g h_1 - dm\, g h_2 = \frac{dm}{\rho} \cdot (p_2 - p_1) \,/ : \frac{dm}{\rho}$$

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = \frac{1}{2}\rho v_2^2 + \rho g h_2 + p_2$$

- Realni fluidi -

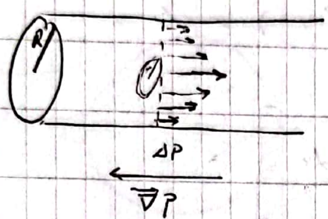
- viskoznost -



$$\frac{F}{S} = \eta \frac{dv}{dy}$$

Površina tijela

Protok fluida kroz cijev -



$$\Delta F = (p_1 - p_2) \cdot r^2 \pi$$

$$\frac{F_v}{2 r \pi L} = \eta \cdot \frac{dv}{dr}$$

Fluid se giba laminarno konstantno po ...

$$F_v = \Delta F$$

$$(p_1 - p_2) \cdot r^2 \pi = \eta \cdot 2\pi r \cdot L \cdot \frac{dv}{dy}$$

Nakon toga dobijemo

$$dv = \frac{\Delta p}{2L\eta} r\,dr$$

$$\int dv' = \int \frac{\Delta p}{2L\eta} r\,dr$$

$$v(r) = \frac{\Delta p}{4L\eta} r^2 + C$$

Dobijemo uvjet $v(R) = 0$

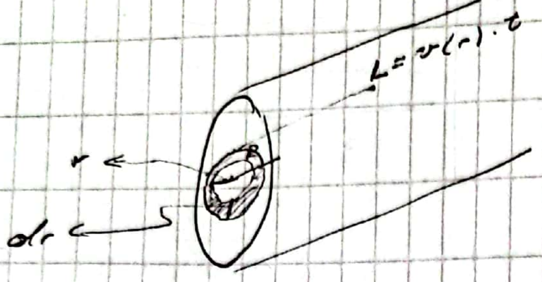
$$C = -\frac{\Delta p}{4L\eta} \cdot R^2$$

Pa je određeni integral

$$\int_0^{v(R)} dv' = \int_0^R \frac{\Delta p}{2L\eta} r\,dr \rightarrow v(r) = ?$$ Određeno integriramo

$$v(r) = \frac{\Delta p}{4\eta L} (R^2 - r^2)$$

– Hagen – Poiseuille-törvény –



gyűrű: $dA = 2\pi r \cdot dr$

$$dV = dA \cdot L = v(r) \cdot t \cdot 2\pi r\,dr$$

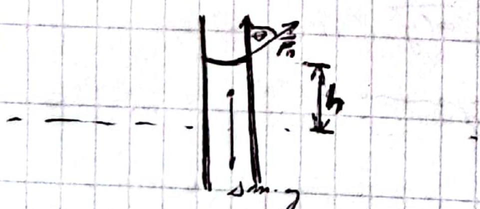
$$V = \int dV = \int_0^R 2\pi t \cdot \frac{\Delta p}{4\eta L}(R^2 - r^2) \cdot r \cdot dr =$$

$$= t \cdot \frac{\pi \cdot \Delta p}{2\eta L}\left[R^2 \int_0^R r\,dr - \int_0^R r^3 dr\right] =$$

$$= \text{ -- '' -- } \left[R^2 \cdot \frac{R^2}{2} - \frac{R^4}{4}\right] =$$

$$= t \cdot \frac{\pi \cdot \Delta p}{8\eta L} \cdot R^4$$

– Kapillaritás –

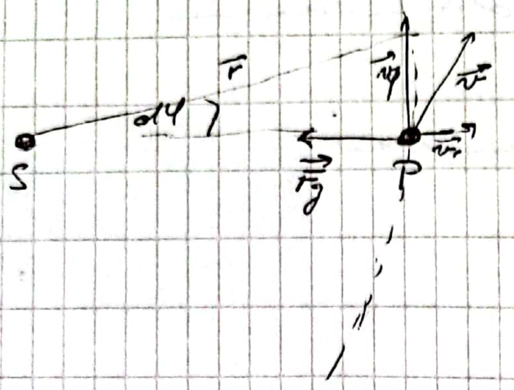


$\gamma = \frac{F}{l}$ $\qquad F_n \cdot \cos\theta = \Delta m \cdot g$

$$\underbrace{\gamma \cdot 2r\pi}_{\text{hez. kapcsolat}} \cdot \cos\theta = r^2\pi \cdot h \cdot \rho \cdot g$$

$$h = \frac{2\gamma \cos\theta}{r\rho g}$$

— Keplerovi zakoni —



$L = m\vec{r}\times\vec{v}(t)$

$L(t+dt) = m\vec{r}(t+dt)\times\vec{v}(t+dt)$

$L(t+dt) = m(\vec{r}(t)+d\vec{r})\times(\vec{v}(t)+d\vec{v})$

$= m\vec{r}(t)\times\vec{v}(t) + m\,d\vec{r}\times\vec{v}$ $\cancel{+ m\vec{r}\,dt}\ldots$

$= L(t)$

$$\vec{v} = \vec{v_r} + \vec{v_\varphi}$$

$$= \frac{dr}{dt}\hat{e}_r + r\frac{d\varphi}{dt}\hat{e}_\varphi$$

$$= \dot{r}\hat{e}_r + r\dot{\varphi}\hat{e}_\varphi$$

$$v^2 = \dot{r}^2 + r^2\dot{\varphi}^2$$

$$\vec{L} = m\vec{r}\times\vec{v} =$$

$$= m r\hat{e}_r \times(\dot{r}\hat{e}_r + r\dot{\varphi}\hat{e}_\varphi) =$$

$$= m r^2\dot{\varphi}\,\hat{e}_r\times\hat{e}_\varphi$$

$$= m r^2\dot{\varphi}\,\hat{e}_L$$

$$L = mr^2\dot{\varphi}$$

Meh. energija

$$E = \frac{1}{2}mv^2 - G\frac{m\,m_S}{r} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - G\frac{m\,m_S}{r}$$

$$= \left| r^2\dot{\varphi}^2 = \frac{L^2}{m^2r^2} \right| = \frac{1}{2}m\left(\dot{r}^2 + \frac{L^2}{m^2r^2}\right) - G\frac{m\,m_S}{r} =$$

$$= \frac{m\dot{r}^2}{2} + \frac{L^2}{2m}\cdot\frac{1}{r^2} - G\,m\,m_S\cdot\frac{1}{r}$$

$$E/m = \frac{\dot{r}^2}{2} + \frac{L^2}{2m^2r^2} - G\frac{m_S}{r}$$

$$\dot{r}^2 = \frac{E}{m} - \frac{L^2}{m^2r^2} + \frac{2Gm_S}{r}$$

$$\dot{\varphi}^2 = \frac{L^2}{m^2r^4}$$

$$\left(\frac{\frac{dr}{dt}}{\frac{d\varphi}{dt}}\right)^2 = \frac{\frac{E}{m} - \frac{L^2}{m^2r^2} + \frac{2Gm_S}{r}}{\frac{L^2}{m^2r^4}}$$

$$\frac{dr}{d\varphi} = \frac{2Emr^4}{L^2} - r^2 + \frac{2Gm_s m^2 r^3}{L^2} \Big/ : r^4$$

$$\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right)^2 = \frac{2Em}{L^2} - \frac{1}{r^2} + \frac{2Gm_s m^2}{L^2} \cdot \frac{1}{r} \qquad \Big/ \; S: u = \frac{1}{r}$$

$$\frac{du}{d\varphi} = \frac{d}{d\varphi}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\varphi}$$

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{2Em}{L^2} - u^2 + \frac{2Gm_s m^2}{L^2} u =$$

$$= -\underbrace{\left(u - \frac{Gm_s m^2}{L^2}\right)}_{s}^2 + \underbrace{\left(\frac{Gm_s m^2}{L^2}\right)}_{C}^2 + \frac{2Em}{L^2}$$

$$\left(\frac{ds}{d\varphi}\right)^2 = -s^2 + C^2$$

$$\frac{ds}{d\varphi} = \pm\sqrt{-s^2 + C^2}$$

$$\frac{ds}{\pm\sqrt{-s^2+C^2}} = d\varphi \Big/ \int$$

$$\mp \arccos\left(\frac{s}{C}\right) = \varphi - \varphi_0$$

$$s(\varphi) = C \cdot \cos(\varphi - \varphi_0)$$

$$\varphi_0 = 0 \implies s(\varphi) = C \cdot \cos\varphi$$

$$u(\varphi) = s(\varphi) + G\frac{m_s m^2}{L^2}$$

$$r(\varphi) = \frac{1}{u(\varphi)} = \frac{1}{C\cos\varphi + \frac{Gm_s m^2}{L^2}} =$$

$$= \frac{\frac{L^2}{Gm_s m^2}}{\frac{L^2}{Gm_s m^2} \cdot C \cdot \cos\varphi + 1}$$

$$= \frac{\frac{L^2}{Gm_s m^2}}{\sqrt{\frac{2EL^2}{(Gm_s m)^2 \cdot m} + 1} \cdot \cos\varphi + 1}$$

$$p = \frac{L^2}{G m_S m^2}$$

$$e = \sqrt{\frac{2EL^2}{(G m_S m)^2 \cdot m} + 1}$$

Sad imamo:

$$r(\varphi) = \frac{p}{e\cos\varphi + 1}$$

Imamo r_{min} i r_{max}. Još ne znamo o kakvoj putanji je riječ.

$r_{max} \to \infty \Rightarrow v \to 0 \Rightarrow E \to 0 \Rightarrow e = 1$

$r_{max} \not\to \infty \Rightarrow E < 0 \Rightarrow e < 1$

$$E = \frac{mv^2}{2} + \frac{L^2}{2mr^2} - G\frac{m_S m}{r}$$

Ali imamo E_{min} za r_{min} i $\dot{r} = 0$ & imamo kružnu putanju.

U tom slučaju:

$$\frac{dE_{min}}{dr} = 0 \Rightarrow -\frac{L^2}{m r_0^3} + \frac{G m_S m}{r_0^2} = 0$$

$$\Downarrow$$

$$r_0 = \frac{L^2}{G m_S m^2} = p$$

$$E_{min}(r_0) = \dots = -\frac{G^2 m_S^2 m^3}{2L^2} \Rightarrow e = 0$$

$$r = \frac{p}{1 + e\cos\varphi} =$$

$$= \left| \cos\varphi = \frac{x}{r} \text{ (slika)} \right| =$$

$$= \frac{rp}{r + xe} \quad / : r$$

$$1 = \frac{p}{r + ex}$$

$$r = p - ex \quad /^2$$

$$x^2+y^2=p^2-2epx+e^2x^2$$

Ako je $e=0$

$$x^2+y^2=p^2 \rightarrow \text{kružnica}$$

-||- $e=1$

$$x^2+y^2=p^2-2px+x^2$$

$$y^2=p^2-2px \rightarrow \text{Parabola}$$

-||- Inače:

$$(1-e^2)x^2+y^2+2epx=p^2 \quad / :(1-e^2)$$

$$x^2+2\frac{ep}{1-e^2}x+\frac{y^2}{1-e^2}=\frac{p^2}{1-e^2}$$

$$\left(x+\frac{ep}{1-e^2}\right)^2-\frac{e^2p^2}{(1-e^2)^2}+\frac{y^2}{1-e^2}=\frac{p^2}{1-e^2}$$

$$\frac{p^2}{1-e^2}+\frac{e^2p^2}{(1-e^2)^2}=\frac{p^2}{(1-e^2)^2}$$

$$/:\frac{p^2}{(1-e^2)^2}: \quad \frac{\left(x+\frac{ep}{1-e^2}\right)^2}{\frac{p^2}{(1-e^2)^2}}+\frac{(1-e^2)}{p^2}y^2=1$$

$$a=\frac{p^2}{(1-e^2)^2}$$

$$b^2=\frac{p^2}{1-e^2}$$

$\Downarrow$

fokus → Sunce

$$\frac{\left(x+\frac{ep}{1-e^2}\right)^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow \text{elipsa}$$

Sustav centra mase

$$r_{cm} = \frac{m_s \vec{r}_s + m_p \vec{r}_p}{m_s + m_p}$$

$$\vec{r}_{rel} = \vec{r}_s - \vec{r}_p$$

$$\Downarrow$$

$$\vec{r}_s = \vec{r}_{rel} + \vec{r}_p$$

$$r_p = r_{cm} + \frac{m_s}{m_s + m_p} \cdot \vec{r}_{rel}$$

Ako izrazimo r_s i pomoću r_{rel}:

$$\vec{r}_s = \vec{r}_{cm} - \frac{m_p}{m_p + m_s} \vec{r}_{rel}$$

Ako je cm u ishodištu, $r_{cm} = 0$:

$$\vec{r}_p = \frac{m_s}{m_s + m_p} \cdot \vec{r}_{rel}$$

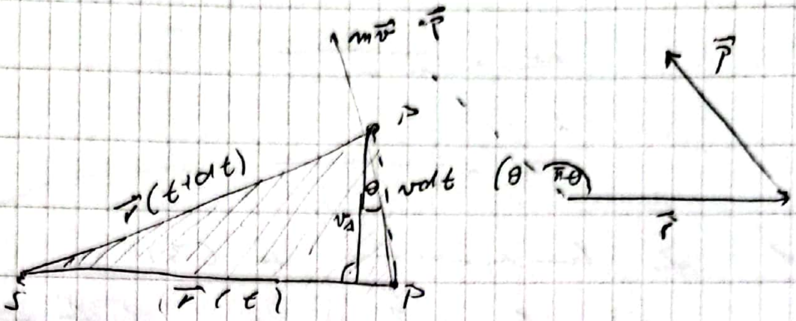
$$\vec{r}_s = \frac{m_p}{m_s + m_p} \cdot \vec{r}_{rel}$$

$$\vec{a}_{rel} = \vec{a}_s - \vec{a}_p = a_s + \frac{m_s}{m_p} a_s =$$

$$= \frac{m_s + m_p}{m_s \cdot m_p} \cdot \underbrace{m_s \cdot \vec{a}_s}_{\vec{F}_g}$$

$$\underbrace{\frac{m_s \cdot m_p}{m_s + m_p}}_{\mu \text{ - efektivna masa}} \cdot \vec{a}_{rel} = \vec{F}_g$$

II. Keplerov zakon:



$$dt \to 0 \Rightarrow \vec{r}(t+dt) \approx \vec{r}(t) = r$$

$$dA = \frac{1}{2} r \cdot v_\Delta$$

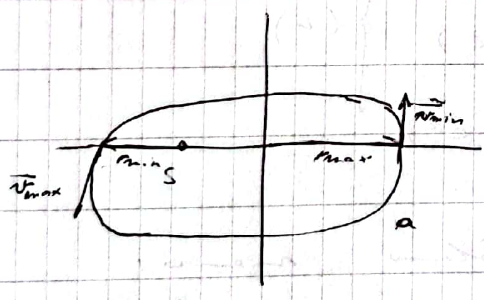
Iz pravokutnog trokuta: $v_\Delta = v\,dt \cdot \sin\theta$

$$dA = \frac{1}{2} r \cdot v\,dt \sin\theta \quad / : dt$$

$$\frac{dA}{dt} = \frac{1}{2} r v \sin\theta \quad / \cdot m$$

$$\frac{dA}{dt} = \frac{1}{2m} r \cdot mv \cdot \sin\theta = \frac{1}{2m} \cdot r \cdot p \cdot \sin(\pi - \theta) = \frac{L}{2m} = konst.$$

III. K.Z.



$$P = ab \cdot \pi$$

$$\int_0^T \left(\frac{dA}{dt}\right) \cdot dt = P$$

konst.

$$\frac{L}{2m} \cdot T = P = ab \cdot \pi \quad (1)$$

$$L = m \cdot r_{max} \cdot v_{min} = m \cdot r_{min} \cdot v_{max} \quad (2)$$

$$E = \frac{1}{2} m v_{min}^2 - G \frac{m M_S}{r_{max}} = \frac{1}{2} m v_{max}^2 - G \frac{m M_S}{r}$$

$$v_{max}^2 - v_{min}^2 = -2 G m_S \left(\frac{1}{r_{min}} - \frac{1}{r_{max}} \right) \quad (3)$$

Iz (2) imamo:

$$v_{min} = \frac{L}{m \cdot r_{max}}$$

$$v_{max} = \frac{L}{m \cdot r_{min}}$$

Kad to kombinujemo sa (3):

$$-\frac{L^2}{m^2}\left(\frac{1}{r_{min}^2} - \frac{1}{r_{max}^2}\right) = -2Gm_s\left(\cancel{\frac{1}{r_{min}} - \frac{1}{r_{max}}}\right)$$

raz. kvadr.

$$\frac{1}{r_{min}} + \frac{1}{r_{max}} = +2G\frac{m_s m^2}{L^2}$$

$$L^2 = \frac{2Gm_s m^2}{\frac{1}{r_{min}} + \frac{1}{r_{max}}}$$

Kad to ubacimo u (1):

$$\frac{L^2T^2}{4m^2} = a^2b^2\pi^2$$

Uz svojstva elipse:

$$\left.\begin{aligned} r_{min} &= a(1-e) \\ r_{max} &= a(1+e) \\ b^2 &= a^2(1-e^2) \end{aligned}\right\} (4)$$

Kad to uvrstimo i sredimo imamo:

$$T^2 = \frac{2(ab\pi)^2}{Gm_s}\underbrace{\left(\frac{1}{r_{min}} + \frac{1}{r_{max}}\right)}_{u} \quad (5)$$

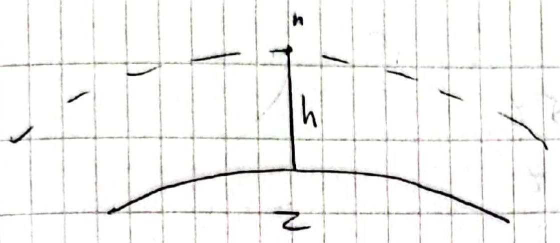
Za u računamo: (4)

$$\frac{1}{r_{min}} + \frac{1}{r_{max}} = \frac{r_{min} + r_{max}}{r_{min} r_{max}} = \frac{2a}{b^2},$$

što uvrštavamo u (5):

$$T^2 = \frac{2a^2\cancel{b^2}\pi^2}{Gm_s} \cdot \frac{2a}{\cancel{b^2}} = \frac{4\pi^2}{Gm_s} \cdot a^3$$

I. kozmická rýchlosť



$$F_g = G\frac{m_1 m_2}{(R_z+h)^2} = \frac{m_1 v_1^2}{R_z+h}$$

$$v_1 = \sqrt{G\frac{m_2}{R_z+h}} \doteq \sqrt{G\frac{m_2}{R_z}} = 7{,}9\ km/s$$

II. — " —

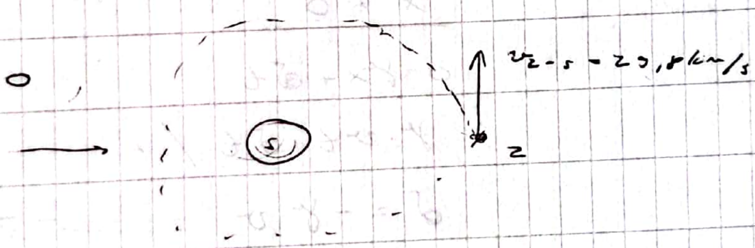
$$E = \frac{1}{2} m v_2^2 - G\frac{m m_2}{R_z} = 0$$

$$v_2 = 11{,}2\ km/s$$

III. — " —

$$\frac{1}{2} m v_s^2 - G\frac{m m_s}{R_s} = 0$$

$$v_s = 42{,}1\ km/s$$



$$v_{s-z} = v_s - v_{z-s} = 12{,}3\ km/s$$

$$\frac{1}{2} m v_{s-z}^2 + \frac{1}{2} m v_3^2 - G\frac{m m_2}{R_z} = 0$$

...

$$v_3 = 16{,}7\ km/s$$

– Metoda kompleksnih brojeva –

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

$$\tilde{u}(t) = \tilde{A} e^{i\omega t}$$
$$\tilde{F}(t) = F_0 e^{i\omega t}$$

Fizikalno rješenje će biti $u(t) \in \mathbb{R}$ pa moramo:

$$\frac{d^2\tilde{u}}{dt^2} + \frac{1}{\tau}\frac{d\tilde{u}}{dt} + \omega_0^2\tilde{u} = \frac{\tilde{F}}{m}$$

$$-\omega^2\tilde{A}e^{i\omega t} + \frac{1}{\tau}i\omega\tilde{A}e^{i\omega t} + \omega_0^2\tilde{A}e^{i\omega t} = \frac{F_0}{m}e^{i\omega t}$$

$$\tilde{A}\left[(\omega_0^2-\omega^2) + \frac{i}{\tau}\omega\right]e^{i\omega t} = \frac{F_0}{m}e^{i\omega t}$$

Po [illegible]

$$\tilde{A} = \frac{F_0/m}{(\omega_0^2-\omega^2)+i\frac{\omega}{\tau}} =$$

$$= \frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)+(\omega/\tau)^2} - i\frac{F_0}{m}\frac{\omega/\tau}{(\omega_0^2-\omega^2)+(\omega/\tau)^2}$$

$$\tilde{A} = A_a - iA_d$$

$$|\tilde{A}| = \sqrt{A_a^2 + A_d^2} \rightarrow \text{Amplituda}$$

$$\operatorname{tg}\varphi = \frac{A_d}{A_a} = \frac{\omega/\tau}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}}$$

$$\tilde{u}(t) = \tilde{A}e^{i\omega t} = |\tilde{A}|e^{i(\omega t-\varphi)}$$

$$\mathrm{Re}[\tilde{u}(t) = \text{rješenje}]$$