Quantum Computing:

Bra-Ket-Notation:

(i) Ket-Notation $|-\rangle$.

(ii) Bra-notation $\langle -|$,

(iii) Ket-Ket $|-\rangle|-\rangle$.

(iv) Bra-Bra $\langle -|\langle -|$,

(v) Ket Bra $|-\rangle\langle -|$.

(vi) Braket $\langle -|-\rangle$,

Ket-Notation:

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{array}{l} \to 0\text{th} \\ \to 1\text{st} \end{array}$ → two dimentional vector (colm)

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|101\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to 0\text{th} \\ \\ \\ \\ \\ \to 5\text{th} \\ \\ \to 7\text{th} \end{array}$

$2^3 = 8D$

$|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$2^2 = 4D$

Bra-Notation:

$\langle 0| = [1, 0]$, $\langle 1| = [0, 1]$

$|\psi\rangle^\dagger = \langle\psi|$, $\langle\psi|^\dagger = |\psi\rangle$

$\dagger$ = conjugate transpose

$|\psi\rangle = (3+5i)|0\rangle + 7|1\rangle$

$\langle\psi| = |\psi\rangle^\dagger = (3-5i)\langle 0| + 7\langle 1|$

$|\psi\rangle = \begin{pmatrix} 3+5i \\ 7 \end{pmatrix}$

$\langle\psi| = [3+5i \cdot 7]$

2D $\{|0\rangle, |1\rangle\}$ Standard Basis

$\begin{pmatrix} 7 \\ 3+5i \end{pmatrix} = 7|0\rangle + (3+5i)|1\rangle$

4D

$\begin{pmatrix} 7 \\ 0 \\ i+3 \\ 0 \end{pmatrix} = 7|00\rangle + (i+3)|10\rangle$,

Basis = $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$[3\ 0\ i\ 7] = 3\langle 00| + i\langle 10| + 7\langle 11|$

Tensor Product:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1 \cdot B & 0 \cdot B \\ 0 \cdot B & 2 \cdot B \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 8 & 14 \end{pmatrix}$$

ket-ket:

$$|\psi\rangle \otimes |\varphi\rangle = |\psi\rangle|\varphi\rangle$$
$$= |\psi\varphi\rangle$$

Ex: $|\psi\rangle = i|0\rangle + 7|1\rangle$

$|\varphi\rangle = |00\rangle + 3|10\rangle + 7|11\rangle$

$|\psi\rangle|\varphi\rangle = |\psi\varphi\rangle$

$= i|000\rangle + 3i|010\rangle + 7i|011\rangle$

$+ 7|100\rangle + 21|110\rangle + 49|111\rangle$

$$|\psi\rangle = \begin{bmatrix} i \\ 7 \end{bmatrix}, \quad \varphi = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \end{bmatrix}$$

$$|\psi\varphi\rangle = \begin{bmatrix} i \\ 0 \\ 3i \\ 7i \\ 7 \\ 0 \\ 21 \\ 49 \end{bmatrix}$$

Bra-Bra-Notation

Ex:

$\langle\psi| = 3\langle 0| + 7\langle 1|$

$\langle\varphi = \langle 0| + i\langle 1|$

$\langle\psi\varphi| = 3\langle 00| + 3i\langle 01|$

$+ 7\langle 10| + 7i\langle 11|$

Ket - Bra

$$|\alpha\rangle\langle\beta| = |\alpha\rangle\langle\beta|$$

column vector $(m\times 1)$

row vector $(1\times n)$

$= m\times n$

Ex:

$|00\rangle\langle 10|$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$|\alpha\rangle = 3|0\rangle + i|1\rangle$

$|\beta\rangle = |00\rangle + 2|10\rangle + 7|11\rangle$

$|\alpha\rangle\langle\beta| =$

$\langle\beta| = \langle 00| + 2\langle 10| + 7\langle 11|$

$|\alpha\rangle\langle\beta| = (3|0\rangle + i|1\rangle)(\langle 00| + 2\langle 10| + 7\langle 11|)$

$= 3|0\rangle\langle 00| + 6|0\rangle\langle 10| + 21|0\rangle\langle 11| + i|1\rangle\langle 00| + 2i|1\rangle\langle 10| + 7i|1\rangle\langle 11|$

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2 rows $\times$ 4 colom

$$\begin{bmatrix} 3 & 0 & 6 & 21 \\ i & 0 & 2i & 7i \end{bmatrix}$$

Ex:

$$A = \begin{array}{c} \begin{pmatrix} 0 & 1 \\ 3 & i \\ 7 & 0 \\ 0 & 13 \end{pmatrix} \\ \begin{matrix} | & | \\ 0 & 1 \end{matrix} \end{array} \begin{matrix} -0 \\ -1 \\ -2 \\ -3 \\ \\ \\ \end{matrix}$$

$= |00\rangle\langle 1| + 3|01\rangle\langle 0| + i|01\rangle\langle 1| + 7|10\rangle\langle 0| + 13|11\rangle\langle 1|$

Bra-Ket:

row $\langle \alpha | \beta \rangle$ colom

$1 \times n$ $m \times 1$

$n \times m$

$1 \times 1 =$ number

inner product

$\langle 0|0 \rangle = [1, 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= 1 \times 1 + 0 \times 0 = 1$

norm:

$|\alpha\rangle$,

is $\sqrt{\langle \alpha | \alpha \rangle}$

if $\sqrt{\langle \alpha|\alpha \rangle} = 1$

unit vector

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orthogonal

when $\langle \alpha | \beta \rangle = 0$

$\langle 0|1 \rangle = [1, 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= 1 \times 0 + 0 \times 1 = 0$

Ex: $|\alpha\rangle = i|0\rangle + 7|1\rangle$

$|\beta\rangle = 3|0\rangle + |1\rangle$

$\langle \alpha | \beta \rangle = ?$

Solve:

$\langle \alpha | = |\alpha\rangle^{\dagger} = -i\langle 0| + 7\langle 1|$

$\langle \alpha | \beta \rangle = (-i\langle 0| + 7\langle 1|)(3|0\rangle + |1\rangle)$

$= -3i \underbrace{\langle 0|0 \rangle}_{=1} - i \underbrace{\langle 0|1 \rangle}_{=0} + 21 \underbrace{\langle 1|0 \rangle}_{=0} + 7 \underbrace{\langle 1|1 \rangle}_{=1}$

$\boxed{= -3i + 7} \rightarrow$ inner product

Real: $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle$

Complex:

$\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^{\dagger}$

Qubits & Measurments

classical computer:

unit of data
↳ Bit

0 or 1

Quantum computer:

unit of data
↳ qubit

states of qubit:

① Pure state:

0 or 1

② superposition:

0 and 1

prob I   prob 1-P

math Representation:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

α, β are amptitude

$\alpha, \beta \in \mathbb{C}$

Normaligation constraint:

$$|\alpha|^2 + |\beta|^2 = 1$$

prob of $0 = |\alpha|^2$

prob of $1 = |\beta|^2$

$$|\alpha|^2 = \alpha^{+} \cdot \alpha$$

$$|\beta|^2 = \beta^{+} \cdot \beta$$

$$|\psi\rangle = \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \leftarrow \text{Unit vector}$$

$$\| |\psi\rangle \| = 1$$

or

$$\langle\psi|\psi\rangle = 1$$

Ex:

$|\psi\rangle = -\frac{4}{5}i|0\rangle + \frac{3}{5}|1\rangle$

ⓐ if $|\psi\rangle$ a valid qubit

ⓑ what is the prob of 0

" " " " of 1

Sol:

Part a:

⊛ $|\alpha|^2 + |\beta|^2 = 1$

⊛ $\||\psi\rangle\| = 1$

⊛ $\langle\psi|\psi\rangle = 1$

$\left|-\frac{4}{5}i\right|^2 + \left|\frac{3}{5}\right|^2 = 1$

$= \left(\frac{4}{5}i \times -\frac{4}{5}i\right) + \left(\frac{3}{5} \times \frac{3}{5}\right)$

$= \frac{16}{25} + \frac{9}{25} = 1$

$\boxed{\text{Proved is validation}}$

Part 2:

Prob of 0 $= \left|-\frac{4}{5}i\right|^2$

$= \frac{16}{25} = 0.64$

Prob of 1 $= \left|\frac{3}{5}\right|^2$

$= \frac{9}{25} = 0.36$

$|\psi\rangle = -\frac{4}{5}i|0\rangle + \frac{3}{5}|1\rangle$

$|\varphi\rangle = \frac{|0\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}}$

$\boxed{|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1}$

$|\psi\rangle \otimes |\varphi\rangle$

$= |\psi\varphi\rangle$

$= |\psi\varphi\rangle = \left(-\frac{4}{5}i|0\rangle + \frac{3}{5}|1\rangle\right) \otimes \left(\frac{|0\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}}\right)$

$= \underbrace{-\frac{4}{5\sqrt{2}}}_{\alpha}|00\rangle - \underbrace{\frac{4}{5\sqrt{2}}}_{\beta}|01\rangle + \underbrace{\frac{3}{5\sqrt{2}}}_{\gamma}|10\rangle + \underbrace{\frac{3}{5\sqrt{2}}}_{\delta}|11\rangle$

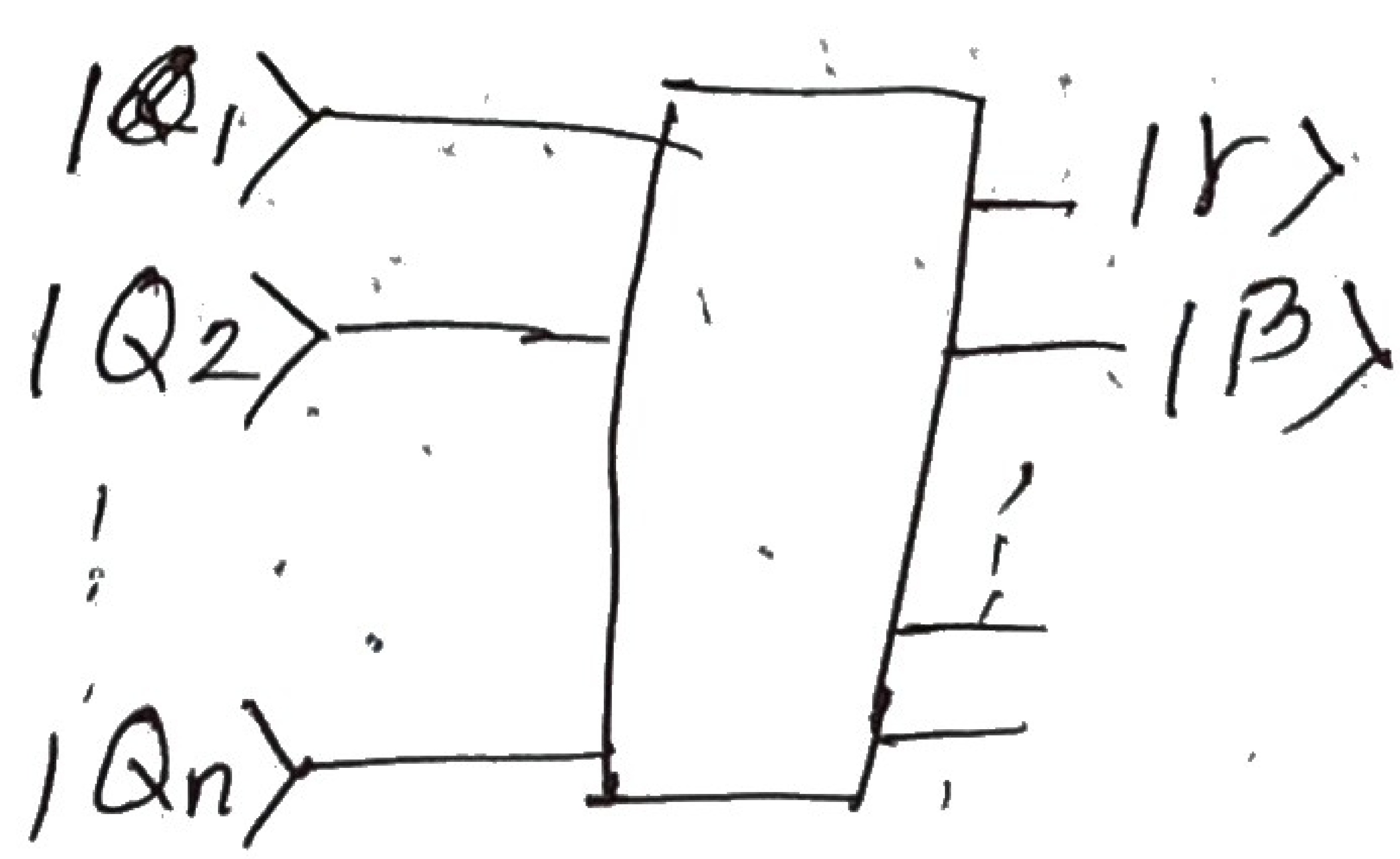
Prob of 00:

$$= |\alpha|^2 = \left|-\frac{4}{5\sqrt{2}}i\right|^2$$

$$= \frac{16}{50} = 0.32$$

operations on qubits:

- Measure
  - Full
  - Partial
- Transform qubits using quantum gate



Qubits Measurement:

Rule:

(I) Superposition: → Pure state

(II) Normalization constraint must always upheld.

Ex: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measurements	Prob	Resultant state
00	$\left\lvert\frac{1}{2}\right\rvert^2 = \frac{1}{4} = 0.25$	$\lvert\psi\rangle = \lvert 00\rangle$
10	$\left\lvert-\frac{i}{2}\right\rvert^2 = \frac{i}{2} \times \frac{-i}{2} = \frac{1}{4} = 0.25$	$\lvert\psi\rangle = \lvert 10\rangle$
11	$\left\lvert\frac{1}{\sqrt{2}}\right\rvert^2 = \frac{1}{2} = 0.5$	$\lvert\psi\rangle = \lvert 11\rangle$

$P(A \cup B) = P(A) + P(B) - \cancel{P(A \cap B)}$

Measurements	Probability	Resultant state
1st qubit = 1	$\left\lvert-\frac{i}{2}\right\rvert^2 + \left\lvert\frac{1}{\sqrt{2}}\right\rvert^2$ $= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$	$\lvert\psi\rangle = \dfrac{-\frac{i}{2}\lvert 10\rangle + \frac{1}{\sqrt{2}}\lvert 11\rangle}{\left\lVert -\frac{i}{2}\lvert 10\rangle + \frac{1}{\sqrt{2}}\lvert 11\rangle \right\rVert}$ $= \dfrac{-\frac{i}{2}\lvert 10\rangle + \frac{1}{2}\lvert 11\rangle}{\sqrt{\frac{3}{4}}}$ $= \frac{-i}{\sqrt{3}}\lvert 10\rangle + \frac{\sqrt{2}}{\sqrt{3}}\lvert 11\rangle$
2nd qubit = 0	$\left\lvert\frac{1}{2}\right\rvert^2 + \left\lvert-\frac{1}{2}\right\rvert^2$ $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$	$\lvert\psi\rangle = \frac{1}{2}\lvert 00\rangle - \frac{1}{2}\lvert 10\rangle$ $= \dfrac{\frac{1}{2}\lvert 00\rangle - \frac{1}{2}\lvert 10\rangle}{\frac{1}{\sqrt{2}}}$ $= \frac{1}{\sqrt{2}}\lvert 00\rangle - \frac{1}{\sqrt{2}}\lvert 10\rangle$

Ex: $\frac{1}{\sqrt{5}}|000\rangle - \sqrt{\frac{2}{5}}|0100\rangle + \sqrt{\frac{1}{5}}|1111\rangle + \sqrt{\frac{1}{5}}|0110\rangle$,

what is the prob and resultant state if 1st and 4th qubits are $= 0$, $P(A \cup B \cup C) = P(A) + P(B) + P(C)$,

Solution:

Prob: $\left|\frac{1}{\sqrt{5}}\right|^2 + \left|-\frac{\sqrt{2}}{5}\right|^2 + \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{4}{5}$

Resultant state: $|\psi\rangle = \dfrac{\frac{1}{\sqrt{5}}|0000\rangle - \sqrt{\frac{2}{5}}|0100\rangle + \frac{1}{\sqrt{5}}|0110\rangle}{\frac{2}{\sqrt{5}}}$

$= \frac{1}{2}|0000\rangle - \frac{1}{\sqrt{2}}|0100\rangle + \frac{1}{2}|0110\rangle$,

Measuring in orthonormal Basis:

(i) Standard Basis:
(ii) Orthonormal Basis:
(iii) Formula:
(iv) Examples:

Formula:

$$\{|\alpha\rangle, |\beta\rangle\}$$

$$|\psi\rangle = \langle\psi|\alpha\rangle |\alpha\rangle + \langle\psi|\beta\rangle |\beta\rangle$$

amplitude

Prob of measuring $|\alpha\rangle = |\langle\psi|\alpha\rangle|^2$

Prob of measring $|\beta\rangle = |\langle\psi|\alpha\rangle|^2$

Ex:

$$|\psi\rangle = |0\rangle$$

Standard Basis:

prob of measuring

$0 = |1|^2 = 1$

prob of measuring

$1 = 0$

Hadamard basis:

$$\{|+\rangle, |-\rangle\}$$

Standard Basis:

$$2D = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= \{|0\rangle, |1\rangle\}$$

$$3D = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$4D = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

Orthonormal Basis:

① Normalized "Unit vector",
② orthonormal "$\perp$",

$$\{|+\rangle, |-\rangle\} \quad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Unit vector:

$$\| |+\rangle \| = \sqrt{\langle + | + \rangle} = 1$$

$$= \sqrt{\left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)}$$

$$\sqrt{\frac{1}{2}\left[\underbrace{\langle 0|0\rangle}_{1} + \underbrace{\langle 0|1\rangle}_{0} + \underbrace{\langle 1|0\rangle}_{0} + \underbrace{\langle 1|1\rangle}_{1}\right]}$$

$$\sqrt{\frac{1}{2} \times 2} = \sqrt{1} = 1$$

Proved "unit vector"

$$\| |-\rangle \| = \sqrt{\langle - | - \rangle} = 1$$

Orthogonal:

$\langle +|0\rangle = \left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right)\cdot\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

$= \frac{1}{2}\left[\underset{1}{\langle 0|0\rangle} - \underset{0}{\langle 0|1\rangle} + \underset{0}{\langle 1|0\rangle} - \underset{1}{\langle 1|1\rangle}\right]$

$= \frac{1}{\cancel{2}} \times \cancel{2}$

$= 1$

Prob of m $|+\rangle$ $\qquad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$= |\langle\psi|+\rangle|^2 = \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$= \langle\psi|+\rangle = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

$= (\langle 0|)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}}$

$\boxed{= |\langle\psi|+\rangle|^2 = \frac{1}{2}}$

Prob m $|-\rangle = 1 - \frac{1}{2}$

$= \frac{1}{2}$

Ex: $|\varphi\rangle = \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\right)|+\rangle + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right)|-\rangle$

Sol: Hadamard Basis:

Prob of measuring $|+\rangle = \left|\left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\right)\right|^2$

$= \frac{1}{6} + \frac{1}{3} - \frac{2}{\sqrt{6}\sqrt{3}}$

$= \frac{1}{6} + \frac{1}{3} - \frac{\sqrt{2}}{3}$

$= \frac{1+2-2\sqrt{2}}{6} = \frac{3-2\sqrt{2}}{6}$

Prob of measuring $|-\rangle$

$$= 1 - \frac{3-2\sqrt{2}}{6} = \frac{3+2\sqrt{2}}{6}$$

$$\langle +|0\rangle = \frac{\langle 0| + \langle 1|}{\sqrt{2}} |0\rangle$$

$$= \frac{1}{\sqrt{2}}$$

Standard Basis:

$$\langle -|0\rangle = \frac{1}{\sqrt{2}}$$

$\{|0\rangle, |1\rangle\}$

Prob measuring $|0\rangle = |\langle\psi|0\rangle|^2$

$$\langle\psi|0\rangle = \left[\left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\right)\langle +| + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right)\langle -|\right]|0\rangle$$

$$= \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\right)\langle +|0\rangle + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right)\langle -|0\rangle$$

$$= \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{6}}$$

$$= \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$$

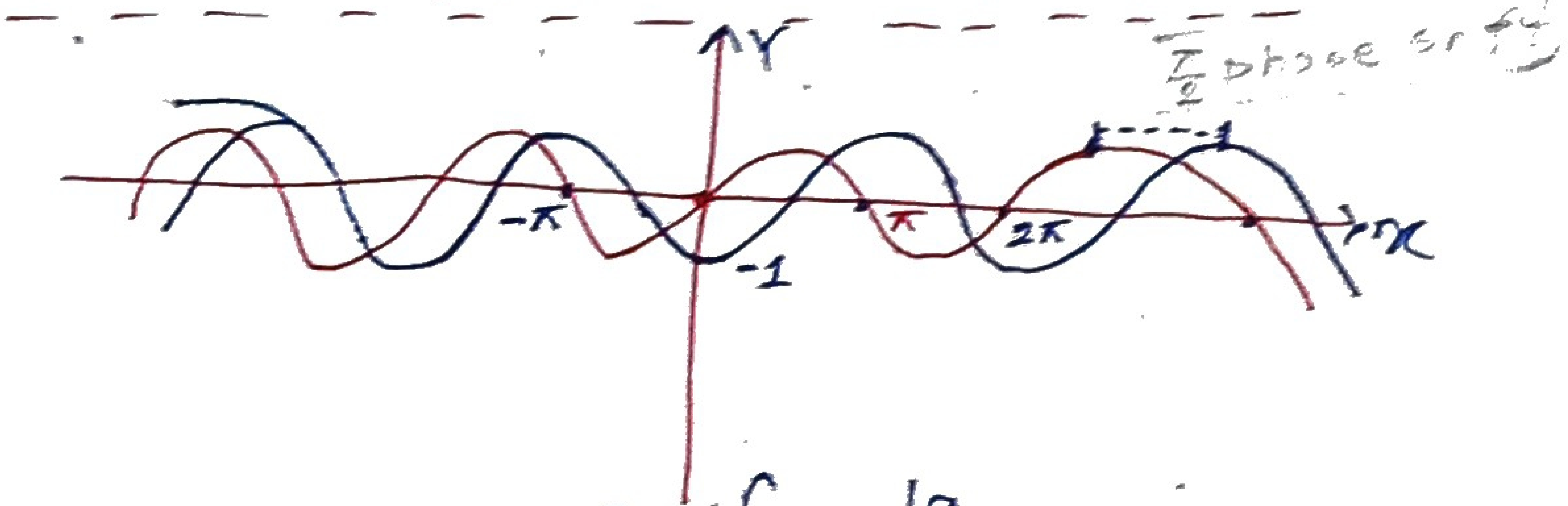
$$\langle\psi|0\rangle = \frac{1}{3}$$

Prob measuring $|-\rangle$

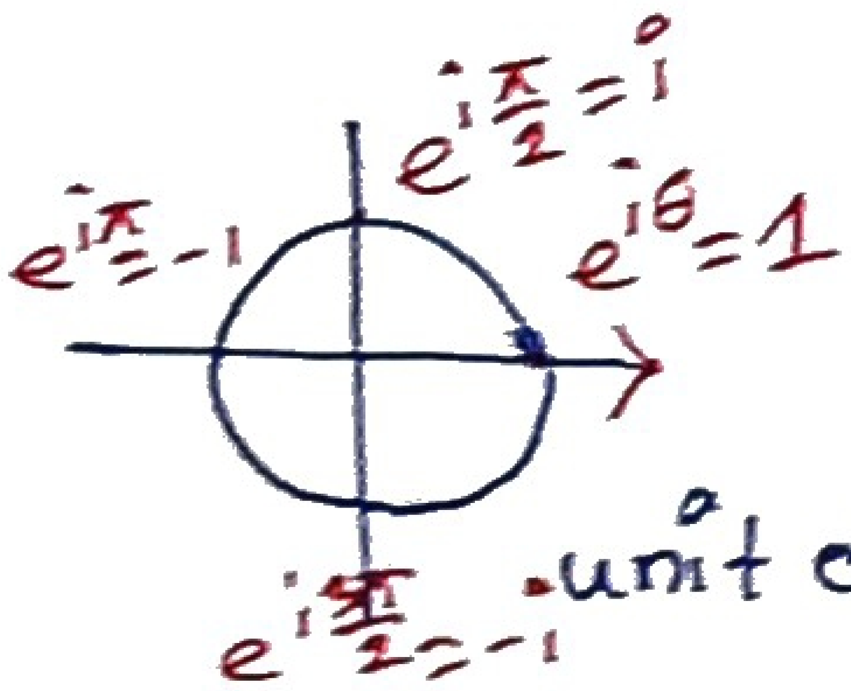
What is phase

1. phase shift in waves
2. Quantum Global phase
3. Proof: Global phase is not (m)
4. Quantum Relative phase
5. Relative phase is (m)

Phase shift in wave functions:



sin function



Quantum global phase shift:

$|\psi\rangle$ = n-qubit register

$e^{i\theta}|\psi\rangle$ = same register with global phase shift of θ

Exp: $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

global phase shift of $\pi/2$ (180°)

$$e^{i\frac{\pi}{2}}|\psi\rangle = \frac{e^{i\pi/2}}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/2}}{\sqrt{2}}|1\rangle$$

$$e^{i\pi/2}|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

$-1|\psi\rangle$ = phase shift of π rad (180°)

Global phase is not distinguishable when (m)

Proof: $|\psi\rangle$ = n-qubit register

$\{\beta_1, \beta_2, ---, \beta_n\}$ = Orthonormal Basis

Prob of (m) $\beta_i = (|\psi\rangle)^{+} m_{\beta_i} |\psi\rangle = \langle\psi| m_{\beta_i} |\psi\rangle$ --- (1)

$e^{i\theta}|\psi\rangle$ = same register with θ global phase shift

Prob of (m) $\beta_i = (e^{i\theta}|\psi\rangle)^{+} m_{\beta_i} (e^{i\theta}|\psi\rangle)$

$= \langle\psi| e^{-i\theta} m_{\beta_i} e^{i\theta} |\psi\rangle$

$= \cancel{e^{-i\theta}} \cdot \cancel{e^{i\theta}} \langle\psi| m_{\beta_i} |\psi\rangle$ --- (2)

= (1) = (2)

Projection operation:

Quantum Relative phase:

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

In $|\psi\rangle$ & $|\varphi\rangle$:

$|0\rangle$ = Same amplitude

$|1\rangle$ = Same magnitude as $\frac{1}{2}$

= Diff relative phase (π red)

Quantum Gates

$U|\alpha\rangle = |\beta\rangle$ ← transform qubit

unitary matrix (U), qubit ($|\alpha\rangle$)

⊛ $U^{+}U = UU^{+} = I$ ← identity

$U^{+} = U^{-1}$

$A^{-1} = A$

⊛ $H^{+} = H$ ← Harmition.

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y^{+} = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

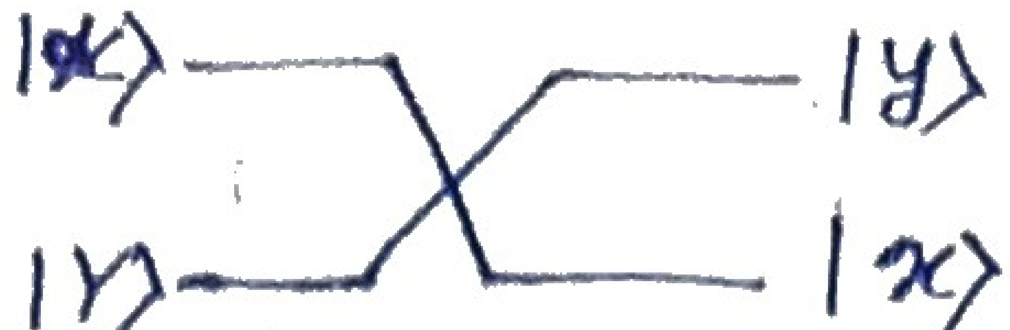
$$Y^{+} = Y$$

$$Y^{+}Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -i \times i & 0 \\ 0 & -i \times i \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

GATE	MATRIX	Examples
$-\boxed{X}-$ Pauli-X U&H	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$X\|0\rangle = \|1\rangle$ $X\|1\rangle = \|0\rangle$ $X\, \alpha\|0\rangle + \beta\|1\rangle$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \beta\|0\rangle + \alpha\|1\rangle$ $= X(\alpha\|0\rangle + \beta\|1\rangle)$ $= \alpha X\|0\rangle + \beta X\|1\rangle$ $= \alpha\|1\rangle + \beta\|0\rangle$
$-\boxed{Z}-$ U&H	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$Z\|0\rangle = \|0\rangle$ $Z\|1\rangle = -\|1\rangle$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\|1\rangle$ $\cancel{X}(\alpha\|0\rangle + \beta\|1\rangle)$ $= \alpha\|0\rangle - \beta\|1\rangle$
$-\boxed{H}-$ U&H Hadamard Gate	$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ $= \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $= \|0\rangle$	$H\|0\rangle = \frac{\|0\rangle + \|1\rangle}{\sqrt{2}} = \|+\rangle$ $H\|1\rangle = \frac{\|0\rangle - \|1\rangle}{\sqrt{2}} = \|-\rangle$ $H\|+\rangle = \|0\rangle, \quad H\|-\rangle = \|1\rangle$
$-\boxed{R_\theta}-$ U but $\cancel{H}$	$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	$R_\theta \neq R_\theta^+$ $R_\theta^+ \cdot R_\theta = 1$



Swap গেট

Swap =

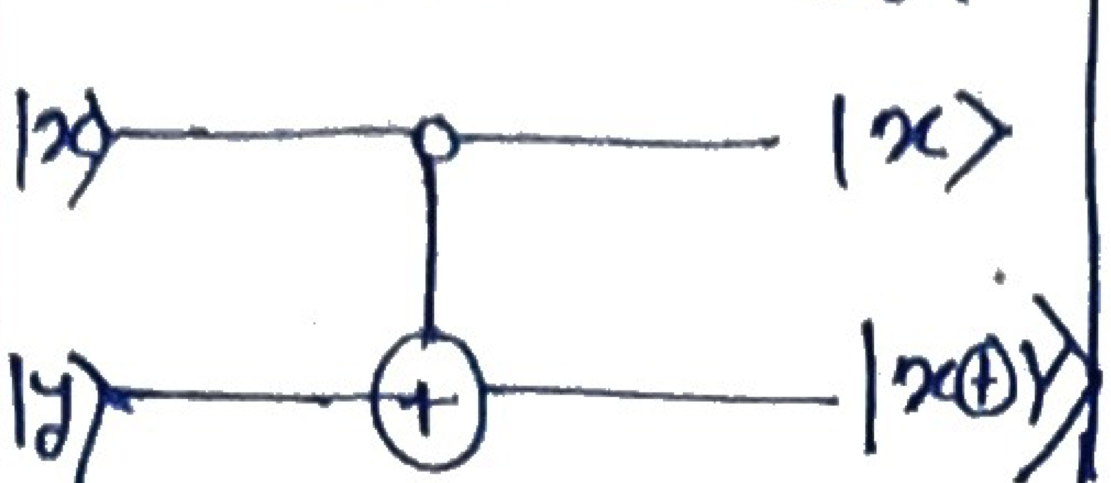
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Swap $|01\rangle = |10\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$= |10\rangle$

CNOT গেট



CNOT =

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CNOT $|10\rangle = |11\rangle$ 2nd bit

CNOT $|01\rangle = |01\rangle$

CU =

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\boxed{U}} \\ 0 & 0 & & \end{pmatrix}$$

C(H) =

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

CZ =

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Hadamard

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Equal superposition

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$H|\psi\rangle = |\psi\rangle'$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1+0 \\ 1+0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

Unitary:

$$HH^{+} = I$$

Hermition:

$H = H^{+}$ ← conjugate trasform.

$$H^{-1} = H$$

$$H\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{H|0\rangle}{\sqrt{2}} - \frac{H|1\rangle}{\sqrt{2}}$$

$$= \frac{\frac{|0\rangle + |1\rangle}{\cancel{\sqrt{2}}}}{\cancel{\sqrt{2}}\,2} - \frac{|0\rangle - |1\rangle}{2}$$

$$= \frac{2|1\rangle}{2} = |1\rangle$$

Hadamard:

$H^{\oplus 2}|00\rangle$

matrix

$H^{\oplus 2}|00\rangle$

$H^{\oplus 2} = H \otimes H$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$H^{\otimes 2}|00\rangle = H^{\otimes 2}[|0\rangle|0\rangle]$$
$$= H|0\rangle H|0\rangle$$
$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \cdot \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$
$$= \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

General Expression

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

$$H^{\otimes n}|0^n\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=1}^{n-1} |y\rangle$$

$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$H|x\rangle = \frac{|0\rangle + (-1)^x|1\rangle}{\sqrt{2}}, \quad x \in \{0,1\}$$

$(-1)^0 = 1 \quad (-1)^1 = 1$

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} (-1)^{x \cdot y} |y\rangle \quad \text{—(1)}$$

$y \in \{0,1\}$

$$H^{\otimes 2}|x\rangle = H|x_1\rangle \otimes H|x_2\rangle$$

$x \in \{0,1\}^2$

$x \in \{00, 01, 10, 11\}$

$x \in \{0, 1, 2, 3\}$

$$= \left(\frac{1}{\sqrt{2}} \sum_{y_1 \in \{0,1\}} (-1)^{x_1 \cdot y_1} |y_1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{y_2 \in \{0,1\}} (-1)^{x_2 \cdot y_2} |y_2\rangle\right)$$

$$= \frac{1}{\sqrt{2^2}} \sum_{y \in \{0,1\}^2} (-1)^{x_1 y_1 + x_2 y_2} |y\rangle$$

Tabular Method

$$H^{\otimes 3}\left(\frac{|000\rangle + |110\rangle - |100\rangle - |111\rangle}{\sqrt{4}}\right)$$

	000	001	010	011	100	101	110	111
000	+	+	+	+	+	+	+	+
110	+	+					+	+
−100	−	+	−	−	+	+	+	+
−111	−	+	+	−	+	−	−	+

$$= \frac{1}{4} \cdot \frac{1}{\sqrt{2^2}}\left(2|001\rangle - 2|011\rangle + 2|100\rangle\ 2|110\rangle + 4|111\rangle\right)$$

CONTROLLED NOT

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{4\times4} \leftarrow \text{matrix}$$

Unitary:

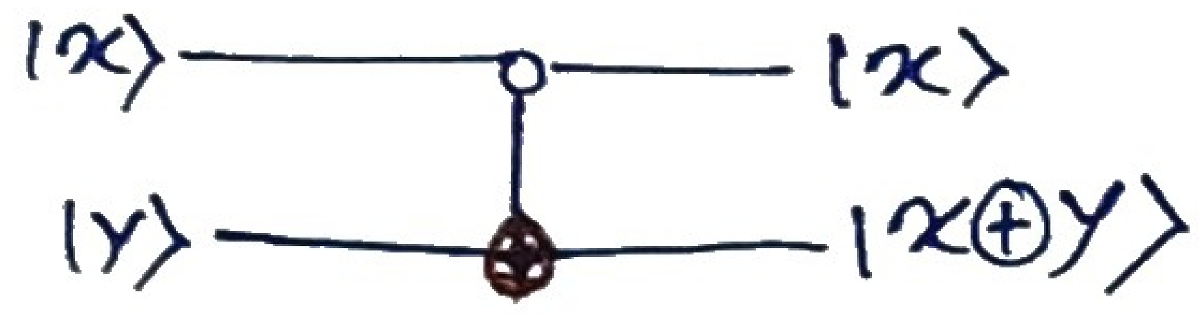
$CNOT \times CNOT^{\dagger} = I$

$CNOT^{\dagger} = CNOT^{-1}$

Hermition:

$CNOT = CNOT^{\dagger}$

$\boxed{CNOT(CNOT|x\rangle) = |x\rangle}$ $\quad CNOT^{-1} = CNOT$



1-qubit $= 2^1 \times 2^1$
2-qubit $= 2^2 \times 2^2$
n-qubit $= 2^n \times 2^n$

$CNOT|00\rangle = |00\rangle$
$CNOT|10\rangle = |11\rangle$
$CNOT|01\rangle = |10\rangle$
$CNOT|11\rangle = |10\rangle$

$$CNOT|11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

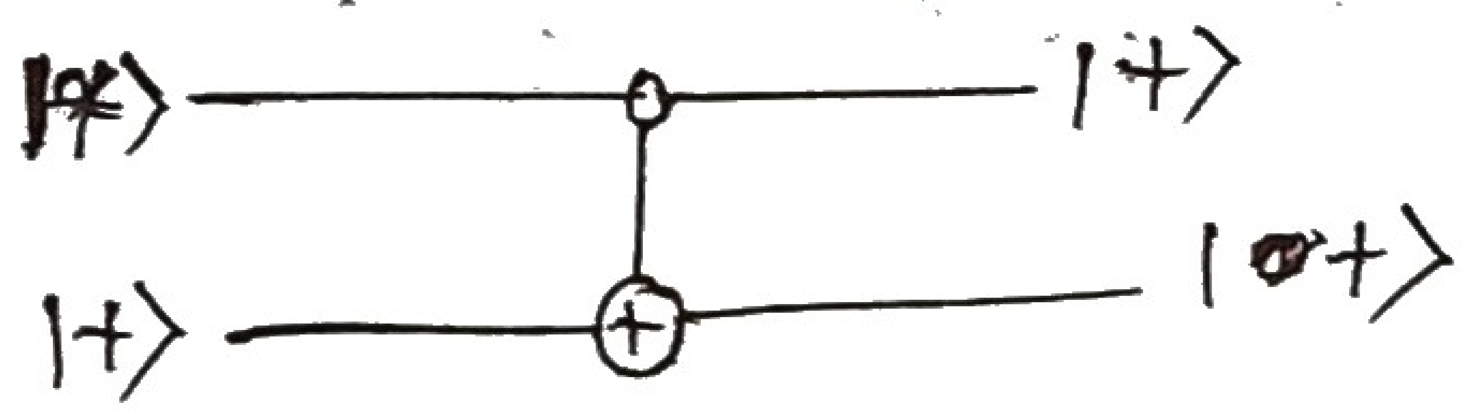
$= |10\rangle$

2

$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$

$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$

$CNOT|++\rangle = |++\rangle$
$CNOT|-+\rangle = |-+\rangle$
$CNOT|--\rangle = |+-\rangle$
$CNOT|+-\rangle = |--\rangle$



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$|--\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \cdot \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

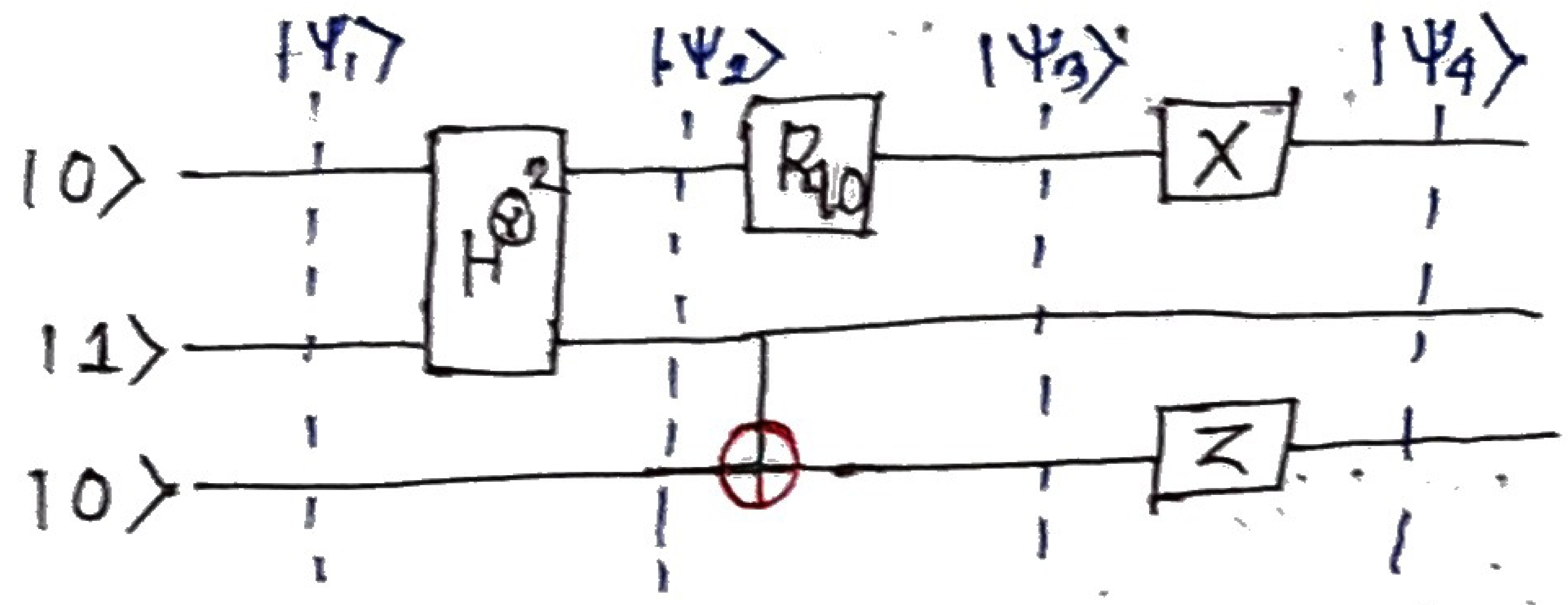
$= \frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ \ldots \end{pmatrix}$

$$|+-\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \cdot \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$= \frac{|00\rangle-|01\rangle+|10\rangle-|11\rangle}{2}$$

Quantum Circuits



(a) Make reverse circuit.

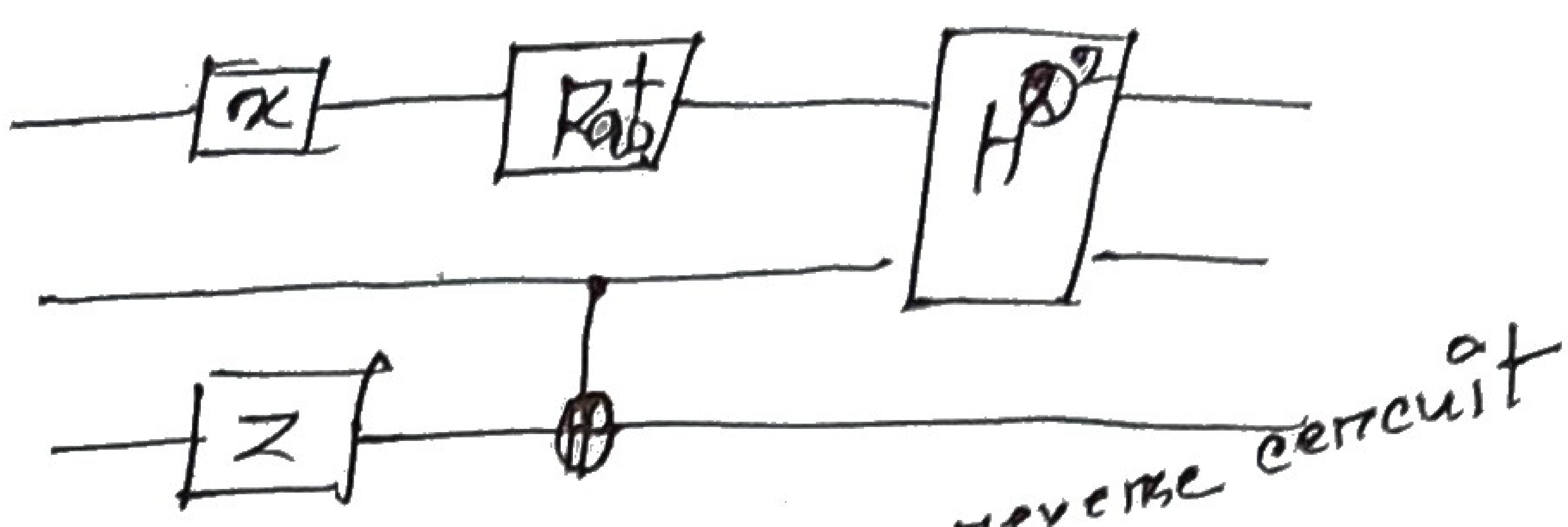
(b) Write output of the circuit.

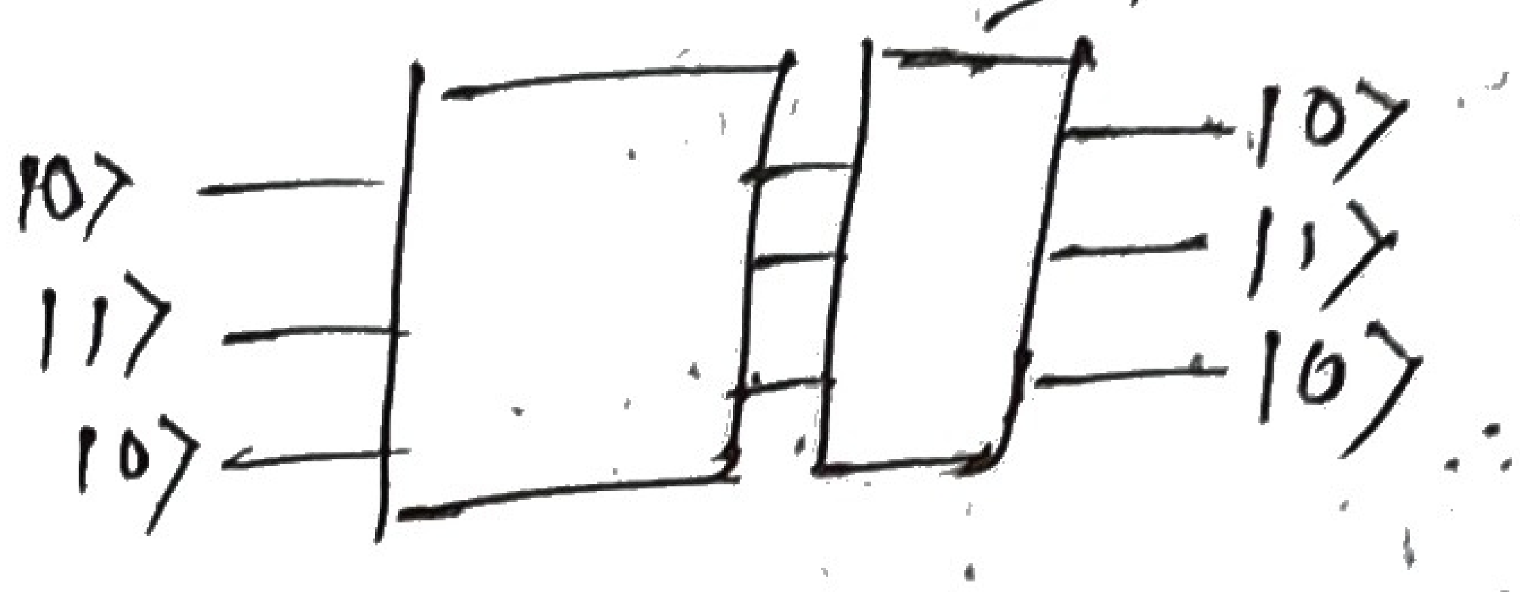
(c) Write complete circuit as unitary matrix.

(d) Write reverse circuit as unitary matrix.

(e) Compute output using unitary matrix of circuit.

(a)





(b) dirac →

$|\psi_1\rangle = |0\rangle|1\rangle|0\rangle$

$|\psi_2\rangle = H^{\otimes 2}(|0\rangle|1\rangle)|0\rangle$

$= \frac{|0\rangle+|1\rangle}{\sqrt{2}} \cdot \frac{|0\rangle-|1\rangle}{\sqrt{2}} |0\rangle$

$= \frac{|00\rangle-|01\rangle+|10\rangle-|11\rangle}{2} \cdot |0\rangle$

$= \frac{|000\rangle-|010\rangle+|100\rangle-|110\rangle}{2}$

$|\psi_3\rangle = R_{90}|\psi_2\rangle$

$= \begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} |\psi_2\rangle$

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

$$|\Psi_3\rangle = |\Psi_2\rangle R_{90}$$

$$= \frac{|100\rangle - |110\rangle - |000\rangle + |010\rangle}{2}$$

$$|\Psi_3\rangle = \frac{|100\rangle - |111\rangle - |000\rangle + |011\rangle}{2}$$

$$|\Psi_4\rangle = \frac{|000\rangle + |011\rangle - |100\rangle - |111\rangle}{2}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

(c) $S_1 = H \otimes H \otimes I_{8\times 8}$

$S_2 = R_{90} \otimes CNOT_{8\times 8}$

$S_3 = X \otimes I \otimes Z_{8\times 8}$

$S_1 \times S_2 \times S_3$

$S_3 \times S_2 \times S_1$

$$\boxed{S_3(S_2(S_1|010\rangle))}$$

$S_1)$ $H \otimes H \otimes I$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0 & \cdots\\ 0 & 1 & \cdots\\ \vdots & & \ddots\end{pmatrix}$$

$$S_2 = R_{90} \otimes CNOT$$

$$= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

$$= \begin{pmatrix}0 & -1\\ & 0\end{pmatrix}$$ → calculation own our

$$S_3 = X \otimes I \otimes Z$$

$$= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \begin{pmatrix} \cdots \end{pmatrix}$$ → calculation

$$S_3 \times S_2 \times S_1 = \frac{1}{2}$$

Entanglement

① Given a state of multiple enttengled qubit, one cannot express individual qubit seperatly.

Ex: Entangled

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

Seperable

$$\frac{|01\rangle + |00\rangle}{\sqrt{2}} = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

② Given a state of multiple entengled qubit, measuring any qubit individually reveals all other qubits.

*Seperable

$$\frac{|01\rangle + |00\rangle}{\sqrt{2}}$$

1st $|0\rangle$

2nd.

$|0\rangle$ --- $\frac{1}{2}$

$|1\rangle$ --- $\frac{1}{2}$

Ex: Entangled

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

1st qubit → m

1/0 → 0   2nd

→ 1

2nd m

1/0 → 0   1st

→ 1

13

0.006873
6.873×10^{-3}

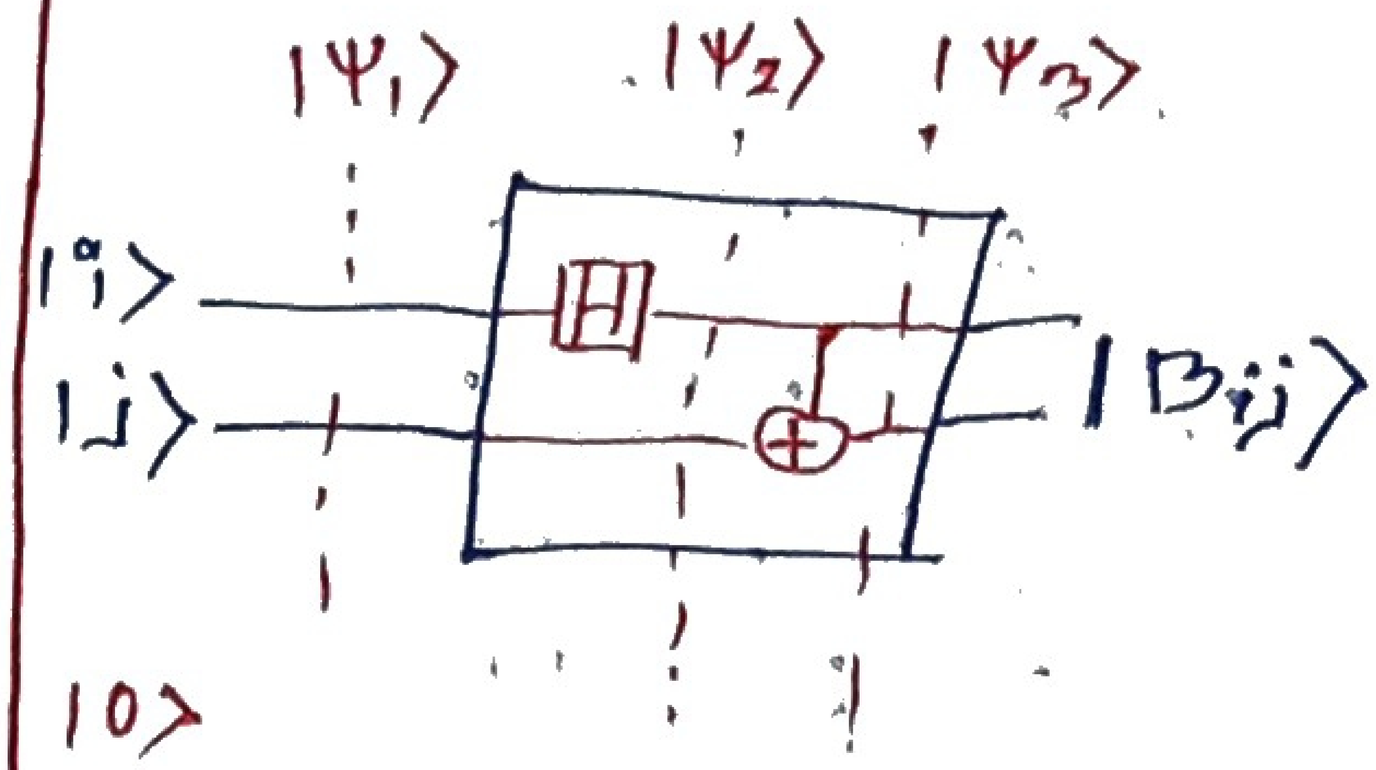
Bell states (EPR state):

$$|\phi^+\rangle = |\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\phi^-\rangle = |\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\psi^+\rangle = |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\psi^-\rangle = |\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$



$|0\rangle$

$|1\rangle$

$$|\psi_1\rangle = |0\rangle|1\rangle = |01\rangle$$

$$|\psi_2\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|1\rangle$$

$$= \frac{|01\rangle + |11\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = |\beta_{01}\rangle$$

Given $|w\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

Find if $|w\rangle$ is entangled or separable?

Proof: PBC

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$\frac{|01\rangle}{\sqrt{2}} - \frac{|10\rangle}{\sqrt{2}} = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

$\alpha\gamma = 0$

$\alpha\delta = \frac{1}{\sqrt{2}}$

$\beta\gamma = -\frac{1}{\sqrt{2}}$

$\beta\delta = 0$ either $\beta = 0$ or $\delta = 0$

$\beta \neq 0$

$\delta \neq 0$

LHS $\neq$ R.H.S

$|w\rangle$ is entangled.