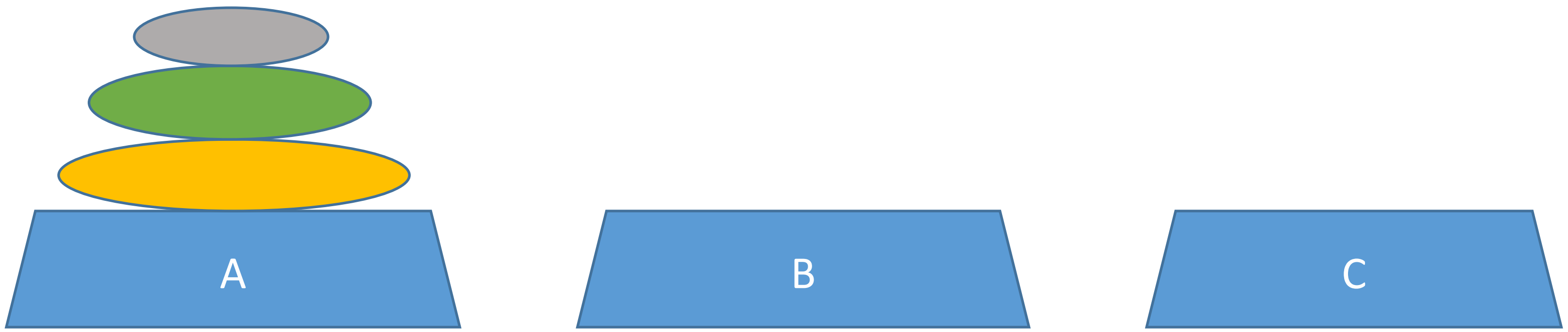
Chapter 1: Recurrent Problems

Tower of Hanoi

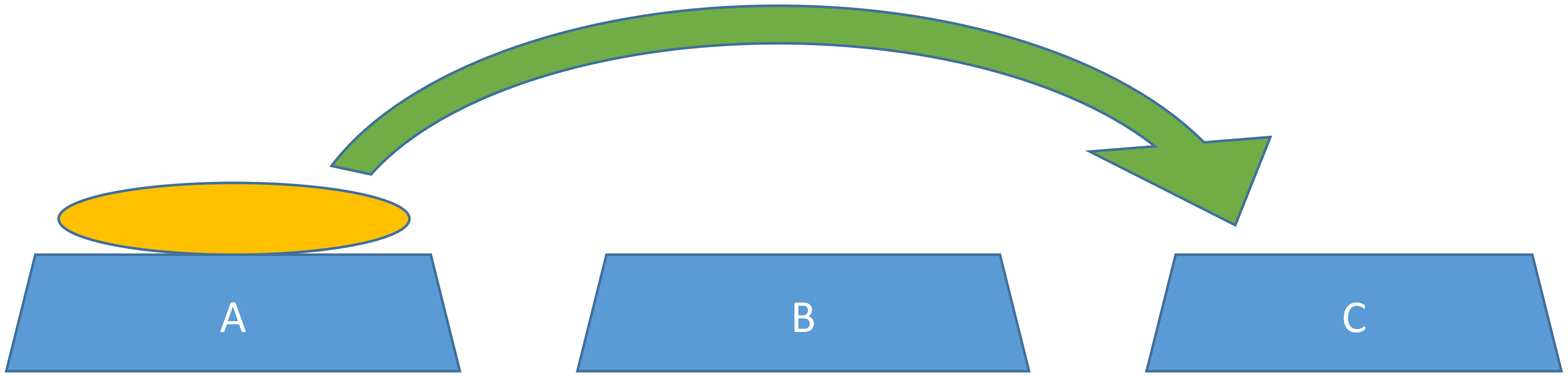
Rules of Game

1. Only top disc can be moved
2. One disk can be moved at a time
3. Larger disk can not be placed on smaller disk

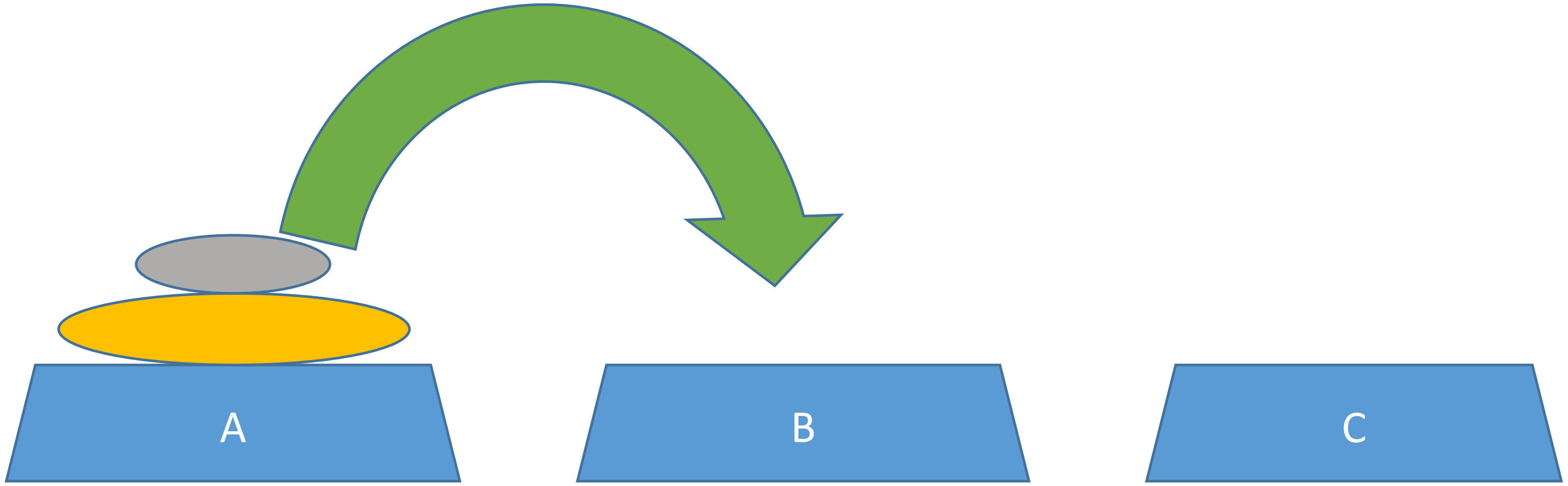


Example with n=1

Disk No.	Move
1	A -> C

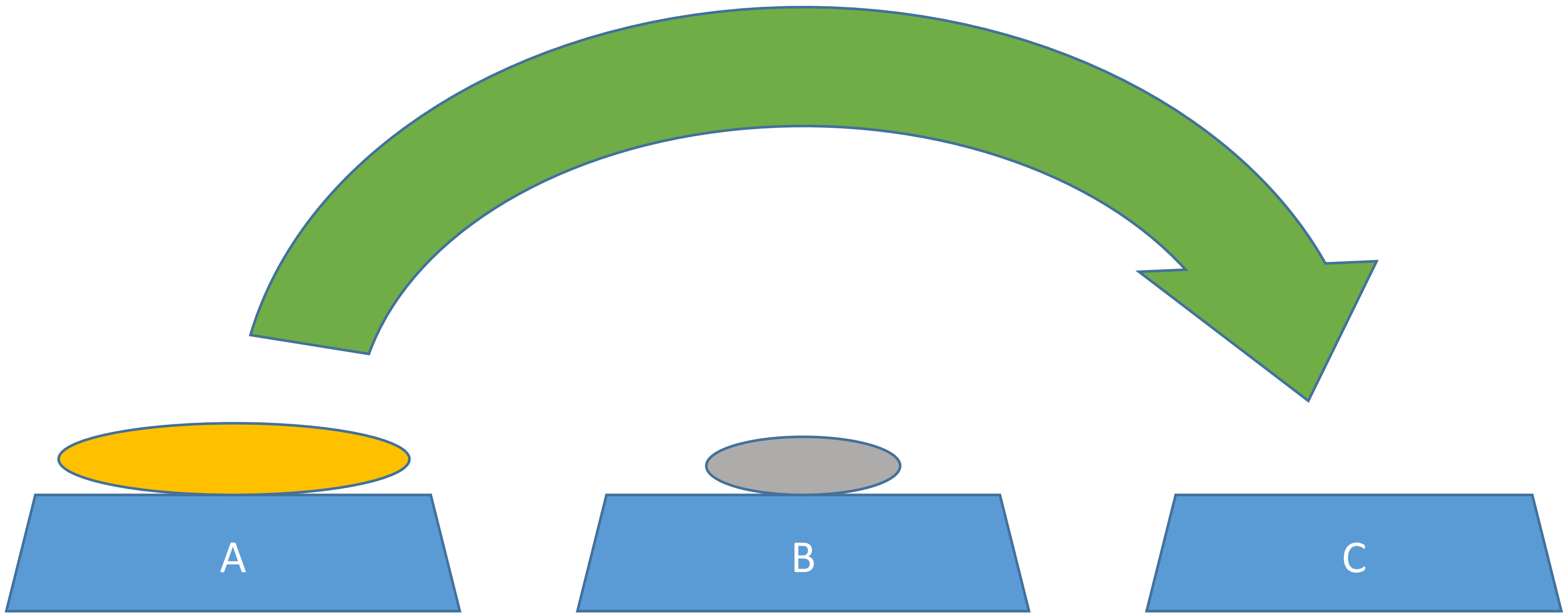


Example with n=2



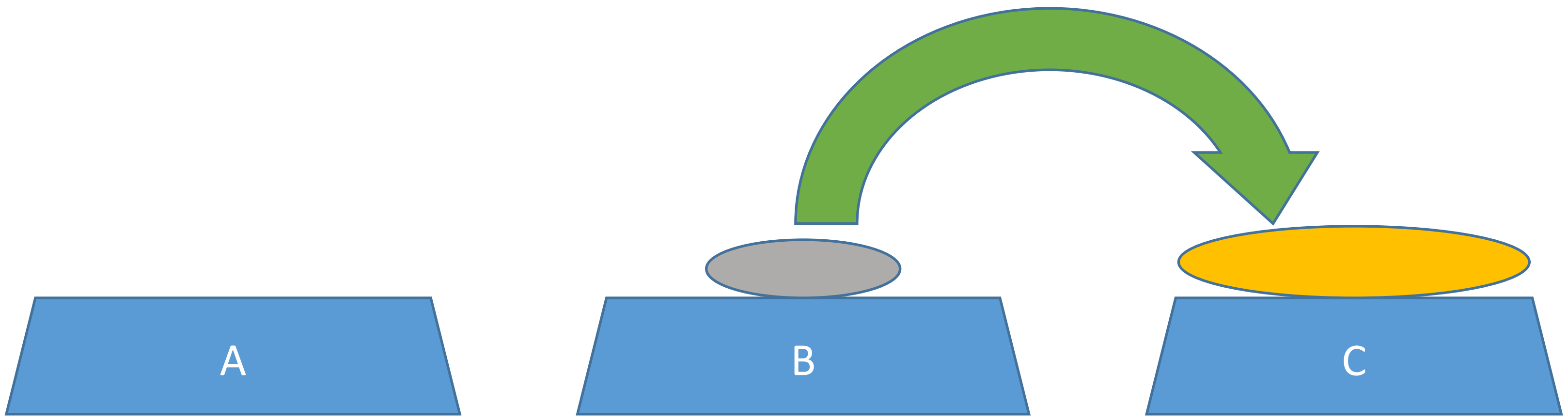
Example with n=2

Disk No.	Move
1	A -> B



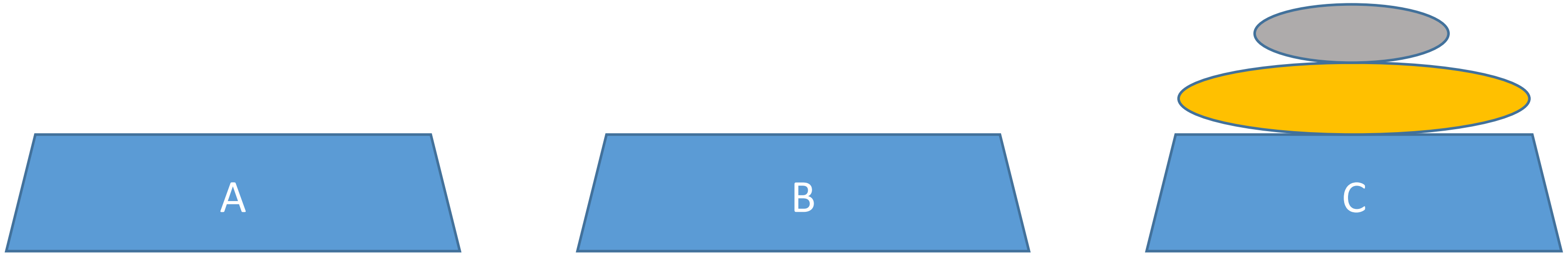
Example with n=2

Disk No.	Move
1	A -> B
2	A -> C



Example with n=2

Disk No.	Move
1	A -> B
2	A -> C
1	B ->C



Generalization

- Total number of moves for 2 disks, $T_2 = 3$
- Total number of moves for 1 disks, $T_1 = 1$
- Total number of moves for 0 disks, $T_0 = 0$
- …
- …
- …
- Total number of moves for n disks, $T_n = ?$

Steps

- Transfer (n-1) smallest disks to the auxiliary peg

  ↓

  T_{n-1} moves

- Transfer the largest disk to the destination peg → 1 move

- Transfer (n-1) smallest disks back onto the largest disk

  ↓

  T_{n-1} moves

Recurrence Formula

- $T_n <= 2 * T_{n-1} + 1$; for $n > 0$
- $T_n >= 2 * T_{n-1} + 1$; for $n > 0$
- Recurrence solution:

$$\left.\begin{aligned} T_0 &= 0 \\ T_n &= 2 * T_{n-1} + 1 \text{ ; for } n > 0 \end{aligned}\right\} \text{.................. (1)}$$

Solve of Recurrence: Guessing

- $T_0 = 0$ $= 2^0 - 1$
- $T_1 = 2 * T_0 + 1 = 2 * 0 + 1 = 1$ $= 2^1 – 1$
- $T_2 = 2 * T_1 + 1 = 2 * 1 + 1 = 3$ $= 2^2 – 1$
- $T_3 = 2 * T_2 + 1 = 2 * 3 + 1 = 7$ $= 2^3 – 1$
- …
- …
- …
- $T_n = 2 * T_{n-1} + 1$ $= 2^n – 1$ … … … … …(2)

Mathematical Proof: Induction

- Trivial Basis:
  $T_0 = 2^0 - 1 = 0$
- Suppose equation (2) holds for (n-1). So,
  $T_{n-1} = 2^{n-1} - 1$ holds
- Now,
  $$\begin{aligned} T_n &= 2 * T_{n-1} + 1 \\ &= 2 * (2^{n-1} - 1) + 1 \\ &= 2^n - 2 + 1 \\ &= 2^n - 1 \end{aligned}$$
- Hence equation (2) holds for n as well

Solve of Recurrence: Without Inductive Leap

- Recurrence solution:

$$\left.\begin{aligned} T_0 &= 0 \\ T_n &= 2 * T_{n-1} + 1 \; ; \text{for } n > 0 \end{aligned}\right\} \text{.................. (1)}$$

$$T_0 + 1 = 1$$

$$T_n + 1 = 2 * T_{n-1} + 2 \; ; \text{for } n > 0$$

Now, if we let, $U_n = T_n + 1$, then we have

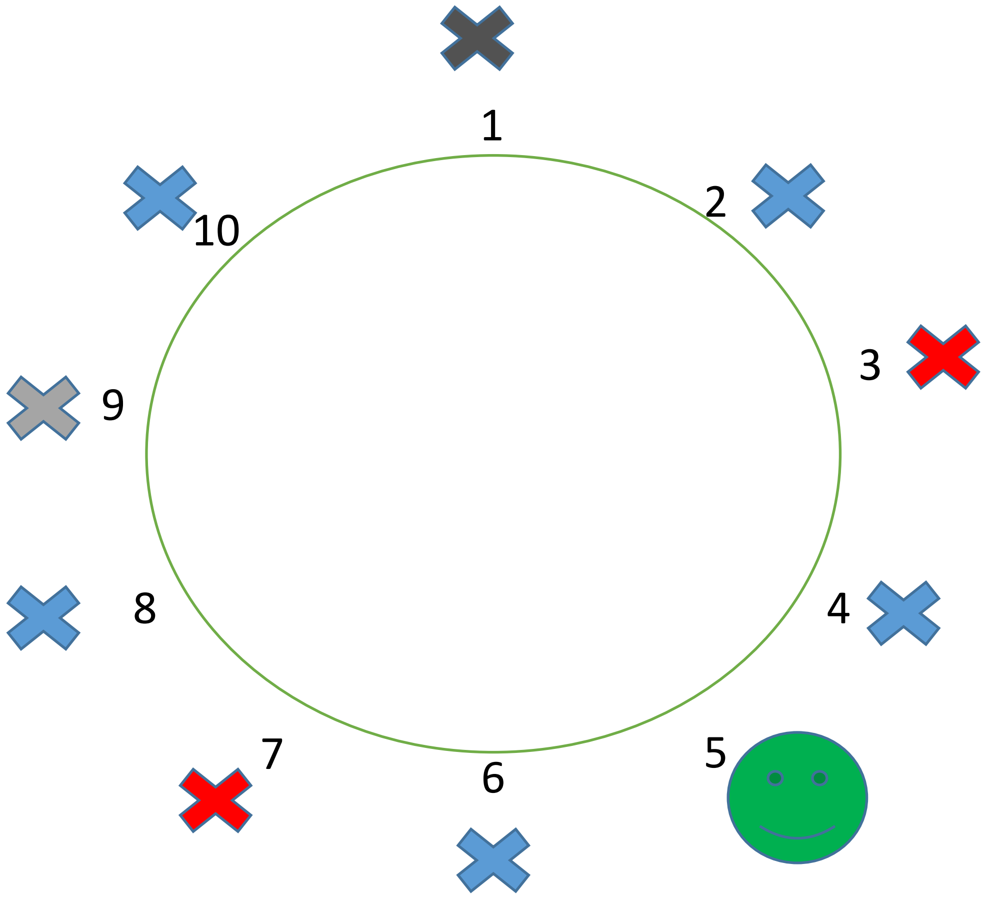
$$\left.\begin{aligned} U_0 &= 1 \\ U_n &= 2 * U_{n-1} \; ; \text{for } n > 0 \end{aligned}\right\} \text{.................. (2)}$$

$$U_n = 2^n \; ; \text{hence } T_n = 2^n - 1$$

The Josephus Problem

Problem Statement

1. n people numbered 1 to n around a circle
2. Eliminate every second person
3. Last person ALIVE!!



Generalization

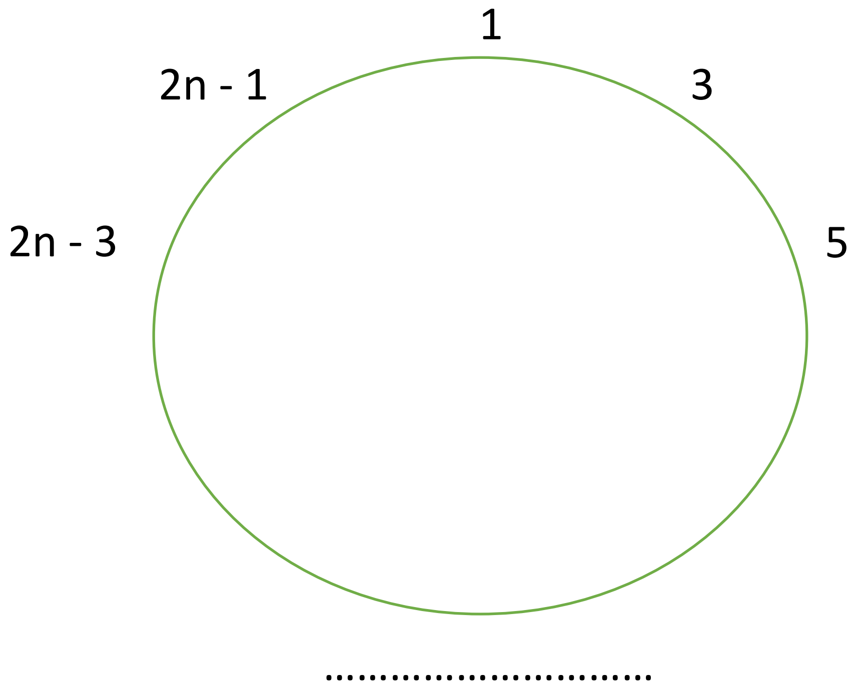
- $J(10) = 5$
- $J(10) = 10/2 = 5$
- $J(2) = 2/2 = 1$
- So……………… $J(n) = n/2$??

Generalization

n	J(n)
1	1
2	1
3	3
4	1
5	3
6	5

Generalization: Even Case

- J(n) is always odd
- If n is an even number, we arrive at a situation similar to what we began with…..only half as many people
- Suppose, we start with 2n people. After 1st iteration, we get:

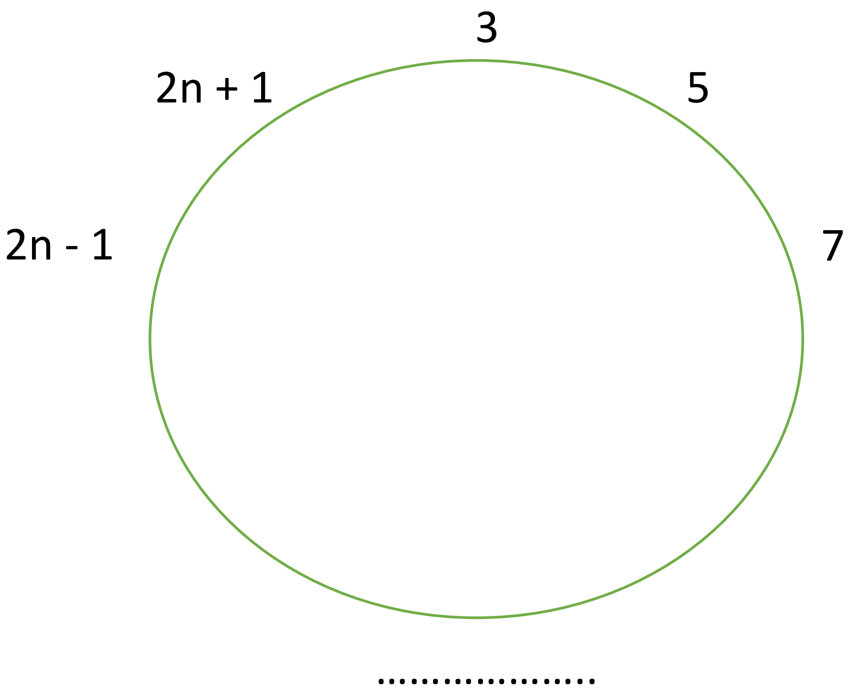


Generalization: Even Case

- Same situation as starting with n people, except…
- $J(2n) = 2\,J(n) - 1$; for $n>=1$
- $J\,(20) = 2\,J(10) - 1$
  $= 2 * 5 - 1$
  $= 10 - 1$
  $= 9$

Generalization: Odd Case

- With (2n + 1) people, person #1 is eliminated just after person #2n
- We are left with:



So, $J(2n + 1) = 2J(n) + 1$; for $n >= 1$

Generalization

$$\left.\begin{aligned} &J(1) = 1 \\ &J(2n) = 2 * J(n) - 1 \text{ ;for } n >= 1 \\ &J(2n + 1) = 2 * J(n) + 1 \text{ ; for } n >= 1 \end{aligned}\right\} \text{.....................} (3)$$

Generalization

n	J(n)
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15
16	1

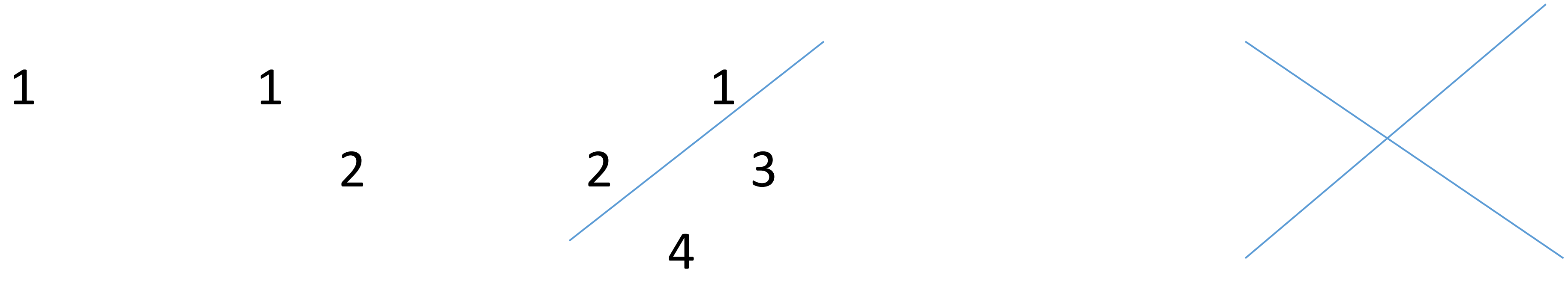
The Josephus Problem: Solution

- $J(2^m + p) = 2 * p + 1$; for $m >= 0$ and $0 <= p < 2^m$
- Let, $n = 2^m + p$
- So we get,
  $J(n) = 2 * p + 1$; where $n = 2^m + p$
- Proof???

Lines in the Plane

Problem Statement

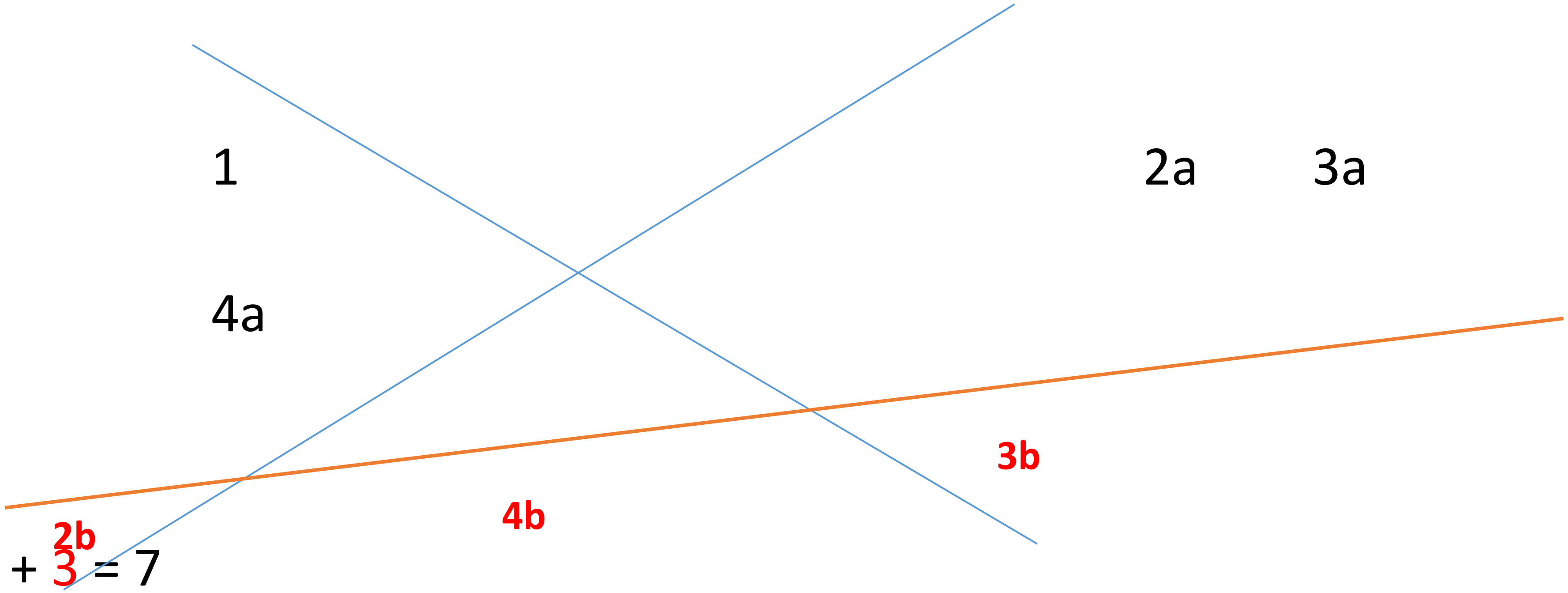
- What is the maximum number of regions (L_n) defined by n lines in a plane?
- Start by looking at small cases-



$L_0 = 1$ $L_1 = 2$ $L_2 = 4$

Generalization

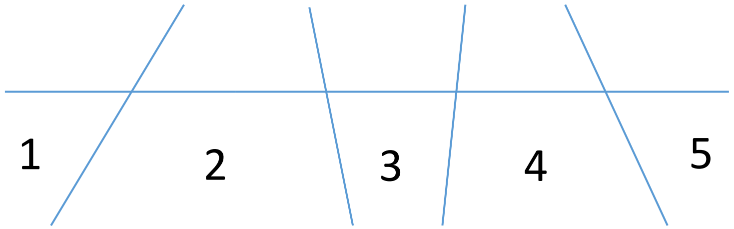
- So, $L_n = 2^n$???
- What happens when we add a third line? (Orange)



$L_3 = 4 + 3 = 7$

Generalization

- So how many new regions for the n^{th} line?
  - Number of intersection + 1



- The n^{th} line will intersect the previous (n-1) line (at most)
- Number of intersections = n – 1
- So number of new regions = n – 1 + 1 = n

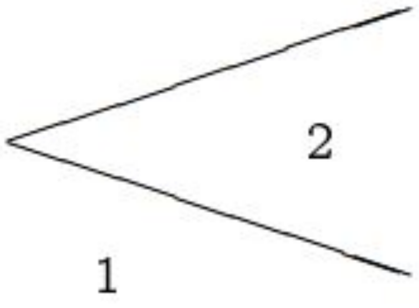
Generalization

- $L_n = L_{n-1} + n$
- So recursive formula:
  - $L_0 = 1$
  - $L_n = n + L_{n-1}$

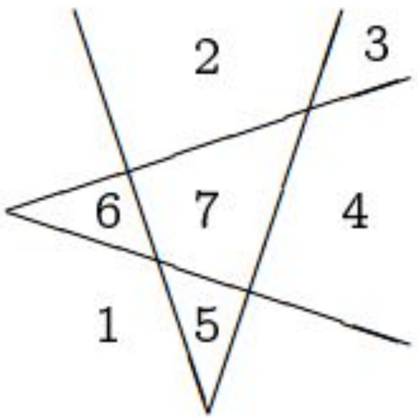
$$= n + (n-1) + L_{n-2}$$
$$= n + (n-1) + (n-2) + L_{n-3}$$
$$= n + (n-1) + (n-2) + \ldots\ldots + 1 + L_0$$
$$= \frac{n(n+1)}{2} + 1$$

Problem Statement: Zig

- What is the maximum number of regions (Z_n) defined by n bent lines (zig) in a plane?
- Start by looking at small cases-



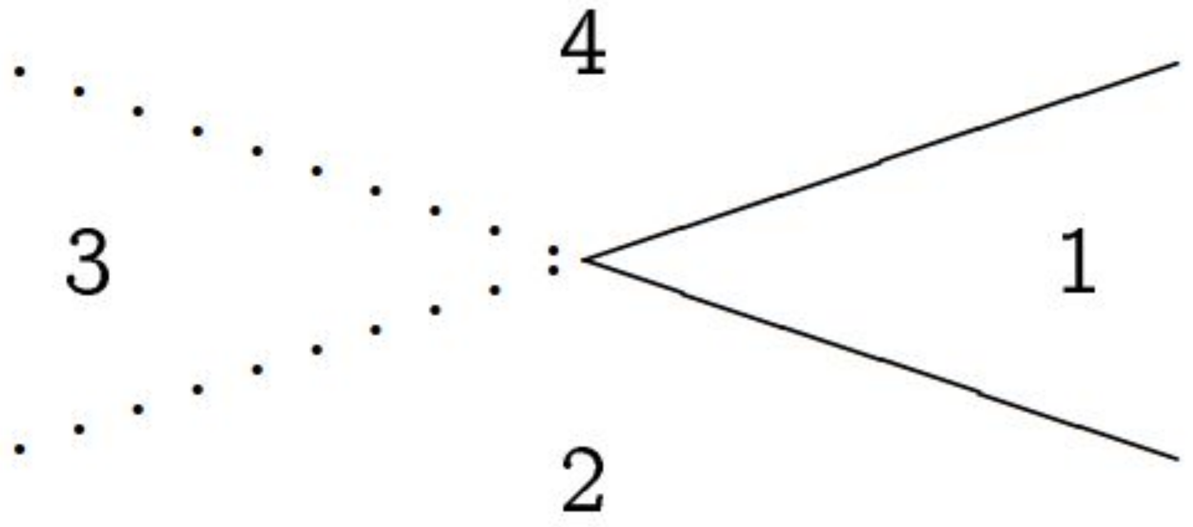
$Z_1 = 2$



$Z_2 = 7$

Generalization

- $Z_n = L_{2n} - 2n$
- How??



Approximation

- $Z_n = L_{2n} - 2n$

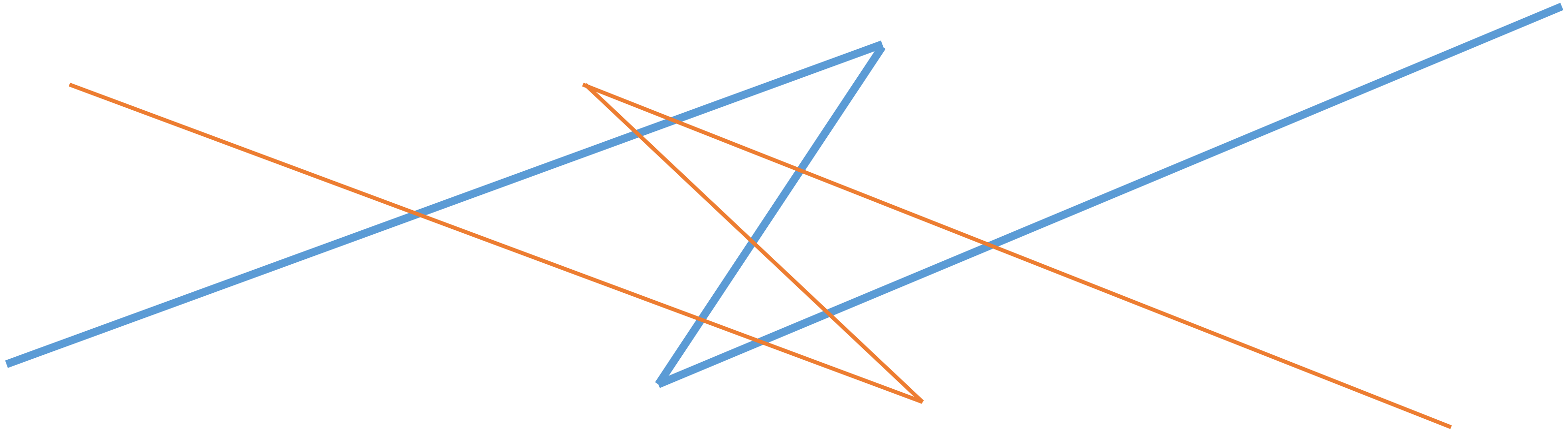
  $= 2n^2 - n + 1$

  ~ 2

- $L_n \sim n^2 / 2$

Assignment

Find out the maximum number of regions (ZZ_n) defined by n Zig-zag lines in a plane?



$ZZ_2 = 12$