## 1. Introduction

The Anderson model is given by a class of discrete analogs of Schrődinger operators $H_\omega$ with real $i.i.d$ potetials $\{V_\omega(n)\}$:

$$(H_\omega\Psi)(n) = \Psi(n+1) + \Psi(n-1) + V_\omega(n)\Psi(n), \tag{1.1}$$

where $\omega = \{\omega_n\}_{n\in\mathbb{Z}} \in \Omega = S^{\mathbb{Z}}$, $S \subset \mathbb{R}$ is the topological support of $\mu$, so compact, and contains at least two points, $\mu$ is a Borel probability on $\mathbb{R}$. *i.e.* for each $n \in \mathbb{Z}$, $V_\omega(n)$ is *i.i.d.* random variables depending on $\omega_n$ in $(S, \mu)$. We will consider $V_\omega$ in the product probability space $(S^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ as a whole instead. Denote $\mu^{\mathbb{Z}}$ as $\mathbb{P}$, and let $\mathbb{P}_{[a,b]}$ be $\mu^{[a,b]^c\cap\mathbb{Z}}$ on $S^{[a,b]^c\cap\mathbb{Z}}$. Also denote Lebesgue measure as $m$.

We say that $H_\omega$ exhibits the spectral localization property in $I$ if for $a.e.\omega$, $H_\omega$ has only pure point spectrum in $I$ and its eigenfunction $\Psi(n)$ decays exponentially in $n$. We are going to give a new proof for Anderson model based on the large deviation estimates and subharmonicity of Lyapunov exponents.

## 2. general setup

**Definition 1.** We call $E$ a generalized eigenvalue (denote as *g.e.*), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H_\omega\Psi = E\Psi$. We call $\Psi(n)$ a generalized eigenfunction.

Since the set of g.e. supports the spectral measure of $H_\omega$, we only need to show:

**Theorem 2.1.** *For a.e. $\omega$, for every g.e. $E$, the corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in $n$.*

For $[a, b]$ an interval, $a, b \in \mathbb{Z}$, define $H_{[a,b],\omega}$ to be operator $H_\omega$ resticted to $[a, b]$ with zero boundary condition outside $[a, b]$. Note that it can be expressed as a "$b - a + 1$"-dimensional matrix. The Green's function defined on $[a, b]$ for $H_\omega$ with energy $E \notin \sigma_{[a,b],\omega}$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}$$

Note that this can also be expressed as a "$b - a + 1$"-dimensional matrix. Denote its $(x, y)$ entry as $G_{[a,b],E,\omega}(x, y)$.

It is well known that

$$\Psi(x) = -G_{[a,b],E,\omega}(x, a)\Psi(a-1) - G_{[a,b],E,\omega}(x, b)\Psi(b+1), \quad x \in [a, b] \tag{2.1}$$

and we know that,

$$\sigma := \sigma(H_\omega) = [-2, 2] + S \quad a.e.\omega \tag{2.2}$$

**Definition 2.** For $c > 0, n \in \mathbb{Z}$, we say $x \in \mathbb{Z}$ is $(c, n, E, \omega)$-regular, if

$$G_{[x-n,x+n],E,\omega}(x, x-n) \leqslant e^{-cn}$$

$$G_{[x-n,x+n],E,\omega}(x, x+n) \leqslant e^{-cn}$$

Otherwise, we call it $(c, n, E, \omega)$-singular.

By (2.1) and definition 2, Theorem 2.1 follows from

**Theorem 2.2.** *There exists $\Omega_0$ with $\mathbb{P}(\Omega_0) = 1$, such that for every $\tilde{\omega} \in \Omega_0$, for any g.e.$\tilde{E}$ of $H_{\tilde{\omega}}$, there exist $N = N(\tilde{E}, \tilde{\omega}), C = C(\tilde{E})$, for every $n > N$, $2n$, $2n+1$ are $(C, n, \tilde{E}, \tilde{\omega})$ regular.*

Some other basic settings are below. Denote

$$P_{[a,b],E,\omega} = det(H_{[a,b],E,\omega} - E)$$

If $a = b$, let $P_{[a,b],E,\omega} = 1$, then

$$\left|G_{[a,b],E,\omega}(x,y)\right| = \frac{\left|P_{[a,x-1],E,\omega}P_{[y+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|}, \quad x \leqslant y \tag{2.3}$$

If we denote the transfer matrix $T_{[a,b],E,\omega}$ as the matrix such that

$$\begin{pmatrix} \Psi(b) \\ \Psi(b-1) \end{pmatrix} = T_{[a,b],E,\omega} \begin{pmatrix} \Psi(a) \\ \Psi(a-1) \end{pmatrix}$$

then

$$T_{[a,b],E,\omega} = \begin{pmatrix} P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega} \\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega} \end{pmatrix}$$

The Lyapunov exponent is given by

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n}\int_0^1 \log \|T_{[0,n],E,\omega}\| d\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|T_{[0,n],E,\omega}\|, \quad a.e.\omega.$$

Let $\nu = \inf_{E\in\sigma} \gamma(E) > 0$.

We introduce the large deviation theorem here without proof. [1]

**Lemma 2.3** (Large deviation estimates)**.** *For any $\epsilon > 0$, there exists $\eta = \eta(\epsilon) > 0$ such that, there exists $N_0 = N_0(\epsilon)$, for every $b - a > N_0$*

$$\mu\left\{\omega : \left|\frac{1}{b-a+1}\log \|P_{[a,b],E,\omega}\| - \gamma(E)\right| \geqslant \epsilon\right\} \leqslant e^{-\eta(b-a+1)}$$

## 3. Main Lemmas

Denote

$$B^{+}_{[a,b],\epsilon} = \left\{(E,\omega) : |P_{[a,b],E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\right\} \tag{3.1}$$

$$B^{-}_{[a,b],\epsilon} = \left\{(E,\omega) : |P_{[a,b],E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\right\} \tag{3.2}$$

and denote $B^{\pm}_{[a,b],\epsilon,E} = \{\omega : (E,\omega) \in B^{\pm}_{[a,b],\epsilon}\}$, $B^{\pm}_{[a,b],\epsilon,\omega} = \{E : (E,\omega) \in B^{\pm}_{[a,b],\epsilon}\}$, $B_{[a,b],*} = B^{+}_{[a,b],*} \cup B^{-}_{[a,b],*}$.

Let $E_{j,(\omega_a,\cdots,\omega_b)}$ be the eigenvalue of $H_{[a,b],\omega}$ with $\omega|_{[a,b]} = (\omega_a,\cdots,\omega_b)$.

Large deviation theorem gives us the estimate that for all $E, a, b, \epsilon$

$$P(B^{\pm}_{[a,b],\epsilon,E}) \leqslant e^{-\eta(b-a+1)} \tag{3.3}$$

Assume $\epsilon = \epsilon_0 < \frac{1}{8}\nu$ is fixed for now, so we omit it from the notations until Lemma 3.4. $\eta_0 = \eta(\epsilon_0)$ is the corresponding parameter from Lemma 2.3

**Lemma 3.1.** *For $n \geqslant 2$, if $x$ is $(\gamma(E) - 8\epsilon_0, n, E, \omega)$-singular, then*

$$(E,\omega) \in B^{-}_{[x-n,x+n]} \cup B^{+}_{[x-n,x]} \cup B^{+}_{[x,x+n]}$$

*Remark* 1. Note that from (3.3), for all $E, x, n \geqslant 2$,

$$P(B^{-}_{[x-n,x+n],E} \cup B^{+}_{[x-n,x],E} \cup B^{+}_{[x,x+n],E}) \leqslant 3e^{-\eta_0(n+1)}$$

*Proof.* Follows imediately from the definition of singularity and (2.3). □

Now we will use the following three lemmas to find the proper $\Omega_0$ for Theorem 2.2.

**Lemma 3.2.** *Let $0 < \delta_0 < \eta_0$. For a.e. $\omega$ (we denote this set as $\Omega_1$), there exists $N_1 = N_1(\omega)$, such that for every $n > N_1$,*

$$\max\{m(B^-_{[n+1,3n+1],\omega}), m(B^-_{[-n,n],\omega})\} \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

*Proof.* By (3.3),

$$m \times \mathbb{P}(B^-_{[n+1,3n+1]}) \leqslant m(\sigma)e^{-\eta_0(2n+1)}$$

$$m \times \mathbb{P}(B^-_{[-n,n]}) \leqslant m(\sigma)e^{-\eta_0(2n+1)}$$

If we denote

$$\Omega_{\delta_0,n,+} = \left\{\omega : m(B^-_{[n+1,3n+1],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\}$$

$$\Omega_{\delta_0,n,-} = \left\{\omega : m(B^-_{[-n,n],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\},$$

We have Tchebyshev,

$$\mathbb{P}(\Omega^c_{\delta_0,n,\pm}) \leqslant m(\sigma)e^{-\delta_0(2n+1)}.$$

By Borel-Cantelli lemma, we get for *a.e.* $\omega$,

$$\max\{m(B^-_{[n+1,3n+1],\omega}), m(B^-_{[-n,n],\omega})\} \leqslant e^{-(\eta_0-\delta_0)(2n+1)},$$

for $n > N_1(\omega)$. □

*Remark* 2. Note that we can actually shift the operator and use center point $l$ instead of 0. Then we will get $\Omega_1(l)$ instead of $\Omega_1$, $N_1(l,\omega)$ instead of $N_1(\omega)$. And if we pick $N_1(l,\omega)$ in the theorem as the smallest interger satisfying the conclusion, we can estimate when will $N_1(l,\omega) \leqslant \ln^2|l|$, which is very useful in the proof for dynamical localization in section 6. In fact, $\mathbb{P}\{\omega : N_1(l,\omega) > \ln^2|l|\} \leqslant C'e^{-\delta_0(2|\ln^2|l|+1)}$, By Borel-Cantelli, for $a.e.\omega$,(We denote this set as $\Omega_{N_1}$,) there exists $L_1(\omega)$, such that for any $|l| > L_1(\omega)$, $N_1(l,\omega) \leqslant \ln^2|l|$.

The next results follows from [2]:

**Theorem 3.3** (Craig-Simon)**.** *For a.e.$\omega$(denote as $\Omega_2$), for all $E$, we have*

$$\max\left\{\overline{\lim_{n\to\infty}}\frac{\log\|T_{[-n,0],E,\omega}\|}{n+1}, \overline{\lim_{n\to\infty}}\frac{\log\|T_{[0,n],E,\omega}\|}{n+1}\right\} \leqslant \gamma(E) \tag{3.4}$$

$$\max\left\{\overline{\lim_{n\to\infty}}\frac{\log\|T_{[n+1,2n+1],E,\omega}\|}{n+1}, \overline{\lim_{n\to\infty}}\frac{\log\|T_{[2n+1,3n+1],E,\omega}\|}{n+1}\right\} \leqslant \gamma(E) \tag{3.5}$$

*Remark* 3. (3.4) is a direct reformulation of Craig-Simon, while (3.5) follows by exactly the same proof.

**Corollary 1.** *for every $\omega \in \Omega_2$, for every $E$, there exists $N_2 = N_2(\omega,E)$, such that for every $n > N_2$,*

$$\max\{\|T_{[-n,0],E,\omega}\|, \|T_{[0,n],E,\omega}\|\} < e^{(\gamma(E)+\epsilon)(n+1)}$$

$$\max\{T_{[n+1,2n+1],E,\omega}\|, \|T_{[2n+1,3n+1],E,\omega}\|\} < e^{(\gamma(E)+\epsilon)(n+1)}$$

**Lemma 3.4.** *Let $\epsilon > 0, K > 1$, For a.e.$\omega$(We denote this set as $\Omega_3 = \Omega_3(\epsilon, K)$), there exists $N_3 = N_3(\omega)$, so that for every $n > N_3$, for every $E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}$, for every $y_1, y_2$ satisfying $-n \leqslant y_1 \leqslant y_2 \leqslant n$, $|-n-y_1| \geqslant \frac{n}{K}$, and $|n-y_2| \geqslant \frac{n}{K}$, we have $E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})} \notin B_{[-n,y_1],\epsilon,\omega} \cup B_{[y_2,n],\epsilon,\omega}$.*

*Remark* 4. Note that $\epsilon$ and $K > 0$ are not fixed yet, we're going to determine them later in section 4.

*Proof.* Let $\bar{P}$ be the probability that there are some $y_1, y_2, j$ with

$$E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})} \in B_{[-n,y_1],\epsilon,\omega} \cup B_{[y_2,n],\epsilon,\omega}.$$

Note that for any fixed $\omega_c, \cdots, \omega_d$, with $[c,d] \cap [a,b] = \varnothing$, by independence,

$$\mathbb{P}(B_{[a,b],\epsilon,E_{j,(\omega_c,\cdots,\omega_d)}}) = \mathbb{P}_{[a,b]}(B_{[a,b],\epsilon,E_{j,(\omega_c,\cdots,\omega_d)}}) \leqslant e^{-\eta_0(b-a+1)}$$

Applying to $[a,b] = [-n, y_1]$ or $[y_2, n]$, $[c,d] = [n+1, 3n+1]$ and integrating over $\omega_{-n}, \cdots, \omega_{y_1}$ or $\omega_{y_2}, \cdots, \omega_n$, we get

$$\mathbb{P}(B_{[-n,y_1],\epsilon,E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}} \cup B_{[y_2,n],\epsilon,E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}}) \leqslant 2e^{-\eta_0(\frac{n}{K}+1)},$$

so

$$\bar{\mathbb{P}} \leqslant (2n+1)^3 2e^{-\eta_0(\frac{n}{K}+1)}$$

Thus by Borel-Cantelli, we can get the result. □

*Remark* 5. Similar to remark 2, we can get $\Omega_3(l)$, $N_3(l,\omega)$ instead, and get that for $a.e.\omega$, (We denote this set as $\Omega_{N_3}$,) there exists $L_3(\omega)$, such that for any $|l| > L_3$, $N_3(l,\omega) \leqslant \ln^2 |l|$.

## 4. Proof of Theorem 2.2

We will only provide a proof that $2n+1$ is $(c, n, E, \omega)$-regular, the argument for $2n$ being similar.

*Proof.* Let $\epsilon$ be small enough such that

$$\epsilon < \min\{(\eta_0 - \delta_0)/3, \nu\}. \tag{4.1}$$

Now let

$$L := e^{(\eta_0 - \delta_0 - \epsilon)} > 1,$$

and note that since $V$ is bounded, by (2.2) we have there exists $M > 0$, such that

$$|P_{[a,b],E,\omega}| < M^{(b-a+1)}, \quad \forall E \in \sigma, \omega$$

Pick $K$ big enough such that

$$M^{\frac{1}{K}} < L$$

Let$\sigma > 0$ be such that

$$M^{\frac{1}{K}} \leqslant L - \sigma < L \tag{4.2}$$

Let $\Omega_0 = \Omega_1 \cap \Omega_2 \cap \Omega_3(\epsilon, K)$. Pick $\tilde{\omega} \in \Omega_0$, and take $\tilde{E}$ a *g.e.* for $H_{\tilde{\omega}}$. Without loss of generality assume $\Psi(0) \neq 0$. Then there exists $N_4$, such that for every $n > N_4$, 0 is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular.

For $n > N_0 = \max\{N_1(\tilde{\omega}), N_2(\tilde{\omega}, \tilde{E}), N_3(\tilde{\omega}), N_4(\tilde{\omega}, \tilde{E})\}$, assume $2n+1$ is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular. Then both 0 and $2n+1$ is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular. So by Lemma 3.1, $\tilde{E} \in B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$. By

Corollary 1 and (3.1), $\tilde{E} \notin B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$, so it can only lie in $B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}}$.

Note that in (3.2), $P_{[n+1,3n+1],\epsilon,\epsilon_0,E,\tilde{\omega}}$ is a polynomial in $E$ that has $2n+1$ real zeros (eigenvalues of $H_{[n+1,3n+1],\tilde{\omega}}$), which are all in $B = B_{[n+1,3n+1],\epsilon,\tilde{\omega}}$. Thus $B$ contains less than $2n+1$ intervals near the eigenvalues. $\tilde{E}$ should lie in one of them. By Theorem 3.2, $m(B) \leqslant Ce^{-(\eta_0-\delta_0)(2n+1)}$. So there is some e.v. $E_{j,[n+1,3n+1],\tilde{\omega}}$ of $H_{[n+1,3n+1],\omega}$ such that

$$|\tilde{E} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

By the same argument, there exists $E_{i,[-n,n],\tilde{\omega}}$, such that

$$|\tilde{E} - E_{i,[-n,n],\tilde{\omega}}| \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

Thus $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$. However, by Theorem 3.4, one has $E_{j,[n+1,3n+1],\tilde{\omega}} \notin B_{[-n,n],\epsilon,\tilde{\omega}}$, while $E_{i,[-n,n],\tilde{\omega}} \in B_{[-n,n],\epsilon,\tilde{\omega}}$ This will give us a contradiction below.
Since $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$ and $E_{i,[-n,n],\tilde{\omega}}$ is the e.v. of $H_{[-n,n],\tilde{\omega}}$,

$$\left\| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}} \right\| \geqslant \frac{1}{2} e^{(\eta_0-\delta_0)(2n+1)}$$

Thus there exist $y_1, y_2 \in [-n,n]$ and such that

$$\left| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}}(y_1,y_2) \right| \geqslant \frac{1}{2n} e^{(\eta_0-\delta_0)(2n+1)}$$

Let $E_j = E_{j,[n+1,3n+1],\tilde{\omega}}$. We have $E_j \notin B_{[-n,n],\epsilon,\tilde{\omega}}$, thus

$$|P_{[-n,n],\epsilon,E_j,\tilde{\omega}}| \geqslant e^{(\gamma(E_j)-\epsilon)(2n+1)}$$

so by (2.3),

$$\left\| P_{[-n,y_1],\epsilon,E_j\tilde{\omega}} P_{[y_2,n],\epsilon,E_j,\tilde{\omega}} \right\| \geqslant \frac{1}{2n} e^{(\eta_0-\delta_0)(2n+1)} e^{(\gamma(E_j)-\epsilon)(2n+1)} \tag{4.3}$$

Then for the left hand side of (4.3), there are three cases:

(1) both $|-n-y_1| > \frac{n}{K}$ and $|n-y_2| > \frac{n}{K}$
(2) one of them is large, say $|-n-y_1| > \frac{n}{K}$ while $|n-y_2| \leqslant \frac{n}{K}$
(3) both small.

For (1),

$$\frac{1}{2n} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant e^{2n(\gamma(E_j)+\epsilon)}$$

Since by our choice (4.1), $\eta_0 - \delta_0 + \gamma(E_j) - \epsilon > \gamma(E_j) + \epsilon$, for $n$ large enough, we get a contradiction.

For (2),

$$\frac{1}{2n} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant e^{(\gamma(E_j)+\epsilon)(2n+1)} (M)^{\frac{n}{K}}$$

is in contradiction with (4.1) and (4.2)

For (3), with (4.1) and (4.2)

$$\frac{1}{2n} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant M^{\frac{2n}{K}} \leqslant (L-\sigma)^{2n} \leqslant (e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)} - \sigma)^{2n}$$

also a contradiction.

Thus our assumption that $2n+1$ is not $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-regular is false. Theorem 2.2 follows. $\square$

## 5. Quantative Craig-Simon

We improve the results of Craig-Simon as following:

**Theorem 5.1.** *For fixed $\epsilon_0 > 0$, for a.e.$\omega$,(We denote this set as $\Omega_2$), there exists $N_2(\omega)$, such that for any $n > N_2(\omega)$, $E \in \sigma$,*

$$\max\left\{|P_{[0,n],E,\omega}|, |P_{[-n,0],E,\omega}|, |P_{[n+1,2n+1],E,\omega}|, |P_{[2n+1,3n+1],E,\omega}|\right\} \leqslant e^{(\gamma(E)+3\epsilon_0)(n+1)}$$

We begin with an elementary Lemma:

**Lemma 5.2.** *If $Q(x)$ is a polynomial of order $n$ on and $x_1, \cdots, x_n$ are distributed like: $x_i = \cos$*

**Lemma 5.3.** *If $Q(x)$ is a polynomial of order $n$, and $x_1, \cdots, x_n$ are $n$ uniformly distributed points in $[x_1, x_n]$. If $Q(x_i) \leqslant a$ for any $i = 1, \cdots, n$, then $Q(x) \leqslant an^c$ for some $c > 0$ and any $x \in [x_1, x_n]$.*

Now we prove the Theorem 5.1.

*Proof.* We know that $\sigma = [-2, 2] + S$, with $S \subset \mathbb{R}$, so $\sigma$ is a finite union of closed intervals. Assume we are dealing with one of them, $[0, A]$. By continuity of $\gamma(E)$ on compact set $\sigma$, for $\epsilon_0$, there exists $\delta_0$ such that

$$|\gamma(E_x) - \gamma(E_y)| \leqslant \epsilon_0, \quad \forall |E_x - E_y| \leqslant \delta_0. \tag{5.1}$$

Devide the interval $[0, A]$ into length $\delta_0$ sub-intervals. There are $K = [A/\delta_0] + 1$ of them. (The last one may be shorter.) Denote them as $I_k$, for $k = 1, \cdots, K$. For $I_k$, devide it into $n - 1$ equal sub-subintervals with end points $E_{k1,n}, \cdots, E_{kn,n}$. Then with any $E_x$, $E_y \in [E_{k1,n}, E_{kn,n}]$, $|\gamma(E_x) - \gamma(E_y)| \leqslant \epsilon_0$.

Since

$$\mathbb{P}\left(\left\{\omega : \exists i = 1, \cdots, n,\ s.t.\ |P_{[0,n],E_{ki,n},\omega}| \geqslant e^{(\gamma(E_{ki,n})+\epsilon_0)(n+1)}\right\}\right) \leqslant ne^{-\eta_0(n+1)},$$

by Borel-Cantelli, for $a.e.\omega$, (We denote this set as $\Omega_k$,) there exists $N(k, \omega)$ for $I_k$, such that for all $n > N(k, \omega)$,

$$|P_{[0,n],E_{ki,n},\omega}| \leqslant e^{(\gamma(E_{ki,n})+\epsilon_0)(n+1)}, \quad \forall i = 1, \cdots, n.$$

If we denote $\gamma_{k,n} = \inf_{E \in [E_{k1,n}, E_{kn,n}]} \gamma(E)$, then by (5.1)

$$|P_{[0,n],E_{ki,n},\omega}| \leqslant e^{(\gamma(E_{ki,n})+\epsilon_0)(n+1)} \leqslant e^{(\gamma_{k,n}+2\epsilon_0)(n+1)}, \quad \forall i = 1, \cdots, n.$$

Let $M$ big enough such that, for any $n > M$, $n^c \leqslant e^{\epsilon_0(n+1)}$. Thus by Lemma 5.2, for $E \in [E_{k1,n}, E_{kn,n}]$, $n > \max\{N(k, \omega), M\}$,

$$|P_{[0,n],E,\omega}| \leqslant n^c e^{(\gamma_{k,n}+2\epsilon_0)(n+1)} \leqslant n^c e^{(\gamma(E)+2\epsilon_0)(n+1)} \leqslant e^{(\gamma(E)+3\epsilon_0)(n+1)}$$

Let $\Omega_2 = \bigcap_k \Omega(k)$, $\tilde{N}(\omega) = \max_k\{N(k, \omega), M\}$, then for any $n > \tilde{N}(\omega)$,

$$|P_{[0,n],E,\omega}| \leqslant e^{(\gamma(E)+3\epsilon_0)(n+1)}, \quad \forall E \in [0, A]$$

Use the same methods for $P_{[-n,0],E,\omega}$, $P_{[n+1,2n+1],E,\omega}$, and $P_{[2n+1,3n+1],E,\omega}$. $N_2(\omega)$ being the maximum of $\tilde{N}(\omega)$ for each of them would work for our theorem. $\square$

*Remark* 6. Similar as remark 2 and 5, we can get $\Omega_2(l)$, $N_2(l,\omega)$ instead. Note $M$ is independent of $l$, and we can then estimate in the same way that, for $a.e.\omega$, (We denote this set as $\Omega_{N_2}$), there exists $L_2 = L_2(\omega)$, such that for any $|l| > L_2$, $N_2(l,\omega) \leqslant \ln^2|l|$

*Remark* 7. For LD implies continuity of $\gamma$.

## 6. Dynamical Localization

Now we have established the spectral localization for 1-d Anderson Model. With some more effort, we can get the Dynamical localization. We say that $H_\omega$ exhibits dynamical localization property if for $a.e.\omega$, for any $\epsilon > 0$, there exists a $\alpha = \alpha(\omega) > 0$, a $C = C(\epsilon,\omega)$, such that for all $x, y \in \mathbb{Z}$:

$$\sup_t |\langle \delta_x, e^{-itH_\omega}\delta_y\rangle| \leqslant C_\epsilon e^{\epsilon|y|}e^{-\alpha|x-y|}$$

According to [3], we only need to prove that for $a.e.\omega$, $H_\omega$ has SULE(Semi-Uniformly Localized Eigenfunction). We say $H$ has SULE if $H$ has a complete set $\{\varphi_E\}$ of orthonormal eigenfunctions, there is $\alpha > 0$, $l = l_E \in \mathbb{Z}$, and for each $\epsilon > 0$, a $C_\epsilon$ such that for any eigenvalue $E$,

$$|\varphi_E(x)| \leqslant C_\epsilon e^{\epsilon|l_E|}e^{-\alpha|x-l_E|}$$

For any central point $l \in \mathbb{Z}$, by remark 2, 6, (We use Quantitative Craig-Simon instead of the original one for estimating $N_2$, $\Omega_2$) remark 5 for $l$ and $l + 2n + 1$, and their natural extension to $l - 2n - 1$ and $l$, (But we keep the original notations, even if now it satisfies both properties.) and the same analysis in section 4, if we let $\Omega(l) = \bigcap_{i=1,2,3}\Omega_i(l)$, then for each $\omega \in \Omega(l)$, there exists $N(l,\omega) = \max\{N_1(l,\omega), N_2(l,\omega), N_3(l,\omega)\}$, such that for any $n > N(l,\omega)$, either $l$ or $l + 2n + 1$, either $l$ or $l - 2n - 1$ are $(\mu - 8\epsilon_0, n, E, \omega)$-regular for all $E \in \sigma$.

Take $\Omega' = \bigcap_l \Omega_l \cap \bigcap_{i=1,2,3}\Omega_{N_i}$ and fix $\omega \in \Omega'$. (We omit $\omega$ from notations from now on.)

By remark 2, remark 5, there exists $L_1$, $L_3$ such that for all $|l| > \max\{L_1, L_2, L_3\}$,

$$N_i(l) \leqslant \ln^2|l|, \quad \forall i = 1, 2, 3$$

for all $E$.

Let $l_E$ be the maximum point of $\varphi_E$. For any $n \geqslant N_4 := \frac{\ln 2}{\mu - 8\epsilon_0}$, $l_E$ is naturally $(\mu - 8\epsilon_0, n, E)$-singular by (2.1). So there exists $L_4$, for any $|l| > L_4$,

$$N_4 < \ln^2|l|$$

for all $E$.

Let $L = \max\{L_1, L_2, L_3, L_4\}$, $N(l) := \max\{N_1(l), N_2(l), N_3(l), N_4\}$, then for any $|l| > L$,

$$N(l) \leqslant \ln^2|l| \tag{6.1}$$

If $|l_E| > L$, then for any $|x - l_E| \geqslant N(l_E)$, $l_E$ is $(\mu - 8\epsilon_0, n, E)$-singular, so $x$ is $(\mu - 8\epsilon_0, n, E)$-regular. By (2.1), for any $|x - l_E| \geqslant N(l_E)$

$$|\varphi_E(x)| \leqslant 2e^{-(\mu-8\epsilon_0)|x-l_E|}$$

Since $\varphi_E$ is normalized, in fact for all $x$,

$$|\varphi_E(x)| \leqslant e^{(\mu-8\epsilon_0)N(l_E)}e^{-(\mu-8\epsilon_0)|x-l_E|}$$

By (6.1), for any $\epsilon$, there exists $C_{1\epsilon}$ such that

$$|\varphi_E(x)| \leqslant e^{(\mu-8\epsilon_0)\ln^2|l_E|}e^{-(\mu-8\epsilon_0)|x-l_E|} \leqslant C_{1\epsilon}e^{\epsilon|l_E|}e^{-(\mu-8\epsilon_0)|x-l_E|}$$

If $|l_E| \leqslant L$, consider all $i \in [-L, L]$, all $|x-i| < N(i)$. For any $\epsilon$, take $M_2 = \max_i\{e^{\epsilon i}e^{-(\mu-8\epsilon_0)|x-i|}\}$, $C_{2\epsilon} = M^{-1}$, then for all $|x-l_E| < N(l_E)$,

$$|\varphi_E(x)| \leqslant 1 \leqslant C_2\epsilon e^{\epsilon|l_E|}e^{-(\mu-8\epsilon_0)|x-l_E|}$$

As for $|x-l_E| \geqslant N(l_E)$,

$$|\varphi_E(x)| \leqslant e^{-(\mu-8\epsilon_0)|x-l_E|} \leqslant e^{\epsilon|l_E|}e^{-(\mu-8\epsilon_0)|x-l_E|}$$

So $C_\epsilon = \max\{C_{1\epsilon}, C_{2\epsilon}, 1\}$ would work.

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