1. Introduction

The Anderson model is given by a class of discrete analogs of Schrӧdinger operators H_ω with real $i.i.d$ potetials $\{V_\omega(n)\}$:

$$(H_\omega\Psi)(n) = \Psi(n+1) + \Psi(n-1) + V_\omega(n)\Psi(n), \tag{1.1}$$

where $\omega = \{\omega_n\}_{n\in\mathbb{Z}} \in \Omega = S^{\mathbb{Z}}$, $S \subset \mathbb{R}$ is the topological support of μ, so compact, and contains at least two points, μ is a Borel probability on $\mathbb{R}$. *i.e.* for each $n \in \mathbb{Z}$, $V_\omega(n)$ is *i.i.d.* random variables depending on ω_n in (S, μ). We will consider V_ω in the product probability space $(S^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ as a whole instead. Denote $\mu^{\mathbb{Z}}$ as $\mathbb{P}$, and let $\mathbb{P}_{[a,b]}$ be $\mu^{[a,b]^c\cap\mathbb{Z}}$ on $S^{[a,b]^c\cap\mathbb{Z}}$. Also denote Lebesgue measure as m.

We say that H_ω exhibits the spectral localization property in I if for $a.e.\omega$, H_ω has only pure point spectrum in I and its eigenfunction $\Psi(n)$ decays exponentially in n. We are going to give a new proof for Anderson model based on the large deviation estimates and subharmonicity of Lyapunov exponents.

2. Preliminaries

Definition 1. We call E a generalized eigenvalue ($g.e.$), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H_\omega\Psi = E\Psi$. We call $\Psi(n)$ a generalized eigenfunction.

Since the set of g.e. supports the spectral measure of H_ω, we only need to show:

Theorem 2.1. *For a.e. ω, for every g.e. E, the corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in n.*

For $[a,b]$ an interval, $a, b \in \mathbb{Z}$, define $H_{[a,b],\omega}$ to be operator H_ω resticted to $[a,b]$ with zero boundary condition outside $[a,b]$. Note that it can be expressed as a "$b-a+1$"-dimensional matrix. The Green's function defined on $[a,b]$ for H_ω with energy $E \notin \sigma_{[a,b],\omega}$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}$$

Note that this can also be expressed as a "$b-a+1$"-dimensional matrix. Denote its (x,y) entry as $G_{[a,b],E,\omega}(x,y)$.

It is well known that

$$\Psi(x) = -G_{[a,b],E,\omega}(x,a)\Psi(a-1) - G_{[a,b],E,\omega}(x,b)\Psi(b+1), \quad x \in [a,b] \tag{2.1}$$

and we have

$$\sigma := \sigma(H_\omega) = [-2,2] + S \quad a.e.\omega. \tag{2.2}$$

Definition 2. For $c > 0, n \in \mathbb{Z}$, we say $x \in \mathbb{Z}$ is (c,n,E,ω)-regular, if

$$G_{[x-n,x+n],E,\omega}(x, x-n) \leqslant e^{-cn}$$

$$G_{[x-n,x+n],E,\omega}(x, x+n) \leqslant e^{-cn}$$

Otherwise, we call it (c,n,E,ω)-singular.

By (2.1) and definition 2, Theorem 2.1 follows from

Theorem 2.2. *There exists Ω_0 with $\mathbb{P}(\Omega_0) = 1$, such that for every $\tilde{\omega} \in \Omega_0$, for any g.e.$\tilde{E}$ of $H_{\tilde{\omega}}$, there exist $N = N(\tilde{E},\tilde{\omega}), C = C(\tilde{E})$, for every $n > N$, $2n$, $2n+1$ are $(C,n,\tilde{E},\tilde{\omega})$ regular.*

Some other basic settings are below. Denote

$$P_{[a,b],E,\omega} = det(H_{[a,b],E,\omega} - E)$$

If $a = b$, let $P_{[a,b],E,\omega} = 1$. Then

$$\left|G_{[a,b],E,\omega}(x,y)\right| = \frac{\left|P_{[a,x-1],E,\omega}P_{[y+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|}, \quad x \leqslant y \tag{2.3}$$

If we denote the transfer matrix $T_{[a,b],E,\omega}$ as the matrix such that

$$\begin{pmatrix} \Psi(b) \\ \Psi(b-1) \end{pmatrix} = T_{[a,b],E,\omega} \begin{pmatrix} \Psi(a) \\ \Psi(a-1) \end{pmatrix}$$

then

$$T_{[a,b],E,\omega} = \begin{pmatrix} P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega} \\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega} \end{pmatrix}$$

The Lyapunov exponent is given by

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 \log \|T_{[0,n],E,\omega}\| d\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|T_{[0,n],E,\omega}\|, \quad a.e.\omega.$$

Let $\nu = \inf\limits_{E\in\sigma} \gamma(E) > 0$.

We introduce the large deviation theorem here without proof. [1]

Lemma 2.3 (Large deviation estimates)**.** *For any $\epsilon > 0$, there exists $\eta = \eta(\epsilon) > 0$ such that, there exists $N_0 = N_0(\epsilon)$, for every $b - a > N_0$*

$$\mu\left\{\omega : \left|\frac{1}{b-a+1} \log \|P_{[a,b],E,\omega}\| - \gamma(E)\right| \geqslant \epsilon\right\} \leqslant e^{-\eta(b-a+1)}$$

Finally, we provide an estimates on polynomials interpolation.

Lemma 2.4. *If $Q(x)$ is a polynomial of degree $n-1$ on and $x_1, \cdots, x_n$ are distributed as $x_i = \cos\frac{2\pi(i+\theta)}{n}$, if $Q(x_i) \leqslant a^n, \forall i$, then $Q(x) \leqslant Cna^n$, where C is a constant, $x \in [x_1, x_n]$.*

Proof. By Lagrange Interpolation, we have

$$Q(x) = \sum_{i=1}^{n} Q(x_i) \prod_{j\neq i} \frac{x - x_j}{x_i - x_j} \tag{2.4}$$

Note that

$$\sum_{j\neq i} \ln|x_i - x_j| = \sum_{j\neq i} \left\{\ln\left|\sin\frac{\pi(i+j+2\theta)}{n}\right| + \ln\left|\sin\frac{\pi(i-j)}{n}\right| + \ln 2\right\} = A + B + C$$

For B, one use lemma 9.6 from [2] and get

$$B \geqslant \ln n + \ln(2/\pi) - (n-1)\ln 2.$$

For A, if $j = j_0$ reach the minimum term of $\ln|\sin\frac{\pi(i+j+2\theta)}{n}|$, then

$$A \geqslant \ln n + \ln(2/\pi) - (n-1)\ln 2 - \ln\left|\sin\frac{\pi(2i+2\theta)}{n}\right| + \ln\left|\sin\frac{\pi(i+j_0+2\theta)}{n}\right|$$

Choose $0 < \theta < 1/2$, then

$$\frac{|\sin\frac{\pi(2i+2\theta)}{n}|}{|\sin\frac{\pi(i+j_0+2\theta)}{n}|} = \frac{|\sin\frac{\pi(2i+2\theta)}{n}|}{|\sin\frac{\pi\cdot 2\theta}{n}|} \leqslant \frac{1}{|\sin\frac{\pi\cdot 2\theta}{n}|} = O(n)$$

So
$$\sum_{j\neq i}\ln|x_i-x_j|\geqslant-(n-1)\ln 2+\ln n+C$$
Write $x=\cos\frac{2\pi a}{n}$ and use the other half result of lemma 9.6 from [2], one get
$$\sum_{j\neq i}\ln|x-x_j|\leqslant-(n-1)\ln 2+2\ln n+C'$$
So
$$\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}\leqslant Cn$$
$$Q(x)\leqslant Cna^n$$
□

3. Main Lemmas

Denote

$$B^+_{[a,b],\epsilon}=\left\{(E,\omega):|P_{[a,b],E,\omega}|\geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\right\}\tag{3.1}$$

$$B^-_{[a,b],\epsilon}=\left\{(E,\omega):|P_{[a,b],E,\omega}|\leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\right\}\tag{3.2}$$

and denote $B^\pm_{[a,b],\epsilon,E}=\{\omega:(E,\omega)\in B^\pm_{[a,b],\epsilon}\}$, $B^\pm_{[a,b],\epsilon,\omega}=\{E:(E,\omega)\in B^\pm_{[a,b],\epsilon}\}$, $B_{[a,b],*}=B^+_{[a,b],*}\cup B^-_{[a,b],*}$.

Let $E_{j,(\omega_a,\cdots,\omega_b)}$ be the eigenvalue of $H_{[a,b],\omega}$ with $\omega|_{[a,b]}=(\omega_a,\cdots,\omega_b)$.

Large deviation theorem gives us the estimate that for all E,a,b,ϵ

$$P(B^\pm_{[a,b],\epsilon,E})\leqslant e^{-\eta(b-a+1)}\tag{3.3}$$

Assume $\epsilon=\epsilon_0<\frac{1}{8}\nu$ is fixed for now, so we omit it from the notations until Lemma 3.4. $\eta_0=\eta(\epsilon_0)$ is the corresponding parameter from Lemma 2.3

Lemma 3.1. *For $n\geqslant 2$, if x is $(\gamma(E)-8\epsilon_0,n,E,\omega)$-singular, then*
$$(E,\omega)\in B^-_{[x-n,x+n]}\cup B^+_{[x-n,x]}\cup B^+_{[x,x+n]}$$

Remark 1. Note that from (3.3), for all $E,x,n\geqslant 2$,
$$P(B^-_{[x-n,x+n],E}\cup B^+_{[x-n,x],E}\cup B^+_{[x,x+n],E})\leqslant 3e^{-\eta_0(n+1)}$$

Proof. Follows imediately from the definition of singularity and (2.3). □

Now we will use the following three lemmas to find the proper Ω_0 for Theorem 2.2.

Lemma 3.2. *Let $0<\delta_0<\eta_0$. For a.e. ω (we denote this set as Ω_1), there exists $N_1=N_1(\omega)$, such that for every $n>N_1$,*
$$\max\{m(B^-_{[n+1,3n+1],\omega}),m(B^-_{[-n,n],\omega})\}\leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

Proof. By (3.3),
$$m\times\mathbb{P}(B^-_{[n+1,3n+1]})\leqslant m(\sigma)e^{-\eta_0(2n+1)}$$
$$m\times\mathbb{P}(B^-_{[-n,n]})\leqslant m(\sigma)e^{-\eta_0(2n+1)}$$
If we denote
$$\Omega_{\delta_0,n,+}=\left\{\omega:m(B^-_{[n+1,3n+1],\omega})\leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\}$$

$$\Omega_{\delta_0,n,-} = \left\{\omega : m(B^-_{[-n,n],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\},$$

We have by Tchebyshev,

$$\mathbb{P}(\Omega^c_{\delta_0,n,\pm}) \leqslant m(\sigma)e^{-\delta_0(2n+1)}. \tag{3.4}$$

By Borel-Cantelli lemma, we get for *a.e.* ω,

$$\max\{m(B^-_{[n+1,3n+1],\omega}), m(B^-_{[-n,n],\omega})\} \leqslant e^{-(\eta_0-\delta_0)(2n+1)},$$

for $n > N_1(\omega)$. □

Remark 2. Note that we can actually shift the operator and use center point l instead of 0. Then we will get $\Omega_1(l)$ instead of Ω_1, $N_1(l,\omega)$ instead of $N_1(\omega)$. And if we pick $N_1(l,\omega)$ in the theorem as the smallest interger satisfying the conclusion, we can estimate when will $N_1(l,\omega) \leqslant \ln^2|l|$, which is very useful in the proof for dynamical localization in section 6. In fact, $\mathbb{P}\{\omega : N_1(l,\omega) > \ln^2|l|\} \leqslant C' e^{-\delta_0(2|\ln^2|l|+1)}$, By Borel-Cantelli, for $a.e.\omega$,(We denote this set as Ω_{N_1},) there exists $L_1(\omega)$, such that for any $|l| > L_1(\omega)$, $N_1(l,\omega) \leqslant \ln^2|l|$.

The next results follows from :

Theorem 3.3 (Craig-Simon [3]). *For a.e. ω(we denote this set as Ω_2), for all E, we have*

$$\max\left\{\overline{\lim_{n\to\infty}} \frac{\log\|T_{[-n,0],E,\omega}\|}{n+1}, \overline{\lim_{n\to\infty}} \frac{\log\|T_{[0,n],E,\omega}\|}{n+1}\right\} \leqslant \gamma(E) \tag{3.5}$$

$$\max\left\{\overline{\lim_{n\to\infty}} \frac{\log\|T_{[n+1,2n+1],E,\omega}\|}{n+1}, \overline{\lim_{n\to\infty}} \frac{\log\|T_{[2n+1,3n+1],E,\omega}\|}{n+1}\right\} \leqslant \gamma(E) \tag{3.6}$$

Remark 3. (3.5) is a direct reformulation of the result of [3], while (3.6) follows by exactly the same proof.

Corollary 1. *For every $\omega \in \Omega_2$, for every E, there exists $N_2 = N_2(\omega, E)$, such that for every $n > N_2$,*

$$\max\{\|T_{[-n,0],E,\omega}\|, \|T_{[0,n],E,\omega}\|\} < e^{(\gamma(E)+\epsilon)(n+1)}$$
$$\max\{\|T_{[n+1,2n+1],E,\omega}\|, \|T_{[2n+1,3n+1],E,\omega}\|\} < e^{(\gamma(E)+\epsilon)(n+1)}$$

Lemma 3.4. *Let $\epsilon > 0, K > 1$, For a.e. ω(We denote this set as $\Omega_3 = \Omega_3(\epsilon, K)$), there exists $N_3 = N_3(\omega)$, so that for every $n > N_3$, for every $E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}$, for every y_1, y_2 satisfying $-n \leqslant y_1 \leqslant y_2 \leqslant n$, $|-n-y_1| \geqslant \frac{n}{K}$, and $|n-y_2| \geqslant \frac{n}{K}$, we have $E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})} \notin B_{[-n,y_1],\epsilon,\omega} \cup B_{[y_2,n],\epsilon,\omega}$.*

Remark 4. Note that ϵ and $K > 0$ are not fixed yet, we're going to determine them later in section 4.

Proof. Let $\bar{P}$ be the probability that there are some y_1, y_2, j with

$$E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})} \in B_{[-n,y_1],\epsilon,\omega} \cup B_{[y_2,n],\epsilon,\omega}.$$

Note that for any fixed $\omega_c, \cdots, \omega_d$, with $[c,d] \cap [a,b] = \varnothing$, by independence,

$$\mathbb{P}(B_{[a,b],\epsilon,E_{j,(\omega_c,\cdots,\omega_d)}}) = \mathbb{P}_{[a,b]}(B_{[a,b],\epsilon,E_{j,(\omega_c,\cdots,\omega_d)}}) \leqslant e^{-\eta_0(b-a+1)}$$

Applying to $[a,b] = [-n, y_1]$ or $[y_2, n]$, $[c,d] = [n+1, 3n+1]$ and integrating over $\omega_{-n}, \cdots, \omega_{y_1}$ or $\omega_{y_2}, \cdots, \omega_n$, we get

$$\mathbb{P}(B_{[-n,y_1],\epsilon,E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}} \cup B_{[y_2,n],\epsilon,E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}}) \leqslant 2e^{-\eta_0(\frac{n}{K}+1)},$$

so

$$\bar{\mathbb{P}} \leqslant (2n+1)^3 2e^{-\eta_0(\frac{n}{K}+1)} \tag{3.7}$$

Thus by Borel-Cantelli, we get the result.

□

Remark 5. Similar to remark 2, we can get $\Omega_3(l)$, $N_3(l,\omega)$ instead, and get that for $a.e.\omega$, (We denote this set as Ω_{N_3},) there exists $L_3(\omega)$, such that for any $|l| > L_3$, $N_3(l,\omega) \leqslant \ln^2 |l|$.

4. Proof of Theorem 2.2

We will only provide a proof that $2n+1$ is (c,n,E,ω)-regular, the argument for $2n$ being similar.

Proof. Let ϵ be small enough such that

$$\epsilon < \min\{(\eta_0 - \delta_0)/3, \nu\}. \tag{4.1}$$

Now let

$$L := e^{(\eta_0-\delta_0-\epsilon)} > 1,$$

and note that since V is bounded, by (2.2) we have there exists $M > 0$, such that

$$|P_{[a,b],E,\omega}| < M^{(b-a+1)}, \quad \forall E \in \sigma, \omega$$

Pick K big enough such that

$$M^{\frac{1}{K}} < L$$

Let $\sigma > 0$ be such that

$$M^{\frac{1}{K}} \leqslant L - \sigma < L \tag{4.2}$$

Let $\Omega_0 = \Omega_1 \cap \Omega_2 \cap \Omega_3(\epsilon, K)$. Pick $\tilde{\omega} \in \Omega_0$, and take $\tilde{E}$ a *g.e.* for $H_{\tilde{\omega}}$. Without loss of generality assume $\Psi(0) \neq 0$. Then there exists N_4, such that for every $n > N_4$, 0 is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular.

For $n > N_0 = \max\{N_1(\tilde{\omega}), N_2(\tilde{\omega}, \tilde{E}), N_3(\tilde{\omega}), N_4(\tilde{\omega}, \tilde{E})\}$, assume $2n+1$ is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular. Then both 0 and $2n+1$ is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular. So by Lemma 3.1, $\tilde{E} \in B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$. By Corollary 1 and (3.1), $\tilde{E} \notin B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$, so it can only lie in $B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}}$.

Note that in (3.2), $P_{[n+1,3n+1],\epsilon,\epsilon_0,E,\tilde{\omega}}$ is a polynomial in E that has $2n+1$ real zeros (eigenvalues of $H_{[n+1,3n+1],\tilde{\omega}}$), which are all in $B = B_{[n+1,3n+1],\epsilon,\tilde{\omega}}$. Thus B contains less than $2n+1$ intervals near the eigenvalues. $\tilde{E}$ should lie in one of them. By Theorem 3.2, $m(B) \leqslant Ce^{-(\eta_0-\delta_0)(2n+1)}$. So there is some e.v. $E_{j,[n+1,3n+1],\tilde{\omega}}$ of $H_{[n+1,3n+1],\omega}$ such that

$$|\tilde{E} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

By the same argument, there exists $E_{i,[-n,n],\tilde{\omega}}$, such that

$$|\tilde{E} - E_{i,[-n,n],\tilde{\omega}}| \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

Thus $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$. However, by Theorem 3.4, one has $E_{j,[n+1,3n+1],\tilde{\omega}} \notin B_{[-n,n],\epsilon,\tilde{\omega}}$, while $E_{i,[-n,n],\tilde{\omega}} \in B_{[-n,n],\epsilon,\tilde{\omega}}$ This will give us a contradiction below.

Since $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$ and $E_{i,[-n,n],\tilde{\omega}}$ is the e.v. of $H_{[-n,n],\tilde{\omega}}$,

$$\left\| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}} \right\| \geqslant \frac{1}{2} e^{(\eta_0-\delta_0)(2n+1)}$$

Thus there exist $y_1, y_2 \in [-n,n]$ and such that

$$\left| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}}(y_1,y_2) \right| \geqslant \frac{1}{2n} e^{(\eta_0-\delta_0)(2n+1)}$$

Let $E_j = E_{j,[n+1,3n+1],\tilde{\omega}}$. We have $E_j \notin B_{[-n,n],\epsilon,\tilde{\omega}}$, thus

$$|P_{[-n,n],\epsilon,E_j,\tilde{\omega}}| \geqslant e^{(\gamma(E_j)-\epsilon)(2n+1)}$$

so by (2.3),

$$\|P_{[-n,y_1],\epsilon,E_j\tilde{\omega}} P_{[y_2,n],\epsilon,E_j,\tilde{\omega}}\| \geqslant \frac{1}{2n} e^{(\eta_0-\delta_0)(2n+1)} e^{(\gamma(E_j)-\epsilon)(2n+1)} \tag{4.3}$$

Then for the left hand side of (4.3), there are three cases:

(1) both $|-n-y_1| > \frac{n}{K}$ and $|n-y_2| > \frac{n}{K}$
(2) one of them is large, say $|-n-y_1| > \frac{n}{K}$ while $|n-y_2| \leqslant \frac{n}{K}$
(3) both small.

For (1),

$$\frac{1}{2n} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant e^{2n(\gamma(E_j)+\epsilon)}$$

Since by our choice (4.1), $\eta_0 - \delta_0 + \gamma(E_j) - \epsilon > \gamma(E_j) + \epsilon$, for n large enough, we get a contradiction.

For (2),

$$\frac{1}{2n} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant e^{(\gamma(E_j)+\epsilon)(2n+1)} (M)^{\frac{n}{K}}$$

is in contradiction with (4.1) and (4.2)

For (3), with (4.1) and (4.2)

$$\frac{1}{2n} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant M^{\frac{2n}{K}} \leqslant (L-\sigma)^{2n} \leqslant (e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)} - \sigma)^{2n},$$

also a contradiction.

Thus our assumption that $2n+1$ is not $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-regular is false. Theorem 2.2 follows. □

5. Quantitative Craig-Simon

We improve the results of Craig-Simon [3], on the one hand, to a quantitative sense such that it allow us to estimates N, while on the other hand, to be derived only from "large deviation type" (LDT) models. So in fact, we can generalize [3] to a larger variety of models. In this section, we are not limited in the Anderson Model, on the contrary, we consider

$$H_\omega u(n) = u(n+1) + u(n-1) + V(\omega)u(n)$$

where the potential V is an ergodic process in the sense that the index ω lies in a probability measure space (Ω, μ), which supports a group τ_x of measure preserving ergodic transformations with $V_\omega(x+y) = V_{\tau_y\omega}(x)$, where $\sup\{|V_\omega(x)|,\ x \in \mathbb{Z}, \omega \in \Omega\} < \infty$.

Definition 3. We say H_ω satisfies a uniform-LDT condition if for every $\epsilon > 0$, there are $\delta > 0$, and $M > 0$, $c > 0$, such that if $\|S - T\| \leqslant \delta$, then

$$\mu\left\{\omega \in \Omega : |\frac{1}{n}\log\|S_{[0,n],\omega}\| - \gamma_n(S)| > \epsilon\right\} \leqslant Me^{-cn}.$$

where $\gamma_n(S) = \int_\Omega \frac{1}{n}\log\|S_{[0,n]}\|d\mu$.

By [4], for those operators that satisfy the uniform-LDT condition, we can conclude the continuity of Lvapunov exponent.

Theorem 5.1. *For those operators H_ω that satisfies the unifotm-LDT condition, and every $\gamma(E)$ is uniformly L_p bounded, the Lyapunov exponent $\gamma(E)$ is continuous.*

Theorem 5.2. *For H_ω satisfying the uniform-LDT condition, for fixed $\epsilon_0 > 0$, for a.e. ω (we denote this set as $\Omega_2 = \Omega_2(\epsilon_0)$), there exists $N_2(\omega)$, such that for any $n > N_2(\omega)$, $E \in \sigma$,*

$$|P_{[0,n],E,\omega}| \leqslant e^{(\gamma(E)+\epsilon_0)(n+1)}$$

Remark 6. It's obvious that we can in fact get the result:

$$\max\left\{|P_{[0,n],E,\omega}|, |P_{[-n,0],E,\omega}|, |P_{[n+1,2n+1],E,\omega}|, |P_{[2n+1,3n+1],E,\omega}|\right\} \leqslant e^{(\gamma(E)+3\epsilon_0)(n+1)}$$

for replacing Theorem 3.3 with a quatitative version.

Now we prove the Theorem 5.2.

Proof. We know that σ is compact, so by (2.2), σ is a finite union of closed intervals. Assume we are dealing with one of them, $[a, a + A]$. By continuity, so uniformly continuity of $\gamma(E)$ on compact set σ, for ϵ_0, there exists δ_0 such that

$$|\gamma(E_x) - \gamma(E_y)| \leqslant \epsilon_0, \quad \forall |E_x - E_y| \leqslant \delta_0. \tag{5.1}$$

Divide the interval $[a, a+A]$ into length δ_0 sub-intervals. There are $K = [A/\delta_0] + 1$ of them (the last one may be shorter). Denote them as I_k, for $k = 1, \cdots, K$. For I_k, divide it into $n - 1$ equal sub-subintervals with end points $E_{k1,n}, \cdots, E_{kn,n}$. Then with any E_x, $E_y \in [E_{k1,n}, E_{kn,n}]$, $|\gamma(E_x) - \gamma(E_y)| \leqslant \epsilon_0$.

Since

$$\mathbb{P}\left(\left\{\omega : \exists i = 1, \cdots, n, \ s.t. \ |P_{[0,n],E_{ki,n},\omega}| \geqslant e^{(\gamma(E_{ki,n})+\epsilon_0)(n+1)}\right\}\right) \leqslant ne^{-\eta_0(n+1)},$$

by Borel-Cantelli, for $a.e.\omega$, (We denote this set as Ω_k,) there exists $N(k, \omega)$ for I_k, such that for all $n > N(k, \omega)$,

$$|P_{[0,n],E_{ki,n},\omega}| \leqslant e^{(\gamma(E_{ki,n})+\epsilon_0)(n+1)}, \quad \forall i = 1, \cdots, n.$$

If we denote $\gamma_{k,n} = \inf_{E \in [E_{k1,n}, E_{kn,n}]} \gamma(E)$, then by (5.1)

$$|P_{[0,n],E_{ki,n},\omega}| \leqslant e^{(\gamma(E_{ki,n})+\epsilon_0)(n+1)} \leqslant e^{(\gamma_{k,n}+2\epsilon_0)(n+1)}, \quad \forall i = 1, \cdots, n.$$

Let M big enough such that, for any $n > M$, $n^c \leqslant e^{\epsilon_0(n+1)}$. Thus by Lemma 2.4, for $E \in [E_{k1,n}, E_{kn,n}]$, $n > \max\{N(k, \omega), M\}$,

$$|P_{[0,n],E,\omega}| \leqslant n^c e^{(\gamma_{k,n}+2\epsilon_0)(n+1)} \leqslant n^c e^{(\gamma(E)+2\epsilon_0)(n+1)} \leqslant e^{(\gamma(E)+3\epsilon_0)(n+1)}$$

Let $\Omega_2 = \bigcap_k \Omega(k)$, $\tilde{N}(\omega) = \max_k\{N(k, \omega), M\}$, then for any $n > \tilde{N}(\omega)$,

$$|P_{[0,n],E,\omega}| \leqslant e^{(\gamma(E)+3\epsilon_0)(n+1)}, \quad \forall E \in [0, A]$$

Use the same methods for $P_{[-n,0],E,\omega}$, $P_{[n+1,2n+1],E,\omega}$, and $P_{[2n+1,3n+1],E,\omega}$. $N_2(\omega)$ being the maximum of $\tilde{N}(\omega)$ for each of them would work for our theorem. □

Remark 7. Similar as remark 2 and 5, we can get $\Omega_2(l)$, $N_2(l,\omega)$ instead. Note M is independent of l, and we can then estimate in the same way that, for $a.e.\omega$, (We denote this set as Ω_{N_2}), there exists $L_2 = L_2(\omega)$, such that for any $|l| > L_2$, $N_2(l,\omega) \leqslant \ln^2|l|$

Remark 8. For LD implies continuity of γ.

6. Dynamical Localization

Now we have established the spectral localization for 1-d Anderson Model. With some more effort, we can get the Dynamical localization. We say that H_ω exhibits dynamical localization property if for $a.e.\omega$, for any $\epsilon > 0$, there exists a $\alpha = \alpha(\omega) > 0$, a $C = C(\epsilon,\omega)$, such that for all $x, y \in \mathbb{Z}$:

$$\sup_t |\langle \delta_x, e^{-itH_\omega}\delta_y\rangle| \leqslant C_\epsilon e^{\epsilon|y|}e^{-\alpha|x-y|}$$

According to [5], we only need to prove that for $a.e.\omega$, H_ω has SULE(Semi-Uniformly Localized Eigenfunction). We say H has SULE if H has a complete set $\{\varphi_E\}$ of orthonormal eigenfunctions, there is $\alpha > 0$, $l = l_E \in \mathbb{Z}$, and for each $\epsilon > 0$, a C_ϵ such that for any eigenvalue E,

$$|\varphi_E(x)| \leqslant C_\epsilon e^{\epsilon|l_E|}e^{-\alpha|x-l_E|}$$

For any central point $l \in \mathbb{Z}$, by remark 2, 7, (We use Quantitative Craig-Simon instead of the original one for estimating N_2, Ω_2) remark 5 for l and $l+2n+1$, and their natural extension to $l-2n-1$ and l, (But we keep the original notations, even if now it satisfies both properties.) and the same analysis in section 4, if we let $\Omega(l) = \bigcap_{i=1,2,3}\Omega_i(l)$, then for each $\omega \in \Omega(l)$, there exists $N(l,\omega) = \max\{N_1(l,\omega), N_2(l,\omega), N_3(l,\omega)\}$, such that for any $n > N(l,\omega)$, either l or $l+2n+1$, either l or $l-2n-1$ are $(\mu-8\epsilon_0, n, E, \omega)$-regular for all $E \in \sigma$.

Take $\Omega' = \bigcap\limits_l \Omega_l \cap \bigcap_{i=1,2,3}\Omega_{N_i}$ and fix $\omega \in \Omega'$. (We omit ω from notations from now on.)

By remark 2, remark 5, there exists L_1, L_3 such that for all $|l| > \max\{L_1, L_2, L_3\}$,

$$N_i(l) \leqslant \ln^2|l|, \quad \forall i = 1,2,3$$

for all E.

Let l_E be the maximum point of φ_E. For any $n \geqslant N_4 := \frac{\ln 2}{\mu - 8\epsilon_0}$, l_E is naturally $(\mu-8\epsilon_0, n, E)$-singular by (2.1). So there exists L_4, for any $|l| > L_4$,

$$N_4 < \ln^2|l|$$

for all E.

Let $L = \max\{L_1, L_2, L_3, L_4\}$, $N(l) := \max\{N_1(l), N_2(l), N_3(l), N_4\}$, then for any $|l| > L$,

$$N(l) \leqslant \ln^2|l| \tag{6.1}$$

If $|l_E| > L$, then for any $|x - l_E| \geqslant N(l_E)$, l_E is $(\mu-8\epsilon_0, n, E)$-singular, so x is $(\mu-8\epsilon_0, n, E)$-regular. By (2.1), for any $|x-l_E| \geqslant N(l_E)$

$$|\varphi_E(x)| \leqslant 2e^{-(\mu-8\epsilon_0)|x-l_E|}$$

Since φ_E is normalized, in fact for all x,

$$|\varphi_E(x)| \leqslant e^{(\mu-8\epsilon_0)N(l_E)}e^{-(\mu-8\epsilon_0)|x-l_E|}$$

By (6.1), for any ϵ, there exists $C_{1\epsilon}$ such that

$$|\varphi_E(x)| \leqslant e^{(\mu-8\epsilon_0)\ln^2|l_E|}e^{-(\mu-8\epsilon_0)|x-l_E|} \leqslant C_{1\epsilon}e^{\epsilon|l_E|}e^{-(\mu-8\epsilon_0)|x-l_E|}$$

If $|l_E| \leqslant L$, consider all $i \in [-L, L]$, all $|x-i| < N(i)$. For any ϵ, take $M_2 = \max\limits_i\{e^{\epsilon i}e^{-(\mu-8\epsilon_0)|x-i|}\}$, $C_{2\epsilon} = M^{-1}$, then for all $|x-l_E| < N(l_E)$,

$$|\varphi_E(x)| \leqslant 1 \leqslant C_2\epsilon e^{\epsilon|l_E|}e^{-(\mu-8\epsilon_0)|x-l_E|}$$

As for $|x-l_E| \geqslant N(l_E)$,

$$|\varphi_E(x)| \leqslant e^{-(\mu-8\epsilon_0)|x-l_E|} \leqslant e^{\epsilon|l_E|}e^{-(\mu-8\epsilon_0)|x-l_E|}$$

So $C_\epsilon = \max\{C_{1\epsilon}, C_{2\epsilon}, 1\}$ would work.

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