1. Introduction

The Anderson model is given by a class of discrete analogs of Schrödinger operators H_ω with real $i.i.d$ potetials $\{V_\omega(n)\}$:

$$(H_\omega\Psi)(n) = \Psi(n+1) + \Psi(n-1) + V_\omega(n)\Psi(n), \tag{1.1}$$

where $\omega = \{\omega_n\}_{n\in\mathbb{Z}} \in \Omega = S^{\mathbb{Z}}$, $S \subset \mathbb{R}$ is the topological support of μ, so compact and contains at least two points, μ is a Borel probability on $\mathbb{R}$. *i.e.* for each $n \in \mathbb{Z}$, $V_\omega(n)$ is *i.i.d.* random variables depending on ω_n in (S, μ), but we will consider V_ω in the product probability space $(S^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ as a whole instead. Denote $\mu^{\mathbb{Z}}$ as $\mathbb{P}$,and $\mathbb{P}_{\omega_a,\cdots,\omega_b}$ be the projection of $\mathbb{P}$ to $S^{[a,b]^c\cap\mathbb{Z}}$. Also denote Lebesgue measure as m.

We say that H_ω exhibits the spectral localization property in I if for $a.e.\omega$, H_ω has only pure point spectrum in I and its eigenfunction $\Psi(n)$ decays exponentially in n. We are going to give a new proof for Anderson model based on the large deviation estimates and subharmonicity of Lvapunov exponents.

2. general setup

We know that

$$\sigma := \sigma(H_\omega) = [-2, 2] + S \quad a.e.\omega \tag{2.1}$$

Definition 1. We call E a generalized eigenvalue (denote as g.e.), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H\Psi = E\Psi$. We call $\Psi(n)$ generalized eigenfunction.

Since the set of g.e. supports the spectral measure of H_ω, we only need to show:

Theorem 2.1. *For a.e. ω, for every g.e. E, the corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in n.*

For $[a, b]$ an interval, $a, b \in \mathbb{Z}$, define $H_{[a,b],\omega}$ to be operator H_ω resticted to $[a, b]$ with zero boundary condition outside $[a, b]$. Note that it can be expressed as a "$b - a + 1$"-dimensional matrix. The Green's function defined on $[a, b]$ for H_ω with energy $E \notin \sigma_{[a,b],\omega}$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}$$

Note that this can also be expressed as a "$b - a + 1$"-dimensional matrix. Denote its (x, y) entry as $G_{[a,b],E,\omega}(x, y)$.

We have

$$\Psi(x) = -G_{[a,b],E,\omega}(x, a)\Psi(a-1) - G_{[a,b],E,\omega}(x, b)\Psi(b+1), \quad x \in [a, b] \tag{2.2}$$

Definition 2. For $c > 0, n \in \mathbb{Z}$, we say $x \in \mathbb{Z}$ is (c, n, E, ω)-regular, if

$$G_{[x-n,x+n],E,\omega}(x, x-n) \leqslant e^{-cn}$$

$$G_{[x-n,x+n],E,\omega}(x, x+n) \leqslant e^{-cn}$$

Otherwise, we call it (c, n, E, ω)-singular.

By (2.2) and definition 2, Theorem 2.1 follows from

Theorem 2.2. *There exists Ω_0 with $P(\Omega_0) = 1$, such that for every $\tilde{\omega} \in \Omega_0$, for any g.e.$\tilde{E}$ of $H_{\tilde{\omega}}$, there exist $N = N(\tilde{E}, \tilde{\omega}), C = C(\tilde{E})$, for every $n > N$, $2n$, $2n+1$ are $(C, n, \tilde{E}, \tilde{\omega})$ regular.*

Some other basic settings are below. Denote

$$P_{[a,b],E,\omega} = det(H_{[a,b],E,\omega} - E)$$

If $a = b$, let $P_{[a,b],E,\omega} = 1$, then

$$\left|G_{[a,b],E,\omega}(x,y)\right| = \frac{\left|P_{[a,x-1],E,\omega} P_{[y+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|}, \quad x \leqslant y \tag{2.3}$$

If we denote the transfer matrix $T_{[a,b],E,\omega}$ as the matrix such that

$$\begin{pmatrix} \Psi(b) \\ \Psi(b-1) \end{pmatrix} = T_{[a,b],E,\omega} \begin{pmatrix} \Psi(a) \\ \Psi(a-1) \end{pmatrix}$$

then

$$T_{[a,b],E,\omega} = \begin{pmatrix} P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega} \\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega} \end{pmatrix}$$

The Lyapunov exponent is given by

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 \log \|T_{[0,n],E,\omega}\| dP(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|T_{[0,n],E,\omega}\|, \quad a.e.\omega.$$

Let $\nu = \inf_{E\in\sigma} \gamma(E) > 0$.

We introduce the large deviation theorem here without proof. [1]

Lemma 1 (Large deviation estimates)**.** *For any $\epsilon > 0$, there exists $\eta = \eta(\epsilon) > 0$ such that, there exists $N_0 = N_0(\epsilon)$, for every $b - a > N_0$*

$$\mu\left\{\omega : \left|\frac{1}{b-a+1} \log \|P_{[a,b],E,\omega}\| - \gamma(E)\right| \geqslant \epsilon\right\} \leqslant e^{-\eta(b-a+1)}$$

for every

3. Main technique

Denote

$$B^+_{[a,b],\epsilon} = \left\{(E,\omega) : |P_{[a,b],E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\right\} \tag{3.1}$$

$$B^-_{[a,b],\epsilon} = \left\{(E,\omega) : |P_{[a,b],E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\right\} \tag{3.2}$$

and denote $B^\pm_{[a,b],\epsilon,E} = \{\omega : (E,\omega) \in B^\pm_{[a,b],\epsilon}\}$, $B^\pm_{[a,b],\epsilon,\omega} = \{E : (E,\omega) \in B^\pm_{[a,b],\epsilon}\}$, $B_{[a,b],*} = B^+_{[a,b],*} \cup B^-_{[a,b],*}$.

Let $E_{j,(\omega_a,\cdots,\omega_b)}$ be the eigenvalue of $H_{[a,b],\omega}$ with $\omega|_{[a,b]} = (\omega_a, \cdots, \omega_b)$.

Large deviation theorem gives us the estimate that for all E, a, b, ϵ

$$P(B^\pm_{[a,b],\epsilon,E}) \leqslant e^{-\eta(b-a+1)} \tag{3.3}$$

Assume $\epsilon = \epsilon_0 < \frac{1}{8}\nu$ is fixed for now, so we omit it from the notations until Theorem 3.3. $\eta_0 = \eta(\epsilon_0)$ is the corresponding parameter from lemma 3.3.

Lemma 2. *For $n \geqslant 2$, if x is $(\gamma(E) - 8\epsilon_0, n, E, \omega)$-singular, then*

$$(E,\omega) \in B^-_{[x-n,x+n]} \cup B^+_{[x-n,x]} \cup B^+_{[x,x+n]}$$

Remark 1. Note that from (3.3), for all $E, x, n \geqslant 2$,

$$P(B^-_{[x-n,x+n],E} \cup B^+_{[x-n,x],E} \cup B^+_{[x,x+n],E}) \leqslant 3e^{-\eta_0(n+1)}$$

Proof. Follows imediately from the definition of singularity and (2.3). ☐

Now we will use three theorems to find the proper Ω_0 for Theorem 2.2.

Theorem 3.1. *Let $0 < \delta_0 < \eta_0$. For a.e. ω (denote as Ω_1), there exists $N_1 = N_1(\omega)$, such that for every $n > N_1$,*

$$\max\{m(B^-_{[n+1,3n+1],\omega}), m(B^-_{[-n,n],\omega})\} \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

Proof. By (3.3),

$$m \times \mathbb{P}(B^-_{[n+1,3n+1]}) \leqslant m(\sigma)e^{-\eta_0(2n+1)}$$

$$m \times \mathbb{P}(B^-_{[-n,n]}) \leqslant m(\sigma)e^{-\eta_0(2n+1)}$$

If we denote

$$\Omega_{\delta_0,n,+} = \left\{\omega : m(B^-_{[n+1,3n+1],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\}$$

$$\Omega_{\delta_0,n,-} = \left\{\omega : m(B^-_{[-n,n],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\}$$

By Tchebyshev,

$$P(\Omega^c_{\delta_0,n,\pm}) \leqslant m(\sigma)e^{-\delta_0(2n+1)}$$

By Borel-Cantelli lemma, we get for *a.e.* ω,

$$\max\{m(B^-_{[n+1,3n+1],\omega}), m(B^-_{[-n,n],\omega})\} \leqslant e^{-(\eta_0-\delta_0)(2n+1)},$$

for $n > N_1(\omega)$. ☐

Theorem 3.2 (Craig-Simon)**.** *For a.e.ω(denote as Ω_2), for all E, we have*

$$\max\{\overline{\lim_{n\to\infty}}\frac{1}{n+1}\log\|T_{[-n,0],E,\omega}\|, \overline{\lim_{n\to\infty}}\frac{1}{n+1}\log\|T_{[0,n],E,\omega}\|\} \leqslant \gamma(E)$$
$$\max\{\overline{\lim_{n\to\infty}}\frac{1}{n+1}\log\|T_{[n+1,2n+1],E,\omega}\|, \overline{\lim_{n\to\infty}}\frac{1}{n+1}\log\|T_{[2n+1,3n+1],E,\omega}\|\} \leqslant \gamma(E)$$

Corollary 1. *for every $\omega \in \Omega_2$, for every E, there exists $N_2 = N_2(\omega, E)$, such that for every $n > N_2$,*

$$\max\{\|T_{[-n,0],E,\omega}\|, \|T_{[0,n],E,\omega}\|\} < e^{(\gamma(E)+\epsilon)(n+1)}$$
$$\max\{T_{[n+1,2n+1],E,\omega}\|, \|T_{[2n+1,3n+1],E,\omega}\|\} < e^{(\gamma(E)+\epsilon)(n+1)}$$

Theorem 3.3. *Let $\epsilon > 0, K > 1$, For a.e.ω(denote as $\Omega_3 = \Omega_3(\epsilon, K)$), there exists $N_3 = N_3(\omega)$, for every $n > N_3$, for every $E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}$, for every y_1, y_2 satisfying $-n \leqslant y_1 \leqslant y_2 \leqslant n$, $|-n - y_1| \geqslant \frac{n}{K}$, and $|n - y_2| \geqslant \frac{n}{K}$, we have $E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})} \notin B_{[-n,y_1],\epsilon,\omega} \cup B_{[y_2,n],\epsilon,\omega}$.*

Remark 2. Note that ϵ and $K > 0$ are not fixed yet, we're going to determine them later in section 4.

Proof. Let $\bar{P}$ be the probability that there are some y_1, y_2, j with

$$E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})} \in B_{[-n,y_1],\epsilon,\omega} \cup B_{[y_2,n],\epsilon,\omega}.$$

Note that for any fixed $\omega_c, \cdots, \omega_d$, with $[c, d] \cap [a, b] = \varnothing$, by independence,

$$\mathbb{P}(B_{[a,b],\epsilon,E_{j,(\omega_c,\cdots,\omega_d)}}) = \mathbb{P}_{[a,b]}(B_{[a,b],\epsilon,E_{j,(\omega_c,\cdots,\omega_d)}}) \leqslant e^{-\eta_0(b-a+1)}$$

Applying to $[a,b]=[-n,y_1]$ or $[y_2,n]$, $[c,d]=[n+1,3n+1]$ and integrating over $\omega_{-n},\cdots,\omega_{y_1}$ or $\omega_{y_2},\cdots,\omega_n$, we get

$$\mathbb{P}(B_{[-n,y_1],\epsilon,E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}}\cup B_{[y_2,n],\epsilon,E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}})\leqslant 2e^{-\eta_0(\frac{n}{K}+1)},$$

so

$$\bar{\mathbb{P}}\leqslant(2n+1)^3 2e^{-\eta_0(\frac{n}{K}+1)}$$

Thus by Borel-Cantelli, we can get the result. □

4. Proof of Theorem 2.2

We will only provide a proof that $2n+1$ is (c,n,E,ω)-regular, the argument for $2n$ being similar.

Proof. Let ϵ be small enough such that

$$\epsilon<\min\{(\eta_0-\delta_0)/3,\nu\}. \tag{4.1}$$

Now let

$$L:=e^{(\eta_0-\delta_0-\epsilon)}>1,$$

and note that since V is bounded, by 2.1 we have there exists $M>0$, such that

$$|P_{[a,b],E,\omega}|<M^{(b-a+1)},\quad \forall E\in\sigma,\omega$$

Pick K big enough such that

$$M^{\frac{1}{K}}<L$$

Let$\sigma>0$ be such that

$$M^{\frac{1}{K}}\leqslant L-\sigma<L \tag{4.2}$$

Let $\Omega_0=\Omega_1\cap\Omega_2\cap\Omega_3(\epsilon,K)$, pick $\tilde{\omega}\in\Omega_0$, take $\tilde{E}$ a g.e. for $H_{\tilde{\omega}}$. WLOG assume $\Psi(0)\neq 0$, then there exists N_4, such that for every $n>N_4$, 0 is $(\gamma(\tilde{E})-8\epsilon_0,n,\tilde{E},\tilde{\omega})$-singular.

For $n>N_0=\max N_1(\tilde{\omega}),N_2(\tilde{\omega},\tilde{E}),N_3(\tilde{\omega}),N_4(\tilde{\omega},\tilde{E})$, assume $2n+1$ is $(\gamma(\tilde{E})-8\epsilon_0,n,\tilde{E},\tilde{\omega})$-singular.

- Both 0 and $2n+1$ is $(\gamma(\tilde{E})-8\epsilon_0,n,\tilde{E},\tilde{\omega})$-singular.
- So by Lemma 2, $\tilde{E}\in B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}}\cup B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}}\cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$.
- By Corollary 1 and (3.1), $\tilde{E}\notin B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}}\cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$, so it can only lie in $B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}}$
- Note that in (3.2), $P_{[n+1,3n+1],\epsilon,\epsilon_0,E,\tilde{\omega}}$ is a polynomial in E that has $2n+1$ real zeros (eigenvalues of $H_{[n+1,3n+1],\tilde{\omega}}$), which are all in $B=B_{[n+1,3n+1],\epsilon,\tilde{\omega}}$. Thus B contains less than $2n+1$ intervals near the eigenvalues. $\tilde{E}$ should lie in one of them. By Theorem 3.1, $m(B)\leqslant Ce^{-(\eta_0-\delta_0)(2n+1)}$. So there is some e.v. $E_{j,[n+1,3n+1],\tilde{\omega}}$ of $H_{[n+1,3n+1],\omega}$ such that

  $$|\tilde{E}-E_{j,[n+1,3n+1],\tilde{\omega}}|\leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

  By the same argument, there exists $E_{i,[-n,n],\tilde{\omega}}$, such that

  $$|\tilde{E}-E_{i,[-n,n],\tilde{\omega}}|\leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

- So $|E_{i,[-n,n],\tilde{\omega}}-E_{j,[n+1,3n+1],\tilde{\omega}}|\leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$. However, by Theorem 3.3, one has $E_{j,[n+1,3n+1],\tilde{\omega}}\notin B_{[-n,n],\epsilon,\tilde{\omega}}$, while $E_{i,[-n,n],\tilde{\omega}}\in B_{[-n,n],\epsilon,\tilde{\omega}}$ This will give us a contradiction below.

Since $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$ and $E_{i,[-n,n],\tilde{\omega}}$ is the e.v. of $H_{[-n,n],\tilde{\omega}}$,

$$\left\| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}} \right\| \geqslant \frac{1}{2}e^{(\eta_0-\delta_0)(2n+1)}$$

So there exists $y_1, y_2 \in [-n,n]$ and such that

$$\left| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}}(y_1,y_2) \right| \geqslant \frac{1}{2n}e^{(\eta_0-\delta_0)(2n+1)}$$

Let $E_j = E_{j,[n+1,3n+1],\tilde{\omega}}$, we have $E_j \notin B_{[-n,n],\epsilon,\tilde{\omega}}$, i.e.

$$|P_{[-n,n],\epsilon,E_j,\tilde{\omega}}| \geqslant e^{(\gamma(E_j)-\epsilon)(2n+1)}$$

so

$$\|P_{[-n,y_1],\epsilon,E_j\tilde{\omega}} P_{[y_2,n],\epsilon,E_j,\tilde{\omega}}\| \geqslant \frac{1}{2n}e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\epsilon)(2n+1)} \tag{4.3}$$

Then for the left hand side of (4.3), there are three cases:

(1) both $|-n-y_1| > \frac{n}{K}$ and $|n-y_2| > \frac{n}{K}$
(2) one of them is large, say $|-n-y_1| > \frac{n}{K}$ while $|n-y_2| \leqslant \frac{n}{K}$
(3) both small.

For (1),

$$\frac{1}{2n}e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant e^{2n(\gamma(E_j)+\epsilon)}$$

Since by our choice (4.1), $\eta_0 - \delta_0 + \gamma(E_j) - \epsilon > \gamma(E_j) + \epsilon$, for n large enough, we get a contradiction.

For (2),

$$\frac{1}{2n}e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant e^{(\gamma(E_j)+\epsilon)(2n+1)}(M)^{\frac{n}{K}}$$

is in contradiction with (4.1) and (4.2)

For (3),with (4.1) and (4.2)

$$\begin{aligned}\frac{1}{2n}e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} &\leqslant M^{\frac{2n}{K}} \\ &\leqslant (L-\sigma)^{2n} \\ &\leqslant (e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)} - \sigma)^{2n}\end{aligned}$$

also a contradiction.

So our assumption that $2n+1$ is not $(\gamma(\tilde{E})-8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-regular is false. Theorem 2.2 follows. $\square$

References

[1] Jhishen Tsay and . Some uniform estimates in products of random matrices. *Taiwanese Journal of Mathematics*, pages 291–302, 1999.