1. Introduction

The Anderson model is given by a class of discrete analogs of Schrődinger operators H_ω with real $i.i.d$ potetials $\{V_\omega(n)\}$:

$$(H_\omega\Psi)(n) = \Psi(n+1) + \Psi(n-1) + V_\omega(n)\Psi(n), \tag{1.1}$$

where $\omega = \{\omega_n\}_{n\in\mathbb{Z}} \in \Omega = S^{\mathbb{Z}}$, $S = supp\{\mu\} \subset \mathbb{R}$ is assumed to be compact and contains at least two points, μ is a borel probability on $\mathbb{R}$. *i.e.* for each $n \in \mathbb{Z}$, $V_\omega(n)$ is *i.i.d.* random variables depending on ω_n in (S, μ), but we will consider V_ω in the product probability space $(S^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ as a whole instead.

We say that H_ω exhibits the pectral localization property in an interval I if for $a.e.\omega$, H_ω has only pure point spectrum in I and its eigenfunction $\Psi(n)$ decays exponentially in n. We are gonna give a new proof for Anderson model based on the large deviation estimates and subharmonicity of Lvapunov exponents.

2. general setup

Definition 1 ($g.e.v.$)**.** We call E a generalized eigenvalue (denote as $g.e.v.$), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H\Psi = E\Psi$. We call $\Psi(n)$ generalized eigenfunction.

Then due to the fact from [1] that: With respect to spectral measure μ,

$$\mu(\{g.e.v\}^c = 0)$$

We only need to show:

Theorem 2.1. *For a.e. ω, $\forall$ g.e.v. E. The corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in n.*

In order to get the dacaying speed of Ψ, We estimates the decaying speed of the Green's functions. Assume $[a, b]$ is an interval, $a, b \in \mathbb{Z}$, define $H_{[a,b],\omega}$ to be the the operator H_ω resticted to $[a, b]$ with zero boundary condition outside $[a, b]$. Note that it can be expressed as a "$b - a + 1$"-dim matrix. The Green's function defined on $[a, b]$ for H_ω with energy $E \notin \sigma(H)$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}$$

Note that this can also be expressed as a "$b - a + 1$"-dim matrix. Denote its x line y column elements as $G_{[a,b],E,\omega}(x, y)$.

By the well-known formula:

$$\Psi(x) = -G_{[a,b],E,\omega}(x, a)\Psi(a-1) - G_{[a,b],E,\omega}(x, b)\Psi(b+1), \quad x \in [a, b] \tag{2.1}$$

If one can get that, the Green's function near n, say, for example on $[n-k, n+k]$, is decaying somehow exponentially in n as n growing, then since Ψ on the right-hand-side is polynomially bounded, $\Psi(n)$ on the left-hand-side will decay exponentially in n, too.

This inspires us to define "regular and singular".

Definition 2. For $c > 0, n \in \mathbb{Z}$, we say $n \in \mathbb{Z}$ is (c, n, E, ω)-regular, if

$$G_{[x-n,x+n],E,\omega}(x, x-n) \leqslant e^{-cn}$$
$$G_{[x-n,x+n],E,\omega}(x, x+n) \leqslant e^{-cn}$$

Otherwise, we call it (c, n, E, ω)-singular.

So we only need to prove

Theorem 2.2. *$\exists\Omega_0$ with $P(\Omega_0) = 1$, s.t. $\forall\tilde{\omega} \in \Omega_0$, for any g.e.v. $\tilde{E}$ of $H_{\tilde{\omega}}$, $\exists N = N(\tilde{E}, \tilde{\omega}), C = C(\tilde{E})$, $\forall n > N$, $2n + 1$ is $(C, n, \tilde{E}, \tilde{\omega})$ regular.*

Remark 1. It's similar for even terms. We omit them only because of notation reasons.

Some other basic settings are below. If we denote

$$P_{[a,b],E,\omega} = det(H_{[a,b],E,\omega} - E)$$

if $a = b$, let $P_{[a,b],E,\omega} = 1$ for next formula. By linear algebra calculation, we get:(if $x \leqslant y$)

$$\left|G_{[a,b],E,\omega}(x, y)\right| = \frac{\left|P_{[a,x-1],E,\omega}P_{[y+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|}, \quad x \leqslant y \tag{2.2}$$

$$\left|G_{[a,b](x,y),E,\omega}\right| = \frac{\left|P_{[a,y-1],E,\omega}P_{[x+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|}, \quad x \geqslant y?$$

If we denote the transfer matrix $T_{[a,b],E,\omega}$ as the matrix such that

$$\begin{pmatrix} \Psi(b) \\ \Psi(b-1) \end{pmatrix} = T_{[a,b],E,\omega} \begin{pmatrix} \Psi(a) \\ \Psi(a-1) \end{pmatrix}$$

We can prove by induction that

$$T_{[a,b],E,\omega} = \begin{pmatrix} P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega} \\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega} \end{pmatrix}$$

Definition 3 (Lyapunov Exponent)**.**

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 ln\|T_{[0,n],E,\omega}\| dP(\omega) = \lim_{n\to\infty} \frac{1}{n} ln\|T_{[0,n],E,\omega}\|$$

$$\nu = \inf_{E\in I} \gamma(E) > 0$$

3. MAIN TECHNIQUE

We introduce the large deviation theorem here without proof. [2]

Lemma 1 (large deviation estimates)**.** *For any $\epsilon > 0$, there exists $\eta = \eta(\epsilon) > 0$ such that, $\exists N_0 = N_0(\epsilon)$, $\forall b - a > N_0$*

$$\mu\left\{\omega : \left|\frac{1}{b-a+1} ln\|P_{[a,b],E,\omega}\| - \gamma(E)\right| \geqslant \epsilon\right\} \leqslant e^{-\eta(b-a+1)}$$

Remark 2. Denote

$$B^+_{[a,b],\epsilon} = \left\{(E, \omega) : |P_{[a,b],E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\right\} \tag{3.1}$$

$$B^-_{[a,b],\epsilon} = \left\{(E, \omega) : |P_{[a,b],E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\right\} \tag{3.2}$$

and denote $B^\pm_{[a,b],\epsilon,E} = \{\omega : (E, \omega) \in B_{[a,b],\epsilon}\}^\pm$. And large deviation theorem gives us the estimates that for all E, a, b, ϵ

$$P(B_{[a,b],\epsilon,E}) \leqslant e^{-\eta(b-a+1)} \tag{3.3}$$

Also, denote $E_{j,[a,b],\omega}$, $j = 1, 2, \cdot, b - a + 1$ as eigenvalues of $H_{[a,b],\omega}$.

Assume $\epsilon = \epsilon_0 < \frac{1}{8}\nu$ is fixed for now, so we omit it from the notations untill Theorem 3.3. And η_0 is the corresponding parameter

Lemma 2. *$n \geqslant 2$, if x is $(\gamma(E)-8\epsilon_0, n, E, \omega)$-singular, then $(E,\omega) \in B^-_{[x-n,x+n]} \cup B^+_{[x-n,x]} \cup B^+_{[x,x+n]}$.*

Remark 3. Note that from (3.3), for all $E, x, n \geqslant 2$,

$$P(B^-_{[x-n,x+n],E} \cup B^+_{[x-n,x],E} \cup B^+_{[x,x+n],E}) \leqslant 3Ce^{-\eta_0 n}$$

Proof. Follows imediately from definition of singularity and (2.2). □

Theorem 3.1. *Let $0 < \delta_0 < \eta_0$, for a.e. ω (denote as Ω_1), $\exists N_1 = N_1(\omega)$, s.t. $\forall n > N_1$, $m(B^-_{[n+1,3n+1],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$ and $m(B^-_{[-n,n],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$*

Proof. By (3.3), $\forall E \in I$, $P(B^-_{[n+1,3n+1],E}) \leqslant e^{-\eta_0(2n+1)}$ and $P(B^-_{[-n,n],E}) \leqslant e^{-\eta_0(2n+1)}$

If we denote

$$\Omega_{\delta_0,n,+} = \left\{\omega : m(B^-_{[n+1,3n+1],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\}$$

$$\Omega_{\delta_0,n,-} = \left\{\omega : m(B^-_{[-n,n],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\}$$

By Tchebyshev,

$$P(\Omega^c_{\delta_0,n,\pm}) \leqslant m(I)e^{-\delta_0(2n+1)}$$

By Borel-Cantelli lemma, we get for *a.e.* ω,

$$\max\{m(B^-_{[n+1,3n+1],\omega}), m(B^-_{[-n,n],\omega})\} \leqslant e^{-(\eta_0-\delta_0)(2n+1)},$$

for $n > N_1(\omega)$. □

Theorem 3.2 (Craig-Simon)**.** *For a.e.ω(denote as Ω_2), for all E, we have*

$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} ln\|T_{[-n,0],E,\omega}\| \leqslant \gamma(E)$$
$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} ln\|T_{[0,n],E,\omega}\| \leqslant \gamma(E)$$
$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} ln\|T_{[n+1,2n+1],E,\omega}\| \leqslant \gamma(E)$$
$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} ln\|T_{[2n+1,3n+1],E,\omega}\| \leqslant \gamma(E)$$

Corollary 1. *$\forall \omega \in \Omega_2$, $\forall E$, $\exists N_2 = N_2(\omega, E)$, s.t. $\forall n > N_2$,*

$$\|T_{[-n,0],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)}$$
$$\|T_{[0,n],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)}$$
$$\|T_{[n+1,2n+1],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)}$$
$$\|T_{[2n+1,3n+1],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)}$$

Remark 4. Basically speaking, the only difference from Theorem 1.5 in [3] is that we are considering restrictions on some different box-sequences, for example $\{[n+1, 2n+1]\}$, instead of the original boxes $\{[0,n]\}$. However, by3, $\gamma(E)$ keeps constant under $\{[n+1, 2n+1]\}$, so subharmonic. While $\gamma_(\bar{E})$ as limsup of $\gamma_(E)_[n+1, 2n+1]$ is still submean since $\gamma_(E)_[n+1, 2n+1]$ are submean. By properties of submean and subharmonic, together with Fustenberg Theorem and Fubini, we can get the results.

Theorem 3.3. *$\epsilon > 0, K > 1$, For a.e.ω(denote as $\Omega_3 = \Omega_3(\epsilon, K)$), $\exists N_3 = N_3(\omega)$, $\forall n > N_3$, $\forall E_{j,[n+1,3n+1],\omega}$, $\forall y_1, y_2$ satisfy $-n \leqslant y_1 \leqslant y_2 \leqslant n$, $|-n - y_1| \geqslant \frac{n}{K}$, and $|n - y_2| \geqslant \frac{n}{K}$, we have $E_{j,[n+1,3n+1],\omega} \notin B_{[-n,y_1],\epsilon,\omega} \cup B_{[y_2,n],\epsilon,\omega}$.*

Remark 5. Note that ϵ and $K > 0$ is not fixed yet, we're gonna determine it later on in section 4.

Proof. Let

$$\bar{P} = P\left(\bigcup_{y_1,y_2} \bigcup_{j=1}^{2n+1} B_{[-n,y_1],\epsilon,E_{j,[n+1,3n+1],\omega}} \cup B_{[y_2,n],\epsilon,E_{j,[n+1,3n+1],\omega}}\right)$$

be the probability that there are some y_1, y_2 and $E_{j,[n+1,3n+1],\omega}$ satisfying the condition. By (3.3), for any E, $P(B_{[a,b],E}) < e^{-\eta_0 - \delta_0(b-a+1)}$. Since $B_{[a,b],E}$ is a cylinder set that depends only on $\omega_i, i \in [a, b]$, we have that for any $[c, d] \cap [a, b] = \varnothing$,

$$P(B_[a, b], E | \{\omega : E = E_{j,[c,d],\omega}\}) = P(B_[a, b], E)$$

Integrating over ? ∎

4. Proof of Theorem 2.2

Proof. Let $\Omega_0 = \Omega_1 \cap \Omega_2 \cap \Omega_3(\epsilon, K)$ (ϵ, K to be determined later), pick $\tilde{\omega} \in \Omega_0$, take $\tilde{E}$ a g.e.v. for $H_{\tilde{\omega}}$. WLOG assume $\Psi(0) \neq 0$, then $\exists N_4$, s.t. $\forall n > N_4$, 0 is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular.

For $n > N_0 = \max N_1, N_2, N_3, N_4$, assume $2n+1$ is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular.

- Both 0 and $2n + 1$ is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular.
- So by lemma 2, $\tilde{E} \in B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$.
- By corollary 1 and (3.1), $\tilde{E} \notin B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$, so it can only lies in $B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}}$
- Note that by (3.2), $P_{[n+1,3n+1],\epsilon,\epsilon_0,E,\tilde{\omega}}$ in $B = B_{[n+1,3n+1],\epsilon,\tilde{\omega}}$ is a polynomial in E that have $2n + 1$ real zeros (eigenvalues of $H_{[n+1,3n+1],\tilde{\omega}}$), which are all in B. Thus B contains less than $2n+1$ intervals near the eigenvalues. $\tilde{E}$ should lie in one of them. By Theorem 3.1, $m(B) \leqslant Ce^{-(\eta_0-\delta_0)(2n+1)}$. So there is some e.v. $E_{j,[n+1,3n+1],\tilde{\omega}}$ of $H_{[n+1,3n+1],\omega}$ s.t.

  $$|\tilde{E} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

  By the same argument, $\exists E_{i,[-n,n],\tilde{\omega}}$, s.t.

  $$|\tilde{E} - E_{i,[-n,n],\tilde{\omega}}| \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

- So $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$. However, by Theorem 3.3, one have $E_{j,[n+1,3n+1],\tilde{\omega}} \notin B_{[-n,n],\epsilon,\tilde{\omega}}$, while $E_{i,[-n,n],\tilde{\omega}} \in B_{[-n,n],\epsilon,\tilde{\omega}}$ This will give us a contradiction below.

Since $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$ and $E_{i,[-n,n],\tilde{\omega}}$ being the e.v. of $H_{[-n,n],\tilde{\omega}}$,

$$\left\| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}} \right\| \geqslant \frac{1}{2} e^{(\eta_0-\delta_0)(2n+1)}$$

So $\exists y_{n1}, y_{n2} \in [-n, n]$ s.t. NEED FIX

$$\left| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}}(y_{n1}, y_{n2}) \right| \geqslant \frac{1}{2} e^{(\eta_0-\delta_0)(2n+1)}$$

But $E_{j,[n+1,3n+1],\omega} \notin B_{[-n,n],\epsilon,\tilde{\omega}}$, i.e.

$$|P_{[-n,n],\epsilon,E_{j,[n+1,3n+1],\omega},\tilde{\omega}}| \geqslant e^{(\gamma(E_j)-\epsilon)(2n+1)}$$

so

$$\|P_{[-n,y_{n1}],\epsilon,E_j} P_{[y_{n2},n],\epsilon,E_j}\| \geqslant \frac{1}{2} e^{(\eta_0-\delta-0)(2n+1)} e^{(\gamma(E_j)-\epsilon)(2n+1)} \tag{4.1}$$

Let $M = sup\{|V| + |E_j| + 2\}$, where $|V|$ is assumed bounded, E_i, E_j are bounded because they are close to $E \in I$.
Then pick ϵ small enough in Theorem 3.3 s.t.

$$\epsilon < \min\{(\eta_0 - \delta_0)/3, \nu\} \tag{4.2}$$

and fix it, then let

$$L := e^{(\eta_0-\delta_0-\epsilon)} > 1$$

Pick K big enough in Theorem 3.3 to be s.t.

$$(3M)^{\frac{1}{K}} < L$$

say, $\exists \sigma > 0$,

$$(3M)^{\frac{1}{K}} \leqslant L - \sigma < L \tag{4.3}$$

then for left hand side of (4.1), there are three cases:

(1) both $|-n-y_{1n}| > \frac{n}{K}$ and $|n-y_{2n}| > \frac{n}{K}$
(2) one of them is large, say $|-n-y_{1n}| > \frac{n}{K}$ and $|x_{2n} - y_2 n| \leqslant \frac{n}{K}$
(3) both small.

for (1),

$$\frac{1}{2} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant e^{2n(\gamma(E_j)+\epsilon)}$$

by our choice (4.2), $\eta - \delta + \gamma(E_j) - \epsilon > \gamma(E_j) + \epsilon$. Then for n large enough, we get contradiction.

for (2), similarly with (4.2) and(4.3)

$$\begin{aligned}
\frac{1}{2} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} &\leqslant e^{(\gamma(E_j)+\epsilon)(2n+1)} (3M)^{\frac{n}{K}} \\
\frac{1}{2} e^{(\eta_0-\delta_0-\epsilon)(2n+1)} &\leqslant e^{\epsilon(2n+1)} L^n \\
&\leqslant e^{\epsilon(2n+1)} e^{(\eta_0-\delta_0-\epsilon)n} \\
\frac{1}{2} e^{(\eta_0-\delta_0-\epsilon)(n+1)} &\leqslant e^{2\epsilon(n+1)}
\end{aligned}$$

We get contradiction.

for (3),with (4.2) and (4.3)

$$\begin{aligned}
\frac{1}{2} e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} &\leqslant (3M)^{\frac{2n}{K}} \\
&\leqslant (L-\sigma)^{2n} \\
&\leqslant (e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)} - \sigma)^{2n}
\end{aligned}$$

Contradiction.

So our assumption that $2n+1$ is not eventually $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-regular is flase. Theorem 2.2 follows. $\square$

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[2] Jhishen Tsay and . Some uniform estimates in products of random matrices. *Taiwanese Journal of Mathematics*, pages 291–302, 1999.

[3] Walter Craig, Barry Simon, et al. Subharmonicity of the lyaponov index. *Duke Mathematical Journal*, 50(2):551–560, 1983.