

PROOF FOR ANDERSON LOCALIZATION

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This paper is dedicated to our advisors.

ABSTRACT. This paper gives a new proof of localization for one dimension Anderson model.

1. INTRODUCTION

The Anderson model is given by a class of discrete analogs of Schrödinger operators H_ω with real *i.i.d* potentials $\{V_\omega(n)\}$:

$$(1.1) \quad (H_\omega \Psi)(n) = \Psi(n+1) + \Psi(n-1) + V_\omega(n)\Psi(n),$$

where $\omega = \{\omega_n\}_{n \in \mathbb{Z}} \in \Omega = S^{\mathbb{Z}}$, $S = \text{supp}\{\mu\} \subset \mathbb{R}$ is assumed to be compact and contains at least two points, μ is a borel probability on \mathbb{R} . *i.e.* for each $n \in \mathbb{Z}$, $V_\omega(n)$ is *i.i.d.* random variables depending on ω_n in (S, μ) , but we will consider V_ω in the product probability space $(S^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ as a whole instead.

We say that H_ω exhibits the pectral localization property in an interval I if for *a.e.* ω , H_ω has only pure point spectrum in I and its eigenfunction $\Psi(n)$ decays exponentially in n . We are gonna give a new proof for Anderson model based on the large deviation estimates and subharmonicity of Lvapunov exponents.

2. GENERAL SETUP

Definition 1 (*g.e.v.*). We call E a generalized eigenvalue (denote as *g.e.v.*), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H\Psi = E\Psi$. We call $\Psi(n)$ generalized eigenfunction.

Then due to the fact from [1] that: *spectrally almost surely*,

$$\sigma(H) = \overline{\{E : E \text{ is g.e.v.}\}},$$

We only need to show:

Theorem 2.1. *For a.e. ω , \forall g.e.v. $E \in I$, I is a closed subset of \mathbb{R} . The corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in n .*

In order to get the dacaying speed of Ψ , We estimates the decaying speed of the Green's functions. Assume $[a, b]$ is an interval, $a, b \in \mathbb{Z}$, define $H_{[a,b],\omega}$ to be the the operator H_ω resticted to $[a, b]$ with zero boundary condition outside $[a, b]$. Note that it can be expressed as a " $b - a + 1$ "-dim matrix. The Green's function defined on $[a, b]$ for H_ω with energy $E \notin \sigma(H)$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}$$

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Note that this can also be expressed as a " $b - a + 1$ "-dim matrix. Denote its x line y column elements as $G_{[a,b],E,\omega}(x,y)$.

By the well-known formula:

$$(2.1) \quad \Psi(x) = G_{[a,b],E,\omega}(x,a)\Psi(a) + G_{[a,b],E,\omega}(x,b)\Psi(b), \quad x \in [a,b]$$

If one can get that, the Green's function near n , say, for example on $[n-k, n+k]$, is decaying somehow exponentially in n as n growing, then since Ψ on the right-hand-side is polynomially bounded, $\Psi(n)$ on the left-hand-side will decay exponentially in n , too.

This, together with another purposes for using large deviation estimates, which you will see in !!!!(one lemma), inspires us to define "regular and singular".

Definition 2. $\mathcal{A}_{[a,b]} = \{[a,b], [a,b-1], [a+1,b], [a+1,b-1]\}$

Definition 3. For $c > 0, k \in \mathbb{Z}$, we say $x \in \mathbb{Z}$ is (c,k,E,ω) -regular, if $\exists A \in \mathcal{A}_{[x-k,x+k]}$, s.t.

$$G_{A,E,\omega}(x, \partial A) \leq e^{-c|x-\partial A|}$$

i.e. if $A = [x_1, x_2]$

$$G_{A,E,\omega}(x, x_i) \leq e^{-c|x-x_i|}, \quad i = 1, 2$$

Otherwise, we call it singular.

So we only need to prove

Theorem 2.2. $\exists \Omega_0$ with $P(\Omega_0) = 1$, s.t. $\forall \tilde{\omega} \in \Omega_0$, take \tilde{E} to be any g.e.v. of $H_{\tilde{\omega}}$, $\exists N = N(\tilde{E}, \tilde{\omega}), C = C(\tilde{E}), \forall n > N, 2n+1$ is $(C, n, \tilde{E}, \tilde{\omega})$ regular.

Remark 1. It's similar for even terms. We omit them only because of notation reasons.

Remark 2. Because if we achieve this, denote the corresponding polynomially bounded generalized eigenfunction as $\Psi(n) = \Psi_{\tilde{\omega}, \tilde{E}}(n) \leq M(1+n)^p$, for $p > 0$. Then $\forall n > N, \exists A_n \in \mathcal{A}_{[n+1, 3n+1]}$, s.t.

$$G_{A,E,\omega}(x, \partial A_n) \leq e^{-C|x-\partial A_n|}$$

so by equation 2.1, and $|x - \partial A_n| \geq n - 1$.

$$|\Psi(2n+1)| \leq M e^{-C(n-1)} (1+3n+1)^p$$

So for large enough n , $\Psi(2n+1)$ decays exponentially in n . Similarly for even terms and we will get Theorem 2.1.

Some other basic settings are below. If we denote

$$P_{[a,b],E,\omega} = \det(H_{[a,b],E,\omega} - E)$$

By linear algebra calculation, we get:(if $x \leq y$)

$$|G_{[a,b](x,y),E,\omega}| = \frac{|P_{[a,x-1],E,\omega} P_{[y+1,b],E,\omega}|}{|P_{[a,b],E,\omega}|}, \quad x \leq y$$

$$|G_{[a,b](x,y),E,\omega}| = \frac{|P_{[a,y-1],E,\omega} P_{[x+1,b],E,\omega}|}{|P_{[a,b],E,\omega}|}, \quad x \geq y$$

If we denote the transfer matrix $T_{[a,b],E,\omega}$ as the matrix such that

$$\begin{pmatrix} \Psi(b) \\ \Psi(b-1) \end{pmatrix} = T_{[a,b],E,\omega} \begin{pmatrix} \Psi(a) \\ \Psi(a-1) \end{pmatrix}$$

We can prove by induction that

$$T_{[a,b],E,\omega} = \begin{pmatrix} P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega} \\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega} \end{pmatrix}$$

Definition 4 (Lyapunov Exponent).

$$\gamma(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \ln \|T_{[0,n],E,\omega}\| dP(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|T_{[0,n],E,\omega}\|$$

3. MAIN TECHNIQUE

We introduce the large deviation theorem here without proof.[2]

Lemma 1 (large deviation estimates). *For any $\epsilon > 0$, there exists $C = C(\epsilon) > 0, \eta = \eta(\epsilon) > 0$ such that*

$$\mu \left\{ \omega : \left| \frac{1}{b-a+1} \ln \|T_{[a,b],E,\omega}\| - \gamma(E) \right| \geq \epsilon \right\} \leq C e^{-\eta(b-a+1)}$$

Remark 3. If we denote

$$B_{[a,b]} = \left\{ (E, \omega) : \exists A \in \mathcal{A}_{[a,b]} \text{ s.t. } |P_{A,E,\omega}| \geq e^{(\gamma(E)+\epsilon)(b-a+1)} \right\} \\ \bigcup \left\{ (E, \omega) : \forall A \in \mathcal{A}_{[a,b]}, |P_{A,E,\omega}| \leq e^{(\gamma(E)-\epsilon)(b-a+1)} \right\}$$

we find that

$$B_{[a,b]} \subseteq \left\{ (E, \omega) : \left| \frac{1}{b-a+1} \ln \|T_{[a,b],E,\omega}\| - \gamma(E) \right| \geq \epsilon \right\}$$

This is the "bad" set. And Large deviation theorem gives us the estimates that for all E, a, b

$$(3.1) \quad P(\{\omega : (E, \omega) \in B_{[a,b]}\}) \leq C e^{-\eta(b-a+1)}$$

Moreover, we denote

$$(3.2) \quad B_{[a,b]}^+ = \left\{ (E, \omega) : \exists A \in \mathcal{A}_{[a,b]} \text{ s.t. } |P_{A,E,\omega}| \geq e^{(\gamma(E)+\epsilon)(b-a+1)} \right\} \\ = \bigcup_{A \in \mathcal{A}_{[a,b]}} \left\{ (E, \omega) : |P_{A,E,\omega}| \geq e^{(\gamma(E)+\epsilon)(b-a+1)} \right\}$$

$$(3.3) \quad B_{[a,b]}^- = \left\{ (E, \omega) : \forall A \in \mathcal{A}_{[a,b]}, |P_{A,E,\omega}| \leq e^{(\gamma(E)-\epsilon)(b-a+1)} \right\} \\ = \bigcap_{A \in \mathcal{A}_{[a,b]}} \left\{ (E, \omega) : |P_{A,E,\omega}| \leq e^{(\gamma(E)-\epsilon)(b-a+1)} \right\}$$

and denote $B_{[a,b],E} = \{\omega : (E, \omega) \in B_{[a,b]}\}$. Others similar.

Lemma 2. $n \geq 2$, if x is $(\gamma(E)) - 8\epsilon, n, E, \omega$ -singular, then $(E, \omega) \in B_{[x-n, x+n]}^- \cup B_{[x-n, x]}^+ \cup B_{[x, x+n]}^+$.

Remark 4. Note that from (3.1), for all $E, x, n \geq 2$,

$$P(B_{[x-n, x+n], E}^- \cup B_{[x-n, x], E}^+ \cup B_{[x, x+n], E}^+) \leq 3C e^{-\eta n}$$

Proof. Assume not,

$$\begin{cases} \exists A_{[x-n, x+n]} \in \mathcal{A}_{[x-n, x+n]} \text{ s.t. } |P_{A_{[x-n, x+n]}, E, \omega}| \geq e^{(\gamma(E)+\epsilon)(2n+1)} \\ \forall A_{[x-n, x]} \in \mathcal{A}_{[x-n, x]}, |P_{A_{[x-n, x]}, E, \omega}| \leq e^{(\gamma(E)-\epsilon)(n+1)} \\ \forall A_{[x, x+n]} \in \mathcal{A}_{[x, x+n]}, |P_{A_{[x, x+n]}, E, \omega}| \leq e^{(\gamma(E)-\epsilon)(n+1)} \end{cases}$$

Pick $A_{[x-n, x+n]}$ as above. If denote it as $[x_1, x_2]$, i.e. $x_1 = x - n$ or $x - n + 1$, $x_2 = x + n$ or $x + n - 1$, then

$$\begin{aligned} |G_{[x_1, x_2], E, \omega}(x, x_1)| &= \frac{|P_{[x+1, x_2], E, \omega}|}{|P_{[x_1, x_2], E, \omega}|} \\ &\leq \frac{e^{(\gamma(E)+\epsilon)(x_2-x+1)}}{e^{(\gamma(E)-\epsilon)(x_2-x_1+1)}} \\ &\leq e^{-\gamma(E)|x-x_1|+\epsilon(|x_2-x+1|+|x_2-x_1+1|)} \\ &\leq e^{-\gamma(E)|x-x_1|+\epsilon(3n+2)} \\ &\leq e^{-\gamma(E)|x-x_1|+8\epsilon|x-x_1|} \\ &\leq e^{-(\gamma(E)-8\epsilon)|x-x_1|} \end{aligned}$$

Similar for $G_{[x_1, x_2], E, \omega}(x, x_2)$. Thus x is $(\gamma(E)) - 8\epsilon, n, E, \omega$ -regular, contradiction. \square

By Theorem 2.2,

Theorem 3.1. Let $0 < \delta < \eta$, Let $E \in I$. For a.e. ω (denote as Ω_1), $\exists N_1 = N_1(\omega)$, s.t. $\forall n > N_1$, $m(B_{[n+1, 3n+1], \omega}^-) \leq Ce^{-(\eta-\delta)(2n+1)}$ and $m(B_{[-n, n], \omega}^-) \leq Ce^{-(\eta-\delta)(2n+1)}$

Proof. We only prove for $m(B_{[n+1, 3n+1], \omega}^-)$.

By (3.1), $\forall E \in I$, $P(B_{[n+1, 3n+1], E}^-) \leq Ce^{-\eta(2n+1)}$.

If we denote

$$\Omega_{\delta, n} = \left\{ \omega : m(B_{[n+1, 3n+1], \omega}^-) \leq e^{-(\eta-\delta)(2n+1)} \right\}$$

By chebyshev,

$$\begin{aligned} P(\Omega_{\delta, n}^c) &\leq e^{(\eta-\delta)n} \int_{\Omega} m(B_{[n+1, 3n+1], \omega}) dP\omega \\ &= e^{(\eta-\delta)n} \int_I P(B_{[n+1, 3n+1], E}) dx \\ &\leq e^{(\eta-\delta)n} Cm(I) e^{-\eta(2n+1)} \\ &= Cm(I) e^{-\delta(2n+1)} \end{aligned}$$

By Borel-Cantelli lemma, we get

for a.e. ω , $\exists N$, s.t. $\forall n > N$, $\omega \in \Omega_{\delta, n}$, i.e. $m(B_{[n+1, 3n+1], \omega}^-) \leq e^{-(\eta-\delta)(2n+1)}$.

Similar for $[-n, n]$, pick insection for Ω_1 , and maximum for N_1 . \square

Theorem 3.2 (Craig-Simon). *For a.e. ω (denote as Ω_2), we have*

$$\begin{aligned}\overline{\lim}_{n \rightarrow \infty} \frac{1}{n+1} \ln \|T_{[-n,0],E,\omega}\| &\leq \gamma(E) \\ \overline{\lim}_{n \rightarrow \infty} \frac{1}{n+1} \ln \|T_{[0,n],E,\omega}\| &\leq \gamma(E) \\ \overline{\lim}_{n \rightarrow \infty} \frac{1}{n+1} \ln \|T_{[n+1,2n+1],E,\omega}\| &\leq \gamma(E) \\ \overline{\lim}_{n \rightarrow \infty} \frac{1}{n+1} \ln \|T_{[2n+1,3n+1],E,\omega}\| &\leq \gamma(E)\end{aligned}$$

Corollary 1. $\forall \omega \in \Omega_2, \exists N_2 = N_2(\omega), \text{ s.t. } \forall n > N_2,$

$$\begin{aligned}\|T_{[-n,0],E,\omega}\| &< e^{(\gamma(E)+\epsilon)(n+1)} \\ \|T_{[0,n],E,\omega}\| &< e^{(\gamma(E)+\epsilon)(n+1)} \\ \|T_{[n+1,2n+1],E,\omega}\| &< e^{(\gamma(E)+\epsilon)(n+1)} \\ \|T_{[2n+1,3n+1],E,\omega}\| &< e^{(\gamma(E)+\epsilon)(n+1)}\end{aligned}$$

Remark 5. The only difference between here and [3] is the restricted intervals. As long as one follows the proof there, one can get the results here.

Theorem 3.3. $K > 0$, *For a.e. ω (denote as $\Omega_3 = \Omega_3(K)$), $\exists N_3 = N_3(\omega), \forall n > N_3,$ $\forall E_{j,[n+1,3n+1],\omega}$ being eigenvalue of $H_{[n+1,3n+1],\omega}, \forall y_1, y_2$ satisfy $-n \leq y_1 \leq y_2 \leq n, |-n-y_1| \geq \frac{n}{K}, \text{ and } |n-y_2| \geq \frac{n}{K}, \text{ we have } E_{j,[n+1,3n+1],\omega} \notin B_{[-n,y_1],\omega} \cup B_{[y_2,n],\omega}.$*

Remark 6. Note that $K > 0$ is not determined yet, we're gonna determine it later on.

Proof. In order to use Borel-Cantelli, one need to estimate

$$P \left(\bigcup_{y_1, y_2} \bigcup_{j=1}^{2n+1} B_{[-n,y_1],E_{j,[n+1,3n+1],\omega}} \cup B_{[y_2,n],E_{j,[n+1,3n+1],\omega}} \right)$$

where y_1, y_2 satisfy assumptions above. Denote it by \bar{P} . So consider

$$\begin{aligned}P \left(B_{[y_2,n],E_{j,[n+1,3n+1],\omega}} \right) &= \int_{\Omega} \chi_{B_{[y_2,n],E_{j,[n+1,3n+1],\omega}}} dP\omega \\ &= \int_{S^{2n+1}} \left(\int_{\tilde{\Omega}} \chi_B d\tilde{\mu} \right) d\mu^{2n+1}(\omega_{n+1}, \dots, \omega_{3n+1}) \\ &= \int_{S^{2n+1}} \tilde{P}(\tilde{B}_{[y_2,n],E_{j,[n+1,3n+1],\omega}}) d\mu^{2n+1}(\omega_{n+1}, \dots, \omega_{3n+1})\end{aligned}$$

where for $\tilde{\Omega}$ and $\tilde{\mu}$ one take away the $[n+1, 3n+1]$ terms from Ω and $\mu^{\mathbb{Z}}$. However, for any fixed E , $B_{[y_2,n],E}$ is of the form

$$\left(\bigotimes_{i \in [y_2,n]} S \right) \times B'_{[y_2,n]}$$

where

$$B'_{[y_2,n],E} = \{ \omega|_{[y_2,n]} : \omega \in B_{[y_2,n],E} \}$$

So,

$$\begin{aligned} P(B_{[y_2, n], E}) &= \int_{S^{2n+1}} \left(\int_{\tilde{\Omega}} \chi_{B_{[y_2, n], E}} d\tilde{\mu} \right) d\mu^{2n+1}(\omega_{n+1}, \dots, \omega_{3n+1}) \\ &= \tilde{P}(\tilde{B}_{[y_2, n], E}) \times 1 \times \dots \times 1 \\ &= \tilde{P}(\tilde{B}_{[y_2, n], E}) \end{aligned}$$

Br (3.1),

$$\tilde{P}(\tilde{B}_{[y_2, n], E}) \leq C e^{-\eta|n-y_2|}, \quad \forall E$$

So

$$\begin{aligned} P(B_{[y_2, n], E_{j, [n+1, 3n+1], \omega}}) &\leq C e^{-|n-y_2|} \leq C e^{-n/K} \\ \bar{P} &\leq C(2n+1)^3 e^{-n/K} \end{aligned}$$

The sum over n is finite, use Borel-Cantelli, we can get the result. \square

4. ???

Let $\Omega_0 = \Omega_1 \cap \Omega_2 \cap \Omega_3(K)$ (K is to be determined later), pick $\tilde{\omega} \in \Omega_0$, take \tilde{E} a g.e.v. for $H_{\tilde{\omega}}$. WLOG assume $\Psi(0) \neq 0$, then $\exists N_4$, s.t. $\forall n > N_4$, 0 is $(\gamma(\tilde{E}), n, \tilde{E}, \tilde{\omega})$ -singular.

Assume $2n+1$ is not eventually $(\gamma(\tilde{E}), n, \tilde{E}, \tilde{\omega})$ -regular, then $\exists \{n_k\}$ with $n_k \rightarrow \infty$, s.t. $2n_k+1$ is $(\gamma(\tilde{E}), n_k, \tilde{E}, \tilde{\omega})$ -singular.

Then, for those $n_k > N_0 = \max N_1, N_2, N_3, N_4$, WLOG use $\{n\}$ instead of $\{n_k\}$, we have that

- Both 0 and n is $(\gamma(\tilde{E}), n, \tilde{E}, \tilde{\omega})$ -singular. (We focus on n below, 0 is similar.)
- So by lemma 2, $\tilde{E} \in B_{[n+1, 3n+1], \tilde{\omega}}^- \cup B_{[n+1, 2n+1], \tilde{\omega}}^+ \cup B_{[2n+1, 3n+1], \tilde{\omega}}^+$.
- By corollary 1 and definition 3.2, $\tilde{E} \notin B_{[n+1, 2n+1], \tilde{\omega}}^+ \cup B_{[2n+1, 3n+1], \tilde{\omega}}^+$, it can only lie in $B_{[n+1, 3n+1], \tilde{\omega}}^-$
- But

$$(4.1) \quad B_{[n+1, 3n+1], \tilde{\omega}}^- \subset \left\{ E : |P_{[n+1, 3n+1], E, \tilde{\omega}}| \leq e^{(\gamma(E)-\epsilon)(2n+1)} \right\}$$

by definition 3.3. Note that $P_{[n+1, 3n+1], E, \tilde{\omega}}$ is indeed a polynomial in E and have $2n+1$ zeros (e.v.), which are all in the bad set (4.1). So this bad set (4.1) contains less than $2n+1$ intervals near the zeros. \tilde{E} should lie in one of them. And by Theorem 3.1, $m(B_{[n+1, 3n+1], \tilde{\omega}}^-) \leq C e^{-(\eta-\delta)(2n+1)}$

$$|\tilde{E} - E_{j, [n+1, 3n+1], \tilde{\omega}}| \leq e^{-(\eta-\delta)(2n+1)}?$$

similarly,

$$|\tilde{E} - E_{i, [-n, n], \tilde{\omega}}| \leq e^{-(\eta-\delta)(2n+1)}$$

- So $|E_{i, [-n, n], \tilde{\omega}} - E_{j, [n+1, 3n+1], \tilde{\omega}}| \leq 2e^{-(\eta-\delta)(2n+1)}$. However, by Theorem ??, one have $E_{j, [n+1, 3n+1], \tilde{\omega}} \notin B_{[-n, n], \tilde{\omega}}$, while $E_{i, [-n, n], \tilde{\omega}} \in B_{[-n, n], \tilde{\omega}}$. This will give us a contradiction.

proof of contradiction. Since $E_{j, [n+1, 3n+1], \tilde{\omega}} \notin B_{[-n, n], \tilde{\omega}}$, i.e. $\exists A_{[-n, n]} \in \mathbb{A}_{[-n, n]}$,

$$|P_{A_{[-n, n]}, E_{j, [n+1, 3n+1], \tilde{\omega}}, \tilde{\omega}}| \geq e^{(\gamma(E_j)-\epsilon)(2n+1)}$$

But $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leq 2e^{-(\eta-\delta)(2n+1)}$ and $E_{i,[-n,n],\tilde{\omega}}$ being the e.v. gives us

$$\|G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}}\| \geq \frac{1}{2}e^{(\eta-\delta)(2n+1)}$$

So $\exists y_{n1}, y_{n2} \in A_{[-n,n]}$ s.t.

$$\left|G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}}(y_{n1}, y_{n2})\right| \geq \frac{1}{2}e^{(\eta-\delta)(2n+1)}$$

Denote $A_{[-n,n]} = [x_{n1}, x_{n2}]$, so

$$(4.2) \quad \|P_{[x_{n1}, y_{n1}], E_j} P_{[y_{n2}, x_{n2}], E_j}\| \geq \frac{1}{2}e^{(\eta-\delta)(2n+1)}e^{(\gamma(E_j)-\epsilon)(2n+1)}$$

Let $M = \sup\{|V| + |E_i| + |E_j| + 2\}$, where $|V|$ is assumed bounded, E_i, E_j are bounded because they are close to $E \in I$. Then pick ϵ small enough s.t.

$$L := e^{(\eta-\delta)}e^{(\gamma(E_j)-\epsilon)} > 1$$

Pick K in Theorem ?? to be s.t.

$$(3M)^{\frac{1}{K}} < L$$

then for left hand side of 4.2, there are three cases:

- (1) both $|x_{1n} - y_{1n}| > \frac{n}{K}$ and $|x_{2n} - y_{2n}| > \frac{n}{K}$
- (2) one of them is large, say $|x_{1n} - y_{1n}| > \frac{n}{K}$ and $|x_{2n} - y_{2n}| \leq \frac{n}{K}$
- (3) both small.

for (1),

$$\frac{1}{2}e^{(\eta-\delta+\gamma(E_j)-\epsilon)(2n+1)} \leq e^{(\gamma(E_j)+\epsilon)(2n+1)}$$

but for ϵ small enough, we can make $\eta - \delta + \gamma(E_j) - \epsilon > \gamma(E_j) + \epsilon$. Then for n large enough, we get contradiction.

for (2), similarly

$$\begin{aligned} \frac{1}{2}e^{(\eta-\delta+\gamma(E_j)-\epsilon)(2n+1)} &\leq e^{(\gamma(E_j)+\epsilon)(n+1)}(3M)^{\frac{n}{K}} \\ &\leq e^{(\gamma(E_j)+\epsilon)(n+1)} \end{aligned}$$

□

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