PROOF FOR ANDERSON LOCALIZATION

...

This paper is dedicated to our advisors.

Abstract. This paper gives a new proof of localization for one dimension Anderson model.

1. Introduction

The Anderson model is given by a class of discrete analogs of Schrödinger operators H_ω with real *i.i.d* potetials $\{V_\omega(n)\}$:

$$(H_\omega\Psi)(n) = \Psi(n+1) + \Psi(n-1) + V_\omega(n)\Psi(n), \tag{1.1}$$

where $\omega = \{\omega_n\}_{n\in\mathbb{Z}} \in \Omega = S^{\mathbb{Z}}$, $S = supp\{\mu\} \subset \mathbb{R}$ is assumed to be compact and contains at least two points, μ is a borel probability on $\mathbb{R}$. *i.e.* for each $n \in \mathbb{Z}$, $V_\omega(n)$ is *i.i.d.* random variables depending on ω_n in (S, μ), but we will consider V_ω in the product probability space $(S^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ as a whole instead.

We say that H_ω exhibits the pectral localization property in an interval I if for $a.e.\omega$, H_ω has only pure point spectrum in I and its eigenfunction $\Psi(n)$ decays exponentially in n. We are gonna give a new proof for Anderson model based on the large deviation estimates and subharmonicity of Lvapunov exponents.

2. main idea

We are gonna introduce the main idea in this section.

Definition 1 (*g.e.v.*). We call E a generalized eigenvalue (denote as *g.e.v.*), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H\Psi = E\Psi$. We call $\Psi(n)$ generalized eigenfunction.

Then due to the fact from [1] that: *spectrally almost surely,*

$$\sigma(H) = \overline{\{E : E \text{ } is \text{ } g.e.v.\}},$$

We only need to show:

Theorem 2.1. *For a.e. ω, $\forall$ g.e.v. E, the corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in n.*

In order to get the dacaying speed of Ψ, We estimates the decaying speed of the Green's functions.

Definition 2. Assume $[a, b]$ is an interval, $a, b \in \mathbb{Z}$, define $H_[a, b], \omega$ to be the the operator H_ω resticted to $[a, b]$ with zero boundary condition outside $[a, b]$.

Note that it can be expressed as a "$b - a + 1$"-dim matrix.

Date: July 23, 2017.

Key words and phrases. Anderson Localization, spectrum.

Definition 3. The Green's function defined on $[a,b]$ for H_ω with energy $E \notin \sigma(H)$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}$$

Note that this can be expressed as a "$b-a+1$"-dim matrix, too. Denote its x line y column elements as $G_{[a,b],E,\omega}(x,y)$.

By the well-known formula:

$$\Psi(x) = G_{[a,b],E,\omega}(x,a)\Psi(a) + G_{[a,b],E,\omega}(x,b)\Psi(b), \quad x \in [a,b] \tag{2.1}$$

If one can get that, the Green's function near n, say, for example on $[n-k, n+k]$, is decaying somehow exponentially in n as n growing, then since Ψ on the right-hand-side is polynomially bounded, $\Psi(n)$ on the left-hand-side will decay exponentially in n, too.

This, together with another purposes for using large deviation estimates, which you will see in !!!!(one lemma), inspires us to define "regular and singular".

Definition 4. $\mathcal{A}_{[a,b]} = \{[a,b], [a,b-1], [a+1,b], [a+1,b-1]\}$

Definition 5. For $c > 0, k \in \mathbb{Z}$, we say $x \in \mathbb{Z}$ is (c,k,E,ω)-regular, if $\exists A \in \mathcal{A}_{[x-k,x+k]}$, s.t.

$$G_{A,E,\omega}(x, \partial A) \leq e^{-c|x-\partial A|}$$

i.e. if $A = [x_1, x_2]$

$$G_{A,E,\omega}(x, x_i) \leq e^{-c|x-x_i|}, \quad i = 1,2$$

Otherwise, we call it singular.

So we only need to prove

Theorem 2.2. *$\exists \Omega_0$ with $P(\Omega_0) = 1$, s.t. $\forall \tilde{\omega} \in \Omega_0$, take $\tilde{E}$ to be any g.e.v. of $H_{\tilde{\omega}}$, $\exists N = N(\tilde{E}, \tilde{\omega}), C = C(\tilde{E})$, $\forall n > N$, $2n+1$ is $(C, n, \tilde{E}, \tilde{\omega})$ regular.*

Remark 1. It's similar for even terms. We omit them only because of notation reasons.

Remark 2. Because if we achieve this, denote the corresponding polynomially bounded generalized eigenfunction as $\Psi(n) = \Psi_{\tilde{\omega},\tilde{E}}(n) \leq M(1+n)^p$, for $p > 0$. Then $\forall n > N$, $\exists A_n \in \mathcal{A}_{[n+1,3n+1]}$, s.t.

$$G_{A,E,\omega}(x, \partial A_n) \leq e^{-C|x-\partial A_n|}$$

so by eqution 2.1, and $|x - \partial A_n| \geq n-1$.

$$|\Psi(2n+1)| \leq Me^{-C(n-1)}(1+3n+1)^p$$

So for large enough n, $\Psi(2n+1)$ decays exponentially in n. Similarly for even terms and we will get Theorem 2.1.

The main technique is to find the proper set Ω_0. Roughly speaking, the idea is: a point x is (C,k,E,ω)-singular means the corresponding pair (E,ω) is in some "bad" "large deviation set" for operators restricted near x, say, $H_{[x-k,x+k]}$. We can then pick proper set Ω_1 such that these bad sets have small measures. Then pick Ω_2 such that E is not only in these bad sets, but also stay very close to eigenvalues of $H_{[x-k,x+k]}$ which are also in these bad sets. In this case, if we pick $\tilde{\omega} \in \Omega_1 \cap \Omega_2$ and if 0 and $2n+1$ are both $(C, n, \tilde{E}, \tilde{\omega})$-singular, then $\tilde{E}$ will be close to both eigenvalues $E_{i,n}$ of $H_{[-n,n]}$ near 0 and eigenvalues $E_{j,n}$ of $H_{[n+1,3n+1]}$ near $2n+1$.

So now $\tilde{E}$ is in bad sets for both intervals and are closed to both e.v., where each e.v. only belongs to bad sets for their own intervals. Which leads to a contradiction because the bad sets are so small that we can pick Ω_3 by Borel Cantelli, such that eventually all the eigenvalues of $[n+1, 3n+1]$ can't stay in bad set for $[-n, n]$,

[1] [2] [3] [4]

References

[1] Barry Simon. Schrödinger semigroups. *Bulletin of the American Mathematical Society*, 7(3):447–526, 1982.

[2] Henrique von Dreifus and Abel Klein. A new proof of localization in the anderson tight binding model. *Communications in Mathematical Physics*, 124(2):285–299, 1989.

[3] Svetlana Ya Jitomirskaya. Metal-insulator transition for the almost mathieu operator. *Annals of Mathematics*, 150(3):1159–1175, 1999.

[4] Hans L Cycon, Richard G Froese, Werner Kirsch, and Barry Simon. *Schrödinger operators: With application to quantum mechanics and global geometry.* Springer, 2009.

Department of Mathematics, University of California, Irvine, Rowland 410C