PROOF FOR ANDERSON LOCALIZATION

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This paper is dedicated to our advisors.

Abstract. This paper gives a new proof of localization for one dimension Anderson model.

1. Introduction

The Anderson model is given by a class of discrete analogs of Schrödinger operators H_ω with real *i.i.d* potetials $\{V_\omega(n)\}$:

$$(H_\omega\Psi)(n) = \Psi(n+1) + \Psi(n-1) + V_\omega(n)\Psi(n), \tag{1.1}$$

where $\omega = \{\omega_n\}_{n\in\mathbb{Z}} \in \Omega = S^{\mathbb{Z}}$, $S = supp\{\mu\} \subset \mathbb{R}$ is assumed to be compact and contains at least two points, μ is a borel probability on $\mathbb{R}$. *i.e.* for each $n \in \mathbb{Z}$, $V_\omega(n)$ is *i.i.d.* random variables depending on ω_n in (S, μ), but we will consider V_ω in the product probability space $(S^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ as a whole instead.

We say that H_ω exhibits the pectral localization property in an interval I if for $a.e.\omega$, H_ω has only pure point spectrum in I and its eigenfunction $\Psi(n)$ decays exponentially in n. We are gonna give a new proof for Anderson model based on the large deviation estimates and subharmonicity of Lvapunov exponents.

2. general setup

Definition 1 (*g.e.v.*)**.** We call E a generalized eigenvalue (denote as *g.e.v.*), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H\Psi = E\Psi$. We call $\Psi(n)$ generalized eigenfunction.

Then due to the fact from [1] that: *spectrally almost surely,*

$$\sigma(H) = \overline{\{E : E \text{ } is \text{ } g.e.v.\}},$$

We only need to show:

Theorem 2.1. *For a.e. ω, $\forall$ g.e.v. $E \in I$,I is a closed subset of $\mathbb{R}$. The corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in n.*

In order to get the dacaying speed of Ψ, We estimates the decaying speed of the Green's functions. Assume $[a, b]$ is an interval, $a, b \in \mathbb{Z}$, define $H_{[a,b],\omega}$ to be the the operator H_ω resticted to $[a, b]$ with zero boundary condition outside $[a, b]$. Note that it can be expressed as a "$b - a + 1$"-dim matrix. The Green's function defined on $[a, b]$ for H_ω with energy $E \notin \sigma(H)$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}$$

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Note that this can also be expressed as a "$b-a+1$"-dim matrix. Denote its x line y column elements as $G_{[a,b],E,\omega}(x,y)$.

By the well-known formula:

$$\Psi(x) = G_{[a,b],E,\omega}(x,a)\Psi(a) + G_{[a,b],E,\omega}(x,b)\Psi(b), \quad x \in [a,b] \tag{2.1}$$

If one can get that, the Green's function near n, say, for example on $[n-k, n+k]$, is decaying somehow exponentially in n as n growing, then since Ψ on the right-hand-side is polynomially bounded, $\Psi(n)$ on the left-hand-side will decay exponentially in n, too.

This, together with another purposes for using large deviation estimates, which you will see in !!!!(one lemma), inspires us to define "regular and singular".

Definition 2. $\mathcal{A}_{[a,b]} = \{[a,b],[a,b-1],[a+1,b],[a+1,b-1]\}$

Definition 3. For $c>0, k\in\mathbb{Z}$, we say $x\in\mathbb{Z}$ is (c,k,E,ω)-regular, if $\exists A \in \mathcal{A}_{[x-k,x+k]}$, s.t.

$$G_{A,E,\omega}(x,\partial A) \leqslant e^{-c|x-\partial A|}$$

i.e. if $A=[x_1,x_2]$

$$G_{A,E,\omega}(x,x_i) \leqslant e^{-c|x-x_i|}, \quad i=1,2$$

Otherwise, we call it singular.

So we only need to prove

Theorem 2.2. *$\exists\Omega_0$ with $P(\Omega_0)=1$, s.t. $\forall\tilde\omega\in\Omega_0$, take $\tilde E$ to be any g.e.v. of $H_{\tilde\omega}$, $\exists N = N(\tilde E,\tilde\omega), C=C(\tilde E)$, $\forall n>N$, $2n+1$ is $(C,n,\tilde E,\tilde\omega)$ regular.*

Remark 1. It's similar for even terms. We omit them only because of notation reasons.

Remark 2. Because if we achieve this, denote the corresponding polynomially bounded generalized eigenfunction as $\Psi(n)=\Psi_{\tilde\omega,\tilde E}(n)\leqslant M(1+n)^p$, for $p>0$. Then $\forall n>N$, $\exists A_n\in\mathcal{A}_{[n+1,3n+1]}$, s.t.

$$G_{A,E,\omega}(x,\partial A_n)\leqslant e^{-C|x-\partial A_n|}$$

so by eqution 2.1, and $|x-\partial A_n|\geqslant n-1$.

$$|\Psi(2n+1)|\leqslant Me^{-C(n-1)}(1+3n+1)^p$$

So for large enough n, $\Psi(2n+1)$ decays exponentially in n. Similarly for even terms and we will get Theorem 2.1.

Some other basic settings are below. If we denote

$$P_{[a,b],E,\omega} = det(H_{[a,b],E,\omega}-E)$$

By linear algebra calculation, we get:(if $x\leqslant y$)

$$\left|G_{[a,b](x,y),E,\omega}\right| = \frac{\left|P_{[a,x-1],E,\omega}P_{[y+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|}, \quad x\leqslant y$$

$$\left|G_{[a,b](x,y),E,\omega}\right| = \frac{\left|P_{[a,y-1],E,\omega}P_{[x+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|}, \quad x\geqslant y$$

If we denote the transfer matrix $T_{[a,b],E,\omega}$ as the matrix such that

$$\begin{pmatrix}\Psi(b)\\ \Psi(b-1)\end{pmatrix} = T_{[a,b],E,\omega}\begin{pmatrix}\Psi(a)\\ \Psi(a-1)\end{pmatrix}$$

We can prove by induction that

$$T_{[a,b],E,\omega} = \begin{pmatrix} P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega} \\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega} \end{pmatrix}$$

Definition 4 (Lyapunov Exponent)**.**

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 ln\|T_{[0,n],E,\omega}\| dP(\omega) = \lim_{n\to\infty} \frac{1}{n} ln\|T_{[0,n],E,\omega}\|$$

3. MAIN TECHNIQUE

We introduce the large deviation theorem here without proof.[2]

Theorem 3.1 (large deviation estimates)**.** *For any* $\epsilon > 0$, *there exists* $C = C(\epsilon) > 0$,$\eta = \eta(\epsilon) > 0$ *such that*

$$\mu\left\{\omega : \left|\frac{1}{b-a+1} ln\|T_{[a,b],E,\omega}\| - \gamma(E)\right| \geqslant \epsilon\right\} \leqslant Ce^{-\eta(b-a+1)}$$

Remark 3. If we denote

$$\begin{aligned} B_{[a,b]} &= \left\{(E,\omega) : \exists A \in \mathcal{A}_{[a,b]} s.t. |P_{A,E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\right\} \\ &\bigcup \left\{(E,\omega) : \forall A \in \mathcal{A}_{[a,b]}, |P_{A,E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\right\} \end{aligned}$$

we find that

$$B_{[a,b]} \subseteq \left\{(E,\omega) : \left|\frac{1}{b-a+1} ln\|T_{[a,b],E,\omega}\| - \gamma(E)\right| \geqslant \epsilon\right\}$$

This is the "bad" set. And Large deviation theorem gives us the estimates that for all E, a, b

$$P(\{\omega : (E,\omega) \in B_{[a,b]}\}) \leqslant Ce^{\eta(b-a+1)} \tag{3.1}$$

Moreover, we denote

$$\begin{aligned} B^+_{[a,b]} &= \left\{(E,\omega) : \exists A \in \mathcal{A}_{[a,b]} \ s.t. |P_{A,E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\right\} \\ &= \bigcup_{A\in\mathcal{A}_{[a,b]}} \left\{(E,\omega) : |P_{A,E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\right\} \end{aligned}$$

$$\begin{aligned} B^-_{[a,b]} &= \left\{(E,\omega) : \forall A \in \mathcal{A}_{[a,b]}, \ ad|P_{A,E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\right\} \\ &= \bigcap_{A\in\mathcal{A}_{[a,b]}} \left\{(E,\omega) : |P_{A,E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\right\} \end{aligned}$$

and denote $B_{[a,b],E} = \{\omega : (E,\omega) \in B_{[a,b]}\}$. Others similar.

Lemma 3.2. $n \geqslant 2$, *if* x *is* $(\gamma(E))-8\epsilon, n, E, \omega)$*-singular, then* $(E,\omega) \in B^-_{[x-n,x+n]} \cup B^+_{[x-n,x]} \cup B^+_{[x,x+n]}$.

Remark 4. Note that from (3.1), for all $E, x, n \geqslant 2$,

$$P(B^-_{[x-n,x+n],E} \cup B^+_{[x-n,x],E} \cup B^+_{[x,x+n],E}) \leqslant 3Ce^{-\eta n}$$

Proof. Assume not,

$$\begin{cases} \exists A_{[x-n,x+n]} \in \mathcal{A}_{[x-n,x+n]} \ s.t. \ |P_{A_{[x-n,x+n]},E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(2n+1)} \\ \forall A_{[x-n,x]} \in \mathcal{A}_{[x-n,x]}, \ |P_{A_{[x-n,x]},E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(n+1)} \\ \forall A_{[x,x+n]} \in \mathcal{A}_{[x,x+n]}, \ |P_{A_{[x,x+n]},E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(n+1)} \end{cases}$$

Pick $A_{[x-n,x+n]}$ as above. If denote it as $[x_1, x_2]$, i.e. $x_1 = x-n$ or $x-n+1$, $x_2 = x+n$ or $x+n-1$, then

$$\begin{aligned} \left|G_{[x_1,x_2],E,\omega}(x,x_1)\right| &= \frac{\left|P_{[x+1,x_2],E,\omega}\right|}{\left|P_{[x_1,x_2],E,\omega}\right|} \\ &\leqslant \frac{e^{(\gamma(E)+\epsilon)(x_2-x+1)}}{e^{(\gamma(E)-\epsilon)(x_2-x_1+1)}} \\ &\leqslant e^{-\gamma(E)|x-x_1|+\epsilon(|x_2-x+1|+|x_2-x_1+1|)} \\ &\leqslant e^{-\gamma(E)|x-x_1|+\epsilon(3n+2)} \\ &\leqslant e^{-\gamma(E)|x-x_1|+8\epsilon|x-x_1|} \\ &\leqslant e^{-(\gamma(E)-8\epsilon)|x-x_1|} \end{aligned}$$

Similar for $G_{[x_1,x_2],E,\omega}(x,x_2)$. Thus x is $(\gamma(E))-8\epsilon, n, E, \omega)$-regular, contradiction. ☐

By Theorem 2.2,

Theorem 3.3. *Let* $0 < \delta < \eta$, *Let* $E \in I$. *For a.e.* ω *(denote as* Ω_1*),* $\exists N_1 = N_1(\omega)$, *s.t.* $\forall n > N_1$, $m(B^-_{[n+1,3n+1],\omega}) \leqslant Ce^{-(\eta-\delta)(2n+1)}$ *and* $m(B^-_{[-n,n],\omega}) \leqslant Ce^{-(\eta-\delta)(2n+1)}$

Proof. We only prove for $m(B^-_{[n+1,3n+1],\omega})$.

By (3.1), $\forall E \in I$, $P(B^-_{[n+1,3n+1],E}) \leqslant Ce^{-\eta(2n+1)}$.

If we denote

$$\Omega_{\delta,n} = \left\{\omega : m(B^-_{[n+1,3n+1],\omega}) \leqslant e^{-(\eta-\delta)(2n+1)}\right\}$$

By chebyshev,

$$\begin{aligned} P(\Omega^c_{\delta,n}) &\leqslant e^{(\eta-\delta)n} \int_\Omega m(B_{[n+1,3n+1],\omega}) dP\omega \\ &= e^{(\eta-\delta)n} \int_I P(B_{[n+1,3n+1],E}) dx \\ &\leqslant e^{(\eta-\delta)n} Cm(I) e^{-\eta(2n+1)} \\ &= Cm(I) e^{-\delta(2n+1)} \end{aligned}$$

By Borel-Cantelli lemma, we get

for *a.e.* ω, $\exists N$, s.t. $\forall n > N$, $\omega \in \Omega_{\delta,n}$, i.e. $m(B^-_{[n+1,3n+1,\omega]}) \leqslant e^{-(\eta-\delta)(2n+1)}$. Similar for $[-n,n]$, pick insection for Ω_1, and maximum for N_1. ☐

Theorem 3.4 (Craig-Simon). *For a.e.ω(denote as Ω_2), we have*

$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} ln\|T_{[-n,0],E,\omega}\| \leqslant \gamma(E)$$
$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} ln\|T_{[0,n],E,\omega}\| \leqslant \gamma(E)$$
$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} ln\|T_{[n+1,2n+1],E,\omega}\| \leqslant \gamma(E)$$
$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} ln\|T_{[2n+1,3n+1],E,\omega}\| \leqslant \gamma(E)$$

Corollary 3.5. $\forall \omega \in \Omega_2$, $\exists N_2 = N_2(\omega)$, *s.t.* $\forall n > N_2$,

$$\|T_{[-n,0],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)}$$
$$\|T_{[0,n],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)}$$
$$\|T_{[n+1,2n+1],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)}$$
$$\|T_{[2n+1,3n+1],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)}$$

Remark 5. The only difference between here and [3] is the restricted intervals. As long as one follows the proof there, one can get the results here.

Theorem 3.6. $K > 0$, *For a.e.ω(denote as $\Omega_3 = \Omega_3(K)$), $\exists N_3 = N_3(\omega)$, $\forall n > N_3$, $\forall E_{j,[n+1,3n+1],\omega}$ being eigenvalue of $H_{[n+1,3n+1],\omega}$, $\forall y_1, y_2$ satisfy $-n \leqslant y_1 \leqslant y_2 \leqslant n$, $|-n-y_1| > \frac{n}{K}$, and $|n-y_2| > \frac{n}{K}$, we have $E_{j,[n+1,3n+1],\omega} \notin B_{[-n,y_1],\omega} \cup B_{[y_2,n],\omega}$.*

Proof. In order to use Borel-Cantelli, one need to estimate

$$P\left(\bigcup_{y_1,y_2}\bigcup_{j=1}^{2n+1} B_{[-n,y_1],E_{j,[n+1,3n+1],\omega}} \cup B_{[y_2,n],E_{j,[n+1,3n+1],\omega}}\right)$$

where y_1, y_2 satisfy assumptions above.Denote it by $\bar{P}$ So consider

$$\begin{aligned} P\left(B_{[y_2,n],E_{j,[n+1,3n+1],\omega}}\right) &= \int_{\Omega} \chi_{B_{[y_2,n],E_{j,[n+1,3n+1],\omega}}} dP\omega \\ &= \int_{S^{2n+1}} \left(\int_{\tilde{\Omega}} \chi_B \ d\tilde{\mu}\right) d\mu^{2n+1}(\omega_{n+1}, \cdots, \omega_{3n+1}) \\ &= \int_{S^{2n+1}} \tilde{P}(\tilde{B}_{[y_2,n],E_{j,[n+1,3n+1],\omega}}) d\mu^{2n+1}(\omega_{n+1}, \cdots, \omega_{3n+1}) \end{aligned}$$

where for $\tilde{\Omega}$ and $\tilde{\mu}$ one take away the $[n+1, 3n+1]$ terms from Ω and $\mu^{\mathbb{Z}}$ However, for any fixed E, $B_{[y_2,n],E}$ is of the form

$$\left(\bigotimes_{i\in[y_2,n]} S\right) \times B'_{[y_2,n]}$$

where

$$B'_{[y_2,n],E} = \left\{\omega|_{[y_2,n]} : \omega \in B_{[y_2,n],E}\right\}$$

So,

$$\begin{aligned} P(B_{[y_2,n],E}) &= \int_{S^{2n+1}} \left(\int_{\tilde{\Omega}} \chi_{B_{[y_2,n],E}} \ d\tilde{\mu}\right) d\mu^{2n+1}(\omega_{n+1}, \cdots, \omega_{3n+1}) \\ &= \tilde{P}(\tilde{B}_{[y_2,n],E}) \times 1 \times \cdots \times 1 \\ &= \tilde{P}(\tilde{B}_{[y_2,n],E}) \end{aligned}$$

Br (3.1),

$$\tilde{P}(\tilde{B}_{[y_2,n],E}) \leqslant Ce^{|n-y_2|}, \quad \forall E$$

So

$$P\left(B_{[y_2,n],E_{j,[n+1,3n+1],\omega}}\right) \leqslant Ce^{|n-y_2|}$$

□

[1] [4] [5] [6] [2] [3]

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