1. Introduction

The Anderson model is given by a class of discrete analogs of Schrödinger operators H_ω with real $i.i.d$ potetials $\{V_\omega(n)\}$:

$$(H_\omega \Psi)(n) = \Psi(n+1) + \Psi(n-1) + V_\omega(n)\Psi(n), \tag{1.1}$$

where $\omega = \{\omega_n\}_{n\in\mathbb{Z}} \in \Omega = S^{\mathbb{Z}}$, $S = supp\{\mu\} \subset \mathbb{R}$ is assumed to be compact and contains at least two points, μ is a borel probability on $\mathbb{R}$. *i.e.* for each $n \in \mathbb{Z}$, $V_\omega(n)$ is *i.i.d.* random variables depending on ω_n in (S, μ), but we will consider V_ω in the product probability space $(S^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ as a whole instead.

We say that H_ω exhibits the pectral localization property in an interval I if for $a.e.\omega$, H_ω has only pure point spectrum in I and its eigenfunction $\Psi(n)$ decays exponentially in n. We are gonna give a new proof for Anderson model based on the large deviation estimates and subharmonicity of Lvapunov exponents.

2. general setup

Definition 1 ($g.e.v.$)**.** We call E a generalized eigenvalue (denote as $g.e.v.$), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H\Psi = E\Psi$. We call $\Psi(n)$ generalized eigenfunction.

Then due to the fact from [1] that: *spectrally almost surely,*

$$\sigma(H) = \overline{\{E : E \text{ } is \text{ } g.e.v.\}},$$

We only need to show:

Theorem 2.1. *For a.e. ω, $\forall$ $g.e.v.$ $E \in I$,I is a closed subset of $\mathbb{R}$. The corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in n.*

In order to get the dacaying speed of Ψ, We estimates the decaying speed of the Green's functions. Assume $[a, b]$ is an interval, $a, b \in \mathbb{Z}$, define $H_{[a,b],\omega}$ to be the the operator H_ω resticted to $[a, b]$ with zero boundary condition outside $[a, b]$. Note that it can be expressed as a "$b - a + 1$"-dim matrix. The Green's function defined on $[a, b]$ for H_ω with energy $E \notin \sigma(H)$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}$$

Note that this can also be expressed as a "$b - a + 1$"-dim matrix. Denote its x line y column elements as $G_{[a,b],E,\omega}(x, y)$.

By the well-known formula:

$$\Psi(x) = G_{[a,b],E,\omega}(x, a)\Psi(a) + G_{[a,b],E,\omega}(x, b)\Psi(b), \quad x \in [a, b] \tag{2.1}$$

If one can get that, the Green's function near n, say, for example on $[n-k, n+k]$, is decaying somehow exponentially in n as n growing, then since Ψ on the right-hand-side is polynomially bounded, $\Psi(n)$ on the left-hand-side will decay exponentially in n, too.

This inspires us to define "regular and singular".

Definition 2. For $c > 0, k \in \mathbb{Z}$, we say $x \in \mathbb{Z}$ is (c, k, E, ω)-regular, if

$$G_{[x-n,x+n],E,\omega}(x, x - n) \leqslant e^{-cn}$$
$$G_{[x-n,x+n],E,\omega}(x, x + n) \leqslant e^{-cn}$$

Otherwise, we call it singular.

So we only need to prove

Theorem 2.2. *$\exists\Omega_0$ with $P(\Omega_0)=1$, s.t. $\forall\tilde{\omega}\in\Omega_0$, take $\tilde{E}>0$ to be any g.e.v. of $H_{\tilde{\omega}}$, $\exists N=N(\tilde{E},\tilde{\omega}), C=C(\tilde{E})$, $\forall n>N$, $2n+1$ is $(C,n,\tilde{E},\tilde{\omega})$ regular.*

Remark 1. It's similar for even terms. We omit them only because of notation reasons.

Remark 2. Because if we achieve this, denote the corresponding polynomially bounded generalized eigenfunction as $\Psi(n)=\Psi_{\tilde{\omega},\tilde{E}}(n)\leqslant M(1+n)^p$, for $p>0$. Then $\forall n>N$, let $A_n=[n+1,3n+1]$

$$|G_{A_n,E,\omega}(x,\partial A_n)|\leqslant e^{-C|x-\partial A_n|}$$

so by eqution 2.1, and $|x-\partial A_n|\geqslant n-1$.

$$|\Psi(2n+1)|\leqslant Me^{-Cn}(1+3n+1)^p$$

So for large enough n, $\Psi(2n+1)$ decays exponentially in n. Similarly for even terms and we will get Theorem 2.1.

Some other basic settings are below. If we denote

$$P_{[a,b],E,\omega}=det(H_{[a,b],E,\omega}-E)$$

if $a=b$, let $P_{[a,b],E,\omega}=1$ for next formula. By linear algebra calculation, we get:(if $x\leqslant y$)

$$\left|G_{[a,b](x,y),E,\omega}\right|=\frac{\left|P_{[a,x-1],E,\omega}P_{[y+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|},\quad x\leqslant y$$

$$\left|G_{[a,b](x,y),E,\omega}\right|=\frac{\left|P_{[a,y-1],E,\omega}P_{[x+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|},\quad x\geqslant y$$

If we denote the transfer matrix $T_{[a,b],E,\omega}$ as the matrix such that

$$\begin{pmatrix}\Psi(b)\\ \Psi(b-1)\end{pmatrix}=T_{[a,b],E,\omega}\begin{pmatrix}\Psi(a)\\ \Psi(a-1)\end{pmatrix}$$

We can prove by induction that

$$T_{[a,b],E,\omega}=\begin{pmatrix}P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega}\\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega}\end{pmatrix}$$

Definition 3 (Lyapunov Exponent)**.**

$$\gamma(E)=\lim_{n\to\infty}\frac{1}{n}\int_0^1 ln\|T_{[0,n],E,\omega}\|dP(\omega)=\lim_{n\to\infty}\frac{1}{n}ln\|T_{[0,n],E,\omega}\|$$

$$\nu=\inf_{E\in I}\gamma(E)>0$$

3. MAIN TECHNIQUE

We introduce the large deviation theorem here without proof. [2]

Lemma 1 (large deviation estimates)**.** *For any $\epsilon>0$, there exists $\eta=\eta(\epsilon)>0$ such that, $\exists N_0$, $\forall b-a>N_0$*

$$\mu\left\{\omega:\left|\frac{1}{b-a+1}ln\|P_{[a,b],E,\omega}\|-\gamma(E)\right|\geqslant\epsilon\right\}\leqslant e^{-\eta(b-a+1)}$$

Remark 3. If we denote

$$B_{[a,b],\epsilon} = \Big\{(E,\omega) : |P_{[a,b],E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\Big\}$$
$$\bigcup \Big\{(E,\omega) : |P_{[a,b],E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\Big\}$$

we find that

$$B_{[a,b],\epsilon} \subseteq \left\{(E,\omega) : \left|\frac{1}{b-a+1} ln\|P_{[a,b],E,\omega}\| - \gamma(E)\right| \geqslant \epsilon\right\}$$

This is the "bad" set. And Large deviation theorem gives us the estimates that for all E, a, b

$$P(\{\omega : (E,\omega) \in B_{[a,b],\epsilon}\}) \leqslant e^{-\eta(b-a+1)} \tag{3.1}$$

Moreover, we denote

$$B^{+}_{[a,b],\epsilon} = \Big\{(E,\omega) : |P_{[a,b],E,\omega}| \geqslant e^{(\gamma(E)+\epsilon)(b-a+1)}\Big\} \tag{3.2}$$

$$B^{-}_{[a,b],\epsilon} = \Big\{(E,\omega) : |P_{[a,b],E,\omega}| \leqslant e^{(\gamma(E)-\epsilon)(b-a+1)}\Big\} \tag{3.3}$$

and denote $B_{[a,b],\epsilon,E} = \{\omega : (E,\omega) \in B_{[a,b]}\}$. Others similar.

Assume $\epsilon = \epsilon_0 < \frac{1}{8}\nu$ is fixed for now,s.t. so we omit it from $B_{[a,b]}$ untill Theorem 3.3. And η_0 is the corresponding parameter

Lemma 2. *$n \geqslant 2$, if x is $(\gamma(E) - 8\epsilon_0, n, E, \omega)$-singular, then $(E,\omega) \in B^{-}_{[x-n,x+n]} \cup B^{+}_{[x-n,x]} \cup B^{+}_{[x,x+n]}$.*

Remark 4. Note that from (3.1), for all $E, x, n \geqslant 2$,

$$P(B^{-}_{[x-n,x+n],E} \cup B^{+}_{[x-n,x],E} \cup B^{+}_{[x,x+n],E}) \leqslant 3Ce^{-\eta_0 n}$$

Proof. Assume not, then

$$\begin{cases} |P_{[x-n,x+n],E,\omega}| \geqslant e^{(\gamma(E)+\epsilon_0)(2n+1)} \\ |P_{[x-n,x],E,\omega}| \leqslant e^{(\gamma(E)-\epsilon_0)(n+1)} \\ |P_{[x,x+n],E,\omega}| \leqslant e^{(\gamma(E)-\epsilon_0)(n+1)} \end{cases}$$

So we can estimate

$$\begin{aligned} \left|G_{[x-n,x+n],E,\omega}(x, x-n)\right| &= \frac{\left|P_{[x,x+n],E,\omega}\right|}{\left|P_{[x-n,x+n],E,\omega}\right|} \\ &\leqslant \frac{e^{(\gamma(E)+\epsilon_0)(n+1)}}{e^{(\gamma(E)-\epsilon_0)(2n+1)}} \\ &\leqslant e^{-\gamma(E)(n)+\epsilon_0(3n+2)} \\ &\leqslant e^{-(\gamma(E)-8\epsilon_0)n} \end{aligned}$$

Similar for $G_{[x-n,x+n],E,\omega}(x, x+n)$. Thus x is $(\gamma(E)) - 8\epsilon_0, n, E, \omega)$-regular, contradiction. $\square$

By Theorem 2.2,

Theorem 3.1. *Let $0 < \delta_0 < \eta_0$, Let $E \in I$. For a.e. ω (denote as Ω_1), $\exists N_1 = N_1(\omega)$, s.t. $\forall n > N_1$, $m(B^{-}_{[n+1,3n+1],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$ and $m(B^{-}_{[-n,n],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$*

Proof. We only prove for $m(B^-_{[n+1,3n+1],\omega})$.

By (3.1), $\forall E \in I$, $P(B^-_{[n+1,3n+1],E}) \leqslant Ce^{-\eta_0(2n+1)}$.

If we denote

$$\Omega_{\delta_0,n} = \left\{\omega : m(B^-_{[n+1,3n+1],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}\right\}$$

By chebyshev,

$$\begin{aligned} P(\Omega^c_{\delta_0,n}) &\leqslant e^{(\eta_0-\delta_0)n} \int_\Omega m(B_{[n+1,3n+1],\omega}) dP\omega \\ &= e^{(\eta_0-\delta_0)n} \int_I P(B_{[n+1,3n+1],E}) dx \\ &\leqslant e^{(\eta_0-\delta_0)n} m(I) e^{-\eta_0(2n+1)} \\ &= m(I) e^{-\delta(2n+1)} \end{aligned}$$

By Borel-Cantelli lemma, we get

for *a.e.* ω, $\exists N$, s.t. $\forall n > N$, $\omega \in \Omega_{\delta,n}$, i.e. $m(B^-_{[n+1,3n+1],\omega}) \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$. Similar for $[-n, n]$, pick insection for Ω_1, and maximum for N_1. □

Theorem 3.2 (Craig-Simon). *For a.e.ω(denote as Ω_2), we have*

$$\begin{aligned} \overline{\lim_{n\to\infty}} \frac{1}{n+1} ln \|T_{[-n,0],E,\omega}\| &\leqslant \gamma(E) \\ \overline{\lim_{n\to\infty}} \frac{1}{n+1} ln \|T_{[0,n],E,\omega}\| &\leqslant \gamma(E) \\ \overline{\lim_{n\to\infty}} \frac{1}{n+1} ln \|T_{[n+1,2n+1],E,\omega}\| &\leqslant \gamma(E) \\ \overline{\lim_{n\to\infty}} \frac{1}{n+1} ln \|T_{[2n+1,3n+1],E,\omega}\| &\leqslant \gamma(E) \end{aligned}$$

Corollary 1. $\forall \omega \in \Omega_2$, $\exists N_2 = N_2(\omega)$, *s.t.* $\forall n > N_2$,

$$\begin{aligned} &\|T_{[-n,0],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)} \\ &\|T_{[0,n],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)} \\ &\|T_{[n+1,2n+1],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)} \\ &\|T_{[2n+1,3n+1],E,\omega}\| < e^{(\gamma(E)+\epsilon)(n+1)} \end{aligned}$$

Remark 5. The only difference between here and [3] is the restrict intervals. As long as one follows the proof there, one can get the results here.

Theorem 3.3. $\epsilon > 0, K > 0$, *For a.e.ω(denote as $\Omega_3 = \Omega_3(\epsilon, K)$), $\exists N_3 = N_3(\omega)$, $\forall n > N_3$, $\forall E_{j,[n+1,3n+1],\omega}$ being eigenvalue of $H_{[n+1,3n+1],\omega}$, $\forall y_1, y_2$ satisfy $-n \leqslant y_1 \leqslant y_2 \leqslant n$, $|-n - y_1| \geqslant \frac{n}{K}$, and $|n - y_2| \geqslant \frac{n}{K}$, we have $E_{j,[n+1,3n+1],\omega} \notin B_{[-n,y_1],\epsilon,\omega} \cup B_{[y_2,n],\epsilon,\omega}$.*

Remark 6. Note that ϵ and $K > 0$ is not determined yet, we're gonna determine it later on in section 4.

Proof. In order to use Borel-Cantelli, one need to estimate

$$P\left(\bigcup_{y_1,y_2} \bigcup_{j=1}^{2n+1} B_{[-n,y_1],\epsilon,E_{j,[n+1,3n+1],\omega}} \cup B_{[y_2,n],\epsilon,E_{j,[n+1,3n+1],\omega}}\right)$$

where y_1, y_2 satisfy assumptions above. Denote it by $\bar{P}$. Consider

$$\begin{aligned}
P\left(B_{[y_2,n],\epsilon,E_{j,[n+1,3n+1],\omega}}\right) &= \int_\Omega \chi_{B_{[y_2,n],\epsilon,E_{j,[n+1,3n+1],\omega}}} dP\omega \\
&= \int_{S^{2n+1}} \left(\int_{\tilde{\Omega}} \chi_B \; d\tilde{\mu}\right) d\mu^{2n+1}(\omega_{n+1}, \cdots, \omega_{3n+1}) \\
&= \int_{S^{2n+1}} \tilde{P}(\tilde{B}_{[y_2,n],\epsilon,E_{j,[n+1,3n+1],\omega}}) d\mu^{2n+1}(\omega_{n+1}, \cdots, \omega_{3n+1})
\end{aligned}$$

where for $\tilde{\Omega}$ and $\tilde{\mu}$, one take away the $[n+1, 3n+1]$ terms from Ω and $\mu^{\mathbb{Z}}$ However, for any fixed E, $B_{[y_2,n],\epsilon,E}$ is of the form

$$\left(\bigotimes_{i\in[y_2,n]} S\right) \times B'_{[y_2,n],\epsilon}$$

where

$$B'_{[y_2,n],\epsilon,E} = \{\omega|_{[y_2,n]} : \omega \in B_{[y_2,n],\epsilon,E}\}$$

So,

$$\begin{aligned}
P(B_{[y_2,n],\epsilon,E}) &= \int_{S^{2n+1}} \left(\int_{\tilde{\Omega}} \chi_{B_{[y_2,n],\epsilon,E}} \; d\tilde{\mu}\right) d\mu^{2n+1}(\omega_{n+1}, \cdots, \omega_{3n+1}) \\
&= \tilde{P}(\tilde{B}_{[y_2,n],E}) \times 1 \times \cdots \times 1 \\
&= \tilde{P}(\tilde{B}_{[y_2,n],E})
\end{aligned}$$

Br (3.1),

$$\tilde{P}(\tilde{B}_{[y_2,n],\epsilon,E}) \leqslant Ce^{-\eta|n-y_2|}, \quad \forall E$$

So

$$P\left(B_{[y_2,n],\epsilon,E_{j,[n+1,3n+1],\omega}}\right) \leqslant Ce^{-|n-y_2-|} \leqslant Ce^{-n/K}$$

$$\bar{P} \leqslant C(2n+1)^3 e^{-n/K}$$

The sum over n is finite, use Borel-Cantelli, we can get the result. $\square$

4. Proof of Theorem 2.2

Proof. Let $\Omega_0 = \Omega_1 \cap \Omega_2 \cap \Omega_3(\epsilon, K)$ (ϵ, K to be determined later), pick $\tilde{\omega} \in \Omega_0$, take $\tilde{E}$ a g.e.v. for $H_{\tilde{\omega}}$. WLOG assume $\Psi(0) \neq 0$, then $\exists N_4$, s.t. $\forall n > N_4$, 0 is $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-singular.

Assume $2n+1$ is not eventually $(\gamma(\tilde{E}) - 8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-regular, then $\exists\{n_k\}$ with $n_k \to \infty$, s.t. $2n_k + 1$ is $(\gamma(\tilde{E}) - 3\epsilon_0, n_k, \tilde{E}, \tilde{\omega})$-singular.

Then, for those $n_k > N_0 = \max N_1, N_2, N_3, N_4$, WLOG use $\{n\}$ istead of $\{n_k\}$, we have that

- Both 0 and n is $(\gamma(\tilde{E})), n, \tilde{E}, \tilde{\omega})$-singular. (We focus on n below, 0 is similar.)
- So by lemma 2, $\tilde{E} \in B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$.
- By corollary 1 and definition 3.2, $\tilde{E} \notin B^+_{[n+1,2n+1],\epsilon_0,\tilde{\omega}} \cup B^+_{[2n+1,3n+1],\epsilon_0,\tilde{\omega}}$, it can only lies in $B^-_{[n+1,3n+1],\epsilon_0,\tilde{\omega}}$

- But

$$B^-_{[n+1,3n+1],\epsilon_0,\tilde\omega} = \left\{E : |P_{[n+1,3n+1],\epsilon,E,\tilde\omega}| \leqslant e^{(\gamma(E)-\epsilon_0)(2n+1)}\right\} \tag{4.1}$$

  by definition 3.3. Note that $P_{[n+1,3n+1],\epsilon,\epsilon_0,E,\tilde\omega}$ is indeed a polynomial in E and have $2n+1$ zeros (e.v.), which are all in the bad set (4.1). So this bad set (4.1) contains less than $2n+1$ intervals near the zeros. $\tilde{E}$ should lie in one of them. And by Theorem 3.1, $m(B^-_{[n+1,3n+1],\epsilon,\tilde\omega}) \leqslant Ce^{-(\eta-\delta)(2n+1)}$. So there is some e.v. $E_{j,[n+1,3n+1],\tilde\omega}$ of $H_{[n+1,3n+1],\omega}$ s.t.

$$|\tilde{E} - E_{j,[n+1,3n+1],\tilde\omega}| \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

  similarly, $\exists E_{i,[-n,n],\tilde\omega}$, s.t.

$$|\tilde{E} - E_{i,[-n,n],\tilde\omega}| \leqslant e^{-(\eta_0-\delta_0)(2n+1)}$$

- So $|E_{i,[-n,n],\tilde\omega} - E_{j,[n+1,3n+1],\tilde\omega}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$. However, by Theorem 3.3, one have $E_{j,[n+1,3n+1],\tilde\omega} \notin B_{[-n,n],\epsilon,\tilde\omega}$, while $E_{i,[-n,n],\tilde\omega} \in B_{[-n,n],\epsilon,\tilde\omega}$ This will give us a contradiction below.

Since $|E_{i,[-n,n],\tilde\omega} - E_{j,[n+1,3n+1],\tilde\omega}| \leqslant 2e^{-(\eta_0-\delta_0)(2n+1)}$ and $E_{i,[-n,n],\tilde\omega}$ being the e.v. of $H_{[-n,n],\tilde\omega}$,

$$\left\|G_{[-n,n],E_{j,[n+1,3n+1],\tilde\omega},\tilde\omega}\right\| \geqslant \frac{1}{2}e^{(\eta_0-\delta_0)(2n+1)}$$

So $\exists y_{n1}, y_{n2} \in [-n,n]$ s.t.

$$\left|G_{[-n,n],E_{j,[n+1,3n+1],\tilde\omega},\tilde\omega}(y_{n1},y_{n2})\right| \geqslant \frac{1}{2}e^{(\eta_0-\delta_0)(2n+1)}$$

But $E_{j,[n+1,3n+1],\omega} \notin B_{[-n,n],\epsilon,\tilde\omega}$, i.e.

$$|P_{[-n,n],\epsilon,E_{j,[n+1,3n+1],\omega},\tilde\omega}| \geqslant e^{(\gamma(E_j)-\epsilon)(2n+1)}$$

so

$$\left\|P_{[-n,y_{n1}],\epsilon,E_j}P_{[y_{n2},n],\epsilon,E_j}\right\| \geqslant \frac{1}{2}e^{(\eta_0-\delta-0)(2n+1)}e^{(\gamma(E_j)-\epsilon)(2n+1)} \tag{4.2}$$

Let $M = sup\{|V| + |E_i| + |E_j| + 2\}$, where $|V|$ is assumed bounded, E_i, E_j are bounded because they are close to $E \in I$.
Then pick ϵ small enough in Theorem 3.3 s.t.

$$2\epsilon < \min\{\eta_0 - \delta_0, \nu\} \tag{4.3}$$

and fix it, then let

$$L := e^{(\eta-\delta)}e^{(\nu-\epsilon)} > 1$$

Pick K big enough in Theorem 3.3 to be s.t.

$$(3M)^{\frac{1}{K}} < L$$

say, $\exists\sigma > 0$,

$$(3M)^{\frac{1}{K}} \leqslant L - \sigma < L \tag{4.4}$$

then for left hand side of (4.2), there are three cases:

(1) both $|-n-y_{1n}| > \frac{n}{K}$ and $|n - y_{2n}| > \frac{n}{K}$
(2) one of them is large, say $|-n-y_{1n}| > \frac{n}{K}$ and $|x_{2n} - y_2n| \leqslant \frac{n}{K}$
(3) both small.

for (1),

$$\frac{1}{2}e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} \leqslant e^{2n(\gamma(E_j)+\epsilon)}$$

by our choice (4.3), $\eta-\delta+\gamma(E_j)-\epsilon>\gamma(E_j)+\epsilon$. Then for n large enough, we get contradiction.

for (2), similarly with (4.3) and(4.4)

$$\begin{aligned}
\frac{1}{2}e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} &\leqslant e^{(\gamma(E_j)+\epsilon)(n+1)}(3M)^{\frac{n}{K}}\\
&\leqslant e^{(\gamma(E_j)+\epsilon)(n+1)}L^n\\
&\leqslant e^{(\gamma(E_j)+\epsilon)(n+1)}e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)n}\\
\frac{1}{2}e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(n+1)} &\leqslant e^{(\gamma(E_j)+\epsilon)(n+1)}
\end{aligned}$$

We get contradiction.

for (3),with (4.3) and (4.4)

$$\begin{aligned}
\frac{1}{2}e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)(2n+1)} &\leqslant (3M)^{\frac{2n}{K}}\\
&\leqslant (L-\sigma)^{2n}\\
&\leqslant (e^{(\eta_0-\delta_0+\gamma(E_j)-\epsilon)}-\sigma)^{2n}
\end{aligned}$$

Contradiction.

So our assumption that $2n+1$ is not eventually $(\gamma(\tilde{E})-8\epsilon_0, n, \tilde{E}, \tilde{\omega})$-regular is flase. Theorem 2.2 follows. $\square$

References

[1] Barry Simon. Schrödinger semigroups. *Bulletin of the American Mathematical Society*, 7(3):447–526, 1982.

[2] Jhishen Tsay and . Some uniform estimates in products of random matrices. *Taiwanese Journal of Mathematics*, pages 291–302, 1999.

[3] Walter Craig, Barry Simon, et al. Subharmonicity of the lyaponov index. *Duke Mathematical Journal*, 50(2):551–560, 1983.