Testing for Dictionary Learning

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Dictionary Learning

What is Dictionary Learning

Sparse Linear Combination of Dictionary Atoms

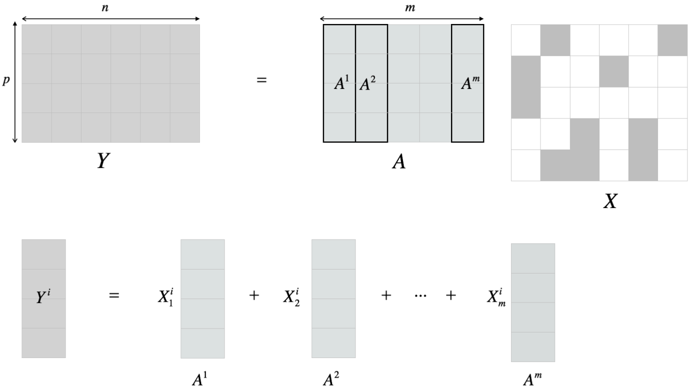

Figure 1: Dictionary Learning

Applications

- Image Restoration
- High Speed Photography
- Automatic De-blurring and digital zoom.
- Brain Imaging

Literature

- Exact Recovery of Sparsely-Used Dictionaries (SWW12).
- Algorithms for Learning Incoherent and Overcomplete Dictionaries (AGM14).
- Learning Sparsely Used Overcomplete Dictionaries (AAJ$^+$14).

Property Testing

Given an object O, and the desired property P, Property testing is a decision problem which distinguishes between following the two cases.

- Object O possesses the property P.
- Object O is ϵ-"far" from the property P.

Consider Object O to be set S of vectors in $\mathbb{R}^n$
Query Complexity $= f(n, |S|, \epsilon)$

Property Testing

- **Completeness**:
  If S has property P, then the testing algorithm outputs YES.
- **Soundness**:
  If S is ϵ-"far" from property P, then the testing algorithm outputs NO.

Objective

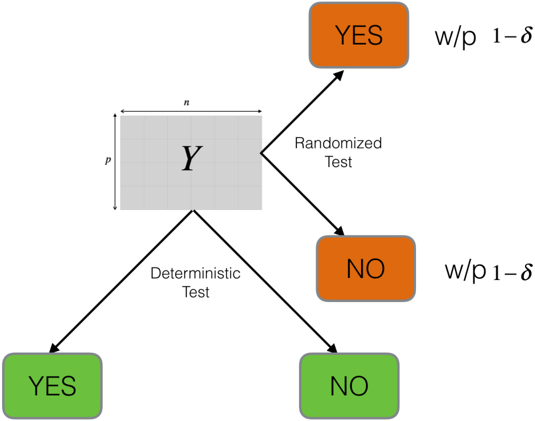

Figure 2: Property Testing

Restricted Isometry Property (RIP)

Restricted Isometry Property (k, δ_k) (CT06)

A matrix $A \in \mathbb{R}^{m \times n}$ is said to satisfy k-RIP with RIP constant δ_k if $\forall\ X \in \mathbb{R}^n$ with $\|X\|_0 \le k$, the following hold:

$$(1 - \delta_k)\|X\|_2 \le \|AX\|_2 \le (1 + \delta_k)\|X\|_2$$

Incoherence

Stronger Condition than RIP

Incoherence

A matrix $A \in \mathbb{R}^{m \times n}$ is said to be $\mu-$incoherence if

$$\langle A_i, A_j \rangle \leq \frac{\mu}{\sqrt{m}} \ \forall \ i \neq j$$

$$\text{Incoherence} \implies (k, \delta_k)\text{-RIP with } \delta_k = \sqrt{\left(1 + (k-1)\frac{\mu}{\sqrt{m}}\right)} - 1$$

Gaussian Width

Gaussian Width

The gaussian width of a set $S \subset \mathbb{R}^d$ is

$$\omega(S) = \mathop{\mathbb{E}}_{g}\left[\sup_{v \in S} \langle g, v \rangle\right]$$

where $g \sim \mathcal{N}(0,1)^d$.

Incoherence on a set of vectors

Incoherence follows a very close relationship to Gaussian width and some of the properties from Gaussian width also hold for incoherence

- $S \subset \mathcal{S}^{d-1}$ then $\mu(S) \leq C\sqrt{d}$
- For a set S such that $dim(S) = k$ then $\mu(S) \leq c\sqrt{k}$
- If $S \subset Sp_k^d$ where Sp_k^d is the collection of all k-sparse vectors in d-dimensional space then $\mu(S) = \mathcal{O}(klog(\frac{p}{d}))$

Outline for testing

- Check for norms of all columns of $\mathbb{Y}$; if any of them is below $(1-\delta)^2$ or greater than $(1+\delta)^2$ then reject.
- Calculate the inner products for all columns of $\mathbb{Y}$ and if any of them is greater than $(1+\delta)^2 \frac{\mu_X}{\sqrt{m}}$ then reject.
- If all the inner products are less than $\mathcal{O}(f(\epsilon, \delta, l, k))$ then accept.

Experiments

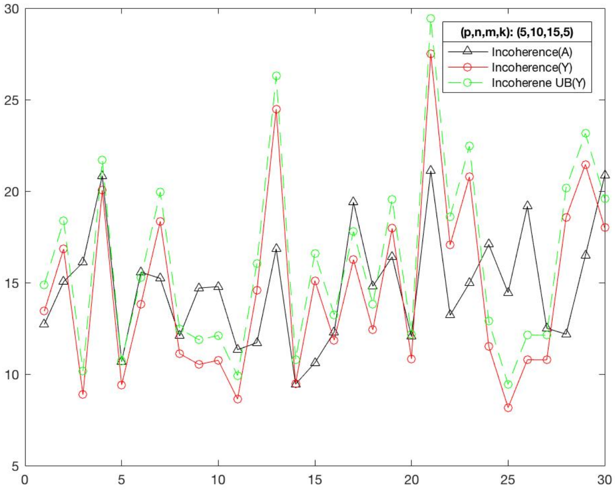

Figure 3: Incoherence of Y and A and Upper Bound

Experiments

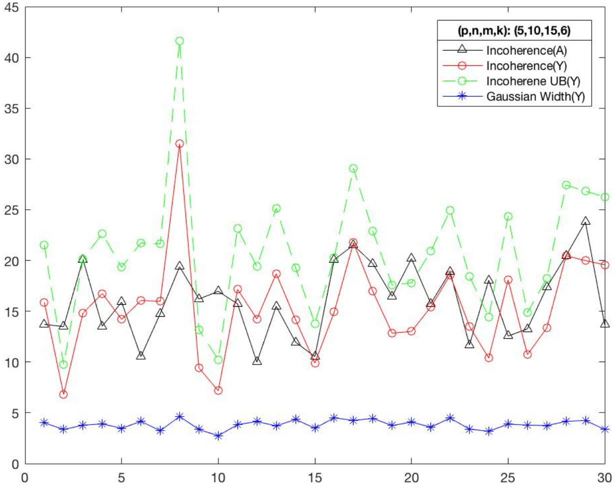

Figure 4: Incoherence of Y and A and Upper Bound, and Gaussian Width

Experiments

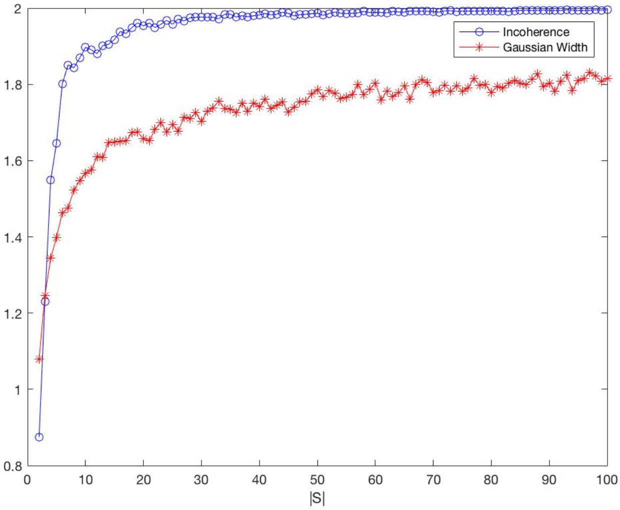

Figure 5: Incoherence and Gaussian Width VS Set Cardinality

Experiments

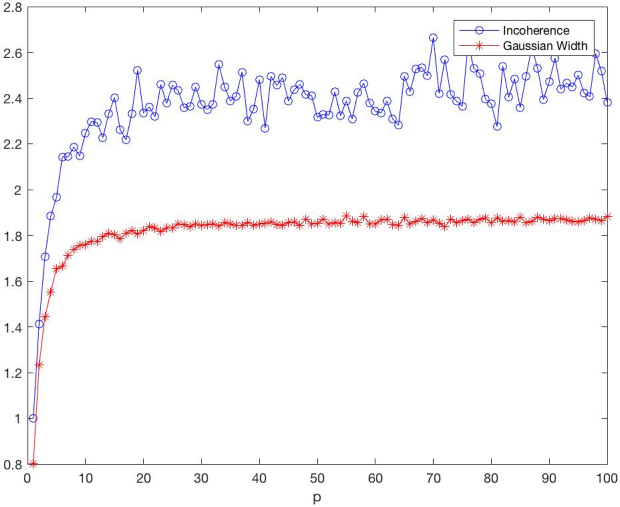

Figure 6: Incoherence and Gaussian Width VS vector dimensionality p

Outline for Proof

Projection: Sets with small incoherence can be almost isometrically embedded into a low dimensional subspace.
Covering: Appropriately sparse point sets on projection to a low dimensional subspace form a cover of the unit sphere on the smaller dimension. (BBG16)

Testing for Rank

Given μ, sparsity k, we can calculate minimum rank of Y.

minrank $= f(\mu, k, m)$

If rank(Y) $<$ minrank $\implies$ Reject.

If rank(Y) $\geq$ minrank, we can give an upper bound on k, under which dictionary learning is possible.

References

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Questions?