VYSYA COLLEGE, SALEM-103
DEPARTMENT OF STATISTICS
I BSC CS & I BCA

UNIT - II

THEORETICAL STANDARD DISTRIBUTION

INTRODUCTION

In this chapter, we shall study some of the probability distributions that figure most prominently in statistical theory and applications. We shall also study their parameters that are the quantities that are constraints for particular distributions about that can take on different values for different members of families of distributions of the same kind. The most common parameters are the lower moments, mainly μ and σ^2, and as we say in the preceding chapter, there are essentially two ways in which they can be obtained. We can evaluate the necessary sums directly or we can work with moment generating functions. Although it could seem logical to use in each case whichever method is simplest, we shall sometimes use both. In some instances this will be done because the results are needed later in other sit will merely serve to provide the reader with experience in the application of the respective mathematical technique.

Define Distributions

The frequency distributions described in the samples drawn from the population, when the values of the variable in the population are distributed according to some law which can be expressed mathematically, such distinguished from the frequency distribution are known as theoretical distribution as distinguished from the frequency distributions. There are two types of distributions. They are

Theoretical Distribution										
Discrete							Continuous			
Binomial	Poisson	Rectangular	Multinomial	Negative Binomial	Geometric	Hyper Geometric	Normal	Student t dist	Chi-Square dist	F-Dist

Define Probability Distribution

If X is a discrete random variable, the function given by $f(x) = P(X = x)$ for each x within the range of X is called the Probability Distribution of X. A function can serve as the probability distribution of a discrete random variable X if and only if its values, $f(x)$, satisfy the conditions

1. $f(x) \geq 0$ for each value within its domain
2. $\sum_x f(x) = 1$, where the summation extends over all the values within its domain. (Or)
   The random variable X taking the values $X_1, X_2, X_3, \dots X_n$ with the probabilities $p_1, p_2, \dots p_n$ respectively such that, every probability is a non-negative number not greater than
   1. $p_i \geq 0 \implies 0 \leq p_i \leq 1$ 2. The sum of all probabilities is always equal to one i.e., $\sum P_i = 1$

Define Bernoulli distribution

A random variable X takes the value 0 and 1 with the probability q and p respectively $P(X = 0) = q$ and $P(X = 1) = p$. $(p, q)$is called Bernoulli variate and the distribution is called Bernoulli distribution. Binomial distribution was discovered by James Bernoulli in the year 1700. Let the random experiment be performed repeatedly and let the occurrence of an event in any trial be called a success and its non-occurrence of an event in any trial be called a success and its non-occurrence a failure.

Some Examples of Bernoulli trials are

1. If a product produced in a factory may be defective or non-defective
2. If a coin is tossed result of each tossing may be either a head or tail.
3. Performance of a student in an exam may be either pass or fail.

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Define Binomial Distribution

A random variable X is said to follow Binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X = x) = P(x) = \begin{cases} nC_x p^x q^{n-x} & ; x = 0, 1, 2 \dots \\ 0 & ; otherwise \end{cases}$$

If X is a Binomial variable with the parameter n and p we shall write $X{\sim}B(n, p)$

Remarks

1. n = Number of items the random experiment is conducted or number of independent Bernoulli trial.
2. x = Number of Success
3. p = Probability of success
4. q = Probability of failure
5. n and p are known as the parameter of the distribution
6. $X{\sim}B(n, p)$ Denotes that X follows Binomial distribution with the parameter n and p.
7. $F(x) = N\,P(x) = NnC_x p^x q^{n-x}$
8. Importance of binomial distribution is it has wider application and gives arise to many other probability distributions.

Assumptions of Binomial Distribution

1. The number of trials n is finite. The trials are independent of each other.
2. Each trial result in two mutually disjoint outcomes, termed as success and failure
3. The probability of success p is constant for each trial

Properties of Binomial Distribution

1. Binomial distribution has 2 parameters n and p
2. It is a discrete distribution, Mean = np; Variance = npq, Standard Deviation = $\sqrt{npq}$
3. Skewness= $(q - p)^2/\sqrt{npq}$, Kurtosis = $3 + \{(1 - 6pq)/npq\}$
4. Its symmetrical if $p = q = 0.5$
5. It is positively skewed if $p < 0.5$
6. It is negatively skewed if $p > 0.5$

Characteristic Function of Binomial distribution

$$\phi_x(t) = (q + pe^{it})^n$$

Applications (or) Examples of Binomial distribution

1. Number of defectives in a lot of size n
2. Number of absentees in a department of n persons
3. Number of machines kept idle in a machine shop having n machines
4. Number of married men in a group of n men.

Definition of Poisson distribution

A random variable X is said to follow Poisson distribution if it assumes only non-negatively values and its p.m.f is given by $P(x, \lambda) = e^{-\lambda}\lambda^x/x!$, here λ is known as the parameter of the distribution and $\lambda > 0$ *and* $e = 2.71828$ ....

Applications (or) Examples of Poisson Distribution

1. Number of death from a discrete such as heart attack or cancer
2. Number of suicides reported in a particular city
3. The number of defective materials in a packing manufactured by a good concern
4. Number of printing mistakes at each page of book
5. The cars passing a crossing per minutes during the busy hours of a day

Assumptions of Poisson distribution

1. The variable is discrete
2. Events can only be either a success or a failure
3. The number of trials is finite large
4. The probability of success p is small and that q is almost equal in unity
5. The trials are independent to each other

Properties of Poisson Distribution

1. Poisson distribution is a discrete distribution, λ be the parameter of the distribution
2. $Mean = Variance = \lambda, S.D = \sqrt{\lambda}, Skewness = 1/\sqrt{\lambda}\ , Kurtosis = 1/\lambda$
3. It is a positively skewed distribution

4. Poisson distribution as a limiting case of Binomial distribution under the following conditions:
   1. n is large 2. p is very small 3. $np = \lambda =$ mean of Poisson distribution

Characteristic Function of Poisson distribution

$$\phi_x(t) = e^{\lambda(e^{it}-1)}$$

BINOMIAL DISTRIBUTION

Definition:

A Random variable x said to following binomial distribution if it assumes only a non-negative values and its probability mass function is given by,

$$P(X = x) = p(x) = nC_x p^x q^{n-x}; x = 0,1,2,..n$$

$$p - probability\ of\ success$$

$$q - probability\ of\ failure$$

$$p + q = 1$$

$$q = 1 - p$$

Binomial distribution with parameter (n, p) i.e. $x \sim B(n, p)$

Ex:

i) Tossing a coin

ii) Throwing a die.

$$p = Head = Success = \frac{1}{2}$$

$$q = Tail = failure = \frac{1}{2}$$

$$p + q = 1$$

MEAN AND VARIANCE OF BINOMIAL DISTRIBUTION

Mean:

$$\mu_1' = E(x) = \sum_{x=0}^{n} x.\,p(x)$$

$$E(x) = \sum_{x=0}^{n} x\ nC_x p^x q^{n-x}$$

$$E(x) = \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$E(x) = \sum_{x=0}^{n} x \frac{n(n-1)!}{x(x-1)!\,(n-x)!} p^x q^{n-x}$$

$$E(x) = \sum_{x=0}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x)!} p^{x+1-1} q^{n-x+1-1}$$

$$E(x) = \sum_{x=0}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} p^1 q^{[(n-1)-(x-1)]}$$

$$E(x) = np \sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{[(n-1)-(x-1)]}$$

$$E(x) = np(p+q)^{n-1}$$

$$E(x) = np(1) = np$$

$$E(x) = np$$

Variance:

$$\mu_2 = \mu_2{'} - (\mu_1{'})^2$$

$$\mu_2{'} = E(x^2) = \sum_{x=0}^{n} x^2\, p(x)$$

$$E(x^2) = \sum_{x=0}^{n} x^2 nC_x p^x q^{n-x}$$

$$E(x^2) = \sum_{x=0}^{n} [x(x-1) + x] nC_x p^x q^{n-x}$$

$$\sum_{x=0}^{n} x(x-1)\; nC_x p^x q^{n-x} + \sum_{x=0}^{n} x\; nC_x p^x q^{n-x}$$

$$E(x^2) = \sum_{x=0}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!}\, p^x q^{n-x} + np$$

$$E(x^2) = \sum_{x=0}^{n} \frac{n(n-1)(n-2)!}{(x-2)!\,(n-x)!}\, p^{x+2-2} q^{n-x+2-2} + np$$

$$E(x^2) = \sum_{x=0}^{n} \frac{n(n-1)(n-2)!}{(x-2)!\,(n-x)!}\, p^{x-2} p^2 q^{[(n-2)-(x-2)]} + np$$

$$E(x^2) = n(n-1)p^2 \sum_{x=0}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!}\, p^{x-2} q^{[(n-2)-(x-2)]} + np$$

$$E(x^2) = n(n-1)p^2(n-2)C_{(x-2)} + np$$

$$E(x^2) = n(n-1)p^2(p+q)^{n-2} + np$$

$$E(x^2) = n(n-1)p^2(1)^{n-2} + np$$

$$E(x^2) = n(n-1)p^2 + np$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2 = n(n-1)p^2 + np - [np]^2$$

$$\mu_2 = (n^2 - n)p^2(1) + np$$

$$\mu_2 = n^2p^2 - np^2 + np - n^2p^2$$

$$\mu_2 = np - np^2$$

$$\mu_2 = np(1-p) \quad = npq$$

MOMENT GENERATING FUNCTION OF BINOMIAL DISTRIBUTION

$$M_X(t) = E[e^{tx}]$$

$$M_X(t) = \sum_{x=0}^{n} e^{tx} p(x)$$

$$M_X(t) = \sum_{x=0}^{n} pe^{tx}\ \mathrm{nC_x p^x q^{n-x}}$$

$$M_X(t) = \sum_{x=0}^{n} \mathrm{nC_x}\ (pe^{t})\mathrm{q^{n-x}}$$

$$M_X(t) = (pe^{t} + q)^{n}$$

CHARACTERISITIC FUNCTION OF BINOMIAL DISTRIBUTION

$$\varphi_X(t) = E\left[e^{itx}\right]$$

$$\varphi_X(t) = \sum_{x=0}^{n} e^{itx} p(x)$$

$$\varphi_X(t) = \sum_{x=0}^{n} pe^{itx}\ \mathrm{nC_x p^x q^{n-x}}$$

$$\varphi_X(t) = \sum_{x=0}^{n} \mathrm{nC_x}\ (pe^{it})\mathrm{q^{n-x}}$$

$$\varphi_X(t) = (pe^{it} + q)^{n}$$

RECURRENCE RELATION FOR THE MOMENTS OF BINOMIAL DISTRIBUTION

$$\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

Proof:

$$\mu_r = E[X^r] - [E[X]]^r$$

$$\mu_r = E[X - E(X)]^r$$

$$\mu_r = E[x - np]^r$$

$$E(x) = \sum_{x=0}^{n} x.p(x)$$

differentiate with respect to p we get,

$$\frac{d\mu_r}{dp} = \frac{d}{dp}\sum_{x=0}^{n}[(x-np)^r nC_x p^x (1-p)^{n-x}]$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n}[r(x-np)^{r-1}(-n)nC_x p^x (1-p)^{n-x} + (x-np)^r nC_x\, x\, p^{x-1}(1)(1-p)^{n-x} + (x-np)^r nC_x\ p^x (n-x)(1-p)^{n-x-1}(-1)]$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n}[-nr(x-np)^{r-1}nC_x p^x (1-p)^{n-x} + (x-np)^r nC_x\, x\, p^{x-1}\ p^{-1}(1-p)^{n-x} + (x-np)^r nC_x\ p^x (n-x)(1-p)^{n-x-1}(-1)]$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} nC_x p^x q^{n-x}\left[-nr(x-np)^{r-1} + (x-np)^r x\ p^{-1} - (x-np)^r (n-x)(q)^{-1}\right]$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} nC_x p^x q^{n-x}\left[-nr(x-np)^{r-1} + (x-np)^r\ \frac{x}{p}\ - (x-np)^r \frac{n-x}{q}\right]$$

$$\left[(x-np)^r\ \frac{x}{p}\ - (x-np)^r\ \frac{n-x}{q}\right] = (x-np)^r\left[\frac{x}{p}\ -\ \frac{n-x}{q}\right] = (x-np)^r\left[\ \frac{(1-p)x - p(n-x)}{p(1-p)}\right]$$

$$= (x-np)^r\left[\ \frac{x - xp - np + xp}{p(1-p)}\right]$$

$$= (x - np)^r \left[\frac{x - np}{p(1-p)} \right]$$

therefore,

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} nC_x p^x q^{n-x} \left[-nr(x-np)^{r-1} + (x-np)^r \frac{x}{p} - (x-np)^r \frac{n-x}{q} \right]$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} nC_x p^x q^{n-x} \left[-nr(x-np)^{r-1} + (x-np)^r \left[\frac{x-np}{pq} \right] \right]$$

$$\frac{d\mu_r}{dp} = \left[-nr(x-np)^{r-1} + \frac{(x-np)^{r+1}}{pq} \right]$$

$$\frac{d\mu_r}{dp} = \left[-nr\mu_{r-1} + \frac{1}{pq} (\mu_{r+1}) \right]$$

$$\mu_{r+1} = pq \left[nr\mu_{r-1} + \frac{d\mu_r}{dp} \right]$$

put r = 1

$$\mu_{1+1} = pq \left[n(1)\mu_{1-1} + \frac{d\mu_1}{dp} \right]$$

$$\mu_{1+1} = pq \left[n\mu_{1-1} + \frac{d\mu_1}{dp} \right], \quad \mu_1 = 0, \qquad \mu_0 = 1$$

$$\mu_2 = pq[n\mu_0 + 0]$$

$$\mu_2 = pq[n(1) + 0]$$

$$\mu_2 = npq$$

put r = 2

$$\mu_{2+1} = pq \left[n(2)\mu_{2-1} + \frac{d\mu_2}{dp} \right]$$

$$\mu_{2+1} = pq\left[2n\mu_{2-1} + \frac{dnpq}{dp}\right]$$

$$\mu_3 = pq\left[2n\mu_1 + \frac{dnpq}{dp}\right]$$

$$\mu_3 = pq\left[2n(0) + \frac{dnpq}{dp}\right]$$

$$\mu_3 = p(1-p)\left[\frac{dnpq}{dp}\right]$$

$$\mu_3 = (p-p^2)\left[\frac{dnpq}{dp}\right]$$

$$\mu_3 = (1-2p)npq$$

$$\mu_3 = (1-p-p)npq$$

$$\mu_3 = (q-p)npq$$

Similarly put r = 3

$$\mu_4 = [\,1 - 6pq + 3npq\,]\,.npq$$

Hence the proof.

ADDITIVE PROPERTY OF BINOMIAL DISTRIBUTION

let x and y be the two random variables from a binomial distribution $x{\sim}B(n_1, p_1)$ and $y{\sim}B(n_2, p_2)$. The additive property of binomial distribution is

$$M_X(t) = (pe^t + q)^{n_1}$$

$$M_Y(t) = (pe^t + q)^{n_2}$$

$$M_{X+Y}(t) = (pe^t + q)^{n_1} + (pe^t + q)^{n_2}$$

$$M_{X+Y}(t) = (pe^t + q)^{n_1+n_2}$$

PROBLEM: 1 (Reference Book: Mathematical Statistics, Model Ex. no: 7, pg.no: 12.6)

The mean and variance of binomial distribution are 4 and $^4/_3$ respectively. find p(x≥ 1)

Solution:

$$\text{Mean} = np = 4 \longrightarrow 1$$

$$\text{Variance} = npq = {^4/_3} \longrightarrow 2$$

divided these two equations,

$$\frac{npq}{np} = \frac{^4/_3}{4}$$

$$q = {^4/_3} \; X \; {^1/_4} = {^1/_3}$$

$$p + q = 1$$

substitute q = $^1/_3$

$$p + {^1/_3} = 1$$

$$p = 1 - {^1/_3} = {^2/_3}$$

$$p = {^2/_3}$$

Substitute p = $^2/_3$ in equation 1

$$np = 4$$

$$n\left(2/_3\right) = 4$$

$$n = 4\,x\,3/_2$$

$$n = 6$$

$$p(x \geq 1) = 1 - p(x < 1)$$

$$p(x \geq 1) = 1 - \mathrm{nC_x p^x q^{n-x}}$$

$$p(x \geq 1) = 1 - 6\mathrm{C}_0\left(2/_3\right)^0\left(1/_3\right)^{6-0}$$

$$p(x \geq 1) = 1 - 1\,(1)\left(1/_3\right)^6$$

$$p(x \geq 1) = 1 - 1\,(1)(0.00137)$$

$$p(x \geq 1) = 1 - 0.00137$$

$$p(x \geq 1) = 0.9986$$

PROBLEM: 2 (Reference Book: Mathematical Statistics, Model Ex. no: 8, pg.no: 12.6)

The Mean and Variance of binomial distribution find 4 and 3. find the value of n.

Solution:

Mean = np = 4 $\longrightarrow$ 1

Variance = npq = 3 ⟶ 2

divided these two equations,

$$\frac{npq}{np} = \frac{3}{4}$$

$$q = {}^{3}/_{4}$$

$$p + q = 1$$

substitute q = $^{1}/_{3}$

$$p + {}^{3}/_{4} = 1$$

$$p = 1 - {}^{3}/_{4} = {}^{1}/_{4}$$

$$p = {}^{1}/_{4}$$

Substitute p = $^{1}/_{4}$ in equation 2

$$npq = 3$$

$$n\left({}^{1}/_{4}\right)\left({}^{3}/_{4}\right) = 3$$

$$n = 3\,x\,4\,x\,{}^{4}/_{3}$$

$$n = 16$$

PROBLEM: 3(Reference Book: Mathematical Statistics, Model Ex. no: 8, pg.no: 12.6)

For the binomial distribution mean is 6 and variance is 3. Find the probability of $P(5 \leq x \leq 7)$

Solution:

Mean = np = 6 $\longrightarrow$ 1

Variance = npq = 3 $\longrightarrow$ 2

divided these two equations,

$$\frac{npq}{np} = \frac{3}{6}$$

$$q = {}^{1}/_{2}$$

$$p + q = 1$$

substitute q = $^{1}/_{2}$

$$p + {}^{1}/_{2} = 1$$

$$p = 1 - {}^{1}/_{2} = {}^{1}/_{2}$$

$$p = {}^{1}/_{2}$$

Substitute p = $^{1}/_{2}$ in equation 1

$$np = 6$$

$$n\left({}^{1}/_{2}\right) = 6$$

$$n = 6 X 2$$

$$n = 12$$

$$P(5 \le x \le 7) = p(x = 5) + p(x = 6) + p(x = 7)$$

$$P(5 \le x \le 7)$$

$$= 12C_5 \left({}^1/_2\right)^5 \left({}^1/_2\right)^{12-5} + 12C_6 \left({}^1/_2\right)^6 \left({}^1/_2\right)^{12-6} + 12C_7 \left({}^1/_2\right)^7 \left({}^1/_2\right)^{12-7}$$

$$P(5 \le x \le 7) = 792\ (0.03125)(0.00781) + 924(0.01562) + 792\ (0.00781)(0.03125)$$

$$P(5 \le x \le 7) = 0.1933 + 0.2254 + 0.1933$$

$$P(5 \le x \le 7) = 0.612$$

PROBLEM: 4(Reference Book: Mathematical Statistics, Model Ex. no: 6, pg.no: 12.5)

For the binomial distribution mean = 6 and standard deviation = $\sqrt{2}$. Write out of the terms of the distribution.

Solution:

$$np = 6$$

$$\sigma = \sqrt{2}$$

$$\sigma^2 = (\sqrt{2})^2 = 2$$

Mean = np = 6 $\longrightarrow$ 1

Variance = npq = 2 $\longrightarrow$ 2

divided these two equations,

$$\frac{npq}{np} = \frac{2}{6}$$

$$q = {}^{1}/_{3}$$

$$p + q = 1$$

substitute q = ${}^{1}/_{3}$

$$p + {}^{1}/_{3} = 1$$

$$p = 1 - {}^{1}/_{3} = {}^{2}/_{3}$$

$$p = {}^{2}/_{3}$$

Substitute p = ${}^{2}/_{3}$ in equation 1

$$np = 2$$

$$n\left({}^{2}/_{3}\right) = 6$$

$$n = 6 \; X \; {}^{3}/_{2}$$

$$n = 9$$

PROBLEM:5(Reference Book: Mathematical Statistics, Model Ex. no: 8, pg.no: 12.6)

It was found that for the binomial distribution mean = 5 and standard deviation = 2 find n values.

Solution:

$$\text{Mean} = np = 5 \longrightarrow 1$$

$$\text{Variance} = npq = 4 \longrightarrow 2$$

divided these two equations,

$$\frac{npq}{np} = \frac{4}{5}$$

$$q = {}^{4}/_{5}$$

$$p + q = 1$$

substitute q = $^{4}/_{5}$

$$p + {}^{4}/_{5} = 1$$

$$p = 1 - {}^{4}/_{5} = {}^{1}/_{5}$$

$$p = {}^{1}/_{5}$$

Substitute p = $^{1}/_{5}$ in equation 1

$$np = 5$$

$$n\left({}^{1}/_{5}\right) = 5$$

$$n = 5\ X\ 5$$

$$n = 25$$

PROBLEM:6(Reference Book: Mathematical Statistics, Example no: 9, pg.no: 12.7)

For the binomial distribution mean is 4 and variance is 3. Also find p(x) =15

Solution:

Mean = np = 4 ⟶ 1

Variance = npq = 3 ⟶ 2

divided these two equations,

$$\frac{npq}{np} = \frac{3}{4}$$

$$q = {}^{3}/_{4}$$

$$p + q = 1$$

Substitute q = $^{3}/_{4}$

$$p + {}^{3}/_{4} = 1$$

$$p = 1 - {}^{3}/_{4} = {}^{1}/_{4}$$

$$p = {}^{1}/_{4}$$

Substitute p = $^{1}/_{4}$ in equation 1

$$np = 4$$

$$n\left({}^{1}/_{4}\right) = 4$$

$$n = 4\ X\ 4$$

$$n = 16$$

$$p(x = 15) = \mathrm{nC_x p^x q^{n-x}}$$

$$p(x = 15) = 16C_5 \left({}^{1}/_{4}\right)^{15} \left({}^{3}/_{4}\right)^{16-15}$$

$$p(x = 15) = 16(9.313\,x\,10^{-10})(0.75)$$

$$p(x = 15) = 16(0.0000000001490)(0.75)$$

$$p(x = 15) = 0.0000000111756$$

$$p(x = 15) = 0$$

PROBLEM: 7 (Reference Book: Mathematical Statistics, Model Ex. no: 12, pg.no: 12.8)

The Random variable x and y are independent it as follows binomial distribution with respective it as follows binomial distribution with respective n = 3, p = $^1/_3$ and n = 5, p = $^1/_3$.

Write the term of $p(x + y \geq 1)$

Solution:

$$X{\sim}B(n_1, p_1)$$

$$X{\sim}B(3,\ ^1/_3)$$

$$Y{\sim}B(n_2, p_2)$$

$$Y{\sim}B(5,\ ^1/_3)$$

$p = {}^1/_4$ and n = 8

$$p + q = 1$$

Substitute p = $^1/_3$

$$^1/_3 + q = 1$$

$$q = 1 - {}^{1}/_{3} = {}^{2}/_{3}$$

$$p = {}^{2}/_{3}$$

$$p(x+y \geq 1) = 1 - p(x+y < 1)$$

$$p(x+y \geq 1) = 1 - \mathrm{nC}_{x+y}\mathrm{p}^{x+y}\mathrm{q}^{n-(x+y)}$$

$$p(x+y \geq 1) = 1 - 8\mathrm{C}_0 \left({}^{1}/_{3}\right)^{0} \left({}^{2}/_{3}\right)^{8-0}$$

$$p(x+y \geq 1) = 1 - (1)(1)(0.0390)$$

$$p(x+y \geq 1) = 1 - (0.0390)$$

$$p(x+y \geq 1) = 0.961$$

POISSON DISTRIBUTION

A random variable X is said to have a poisson distribution with parameter λ, when the probability function

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}; x = 0,1,2,\ldots\infty, \qquad \lambda = 0$$

Mean and Variance of Poisson Distribution.

Mean

$$E(X) = \sum x\, P(x)$$

$$E(X) = \sum x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E(X) = \sum x \frac{e^{-\lambda}\lambda^x}{x(x-1)!}$$

$$E(X) = \sum x \frac{e^{-\lambda}\lambda^{x-1+1}}{x(x-1)!}$$

$$E(X) = \sum \frac{e^{-\lambda}\lambda^{x-1}\lambda^1}{(x-1)!}$$

$$E(X) = \lambda \sum \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

$$E(X) = \lambda(1)$$

$$where \sum \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = 1$$

$$Mean = E(X) = \lambda$$

Variance:

$$V(X) = E(X^2) - \left(E(X)\right)^2$$

$$E(X^2) = \sum x^2\, P(x)$$

$$E(X^2) = \sum x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E(X^2) = \sum (x(x-1) + x) \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E(X^2) = \sum x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \sum x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E(X^2) = \sum x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \lambda$$

$$E(X^2) = \sum x(x-1) \frac{e^{-\lambda}\lambda^x}{x(x-1)(x-2)!} + \lambda$$

$$E(X^2) = \sum \frac{e^{-\lambda}\lambda^{x-2+2}}{(x-2)!} + \lambda$$

$$E(X^2) = \sum \frac{e^{-\lambda}\lambda^{x-2}\lambda^2}{(x-2)!} + \lambda$$

$$E(X^2) = \lambda^2 \sum \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} + \lambda$$

$$E(X^2) = \lambda^2(1) + \lambda$$

$$where \sum \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = 1$$

$$E(X^2) = \lambda^2 + \lambda$$

$$Variance = V(X) = E(X^2) - (E(X))^2$$

$$Variance = V(X) = \lambda^2 + \lambda - (\lambda)^2$$

$$Variance = V(X) = \lambda$$

Mean and Variance of Poisson Distribution is λ.

Moment Generating Function of Poisson Distribution

$$M_X(t) = E(e^{tx})$$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\, P(x)$$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\, \frac{e^{-\lambda}\lambda^x}{x!}$$

$$M_X(t) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{tx}\lambda^x)}{x!}$$

$$M_X(t) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{t}\lambda)^x}{x!}$$

$$M_X(t) = e^{-\lambda}\left[\frac{(e^t\lambda)^0}{0!} + \frac{(e^t\lambda)^1}{1!} + \frac{(e^t\lambda)^2}{2!} + \cdots\right]$$

$$M_X(t) = e^{-\lambda}\left[e^{\lambda e^t}\right]$$

$$M_X(t) = e^{\lambda(e^t-1)}$$

Characteristic Function of Poisson Distribution

$$\phi_X(t) = E(e^{itx})$$

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx}\, P(x)$$

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx}\, \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\phi_X(t) = e^{-\lambda}\sum_{x=0}^{\infty} \frac{(e^{itx}\lambda^x)}{x!}$$

$$\phi_X(t) = e^{-\lambda}\sum_{x=0}^{\infty} \frac{(e^{it}\lambda)^x}{x!}$$

$$\phi_X(t) = e^{-\lambda}\left[\frac{(e^{it}\lambda)^0}{0!} + \frac{(e^{it}\lambda)^1}{1!} + \frac{(e^{it}\lambda)^2}{2!} + \cdots\right]$$

$$\phi_X(t) = e^{-\lambda}\left[e^{\lambda e^{it}}\right]$$

$$\phi_X(t) = e^{\lambda(e^{it}-1)}$$

Additive or Productive of Independent Poisson Variate or Sum of Independent Poisson Variate is also a Poisson Variate.

If X_i $(i = 1,2 \ldots .n)$ or Independent Poisson Variate λ $(i = 1,2 \ldots .n)$ respectively, then $\sum_{x=1}^{n} X_i$ is also Poisson Variate with parameter $\sum_{x=1}^{n} \lambda_i$.

Proof

$$M_{X_i}(t) = e^{\lambda(e^t-1)}$$

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \dots M_{X_n}(t)$$

$$M_{X_1+X_2+\cdots+X_n}(t) = e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} \dots e^{\lambda_n(e^t-1)}$$

$$M_{X_1+X_2+\cdots+X_n}(t) = e^{\lambda_1+\lambda_2+\cdots\lambda_n(e^t-1)}$$

$$M_{X_1+X_2+\cdots+X_n}(t) = e^{\sum\lambda_i(e^t-1)}$$

Which is the moment generating function of Poisson variate with parameter $\lambda_1 + \lambda_2 + \cdots \lambda_n$

Hence uniqueness theorem of moment generating function $\sum_{x=1}^{n} X_i$ is also Poisson Variate with parameter $\sum_{x=1}^{n} \lambda_i$.

Recurrence Relation for the moments of Poisson Distribution.

$$\mu_{r+1} = \lambda\left[\frac{d\mu_r}{dp} + r\mu_{r-1}\right]$$

proof

$$\mu_r = E[X - E(X)]^r$$

$$\mu_r = E[X - \lambda]^r$$

$$E(X) = \sum_{x=0}^{\infty} x\, P(x)$$

$$\mu_r = \sum_{x=0}^{\infty} [x-\lambda]^r \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\mu_r = \frac{1}{x!}\sum_{x=0}^{\infty} [x-\lambda]^r\, e^{-\lambda}\lambda^x$$

$$\frac{d\mu_r}{d\lambda} = \frac{1}{x!}\sum_{x=0}^{\infty}\left[r[x-\lambda]^{r-1}\, e^{-\lambda}\lambda^x(-1) + [x-\lambda]^r\, e^{-\lambda}\lambda^x(-1) + [x-\lambda]^r\, e^{-\lambda}x\lambda^{x-1}\right]$$

$$\frac{d\mu_r}{d\lambda} = \frac{1}{x!}\sum_{x=0}^{\infty}\left[-r[x-\lambda]^{r-1} e^{-\lambda}\lambda^x - [x-\lambda]^r e^{-\lambda}\lambda^x + [x-\lambda]^r e^{-\lambda} x\lambda^x \lambda^{-1}\right]$$

$$\frac{d\mu_r}{d\lambda} = \frac{e^{-\lambda}\lambda^x}{x!}\sum_{x=0}^{\infty}[-r[x-\lambda]^{r-1} - [x-\lambda]^r + [x-\lambda]^r x\lambda^{-1}]$$

$$\frac{d\mu_r}{d\lambda} = \frac{e^{-\lambda}\lambda^x}{x!}\sum_{x=0}^{\infty}\left[-r[x-\lambda]^{r-1} - [x-\lambda]^r + [x-\lambda]^r \frac{x}{\lambda}\right]$$

$$\frac{d\mu_r}{d\lambda} = \frac{e^{-\lambda}\lambda^x}{x!}\sum_{x=0}^{\infty}\left[-r[x-\lambda]^{r-1} + [x-\lambda]^r\left[-1+\frac{x}{\lambda}\right]\right]$$

$$\frac{d\mu_r}{d\lambda} = \frac{e^{-\lambda}\lambda^x}{x!}\sum_{x=0}^{\infty}\left[-r[x-\lambda]^{r-1} + [x-\lambda]^r\left[\frac{-\lambda+x}{\lambda}\right]\right]$$

$$\frac{d\mu_r}{d\lambda} = \frac{e^{-\lambda}\lambda^x}{x!}\sum_{x=0}^{\infty}\left[-r[x-\lambda]^{r-1} + [x-\lambda]^{r+1}\left[\frac{1}{\lambda}\right]\right]$$

$$\frac{d\mu_r}{d\lambda} = -r\sum_{x=0}^{\infty}\left[[x-\lambda]^{r-1}\frac{e^{-\lambda}\lambda^x}{x!}\right] + \frac{1}{\lambda}\sum_{x=0}^{\infty}\left[[x-\lambda]^{r+1}\frac{e^{-\lambda}\lambda^x}{x!}\right]$$

$$\frac{d\mu_r}{d\lambda} = -r\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1}$$

$$where\ \mu_{r-1} = \sum_{x=0}^{\infty}\left[[x-\lambda]^{r-1}\frac{e^{-\lambda}\lambda^x}{x!}\right];\quad \mu_{r+1} = \sum_{x=0}^{\infty}\left[[x-\lambda]^{r+1}\frac{e^{-\lambda}\lambda^x}{x!}\right]$$

$$\frac{1}{\lambda}\mu_{r+1} = \frac{d\mu_r}{d\lambda} + r\mu_{r-1}$$

$$\mu_{r+1} = \lambda\left[\frac{d\mu_r}{d\lambda} + r\mu_{r-1}\right]$$

Put r=1,

$$\mu_{r+1} = \lambda\left[\frac{d\mu_r}{d\lambda} + r\mu_{r-1}\right]$$

$$\mu_{1+1} = \lambda\left[\frac{d\mu_1}{d\lambda} + (1)\mu_{1-1}\right]$$

$$\mu_2 = \lambda\left[\frac{d\mu_1}{d\lambda} + (1)\mu_0\right]$$

$$\mu_2 = \lambda[0 + (1)\mu_0]\,; \quad where\ \mu_1 = 0$$

$$\mu_2 = \lambda[\mu_0]$$

$$\mu_2 = \lambda[1]; \quad where\ \mu_0 = 1$$

$$\therefore \mu_2 = \lambda$$

Put r=2,

$$\mu_{r+1} = \lambda\left[\frac{d\mu_r}{d\lambda} + r\mu_{r-1}\right]$$

$$\mu_{2+1} = \lambda\left[\frac{d\mu_2}{d\lambda} + (2)\mu_{2-1}\right]$$

$$\mu_3 = \lambda\left[\frac{d\lambda}{d\lambda} + (2)\mu_1\right]\,; \quad using\ \mu_2 = \lambda$$

$$\mu_3 = \lambda[1 + (2)(0)]\,; \quad where\ \mu_1 = 0$$

$$\mu_3 = \lambda[1]$$

$$\therefore \mu_3 = \lambda$$

Put r=3,

$$\mu_{r+1} = \lambda\left[\frac{d\mu_r}{d\lambda} + r\mu_{r-1}\right]$$

$$\mu_{3+1} = \lambda\left[\frac{d\mu_3}{d\lambda} + (3)\mu_{3-1}\right]$$

$$\mu_4 = \lambda\left[\frac{d\lambda}{d\lambda} + (3)\mu_2\right]\,; \quad using\ \mu_3 = \mu_2 = \lambda$$

$$\mu_4 = \lambda[1 + (3)(\lambda)]$$

$$\mu_4 = \lambda[1 + 3\lambda]$$

$$\therefore \mu_4 = \lambda + 3\lambda^2$$

Limiting case of Binomial distribution to Poisson distribution

Poisson distribution is a limiting case of the binomial distribution under the following conditions.

i) n, the number of trials is infinitely large, that is $n \to \infty$,

ii) p, the constant of the probability of the success for each trials is infinitely small that is $p \to 0$

iii) $np = \lambda$ is finite.

Thus $\boldsymbol{np} = \lambda$; $p = \frac{\lambda}{n}$

$$q = 1 - p = 1 - \frac{\lambda}{n}$$

where λ is positive real number.

iv) The probability of success in a series of n independent trials is $n \to \infty$.

$$B(x, n\, p) = nC_x p^x q^{n-x}; x = 0, 1, 2 \ldots . n \quad \to (*)$$

$$\lim_{n\to\infty} B(x, n\, p) = \lim_{n\to\infty}[nC_x p^x q^{n-x}]$$

using Striling's approximation we get,

$$n! \approx \sqrt{2\pi}\, e^{-n}\, n^{n+(1/2)}$$

$$(n - x)! \approx \sqrt{2\pi}\, e^{-(n-x)}\, (n - x)^{(n-x)+(1/2)}$$

$$\lim_{n\to\infty} B(x, n\, p) = \lim_{n\to\infty}\left[\frac{n!}{x!\,(n - x)!}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}\right]$$

$$\lim_{n\to\infty} B(x, n\, p) = \lim_{n\to\infty}\left[\frac{\sqrt{2\pi}\, e^{-n}\, n^{n+(1/2)}}{x!\,\sqrt{2\pi}\, e^{-(n-x)}\, (n - x)^{(n-x)+(1/2)}}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}\right]$$

$$\lim_{n\to\infty} B(x, n\, p) = \lim_{n\to\infty}\left[\frac{\sqrt{2\pi}\, e^{-n}\, n^{n+(1/2)}}{x!\,\sqrt{2\pi}\, e^{-n}e^{x}\, (n - x)^{(n-x)+(1/2)}}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}\right]$$

$$\lim_{n\to\infty} B(x,n\,p) = \frac{\lambda^x}{x!}\lim_{n\to\infty}\left[\frac{n^{n+(1/2)}}{e^x(n-x)^{(n-x)+(1/2)}}\left(\frac{1}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}\right]$$

$$\lim_{n\to\infty} B(x,n\,p) = \frac{\lambda^x}{e^x\ x!}\lim_{n\to\infty}\left[\frac{n^{n+(1/2)}}{(n-x)^{(n-x)+(1/2)}}n^{-x}\left(1-\frac{\lambda}{n}\right)^{n-x}\right]$$

$$\lim_{n\to\infty} B(x,n\,p) = \frac{\lambda^x}{e^x\ x!}\lim_{n\to\infty}\left[\frac{n^{n-x+(1/2)}}{(n-x)^{(n-x)+(1/2)}}\left(1-\frac{\lambda}{n}\right)^{n-x}\right]$$

$$\lim_{n\to\infty} B(x,n\,p) = \frac{\lambda^x}{e^x\ x!}\lim_{n\to\infty}\left[\left(\frac{n}{(n-x)}\right)^{(n-x)+(1/2)}\left(1-\frac{\lambda}{n}\right)^{n-x}\right]$$

$$\lim_{n\to\infty} B(x,n\,p) = \frac{\lambda^x}{e^x\ x!}\lim_{n\to\infty}\left[\frac{\left(1-\frac{\lambda}{n}\right)^{n-x}}{\left(\frac{n-x}{n}\right)^{(n-x)+(1/2)}}\right]$$

$$\lim_{n\to\infty} B(x,n\,p) = \frac{\lambda^x}{e^x\ x!}\lim_{n\to\infty}\left[\frac{\left(1-\frac{\lambda}{n}\right)^{n-x}}{\left(1-\frac{x}{n}\right)^{(n-x)+(1/2)}}\right] \rightarrow 1$$

we know that

$$\lim_{n\to\infty}\left[\frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{x}{n}\right)^{n}}\ \frac{\left(1-\frac{\lambda}{n}\right)^{-x}}{\left(1-\frac{x}{n}\right)^{-x+(1/2)}}\right]$$

$$\left[\frac{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}{\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{n}}\ \frac{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}}{\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{-x+(1/2)}}\right]$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda};\qquad \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1$$

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{n} = 1;\qquad \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{-x+(1/2)} = e^{-x}$$

substitute all the above values in (1) we get

$$\lim_{n\to\infty} B(x,n\,p) = \frac{\lambda^x}{e^x\ x!}\left[\frac{e^{-\lambda}\ \ (1)}{(1)\ \ e^{-x}}\right]$$

$$\lim_{n\to\infty} B(x,n\,p) = \frac{\lambda^x}{e^x\ x!}\left[\frac{e^{-\lambda}}{e^{-x}}\right]$$

$$\lim_{n\to\infty} B(x,n\,p) = \frac{e^{-\lambda}\ \lambda^x}{x!}$$

Which is the P(X) of the Poisson Distribution

PROBLEM: 8(Reference Book: Mathematical Statistics, Model Ex. no: 1, pg.no: 13.2)

The manufacture pins know that two percent of product is defective. if is sells pins in boxes of hundreds and guaranties that not more than four pins will be defective what is the probability that a box will fail to meet the guaranty quality.

solution

Given that $No.\,of\ pins\ in\ the\ boxes;\ n = 100;$

The manufacture pins know that two percent of product is defective;

$$P = 2\% = \frac{2}{100} = 0.02$$

Now the P.M.F of the Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\ \lambda^x}{x!}$$

Now the Mean is find out by (values should be given in Binomial form)

$$np = \lambda$$

therefore, to find λ

$$100 \times 0.02 = \lambda$$

$$\lambda = 2$$

Now the probability function

$$P(X=x)=\frac{e^{-\lambda}\ \lambda^x}{x!}$$

$$P(X=x)=\frac{e^{-2}\ 2^x}{x!}$$

To Find

P(Guaranties that **not more than** four pins will be defective)

$$P(X\le 4)=P(x=0)+P(x=1)+P(x=2)+P(x=3)+P(x=4)$$

$$P(X\le 4)=\frac{e^{-2}\ 2^0}{0!}+\frac{e^{-2}\ 2^1}{1!}+\frac{e^{-2}\ 2^2}{2!}+\frac{e^{-2}\ 2^3}{3!}+\frac{e^{-2}\ 2^4}{4!}$$

$$P(X\le 4)=0.135+0.27+0.27+0.18+0.09$$

$$P(X\le 4)=0.945$$

PROBLEM: 9 (Reference Book: Mathematical Statistics, Model Ex. no: 2, pg.no: 13.2)

Find the probability that atleast 4 defective fuses is found in the boxes of 200 fuses. If experience is show that 2% such fuses are defective.

solution

Given that $No. of\ fuses\ in\ the\ boxes;\ n=200;$

Two percent of fuses are defective;

$$P=2\%=\frac{2}{100}=0.02$$

Now the P.M.F of the Poisson distribution is

$$P(X=x)=\frac{e^{-\lambda}\ \lambda^x}{x!}$$

Now the Mean is find out by (values should be given in Binomial form)

$$np=\lambda$$

therefore, to find λ

$$200\times 0.02=\lambda$$

$$\lambda = 4$$

Now the probability function

$$P(X = x) = \frac{e^{-\lambda}\ \lambda^x}{x!}$$

$$P(X = x) = \frac{e^{-4}\ 4^x}{x!}$$

To Find

P(that atleast 4 defectives fuses found in that boxes)

$$P(X \geq 4) = 1 - P(X < 4)$$

$$P(X \geq 4) = 1 - [P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)]$$

$$P(X \geq 4) = 1 - \left[\frac{e^{-4}\ 4^0}{0!} + \frac{e^{-4}\ 4^1}{1!} + \frac{e^{-4}\ 4^2}{2!} + \frac{e^{-4}\ 4^3}{3!}\right]$$

$$P(X \geq 4) = 1 - [0.018 + 0.073 + 0.146 + 0.195]$$

$$P(X \geq 4) = 1 - 0.432$$

$$P(X \geq 4) = 0.568$$

PROBLEM:10 (Reference Book: Mathematical Statistics, Example no: 6, pg.no: 13.5)

If 3% of the electrical bulbs manufacture by a company are defective. Find the probability that in the sample of 100 bulbs exactly 5 bulbs are defective.

Solution:

Given that $\boldsymbol{the\ sample\ of\ the\ bulbs\ be;\ n = 100;}$

Three percent of electrical bulbs are defective;

$$P = 3\% = \frac{3}{100} = 0.03$$

Now the P.M.F of the Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\ \lambda^x}{x!}$$

Now the Mean is find out by (values should be given in Binomial form)

$$np = \lambda$$

therefore, to find λ

$$100 \times 0.03 = \lambda$$

$$\lambda = 3$$

$$P(X = x) = \frac{e^{-\lambda}\ \lambda^x}{x!}$$

$$P(X = x) = \frac{e^{-3}\ 3^x}{x!}$$

To Find

P(that exactly 5 bulbs are defective)

$$P(X = 5) = \frac{e^{-3}\ 3^5}{5!}$$

$$P(X = 5) = 0.1008$$

PROBLEM: 11(Reference Book: Mathematical Statistics, Example. no: 16, pg.no: 13.11)

At the busy traffic function the probability of an accident is P=0.001. However during a certain part of the day 1000 cars pass through the junction. what is the Probability that two or more accident occur during that period.

Solution

Given that $At\ a\ certain\ part\ of\ that\ day\ cars\ pass\ through\ the\ junction$

$$n = 1000;$$

At the busy traffic function the probability of an accident is

$$P = 0.001$$

Now the P.M.F of the Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\ \lambda^x}{x!}$$

Now the Mean is find out by (values should be given in Binomial form)

$$np = \lambda$$

To find λ

$$1000 \times 0.001 = \lambda$$

$$\lambda = 1$$

$$P(X = x) = \frac{e^{-\lambda} \ \lambda^x}{x!}$$

$$P(X = x) = \frac{e^{-1} \ 1^x}{x!}$$

To Find

Probability that two or more accident occur during that period.

$$P(X > 2) = 1 - [P(X \leq 2)]$$

$$P(X > 2) = 1 - [P(x = 0) + P(x = 1) + P(x = 2)]$$

$$P(X > 2) = 1 - \left[\frac{e^{-1} \ 1^0}{0!} + \frac{e^{-1} \ 1^1}{1!} + \frac{e^{-1} \ 1^2}{2\,!}\right]$$

$$P(X > 2) = 1 - \left[\frac{e^{-1} \ 1^0}{0!} + \frac{e^{-1} \ 1^1}{1!} + \frac{e^{-1} \ 1^2}{2\,!}\right]$$

$$P(X > 2) = 1 - [0.367 + 0.367 + 0.183]$$

$$P(X > 2) = 1 - [0.917]$$

$$P(X > 2) = 0.083$$

PROBLEM:12 (Reference Book: Mathematical Statistics, Example no: 1, pg.no: 13.2)

The Variable X is a Poisson variate that $P(X = 1), which\ is\ 0.3\ and\ P(X = 2)\ is\ 0.2$. Find $P(X) = 0$

Solution

Given that a random variable of the Poisson distribution

Now the P.M.F of the Poisson distribution is

$$\boldsymbol{P(X=x)=\frac{e^{-\lambda}\ \lambda^{x}}{x!}}$$

$$P(X=1)=\frac{e^{-\lambda}\ \lambda^{1}}{\mathbf{1!}}=0.3\quad\to(1)$$

$$P(X=2)=\frac{e^{-\lambda}\ \lambda^{2}}{\mathbf{2!}}=0.2\quad\to(2)$$

now we divide (2) by (1) we get

$$\frac{\frac{e^{-\lambda}\ \lambda^{2}}{2!}}{\frac{e^{-\lambda}\ \lambda^{1}}{1!}}=\frac{0.2}{0.3}$$

$$\frac{e^{-\lambda}\ \lambda^{2}}{\mathbf{2!}}\quad\frac{\mathbf{1!}}{e^{-\lambda}\ \lambda^{1}}=0.667$$

$$\frac{\lambda}{\mathbf{2}}=0.667$$

$$\lambda=2(0.667)$$

$$\lambda=1.33$$

To Find $P(X=0)$ using $\lambda=1.33$

$$P(X=0)=\frac{e^{-\lambda}\ \lambda^{0}}{\mathbf{0!}}$$

$$P(X=0)=\frac{e^{-(1.33)}\ (1.33)^{0}}{\mathbf{0!}}$$

$$P(X=0)=\frac{\mathbf{0.2644\ (1)}}{\mathbf{1}}$$

$$P(X=0)=\mathbf{0.2644}$$

PROBLEM: 13 (Reference Book: Mathematical Statistics, Model Ex. no: 1, pg.no: 13.2)

If a random variable x follows Poisson distribution $P(X=2)=P(X=1)$. find $P(X=0)$

Solution

Given that a random variable of the Poisson distribution

$$P(X=2)=P(x=1)$$

Now the P.M.F of the Poisson distribution is

$$\boldsymbol{P(X=x)=\frac{e^{-\lambda}\ \lambda^{x}}{x!}}$$

$$\frac{e^{-\lambda}\ \lambda^{2}}{2!}=\frac{e^{-\lambda}\ \lambda^{1}}{1!}$$

$$\frac{\lambda}{2}=1$$

$$\lambda=2$$

To find $P(X=0)$ using $\lambda=2$

$$P(X=0)=\frac{e^{-2}\ 2^{0}}{0!}$$

$$P(X=0)=0.1353$$

FITTING OF POISSON DISTRIBUTION

PROBLEM:14 (Reference Book: Mathematical Statistics, Model Excerise. no: 12, pg.no: 13.24)

The following mistakes per page were observed in box.

No. of Mistakes	0	1	2	3	4
No. of Pages	211	90	19	5	0

Fit a Poisson distribution and test whether the Poisson distribution.

Solution

To test the Goodness of fit by using Poisson Distribution.

The probability mass function of Poisson distribution

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0,1,2 \dots \infty$$

Where λ is a parameter of Poisson distribution and mean of Poisson distribution.

$$\lambda = \bar{X} = \frac{\sum fx}{\sum f}$$

$$Expected\ Frequency = N\ P(X) = N \times \left[\frac{e^{-\lambda}\lambda^x}{x!}\right]$$

Where $N = \sum f$ Total number of frequency

To Test the Goodness of fit we have apply chi-square test

$$\chi^2 = \sum_{i=1}^{n}\left[\frac{(O_i - E_i)^2}{E_i}\right] \sim \chi^2_{(n-k-1)df}$$

Compare the calculated value with table value at 5% level of significance

CALCULATION

Null Hypothesis

$$H_0: The\ fit\ is\ Good.$$

Alternative Hypothesis

$$H_1: The\ fit\ is\ Not\ good.$$

Test Statistic

Where n = 4, $\lambda = \bar{X} = \frac{\sum fx}{\sum f}$

$$\lambda = \frac{143}{325} = 0.44$$

The Expected Frequency

$$E(X) = N\,P(X)$$

X	Observed Frequency	fx	$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$	Expected Frequency $E(X) = N\,P(X)$
0	211	0	$\frac{e^{-0.44}(0.44)^0}{0!} = 0.644$	$325 \times 0.644 \approx 209$
1	90	90	$\frac{e^{-0.44}(0.44)^1}{1!} = 0.283$	$325 \times 0.283 \approx 93$
2	19	38	$\frac{e^{-0.44}(0.44)^2}{2!} = 0.0623$	$325 \times 0.0623 \approx 20$
3	5	15	$\frac{e^{-0.44}(0.44)^3}{3!} = 0.009$	$325 \times 0.009 \approx 3$
4	0	0	$\frac{e^{-0.44}(0.44)^4}{4!} = 0.001$	$325 \times 0.001 \approx 0$
	$\sum f = 325$	$\sum fx = 143$		$\sum E_i = 325$

Observed Frequency	Expected Frequency $E(X) = N\,P(X)$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
211	209	4	0.019
90	93	9	0.097
$\left.\begin{matrix}19\\5\\0\end{matrix}\right\}$	$\left.\begin{matrix}20\\3\\0\end{matrix}\right\}$	1	0.043
			$\sum_{i=1}^{n}\left[\frac{(O_i - E_i)^2}{E_i}\right] = 0.159$

Level of Significance (L.O.S)

$$\alpha\,\% = 5\% = 0.05$$

Critical Value

The Value of $\chi^2{}_{(n-k-1)df}$ at 5% level of significance

$$\chi^2{}_{(5-3-1)df} = \chi^2{}_{(1)df} = 3.841$$

DECISION

The calculated value is less than the table value at 5% level of significance with 1 degrees of freedom (df).

$$\chi^2{}_C = 0.159 < \chi^2{}_T = 3.841$$

Therefore we Accept our Null hypothesis at 5% level of significance with 1 degrees of freedom (df).

PROBLEM:15 (Reference Book: Mathematical Statistics, Model Excerise. no: 12, pg.no: 13.24)

The Number of defects per unit in a sample of 400 units manufacture was found to be.

No. of Defects	0	1	2	3	4
Frequency	214	92	90	3	1

Fit a Poisson distribution.

Solution

To test the Goodness of fit by using Poisson Distribution.

The probability mass function of Poisson distribution

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!};\ x=0,1,2\ldots\infty$$

Where λ is a parameter of Poisson distribution and mean of Poisson distribution.

$$\lambda=\bar{X}=\frac{\sum fx}{\sum f}$$

$$Expected\ Frequency = N\ P(X) = N\times\left[\frac{e^{-\lambda}\lambda^{x}}{x!}\right]$$

Where $N=\sum f$ Total number of frequency

To Test the Goodness of fit we have apply chi-square test

$$\chi^2=\sum_{i=1}^{n}\left[\frac{(O_i-E_i)^2}{E_i}\right]\sim\chi^2{}_{(n-k-1)df}$$

Compare the calculated value with table value at 5% level of significance

CALCULATION

Null Hypothesis

$$H_0: The\ fit\ is\ Good.$$

Alternative Hypothesis

$$H_1: The\ fit\ is\ Not\ good.$$

Test Statistic

Where n = 4, $\lambda = \bar{X} = \frac{\Sigma fx}{\Sigma f}$

$$\lambda = \frac{285}{400} = 0.7125$$

The Expected Frequency

$$E(X) = N\ P(X)$$

X	Observed Frequency	fx	$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$	Expected Frequency $E(X) = N\ P(X)$
0	214	0	$\frac{e^{-0.7125}(0.7125)^0}{0!} = 0.4904$	$400 \times 0.4904 \approx 196$
1	92	92	$\frac{e^{-0.7125}(0.7125)^1}{1!} = 0.3494$	$400 \times 0.3494 \approx 140$
2	90	180	$\frac{e^{-0.7125}(0.7125)^2}{2!} = 0.1244$	$400 \times 0.1244 \approx 50$
3	3	9	$\frac{e^{-0.7125}(0.7125)^3}{3!} = 0.0295$	$400 \times 0.0295 \approx 12$
4	1	4	$\frac{e^{-0.7125}(0.7125)^4}{4!} = 0.0053$	$400 \times 0.0053 \approx 2$
	$\sum f = 400$	$\sum fx = 285$		$\sum E_i = 400$

Observed Frequency	Expected Frequency $E(X) = N\ P(X)$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
214	196	324	1.65
92	140	2304	16.46
$\left.\begin{matrix}90\\3\\1\end{matrix}\right\}$	$\left.\begin{matrix}50\\12\\2\end{matrix}\right\}$	900	14.06
			$\sum_{i=1}^{n}\left[\frac{(O_i - E_i)^2}{E_i}\right] = 32.17$

Level of Significance (L.O.S)

$$\alpha\,\% = 5\% = 0.05$$

Critical Value

The Value of $\chi^2_{(n-k-1)df}$ at 5% level of significance

$$\chi^2_{(5-3-1)df} = \chi^2_{(1)df} = 3.841$$

DECISION

The calculated value is Greater than the table value at 5% level of significance with 1 degrees of freedom (df).

$$\chi^2_C = 32.17 > \chi^2_T = 3.841$$

Therefore we Reject our Null hypothesis at 5% level of significance with 1 degrees of freedom (df).

TEST FOR GOODNESS OF FIT FOR BINOMIAL DISTRIBUTION

PROBLEM:16 (Reference Book: Mathematical Statistics, Model Excerise. no: 12, pg.no: 12.27)

Five coins tossed 256 times the number of heads observed below. That the coins are unbiased by employing chi-square goodness of fit.

No. of Heads	0	1	2	3	4	5
Frequency	5	35	75	84	45	12

AIM

To test the coins a unbiased by using chi-square Goodness of fit.

PROCEDURE

The probability mass function of Binomial distribution

$$P(X = x) = nC_x\, p^x\, q^{n-x}$$

$$where;\, x = 0,1,2 \dots n; \quad p + q = 1$$

We have to find the expected frequency by using the formula

$$Expected\ Frequency = N\, P(X)$$

Where $N = \sum f$ Total number of frequency

To Test the Goodness of fit

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i}\right] \sim \chi^2{}_{(n-1)df}$$

Compare the calculated value with table value at 5% level of significance

CALCULATION

Unbiased form p = q

Probability of getting Head $p = \frac{1}{2}$

Probability of getting Tail $q = \frac{1}{2}$

Null Hypothesis

$H_0: The\ coins\ are\ unbiased.$

Alternative Hypothesis

$H_1: The\ coins\ are\ biased.$

Test Statistic

Where n = 5, $p = \frac{1}{2}$ and $q = \frac{1}{2}; N = 256$

The Expected Frequency $E(X) = N\ P(X)$

X	Observed Frequency	$P(X = x) = nC_x\ p^x\ q^{n-x}$	Expected Frequency $E(X) = N\ P(X)$
0	5	$5C_0\ (0.5)^0\ (0.5)^{5-0} = \frac{1}{32}$	$256\ \times \frac{1}{32} \approx 8$
1	35	$5C_1\ (0.5)^1\ (0.5)^{5-1} = \frac{5}{32}$	$256\ \times \frac{5}{32} \approx 40$
2	75	$5C_2\ (0.5)^2\ (0.5)^{5-2} = \frac{10}{32}$	$256\ \times \frac{10}{32} \approx 80$
3	84	$5C_3\ (0.5)^3\ (0.5)^{5-3} = \frac{10}{32}$	$256\ \times \frac{10}{32} \approx 80$
4	45	$5C_4\ (0.5)^4\ (0.5)^{5-4} = \frac{5}{32}$	$256\ \times \frac{5}{32} \approx 40$
5	12	$5C_5\ (0.5)^5\ (0.5)^{5-5} = \frac{1}{32}$	$256\ \times \frac{1}{32} \approx 8$
	$\sum O_i = 256$		$\sum E_i = 256$

Observed Frequency	Expected Frequency $E(X) = N\,P(X)$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
5	8	9	1.125
35	40	25	0.625
75	80	25	0.3125
84	80	16	0.2
45	40	25	0.625
12	8	16	2
			$\sum_{i=1}^{n}\left[\frac{(O_i - E_i)^2}{E_i}\right] = 4.8875$

Level of Significance (L.O.S)

$$\alpha\,\% = 5\% = 0.05$$

Critical Value

The Value of $\chi^2_{\ (6-1)df}$ at 5% level of significance

$$\chi^2_{\ (5)df} = 11.070$$

DECISION

The calculated value is less than the table value at 5% level of significance with 5degrees of freedom (df).

$$\chi^2_{\ C} = 4.88 < \chi^2_{\ T} = 11.070$$

There fore we Accept our Null hypothesis at 5% level of significance with 5degrees of freedom (df).

RESULT

The Coins are unbiased.

TEST FOR GOODNESS OF FIT FOR BINOMIAL DISTRIBUTION

PROBLEM (Reference Book: Mathematical Statistics, Model Excerise. no: 13, pg.no: 12.27)

Test the goodness and fit Binomial distribution four coins tossed simultaneously and the number of heads occurring at each throw was noted. This is repeated 100 times with following.

No. of Heads	0	1	2	3	4
No. of throws	18	35	30	13	4

AIM

To test the coins a unbiased by using chi-square Goodness of fit.

PROCEDURE

The probability mass function of Binomial distribution

$$P(X = x) = nC_x\, p^x\, q^{n-x}$$

$$where; x = 0,1,2 \dots n;$$

$$\bar{X} = \frac{\sum fx}{\sum f}$$

We have to find the expected frequency by using the formula

$$Expected\ Frequency = N\, P(X)$$

Where $N = \sum f$ Total number of frequency

To Test the Goodness of fit

$$\chi^2 = \sum_{i=1}^{n}\left[\frac{(O_i - E_i)^2}{E_i}\right] \sim \chi^2_{\ (n-1)df}$$

Compare the calculated value with table value at 5% level of significance

CALCULATION

Null Hypothesis

$$H_0: \text{The coins are unbiased.}$$

Alternative Hypothesis

$$H_1: \text{The coins are biased.}$$

Test Statistic

Where n = 4, $\bar{X} = \frac{\sum fx}{\sum f} = \frac{150}{100} = 1.5$

$$\bar{X} = np$$

$$p = \frac{\bar{X}}{n} = \frac{1.5}{4} = 0.375$$

$p = 0.375$ *and* $q = 0.625;\ N = 100$

The Expected Frequency $E(X) = N\,P(X)$

X	Observed Frequency	fx	$P(X = x) = nC_x\, p^x\, q^{n-x}$	Expected Frequency $E(X) = N\,P(X)$
0	18	0	$4C_0\,(0.375)^0\,(0.625)^{4-0} = 0.1525$	$100 \times 0.1525 \approx 15$
1	35	35	$4C_1\,(0.375)^1\,(0.625)^{4-1} = 0.366$	$100 \times 0.366 \approx 36$
2	30	60	$4C_2\,(0.375)^2\,(0.625)^{4-2} = 0.3294$	$100 \times 0.3294 \approx 33$
3	13	39	$4C_3\,(0.375)^3\,(0.625)^{4-3} = 0.1316$	$100 \times 0.1316 \approx 14$
4	4	16	$4C_4\,(0.375)^4\,(0.625)^{4-4} = 0.0197$	$100 \times 0.0197 \approx 2$
	$\sum O_i = 100$	$\sum fx = 150$		$\sum E_i = 100$

Observed Frequency	Expected Frequency $E(X) = N\,P(X)$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
18	15	9	0.6
35	36	1	0.028
30	33	9	0.273
$\left.\begin{matrix}13\\4\end{matrix}\right\}$	$\left.\begin{matrix}14\\2\end{matrix}\right\}$	1	0.0625
			$\sum_{i=1}^{n}\left[\frac{(O_i - E_i)^2}{E_i}\right] = 0.9635$

Level of Significance (L.O.S)

$$\alpha\,\% = 5\% = 0.05$$

Critical Value

The Value of $\chi^2_{(6-2-1)df}$ at 5% level of significance

$$\chi^2_{(3)df} = 7.815$$

DECISION

The calculated value is less than the table value at 5% level of significance with 5degrees of freedom (df).

$$\chi^2_{\,C} = 0.9635 < \chi^2_{\,T} = 7.815$$

There fore we Accept our Null hypothesis at 5% level of significance with 5degrees of freedom (df).

VYSYA COLLEGE, SALEM-103
DEPARTMENT OF STATISTICS
I BSC CS & I BCA

POSSIBLE QUESTIONS FOR UNIVERSITY EXAMINATION

Part – A (1 Marks)

Answer all the questions

1. **The parameter of the binomial distribution is __________**

   a) B(x; n, p) b) B(x: n_1, n_2) c) B(x; λ) d) None of the above

2. **In which distribution having the same mean and variance?**

   a) Binomial distribution **b)Poisson distribution**

   c) Normal distribution d) None of the above

3. **A variable that can assume any value between two given points is called __________**

   a) Continuous random variable b) Discrete random variable

   c) Irregular random variable d) Uncertain random variable

4. **Mean value of a binomial distribution is ____________**

   a) npq **b) np** c) p + q = 1 d) np^2

5. **If a coin is tossed once, the probability of success and failure value is __________**

   a) p + q = 1 b) p – q = 1 c) $p_1 + p_2 = 1$ d) $q_1 + q_2 = 1$

6. **Variance of a binomial distribution is __________**

   a) npq b) np c) p + q = 1 d) np^2

7. **Moment Generating function of a binomial distribution is ____________**

   a) $(pe^{it}+q)^n$ b) $(pe^{it}+q)^{n_1+n_2}$ **c)** $(pe^t+q)^n$ d) $(pe^t+q)^{n_1+n_2}$

8. **Additive property of a binomial distribution is ____________**

   a) $M_X(t) = (pe^t+q)^{n_1}$ b) $M_X(t) = (pe^t+q)^{n_2}$

   c) $M_X(t) = (pe^t+q)^{n_1+n_2}$ d) $M_X(t) = (pe^t+q)^{n_1/n_2}$

9. **Which of the following mentioned standard Probability density functions is applicable to discrete Random Variables?**

   a) Gaussian Distribution **b) Poisson Distribution**

   c) Rayleigh Distribution d) Exponential Distribution

10. The mean value of a poisson distribution is ________

a) λ b) λ^2 c) $\lambda^2 + \lambda$ d) λ^3

11. The probability mass function of a poisson distribution is ____________

a) $p(x) = \frac{e^{-\lambda}\lambda^{x+1}}{x+1!}$ **b)** $\boldsymbol{p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}}$

c) $p(x) = \frac{e^{-\lambda}\lambda^{x-1}}{x-1!}$ d) $p(x) = \frac{e^{-\lambda}\lambda^{x-2}}{x-2}$

12. The moment generating function of a poisson variate is __________

a) $\boldsymbol{e^{\lambda(e^t-1)}}$ b) $e^{-\lambda(e^t-1)}$ c) $e^{\lambda(e^t+1)}$ d) $e^{-\lambda(e^t+1)}$

13. Additive property of poisson variate is __________

a) $e^{\Sigma x_i(e^t-1)}$ b) $e^{\Sigma \lambda_i(e^t+1)}$ **c)** $\boldsymbol{e^{\Sigma \lambda_i(e^t-1)}}$ d) $e^{\Sigma x_i(e^t+1)}$

14. If np = λ, here n is ____________

a) Finite b) Infinite c) Trail d) None of the above

15. $\boldsymbol{\lim_{n\to\infty} B(x; n, p)}$ = ________

a) $\boldsymbol{\frac{e^{-\lambda}\lambda^{x}}{x!}}$ b) $\frac{e^{-\lambda}\lambda^{x+1}}{x+1!}$ c) $\frac{e^{-\lambda}\lambda^{x-2}}{x-2!}$ d) $\frac{e^{-\lambda}\lambda^{n-x}}{n-x!}$

PART – B (5 marks)

Answer any two of the following

16. State and prove mean and variance of binomial distribution.

17. State and prove additive property of poisson distribution.

18. The mean of a binomial distribution is 5 and standard deviation is 2. Determine the distribution.

19. Find the probability that at most 5 defective bolts will be found in a box of 200 bolts, if it is known that 2% of such bolts are expected to be defective. ($e^{-4} = 0.0183$)

20. At a busy traffic junction the probability p of an individual having an accident is p=0.001. However, during a certain part of the day 100 cars pass through the junction. What is the Probability that two or more accident occur during that period.? ($e^{-1} = 0.9048$)

PART – C (10 marks)

Answer all the questions

21.State and prove recurrence relation for the moments of binomial distribution.

22.State and prove recurrence relation for the moments of poisson distribution.

23.Derive limiting case of binomial distribution to poisson distribution.

24.The manufacture pins know that two percent of product is defective. If he sells pins in boxes of hundreds and guaranties that not more than four pins will be defective. What is the probability that a box will fail to meet the guaranty quality.($e^{-2} = \mathbf{0.13534}$)

25.Fit a Poisson distribution to the following data and calculate the theoretical frequencies:

x	0	1	2	3	4
f	123	59	14	3	1

Note: Problems are solved in the class rooms.

Reference Books:

Fundamentals of Mathematical Statistics – S.C.Gupta & V.K.Kapoor

***************END OF THE UNIT II *****************