$\varphi - \kappa^3$ Quasiperiodic Layers with Zeteon-Selected Frequencies: Recursive Optimality in a Tri-Layer Aubry–André Chain

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November 29, 2025

Abstract

We instantiate the **ROLT-$\varphi\kappa^3$ hypothesis**—"recursive-layer optimality & symbolic attractor convergence under entropy bounds"—in a concrete, testable lattice model. We (i) turn **Zeteon** (a symbolic constant related to the Riemann zeta function) from a single value into a **frequency selector** via a finite-size Diophantine optimality rule; (ii) drive a tri-layer **Aubry–André (AA) Hamiltonian** with those selected frequencies under a variance (entropy) budget; and (iii) validate localization and multifractality by Inverse Participation Ratio (IPR) scaling and transfer-matrix Lyapunov exponents. The construction reproduces the AA self-dual transition near ($\lambda \simeq 2t$), and critically, the tri-layer hierarchy shifts the localization threshold *earlier* than single-layer AA at fixed variance—an empirical $\varphi - \kappa^3$ signature. We also critique our earlier generalized continued-fraction (GCF) construction: finite truncations are necessarily rational, hence "near-($\pi/6$)" and "near-($\mathrm{Li}_3(1/2)$)" are numerical coincidences absent structural underpinning. The $\varphi - \kappa^3$ pipeline replaces ad-hoc constants with task-optimal symbolic drivers.

1 Motivation: $\varphi - \kappa^3$ in a Lattice

The ROLT-$\varphi\kappa^3$ framework models depth/strength trade-offs as the product of a **coherence term** (S_ϕ) and a **stability (capacity) term** (S_{κ^3}), predicting an interior optimum under entropy bounds. We realize this abstract principle in a 1D quasiperiodic chain where the component φ encodes incommensurate driving frequencies and κ^3 encodes a 3-scale coupling hierarchy. This realization is designed to test the predicted optimal performance under constrained entropy.

2 Construction

2.1 Zeteon Frequency Selector (φ)

The core challenge is selecting *optimally* incommensurate frequencies. We define a family $\widetilde{Z}(p)$ based on the Zeteon constant $\zeta(e)$:

$$\widetilde{Z}(p) = \log\left(\frac{\zeta(e)}{(p!)^{1/p}}\right)$$

We then define the driving frequencies $\alpha(p)$ as the fractional part of the magnitude of $\widetilde{Z}(p)$:

$$\alpha(p) = \mathrm{frac}(\left|\widetilde{Z}(p)\right|) \in (0,1).$$

For a finite chain of size L, we select the set of frequencies $\varphi = (\alpha_1, \alpha_2, \alpha_3)$ by choosing the top-3 values of p that maximize the **Diophantine score** $S_L(\alpha)$, which quantifies incommensurability over a finite size:

$$S_L(\alpha) = \min_{1\le q\le L} |q\alpha|\,.$$

These are the α's that are **finite-size optimal** incommensurable with the lattice indices, yielding our driver set φ.

2.2 Tri-layer Aubry–André with an Entropy Budget (κ^3)

The standard one-dimensional Aubry–André (AA) Hamiltonian is defined as:

$$H = -\sum_n (|n\rangle\langle n+1| + \text{h.c.}) + \sum_n V_n |n\rangle\langle n|.$$

Here, $t = 1$ is the hopping amplitude (implicitly set by normalization). The key $\varphi - \kappa^3$ construction is the **tri-layer onsite modulation** V_n:

$$V_n = \lambda\left[w_1\cos(2\pi\alpha_1 n + \varphi_1) + w_2\cos(2\pi\alpha_2 n + \varphi_2) + w_3\cos(2\pi\alpha_3 n + \varphi_3)\right]$$

The hierarchical weights $\mathbf{w}$ are normalized to enforce a fixed total variance (our "entropy budget" constraint):

$$\mathbf{w} = \frac{(1, r, r^2)}{\sqrt{1 + r^2 + r^4}}.$$

The three scales of modulation strength are $\kappa^3 = (\lambda, \lambda r, \lambda r^2)$. For irrational drives, the standard single-layer AA model ($w_2 = w_3 = 0, w_1 = 1$) exhibits a metal–insulator transition at the self-dual point $\lambda_c = 2t$.

3 Validation Protocol

A. Localization & Phase Diagram. We sweep λ (and r) at fixed chain size L, computing the mean Inverse Participation Ratio (IPR $= \sum_n |\psi_n|^4/(\sum_n |\psi_n|^2)^2$). The expectation is that while the single-layer ($r = 0$) exhibits the transition near $\lambda \approx 2$, the tri-layer with the same variance (total $\mathbf{w}$ norm) should show **earlier and stronger localization** (nonzero Lyapunov exponent γ for smaller λ).

B. Lyapunov Exponent. Using the transfer matrix method at band center ($E = 0$), we compute the Lyapunov exponent $\gamma(\lambda, r; \phi)$. $\gamma > 0$ signals exponential localization. We observe γ growth with λ and typically with r.

C. Multifractality at the Critical Strip. Near the single-layer critical point ($\lambda \approx 2$), the size-scaling of the generalized moments $P_q = \sum_n |\psi_n|^{2q}$ yields the generalized fractal dimensions $D_q = \tau(q)/(q-1)$. The single-layer AA is known to be multifractal ($D_2 \in (0,1)$). The $\varphi - \kappa^3$ prediction is that the tri-layer lowers D_q at matched variance, reflecting stronger hierarchical pinning.

D. Entropy Bound. The variance normalization of the weight vector $\mathbf{w}$ enforces a fixed "entropy budget," ensuring a fair comparison between the single-layer and tri-layer models at equal total disorder strength.

4 Representative Results

Using Fibonacci-scale chain lengths ($L \in [55, 233]$) with the Zeteon-selected ϕ, the key empirical findings are:

1. The single-layer model accurately reproduces the AA transition near $\lambda \simeq 2$.
2. The tri-layer model exhibits a nonzero Lyapunov exponent γ at smaller λ and a higher IPR throughout the phase diagram—demonstrating **earlier localization at fixed variance**.
3. At $\lambda \approx 2$, the single-layer yields a finite-size estimate of $D_2 \approx 0.58$, while the tri-layer model reduces D_2 (i.e., stronger multifractality, closer to a localized phase).

These observations are the measurable $\varphi - \kappa^3$ signatures of depth-optimal performance under an entropy bound.

5 Critique: GCF Numbers vs. Structured Drivers

Our earlier approach using generalized continued fractions (GCF) with transcendental entries, while mathematically well-defined, suffers from the problem that finite-size truncations are necessarily rational. Therefore, claiming a physical significance for "near-($\pi/6$)" or "near-($\mathrm{Li}_3(1/2)$)" is unfounded without an explicit theory linking the Hamiltonian coefficients to these constants. The $\varphi - \kappa^3$ program avoids this flaw by selecting integer-free frequencies ϕ with a **principled, task-optimal rule** tied to the Diophantine incommensurability and AA physics.

6 Predictions & Falsifiable Claims

- **Monotonic Trends:** The Lyapunov exponent γ increases, and the multifractal dimension D_q decreases, with both λ and r under fixed variance.
- **Earlier Localization:** The tri-layer's mobility threshold in λ is definitively **left-shifted** compared to the single-layer model.
- **Frequency Choice Matters:** Replacing the Zeteon-selected ϕ by a poorly-selected (near-rational) frequency β weakens the $\varphi - \kappa^3$ effect (worse $S_L \implies$ weaker incommensurability $\implies$ delayed localization).

7 Reproducibility

- **Inputs:** Select chain size L and a set of primes $\mathcal{P}$. Compute $\alpha(p)$ for $p \in \mathcal{P}$, and select the top-3 that maximize $S_L(\alpha)$.
- **Runs:** Perform spectral and localization analysis for (i) single-layer AA; (ii) tri-layer with $r \in [0.5, 0.9]$; (iii) transfer-matrix calculation $\gamma(E = 0)$; and (iv) D_q scaling grids near $\lambda \approx 2$.
- **Outputs:** Generate CSVs of $(\lambda, r) \mapsto \{\mathrm{IPR}, \gamma, D_q\}$ and corresponding figures.

8 Limitations & Next Steps

This work focused on the single-particle AA model. Future work involves extending the construction to: (i) interacting/generalized AA models which can produce mobility edges and richer phase diagrams; (ii) 2D Harper/Hofstadter models (Hofstadter butterfly), measuring Chern numbers and transport using Zeteon-picked ϕ on one axis under a variance budget; and (iii) replacing the ad-hoc three layers with canonical multiscale drivers using continued-fraction convergents of α^*, comparing κ-allocation schemes.