SDSE Fall 2023
Midterm exam

SOLUTION

Name:

SID:

Points:

# 1	# 2	# 3	# 4	# 5	Total
10	12	12	16	20	70

Instructions:

- Closed notes.
- The cheat sheet and lookup tables are included.
- When using the lookup tables, you may round the input to the nearest lookup value. However you may not round the values from the table.
- No computers or devices allowed. Only simple calculators. Please let me know if you forgot to bring one.
- Please be neat in your answers. Wherever relevant, enclose your final answers in a box. This will help us to follow your work and award you maximum points.
- Show your work!
- If you have a question, raise your hand and someone will approach you.
- If you need more paper, raise your hand and someone will bring it.

Problem 1 (10 points)

In a laboratory, 4 measurements were made of the gravitational constant g. The sample mean of these measurements was $\hat{\mu}_N = 972$ and the sample variance was $\hat{\sigma}_N^2 = 36$. We wish to test the hypothesis that the true value of g is 981, versus the hypothesis that it is not 981, with a significance level of $\alpha = 0.05$. Assume that Y is Gaussian.

1) (3 pts) Write the null and alternate hypotheses for this test.

2) (4 pts) What is the p-value of the test?

3) (3 pts) State the finding of the test?

1) $H_0 : \quad g = 981$

$H_1 : \quad g \neq 981$

2)
$$p = 2\,\Phi_{\bar{Y}_N}(\hat{\mu}_N) = 2\,\Phi_{t(N-1)}\left(\frac{\hat{\mu}_N - 981}{\hat{\sigma}_N / \sqrt{N}}\right)$$

$$= 2\,\Phi_{t(3)}\left(\frac{972 - 981}{\sqrt{36}/\sqrt{4}}\right) = 2\,\Phi_{t(3)}\left(-9/3\right)$$

$$= 2\,\Phi_{t(3)}(-3) = 2 \cdot 0.029 = 0.058$$

3) $p = 0.058$ is greater than $\alpha = 0.05$

The test does not provide sufficient evidence to reject the hypothesis $g = 981$

Problem 2 (12 points)

In this problem we will consider a joint random variable (X, Y) where both X and Y are discrete-valued with sample space: $\Omega_X = \Omega_Y = \{1, 2, 3\}$. A partially completed table of values of the joint pdf is shown below.

Y \ X	1	2	3	
1	0.2	0.1	0.1	0.4
2	0.1	0	0.2	0.3
3	0.2	0.1	0	0.3
	0.5	0.2		

Additionally, the following facts are known:

- $P(Y = 1) = 0.4$
- $P(Y = 2) = P(Y = 3)$
- $P(X = 1) = 0.5$
- $P(X = 2) = 0.2$
- The events $X = 1$ and $Y = 1$ are independent.

1) (6 pts) Fill in the 6 empty cells of the table.

2) (3 pts) Are the events $X = 1$ and $Y = 3$ independent?

3) (3 pts) What is the expected value of Y?

2) $P(X=1 \,\&\, Y=3) \stackrel{?}{=} P(X=1)\, P(Y=3)$

$0.2 \stackrel{?}{=} 0.5 \cdot 0.3$ ... No.

3) $E[Y] = 0.4 \cdot (1) + 0.3(2) + 0.3(3) = 1.9$

Problem 3 (12 points)

A robot is designed to catch a ball, and we are interested in estimating the probability that it makes a catch. To this end we throw the ball at it 50 times and record 39 catches and 11 misses. Define a random variable Y for the robot's success or failure in catching the ball. $Y = 1$ means that the ball was caught. $Y = 0$ means that the ball was not caught.

1) (2 pts) Which distribution family best characterizes Y?

2) (2 pts) Estimate the mean and standard deviation of Y.

3) (4 pts) Compute a 90% confidence interval for the mean of Y.

4) (4 pts) How many throws are needed to estimate a 90% confidence interval with a radius of at most 0.08?

$N = 50$, $N_+ = 39$, $N_- = 11$

1) Bernoulli

2) $\hat{\mu}_N = \frac{N_+}{N} = 39/50 = 0.78$

$$\hat{\sigma}_N = \sqrt{\hat{\mu}_N(1-\hat{\mu}_N)} = 0.4142$$

3) $$\rho = \frac{\hat{\sigma}_N}{\sqrt{N}} \left| \Phi_N^{-1}\left(\frac{1-\gamma}{2}\right) \right| = \frac{0.4142}{\sqrt{50}} \left| \Phi_N^{-1}(0.05) \right|$$

$$= \frac{0.4142}{\sqrt{50}} \cdot 1.645 = 0.09637$$

4) $$\frac{\hat{\sigma}_N}{\sqrt{N}} \cdot 1.645 \leq 0.08$$

$$\Leftrightarrow \quad N \geq \left(\frac{\hat{\sigma}_N \; 1.645}{0.08} \right)^2 = 72.55 \quad \rightsquigarrow \quad N \geq 73$$

Problem 4 (16 points)

An autonomous drone is programmed to land on a target. It is known that, with each attempt, it has a 70% chance of a succesful land. However it has only 2 attempts. If it fails the first time, it can try once more. If it fails the second time, it is destroyed.

Define a random variable X for the outcome of this experiment. $X = 1$ means that the drone lands on the first attempt. $X = 2$ means it lands on the second attempt. $X = \emptyset$ is a label signifying that it failed twice.

1) (4 pts) Compute $P(X = 1)$, $P(X = 2)$, and $P(X = \emptyset)$.

Now assume that the drone is awarded points depending on its performance. If it lands succesfully on the first try, it gets 10 points. If it lands on the second try, it gets 5 points. If it fails ($X = \emptyset$), it gets 0 points. Denote with V the random variable for the points awarded.

2) (3 pts) Compute $E[V]$.

3) (3 pts) Compute $Var[V]$.

Consider now a swarm of 10 such drones. Define as A the average points awarded to the swarm.

$$A = \frac{1}{10} \sum_{i=1}^{10} V_i$$

where $\{V_i\}_{10} \overset{\text{iid}}{\sim} V$

4) (3 pts) Compute $E[A]$.

5) (3 pts) Compute $Var[A]$.

1. $P(X=1) = 0.7$

$P(X=2) = 0.3 \cdot 0.7 = 0.21$

$P(X=\phi) = 1 - 0.7 - 0.21 = 0.09$

2. $E[V] = \int_{\Omega_x} P_x(x) \cdot V(x)\, dx$

$= 0.7 \cdot 10 + 0.21 \cdot 5 + 0.09 \cdot 0 = 8.05$

3. $Var[V] = E[(V - 8.05)^2]$

$= 0.7(10-8.05)^2 + 0.21(5-8.05)^2 + 0.09(0-8.05)^2$

$= 10.448$

4. $A = \bar{V}_{10}$

$\therefore E[A] = E[V] = 10.448$

5. $Var[A] = \frac{1}{10} Var[V] = 1.0448$

Problem 5 (20 points)

You have a job interview in San Francisco and decide to take the subway to get there. You will spend some time on the platform waiting for the train, followed by some time riding on the train. Suppose that the wait time W is uniformly distributed between 0 and 20 minutes, and that the riding time R is uniformly distributed between 10 minutes and 30 minutes. Assume also that W and R are independent.

1) (2 pts) What is the probability that your wait time is less than 5 minutes?

2) (2 pts) Find the mean and variance of W.

3) (2 pts) Find the mean and variance of R.

4) (2 pts) Neatly sketch the sample space for the joint random variable (W, R) as a subset of $\mathbb{R}^2$.

5) (3 pts) Neatly sketch the event that the total travel time $W + R$ exceeds 40 minutes.

6) (3 pts) What is the probability of this event?

7) (3 pts) Neatly sketch the event that your wait time exceeds your ride time.

8) (3 pts) What is the probability of this event?

$$W \sim \mathcal{U}(0,20) \quad \cdots \quad P_W(x) = \begin{cases} 1/20 & x \in [0,20] \\ 0 & \text{otherwise.} \end{cases}$$

$$R \sim \mathcal{U}(10,30) \quad \cdots \quad P_R(x) = \begin{cases} 1/20 & x \in [10,30] \\ 0 & \text{otherwise.} \end{cases}$$

1) $P(W<5) = \int_0^5 P_W(x)\,dx = 5 \cdot 1/20 = \underline{0.25}$

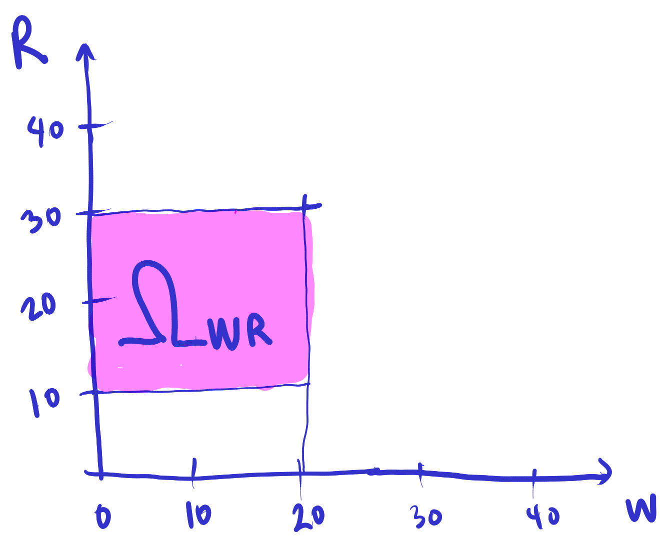
2) $E[W] = \frac{1}{2}(0+20) = 10$

$$Var[W] = \frac{(20-0)^2}{12} = \frac{400}{12} = \frac{100}{3}$$

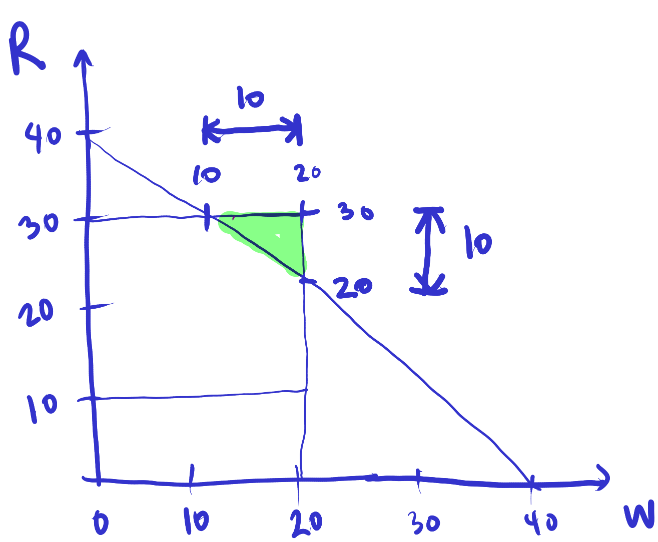
3) $E[R] = \frac{1}{2}(30+10) = 20$

$$Var[R] = \frac{(30-10)^2}{12} = \frac{400}{12} = \frac{100}{3}$$

4)



5)

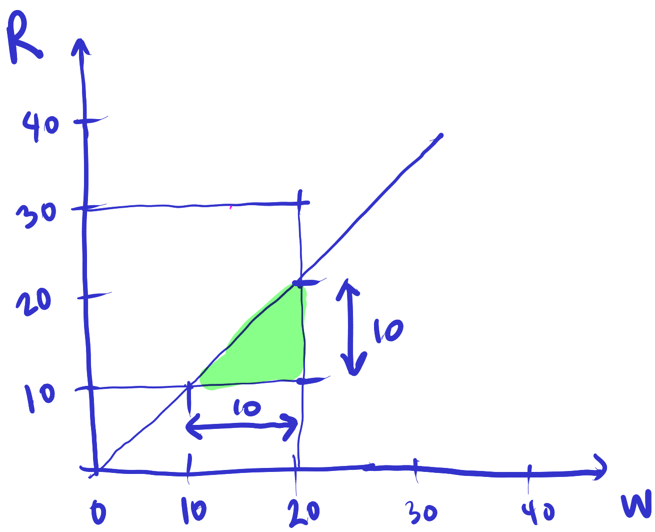


6) $P(W+R>40)$

$$= \begin{pmatrix}\text{Area of the green} \\ \text{triangle}\end{pmatrix} \times \begin{pmatrix}\text{Height of the} \\ \text{distribution.}\end{pmatrix}$$

$$= \left(\frac{1}{2}\cdot 10\cdot 10\right)\left(\frac{1}{20}\cdot\frac{1}{20}\right) = \frac{50}{400} = 1/8$$

7)



8) $P(W>R)$

$$= \begin{pmatrix}\text{Area of the} \\ \text{green} \\ \text{triangle}\end{pmatrix} \times \begin{pmatrix}\text{Height of the} \\ \text{distribution}\end{pmatrix}$$

$=$ same as before $= 1/8$