SDSE Homework 1

Due date: September 19th, 2023 (by 3:30pm)

Note: If you use a computer please include a screenshot of the code.

Problem 1 (15 points)

Find whether the following are valid PDFs. *Note*: You may do the proofs by hand, or with a symbolic solver. Also, you may use a plot of the function over the sample space to show positivity.

(a) $p(x) = \frac{1}{2}\cos(x) \quad , \quad \Omega = [-\pi/2, \pi/2]$

(b) $p(x) = \dfrac{1}{1+e^{-x}} \quad , \quad \Omega = \mathbb{R}$

(c) $p(x) = \frac{1}{3}\,x^3 \quad , \quad \Omega = [0, 1]$

(d) $p(x) = 1/x \quad , \quad \Omega = \mathbb{N}$ (the natural numbers $\{1, 2, 3, \ldots\}$)

(e) $p(x) = 2^{-x} \quad , \quad \Omega = \mathbb{N}$

Problem 2 (4+4+2 points)

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = \frac{x_1^3}{3} - 4x_1 + \frac{x_2^3}{3} - 16x_2$$

(a) Find the set of stationary points of f.

(b) Plot the function using one of Matplotlib's surface plotters (plot_surface, plot_wireframe, or plot_trisurf). Indicate the local minima, local maxima, and saddle points.

(c) What is the solution of the problem: minimize $f(x_1, x_2)$ over $\mathbb{R}^2$?

Problem 3 (2+2+3+3+4+4 points)

A study of washing machines found their lifetime T to follow an exponential distribution:

$$p_T(t) = 0.1e^{-0.1t}$$

with t measured in years.

(a) What is the mean lifetime of washing machines?

(b) What is the standard deviation?

(c) What percentage of washing machine are expected to fail in 10 years?

(d) What is the median life of washing machines?

(e) A small hotel owns five washing machines. All operate independently. What is the probability that none of the machines will fail within the next 3 years?

(f) What is the probability that all five machines will fail within the next 15 years?

Problem 4 (2+3+3 points)

Measurements are made of the length L and width W of a rectangular component. Assume that $L \sim \mathcal{U}(9.95, 10.05)$ and $W \sim \mathcal{U}(4.9, 5.1)$, and that they are independent random variables.

(a) Find $P(L < 9.98)$

(b) Draw the region in the LW plane corresponding to the event $(L \in [9.96, 9.98] \cap W \in [5.0, 5.05])$ and compute its probability.

(c) Draw the region in the LW plane corresponding to the event $(L \in [9.96, 9.98] \cup W \in [5.0, 5.05])$ and compute its probability.

Problem 5 (3 points)

Certain pipes are specified to have a roughness coefficient of between 0.2 and 0.3. The manufacturing process is known to produce pipes with normally distributed roughnesses with mean 0.25 and a standard deviation of 0.03. What percentage of pipes is expected to meet the specification? Hint: `stats.norm`.

Problem 6 (4 points)

3D printers create items by printing layers of material one on top of the other. Different materials have different mean thicknesses and standard deviations. A particular part is made by stacking 10 layers of material A, followed by 15 layers of material B, and 30 layers of material C. The characteristics of these materials is shown in the table below (the standard deviation is specified as a percentage of the average thickness). Assume that the layers are mutually independent. Compute the mean and standard deviation of the total thickness of the part.

Material	Average thickness [mm]	Standard deviation [%]
A	0.2	2%
B	0.1	4%
C	0.05	1%