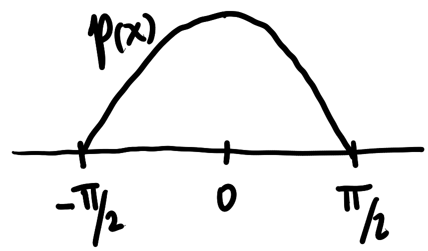
Homework #1

SOLUTION

Problem 1

(a) $p(x) = \frac{1}{2}\cos x$, $\Omega = [-\pi/2, \pi/2]$

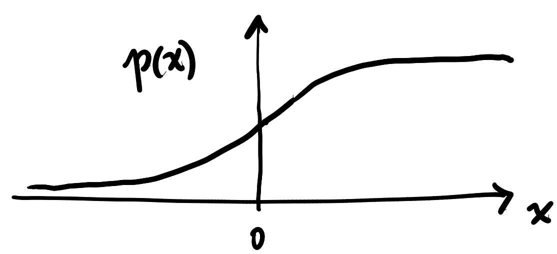


1. $p(x) \geq 0 \quad \forall x$ $\checkmark$

2. $$\int_{-\pi/2}^{\pi/2} p(x)\,dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos x\,dx = \frac{1}{2}\sin x\Big|_{-\pi/2}^{\pi/2} = 1 \quad \checkmark$$

$\therefore$ $p(x)$ is a valid pdf.

(b) $p(x) = \dfrac{1}{1+e^{-x}}$; $\Omega = \mathbb{R}$



1. $p(x) \geq 0 \quad \forall x$ $\checkmark$

2. $$\int_{-\infty}^{\infty} \frac{1}{1+e^{-x}}\,dx$$

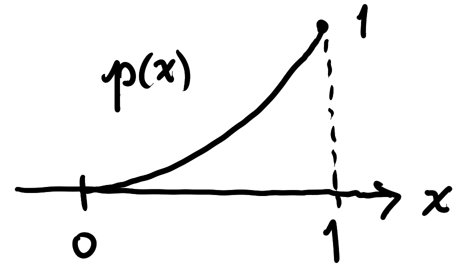
From the plot it is clear that

$$\int_{-\infty}^{a} p(x)\,dx \to \infty \quad \text{as} \quad a \to \infty$$

$\therefore$ The limit does not exist.

$\therefore$ $p(x)$ is not a valid pdf.

(c) $p(x) = \frac{1}{3}x^3 \quad ; \quad \Omega = [0,1]$

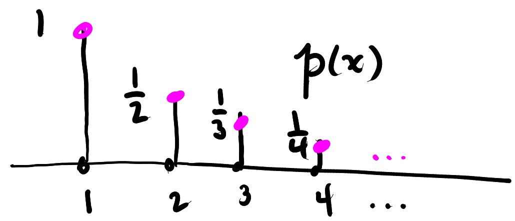


1. $p(x) \geq 0 \quad \forall x \in \Omega$? Yes.

2. $\int_0^1 p(x)\,dx = \frac{1}{3}\int_0^1 x^3\,dx = \frac{1}{3}\frac{x^4}{4}\Big|_0^1 = \frac{1}{12} \neq 1$

$\therefore$ This is not a pdf.

(d) $p(x) = \frac{1}{x} \quad ; \quad \Omega = \mathbb{N}$

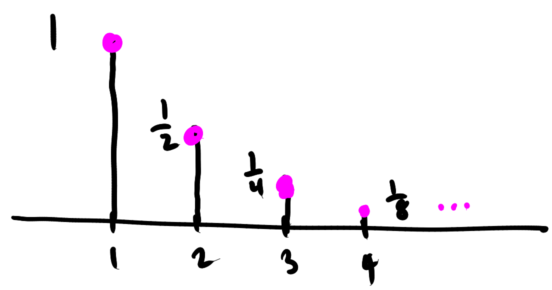


1. $p(x) \geq 0 \quad \forall x \in \Omega$ ✓

2. $\int_{\mathbb{N}} p(x)\,dx = \sum_{x=1}^{\infty} \frac{1}{x}$ This series does not converge.

$\therefore$ This is not a valid pdf.

(e) $p(x) = \frac{1}{2^x} \quad ; \quad \Omega = \mathbb{N}$



1. $p(x) \geq 0 \quad \forall x \in \Omega$ ✓

2. $\int_{\mathbb{N}} p(x)\,dx = \sum_{x=1}^{\infty} \frac{1}{2^x} = 1$ ✓ This is a valid pdf.

Problem 2

$$f(x_1, x_2) = \frac{x_1^3}{3} - 4x_1 + \frac{x_2^3}{3} - 16x_2$$

(a)

$$\nabla f(x_1, x_2) = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 - 4 \\ x_2^2 - 16 \end{bmatrix}$$

$$\nabla f(x_1, x_2) = 0 \iff \begin{cases} x_1^2 - 4 = 0 \\ x_2^2 - 16 = 0 \end{cases} \iff \begin{cases} x_1 = \pm 2 \\ x_2 = \pm 4 \end{cases}$$

$$\therefore \text{Stationary points} = \{ \underbrace{(2,4)}_{s_1}, \underbrace{(2,-4)}_{s_2}, \underbrace{(-2,4)}_{s_3}, \underbrace{(-2,-4)}_{s_4} \}$$

(b)

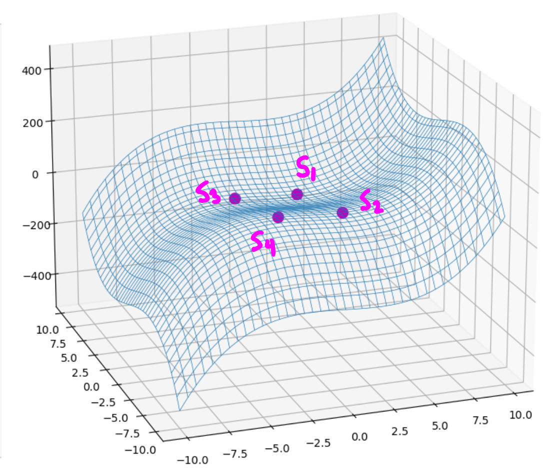
```python
import matplotlib.pyplot as plt
import numpy as np
import matplotlib as mpl

x1 = np.arange(-10, 10, 0.25)
x2 = np.arange(-10, 10, 0.25)
X1, X2 = np.meshgrid(x1, x2)

def f(x1, x2):
    return (x1**3)/3 -4*x1 + (x2**3)/3 -16*x2

fig, ax = plt.subplots( figsize=(10,10), subplot_kw={"projection": "3d"})
ax.plot_wireframe(X1, X2, f(X1,X2), linewidth=0.5)

s1 = np.array([2, -2, 2, -2])
s2 = np.array([4, 4, -4, -4])
ax.scatter3D(s1, s2, f(s1, s2), c='m', marker='o', s=100, alpha=1)
```

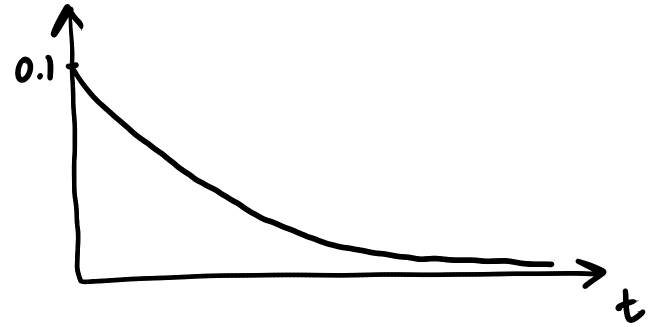


(c) From the figure it is clear that f is unbounded from below.

$\therefore$ The minimization problem has no solution.

Problem 3

$$p_T(t) = 0.1\, e^{-0.1t}$$



(a)

$$E[T] = \int_0^\infty t\, p_T(t)\, dt = \int_0^\infty t\; 0.1\, e^{-0.1t}\, dt$$

Integration by parts:

$u = t$ $\qquad du = dt$

$dv = 0.1\, e^{-0.1t}\, dt$ $\qquad v = -e^{-0.1t}$

$$\therefore\ E[T] = t\left(-e^{-0.1t}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-0.1t}\right) dt$$

$$= 0 + \left(-\frac{1}{0.1}\, e^{-0.1t}\right)\Big|_0^\infty$$

$$= -\frac{1}{0.1}\left(0 - 1\right) = 10 \text{ years.}$$

b) $Var[T] = E[(T-10)^2]$

$$= \int_0^\infty (t-10)^2 \, 0.1 \, e^{-0.1t} \, dt$$

IBP:

$u = (t-10)^2$ $\qquad du = 2(t-10)\,dt$

$dv = 0.1\, e^{-0.1t}\,dt$ $\qquad v = -e^{-0.1t}$

$$\therefore Var[T] = -(t-10)^2 e^{-0.1t}\Big|_0^\infty - \int_0^\infty (-e^{-0.1t})\, 2(t-10)\,dt$$

$$= 0 + 10^2 + 2\int_0^\infty (t-10)\, e^{-0.1t}\,dt$$

$$= 100 + 2\left(\int_0^\infty t\, e^{-0.1t}\,dt - 10\int_0^\infty e^{-0.1t}\,dt\right)$$

$$= 100 + 2\left(\frac{10}{0.1} - 10\left(\frac{1}{-0.1} e^{-0.1t}\right)\Big|_0^\infty\right)$$

$$= 100 + 2\left(100 + 100(0-1)\right)$$

$$= 100.$$

$\therefore$ Standard deviation $= \sqrt{100} = 10$ years.

(c) $P(T<10) = \int_0^{10} p_T(t)\,dt = \int_0^{10} 0.1\, e^{-0.1t}\, dt$

$$= 0.1 \cdot \frac{1}{-0.1} e^{-0.1t} \Big|_0^{10} = 1 - e^{-1} = 0.632$$

$\therefore$ We expect 63.2% of washing machines to fail in 10 years.

(d) $P(T < t_{med}) = 0.5$

$\therefore \quad 1 - e^{-0.1 t_{med}} = 0.5$

$\therefore \quad t_{med} = \frac{\ln(0.5)}{-0.1} = 6.93$ years

(e) Five independent washing machines : $T_1 \quad T_5$

Probability that none will fail in 3 years:

$$P(T_1 > 3 \ \& \ T_2 > 3 \ \& \ T_3 > 3 \ \& \ T_4 > 3 \ \& \ T_5 > 3)$$

$$\underset{\text{by independence}}{=} P(T_1 > 3)\, P(T_2 > 3)\, P(T_3 > 3)\, P(T_4 > 3)\, P(T_5 > 3)$$

$$= \left(e^{-0.1 \times 3}\right)^5 = e^{-1.5} = 0.223$$

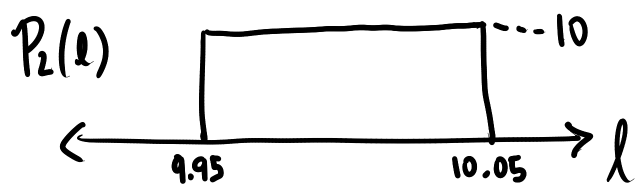
(f) $P(T_1 < 15 \ \& \ T_2 < 15 \ \& \ T_3 < 15 \ \& \ T_4 < 15 \ \& \ T_5 < 15)$

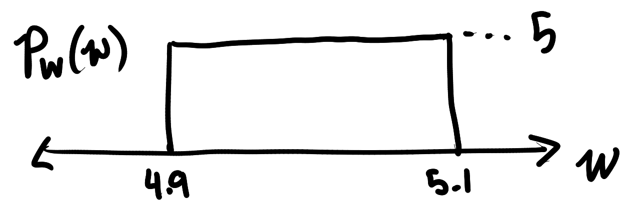
$$= \left(P(T<15)\right)^5$$

$$= \left(1 - e^{-0.1 \times 15}\right)^5 = 0.283$$

Problem 4

$$= \frac{1}{10.05 - 9.95} \qquad \qquad \frac{1}{5.1 - 4.9} =$$





(a) $$P(L < 9.98) = \int_{9.95}^{9.98} P_L(l)\, dl = 10 \cdot (9.98 - 9.95) = 0.3$$

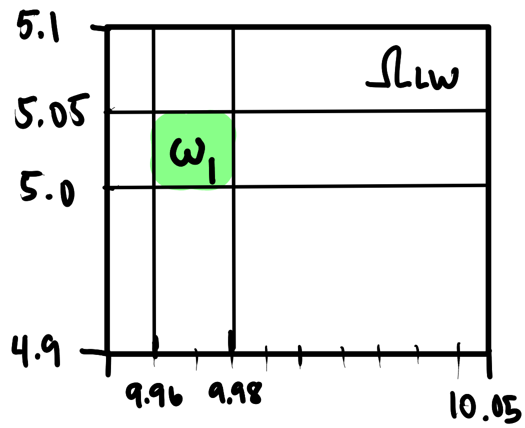
(b) The joint distribution of LW is:

$$\begin{cases} \Omega_{LW} = \Omega_L \times \Omega_W \\ P_{LW}(l,w) \underset{\uparrow}{=} P_L(l) \cdot P_W(w) = 10 \cdot 5 = 50 \end{cases}$$

(independence)

Event ω_1 : $L \in [9.96, 9.98]$ and $W \in [5.0, 5.05]$

$$P(\omega_1) = \int_{\omega} p_{LW}(l,w)\,dl\,dw = (9.98-9.96)(5.05-5)\cdot 50 = 0.05$$



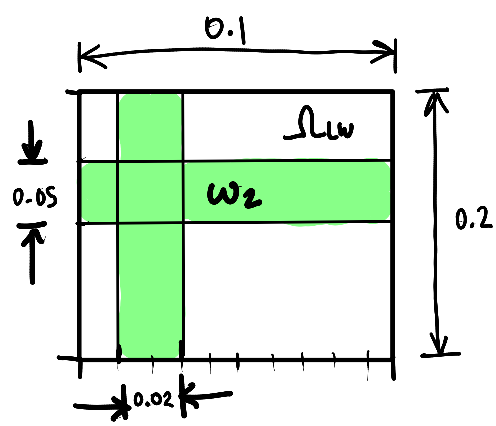
(c)

Event ω_2 : $L \in [9.96, 9.98]$ $\vee$ $W \in [5.0, 5.05]$

$$P(\omega_2) = (\text{green area}) \times 50$$

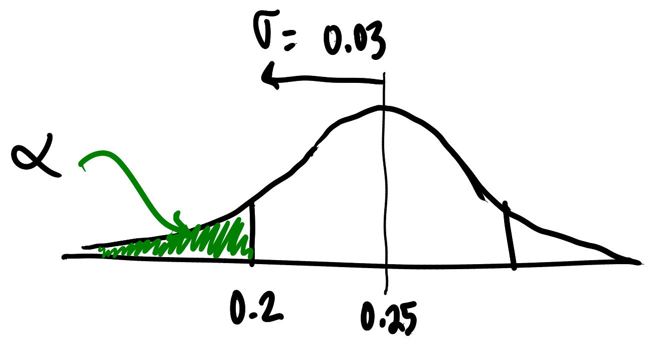
$$= \Big((0.02 \times 0.2) + (0.05 \times 0.1) - (0.02 \times 0.05) \Big) \times 50$$

$$= 0.4$$



Problem 5

$$X \sim \mathcal{N}(0.25\ ,\ 0.03^2)$$



$$\alpha = \Phi_X(0.2)$$

Probability of meeting the spec: $1 - 2\alpha$

Python code:

```python
import scipy.stats as stats
X = stats.norm(loc=0.25,scale=0.03)
alpha = X.cdf(0.2)
print(1-2*alpha)
```

Solution: 90.4%

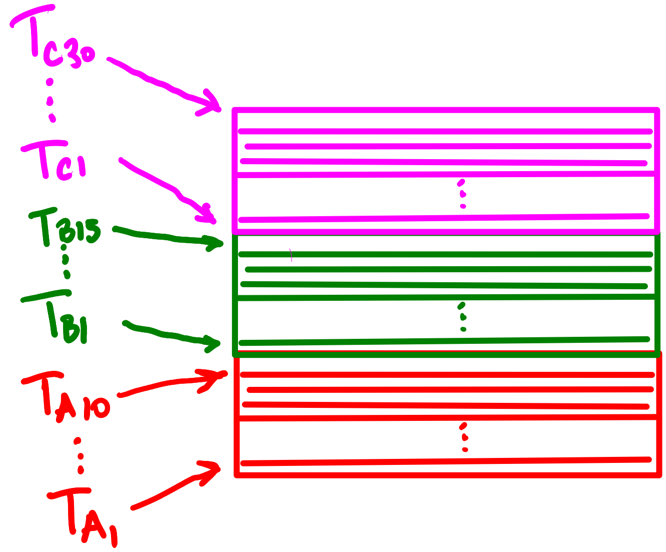
Problem 6

T_A ... thickness material A

T_B ... thickness material B

T_C ... thickness material C

Z ... Total thickness



Layers:

$$\{T_{Ai}\}_{10} \overset{iid}{\sim} T_A$$

$$\{T_{Bi}\}_{15} \overset{iid}{\sim} T_B$$

$$\{T_{Ci}\}_{30} \overset{iid}{\sim} T_C$$

Total thickness:

$$Z = \sum_{i=1}^{10} T_{Ai} + \sum_{i=1}^{15} T_{Bi} + \sum_{i=1}^{30} T_{ci}$$

Expected total thickness.

$$E[Z] = E\left[\sum_{i=1}^{10} T_{Ai} + \sum_{i=1}^{15} T_{Bi} + \sum_{i=1}^{30} T_{ci} \right]$$

$$= \sum_{i=1}^{10} E[T_{Ai}] + \sum_{i=1}^{15} E[T_{Bi}] + \sum_{i=1}^{30} E[T_{ci}] \quad \text{... by linearity of expectation}$$

$$= 10\,E[T_A] + 15\,E[T_B] + 30\,E[T_C] \quad \text{... by iid}$$

$$= 10 \cdot 0.2 + 15 \cdot 0.1 + 30 \cdot 0.05$$

$$= 5 \text{ mm.}$$

<u>Variance in total thickness</u>

$$Var[Z] = Var\left[\sum_{i=1}^{10} T_{Ai} + \sum_{i=1}^{15} T_{Bi} + \sum_{i=1}^{30} T_{Ci}\right]$$

$$= \sum_{i=1}^{10} Var[T_{Ai}] + \sum_{i=1}^{15} Var[T_{Bi}] + \sum_{i=1}^{30} Var[T_{Ci}]$$

$$= 10 \cdot (0.02 \times 0.2)^2 + 15\,(0.04 \times 0.1)^2 + 30\,(0.01 \times 0.05)^2$$

$$= 4.075 \cdot 10^{-4}$$

$$Std[Z] = \sqrt{Var[Z]} = 0.02 \text{ mm.}$$