Automata Theory HW5

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5.7

This is a kind of PERFECT MATCHING, in which, rather than asking that the edges in the matching do not overlap, we ask that they be a distance at least 2 apart from each other. This constraint feels like INDEPENDENT SET, and indeed we will prove that CELLPHONE CAPACITY is NP-Complete by reducing INDEPENDENT SET to it.

Given an instance (G, k) of INDEPENDENT SET where G=(V, E), we construct an instance (G', k') of CELLPHONE CAPACITY as follows. We set k'=k. To construct G', for each vertex $v \in V$ we add a new vertex v', and add an edge (v, v'). We now claim that (G, k) has an independent set of size k iff G' has a set of conversations of size k.

In one direction, if $S \subseteq V$ is independent, we can construct a set of conversations in G' of the same size by having v talk to v' for each $v \in S$.

Conversely, suppose that there is a set C of conversations in G'. If we choose one endpoint from each edge (v, w) in C, the resulting set is independent. If $v, w \in V$, i.e., the conversation is between two of the original vertices of G, we can choose either one. If $v \in V$ and w=v', we choose v. This gives an independent set $S \subseteq V$ of size k, so G has an independent set of size k. Thus, the reduction maps yes-instances to yes-instances and no-instances to no-instances, and the proof is complete.

Clearly, we can carry out this reduction in polynomial time, since we are simply replacing each vertex with a gadget of constant size.

5.11

We will reduce NAE-3-SAT to HYPERGRAPH 2-COLORING. However, since we can't "negate" the color of a vertex, we include two vertices for every variable x, which we call v_x and $v_{\bar{x}}$. The idea is that x is true if v_x is black and $v_{\bar{x}}$ is white, and vice versa. Then our hyperedge will have an edge for each NAE-3-SAT clause, which includes v_x or $v_{\bar{x}}$ if the clause includes x or $\bar{x}$

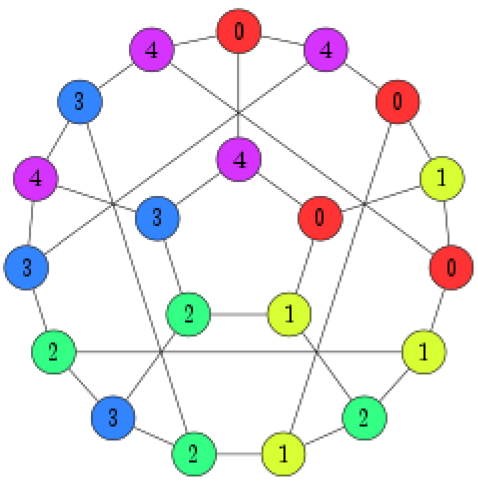
However, we need to make sure that v_x and $v_{\bar{x}}$ are forced to have opposite colors. There are many ways to do this. If vertices are allowed to appear twice in an edge, we can simply include the edge { $v_x, v_x, v_{\bar{x}}$ } in the hypergraph. If repeated vertices are not allowed, we can add three vertices s, t, u and the edge {s, t, u}. Then for each x, we include the edges $\{v_x, v_{\bar{x}}, s\}$, $\{v_x, v_{\bar{x}}, t\}$, $\{v_x, v_{\bar{x}}, u\}$,

Since s, t, u can't all be the same color, v_x *and* $v_{\bar{x}}$ are forced to have opposite colors, and we're done.

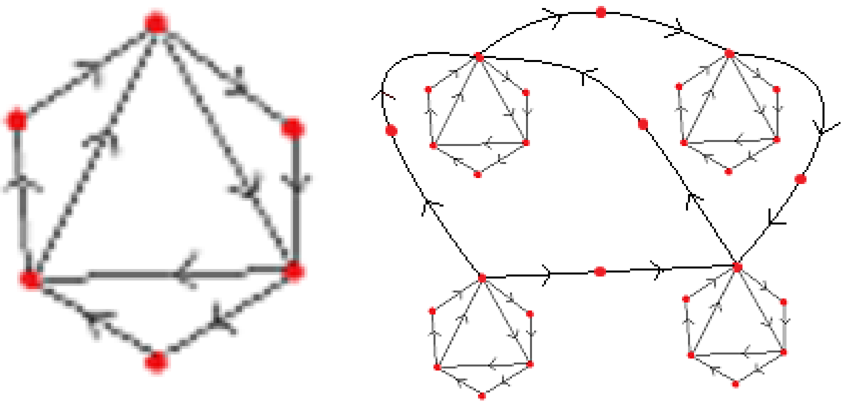
Clearly, we can carry out this reduction in polynomial time, since we are simply replacing each clause with a gadget of constant size.

5.13

The following diagram is showing WHEEL 5-coloring example.



At first, let's look at the example.



If you have a graph that you want to 3-color, create one copy of the above hexagon for each of its nodes, and represent each of its edges by *two* successive edges -- e.g. like this if the original graph is a square with a diagonal:

The direction of the links that represent the original edges doesn't matter --
either way it allows neighboring hexagons to embed into GG in any
two *different* ways, but not in the *same* way.

Thus, there is a homomorphism from the expanded graph to G exactly if the original graph was 3-colorable.

Similarly, the above method can be applied to WHEEL 5-coloring problem.

Hence, WHEEL 5-coloring is NP-complete by the reduction from 3-colorability.

<u>5.27</u>

We can write the problem as follows:

Prove that flipping all edges of an alternating path or cycle strictly increase the size of a matching iff it’s an augmenting path.

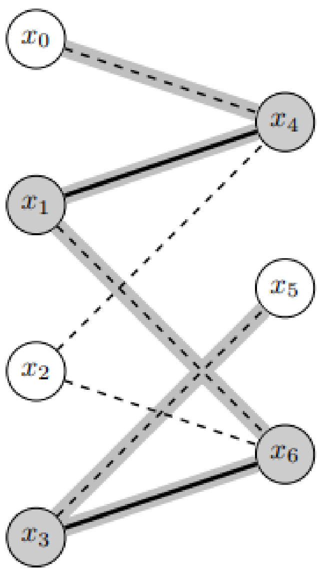


Figure 2: A maximal matching and an augmenting path

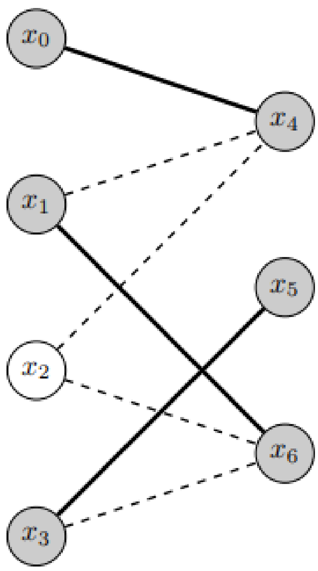


Figure 3: A maximum matching obtained by flipping the augmenting path in figure 2

Pf)

Suppose we have a graph G, a matching M and an M-augmenting path P. We need to prove that the new matching $M' = M \oplus P = (M \setminus P) \cup (P \setminus M) = (M \cup P) \setminus (M \cap P)$ is a valid matching of strictly greater cardinality than M.

Since P is an M-augmenting path, we know that it is an alternating path starting and ending in an unmatched vertex. Suppose we have a vertex x. If $x \notin P$, x is still matched at most once, like it was with respect to M. If x is an endpoint of P, then x was unmatched in M, and is matched exactly once in M'. Otherwise, x lies somewhere in the middle of P, which means that it was matched in M, and is matched to another vertex in M'. In all cases, x is matched at most once with respect to M', which means M' is a valid matching.

As P is an M-augmenting path, P consists of 2n + 1 edges, with $n \in Z_{\geq 0}$. Of these 2n + 1 edges, n are matched in M and the other n + 1 are matched in M', hence |M'| = |M| + 1.

Now suppose we have an M-alternating path or cycle P, such that M' = M $\oplus$ P is a valid matching and |M'| > |M|. For the cardinality of M to increase by flipping P, P must contain more unmatched edges than matched edges. Because P is an alternating path or cycle, this can only be the case if |P\M| = |P∩M|+ 1. Therefore |P| is odd and P can't be an alternating cycle. It follows that P must be a path and both the first and last edge must be unmatched with respect to M. For M' to be a valid matching, the first and last vertex of P must also be unmatched with respect to M. It follows that P is an M-augmenting path.

Hence, adding flow along an augmenting path is the same as flipping the edges along an alternating path, and increasing the size of the matching.