Automata Theory (CSE4058-01)
Homework # 06 - Solution

5.7 This is a variant of PERFECT MATCHING, in which, rather than asking that the edges in the matching don't overlap, we ask that they be a distance at least 2 apart from each other. This constraint feels like INDEPENDENT SET, and indeed we will prove that CELLPHONE CAPACITY is NP-complete by reducing INDEPENDENT SET to it.

Given an instance (G, k) of INDEPENDENT SET where $G = (V, E)$, we construct an instance (G', k') of CELLPHONE CAPACITY as follows. We set $k' = k$. To construct G', for each vertex $v \in V$ we add a new vertex v', and add an edge (v, v'). We now claim that (G, k) has an independent set of size k if and only if G' has a set of conversations of size k.

In one direction, if $S \subseteq V$ is independent, we can construct a set of conversations in G' of the same size by having v talk to v' for each $v \in S$.

Conversely, suppose that there is a set C of conversations in G'. If we choose one endpoint from each edge (v, w) in C, the resulting set is independent. If $v, w \in V$, i.e., the conversation is between two of the original vertices of G, we can choose either one. If $v \in V$ and $w = v'$, we choose v. This gives an independent set $S \subseteq V$ of size k, so G has an independent set of size k. Thus the reduction maps yes-instances to yes-instances and no-instances to no-instances, and the proof is complete.

Clearly we can carry out this reduction in polynomial time, since we are simply replacing each vertex with a gadget of constant size.

5.11 We will reduce NAE-3-SAT to HYPERGRAPH 2-COLORING. However, since we can't "negate" the color of a vertex, we include two vertices for every variable x, which we call v_x and $v_{\overline{x}}$. The idea is that x is true if v_x is black and $v_{\overline{x}}$ is white, and vice versa. Then our hyperedge will have an edge for each NAE-3-SAT clause, which includes v_x or $v_{\overline{x}}$ if the clause includes x or $\overline{x}$.

However, we need to make sure that v_x and $v_{\overline{x}}$ are forced to have opposite colors. There are many ways to do this. If vertices are allowed to appear twice in an edge, we can simply include the edge $\{v_x, v_x, v_{\overline{x}}\}$ in the hypergraph. If repeated vertices are not allowed, we can add three vertices s, t, u and the edge $\{s, t, u\}$. Then for each x, we include the edges

$$\{v_x, v_{\overline{x}}, s\}, \{v_x, v_{\overline{x}}, t\}, \{v_x, v_{\overline{x}}, u\}.$$

Since s, t, u cannot all be the same color, v_x and $v_{\overline{x}}$ are forced to have opposite colors, and we're done.

Clearly we can carry out this reduction in polynomial time, since we are simply replacing each clause with a gadget of constant size.

5.13 WHEEL 5-COLORING is clearly in NP, since we can check a coloring to see if it is valid in polynomial time.

We know that GRAPH k-COLORING is NP-complete for any $k \geq 3$. We will reduce GRAPH 5-COLORING to WHEEL 5-COLORING. Our reduction replaces each edge (u, v) of G with a gadget consisting of a path of length 3 between u and v. You can check that this gadget is wheel-5-colorable if and only if u and v have different colors, so it simulates an edge in traditional GRAPH 5-COLORING. Replacing each edge of G with this gadget gives a new graph G', such that G' is wheel-5-colorable if and only if G is 5-colorable. We can clearly carry out this reduction in polynomial time, and our proof is complete.

5.27 In the reduction from MAX BIPARTITE MATCHING to MAX FLOW, a partial matching M corresponds to a flow f where $f(e)=1$ for each edge included in M, along with $f(e)=1$ from s to the left endpoints of these edges and from their right endpoints to t, and $f(e)=0$ for all other edges. According to the definition of the residual graph G_f, the edges in M become reverse edges with capacity 1. The value of the flow is $|M|$, the number of edges in M.

An augmenting path in G_f starts at s and zig-zags back and forth between the left and right vertices, alternating between forward and reverse edges, starting and ending with forward edges, and then arriving at t. Adding a unit of flow along this path adds the forward edges to M, and removes the reverse edges. The vertices immediately after s and immediately before t in this path are not covered by M. Thus if we remove the first edge out of s and the last edge into t, we are left with an alternating path in the bipartite graph. By adding 1 to the value of the flow, we increase the size of the matching to $|M|+1$.