Homework 6

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Problem 1

Let

$$A = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

A

Find bases for each of the eigenspaces.

I will begin by finding the eigenvalues. So I will compute $Det(A - \lambda I)$.

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 2 & 4 \\ 2 & 1-\lambda & 2 \\ 4 & 2 & 4-\lambda \end{bmatrix}$$

I will compute the matrix using Laplace expansion.

$$\begin{vmatrix} 4-\lambda & 2 & 4 \\ 2 & 1-\lambda & 2 \\ 4 & 2 & 4-\lambda \end{vmatrix} = (-1)^2(4-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} + (-1)^3(2)\begin{vmatrix} 2 & 4 \\ 2 & 4-\lambda \end{vmatrix} + (-1)^4(4)\begin{vmatrix} 2 & 4 \\ 1-\lambda & 2 \end{vmatrix}$$

$$= (4-\lambda)((1-\lambda)(4-\lambda) - 4) - 2(2(4-\lambda) - 8) + 4(4 - 4(1-\lambda))$$

$$= (4-\lambda)(\lambda^2 - 5\lambda) + 4\lambda + 16\lambda$$

$$= -\lambda^3 + 9\lambda^2 - 20\lambda + 20\lambda$$

$$= 9\lambda^2 - \lambda^3$$

$$= \lambda^2(9-\lambda)$$

Now I need to solve for $\lambda^2(9-\lambda) = 0$. This produces a value of $\lambda_1 = 0$ and $\lambda_2 = 9$. Now I need to solve for the eigenspace. The eigenspace is equal to $ker(A - \lambda I)$.

$$A - \lambda_1 I = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

Finding the REF produces

$$\begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can find the kernel by solving this system of equations.

$$x_1 + \frac{1}{2}x_2 + x_3 = 0$$

Which gives us,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 + x_3 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So the basis for $ker(A - \lambda_1 I)$ and thus the basis of the eigenspace E_{λ_1} is given by,

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Similarly,

$$A - \lambda_2 I = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}$$

Getting the REF gives us

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

We can find the kernel by solving this system of equations.

$$\begin{cases} x_1 + 0x_2 - x_3 = 0 \\ 0x_1 + x_2 - \frac{1}{2}x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 = x_3 \\ x_2 = \frac{1}{2}x_3 \end{cases}$$

Which gives us,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

So the basis for $ker(A - \lambda_2 I)$ and thus the basis for the eigenspace E_{λ_2} is given by,

$$\left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

B

λ_1 has an algebraic multiplicity of 2 because the multiplicity of the root in the characteristic polynomial was equal to two. The dimension of its eigenspace and thus its geometric multiplicity was 2 because its basis has 2 vectors. λ_2 has an algebraic multiplicity of 1 because the multiplicity of the root in the characteristic polynomial was one. The dimension of its eigenspace and thus its geometric multiplicity was 1 because its basis has 1 vector.

Problem 2

To compute this we must first find the eigenvectors and eigenvalues of this matrix. The eigenvectors are given by

$$\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix}$$

This determinant equals $(\lambda - 5)(\lambda - 1)$ giving us that $\lambda_1 = 5$ and $\lambda_2 = 1$.

$$A - \lambda_1 I = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$

$$\text{getting the REF gives us } \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\text{So the null space of } \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \text{ is given by } \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\text{getting the REF gives us } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{So the null space of } \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \text{ is given by } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So then we can put these values together to form the matrices

$$Q = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

So A is given by

$$A = QDQ^{-1}$$

And so,

$$A^n = QD^nQ^{-1}$$

so,

$$A^{2021} = QD^{2021}Q^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5^{2021} & 0 \\ 0 & 1^{2021} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

Problem 3

Let $\mathcal{B}$ be the standard basis for $M_{2x2}(\mathbb{R})$ and $b_1 ... b_4$ equal the corresponding vectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in $\mathcal{B}$. $T : M_{2x2}(\mathbb{R}) \to M_{2x2}(\mathbb{R})$ be given by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & c \\ b & a+d \end{bmatrix}$$

A

Let

$$v = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$Tv = \begin{bmatrix} 3 & 3 \\ 2 & 5 \end{bmatrix}$$

$$Tv = 3b_1 + 3b_2 + 2b_3 + 5b_4$$

So, $[Tv]_{\mathcal{B}}$ is given by,

$$\begin{bmatrix} 3 \\ 3 \\ 2 \\ 5 \end{bmatrix}$$

$[T]_{\mathcal{B}}$ is given by,

$$\begin{cases} a+b & & \\ & c & \\ b & & \\ a & & +d \end{cases}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$v_{\mathcal{B}} = b_1 + 2b_2 + 3b_3 + 4b_4$$

$$[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$[T]_{\mathcal{B}}[v]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 5 \end{bmatrix} = [Tv]_{\mathcal{B}}$$

B

The determinant $det([T]_{\mathcal{B}})$ is given by the characteristic polynomial

$$(\lambda - 1)^3(\lambda + 1)$$

For $\lambda_1 = 1$,

$$T - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Getting the REF gives us } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$ker(T - \lambda_1 I) = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] = span \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For $\lambda_2 = -1$,

$$T - \lambda_1 I = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$\text{Getting the REF gives us } \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$ker(T - \lambda_2 I) = \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \end{array}\right] = span \left\{ \begin{bmatrix} -2 \\ 4 \\ -4 \\ 1 \end{bmatrix} \right\}$$

So $\lambda_1 = 1$ has the corresponding eigenvector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\lambda_2 = -1$ has the corresponding eigenvector $\begin{bmatrix} -2 \\ 4 \\ -4 \\ 1 \end{bmatrix}$.

Problem 4

For a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ with $a_1 \in \mathbb{F}$ and a square matrix, A with entries in $\mathbb{F}$, define $p(A)$ by

$$p(A) = a_n A^n + a_{n-1}A^{n-1} + ... + a_1 A + a_0 I.$$

Let $A \in \mathbb{M}_{nxn}(\mathbb{C})$ be diagonalizable and $c(x)$ the characteristic polynomial of A. Then $c(A) = O$ where O is the nxn zero matrix.

Proof. Because A is diagonalizable, $\exists Q, D \in \mathbb{M}_{2x2}$ such that $A = QDQ^{-1}$ where Q's columns are eigenvectors of A and D is a diagonal matrix containing A's eigenvalues. Plugging this into our polynomial, we find that,

$$\begin{gathered} a_n QD^nQ^{-1} + a_{n-1}QD^{n-1}Q^{-1} + ... + a_1 QDQ^{-1} + a_0 I \\ = Q[a_n D^n + a_{n-1}D^{n-1} + ... + a_1 D + a_0 I]Q^{-1} \end{gathered}$$

$$= Q \begin{bmatrix} a_n\lambda_1^n + a_{n-1}\lambda_1^{n-1} + ... + a_1\lambda_1 + a_0 & & & \\ & . & & 0 \\ & & . & \\ 0 & & & . \\ & & & a_n\lambda_n^n + a_{n-1}\lambda_n^{n-1} + ... + a_1\lambda_n + a_0 \end{bmatrix} Q^{-1}$$

$$= Q \begin{bmatrix} c(\lambda_1) & & & \\ & . & & 0 \\ & & . & \\ 0 & & & . \\ & & & c(\lambda_n) \end{bmatrix} Q^{-1}$$

And by the definition of the characteristic polynomial, $c(\lambda_i) = 0 \ \forall i \leq n$. So,

$$c(A) = Q \begin{bmatrix} 0 & & & \\ & . & & 0 \\ & & . & \\ 0 & & & . \\ & & & 0 \end{bmatrix} Q^{-1} = O$$

$\square$

Problem 5

Let $C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Both matrices have the characteristic polynomial $p(x) = (x-1)^2(x-2)$. This gives both of them the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. So then, $\exists Q \in \mathbb{M}_{3x3}$ such that

$$C = Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} Q^{-1}$$

And similarly, $\exists P \in \mathbb{M}_{3x3}$ such that,

$$D = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}$$

So,

$$Q^{-1}CQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = P^{-1}DP$$

$$=> Q^{-1}CQ = P^{-1}DP$$
$$=> PQ^{-1}CQP^{-1} = D$$

Now let $U = QP^{-1}$, we have that

$$U^{-1}CU = D$$

Problem 6

A

This operation fails the non-degeneracy property, and thus is not a inner product on $\mathbb{R}^2$.

$$< \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} >= 1 - 1 = 0$$

B

This operation also fails the non-degeneracy property.

$$< \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} >= trace(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = 0$$

C

This operation also fails the non-degeneracy property.

$$< 1, 1 >= \int_0^1 0 * 1 = 0$$