Homework 6

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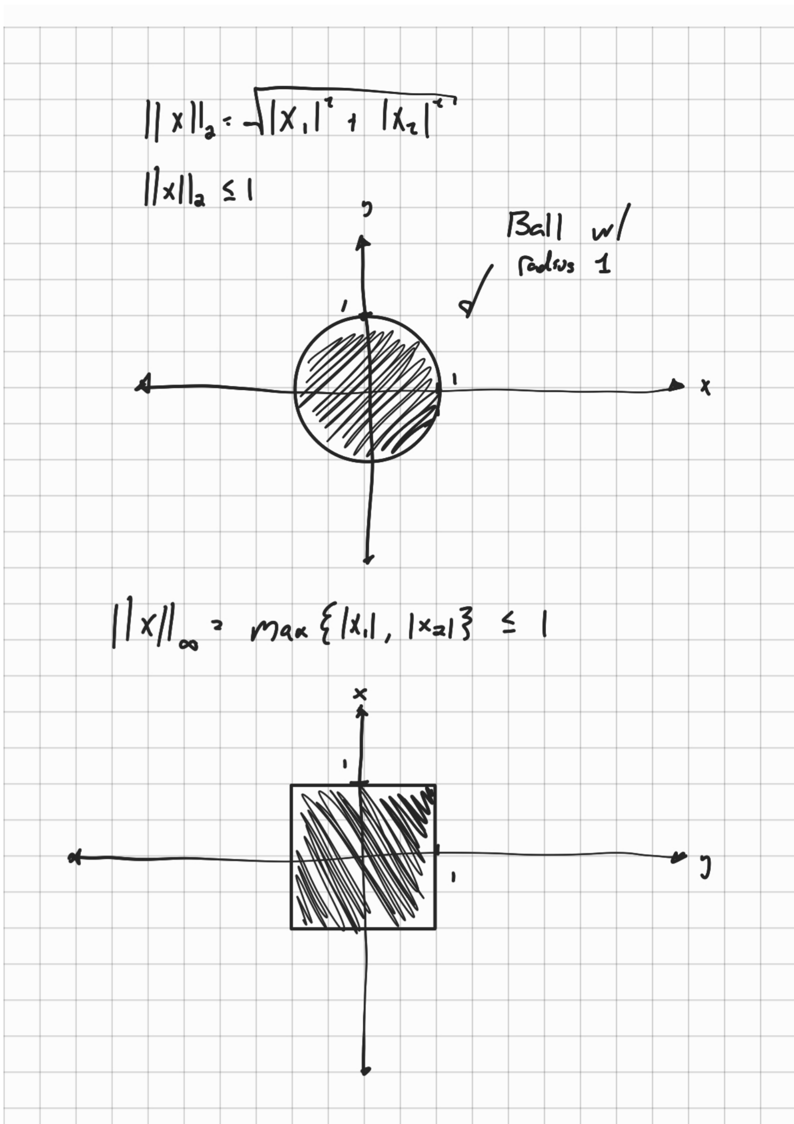
Problem 1



Problem 2

Suppose $P : V \to V$ satisfies both $P^* = P$ and $P^2 = P$. Then P is an orthogonal projection onto $Ran(P)$.

Proof. An orthogonal projection is defined as $P_E v = w$ such that $w \in E$ and $v - w \perp E$. In this case, $E = Ran(P)$. By definition of range, we know the first condition to be true. For the second condition, we must consider $x \in V$, $y \in Ran(P)$ such that $Px = y$, and another arbitrary vector $z \in Ran(P)$. So,

$$\langle x - y, z \rangle$$

Because $z \in Ran(P)$, $\exists v \in V$ such that $Pv = z$. So,

$$\begin{gathered}
\langle x - Px, Pv \rangle \\
= \langle x, Pv \rangle - \langle Px, Pv \rangle \\
= \langle x, Pv \rangle - \langle x, P^* Pv \rangle \\
= \langle x, Pv \rangle - \langle x, P^2 v \rangle \\
= \langle x, Pv \rangle - \langle x, Pv \rangle \\
\langle x, Pv \rangle = \langle x, Pv \rangle
\end{gathered}$$

Because these two inner products equal each other, we can then say that

$$\begin{gathered}
\langle x, pv \rangle - \langle Px, Pv \rangle = 0 \\
=> \langle x - y, z \rangle = 0
\end{gathered}$$

So therefore, $x - y \perp z$. Because z is an arbitrary vector $\in Ran(P)$, this then works for all vectors $\in Ran(P)$, so therefore, $x - y \perp Ran(P)$. So therefore, P is an orthogonal projection onto $Ran(P)$.

$\square$

Problem 3

Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthonormal basis in V.

A

For any $x = \sum_{k=1}^{n} \alpha_k v_k$ and $y = \sum_{k=1}^{n} \alpha_k v_k$, we have that

$$\langle x, y\rangle = \sum_{k=1}^{n} \alpha_k \overline{\beta_k}$$

Proof. Consider,

$$\begin{aligned}
\langle x, y\rangle &= \langle \sum_{k=1}^{n} \alpha_k v_k, y\rangle \\
&= \sum_{k=1}^{n} \alpha_k \langle v_k, y\rangle \\
&= \sum_{k=1}^{n} \alpha_k \langle v_k, \sum_{j=1}^{n} \beta_j v_j\rangle \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k \overline{\beta_j} \langle v_k, v_j\rangle
\end{aligned}$$

Now, because $v_k, v_j \in \mathcal{B}$ and $\mathcal{B}$ is an orthonormal basis, v_k and v_j are orthogonal to each other, so $\langle v_k, v_j\rangle = 0$. Except in the case that $v_k = v_j$, in which case, $\langle v_k, v_j\rangle = 1$. So the value inside the summation is equal to $\alpha_k \beta_j$ only where $k = j$, so,

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k \overline{\beta_j} \langle v_k, v_j\rangle = \sum_{k=1}^{n} \alpha_k \overline{\beta_k}$$

□

B

Consider $\langle x, v_k\rangle$. Based on the formula found in the last problem, we know that

$$\sum_{k=1}^{n} \langle x, v_k\rangle = \sum_{k=1}^{n} \alpha_k * 1$$

Now consider $\langle v_k, y\rangle$. Again,

$$\sum_{k=1}^{n} \langle v_k, y\rangle = \sum_{k=1}^{n} \overline{\langle y, v_k\rangle} = \sum_{k=1}^{n} \overline{\beta_k} * 1$$

So then we can say,

$$\sum_{k=1}^{n} \langle x, v_k \rangle \langle v_k, y \rangle = \sum_{k=1}^{n} \alpha_k \overline{\beta_k} = \langle x, y \rangle$$

$$\therefore \langle x, y \rangle = \sum_{k=1}^{n} \langle x, v_k \rangle \overline{\langle y, v_k \rangle}$$

Problem 4

Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. Define $\{v_1, v_2, v_3\}$ as an orthogonal set with the same span as $\{x_1, x_2, x_3\}$. Let $v_1 = x_1$. Then,

$$v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{< 2, 0, 1 > \cdot < 1, 1, 0 >}{|| < 1, 1, 0 > ||^2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{< 2, 2, 1 > \cdot < 1, 1, 0 >}{|| < 1, 1, 0 > ||^2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{< 2, 2, 1 > \cdot < 1, -1, 1 >}{|| < 1, -1, 1 > ||^2} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

This set is orthogonal, but to also make them normal,

$$\text{let } v_1 = \frac{\begin{bmatrix}1 & 1 & 0\end{bmatrix}^T}{|| < 1, 1, 0 > ||}$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix}1 \\ 1 \\ 0\end{bmatrix}$$

$$\text{let } v_2 = \frac{\begin{bmatrix}1 & -1 & 1\end{bmatrix}^T}{|| < 1, -1, 1 > ||}$$

$$v_2 = \frac{1}{\sqrt{3}} \begin{bmatrix}1 \\ -1 \\ 1\end{bmatrix}$$

$$\text{let } v_3 = \frac{\frac{1}{3}\begin{bmatrix}-1 & 1 & 2\end{bmatrix}^T}{||\frac{1}{3} < -1, 1, 2 > ||}$$

$$v_3 = \frac{1}{3\sqrt{3}} \begin{bmatrix}-1 \\ 1 \\ 2\end{bmatrix}$$

Problem 5

Let W be a finite-dimensional subspace of an inner product space V. If P is orthogonal projection onto W, then $I - P$ is orthogonal projection onto $W^{\perp}$.

Proof. Let $v \in V$ be such that $Pv \in W$. We know that by definition of an orthogonal projection, $v - Pv \in W^{\perp}$. So consider,

$$(I - P)v$$
$$=> v - Pv$$

So therefore, $(I - P) \in W^{\perp}$, so $I - P$ fulfills the first property of the definition of orthogonal projection. Now,

$$v - (I - P)v$$
$$v - v - Pv$$

$$(v - Pv) \perp W^{\perp}$$

Which satisfies the second property of the definition of orthogonal projection.

$\square$

Problem 6

Let $A \in M_{nxn}(\mathbb{F})$. $ker(A^*A + I) = \{0\}$.

Proof. Let v be an arbitrary vector $\in ker(A^*A + I)$. Then, $(A^*A + I)v = 0$, so

$$\langle (A^*A + I)v, v \rangle = 0$$
$$=> \langle A^*Av + Iv, v \rangle = 0$$
$$=> \langle A^*Av, v \rangle + ||v||^2 = 0$$
$$=> \langle Av, Av \rangle + ||v||^2 = 0$$

$$||Av||^2 + ||v||^2 = 0$$

By the inner product's non-negativity property, $||Av||^2 \geq 0$ and $||v||^2 \geq 0$, so therefore,

$$||v|| = 0$$
$$\text{so } v = 0 \text{ by non-degeneracy}$$

$\square$