Homework 8

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March 4th, 2022

Problem 1

To find the plane $z = a+bx+cy$ of best fit to the points $(1, 1, 3), (0, 3, 6), (2, 1, 5),$ and $(0, 0, 0)$, let

$$Ax = b \text{ be defined by } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}.$$

We can solve this system using the following equation, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^*A)^{-1}A^*b.$

So,

$$\text{Because this is } \mathbb{R},\ A^* = A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix}$$

So,

$$A^*A = \begin{bmatrix} 4 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 11 \end{bmatrix}$$

Which we can use to get $(A^*A)^{-1}$ by

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 5 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 5 & 3 & 11 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{REF}} \frac{1}{25}\left[\begin{array}{ccc|ccc} 25 & 0 & 0 & 23 & -9 & -8 \\ 0 & 25 & 0 & -9 & \frac{19}{2} & \frac{3}{2} \\ 0 & 0 & 25 & -8 & \frac{3}{2} & \frac{11}{2} \end{array}\right]$$

So now,

$$(A^*A)^{-1}A^* = \frac{1}{25}\begin{bmatrix} 6 & -1 & -3 & 23 \\ 2 & -\frac{9}{2} & \frac{23}{2} & -9 \\ -1 & \frac{17}{2} & \frac{1}{2} & -8 \end{bmatrix}$$

$$(A^*A)^{-1}A^* \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} -6 \\ 73 \\ 101 \end{bmatrix}$$

So our best approximate solution is given by $z = -\frac{3}{25} + \frac{73}{50}x + \frac{101}{50}y$.

Problem 2

Let

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)dt$$

be the standard inner product on $\mathbb{P}_2(\mathbb{R})$ and let $D : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ be given by $D(p) = p'$.

A)

We can get an orthonormal basis for $\mathbb{P}_2(\mathbb{R})$ by performing Gram-Schmidt orthogonalization on the standard basis for $\mathbb{P}_2(\mathbb{R})$ which is given by $\{1, x, x^2\}$. And then dividing them by their norm. Let $x_1 = 1, x_2 = x, x_3 = x^2$.

$$\text{Let } v_1 = x_1.$$

$$\text{Let } v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{||v_1||^2} = x$$

$$\text{Let } v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{||v_1||^2} - \frac{\langle x_3, v_2 \rangle}{||v_2||^2}x = x^2 - \frac{1}{3} - 0 = x^2 - \frac{1}{3}$$

Now to make them normal,

$$\frac{v_1}{||v_1||} = \frac{1}{\sqrt{2}}$$

$$\frac{v_2}{||v_2||} = \frac{x}{\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}x}{\sqrt{2}}$$

$$\frac{v_3}{||v_3||} = \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \frac{3\sqrt{5}(x^2 - \frac{1}{3})}{2\sqrt{2}}$$

So the orthonormal basis is given by $\{\mathcal{B} = \frac{1}{\sqrt{2}}, \frac{\sqrt{3}x}{\sqrt{2}}, \frac{3\sqrt{5}(x^2-\frac{1}{3})}{2\sqrt{2}}\}$.

B)

To find $[D^*]_B$, let's start by finding $[D]_{\mathcal{B}}$.

$$D(v_1) = 0 = 0v_1 + 0v_2 + 0v_3$$

$$D(v_2) = \sqrt{\frac{3}{2}} = \sqrt{3}v_1 + 0v_2 + 0v_3$$

$$D(v_3) = \frac{3\sqrt{5}x}{\sqrt{2}} = 0v_1 + \sqrt{15}v_2 + 0v_3$$

So,

$$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{bmatrix}$$

Because we are in $\mathbb{P}_2(\mathbb{R})$ and not $\mathbb{P}_2(\mathbb{C})$, $D^* = D^T$, so

$$[D^*]_{\mathcal{B}} = [D^T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{bmatrix}$$

C)

Let $p = av_1 + bv_2 + cv_3$. So then,

$$D^*(p) = \frac{\sqrt{3}}{\sqrt{2}}b + c\sqrt{15}\left(\frac{3\sqrt{5}(x^2 - \frac{1}{3})}{2\sqrt{2}}\right)$$

Problem 3

Let V be the inner product space of complex-valued continuous functions on $[0,1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt.$$

Let $h \in V$ and define $T : V \to V$ by $T(f) = h(f)$. Then T is a unitary operator if and only if $|h(t)| = 1$ for $0 \leq t \leq 1$.

Proof. (=>) Let $|h(t)| = 1$ for $0 \leq t \leq 1$.

$$||Tf||^2 = \langle hf, hf\rangle = \int_0^1 hf(t)\overline{hf(t)}dt$$
$$= \int_0^1 |hf(t)|^2 dt$$

Because $h(t) = 1$,

$$\int_0^1 |hf(t)|^2 dt = \int_0^1 |f(t)^2| = \int_0^1 f(t)\overline{f(t)}dt = ||f(t)||^2$$

So therefore, $||Tf||^2 = ||f||^2$ and T is an isometry. Because $TV \to V$, we know that T is square dimensional. Now consider,

$$\langle T^*Tf, f\rangle = \langle Tf, Tf\rangle = \langle f, f\rangle$$

Therefore, T^*T is the identity matrix. Therefore T is a unitary operator.

(<=) Let T be a unitary operator. Let $f, g \in V$ such that $g = T^*f - f\overline{h}$ and f is arbitrary. Then,

$$||g||^2 = \int_0^1 (T^*f - f\overline{h})\overline{T^*f - f\overline{h})}dt = \int_0^1 |T^*f - f\overline{h}|^2 dt$$

This implies that

$$T^*f - f\overline{h} = 0 => T^*f = f\overline{h}$$

Which then gives you,

$$TT^*f = h\overline{h}f = |h|^2 f$$

And because $TT^*f = f$ due to properties of unitary operators, we then get

$$|h|^2 f = f$$
$$|h|^2 = 1$$

Therefore, $|h| = 1$ along $0 \leq t \leq 1$.

$\square$

Problem 4

A)

Let λ be an arbitrary eigenvalue of A^*A and v be its eigenvector. Then,

$$\langle A^*Av, v\rangle = \langle v, (A^*A)^*v\rangle$$
$$=> \langle \lambda v, v\rangle = \langle v, (A^*)(A^*)^*v\rangle$$
$$=> \langle \lambda v, v\rangle = \langle v, A^*Av\rangle$$
$$=> \langle \lambda v, v\rangle = \langle v, \lambda v\rangle$$
$$\therefore \lambda = \overline{\lambda}$$

And therefore, all eigenvalues are real.

B)

$$||\lambda v||^2 = \langle \lambda v, \lambda v\rangle$$
$$=> ||\lambda v||^2 = \lambda\overline{\lambda}\langle v, v\rangle$$
$$=> \lambda = \frac{||\lambda v||}{||v||}$$

And by the non-negativity property of inner products, λ must be non-negative.