Homework 9

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Problem 1

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = -\lambda^3 + 12\lambda + 16 = -(\lambda + 2)^2(\lambda - 4)$$

So we have that A has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$. With λ_1 having a algebraic multiplicity of 2. Now, to calculate the eigenvectors.

$$A - \lambda_1 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which gives us the eigenvectors $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$A - \lambda_2 = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Which gives us the eigenvector $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now we must normalize the eigenvectors. Since all the values we have seen thus far are real, I will assume we are in the space of real numbers and will use the dot product.

$$||v_1||^2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 2$$

So our first orthonormal eigenvector $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$||v_2||^2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$||v_3||^2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$$

So then we have that $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $u_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

So then we have that

$$U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

And because this is real numbers

$$U^* = U^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

So,

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Problem 2

Let A and B be unitarily equivalent $n \times n$ matrices.

A)

trace$(A^*A) = $ trace(B^*B).

Proof. If A and B are unitarily equivalent, then $\exists U$ such that U is unitary, $A = UBU^*$, and similarly $B = U^*AU$. So then we have that

$$\text{trace}(A^*A) = \text{trace}(A^*UBU^*) = \text{trace}(U^*A^*UB) = \text{trace}(B^*B)$$

□

B)

$$\sum_{j,k=1}^{n} |A_{j,k}|^2 = \sum_{j,k=1}^{n} |B_{j,k}|^2$$

Proof. Consider the matrix representation of A^*A and its two first elements on its main diagonal, $A_{1,1}$, $A_{2,2}$. By matrix multiplication, we can write them as

$$A_{1,1} = \sum_{k=1}^{n} A_{k,1}\overline{A_{k,1}}$$

$$A_{2,2} = \sum_{k=1}^{n} A_{k,2}\overline{A_{k,2}}$$

The coordinates are not flipped due to A^* being the complex conjugate of the **transpose**. So then, the sum of A^*A's elements on the main diagonal can be written as

$$\text{trace}(A^*A) = \sum_{j,k=1}^{n} A_{k,j}\overline{A_{k,j}} = \sum_{j,k=1}^{n} |A_{k,j}|^2$$

The same can be said for B. A and B have already been defined to be unitarily equivalent, and thus

$$\text{trace}(A^*A) = \sum_{j,k=1}^{n} |A_{k,j}|^2$$

$$\text{trace}(B^*B) = \sum_{j,k=1}^{n} |B_{k,j}|^2$$

$$\sum_{j,k=1}^{n} |A_{k,j}|^2 = \text{trace}(A^*A) = \text{trace}(B^*B) = \sum_{j,k=1}^{n} |B_{k,j}|^2$$

$$\sum_{j,k=1}^{n} |A_{k,j}|^2 = \sum_{j,k=1}^{n} |B_{k,j}|^2$$

$\square$

C)

$$\text{let } A = \begin{bmatrix} 1 & 2 \\ 2 & i \end{bmatrix}$$

$$\text{let } B = \begin{bmatrix} i & 4 \\ 1 & 1 \end{bmatrix}$$

$$\sum_{j,k=1}^{n} |A_{k,j}|^2 = 1^2 + 2^2 + 2^2 + |i|^2 = 10$$

$$\sum_{j,k=1}^{n} |B_{k,j}|^2 = |i|^2 + 4^2 + 1^2 + 1^2 = 19$$

$$\sum_{j,k=1}^{n} |A_{k,j}|^2 \neq \sum_{j,k=1}^{n} |B_{k,j}|^2$$

$\therefore$ A and B are not unitarily equivalent.

Problem 3

A)

$\text{trace}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$ and $\text{trace}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0$. So these are not unitarily equivalent.

B)

$$\text{let } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{let } B = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$\sum_{j,k=1}^{n} |A_{k,j}|^2 = 0^2 + 1^2 + 1^2 + 0^2 = 2$$

$$\sum_{j,k=1}^{n} |B_{k,j}|^2 = 0^2 + \frac{1}{2}^2 + \frac{1}{2}^2 + 0^2 = \frac{1}{2}$$

$$\sum_{j,k=1}^{n} |A_{k,j}|^2 \neq \sum_{j,k=1}^{n} |B_{k,j}|^2$$

$\therefore A$ and B are not unitarily equivalent.

C)

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Unitarily equivalent matrices share determinants, so these cannot be unitarily equivalent.

D)

$$\text{let } A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{let } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -(\lambda - 1)(\lambda^2 + 1)$$

So we have the eigenvalues of A, $\lambda_1 = 1, \lambda_2 = -i$, and $\lambda_3 = i$. Solving $A - \lambda_i I = 0$ for all λs gives us the eigenvectors,

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

$$\langle v_1, v_2 \rangle = 0$$

$$\langle v_1, v_3 \rangle = 0$$

$$\langle v_2, v_3 \rangle = 0$$

So the set of these eigenvectors is orthogonal, and thus are also linearly independent.

$$||v_1|| = \sqrt{1^2} = 1$$

$$||v_2|| = \sqrt{(i \times -i)^2 + 1^2 + 0} = \sqrt{2}$$

$$||v_3|| = \sqrt{(-i \times i)^2 + 1^2 + 0} = \sqrt{2}$$

We must normalize these vectors to get our orthonormal basis, so we have that

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} \right\}$$

with the corresponding vectors u_1, u_2, u_3. This proves that A is unitarily equivalent to a diagonal matrix, and so we have that

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i & -i \\ 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \sqrt{2} \\ -i & 1 & 0 \\ i & 1 & 0 \end{bmatrix}$$

And therefore, A is unitarily equivalent to B.

E)

$$\text{let } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{let } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\sum_{j,k=1}^{n} |A_{k,j}|^2 = 1^2 + 1^2 + 2^2 + 2^2 + 3^2 = 19$$

$$\sum_{j,k=1}^{n} |B_{k,j}|^2 = 1^2 + 2^2 + 3^2 = 14$$

$$\sum_{j,k=1}^{n} |A_{k,j}|^2 \neq \sum_{j,k=1}^{n} |B_{k,j}|^2$$

$\therefore A$ and B are not unitarily equivalent.

Problem 4

Suppose A is a normal operator on an inner product space V and that 3 and 4 are eigenvalues of A. Then there exists a vector $v \in V$ such that $||v|| = \sqrt{2}$ and $||Av|| = 5$.

Proof. Let $v_1, v_2 \in V$ be eigenvectors corresponding to 3 and 4. Because A is a normal operator, A has an orthonormal basis of eigenvectors which means

$$||v_1|| = 1$$

$$||v_2|| = 1$$

By the Generalized Pythagorean identity,

$$||v_1 + v_2|| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

From this, we get

$$||Av_1 + Av_2|| = ||3v_1 + 4v_2|| = \sqrt{3^2 + 4^2} = 5$$

□

Problem 5

Suppose V is a complex inner product space and T is a normal operator on V such that $T^7 = T^6$. Then T is self-adjoint and $T^2 = T$.

Proof. By the Spectral Theorem, V has a orthonormal basis of eigenvectors of T, $\mathcal{B} = e_1, ..., e_n$. Let $\lambda_1, ..., \lambda_n$ be the corresponding eigenvalues. Then

$$Te_i = \lambda_i e_i \forall i = 1, ..., n$$

Then we have that

$$T^7 e_i = \lambda_i^7 e_i \text{ and } T^6 e_i = \lambda_i^6 e_i$$

Which gives λ_i the possibility of either being 0 or 1. Because of this, $[T]_{\mathcal{BB}}$ is a diagonal matrix with eigenvalues on the diagonal. Because the $[T]^*_{\mathcal{BB}}$ is the conjugate of the transpose, and $[T]_{\mathcal{BB}}$ is a diagonal matrix with only real values, $[T]^*_{\mathcal{BB}} = [T]_{\mathcal{BB}}$. Knowing this,

$$T^2 e_i = \lambda_i^2 e_i \rightarrow \text{ because } \lambda_i \text{ is either 0 or 1, } \lambda_i^2 \text{ is either 0 or 1, so...}$$

$$T^2 e_i = \lambda_i e_i = Te_i$$

□