Final Review

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Problem 2

A)

Let $v_1 = \begin{bmatrix}1 & 2 & -2 & 1\end{bmatrix}^T, v_2 = \begin{bmatrix}9 & 12 & -4 & 9\end{bmatrix}^T, v_3 = \begin{bmatrix}1 & 3 & -3 & 1\end{bmatrix}^T, v_4 = \begin{bmatrix}1 & 3 & -2 & 1\end{bmatrix}^T, v_5 = \begin{bmatrix}4 & 3 & -1 & 4\end{bmatrix}^T, v_6 = \begin{bmatrix}1 & 3 & -2 & -1\end{bmatrix}$ be vectors in $\mathbb{R}^4$. To find a basis for the span of these vectors, I'll place them into columns of a matrix and find the REF.

$$\begin{bmatrix} 1 & 9 & 1 & 1 & 4 & 6 \\ 2 & 12 & 3 & 3 & 3 & 3 \\ -2 & -4 & -3 & -2 & -1 & -2 \\ 1 & 9 & 1 & 1 & 4 & -1 \end{bmatrix} \xrightarrow{Semi-REF} \begin{bmatrix} 1 & 9 & 1 & 1 & 4 & 6 \\ 0 & -6 & 1 & 1 & -5 & 3 \\ 0 & 0 & -4/3 & -1/3 & 5/3 & -10/3 \\ 0 & 0 & 0 & 0 & 0 & -7 \end{bmatrix}$$

Which then gives us the basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 9 \\ 12 \\ -4 \\ 9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \\ -1 \end{bmatrix} \right\}$$

B)

Let $v_1 = \begin{bmatrix}2 & -1 & 1 & 5 & -3\end{bmatrix}^T, v_2 = \begin{bmatrix}3 & -2 & 0 & 0 & 0\end{bmatrix}^T, v_3 = \begin{bmatrix}1 & 1 & 50 & -921 & 0\end{bmatrix}^T$. We can determine if these vectors are linearly independent by putting them as columns of a matrix and finding its REF.

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 1 & 0 & 50 \\ 5 & 0 & -921 \\ -3 & 0 & 0 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So they are linearly independent. If we were to add the two vectors $v_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ and $v_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$. This set would be a basis for $\mathbb{R}^5$.

Problem 3

Let $n > 0$ be odd and $A, B \in M_{nxn}(\mathbb{C})$ be such that $AB = -BA$. A and B are not invertible.

Proof. If A and B are invertible,

$$\mathrm{Det}(A^{-1}) = \frac{1}{\mathrm{Det}(A)}$$
$$\mathrm{Det}(B^{-1}) = \frac{1}{\mathrm{Det}(B)}$$

So,

$$\mathrm{Det}((AB)^{-1}) = \frac{1}{\mathrm{Det}(-BA)}$$
$$\mathrm{Det}(B^{-1}A^{-1}) = \frac{1}{\mathrm{Det}(-B)\mathrm{Det}(A)}$$
$$\mathrm{Det}(B^{-1}A^{-1}) = -1^n\mathrm{Det}(B^{-1})\mathrm{Det}(A^{-1})$$

And because n is odd,

$$\mathrm{Det}(B^{-1}A^{-1}) = -1 \times \mathrm{Det}(B^{-1}A^{-1})$$

So $\mathrm{Det}((AB)^{-1})$ has to be equal to 0. Thus they are not invertible. $\square$

Problem 4

Suppose A is an $n \times n$ diagonalizable matrix that has only $\lambda = 1$ as an eigenvalue. Every nonzero vector in $\mathbb{R}^n$ is an eigenvector of A corresponding to eigenvalue $\lambda = 1$.

Proof. If A is a square diagonalizable matrix and has one eigenvalue, then it can be said that $\lambda = 1$ has a algebraic multiplicity of n. And because A is diagonalizable, then it can also be said that $\dim(E_\lambda(\mathbb{R})) = n$. And thus, every nonzero vector in $\mathbb{R}^n$ is an eigenvector of A.
OR

$$A = QDQ^{-1}$$
$$A = QIQ^{-1}$$
$$A = QQ^{-1}$$

And because Q is invertible, it's linearly independent, and because it's n-dim'd, span$(Q) = \mathbb{R}^n$. and beacuse Q contains a basis for the eigenvectors of A, blah blah. $\square$

Problem 5