Comments on two counting formulas

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Please visit http://baihacker.github.io/main/ for the latest version.
[optional] read the first 3 chapters of generatingfunctionology [1] to help understand this article.

Two counting formulas

In generatingfunctionology [1] [p.81,p.93], we have two counting formulas
Theorem 3.4.1 (The exponential formula). Let $\mathcal{F}$ be an exponential $\mathcal{F}$ be an exponential family whose deck and hand enumerators are $\mathcal{D}(x)$ and $\mathcal{H}(x,y)$, respectively. Then

$$\mathcal{H}(x,y) = e^{y\mathcal{D}(x)}$$

In detail, the number of hands of weight n and k cards is

$$\mathrm{h}(n,k) = \left[\frac{x^n}{n!}\right]\left\{\frac{\mathcal{D}(x)^k}{k!}\right\}.$$

Theorem 3.14.1. In a prefab P whose hand enumerator is H(x,y) we have

$$\mathcal{H}(x,y) = \prod_{n=1}^{\infty}\frac{1}{(1-yx^n)^{d_n}},$$

where d_n is the number of cards in the nth deck (n ≥ 1).

The **deck** can be viewed as standard **structure pattern** and **hand** can be viewed as the target with **structure pattern**. These two theorems tell us how to compute the hand if the deck enumerator is known. Moreover, inside the target, 3.4.1 is for the target whose element has labels while 3.14.1 is for the target whose element doesn't have labels.

Labeled ball unlabeled box

Based on theorem 3.4.1, the generating function is

$$\mathcal{H}(x,y) = e^{y(\sum_{i=1}^{\infty}\frac{x^i}{i!})} = e^{y(e^x-1)}$$

And if there are n balls m boxes, the answer is $\left[\frac{x^n}{n!}y^m\right]\{\mathcal{H}(x,y)\} = S(n,m)$
(Stirling_numbers_of_the_second_kind).

Unlabeled ball unlabeled box

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$$\mathcal{H}(x,y) = \prod_{i=1}^{\infty}\frac{1}{1-yx^i}$$

And if there are n balls m boxes, the answer is $\left[\frac{x^n}{n!}y^m\right]\{\mathcal{H}(x,y)\}$
(Partition_function_(number_theory)).

Consider labeled boxes

Actually, the two formulas are applied to the problems where the box is not labeled. (Each structure in the target has a label). So what's the formula? The answer is simpler.

Labeled ball labeled box

Theorem 3.4.1'.

$$\mathcal{H}(x, y) = \sum_{i=0}^{\infty}(y\mathcal{D}(x))^i$$

(H,D are exponential generating function)

So, for the problem we have

$$\mathcal{H}(x,y) = \sum_{i=0}^{\infty}(y\sum_{j=0}^{\infty}\frac{x^j}{j!})^i = \sum_{i=0}^{\infty}e^{ix}y^i.$$

And if there are n balls m boxes, the answer is $\left[\frac{x^n}{n!}y^m\right]\{\mathcal{H}(x,y)\} = m^n$.

Unlabeled ball labeled box

Theorem 3.14.1'.

$$\mathcal{H}(x,y) = \sum_{i=0}^{\infty}(y\mathcal{D}(x))^i$$

(H,D are normal generating function)

So, for this problem we have

$$\mathcal{H}(x,y) = \sum_{i=0}^{\infty}(y\sum_{j=0}^{\infty}x^j)^i = \sum_{i=0}^{\infty}(\frac{y}{1-x})^i.$$

And if there are n balls m boxes, the answer is $[x^n y^m]\{\mathcal{H}(x,y)\} = [x^n]\left\{\frac{1}{(1-x)^m}\right\} = \binom{m+n-1}{n}$ (Number_of_combinations_with_repetition).

A trick of theorem 3.14.1.

Since

$$F(x) = \prod\frac{1}{(1-x^k)^{g(k)}},$$

we have

$$\log(F(x)) = \sum g(k)\log(\frac{1}{1-x^k}) = \sum g(k)\sum\frac{x^{km}}{m} = \sum\frac{1}{m}G(x^m)$$

then

$$F(x) = \exp^{\sum\frac{1}{k}G(x^k)}$$

It gives a way to compute $F(x)$ fast if G is known.

Another trick of both theorems

Let $y = 1$ (we don't care the number of used patterns), and use $x\mathrm{D}\log$ trick, we can get the relationship between $\mathcal{H}$ and $\mathcal{D}$。 If $\mathcal{H}, \mathcal{D}$ are the same, maybe they are off by x (usually off by x), we can get the formula which can be used to compute $\mathcal{H}$. It happens when the problem itself has an off by one structure, e.g. forest and tree. (generatingfunctionology [1], [p87,pp.89-90,pp.93-94,pp.102-103]).

References

[1] Herbert S. Wilf, 1992, generatingfunctionology (second edition)