Differential Equations Project: Phase 3

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A stock's price S_t (in USD) can be predicted with respect to time t (in days from an initial time) using a growth function; that is, by relating the rate at which the stock price is changing to the current stock price as a proportion:

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} \propto S_t$$

The constant of proportionality is the drift μ of the stock, which is the rate of change of the expected value of the stock price (which is not the expected value of its rate of change):

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \mu S_t \qquad \text{where} \qquad \mu = \frac{\Delta \mathbb{E}[S_t]}{\Delta t}$$

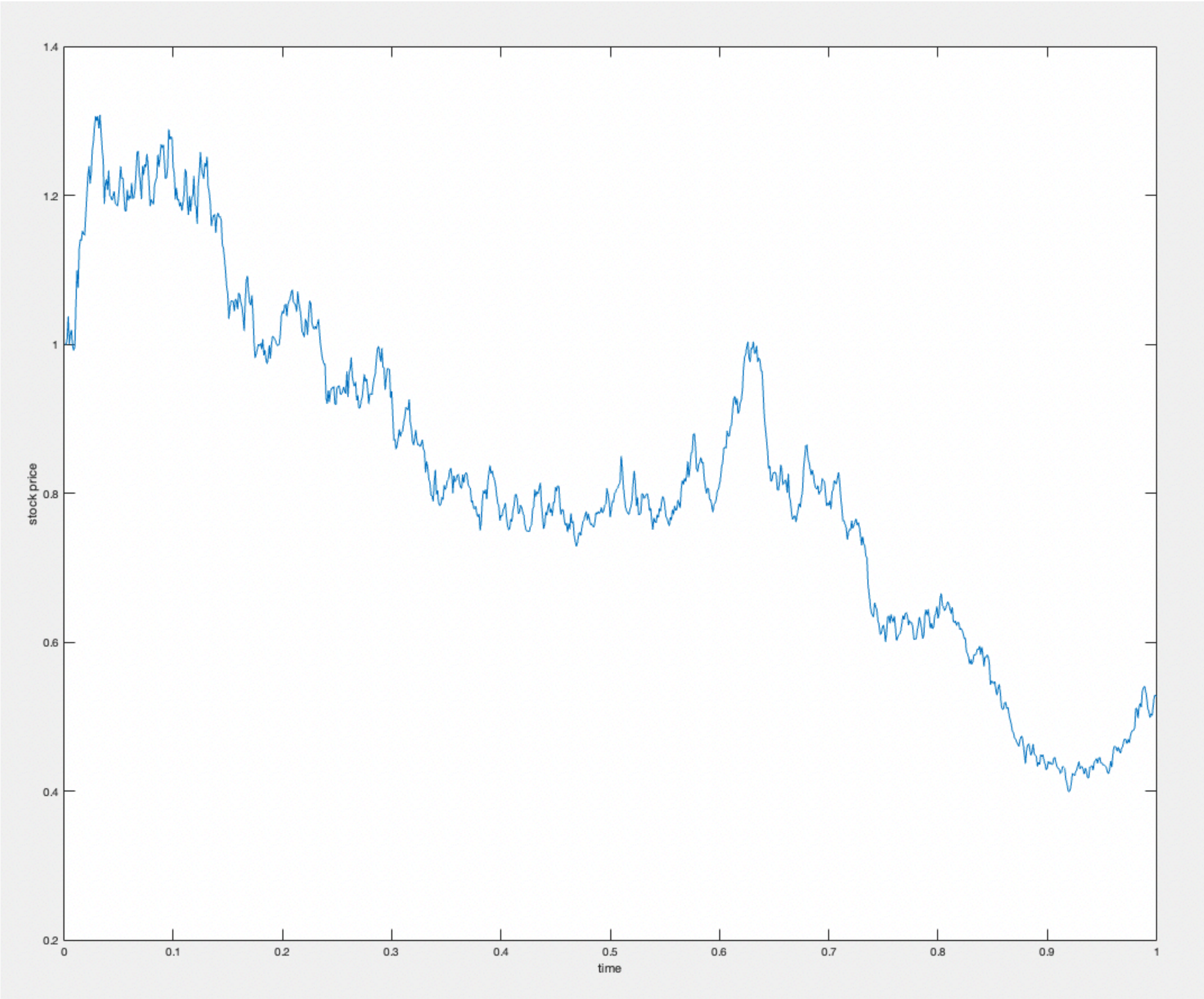
The stock market is constantly volatile, though, changing in unpredictable ways. This randomness element can be modeled by a randomness term, the rate of change of the standard Wiener process W_t:

$$\text{rate of randomness} \propto \frac{\mathrm{d}W_t}{\mathrm{d}t}$$

The proportionality constant is the volatility σ of the stock, which is simply its standard deviation. Adding this term,

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \mu S_t + \sigma \frac{\mathrm{d}W_t}{\mathrm{d}t} \qquad \text{where} \qquad \sigma = \frac{\sum (S_{t,i} - \bar{S}_t)}{n-1}$$

The randomness term ensures that each trial of the model yields a different graph. The following is a sample graph:



The DE can be rewritten as

$$\mathrm{d}S_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t$$

where $\mu_t = \mu S_t$ and $\sigma_t = \sigma S_t$. Integrating and reparameterizing μ_t, σ_t and W_t with s,

$$S_t = \int_0^t \mu_s\,\mathrm{d}s + \int_0^t \sigma_s\,\mathrm{d}W_s + S_0$$

where S_0 is the constant of integration (the initial stock price). This makes S_t an Itô process, a stochastic process expressible as the sum of two integrals, one with respect to a stochastic process and another with respect to time, and a constant.

The Taylor expansion of a twice-differentiable scalar function $f(t, s)$ is

$$\mathrm{d}f = \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial s}\,\mathrm{d}s + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\,\mathrm{d}^2 s + \cdots$$

Substituting S_t for s and appropriately substituting for $\mathrm{d}s$ yields

$$\mathrm{d}f + \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial s}(\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t) + \frac{\partial^2 f}{\partial s^2}\left(\mu_t^2\,\mathrm{d}t^2 + 2\mu_t\sigma_t\,\mathrm{d}t\,\mathrm{d}W_t + \sigma_t^2\,\mathrm{d}{W_t}^2\right) + \cdots$$

As $\mathrm{d}t$ approaches 0, $\mathrm{d}t^2$ and $\mathrm{d}t\,\mathrm{d}W_t$ tend to zero faster than $\mathrm{d}W_t^2$. Substituting 0 for $\mathrm{d}t^2$ and $\mathrm{d}t\,\mathrm{d}W_t$ and $\mathrm{d}t$ for $\mathrm{d}W_t^2$ yields

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial s} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial s^2}\right)\mathrm{d}t + \sigma_t \frac{\partial f}{\partial s}\,\mathrm{d}W_t$$

This is itself an Itô process. Itô's lemma states that for any Itô process S_t and any twice-differentiable function $f(t, s)$, $f(t, S_t)$ is an Itô process.
Let $f(S_t) = \ln S_t$. Applying Itô's lemma,

$$\begin{aligned}
\mathrm{d}f &= f'(S_t)\,\mathrm{d}S_t + \frac{1}{2}f''(S_t)(\mathrm{d}S_t)^2 \\
&= \frac{1}{S_t}\,\mathrm{d}S_t - \frac{1}{2S_2^2}\left(S_t^2\sigma^2\,\mathrm{d}t\right) \\
&= \frac{1}{S_t}(\sigma S_1\,\mathrm{d}W_t + \mu S_t\,\mathrm{d}t) - \frac{\sigma^2}{2}\,\mathrm{d}t \\
&= \sigma\,\mathrm{d}W_t + \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t
\end{aligned}$$

Integrating the separable DE,

$$f(t) = \ln S_t = \ln S_0 + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t$$

This can finally be exponentiated, yielding S_t:

$$S_t = S_0\,\mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

Substituting for S_0, μ, and σ,

$$S_t = 141.501\,\mathrm{e}^{\left(0.00064 + \frac{4.356^2}{2}\right) + 4.356W_t}$$

The following are the Matlab code, solution (for the deterministic term), and graph:

Solution.m

```
close all;
clear all;

dom = [0, 30];
n = 1000;
dt = (dom(2) - dom(1))/50;
S0 = 141.501;

ndist = makedist('Normal', 0 , sqrt(dt));

mu = 0.00064;
sigma = 4.356;

tvals = linspace(dom(1), dom(2), n);
Svals = zeros(1, n);
Svals(1) = S0;
for i = 2:n
    t = dom(1) + (i - 1) .* dt;
    S = Svals(i - 1);
    dW = random(ndist);
    Svals(i) = S + mu .* dt + sigma .* dW
end
figure()
hold on;
plot(tvals, Svals);
xlabel('time (days from 9/27/22)')
ylabel('stock price (USD)')

% Solving for the non-stochastic term
syms s(t)
ode = diff(s, t) == s;
sol(t) = dsolve(ode);
```

```
sol
1x1 symfun

val(t) =

C1*exp(t)
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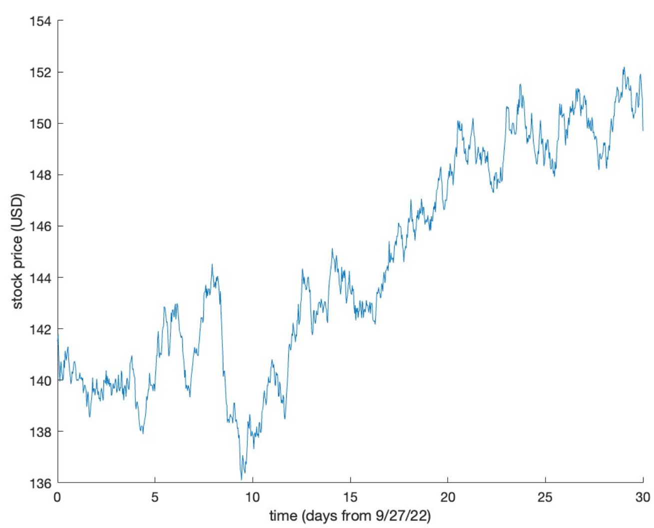

The following is the Excel spreadsheet.