Differential Equations Project: Phase 3

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A stock's price S_t (in USD) can be predicted with respect to time t (in days from an initial time) using a growth function; that is, by relating the rate at which the stock price is changing to the current stock price as a proportion:

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} \propto S_t$$

The constant of proportionality is the drift μ of the stock, which is the rate of change of the expected value of the stock price (which is not the expected value of its rate of change):

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \mu S_t \qquad \text{where} \qquad \mu = \frac{\Delta\mathbb{E}[S_t]}{\Delta t}$$

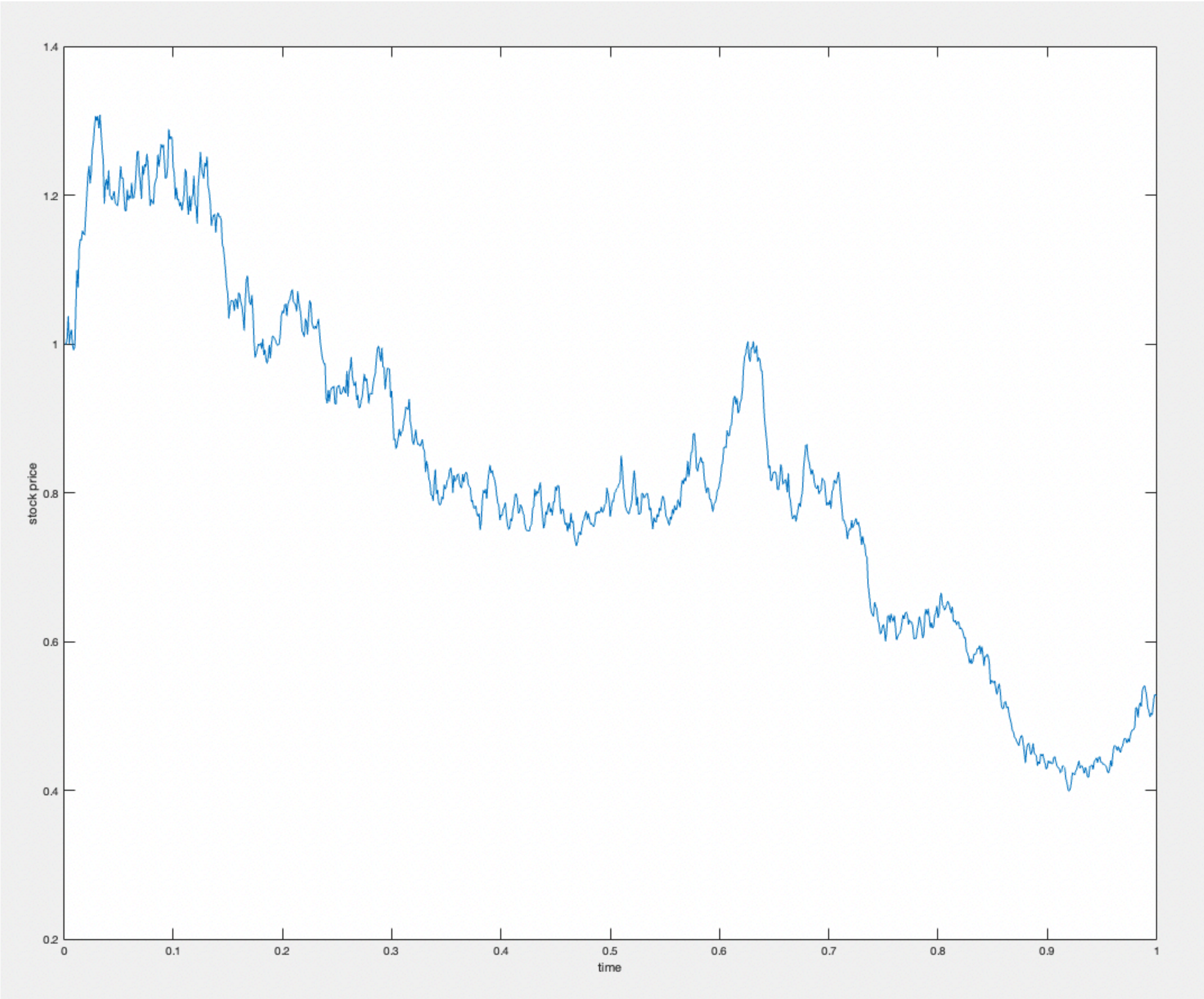
The stock market is constantly volatile, though, changing in unpredictable ways. This randomness element can be modeled by a randomness term, the rate of change of the standard Wiener process W_t:

$$\text{rate of randomness} \propto \frac{\mathrm{d}W_t}{\mathrm{d}t}$$

The proportionality constant is the volatility σ of the stock, which is simply its standard deviation. Adding this term,

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \mu S_t + \sigma\frac{\mathrm{d}W_t}{\mathrm{d}t} \qquad \text{where} \qquad \sigma = \frac{\sum(S_{t,i} - \bar{S}_t)}{n-1}$$

The randomness term ensures that each trial of the model yields a different graph. The following is a sample graph:



The DE can be rewritten as

$$\mathrm{d}S_t = \mu_t \, \mathrm{d}t + \sigma \, \mathrm{d}W_t$$

where $\mu_t = \mu S_t$. Integrating and reparameterizing μ_t with and W_t with s,

$$S_t = \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma \, \mathrm{d}W_s + S_0$$

where S_0 is the constant of integration (the initial stock price). This makes S_t an Itô process, a stochastic process expressible as the sum of two integrals, one with respect to a stochastic process and another with respect to time, and a constant.
The Taylor expansion of a twice-differentiable scalar function $f(t, s)$ is

$$\mathrm{d}f = \frac{\partial f}{\partial t} \, \mathrm{d}t + \frac{\partial f}{\partial s} \, \mathrm{d}s + \frac{1}{2} \frac{\partial^2 f}{\partial s^2} \, \mathrm{d}^2 s + \cdots$$

Substituting S_t for s and appropriately substituting for $\mathrm{d}s$ yields

$$\mathrm{d}f + \frac{\partial f}{\partial t} \, \mathrm{d}t + \frac{\partial f}{\partial s}(\mu_t \, \mathrm{d}t + \sigma \, \mathrm{d}W_t) + \frac{\partial^2 f}{\partial s^2} \left(\mu_t^2 \, \mathrm{d}t^2 + 2\mu_t \sigma_t \, \mathrm{d}t \, \mathrm{d}W_t + \sigma^2 \, \mathrm{d}W_t{}^2 \right) + \cdots$$

As $\mathrm{d}t$ approaches 0, $\mathrm{d}t^2$ and $\mathrm{d}t\,\mathrm{d}W_t$ tend to zero faster than $\mathrm{d}W_t^2$. Substituting 0 for $\mathrm{d}t^2$ and $\mathrm{d}t\,\mathrm{d}W_t$ and $\mathrm{d}t$ for $\mathrm{d}W_t^2$ yields

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial s^2}\right)\mathrm{d}t + \sigma \frac{\partial f}{\partial s}\,\mathrm{d}W_t$$

This is itself an Itô process Itô's lemma states that this is true of any Itô process S_t and any twice-differentiable function $f(t, s)$.
Let $f(S_t) = \ln S_t$. Applying Itô's lemma,

$$\begin{aligned}
\mathrm{d}f &= f'(S_t)\,\mathrm{d}S_t + \frac{1}{2}f''(S_t)(\mathrm{d}S_t)^2 \\
&= \frac{1}{S_t}\,\mathrm{d}S_t - \frac{1}{2S_2^2}\left(S_t^2\sigma^2\,\mathrm{d}t\right) \\
&= \frac{1}{S_t}(\sigma S_1\,\mathrm{d}W_t + \mu S_t\,\mathrm{d}t) - \frac{\sigma^2}{2}\,\mathrm{d}t \\
&= \sigma\,\mathrm{d}W_t + \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t
\end{aligned}$$

Integrating the separable DE,

$$f(t) = \ln S_t = \ln S_0 + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t$$

This can finally be exponentiated, yielding S_t:

$$S_t = S_0\,\mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$