9.3

Integral Test

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^2+1}\right]$$

(is always positive, continuous, and decreases as n grows)

$$\int_1^{\infty}\left[\frac{1}{x^2=1}\right]dx = \lim_{a\to\infty}[\arctan x]_1^a = \lim_{a\to\infty}[\arctan a - \arctan 1] = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4} \therefore \sum_{n=1}^{\infty}\left[\frac{1}{n^2+1}\right]$$

p-series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \text{ diverges}$$

($p = \frac{1}{2} \ \le 1 \therefore$ diverges)

$$\sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges}$$

($p = 1 \le 1 \therefore$ diverges)

9.4 Comparison Tests

Direct Comparison Test

$$\sum_{n=1}^{\infty}\left[\frac{1}{2+3^n}\right]$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{3^n}\right] = \sum_{n=1}^{\infty}\left[\frac{1}{3}\right]^n$$

(converges)

$$\frac{1}{2+3^n} \le \frac{1}{3^n}$$

(is always true $\wedge$ larger series diverges $\therefore$ original converges)

$$\sum_{n=1}^{\infty}\left[\frac{1}{10+\sqrt{n}}\right]$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{n}}\right]$$

(diverges)

$$\frac{1}{\sqrt{n}} \le \frac{1}{10+\sqrt{n}}$$

(false)

$$\sum_{n=1}^{\infty}\left[\frac{1}{n}\right]$$

(diverges)

$$\frac{1}{n} \le \frac{1}{10+\sqrt{n}}$$

n	1	9	16	25
$\frac{1}{n}$	1	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$
$\frac{1}{10+\sqrt{n}}$	$\frac{1}{11}$	$\frac{1}{13}$	$\frac{1}{14}$	$\frac{1}{15}$
$\frac{1}{n} \le \frac{1}{10+\sqrt{n}}$	False	False	True	True

$$\frac{1}{n} \le \frac{1}{10+\sqrt{n}} \text{ as } n \text{ grows larger} \wedge \frac{1}{n} \text{ diverges} \therefore \frac{1}{10+\sqrt{n}} \text{ diverges}$$

9.5

Alternating Series Test

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left[\frac{n}{(-2)^{n-1}}\right] &= \sum_{n=1}^{\infty}\left[\frac{n}{(-1\times 2)^{n-1}}\right] \\
&= \sum_{n=1}^{\infty}\left[\frac{1}{(-1)^{n-1}}\times\frac{n}{2^{n-1}}\right] \\
\lim_{n\to\infty}\left[\frac{n}{2^{n-1}}\right] &= \frac{\text{slow}}{\text{fast}} = 0 \\
a_{n+1} &\le a_n \\
\frac{n+1}{2^n} &\le \frac{n}{2^{n-1}} && \text{(larger denominator} \therefore \text{true} \therefore \text{converges)}
\end{aligned}
$$

9.6

Ratio Test

$$
\sum_{n=1}^{\infty}\left[\frac{2^n}{n!}\right]
$$

$$
\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\times\frac{n!}{2^n}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0 < 1 \therefore \text{converges}
$$

Factorials

$$
\begin{aligned}
(n+1)! &= n!(n+1) \\
(3n+4)! &= (3n)!(3n+4)(3n+3)(3n+2)(3n+1) \\
(an+b)! &= (an)!(an+b)(an+b-1)(an+b-2)\cdots = (an)!\prod_{i=0}^{b-1}(an+b-i) = (an)!\prod_{i=1}^{b}(an+i) \\
(0+1)! &= 0!(0+1) \\
1! &= 0!(1) \\
1 &= 0!
\end{aligned}
$$

Root Test

$$
\sum_{n=1}^{\infty}\left[\frac{e^{2n}}{n^n}\right]
$$

$$
\lim_{n\to\infty}\left(\frac{e^{2n}}{n^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{e^2}{n}\right) = 0 < 1 \therefore \text{converges}
$$

9.7 Power Series

$$\sum_{n=1}^{\infty}\left[\frac{(x-2)^n}{n}\right]$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n+1}\times\frac{n}{(x-2)^n}\right| < 1$$

$$\lim_{n\to\infty}|(x-2)\times 1| < 1$$

$$|x-2| < 1$$

$$x-2 < 1 \qquad\qquad x-2 > -1$$

$$x < 3 \qquad\qquad x > 1$$

$$1 < x < 3$$

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}\right]$$

9.8 Taylor and Maclaurin Polynomials

$f(x) = \sqrt{x+1}$

$f(x) = \sqrt{x+1}$	$f'(x) = \frac{1}{2}(x+1)^{-1/2}$	$f''(x) = \frac{-1}{4}(x+1)^{-3/2}$	$f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2}$	$f^{(4)}(x) = \frac{-15}{16}(x+1)^{-7/2}$
$f(0) = 1$	$f'(1) = \frac{1}{2}$	$f''(0) = \frac{-1}{4}$	$f^{(3)}(0) = \frac{3}{8}$	$f^{(4)}(0) = \frac{-15}{16}$

$$P_4 = 1 + \frac{1}{2}x - \frac{\frac{1}{4}x^2}{2!} + \frac{\frac{3}{8}x^3}{3!} - \frac{\frac{15}{16}x^4}{4!} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$$

$$y = \ln(2+x)$$

$f(x) = \ln(2+x)$	$\ln(1) = 0$
$f'(x) = (2+x)^{-1}$	$(1)^{-1} = 1$
$f''(x) = -1(2+x)^{-2}$	$-1(-1)^2 = -1$
$f^{(3)}(x) = 2(2+x)^{-3}$	$2(-1)^{-3} = 2$
$f^{(4)}(x) = -6(2+x)^{-4}$	$-6(-1)^{-4} = -6$

$$\begin{aligned}
P_4 &= 0 + (1)(x+1) + \frac{-1(x+1)^2}{2!} + \frac{2(x+1)^3}{3!} + frac{-6}(x+1)^4 4! \\
&= x + 1 - \frac{(x+1)^2}{2} + \frac{x+1}{3} - \frac{(x+1)^4}{4} \\
&= \sum_{i=1}^{4}\left[\frac{(-1)^{n+1}(x+1)^n}{n}\right] \\
y &= \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}(x+1)^n}{n}\right]
\end{aligned}$$

9.9 Manipulating Known Maclaurin Polynomials

$$\frac{x^2}{1+x^2} = x^2 \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$$

$$\tan x = \frac{\sin x}{\cos x} \approx \frac{P_2(\sin x)}{P_2 \cos(x)} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{1 - \frac{x^2}{2!} + \frac{x^4}{4!}} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120}}{1 - \frac{x^2}{2} + \frac{x^4}{24}} = x$$

$$P_3(\arctan x) = P_3\left(\int \left[\frac{1}{1+x^2}\right] dx\right) = \int \left[1 - x^2 + x^4 - x^6\right] = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$P_2\left(e^{-x^2} \arctan x\right) = \left(1 - x^2 + \frac{x^4}{2}\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - x^3 + \frac{x^5}{3} - \frac{x^7}{5} + \frac{x^5}{3} - \frac{x^7}{6} + \frac{x^9}{10} = x - \frac{4x^3}{3} + \frac{31x^5}{30}$$

9.10 Error

Actual Error

Find the error when using the first 5 terms of the maclaurin polynomial to find $f(0.2)$.

$$f(x) = \frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4 = P_4(x)$$

$$f(0.2) - P_4(0.2) \approx 1.25 - 1.2496 = 0.0004$$

Alternating Series Error

Approximate the sum of the series using the first 6 terms and find the error bound.

$$\sum_{n=1}^{\infty} \left[(-1)^{n+1}\left(\frac{1}{n!}\right)\right]$$

$$\lim_{n\to\infty} \left[\frac{1}{n!}\right] = 0$$

$$\frac{1}{(n+1)!} \leq \frac{1}{n!}$$

$$\sum_{n=1}^{6} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}$$

$$\sum_{n=1}^{\infty} \left[\frac{(1)^{n+1}}{n!}\right] = 1-$$

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{2n^3}\right]$$

$$\lim_{n\to\infty} \left[\frac{1}{2n^3}\right] = 0$$

$$\frac{1}{(2n+1)^3} \leq \frac{1}{2n^3}$$