Volume: Disk/Washer

Sources

Calculus: Early Transcendentals 9$^{\text{th}}$ Edition

1. 6.2 Exercise 12
2. 6.2 Exercise 15
3. 6.2 Exercise 1
4. 6.2 Exercise 17
5. 6.2 Exercise 24

AP Calculus Exams

6. 2021 AB FRQ 3(c)

Problems

If no instructions are given, evaluate the volume of the solid generated by revolving the region bounded by the given equations about the specified line using the disk/washer method.

1.
$$\left.\begin{array}{ll} y=0 & y=\dfrac{1}{x} \\ x=1 & x=4 \end{array}\right| y=0$$

2.
$$\left.\begin{array}{ll} y=\dfrac{x^2}{4} & y=9 \\ x=0 & \end{array}\right| x=0$$

3.
$$\left.\begin{array}{ll} y=0 & y=x^2+5 \\ x=0 & x=3 \end{array}\right| y=0$$

4.
$$\left.\begin{array}{l} y=x^2 \\ y=2x \end{array}\right| x=0$$

5.
$$\left.\begin{array}{ll} y=\sin x & y=\cos x \\ x\geq 0 & x\leq \dfrac{\pi}{4} \end{array}\right| y=-1$$

6.
$$f(x)=cx\sqrt{4-x^2}$$

The solid of revolution generated by rotating the area bounded by f and the x-axis in the first quadrant about the x-axis is equal to 2π. Solve for c, given that it is a positive constant.

7.
$$x=(y-k)^2 \quad y=x+k \mid k\leq 0 \text{ or } k\geq 1 \,\Big|\, x=k$$

8.
$$\left.\begin{array}{ll} \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \mid a,b\neq 0 & x=\left|\dfrac{ay}{b}\right| \\ x=a^2 & x=\sqrt{a^2+b^2} \end{array}\right| x=0$$

9.
$$\frac{x^2}{a^2}-\frac{y^4}{b^2} \mid a,b\neq 0 \quad y=\Big| x=0$$

10.
$$\left.\begin{array}{ll} x^2+y^2=r^2 & \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \mid a,b>r>0 \\ \text{(a) } y=0 & \end{array}\right| \begin{array}{l} \text{(a) } y=0 \\ \text{(b) } y=b \end{array}$$

Solutions

1.

$$V = \pi \int_1^4 \left(\frac{1}{x}\right)^2 \mathrm{d}x = \pi \left[-\frac{1}{x}\right]_1^4 = \pi \left[-\frac{1}{4} - \left(-\frac{1}{1}\right)\right] = \frac{3\pi}{4}$$

2.

$$\begin{aligned} y &= \frac{x^2}{4} \implies x = 2\sqrt{y} \\ y_1 &= 2\sqrt{0} = 0 \\ V &= \pi \int_0^9 (2\sqrt{y})^2 \,\mathrm{d}y = \pi \left[2y^2\right]_0^9 = 2(81)\pi = 162\pi \end{aligned}$$

3.

$$\begin{aligned} V &= \pi \int_0^3 \left(x^2+5\right)^2 \mathrm{d}x = \pi \int_0^3 \left[x^4 + 10x^2 + 25\right] \mathrm{d}x = \pi \left[\frac{x^5}{5} + \frac{10x^3}{3} + 25x\right]_0^3 \\ &= \pi \left[\frac{3^5}{5} + \frac{10(3)^3}{3} + 25(3) - (0)\right] = \pi \left[\frac{243}{5} + 90 + 75\right] = \frac{\pi(243+825)}{5} = \frac{1068\pi}{5} \end{aligned}$$

4.

$$\begin{aligned} y = x^2 \implies x &= \sqrt{y} \qquad y = 2x \implies x = \frac{y}{2} \\ \sqrt{y} &= \frac{y}{2} \implies 4y = y^2 \implies 0 = y(y-4) \implies y_1 = 0, y_2 = 4 \\ V &= \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right] \mathrm{d}y = \pi \int_0^4 \left[y - \frac{y^2}{4}\right] \mathrm{d}y = \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 \\ &= \pi \left[\frac{y^2(6-y)}{12}\right]_0^4 = \pi \left[\frac{4^2(6-4)}{12} - (0)\right] = \pi \left[\frac{16(2)}{12}\right] = \frac{8\pi}{3} \end{aligned}$$

5.

$$\begin{aligned} \sin x = \cos x &\implies x = \frac{\pi}{4} \\ V &= \pi \int_0^{\pi/4} \left[(\cos x + 1)^2 - (\sin x + 1)^2\right] \mathrm{d}x \\ &= \pi \int_0^{\pi/4} \left[\cos^2 x + 2\cos x + 1 - \sin^2 x - 2\sin x - 1\right] \mathrm{d}x \\ &= \pi \int_0^{\pi/4} \left[\cos(2x) + 2\cos x - 2\sin x\right] \mathrm{d}x = \pi \left[\frac{\sin(2x)}{2} + 2\sin x + 2\cos x\right]_0^{\pi/4} \\ &= \pi \left[\frac{1}{2} + \sqrt{2} + \sqrt{2} - (0+0+2)\right] = \frac{(4\sqrt{2}-3)\pi}{2} \end{aligned}$$

6.

$$0 = cx\sqrt{4-x^2} \implies \begin{cases} x = 0 \\ \sqrt{4-x^2} = 0 \end{cases} \implies 4 - x^2 = 0 \implies 4 = x^2 \implies x = 2$$

$$\begin{aligned}
2\pi &= \pi \int_0^2 \left(cx\sqrt{4-x^2}\right)^2 \mathrm{d}x = \pi \int_0^2 \left[c^2x^2(4-x^2)\right] \mathrm{d}x = \pi \int_0^4 \left[4c^2x^2 - c^2x^4\right] \mathrm{d}x \\
&= \pi \left[\frac{4c^2x^3}{3} - \frac{c^2x^5}{5}\right]_0^2 = \pi \left[\frac{4c^2(2)^3}{3} - \frac{c^2(2)^5}{5} - (0)\right] = \pi \left[\frac{32c^2}{3} - \frac{32c^2}{5}\right] \\
2 &= 32c^2\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{64c^2}{15} \\
c &= \sqrt{\frac{30}{64}} = \sqrt{\frac{15}{32}}
\end{aligned}$$

7.

$$y = x + k \implies x = y - k$$

For both equations, k only controls a vertical shift. As this shift is the same for both equations, the area between the curves is constant regardless of the value of k. The volume when revolved about the same axis will therefore also be the same.

$$y = y^2 \implies 0 = y(y-1) \implies y_1 = 0, y_2 = 1 \qquad (k = 0)$$

$$\begin{aligned}
V &= \pi \int_0^1 \left[(y-k)^2 - (y^2-k)^2\right] \mathrm{d}y = \pi \int_0^1 \left[y^2 - 2ky + k^2 - y^4 + 2ky^2 - k^2\right] \mathrm{d}y \\
&= \pi \int_0^1 \left[-y^4 + y^2(2k+1) - 2ky\right] \mathrm{d}y = \pi \left[-\frac{y^5}{5} + \frac{y^3(2k+1)}{3} - ky^2\right]_0^1 \\
&= \pi \left[\frac{-3y^5 + 5y^3(2k+1) - 15ky^2}{15}\right]_0^1 = \pi \left[\frac{-3 + 5(2k+1) - 15k}{15} - (0)\right] \\
&= \frac{\pi(-3 + 10k + 5 - 15k)}{15} = \frac{\pi(2-5k)}{15}
\end{aligned}$$

8.

$$\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \implies x = a\sqrt{1 + \frac{y^2}{b^2}} = \frac{a\sqrt{b^2+y^2}}{b} \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \implies y = \frac{b\sqrt{x^2-a^2}}{a} \\
y_1 &= \frac{b\sqrt{a^4-a^2}}{a} = b\sqrt{a^2-1} \qquad y_2 = \frac{b\sqrt{a^2+b^2-a^2}}{a} = \frac{b^2}{a} \\
V &= 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} \left[\left(\frac{a\sqrt{b^2+y^2}}{b}\right)^2 - \left(\frac{ay}{b}\right)^2\right] \mathrm{d}y \right| = 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} \left[\frac{a^2(b^2+y^2-y^2)}{b^2}\right] \mathrm{d}y \right| \\
&= 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} \left[a^2\right] \mathrm{d}y \right| = 2\pi \left| \left[a^2y\right]_{b\sqrt{a^2-1}}^{b^2/a} \right| = 2\pi \left| \frac{a^2b^2}{a} - \left(a^2b\sqrt{a^2-1}\right) \right| \\
&= \left| 2\pi ab\left(b - a\sqrt{a^2-1}\right) \right|
\end{aligned}$$

9.

$$V = 2\pi \int_0^\infty \left[\frac{a\sqrt{b^2+y^2}}{b} - \frac{ay}{b}\right] \mathrm{d}y = 2\pi \int_0^\infty \left[\frac{a\left(\sqrt{b^2+y^2}-y\right)}{b}\right] \mathrm{d}y$$

$$I(\alpha) = \int \left[\sqrt{\alpha^2+y^2}\right] \mathrm{d}y$$

$$y = \alpha\tan\theta \implies \mathrm{d}y = \alpha\sec^2\theta \mathrm{d}\theta$$

$$I(\alpha) = \int \left[\alpha\sec^2\theta\sqrt{\alpha^2+\alpha^2\tan^2\theta}\right] \mathrm{d}\theta = \int \left[\alpha^2\sec^3\theta\right] \mathrm{d}\theta$$

$$u = \sec\theta \implies \mathrm{d}u = \sec\theta\tan\theta \qquad \mathrm{d}v = \sec^2\theta \implies v = \tan\theta$$

$$\int \left[\sec^3\theta\right] \mathrm{d}\theta = uv - \int v\,\mathrm{d}u = \sec\theta\tan\theta - \int \left[\sec\theta\tan^2\right] \mathrm{d}\theta$$

$$\int \left[\sec\theta\tan^2\theta\right] \mathrm{d}\theta = \int \left[\sec\theta\left(\sec^2\theta - 1\right)\right] \mathrm{d}\theta = \int \left[\sec^3\theta - \sec\theta\right]$$

$$= \int \left[\sec^3\theta\right] \mathrm{d}\theta - \ln|\sec\theta+\tan\theta|$$

$$\int \left[\sec^3\theta\right] \mathrm{d}\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| - \int \left[\sec^3\theta\right] \mathrm{d}\theta$$

$$2\int \left[\sec^3\theta\right] \mathrm{d}\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|$$

$$\int \left[\sec^3\theta\right] \mathrm{d}\theta = \frac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|}{2} + C$$

$$I(\alpha) = \alpha^2\int \left[\sec^3\theta\right] \mathrm{d}\theta = \frac{\alpha^2(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)}{2}$$

$$\tan\theta = \frac{y}{\alpha} \qquad \sec\theta = \frac{\sqrt{\alpha^2+y^2}}{\alpha}$$

$$I(\alpha) = \frac{\alpha^2}{2}\left(\frac{y\sqrt{\alpha^2+y^2}}{\alpha^2} + \ln\left|\frac{y+\sqrt{\alpha^2+y^2}}{\alpha}\right|\right)$$

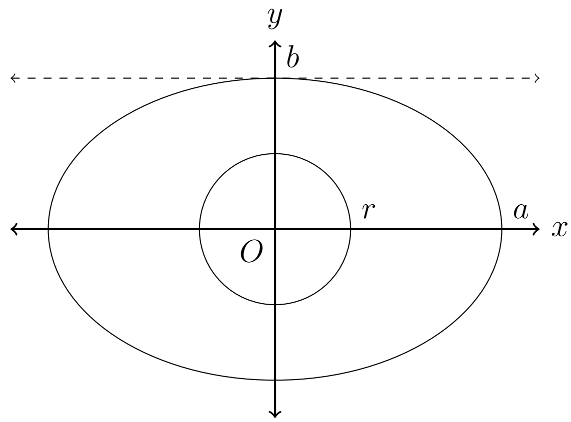
$$= \frac{1}{2}\left(y\sqrt{\alpha^2+y^2} + \alpha^2\ln\left|\frac{y+\sqrt{\alpha^2+y^2}}{\alpha}\right|\right) + C$$

$$V = \frac{2a\pi}{b}\left[I(b) - \frac{y^2}{2}\right]_0^\infty = \frac{2a\pi}{b}\left[\frac{1}{2}\left(y\sqrt{b^2+y^2} + b^2\ln\left|\frac{y+\sqrt{b^2+y^2}}{b}\right| - y^2\right)\right]_0^\infty$$

$$= \frac{a\pi}{b}\lim_{c\to\infty}\left[c\sqrt{b^2+c^2} + b^2\ln\left|\frac{c+\sqrt{b^2+c^2}}{b}\right| - c^2 - (0)\right] \implies \infty + \infty - \infty$$

$$= \frac{a\pi}{b}\lim_{c\to\infty}\left[\frac{\sqrt{b^2+c^2}-c}{1/c}\right]$$

10.



This graph assumes $a > b$, which need not be the case.

$$x^2 + y^2 = r^2 \implies y = \sqrt{r^2 - x^2} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b\sqrt{a^2 - x^2}}{a}$$

(a)

$$\begin{aligned}
V_1 &= \pi \int_0^r \left[\left(\frac{b\sqrt{a^2 - x^2}}{a}\right)^2 - \left(\sqrt{r^2 - x^2}\right)^2\right] \mathrm{d}x = \pi \int_0^r \left[\frac{b^2(a^2 - x^2)}{a^2} - (r^2 - x^2)\right] \mathrm{d}x \\
&= \pi \int_0^r \left[\frac{a^2(b^2 - r^2) + x^2(a^2 - b^2)}{a^2}\right] \mathrm{d}x = \pi \left[\frac{a^2x(b^2 - r^2)}{a^2} + \frac{x^3(a^2 - b^2)}{3a^2}\right]_0^4 \\
&= \pi \left[\frac{3a^2x(b^2 - r^2) + x^3(a^2 - b^2)}{3a^2}\right]_0^r = \pi \left[\frac{3a^2r(b^2 - r^2) + r^3(a^2 - b^2)}{3a^2}\right] \\
&= \pi \left[\frac{3a^2b^2r - 3a^2r^3 + a^2r^3 - b^2r^3}{3a^2}\right] = \pi \left(\frac{3a^2b^2r - 2a^2r^3 - b^2r^3}{3a^2}\right) \\
V_2 &= \pi \int_r^a \left(\frac{b\sqrt{a^2 - x^2}}{a}\right)^2 \mathrm{d}x = \pi \int_r^a \left[\frac{b^2(a^2 - x^2)}{a^2}\right] \mathrm{d}x = \pi \int_r^a \left[\frac{a^2b^2 - b^2x^2}{a^2}\right] \mathrm{d}x \\
&= \pi \left[\frac{a^2b^2x}{a^2} - \frac{b^2x^3}{3a^2}\right]_r^a = \pi \left[\frac{3a^2b^2x - b^2x^3}{3a^2}\right]_r^a = \left[\frac{3a^3b^2 - b^2a^3}{3a^2} - \left(\frac{3a^2b^2r - b^2r^3}{3a^2}\right)\right] \\
&= \pi \left(\frac{2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2}\right) \\
V &= 2(V_1 + V_2) = 2\pi \left(\frac{3a^2b^2r - 2a^2r^3 - b^2r^3 + 2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2}\right) \\
&= 2\pi \left(\frac{2a^3b^2 - 2a^2r^3}{3a^2}\right) = 4\pi \left(\frac{ab^2 - r^3}{3}\right)
\end{aligned}$$

(b)

$$
\begin{aligned}
V_1 &= \pi \int_0^r \left[\left(\sqrt{r^2 - x^2} - b \right)^2 - \left(\frac{b\sqrt{a^2 - x^2}}{a} - b \right)^2 \right] \mathrm{d}x \\
&= \pi \int_0^r \left[(r^2 - x^2) - 2b\sqrt{r^2 - x^2} + b^2 - \left(\frac{b^2(a^2 - x^2)}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right) \right] \mathrm{d}x \\
&= \pi \int_0^r \left[r^2 - x^2 - 2b\sqrt{r^2 - x^2} + b^2 - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2 - x^2}}{a} - b^2 \right] \mathrm{d}x \\
&= \pi \int_0^r \left[r^2 - x^2 - 2b\sqrt{r^2 - x^2} - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2 - x^2}}{a} \right] \mathrm{d}x \\
&= \pi \left(\left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right]_0^r + 2b \int_0^r \left[\frac{b\sqrt{a^2 - x^2}}{a} - \sqrt{r^2 - x^2} \right] \mathrm{d}x \right) \\
I(\alpha) &= \int \left[\sqrt{\alpha^2 - x^2} \right] \mathrm{d}x \implies x = \alpha \sin\theta \implies \mathrm{d}x = \alpha \cos\theta \mathrm{d}\theta \\
&= \int \left[\cos\theta \sqrt{\alpha^2 - \alpha^2 \sin^2\theta} \right] \mathrm{d}\theta = \int \left[\alpha^2 \cos^2\theta \right] \mathrm{d}\theta = \alpha^2 \int \left[\frac{\cos(2\theta) + 1}{2} \right] \mathrm{d}\theta \\
&= \alpha^2 \left(\frac{\sin(2\theta)}{4} + \frac{\theta}{2} \right) + C = \alpha^2 \left(\frac{2\sin\theta\cos\theta}{4} + \frac{\theta}{2} \right) + C \\
&= \alpha^2 \left(\frac{\left(\frac{x}{\alpha}\right)\left(\frac{\sqrt{\alpha^2 - x^2}}{\alpha}\right)}{2} + \frac{\arcsin(x/\alpha)}{2} \right) + C = \frac{x\sqrt{\alpha^2 - x^2} + \alpha^2 \arcsin(x/\alpha)}{2} + C \\
V_1 &= \pi \left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - a \right) \frac{x^3}{3} + 2b \left(\frac{bI(a)}{a} - I(r) \right) \right]_0^r \\
&= \pi \Bigg[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \\
&\quad + 2b \left(\frac{bx\sqrt{a^2 - x^2} + a^2 b \arcsin(x/a)}{2a} - \frac{x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r)}{2} \right) \Bigg]_0^r \\
&= \pi \Bigg[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \\
&\quad + 2b \left(\frac{xb\sqrt{a^2 - x^2} + a^2 b \arcsin(x/a)}{2a} - \frac{x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r)}{2} \right) \Bigg]_0^r \\
&= \pi \Bigg[x \left(r^2 - b^2 + \frac{x^2}{3} \left(\frac{b^2}{a^2} - 1 \right) \right) \\
&\quad + \frac{b}{a} \left(x \left(b\sqrt{a^2 - x^2} - a\sqrt{r^2 - x^2} \right) + a^2 b \arcsin\left(\frac{x}{a} \right) - r^2 \arcsin\left(\frac{x}{r} \right) \right) \Bigg]_0^r \\
&= \pi \Bigg[r \left(r^2 - b^2 + \frac{r^2}{3} \left(\frac{b^2}{a^2} - 1 \right) \right) \\
&\quad + \frac{b}{a} \left(r \left(b\sqrt{a^2 - r^2} - 0 \right) + a^2 b \arcsin\left(\frac{r}{a} \right) - \frac{a\pi r^2}{2} \right) - (0) \Bigg] \\
&= \pi \left(r^3 - b^2 r + \frac{b^2 r^3}{3a^2} - \frac{r^3}{3} + \frac{b^2 r\sqrt{a^2 - r^2}}{a} + ab^2 \arcsin\left(\frac{r}{a} \right) - \frac{b\pi r^2}{2} \right) \\
&= \pi \left(\frac{2r^3(2a^2 + b^2) - 3a^2 b\pi r^2 + 6ab^2 r\left(\sqrt{a^2 - r^2} - a \right) + 6a^3 b^2 \arcsin(r/a)}{6a^2} \right)
\end{aligned}
$$

(b) (cont.)

$$
\begin{aligned}
V_2 &= \pi \int_r^a \left(\frac{b\sqrt{a^2 - x^2}}{a} - b \right)^2 \mathrm{d}x = \pi \int_r^a \left[\frac{b^2(a^2 - x^2)}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right] \mathrm{d}x \\
&= \pi \int_r^a \left[b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right] \mathrm{d}x = \pi \int_r^a \left[2b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} \right] \mathrm{d}x \\
&= \pi \left[2b^2x - \frac{b^2x^3}{3a^2} - \frac{2b^2 I(a)}{a} \right]_r^a = \pi \left[2b^2x - \frac{b^2x^3}{3a^2} - \frac{b^2x\sqrt{a^2 - x^2} + a^2b^2 \arcsin(x/a)}{a} \right]_r^a \\
&= \pi \left[2b^2a - \frac{b^2a^3}{3a^2} - \frac{0 + a^2b^2(\pi/2)}{a} \right. \\
&\qquad \left. - \left(2b^2r - \frac{b^2r^3}{3a^2} - \frac{b^2r\sqrt{a^2 - r^2} + a^2b^2 \arcsin(r/a)}{a} \right) \right] \\
&= \pi \left[2ab^2 - 2b^2r + \frac{b^2r^3 - a^3b^2}{3a^2} + \frac{b^2r\sqrt{a^2 - r^2} + a^2b^2 \arcsin(r/a) - 0.5a^2b^2\pi}{a} \right] \\
&= \frac{b^2\pi \left(6a^3 - 6a^2r + r^3 - a^3 + 3ar\sqrt{a^2 - r^2} + 3a^3 \arcsin(r/a) - 1.5a^3\pi \right)}{3a^2} \\
&= \frac{b^2\pi \left(2r^3 + 6ar \left(\sqrt{a^2 - r^2} - 2a \right) + a^3(10 + 6 \arcsin(r/a) - 3\pi) \right)}{6a^2} \\
V &= 4\left(V_1 + V_2 \right) \\
&= \frac{2\pi}{3a^2} \left(4r^3(2a^2 + b^2) - 3a^2b\pi r^2 + 6ab^2r \left(2\sqrt{a^2 - r^2} - 3a \right) \right. \\
&\qquad \left. + a^3b^2 \left(12 \arcsin\left(\frac{r}{a} \right) + 10 - 3\pi \right) \right)
\end{aligned}
$$

Indeterminate Exponents (Type 3)

Sources

Calculus: Early Transcendentals 9^{th} Edition

1. 4.4 Exercise 61
3. 4.4 Exercise 57
4. 4.4 Exercise 60
5. 4.4 Exercise 65
6. 4.4 Exercise 66

Problems

Evaluate the following limits.

1. $$\lim_{x\to 1^+} \left[x^{1/(1-x)}\right]]$$

2. $$\lim_{x\to\infty} (\ln x)^{1/x}$$

3. $$\lim_{x\to 0^+} \left[x^{\sqrt{x}}\right]$$

4. $$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}$$

5. $$\lim_{x\to 0^+} (4x+1)^{\cot x}$$

6. $$\lim_{x\to 0^+} (1-\cos x)^{\sin x}$$

7. $$\lim_{x\to\infty} \left(\sum_{n=0}^{\infty} \left[\frac{1}{x^n}\right]\right)^x$$

8. $$\lim_{n\to 1^+} \left(\int_n^{\infty} \left[\frac{1}{x^n}\right] \mathrm{d}x\right)^{n-1}$$

Solutions

1.

$$\begin{aligned}
L &= \lim_{x\to 1^+}\left[x^{1/(1-x)}\right] && \implies 1^\infty \\
\ln L &= \lim_{x\to 1^+}\left[\frac{\ln x}{1-x}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to 1^+}\left[-\frac{1/x}{1}\right] = -\frac{1/1}{1} = -1 \\
L &= \frac{1}{e}
\end{aligned}$$

2.

$$\begin{aligned}
L &= \lim_{x\to\infty}(\ln x)^{1/x} && \implies \infty^0 \\
\ln L &= \lim_{x\to\infty}\left[\frac{\ln x}{x}\right] && \implies \frac{\infty}{\infty} \\
&= \lim_{x\to\infty}\left[\frac{1/x}{1}\right] = 0 \\
L &= e^0 = 1
\end{aligned}$$

3.

$$\begin{aligned}
L &= \lim_{x\to 0^+}\left[x^{\sqrt{x}}\right] && \implies 0^0 \\
\ln L &= \lim_{x\to 0^+}\left[\sqrt{x}\ln x\right] && \implies 0\times(-\infty) \\
&= \lim_{x\to 0^+}\left[\frac{\ln x}{x^{-1/2}}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{x\to 0^+}\left[-\frac{1/x}{0.5x^{-3/2}}\right] = \lim_{x\to 0^+}\left[-2\sqrt{x}\right] = 0 \\
L &= e^0 = 1
\end{aligned}$$

4.

$$\begin{aligned}
L &= \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} && \implies 1^\infty \\
\ln L &= \lim_{x\to\infty}\left[bx\ln\left(1+\frac{a}{x}\right)\right] && \implies \infty\times 0 \\
&= \lim_{x\to\infty}\left[\frac{b\ln\left(1+\frac{a}{x}\right)}{x^{-1}}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to\infty}\left[\frac{\frac{-bax^{-2}}{1+\frac{a}{x}}}{-x^{-2}}\right] = \lim_{x\to\infty}\left[\frac{ab}{1+\frac{a}{x}}\right] = \frac{ab}{1+0} = ab \\
L &= e^{ab}
\end{aligned}$$

5.

$$
\begin{aligned}
L &= \lim_{x\to 0^+} (4x+1)^{\cot x} && \implies 1^\infty \\
\ln L &= \lim_{x\to 0^+} [(\cot x)\ln(4x+1)] && \implies \infty \times 0 \\
&= \lim_{x\to 0^+} \left[\frac{\ln(4x+1)}{\tan x}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to 0^+} \left[\frac{4/(x+1)}{\sec^2 x}\right] = \frac{4/1}{1} = 4 \\
L &= e^4
\end{aligned}
$$

6.

$$
\begin{aligned}
L &= \lim_{x\to 0^+} (1-\cos x)^{\sin x} && \implies 0^0 \\
\ln L &= \lim_{x\to 0^+} [(\sin x)\ln(1-\cos x)] && \implies 0 \times (-\infty) \\
&= \lim_{x\to 0^+} \left[\frac{\ln(1-\cos x)}{\csc x}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{x\to 0^+} \left[-\frac{\sin x/(1-\cos x)}{\csc x \cot x}\right] = \lim_{x\to 0^+} \left[-\frac{\sin^2 x \tan x}{1-\cos x}\right] && \implies -\frac{0}{0} \\
&= \lim_{x\to 0^+} \left[\frac{2\sin x\cos x\tan x + \sin^2 x\sec^2 x}{\sin x}\right] \\
&= \lim_{x\to 0^+} [2\cos x\tan x + \sin x\sec x] \\
&= 2(1)(0) + (0)(1) = 0 \\
L &= e^0 = 1
\end{aligned}
$$

7.

$$
\sum_{n=0}^{\infty} \left[\frac{1}{x^n}\right] = \frac{1}{1-1/x} = \frac{1}{(x-1)/x} = \frac{x}{x-1}
$$

$$
\begin{aligned}
L &= \lim_{x\to\infty} \left(\sum_{n=0}^{\infty}\left[\frac{1}{x^n}\right]\right)^x = \lim_{x\to\infty}\left(\frac{x}{x-1}\right)^x && \implies 1^\infty \\
\ln L &= \lim_{x\to\infty} \left[x\ln\left(\frac{x}{x-1}\right)\right] && \implies \infty \times 0 \\
&= \lim_{x\to\infty} \left[\frac{\ln(x/(x-1))}{1/x}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to\infty} \left[\frac{1}{-1/x^2} \times \frac{1}{x/(x-1)} \times \frac{1(x-1)-1(x)}{(x-1)^2}\right] \\
&= \lim_{x\to\infty} \left[-x^2 \times \frac{x-1}{x} \times \frac{x-1-x}{(x-1)^2}\right] = \lim_{x\to\infty}\left[\frac{-x^2(x-1)(-1)}{x(x-1)^2}\right] \\
&= \lim_{x\to\infty} \left[\frac{x}{x-1}\right] = 1 \\
L &= e^1 = e
\end{aligned}
$$

8.

$$\begin{aligned}
\int_n^\infty \left[\frac{1}{x^n}\right] \mathrm{d}x &= \left[-\frac{1}{(n-1)x^{n-1}}\right]_n^\infty && \mid n \neq 1 \\
&= \lim_{b\to\infty}\left[-\frac{1}{(n-1)b^{n-1}} - \left(-\frac{1}{(n-1)n^{n-1}}\right)\right] \\
&= \frac{1}{(n-1)n^{n-1}} \\
L &= \lim_{n\to 1^+}\left(\int_n^\infty \left[\frac{1}{x^n}\right] \mathrm{d}x\right)^{n-1} = \lim_{n\to 1^+}\left(\frac{1}{(n-1)n^{n-1}}\right)^{n-1} && \implies \infty^0 \\
\ln L &= \lim_{n\to 1^+}\left[(n-1)\ln\left(\frac{1}{(n-1)n^{n-1}}\right)\right] \\
&= \lim_{n\to 1^+}\left[(n-1)\left(\ln(1) - \ln\left((n-1)n^{n-1}\right)\right)\right] \\
&= \lim_{n\to 1^+}\left[(n-1)\ln\left((n-1)n^{n-1}\right)\right] && \implies 0\times(-\infty) \\
&= \lim_{n\to 1^+}\left[\frac{\ln\left((n-1)n^{n-1}\right)}{1/(n-1)}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{n\to 1^+}\left[\frac{\frac{1}{n-1} + \left(\ln n + \frac{n-1}{n}\right)}{-1/(n-1)^2}\right] \\
&= \lim_{n\to 1^+}[-(n-1)((1+n-1)+(n-1)\ln n] \\
&= \lim_{n\to 1^+}[(1-n)(n+(n-1)\ln n)] = (0)(1+0(0)) = 0 \\
L &= e^0 = 1
\end{aligned}$$