

15.5.13. (p. 1082)

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Find the area of the surface of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies within the cylinder $x^2 + y^2 = ax$ and above the xy -plane.

$$\begin{aligned}
 ax &= x^2 + y^2 \\
 ar \cos \theta &= r^2 \\
 r &= a \cos \theta \\
 D &= \{(r, \theta) \mid 0 \leq r \leq a \cos \theta, 0 \leq \theta \leq \pi\} \\
 x^2 + y^2 + z^2 &= a^2 \\
 z &= \sqrt{a^2 - x^2 - y^2} \\
 \frac{\partial z}{\partial x} &= \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}} \\
 A &= \iint_D \left[\sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \right] dA = \iint_D \left[\sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}} \right] dA \\
 &= \iint_D \left[\sqrt{\frac{a^2}{a^2 - x^2 - y^2}} \right] dA = \int_0^\pi \int_0^{a \cos \theta} \left[r \sqrt{\frac{a^2}{a^2 - r^2}} \right] dr d\theta \\
 u &= a^2 - r^2 \quad u_1 = a^2 - 0 = a^2 \quad u_2 = a^2 - a^2 \cos^2 \theta = a^2 \sin^2 \theta \\
 du &= -2r dr \\
 A &= -\frac{a}{2} \int_0^\pi \int_{a^2}^{a^2 \sin^2 \theta} \left[\sqrt{\frac{1}{u}} \right] du d\theta = \frac{a}{2} \int_\pi^0 [2\sqrt{u}]_{a^2}^{a^2 \sin^2 \theta} d\theta \\
 &= a \int_\pi^0 [a \sin \theta - a] d\theta = a[-a \cos \theta - a\theta]_\pi^0 = a[-a - 0 - a + a\pi] = a^2(\pi - 2)
 \end{aligned}$$