MATH 135 Assignments

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Contents

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Assignment 1

1. a) The smallest prime number can either be 2 or not 2, making “The smallest prime number is 2” a valid statement.

   b) The sum of $\cos^2\theta$ and $\sin^2\theta$ could be 1 or not 1, so “$\cos^2\theta + \sin^2\theta = 1$” is a valid statement.

   c) It is either possible for every integer to be of the form $2k$ or $2k+1$ or there exists at least one exception, so “Every integer x is of the form $2k$ or $2k+1$” is a valid statement.

   d) 0 can either be even or odd or it could not be either, making “The number 0 is neither even nor odd” a valid statement.

   e) A question is not true or false; therefore, “Is $3 > 2$ true?” is not a valid statement.

2. a) $\forall x \in \mathbb{Z}, x^2 > 0$ b) $\forall x \in \mathbb{R}, x^3 \in \mathbb{R}$

3. a) $\exists x \in Z, \forall y \in \mathbb{R}, x + y \geq 3\sqrt{2}$ b) $\forall a \in \mathbb{N}, \exists b \in \mathbb{Q}, \forall c \in \mathbb{Z}, a = b - c$

4. a) $P(2,4) = \exists y \in \mathbb{Z}, 2(2) + 4y = 4 \implies y = 0 \in \mathbb{Z} \implies$ true
   $P(2,5) = \exists y \in \mathbb{Z}, 2(2) + 4y = 5 \implies y = 0.25 \notin \mathbb{Z} \implies$ false

   b) There is no condition given for n, meaning that the truth value of $P(x,n)$ cannot be determined, so “$\exists x \in \mathbb{Z}, P(x,n)$” is an open sentence depending on n.

   c) As all variables are specified, “$\forall n \in \mathbb{Z}, \exists x \in \mathbb{Z}, P(x,n)$” is a mathematical statement. As the $2(x+2y)$ must be even given that x and y are integers, this statement is false for all odd values of n, meaning that the statement as a whole is false.

5. a)

$$\left(8^{k^2}\right)\left(4^k\right) = \left(2^{3k^2}\right)\left(2^{2k}\right) = 2^{3k^2+2k} = 2 \implies 3k^2 + 2k = 1 \implies 3k^2 + 2k - 1 = 0$$
$$\implies 0 = (3k-1)(k+1) \implies k = -1 \in \mathbb{Z} \implies \text{true}$$

b)

$$x^2 - x + \frac{1}{4} > 0 \implies \neg\exists x = \frac{1 \pm \sqrt{1-1}}{2} = \frac{1}{2} \in \mathbb{R} \implies \text{false}$$

c)

$$\forall x \in \{0,1,2,3\}, \forall y \in \{0,1,2,3\}, (x+y) \in \mathbb{Z}, (x^2+y^2) \in \mathbb{Z} \implies \frac{x+y}{x^2+y^2} \in \mathbb{Q} \implies \text{true}$$

d)

$$4^x + (\ln x)^2 \geq 2x\ln\left(x^2\right) = 4x\ln x$$

As 4^x grows faster than $x \ln x$, for all $x \in \mathbb{N}$ and $4 \geq 0$, the the statement is true.

e)

$$\begin{aligned} x + 2xy &= 4 \\ 1 + 2y &= \frac{2}{x} \\ y &= \frac{4}{x} - \frac{1}{2} \\ x \in \mathbb{Q} &\implies \frac{4}{x} - \frac{1}{2} \in \mathbb{Q} \implies \text{true} \end{aligned}$$

6.

a) $\forall x \in \{1,2,3\}, \forall y \in \{1,2,3\}, \dfrac{4680}{x^2+y^2} \in \mathbb{Z}$

b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^3 = 135$

c) $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x \leq y$

d) $\exists x \in \mathbb{Q}, \exists y \in \mathbb{Q}, x^3y^2 = 108$

Assignment 2

1. a) The hypothesis is "$xy < 0$".

   b) The conclusion is "$x > 0$ and $y < 0$".

   c) The converse is "If $x > 0$ and $y < 0$, then $xy > 0$".

   d) The contrapositive is "If $x \leq 0$ or $y \geq 0$, then $xy \leq 0$".

   e) The negation is "$xy < 0$ or $x \leq 0$ or $y \geq 0$".

   f) In order for the product of two numbers to be negative, one number must be positive and the other negative. If x is negative and y is positive, though, their product is still negative. Therefore, $\forall x, y \in \mathbb{R}, S(x, y)$ is false.

2. a) This fails to restrict the domain of x.

   b) This fails to consider the case $n = 0$.

3. a) The truth table for this is

P	Q	R	$P \implies Q$	$\neg R$	$P \wedge (\neg R)$	$(P \implies Q) \vee (P \wedge (\neg R)$
T	T	T	T	F	F	T
T	T	F	T	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	T	F	F	T
F	T	F	T	T	F	T
F	F	T	T	F	F	T
F	F	F	T	T	F	T

   The only cases where the statement is true are when P and Q are both true, P is true and R is false, or when P is false.

   b) The truth table these statements is

A	B	C	$A \implies B$	$(A \implies B) \implies C$	$C \vee A$	$B \implies C$	$(C \vee A) \wedge (B \implies C)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	F	F	T	F

   As the columns or $(A \implies B) \implies C)$ and $(C \vee A) \wedge (B \implies C)$ are identical, they are logically equivalent.

4.

$$\begin{aligned}
P \vee \neg\big((\neg Q) \vee R\big) &\equiv P \vee \big(Q \wedge (\neg R)\big) && \text{(DeM/def. of } \neg) \\
&\equiv (P \vee Q) \wedge \big(P \vee (\neg R)\big) && \text{(dist. law)} \\
&\equiv (R \implies P) \wedge (P \vee Q) && \text{(comm. law/def. of } \implies)
\end{aligned}$$

5. **Claim.** $\forall a \in \mathbb{Z}, 2 \nmid (a^3 - 6a^2 + 5a + 1)$.
Proof. Let $a \in \mathbb{Z}$. a must be either even or odd.
Case 1. $2 \mid a$. This means that for some $k \in \mathbb{Z}$, $a = 2k$. Substituting with this yields

$$\begin{aligned} a^3 - 6a^2 + 5a + 1 &= (2k)^3 - 6(2k)^2 + 5(2k) + 1 \\ &= 8k^3 - 24k^2 + 10k + 1 \\ &= 2(4k^3 - 12k^2 + 5k) + 1 \end{aligned}$$

As $4k^3 - 12k^2 + 5k \in \mathbb{Z}$, by the definition of divisibility $2 \nmid \big(a^3 - 6a^2 + 5a + 1\big)$.
Case 2. $2 \nmid a$. This means that for some $k \in \mathbb{Z}$, $a = 2k + 1$. Substituting yields

$$\begin{aligned} a^3 - 6a^2 + 5a + 1 &= (2k+1)^3 - 6(2k+1)^2 + 5(2k+1) + 1 \\ &= \big((2k)^3 + 3(2k)^2 + 3(2k) + 1\big) - 6\big((2k)^2 + 2(2k) + 1\big) + 10k + 5 + 1 \\ &= 8k^3 + 12k^2 + 6k + 1 - 24k^2 - 24k - 6 + 10k + 6 \\ &= 8k^3 - 12k^2 - 8k + 1 \\ &= 2\big(4k^3 - 6k^2 - 4k\big) + 1 \end{aligned}$$

As $4k^3 - 6k^2 - 4k \in \mathbb{Z}$, by the definition of divisibility $2 \nmid \big(a^3 - 6a^2 + 5a + 1\big)$.
Regardless of whether a is even or odd, $2 \nmid \big(a^3 - 6a^2 + 5a + 1\big)$. $\square$

6. **Claim.** $\forall x \in \mathbb{R}, 1 + 99\sin^2 x \geq 10\sin(2x)$.
Proof. Let $x \in \mathbb{R}$. Suppose that both sides of the inequality are equal for this x:

$$1 + 99\sin^2 x = 10\sin(2x)$$

The double angle formula and Pythagorean identity yield

$$\begin{aligned} \sin^2 x + \cos^2 x + 99\sin^2 x &= 20\sin x\cos x \\ 0 &= 100\sin^2 x - 20\sin x\cos x + \cos^2 x \end{aligned}$$

This formula may only have solutions if for some $x, y \leq 1$,

$$0 = 100x^2 - 20xy + y^2$$

Applying the quadratic formula yields

$$\begin{aligned} x &= \frac{20y \pm \sqrt{400y^2 - 400y^2}}{200} \\ &= \frac{y}{10} \end{aligned}$$

This means that the equality is only valid when

$$\begin{aligned} \sin x &= \frac{\cos x}{10} \\ x &= \arctan\left(\frac{1}{10}\right) \end{aligned}$$

Substituting this into the equality with the double angle formula applied,

$$\begin{aligned}
1 + 99\sin^2\left(\arctan\left(\frac{1}{10}\right)\right) &= 20\sin\left(\arctan\left(\frac{1}{10}\right)\right)\cos\left(\arctan\left(\frac{1}{10}\right)\right) \\
1 + \frac{99}{101} &= 20\frac{100}{101} \\
\frac{200}{101} &= \frac{200}{101}
\end{aligned}$$

Pythagorean theorem was used to obtain the values of $\sin x$ and $\cos x$. As this is the only solution in the first period (note that the period of $\sin(2x)$ is π, meaning that $x = \arctan(1/10)+\pi$ is in the second period), the sign of

$$y = 1 + 99\sin^2 x - 10\sin(2x)$$

must not change at any other point in that period. At $x = 0$,

$$y = 1 + 0 - 0 = 1$$

At $x = \pi$,

$$y = 1 + 0 - 0 = 1$$

This means that $y \geq 0$ for $x \in [0, \pi]$; that is,

$$\begin{aligned}
1 + 99\sin^2 x - 10\sin(2x) &\geq 0 \\
1 + 99\sin^2 x &\geq 10\sin(2x)
\end{aligned}$$

As sine is a periodic function, this inequality is true for all $x \in \mathbb{R}$. $\square$.

Assignment 4

1.
   a) If $m \geq n$, $(m-n)! = 0$, meaning that the denominator of $\binom{n}{m}$ would be 0.

   b) The bounds of summation were changed so that in the inductive step, the sums from $P(k)$ and $P(k+1)$ could be combined, enabling PI to be applied before changing the bounds again to prove $P(k+1)$.

   c) The upper bound of the summation for the case with $a = 0$ was $m = n$, making $a^{m-n} = 0^0$.

d) **Claim.**

$$\forall a \in \mathbb{Z}, \forall n \in \mathbb{N}, a \mid \big((2+a)^n - 2^n\big)$$

Proof. This is a proof by induction on n, where $P(x)$ is

$$\forall a \in \mathbb{Z}, a \mid \big((2+a)^n - 2^n\big)$$

Base Case: Let $n = 1$. Then

$$2 + a - 2 = a$$

By the definition of divisibility, for all $a \in \mathbb{Z}$, $a \mid a$, so $P(1)$ is true.
Inductive Step: Assume that for some $k \in \mathbb{N}$, $P(k)$ is true; that is,

$$\forall a \in \mathbb{Z}, a \mid \big((2+a)^k - 2^k\big)$$

By BT2 and the definition of divisibility, the expression becomes

$$\begin{aligned}(2+a)^k - 2^k &= \sum_{i=0}^{k}\left[\binom{k}{i}2^{k-i}a^i\right] - 2^k \\ &= xa\end{aligned}$$

for some $x \in \mathbb{Z}$.
The expression for $P(k+1)$ is

$$(2+a)^{k+1} - 2^{k+1} = \sum_{i=0}^{k+1}\left[\binom{k+1}{i}2^{k+1-i}a^i\right] - 2^{k+1}$$

by BT2. By Pascal's identity and the induction hypothesis,

$$\begin{aligned}(2+a)^{k+1} - 2^{k+1} &= \binom{k+1}{0}2^{k+1}a^0 + \sum_{i=1}^{k+1}\left[\left(\binom{k}{i-1} + \binom{k}{i}\right)2^{k+1-i}a^i\right] - 2^{k+1} \\ &= 2^{k+1} + a\sum_{i=0}^{k}\left[\binom{k}{i}2^{k-i}a^i\right] \\ &\quad + 2\sum_{i=0}^{k}\left[\binom{k}{i}2^{k-i}a^i\right] - \binom{k}{0}2^{k+1}a^0 + \binom{k}{k+1}2^0a^{k+1} - 2^{k+1} \\ &= a\big(xa + 2^k\big) + 2\big(xa + 2^k\big) - 2^{k+1} + 0 \\ &= a\big(xa + 2^k\big) + 2xa + 2^{k+1} - 2^{k+1} \\ &= a\big(xa + 2^k + 2x\big)\end{aligned}$$

As $xa + 2^k + 2x \in \mathbb{Z}$, $(2+a)^{k+1} - 2^{k+1}$ is by definition divisible by a, making $P(k+1)$ true. By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Alternate Proof. Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. By BT2,

$$\begin{aligned}
(2+a)^n - 2^n &= \sum_{i=0}^{n} \left[\binom{n}{i} 2^{n-i} a^i \right] - 2^n \\
&= 2^n + \sum_{i=1}^{n} \left[\binom{n}{i} 2^{n-i} a^i \right] - 2^n \\
&= a \sum_{i=1}^{n} \binom{n}{i} 2^{n-i} a^i
\end{aligned}$$

As $\sum\limits_{i=1}^{n} \binom{n}{i} 2^{n-i} a^i \in \mathbb{Z}$, $a \mid ((2+a)^n - 2^n)$.

2. **Claim.**

$$\forall n \in \mathbb{Z}, \big(4 \nmid \big(n^4 + 3\big)\big) \iff (n \mid 2)$$

Proof. Let $n \in \mathbb{Z}$.
Case 1. $n \nmid 2$.
This means that for some $k \in \mathbb{Z}$, $n = 2k+1$, so

$$\begin{aligned}
n^4 + 3 &= (2k+1)^4 + 3 \\
&= (2k)^4 + 4(2k)^3 + 6(2k)^2 + 4(2k) + 1 + 3 \\
&= 16k^2 + 24k^3 + 24k^2 + 8k + 4 \\
&= 4\big(4k^2 + 6k^3 + 6k^2 + 2k + 1\big)
\end{aligned}$$

$4k^2 + 6k^3 + 6k^2 + 2k + 1 \in \mathbb{Z}$, so by the definition of divisibility, $(n \nmid 2) \implies \big(4 \mid (n^4+3)\big)$.
Case 2. $n \mid 2$.
In this case, for some $k \in \mathbb{Z}$, $n = 2k$, so

$$\begin{aligned}
n^4 + 3 &= (2k)^4 + 3 \\
&= 16k^4 + 3 \\
&= 4(4k^4) + 3
\end{aligned}$$

As $4k^4 \in \mathbb{Z}$ and $4 \nmid 3$, $(n \mid 2) \implies \big(4 \nmid (n^4+3)\big)$.
This means that $\big(4 \nmid (n^4+3)\big) \iff (n \mid 2)$. $\square$

3. **Claim.** $\forall a \in \mathbb{N}, \forall n \in \mathbb{Z}, (a^2 - 10) \neq n^2$.
Proof. Let $a \in \mathbb{N}$, and assume that for some $n \in \mathbb{Z}$,

$$a^2 - 10 = n^2$$

This means that

$$a^2 - n^2 = 10$$

This implies that there are two perfect squares separated by distance 10. The sequence of the difference of the perfect squares is given by

$$\begin{aligned}
c_i &= (i+1)^2 - i^2 \\
&= i^2 + 2i + 1 - i^2 \\
&= 2i + 1
\end{aligned}$$

It is clear that this difference is always odd, meaning that it can never be equal to 10. There is therefore no $a \in \mathbb{N}$ that is 10 more than a perfect square. $\square$

4. **Claim.**

$$\forall (w, x, y, z \in \mathbb{Z}), \big((w \neq y) \wedge (wz - xy \neq 0)\big) \implies \exists! r \in \mathbb{Q}, \frac{wr + x}{yr + z} = 1$$

Proof. Let $w, x, y, z \in \mathbb{Z}$ with $w \neq y$ and $wz \neq xy$. Assume that for some $r \in \mathbb{Q}$,

$$\begin{aligned} \frac{wr + x}{yr + z} &= 1 \\ wr + x &= yr + z \\ wr - yr &= z - x \\ r &= \frac{z - x}{w - y} \end{aligned}$$

As $w \neq y$, the denominator is not 0. Substituting this back into the equation,

$$\begin{aligned} 1 &= \frac{wr + x}{yr + z} \\ &= \frac{w\frac{z-x}{w-y} + x}{y\frac{z-x}{w-y} + z} \\ &= \frac{w(z - x) + x(w - y)}{y(z - x) + z(w - y)} \\ &= \frac{wz - wx + wx - xy}{yz - xy + wz - yz} \\ &= \frac{wz - xy}{wz - xy} \\ &= 1 \end{aligned}$$

meaning that this solution is not extraneous. ($wz \neq xy$, meaning that the second to last expression is not indeterminate.) The expression for r is dependent on the values of the parameters alone, meaning that it is simply the ratio of two fixed differences. There is therefore only one such r that may exist. □

5. **Claim**

$$\forall n \in \mathbb{N}, \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n + 1)}{2}$$

Proof. This is a proof by induction on n, where $P(n)$ is the statement

$$\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n + 1)}{2}$$

Base Case. For $n = 1$,

$$\begin{aligned} \sum_{i=1}^{1} (-1)^i i^2 &= (-1)^1 1^2 \\ &= -1 \\ &= -\frac{2}{2} \\ &= \frac{(-1)^1 1(1 + 2)}{2} \end{aligned}$$

so $P(1)$ is true.
Inductive Step Assume that for some $k \in \mathbb{N}$, $P(k)$ is true; that is,

$$\sum_{i=1}^{k}(-1)^i i^2 = \frac{(-1)^k k(k+1)}{2}$$

Adding the next term to the sum yields

$$\begin{aligned}
\sum_{i=1}^{k+1}(-1)^i i^2 &= (-1)^{k+1}(k+1)^2 + \sum_{i=1}^{k}(-1)^i i^2 \\
&= (-1)^{k+1}(k+1)^2 + \frac{(-1)^k k(k+1)}{2} \\
&= \frac{2(-1)^{k+1}(k+1)^2 + (-1)^k k(k+1)}{2} \\
&= \frac{(-1)^{k+1}\big(2(k^2+2k+1) - (k^2+k)\big)}{2} \\
&= \frac{(-1)^{k+1}(2k^2+4k+2-k^2-k)}{2} \\
&= \frac{(-1)^{k+1}(k^2+3k+2)}{2} \\
&= \frac{(-1)^{k+1}(k+1)(k+2)}{2} \\
&= \frac{(-1)^{k+1}(k+1)\big((k+1)+1\big)}{2}
\end{aligned}$$

so $P(k+1)$ is also true.
By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

6. **Claim.**

$$\forall x, y \in \mathbb{R}, (0 < y < x) \implies \forall n \in \mathbb{N}, x^n - y^n \leq nx^{n-1}(x-y)$$

Proof. Let $x, y \in \mathbb{R}$ with $0 < y < x$. This is a proof by induction on n, where $P(n)$ is

$$x^n - y^n \leq nx^{n-1}(x-y)$$

Base Case. Let $n = 1$. Then

$$\begin{aligned}
x^1 - y^1 &\leq (1)x^{1-1}(x-y) \\
x - y &\leq x - y
\end{aligned}$$

so $P(1)$ is true.
Inductive Step. Assume that for some $k \in \mathbb{N}$, $P(k)$ is true; that is,

$$x^k - y^k \leq kx^{k-1}(x-y)$$

Multiplying by x on both sides and expanding

$$x^{k+1} - xy^k \leq kx^{k+1} - kx^k y$$

Adding $xy^k - y^{k+1}$ to both sides and recalling that $0 < y < x$,

$$\begin{aligned} x^{k+1} - y^{k+1} &\leq kx^{k+1} - kx^k y + xy^k - y^{k+1} \\ &= kx^{k+1} - kx^n y + xy^k \left(1 - \frac{y}{x}\right) \\ &< kx^{k+1} - kx^k y + x^{k+1} \left(1 - \frac{y}{x}\right) \\ &= kx^{k+1} - kx^k y + x^{k+1} - x^k y \\ &= (k+1)x^k (x - y) \end{aligned}$$

making $P(k+1)$ true as well.
By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$