9.3

Integral Test

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^2+1}\right] \qquad \text{(is always positive, continuous, and decreases as } n \text{ grows)}$$

$$\int_1^{\infty}\left[\frac{1}{x^2=1}\right]dx = \lim_{a\to\infty}[\arctan x]_1^a = \lim_{a\to\infty}[\arctan a - \arctan 1] = \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}\therefore \sum_{n=1}^{\infty}\left[\frac{1}{n^2+1}\right]$$

p-series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \text{ diverges} \qquad (p=\tfrac{1}{2} \le 1 \therefore \text{diverges})$$

$$\sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges} \qquad (p=1\le 1 \therefore \text{diverges})$$

9.4 Comparison Tests

Direct Comparison Test

$$\sum_{n=1}^{\infty}\left[\frac{1}{2+3^n}\right] \qquad \sum_{n=1}^{\infty}\left[\frac{1}{3^n}\right]=\sum_{n=1}^{\infty}\left[\frac{1}{3}\right]^n$$

(converges)

$$\frac{1}{2+3^n}\le\frac{1}{3^n} \qquad \text{(is always true} \wedge \text{larger series diverges} \therefore \text{original converges)}$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{10+\sqrt{n}}\right] \qquad \sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{n}}\right]$$

(diverges)

$$\frac{1}{\sqrt{n}}\le\frac{1}{10+\sqrt{n}} \qquad \text{(false)}$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{n}\right] \qquad \text{(diverges)}$$

$$\frac{1}{n}\le\frac{1}{10+\sqrt{n}}$$

n	1	9	16	25
$\frac{1}{n}$	1	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$
$\frac{1}{10+\sqrt{n}}$	$\frac{1}{11}$	$\frac{1}{13}$	$\frac{1}{14}$	$\frac{1}{15}$
$\frac{1}{n}\le\frac{1}{10+\sqrt{n}}$	False	False	True	True

$$\frac{1}{n}\le\frac{1}{10+\sqrt{n}} \text{ as } n \text{ grows larger} \wedge \frac{1}{n} \text{ diverges} \therefore \frac{1}{10+\sqrt{n}} \text{ diverges}$$

9.5

Alternating Series Test

$$\begin{aligned}
\sum_{n=1}^{\infty}\left[\frac{n}{(-2)^{n-1}}\right] &= \sum_{n=1}^{\infty}\left[\frac{n}{(-1\times 2)^{n-1}}\right] \\
&= \sum_{n=1}^{\infty}\left[\frac{1}{(-1)^{n-1}}\times\frac{n}{2^{n-1}}\right] \\
\lim_{n\to\infty}\left[\frac{n}{2^{n-1}}\right] &= \frac{\text{slow}}{\text{fast}} = 0 \\
a_{n+1} &\le a_n \\
\frac{n+1}{2^n} &\le \frac{n}{2^{n-1}} \qquad \text{(larger denominator $\therefore$ true $\therefore$ converges)}
\end{aligned}$$

9.6

Ratio Test

$$\sum_{n=1}^{\infty}\left[\frac{2^n}{n!}\right]$$

$$\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\times\frac{n!}{2^n}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0 < 1 \therefore \text{converges}$$

Factorials

$$\begin{aligned}
(n+1)! &= n!(n+1) \\
(3n+4)! &= (3n)!(3n+4)(3n+3)(3n+2)(3n+1) \\
(an+b)! &= (an)!(an+b)(an+b-1)(an+b-2)\cdots = (an)!\prod_{i=0}^{b-1}(an+b-i) = (an)!\prod_{i=1}^{b}(an+i) \\
(0+1)! &= 0!(0+1) \\
1! &= 0!(1) \\
1 &= 0!
\end{aligned}$$

Root Test

$$\sum_{n=1}^{\infty}\left[\frac{e^{2n}}{n^n}\right]$$

$$\lim_{n\to\infty}\left(\frac{e^{2n}}{n^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{e^2}{n}\right) = 0 < 1 \therefore \text{converges}$$

9.7 Power Series

$$\sum_{n=1}^{\infty}\left[\frac{(x-2)^n}{n}\right]$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n+1} \times \frac{n}{(x-2)^n}\right| < 1$$

$$\lim_{n\to\infty}|(x-2)\times 1| < 1$$

$$|x-2| < 1$$

$$x-2 < 1 \qquad\qquad x-2 > -1$$

$$x < 3 \qquad\qquad x > 1$$

$$1 < x < 3$$

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}\right]$$