

9.3

Integral Test

$$\sum_{n=1}^{\infty} \left[\frac{1}{n^2 + 1} \right] \quad (\text{is always positive, continuous, and decreases as } n \text{ grows})$$

$$\int_1^{\infty} \left[\frac{1}{x^2 + 1} \right] dx = \lim_{a \rightarrow \infty} [\arctan x]_1^a = \lim_{a \rightarrow \infty} [\arctan a - \arctan 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \therefore \sum_{n=1}^{\infty} \left[\frac{1}{n^2 + 1} \right]$$

p -series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges} \quad (p = \frac{1}{2} \leq 1 \therefore \text{diverges})$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \quad (p = 1 \leq 1 \therefore \text{diverges})$$

9.4 Comparison Tests

Direct Comparison Test

$$\sum_{n=1}^{\infty} \left[\frac{1}{2 + 3^n} \right] \quad \sum_{n=1}^{\infty} \left[\frac{1}{3^n} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{3} \right]^n$$

(converges)

$$\frac{1}{2 + 3^n} \leq \frac{1}{3^n} \quad (\text{is always true} \wedge \text{larger series diverges} \therefore \text{original converges})$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{10 + \sqrt{n}} \right] \quad \sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{n}} \right]$$

(diverges)

$$\frac{1}{\sqrt{n}} \leq \frac{1}{10 + \sqrt{n}} \quad (\text{false})$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} \right] \quad (\text{diverges})$$

n	1	9	16	25
$\frac{1}{n}$	1	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$
$\frac{1}{10 + \sqrt{n}}$	$\frac{1}{11}$	$\frac{1}{13}$	$\frac{1}{14}$	$\frac{1}{15}$
$\frac{1}{n} \leq \frac{1}{10 + \sqrt{n}}$	False	False	True	True

$$\frac{1}{n} \leq \frac{1}{10 + \sqrt{n}} \text{ as } n \text{ grows larger} \wedge \frac{1}{n} \text{ diverges} \therefore \frac{1}{10 + \sqrt{n}} \text{ diverges}$$

9.5

Alternating Series Test

$$\begin{aligned}
\sum_{n=1}^{\infty} \left[\frac{n}{(-2)^{n-1}} \right] &= \sum_{n=1}^{\infty} \left[\frac{n}{(-1 \times 2)^{n-1}} \right] \\
&= \sum_{n=1}^{\infty} \left[\frac{1}{(-1)^{n-1}} \times \frac{n}{2^{n-1}} \right] \\
\lim_{n \rightarrow \infty} \left[\frac{n}{2^{n-1}} \right] &= \frac{\text{slow}}{\text{fast}} = 0 \\
a_{n+1} &\leq a_n \\
\frac{n+1}{2^n} &\leq \frac{n}{2^{n-1}} \quad (\text{larger denominator } \therefore \text{true } \therefore \text{converges})
\end{aligned}$$

9.6

Ratio Test

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[\frac{2^n}{n!} \right] \\
\lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2}{n+1} \right| = 0 < 1 \therefore \text{converges}
\end{aligned}$$

Factorials

$$\begin{aligned}
(n+1)! &= n!(n+1) \\
(3n+4)! &= (3n)!(3n+4)(3n+3)(3n+2)(3n+1) \\
(an+b)! &= (an)!(an+b)(an+b-1)(an+b-2) \cdots = (an)! \prod_{i=0}^{b-1} (an+b-i) = (an)! \prod_{i=1}^b (an+i) \\
(0+1)! &= 0!(0+1) \\
1! &= 0!(1) \\
1 &= 0!
\end{aligned}$$

Root Test

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[\frac{e^{2n}}{n^n} \right] \\
\lim_{n \rightarrow \infty} \left(\frac{e^{2n}}{n^n} \right)^{1/n} &= \lim_{n \rightarrow \infty} \left(\frac{e^2}{n} \right) = 0 < 1 \therefore \text{converges}
\end{aligned}$$

9.7 Power Series

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \left[\frac{(x-2)^n}{n} \right] \\
 & \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \\
 & \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{n+1} \times \frac{n}{(x-2)^n} \right| < 1 \\
 & \lim_{n \rightarrow \infty} |(x-2) \times 1| < 1 \\
 & |x-2| < 1 \\
 & x-2 < 1 \\
 & x < 3 \\
 & 1 < x < 3 \\
 & \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n} \right]
 \end{aligned}
 \qquad
 \begin{aligned}
 & x-2 > -1 \\
 & x > 1
 \end{aligned}$$

9.8 Taylor and Maclaurin Polynomials

$$\begin{aligned}
 f(x) &= \sqrt{x+1} \\
 \begin{array}{c|c|c|c|c}
 f(x) = \sqrt{x+1} & f'(x) = \frac{1}{2}(x+1)^{-1/2} & f''(x) = \frac{-1}{4}(x+1)^{-3/2} & f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} & f^{(4)}(x) = \frac{-15}{16}(x+1)^{-7/2} \\
 \hline
 f(0) = 1 & f'(1) = \frac{1}{2} & f''(0) = \frac{-1}{4} & f^{(3)}(0) = \frac{3}{8} & f^{(4)}(0) = \frac{-15}{16}
 \end{array} \\
 P_4 &= 1 + \frac{1}{2}x - \frac{\frac{1}{4}x^2}{2!} + \frac{\frac{3}{8}x^3}{3!} - \frac{\frac{15}{16}x^4}{4!} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4
 \end{aligned}$$

$$y = \ln(2+x)$$

$$\begin{array}{l|l}
 f(x) = \ln(2+x) & \ln(1) = 0 \\
 f'(x) = (2+x)^{-1} & (1)^{-1} = 1 \\
 f''(x) = -1(2+x)^{-2} & -1(-1)^2 = -1 \\
 f^{(3)}(x) = 2(2+x)^{-3} & 2(-1)^{-3} = 2 \\
 f^{(4)}(x) = -6(2+x)^{-4} & -6(-1)^{-4} = -6
 \end{array}$$

$$\begin{aligned}
 P_4 &= 0 + (1)(x+1) + \frac{-1(x+1)^2}{2!} + \frac{2(x+1)^3}{3!} + \frac{-6(x+1)^4}{4!} \\
 &= x+1 - \frac{(x+1)^2}{2} + \frac{x+1}{3} - \frac{(x+1)^4}{4} \\
 &= \sum_{i=1}^4 \left[\frac{(-1)^{n+1}(x+1)^n}{n} \right] \\
 y &= \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}(x+1)^n}{n} \right]
 \end{aligned}$$

9.9 Manipulating Known Maclaurin Polynomials

$$\begin{aligned}\frac{x^2}{1+x^2} &= x^2 \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \\ \tan x &= \frac{\sin x}{\cos x} \approx \frac{P_2(\sin x)}{P_2 \cos(x)} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{1 - \frac{x^2}{2!} + \frac{x^4}{4!}} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120}}{1 - \frac{x^2}{2} + \frac{x^4}{24}} = x \\ P_3(\arctan x) &= P_3\left(\int \left[\frac{1}{1+x^2}\right] dx\right) = \int [1 - x^2 + x^4 - x^6] = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \\ P_2\left(e^{-x^2} \arctan x\right) &= \left(1 - x^2 + \frac{x^4}{2}\right) \left(x - \frac{x^3}{3} + \frac{x^5}{5}\right) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - x^3 + \frac{x^5}{3} - \frac{x^7}{5} + \frac{x^5}{3} - \frac{x^7}{6} + \frac{x^9}{10} = x - \frac{4x^3}{3} + \frac{31x^5}{30}\end{aligned}$$

9.10 Error

Actual Error

Find the error when using the first 5 terms of the maclaurin polynomial to find $f(0.2)$.

$$\begin{aligned}f(x) &= \frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4 = P_4(x) \\ f(0.2) - P_4(0.2) &\approx 1.25 - 1.2496 = 0.0004\end{aligned}$$

Alternating Series Error

Approximate the sum of the series using the first 6 terms and find the error bound.

$$\begin{aligned}\sum_{n=1}^{\infty} \left[(-1)^{n+1} \left(\frac{1}{n!} \right) \right] \\ \lim_{n \rightarrow \infty} \left[\frac{1}{n!} \right] &= 0 \\ \frac{1}{(n+1)!} &\leq \frac{1}{n!} \\ \sum_{n=1}^6 &= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} \\ \sum_{n=1}^{\infty} \left[\frac{(1)^{n+1}}{n!} \right] &= 1 - \\ \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{2n^3} \right] \\ \lim_{n \rightarrow \infty} \left[\frac{1}{2n^3} \right] &= 0 \\ \frac{1}{(2n+1)^3} &\leq \frac{1}{2n^3}\end{aligned}$$

Lagrange Error

$$e^x = \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} \right]$$

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$P_3(-0.2) = 1 - 0.2 + \frac{0.2^2}{2} - \frac{0.2^3}{3!} \approx -1.812$$

$$f^4(x) = e^x$$

$$\begin{aligned} \text{error} &= \frac{|f^{(4)}(z)|(-0.2 - 0)^4}{4!} \\ &= \frac{|e^0|(-0.2)^4}{4!} = \frac{(-0.2)^4}{4!} \approx 0.007 \end{aligned}$$