AP Calculus

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Chapter 1

Parametric, Polar, and Vector-Valued Functions

1.1 Parametric Equations

A **parametric equation** defines its independent and dependent variables in terms of the same parameter, often t.

A parametric graph should have arrows going through it showing the direction of t. It should not have arrows showing end behavior.

To graph a parametric function, t can be swept or the parameter can be eliminated by solving for it in terms of one of the variables.

1.1.1 Parametric Derivatives

The derivative of a parametric equation is still taken, unless otherwise specified, with respect to time.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{dt}}$$

The second derivative of a parametric equation is the time is the first derivative differentiated with respect to t divided by $\mathrm{d}x/\mathrm{d}t$.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

The n^{th} derivative of a parametric equation takes the following form:

1.1.2 Parametric Arc Length

Rectangular arc length is defined as such:

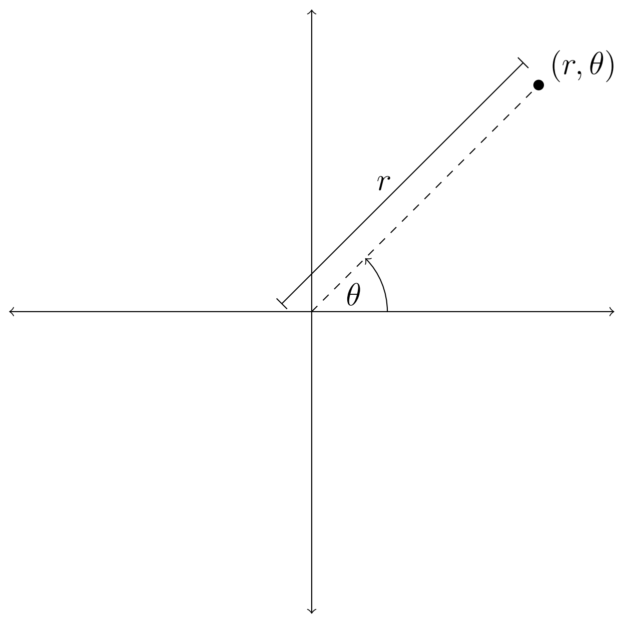
$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Parametric arc length is differentiated with respect to t:

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

1.2 Polar Equations

A **polar** function is written using **Circular coordinates** denote values as an ordered pair of its distance from the origin (the **radial line** and its angle relative to what would be the positive x-axis in the cartesian plane.



Polar coordinates can be written in infinite ways by adding or subtracting integer multiples of 2π.

$$(r, \theta) \equiv (r, \theta \pm 2z\pi) \forall z \in \mathbb{Z}$$

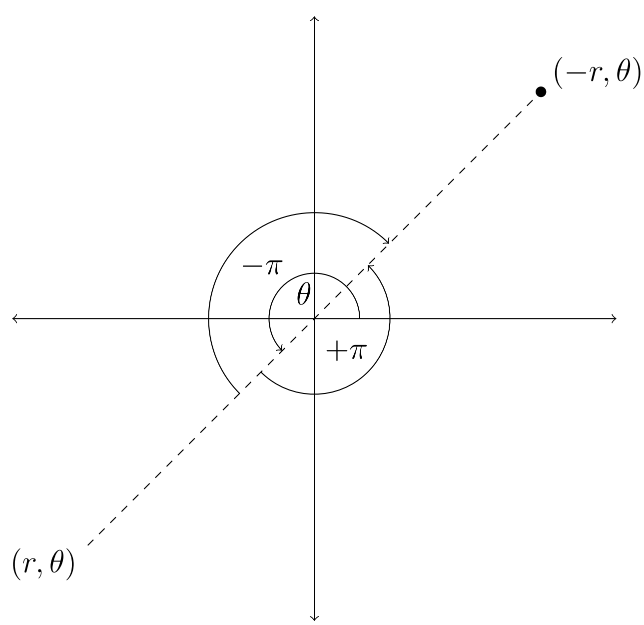
To convert from polar to cartesian coordinates, trigonometric functions can be used.

$$\overset{\text{polar}}{(r, \theta)} \implies \overset{\text{cartesian}}{(r\cos\theta, r\sin\theta)}$$

Inverse tangent and the Pythagorean theorem can be used perform the converse process.

$$\overset{\text{cartesian}}{(x, y)} \implies \overset{\text{polar}}{\left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right)}$$

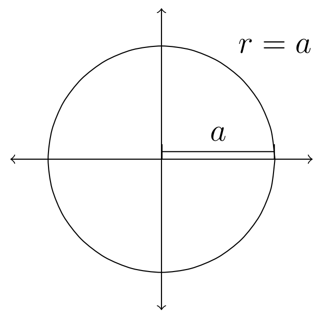
Radii can be negative, reflecting it over the origin. A negative radius is equivalent to an additional rotation of π.



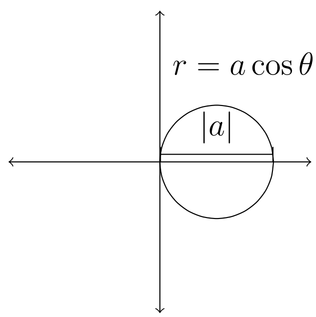
1.2.1 General Forms

Polar equations come in four general forms:

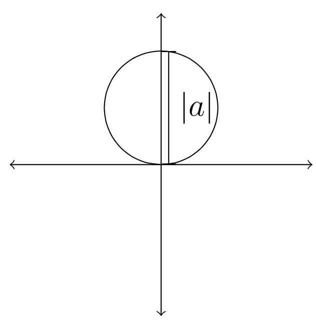
- There are three types of **circles**:
  - $r = a$ is a circle centered about the origin with radius $|a|$.



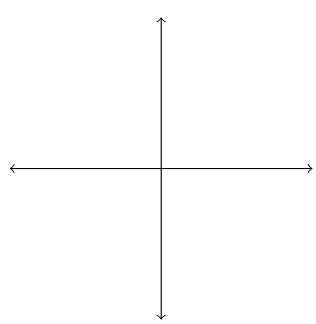
  - $r = a\cos\theta$ is a circle tangent to the origin on the horizontal axis with radius $\frac{|a|}{2}$.



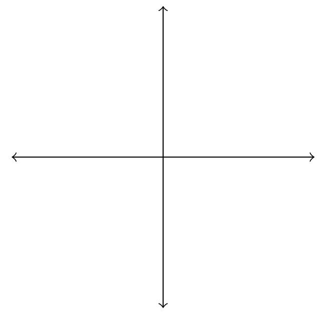
  - $r = a\sin\theta$ is a circle tangent to the origin on the vertical axis with radius $\frac{|a|}{2}$.



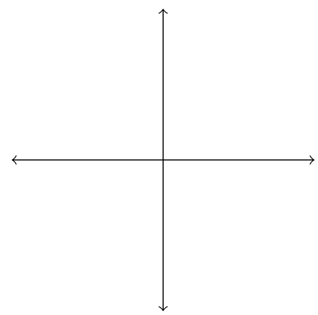
- **Limacons** take the form of $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$ $(a, b > 0)$, which are symmetrical about the vertical or horizontal axis respectively. There are four cases for them:
  - If $\frac{a}{b} < 1$, there is a loop passing through the origin and $(a + b, 0)$ in addition to another loop passing through the origin and $(b - a, 0)$.



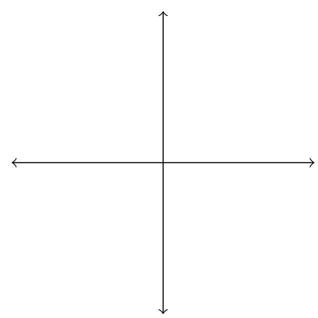
- If $\frac{a}{b} = 1$, the graph is a **cardioid**, passing through the origin, $\left(\pm a, \frac{\pi}{2}\right)$, and $(2a, 0)$.



- If $1 < \frac{a}{b} < 2$, the graph is **dimpled**, passing through $(a + b, 0)$, $\left(\pm a, \frac{\pi}{2}\right)$.

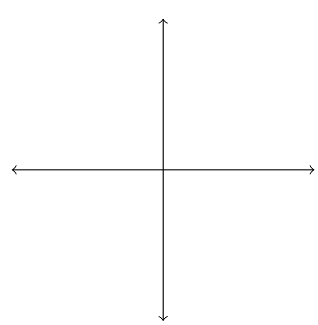


- If $\frac{a}{b} > 2$, the graph is **convex**.

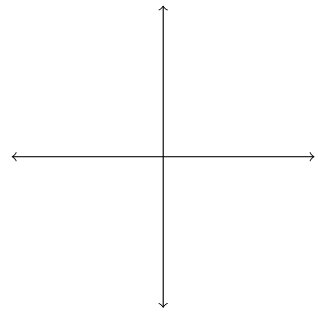


- **Rose curves** take the form of $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$, which are symmetrical about the vertical or horizontal axis respectively. The location of a petal can be found by equation the trigonometric term to 1 while the distance between from the origin to the tip of any given petal is a. There are two cases for rose curves:

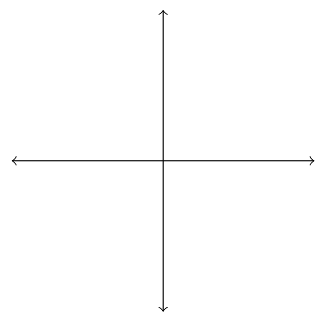
  - If n is odd, there are n evenly spaced petals.



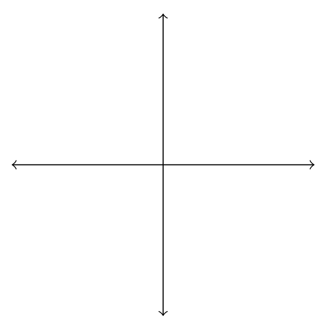
  - If n is even and at least 2, there are $2n$ evenly spaced petals.



- **Lemniscates** take the form of $r^2 = a^2 \sin(n\theta)$ or $r^2 = a^2 \cos(n\theta)$, are symmetrical about $\theta = \frac{\pi}{4}$ or the horizontal axis respectively.
  - If n is odd, there are $2n$ petals.



  - If n is even, there are n petals.



Chapter 2

Infinite Series

Tests for Convergence/Divergence

2.1 Power Series

A **power** series is an infinite series that produces a polynomial.
The **interval of convergence** is the interval that contains all values of x for which the series converges. This can be found by using the ratio test to find the criterion for convergence as an inequality. From there, the values of the bounds can be plugged into the series, and any test can be used to verify convergence, and the bounds' inclusivity appropriately adjusted.
The value subtracted from x within the series is the location of the **center**. It should always be equidistant from the bounds of the interval of convergence. The **radius of convergence** is the difference between either of the bounds and the center.
If the limit test produces ∞, the series only converges at its center, and its radius is 0.
If the limit test produces 0, the series converges for all x values, its radius is ∞.

2.2 Taylor and Maclaurin Polynomials

If f has n derivatives at $x = c$, then the following polynomial is the $\boldsymbol{n^{th}}$ **Taylor polynomial** of f and can be used to approximate f centered at c.

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f^{(3)}(c)(x-c)^3}{3!} + \ldots + \frac{f^{(n)}(c)(x-c)^n}{n!} = \sum_{i=0}^{n} \left[\frac{f^{(i)}(c)(x-c)^i}{i!}\right]$$

The more terms in the series, the more accurate the approximation.
If $c = 0$, it is a Maclaurin polynomial.

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \ldots + \frac{f^{(i)}(0)x^n}{n!} = \sum_{i=0}^{n} \left[\frac{f^{(i)}(0)x^i}{i!}\right]$$

Manipulating Known Maclaurin Polynomials

There are five Maclaurin series that are assumedly known.

$$
\begin{array}{l|l|l}
e^x & 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots & \displaystyle\sum_{n=0}^{\infty} \left[\frac{x^n}{n!} \right] \\
\sin x & x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dots & \displaystyle\sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+1}}{(2n+1)!} \right] \\
\cos x & 1 - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dots & \displaystyle\sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n}}{(2n)!} \right]
\end{array}
$$

Maclaurin series can be derived from these five basic series via a substitution into the known series. Power series can be directly manipulated for monomials.

2.3 Error

2.3.1 Actual Error

The **actual error** at x is the absolute value of the difference between a function's actual value at x and that of its n^{th} Taylor/Maclaurin polynomial.

$$\text{error} = |f(x) - P_n(x)|$$

2.3.2 Alternating Series Error

The error for a convergent alternating series's first n terms cannot exceed the first unused term.

$$\text{error} \leq \left|(-1)^{c+1} a_c\right| \,\middle|\, \sum_{n=1}^{\infty} \left[(-1)^n a_n\right] \text{ converges}$$

2.3.3 Lagrange Error

The error at x when using a Taylor polynomial cannot be greater than the first unused term's maximum value within the closed interval from x to c.

$$\text{error} \leq \frac{\max\left\{\left|f^{(n+1)}(z)\right| \mid z \in [x, c]\right\} (x - c)^{(n+1)}}{(n+1)!} \text{ for the } n^{\text{th}} \text{ Taylor polynomial about } c$$