Discussion 10: Ordinary Points and Singular Points

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1. Consider the linear second-order homogenous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

This can be rewritten in standard form by dividing by $a_2(x)$ as

$$y'' + P(x)y' + Q(x)y = 0$$

where

$$P(x) = \frac{a_1(x)}{a_2(x)} \qquad \text{and} \qquad Q(x) = \frac{a_0(x)}{a_2(x)}$$

A function is said to be analytic at a point if it can be represented by a power series with a radius of convergence that is positive or infinite.
A point $x = x_0$ is an ordinary point of the above DE if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not an ordinary point is a singular point of the DE.

2. Consider the same DE. Let $x = x_0$ be a singular point of it. It is said to be a regular singular point if

$$p(x) = (x - x_0)P(x) \qquad \text{and} \qquad q(x) = (x - x_0)Q(x)$$

are both analytic at x_0. It is said to be irregular if at least one is not analytic.

3. 1)

$$x^3y'' + 4x^2y' + 6y = 0$$

Dividing by x^3 yields the standard form

$$y'' + \frac{4}{x}y' + \frac{6}{x^3}y = 0$$

making

$$P(x) = \frac{4}{x} \qquad \text{and} \qquad Q(x) = \frac{6}{x^3}$$

For both denominators of $P(x)$ and $Q(x)$, the only factor is x, making the only singular point $x_0 = 0$, so $x - x_0 = x$. As there is an x^3 term in the denominator of $Q(x)$, though, and $3 > 2$, $x = 0$ is an irregular singular point.

2)

$$\left(x^2-4\right)y''+(x+2)y'+7y=0$$

It should be noted that $a_2(x)=x^2-4$ can be rewritten as $(x+2)(x-2)$. Dividing by $(x+2)(x-2)$ yields the standard form

$$y''+\frac{1}{x-2}y'+\frac{7}{(x+2)(x-2)}y=0$$

so

$$P(x)=\frac{1}{x-2}\qquad\text{and}\qquad Q(x)=\frac{7}{(x+2)(x-2)}$$

The only factor of the denominator of $P(x)$ is $x-2$ while that of $Q(x)$ has factors $x+2$ and $x-2$. The singular points are therefore $x_0=\pm 2$.
$x-2$ appears only to the first power in the denominators of both $P(x)$ and $Q(x)$, and $1\leq 1\leq 2$, making $x=2$ a regular singular point.
$x+2$ appears only to the first power in only the denominator of $Q(x)$, and $1\leq 2$, making $x=-2$ a regular singular point as well.

3)

$$\left(x^3+4x\right)y''-2xy'+7y=0$$

It should be noted that $a_2(x)=x^3+4x=x\left(x^2+4\right)$. Dividing by $x\left(x^2+4\right)$ yields the standard form

$$y''-\frac{2}{x^2+4}y'+\frac{7}{x\left(x^2+4\right)}y=0$$

so

$$P(x)=-\frac{2}{x^2+4}\qquad\text{and}\qquad Q(x)=\frac{7}{x\left(x^2+4\right)}$$

The only factor of the denominator of $P(x)$ is x^2+4 while that of $Q(x)$ has factors x and x^2+4. The only singular point is therefore $x_0=0$.
x appears only as a factor to the first power in the denominator of $Q(x)$, and $1\leq 2$, making $x=0$ a regular singular point.

4)

$$\left(x^2+x-2\right)y''+(x+2)xy'+(x-1)y=0$$

It should be noted that $a_2(x)=x^2+x-2=(x+2)(x-1)$. Dividing by $(x+2)(x-1)$ yields the standard form

$$y''+\frac{x}{x-1}y'+\frac{1}{x+2}y=0$$

so

$$P(x)=\frac{x}{x-1}\qquad\text{and}\qquad Q(x)=\frac{1}{x+2}$$

The only factor of the denominator of $P(x)$ is $x-1$ while the only factor of that of $Q(x)$ is $x+2$, making the singular points $x_0=-2,1$.
$x+2$ appears only to the first power and only in the denominator of $Q(x)$, and $1\leq 2$, making $x=-2$ a regular singular point.
$x-1$ appears only to the first power and only in the denominator of $P(x)$, and $1\leq 1$, making $x=1$ a regular singular point.