

Volume: Disk/Washer

Sources

Calculus: Early Transcendentals 9th Edition

1. 6.2 Exercise 12
2. 6.2 Exercise 15
3. 6.2 Exercise 1
4. 6.2 Exercise 17
5. 6.2 Exercise 24

AP Calculus Exams

6. 2021 AB FRQ 3(c)

Problems

If no instructions are given, evaluate the volume of the solid generated by revolving the region bounded by the given equations about the specified line using the disk/washer method.

1.

$$\left. \begin{array}{l} y = 0 \quad y = \frac{1}{x} \\ x = 1 \quad x = 4 \end{array} \right| y = 0$$

2.

$$\left. \begin{array}{l} y = \frac{x^2}{4} \quad y = 9 \\ x = 0 \end{array} \right| x = 0$$

3.

$$\left. \begin{array}{l} y = 0 \quad y = x^2 + 5 \\ x = 0 \quad x = 3 \end{array} \right| y = 0$$

4.

$$\left. \begin{array}{l} y = x^2 \\ y = 2x \end{array} \right| x = 0$$

5.

$$\left. \begin{array}{l} y = \sin x \quad y = \cos x \\ x \geq 0 \quad x \leq \frac{\pi}{4} \end{array} \right| y = -1$$

6.

$$f(x) = cx\sqrt{4 - x^2}$$

The solid of revolution generated by rotating the area bounded by f and the x -axis in the first quadrant about the x -axis is equal to 2π . Solve for c , given that it is a positive constant.

7.

$$\left. \begin{array}{l} \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \mid a, b > 0 \text{ and } b > \sqrt{a^4 - a^2} \\ x = \sqrt{a^2 + b^2} \end{array} \right| \begin{array}{l} y = \pm \frac{bx}{a} \text{ (a) } x = 0 \\ x = a^2 \text{ (b) } x = a \end{array}$$

8.

$$\left. \begin{array}{l} x^2 + y^2 = r^2 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \mid a, b > r > 0 \\ \text{(a) } y = 0 \end{array} \right| \begin{array}{l} \text{(a) } y = 0 \\ \text{(b) } y = b \end{array}$$

Solutions

1.

$$V = \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx = \pi \left[-\frac{1}{x}\right]_1^4 = \pi \left[-\frac{1}{4} - \left(-\frac{1}{1}\right)\right] = \frac{3\pi}{4}$$

2.

$$\begin{aligned} y &= \frac{x^2}{4} \implies x = 2\sqrt{y} \\ y_1 &= 2\sqrt{0} = 0 \\ V &= \pi \int_0^9 (2\sqrt{y})^2 dy = \pi [2y^2]_0^9 = 2(81)\pi = 162\pi \end{aligned}$$

3.

$$\begin{aligned} V &= \pi \int_0^3 (x^2 + 5)^2 dx = \pi \int_0^3 [x^4 + 10x^2 + 25] dx = \pi \left[\frac{x^5}{5} + \frac{10x^3}{3} + 25x\right]_0^3 \\ &= \pi \left[\frac{3^5}{5} + \frac{10(3)^3}{3} + 25(3) - (0)\right] = \pi \left[\frac{243}{5} + 90 + 75\right] = \frac{\pi(243 + 825)}{5} = \frac{1068\pi}{5} \end{aligned}$$

4.

$$\begin{aligned} y &= x^2 \implies x = \sqrt{y} & y &= 2x \implies x = \frac{y}{2} \\ \sqrt{y} &= \frac{y}{2} \implies 4y = y^2 \implies 0 = y(y - 4) \implies y_1 = 0, y_2 = 4 \\ V &= \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right] dy = \pi \int_0^4 \left[y - \frac{y^2}{4}\right] dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 \\ &= \pi \left[\frac{y^2(6 - y)}{12}\right]_0^4 = \pi \left[\frac{4^2(6 - 4)}{12} - (0)\right] = \pi \left[\frac{16(2)}{12}\right] = \frac{8\pi}{3} \end{aligned}$$

5.

$$\begin{aligned} \sin x &= \cos x \implies x = \frac{\pi}{4} \\ V &= \pi \int_0^{\pi/4} [(\cos x + 1)^2 - (\sin x + 1)^2] dx \\ &= \pi \int_0^{\pi/4} [\cos^2 x + 2\cos x + 1 - \sin^2 x - 2\sin x - 1] dx \\ &= \pi \int_0^{\pi/4} [\cos(2x) + 2\cos x - 2\sin x] dx = \pi \left[\frac{\sin(2x)}{2} + 2\sin x + 2\cos x\right]_0^{\pi/4} \\ &= \pi \left[\frac{1}{2} + \sqrt{2} + \sqrt{2} - (0 + 0 + 2)\right] = \frac{(4\sqrt{2} - 3)\pi}{2} \end{aligned}$$

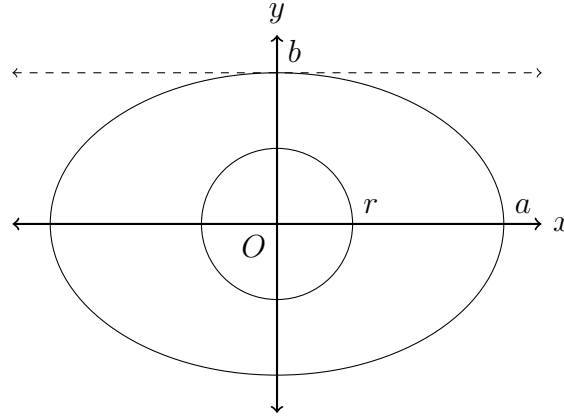
6.

$$\begin{aligned}
0 = cx\sqrt{4-x^2} &\implies \begin{cases} x = 0 \\ \sqrt{4-x^2} = 0 \end{cases} \implies 4-x^2 = 0 \implies 4 = x^2 \implies x = 2 \\
2\pi &= \pi \int_0^2 \left(cx\sqrt{4-x^2}\right)^2 dx = \pi \int_0^2 [c^2 x^2 (4-x^2)] dx = \pi \int_0^4 [4c^2 x^2 - c^2 x^4] dx \\
&= \pi \left[\frac{4c^2 x^3}{3} - \frac{c^2 x^5}{5} \right]_0^2 = \pi \left[\frac{4c^2 (2)^3}{3} - \frac{c^2 (2)^5}{5} - (0) \right] = \pi \left[\frac{32c^2}{3} - \frac{32c^2}{5} \right] \\
2 &= 32c^2 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{64c^2}{15} \\
c &= \sqrt{\frac{30}{64}} = \sqrt{\frac{15}{32}}
\end{aligned}$$

7.

$$\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 &\implies x = a\sqrt{1 + \frac{y^2}{b^2}} = \frac{a\sqrt{b^2 + y^2}}{b} \\
y = \frac{bx}{a} &\implies x = \frac{ay}{b} \\
V &= 2\pi \int
\end{aligned}$$

8.



This graph assumes $a > b$, which need not be the case.

$$x^2 + y^2 = r^2 \implies y = \sqrt{r^2 - x^2} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b\sqrt{a^2 - x^2}}{a}$$

(a)

$$\begin{aligned}
 V_1 &= \pi \int_0^r \left[\left(\frac{b\sqrt{a^2 - x^2}}{a} \right)^2 - \left(\sqrt{r^2 - x^2} \right)^2 \right] dx = \pi \int_0^r \left[\frac{b^2(a^2 - x^2)}{a^2} - (r^2 - x^2) \right] dx \\
 &= \pi \int_0^r \left[\frac{a^2(b^2 - r^2) + x^2(a^2 - b^2)}{a^2} \right] dx = \pi \left[\frac{a^2x(b^2 - r^2)}{a^2} + \frac{x^3(a^2 - b^2)}{3a^2} \right]_0^r \\
 &= \pi \left[\frac{3a^2x(b^2 - r^2) + x^3(a^2 - b^2)}{3a^2} \right]_0^r = \pi \left[\frac{3a^2r(b^2 - r^2) + r^3(a^2 - b^2)}{3a^2} \right] \\
 &= \pi \left[\frac{3a^2b^2r - 3a^2r^3 + a^2r^3 - b^2r^3}{3a^2} \right] = \pi \left(\frac{3a^2b^2r - 2a^2r^3 - b^2r^3}{3a^2} \right) \\
 V_2 &= \pi \int_r^a \left(\frac{b\sqrt{a^2 - x^2}}{a} \right)^2 dx = \pi \int_r^a \left[\frac{b^2(a^2 - x^2)}{a^2} \right] dx = \pi \int_r^a \left[\frac{a^2b^2 - b^2x^2}{a^2} \right] dx \\
 &= \pi \left[\frac{a^2b^2x}{a^2} - \frac{b^2x^3}{3a^2} \right]_r^a = \pi \left[\frac{3a^2b^2x - b^2x^3}{3a^2} \right]_r^a = \left[\frac{3a^3b^2 - b^2a^3}{3a^2} - \left(\frac{3a^2b^2r - b^2r^3}{3a^2} \right) \right] \\
 &= \pi \left(\frac{2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2} \right) \\
 V &= 2(V_1 + V_2) = 2\pi \left(\frac{3a^2b^2r - 2a^2r^3 - b^2r^3 + 2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2} \right) \\
 &= 2\pi \left(\frac{2a^3b^2 - 2a^2r^3}{3a^2} \right) = 4\pi \left(\frac{ab^2 - r^3}{3} \right)
 \end{aligned}$$

(b)

$$\begin{aligned}
V_1 &= \pi \int_0^r \left[\left(\sqrt{r^2 - x^2} - b \right)^2 - \left(\frac{b\sqrt{a^2 - x^2}}{a} - b \right)^2 \right] dx \\
&= \pi \int_0^r \left[(r^2 - x^2) - 2b\sqrt{r^2 - x^2} + b^2 - \left(\frac{b^2(a^2 - x^2)}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right) \right] dx \\
&= \pi \int_0^r \left[r^2 - x^2 - 2b\sqrt{r^2 - x^2} + b^2 - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2 - x^2}}{a} - b^2 \right] dx \\
&= \pi \int_0^r \left[r^2 - x^2 - 2b\sqrt{r^2 - x^2} - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2 - x^2}}{a} \right] dx \\
&= \pi \left(\left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right]_0^r + 2b \int_0^r \left[\frac{b\sqrt{a^2 - x^2}}{a} - \sqrt{r^2 - x^2} \right] dx \right) \\
I(\alpha) &= \int \left[\sqrt{\alpha^2 - x^2} \right] dx \implies x = \alpha \sin \theta \implies dx = \alpha \cos \theta d\theta \\
&= \int \left[\cos \theta \sqrt{\alpha^2 - \alpha^2 \sin^2 \theta} \right] d\theta = \int \left[\alpha^2 \cos^2 \theta \right] d\theta = \alpha^2 \int \left[\frac{\cos(2\theta) + 1}{2} \right] d\theta \\
&= \alpha^2 \left(\frac{\sin(2\theta)}{4} + \frac{\theta}{2} \right) + C = \alpha^2 \left(\frac{2 \sin \theta \cos \theta}{4} + \frac{\theta}{2} \right) + C \\
&= \alpha^2 \left(\frac{\left(\frac{x}{\alpha} \right) \left(\frac{\sqrt{\alpha^2 - x^2}}{\alpha} \right)}{2} + \frac{\arcsin(x/\alpha)}{2} \right) + C = \frac{x\sqrt{\alpha^2 - x^2} + \alpha^2 \arcsin(x/\alpha)}{2} + C \\
V_1 &= \pi \left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - a \right) \frac{x^3}{3} + 2b \left(\frac{bI(a)}{a} - I(r) \right) \right]_0^r \\
&= \pi \left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right. \\
&\quad \left. + 2b \left(\frac{bx\sqrt{a^2 - x^2} + a^2b \arcsin(x/a)}{2a} - \frac{x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r)}{2} \right) \right]_0^r \\
&= \pi \left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right. \\
&\quad \left. + 2b \left(\frac{xb\sqrt{a^2 - x^2} + a^2b \arcsin(x/a)}{2a} - \frac{x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r)}{2} \right) \right]_0^r \\
&= \pi \left[x \left(r^2 - b^2 + \frac{x^2}{3} \left(\frac{b^2}{a^2} - 1 \right) \right) \right. \\
&\quad \left. + \frac{b}{a} \left(x \left(b\sqrt{a^2 - x^2} - a\sqrt{r^2 - x^2} \right) + a^2b \arcsin \left(\frac{x}{a} \right) - r^2 \arcsin \left(\frac{x}{r} \right) \right) \right]_0^r \\
&= \pi \left[r \left(r^2 - b^2 + \frac{r^2}{3} \left(\frac{b^2}{a^2} - 1 \right) \right) \right. \\
&\quad \left. + \frac{b}{a} \left(r \left(b\sqrt{a^2 - r^2} - 0 \right) + a^2b \arcsin \left(\frac{r}{a} \right) - \frac{a\pi r^2}{2} \right) - (0) \right] \\
&= \pi \left(r^3 - b^2r + \frac{b^2r^3}{3a^2} - \frac{r^3}{3} + \frac{b^2r\sqrt{a^2 - r^2}}{a} + ab^2 \arcsin \left(\frac{r}{a} \right) - \frac{b\pi r^2}{2} \right) \\
&= \pi \left(\frac{2r^3(2a^2 + b^2) - 3a^2b\pi r^2 + 6ab^2r(\sqrt{a^2 - r^2} - a) + 6a^3b^2 \arcsin(r/a)}{6a^2} \right)
\end{aligned}$$

(b) (cont.)

$$\begin{aligned}
V_2 &= \pi \int_r^a \left(\frac{b\sqrt{a^2 - x^2}}{a} - b \right)^2 dx = \pi \int_r^a \left[\frac{b^2(a^2 - x^2)}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right] dx \\
&= \pi \int_r^a \left[b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right] dx = \pi \int_r^a \left[2b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} \right] dx \\
&= \pi \left[2b^2x - \frac{b^2x^3}{3a^2} - \frac{2b^2I(a)}{a} \right]_r^a = \pi \left[2b^2x - \frac{b^2x^3}{3a^2} - \frac{b^2x\sqrt{a^2 - x^2} + a^2b^2 \arcsin(x/a)}{a} \right]_r^a \\
&= \pi \left[2b^2a - \frac{b^2a^3}{3a^2} - \frac{0 + a^2b^2(\pi/2)}{a} \right. \\
&\quad \left. - \left(2b^2r - \frac{b^2r^3}{3a^2} - \frac{b^2r\sqrt{a^2 - r^2} + a^2b^2 \arcsin(r/a)}{a} \right) \right] \\
&= \pi \left[2ab^2 - 2b^2r + \frac{b^2r^3 - a^3b^2}{3a^2} + \frac{b^2r\sqrt{a^2 - r^2} + a^2b^2 \arcsin(r/a) - 0.5a^2b^2\pi}{a} \right] \\
&= \frac{b^2\pi (6a^3 - 6a^2r + r^3 - a^3 + 3ar\sqrt{a^2 - r^2} + 3a^3 \arcsin(r/a) - 1.5a^3\pi)}{3a^2} \\
&= \frac{b^2\pi (2r^3 + 6ar(\sqrt{a^2 - r^2} - 2a) + a^3(10 + 6 \arcsin(r/a) - 3\pi))}{6a^2}
\end{aligned}$$

$$\begin{aligned}
V &= 4(V_1 + V_2) \\
&= \frac{2\pi}{3a^2} \left(4r^3(2a^2 + b^2) - 3a^2b\pi r^2 + 6ab^2r \left(2\sqrt{a^2 - r^2} - 3a \right) \right. \\
&\quad \left. + a^3b^2 \left(12 \arcsin \left(\frac{r}{a} \right) + 10 - 3\pi \right) \right)
\end{aligned}$$

Indeterminate Exponents (Type 3)

Sources

Calculus: Early Transcendentals 9th Edition

1. 4.4 Exercise 61
2. 4.4 Exercise 57
3. 4.4 Exercise 60
4. 4.4 Exercise 65
5. 4.4 Exercise 66

Problems

Evaluate the following limits.

1.

$$\lim_{x \rightarrow 1^+} [x^{1/(1-x)}]$$

2.

$$\lim_{x \rightarrow 0^+} [x^{\sqrt{x}}]$$

3.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{bx}$$

4.

$$\lim_{x \rightarrow 0^+} (4x + 1)^{\cot x}$$

5.

$$\lim_{x \rightarrow 0^+} (1 - \cos x)^{\sin x}$$

Solutions

1.

$$\begin{aligned}
 L &= \lim_{x \rightarrow 1^+} [x^{1/(1-x)}] && \Rightarrow 1^\infty \\
 \ln L &= \lim_{x \rightarrow 1^+} \left[\frac{\ln x}{1-x} \right] && \Rightarrow \frac{0}{0} \\
 &= \lim_{x \rightarrow 1^+} \left[-\frac{1/x}{1} \right] = -\frac{1/1}{1} = -1 \\
 L &= \frac{1}{e}
 \end{aligned}$$

2.

$$\begin{aligned}
 L &= \lim_{x \rightarrow 0^+} [x^{\sqrt{x}}] && \Rightarrow 0^0 \\
 \ln L &= \lim_{x \rightarrow 0^+} [\sqrt{x} \ln x] && \Rightarrow 0 \times (-\infty) \\
 &= \lim_{x \rightarrow 0^+} \left[\frac{\ln x}{x^{-1/2}} \right] && \Rightarrow -\frac{\infty}{\infty} \\
 &= \lim_{x \rightarrow 0^+} \left[-\frac{1/x}{0.5x^{-3/2}} \right] = \lim_{x \rightarrow 0^+} [-2\sqrt{x}] = 0 \\
 L &= e^0 = 1
 \end{aligned}$$

3.

$$\begin{aligned}
 L &= \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} \right)^{bx} && \Rightarrow 1^\infty \\
 \ln L &= \lim_{x \rightarrow \infty} \left[bx \ln \left(1 + \frac{a}{x} \right) \right] && \Rightarrow \infty \times 0 \\
 &= \lim_{x \rightarrow \infty} \left[\frac{b \ln \left(1 + \frac{a}{x} \right)}{x^{-1}} \right] && \Rightarrow \frac{0}{0} \\
 &= \lim_{x \rightarrow \infty} \left[\frac{\frac{-bax^{-2}}{1+\frac{a}{x}}}{-x^{-2}} \right] = \lim_{x \rightarrow \infty} \left[\frac{ab}{1+\frac{a}{x}} \right] = \frac{ab}{1+0} = ab \\
 L &= e^{ab}
 \end{aligned}$$

4.

$$\begin{aligned}
 L &= \lim_{x \rightarrow 0^+} (4x+1)^{\cot x} && \Rightarrow 1^\infty \\
 \ln L &= \lim_{x \rightarrow 0^+} [(\cot x) \ln(4x+1)] && \Rightarrow \infty \times 0 \\
 &= \lim_{x \rightarrow 0^+} \left[\frac{\ln(4x+1)}{\tan x} \right] && \Rightarrow \frac{0}{0} \\
 &= \lim_{x \rightarrow 0^+} \left[\frac{4/(x+1)}{\sec^2 x} \right] = \frac{4/1}{1} = 4 \\
 L &= e^4
 \end{aligned}$$

5.

$$\begin{aligned}
L &= \lim_{x \rightarrow 0^+} (1 - \cos x)^{\sin x} && \Rightarrow 0^0 \\
\ln L &= \lim_{x \rightarrow 0^+} [(\sin x) \ln(1 - \cos x)] && \Rightarrow 0 \times (-\infty) \\
&= \lim_{x \rightarrow 0^+} \left[\frac{\ln(1 - \cos x)}{\csc x} \right] && \Rightarrow -\frac{\infty}{\infty} \\
&= \lim_{x \rightarrow 0^+} \left[-\frac{\sin x / (1 - \cos x)}{\csc x \cot x} \right] = \lim_{x \rightarrow 0^+} \left[-\frac{\sin^2 x \tan x}{1 - \cos x} \right] && \Rightarrow -\frac{0}{0} \\
&= \lim_{x \rightarrow 0^+} \left[\frac{2 \sin x \cos x \tan x + \sin^2 x \sec^2 x}{\sin x} \right] = \lim_{x \rightarrow 0^+} [2 \cos x \tan x + \sin x \sec x] \\
&= 2(1)(0) + (0)(1) = 0 \\
L &= 1
\end{aligned}$$