

Contemporary Physics

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January 22, 2023

Contents

1	The Failures of Classical Physics	2
2	The Special Theory of Relativity	3
2.1	Classical Relativity	3
2.3	Einstein's Postulates	3
2.4	Consequences of Einstein's Postulates	4
	The Relativity of Time	4
	The Relativity of Length	4
	Relativistic Velocity Addition	5
	The Relativistic Doppler Effect	6
2.5	The Lorentz Transformation	6
	Length Contraction	7
	Velocity Transformation	8
2.6	The Twin Paradox	8
	Spacetime Diagrams	9
2.7	Relativistic Dynamics	9
	Relativistic Kinetic Energy	10
	Relativistic Total Energy and Rest Energy	10
2.8	Conservation Laws in Relativistic Decays and Collisions	11
2.9	Extra	11

Chapter 1

The Failures of Classical Physics

Under classical physics,

$$\vec{v} = \frac{\Delta \vec{r}}{t} \qquad K = \frac{1}{2}mv^2 \qquad \vec{p} = m\vec{v} \qquad \vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$$

where \vec{v} is velocity, \vec{r} is position, m is mass, \vec{p} is momentum, \vec{L} is angular momentum, and I is rotational inertia.

Working on very small scales, the unit of charge used is often the magnitude of that of an electron (or proton)

$$1\text{ e} \approx 1.602 \times 10^{-19} \text{ C}$$

and that of energy is the **electron-volt (eV)**, which is the energy of an electron subjected to a 1 V potential difference:

$$1\text{ eV} \approx 1.602 \times 10^{-19} \text{ J}$$

An **atomic mass unit (amu)** is defined such that

$$12\text{ amu} = \text{carbon atom}$$

The speed of light c and Planck's constant h are

$$c \approx 3 \times 10^8 \frac{\text{m}}{\text{s}} \qquad h \approx 6.63 \times 10^{-34} \text{ J s} \approx 4.14 \times 10^{-15} \text{ eV s}$$

The product of these two constants is

$$hc \approx 1240 \text{ eV nm}$$

Failures of space and time require relativity while those of particle theories require quantum mechanics.

A **theory** is an organized body of facts. It provides a model of explanation.

A **hypothesis** is a prediction as to how something works. (With data to back up a hypothesis, it can become a theory.)

A **law** is a correlation between variables.

Chapter 2

The Special Theory of Relativity

2.1 Classical Relativity

A “theory of relativity” is simply a way for observers in different reference frames to compare their observations.

The mathematical basis for converting between reference frames is called a *transformation*.

Consider two observers O and O' observing the same event, O' moving relative to O with constant velocity \vec{u} . In their own reference frames O and O' are both at rest, but they move relative to one another with constant velocity \vec{u} . According to O , the space and time coordinates are x , y , z , and t , while according to O' , those of the *same event* are x' , y' , z' , and t' . If the relative velocity is only in the common xx' direction, \vec{u} can represent the velocity of O' as measured by O .

It is assumed that each observer is able to test and verify Newton’s laws in their reference frames. A reference frame that follows Newton’s first law (the law of inertia) is said to be an *inertial frame*. In order for a frame to be inertial, it must not be accelerating.

The classical *Galilean* transformation relating the coordinates assumes as a postulate of classical physics that $t = t'$; that is, time is the same for all observers. It is also assumed for simplicity that the coordinate systems are chosen such that their origins overlap at $t = 0$. Consider an object in O' at coordinates (x', y', z') . According to O , the y and z coordinates are the same as those observed in O' . Along the x direction, though, O would observe the object at $x = x' + ut$. This yields the *Galilean coordinate transformation*

$$x' = x - ut \quad y' = y \quad z' = z \quad t' = t \quad (\text{Galilean coordinate transformation})$$

The velocities as observed by O can be found simply by differentiating the prior results:

$$v'_x = v_x - u \quad v'_y = v_y \quad v'_z = v_z \quad (\text{Galilean velocity transformation})$$

Again differentiating yields the relationships between the accelerations:

$$a'_x = a_x \quad a'_y = a_y \quad a'_z = a_z$$

This shows that Newton’s laws hold for both observers so long as u is constant, as the observers measure identical accelerations, meaning that the results of applying $\vec{F} = m\vec{a}$ are also identical.

2.3 Einstein’s Postulates

1. **Principle of Relativity** The laws of physics are identical for all inertial reference frames.
2. The speed of light is constant for all inertial reference frames.

2.4 Consequences of Einstein's Postulates

As a consequence of these two postulates, the motion of the observer determines their measurements.

The Relativity of Time

Consider a beam of light traveling up to a mirror and back down (in the y direction) in a reference frame moving at speed u in the x direction. An observer O at rest and an observer O' moving in the same reference frame as the mirror.

From the perspective of O' , the time interval between the light leaving and returning from its source is

$$\Delta t_0 = \frac{2L_0}{c}$$

where L_0 is the distance between the origin of the light and the mirror.

From the perspective of O , the light beam travels a distance $2L$, where

$$L = \sqrt{L_0^2 + \left(\frac{u\Delta t}{2}\right)^2}$$

as derived by Pythagorean theorem. As O observes the beam of light to travel at c , the time interval measured is

$$\Delta t = \frac{2L}{c} = \frac{2\sqrt{L_0^2 + (u\Delta t/2)^2}}{c}$$

Substituting for L_0 and solving for Δt yields

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} \quad (\text{time dilation})$$

As the $u \leq c$, the denominator of this expression is at most 1, meaning that $\Delta t \leq \Delta t_0$; that is, O measures a longer time interval than O' . For observer O , the beginning and ending of the time interval occur at the same location, so the interval Δt_0 observed by them is known as *proper time*.

The Relativity of Length

Suppose the beam of light instead travels in the same direction as that of the motion of O' . According to O , the distance traveled is L , while from the perspective of O' (from which the clock is at rest), it is L_0 .

The light takes time interval Δt_1 to reach the mirror, in which it travels distance $c\Delta t_1$, which is equal to the length L plus the distance $u\Delta t_1$ traveled by O' :

$$c\Delta t_1 = L + u\Delta t_1$$

The light then takes time interval Δt_2 to return to its source, in which it travels distance $c\Delta t_2$, equal to the length L minus the distance $u\Delta t_2$:

$$c\Delta t_2 = L - u\Delta t_2$$

Adding these times yields Δt :

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{L}{c - u} + \frac{L}{c + u} = \frac{2L}{c} \frac{1}{1 - u^2/c^2}$$

Applying the formula for time dilation,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} = \frac{2L_0}{c} \frac{1}{\sqrt{1 - u^2/c^2}}$$

Setting the above two equations equal to each other yields

$$L = L_0 \sqrt{1 - \frac{u^2}{c^2}} \quad (\text{length contraction})$$

Observer O' , who is at rest relative to the object, measures the *rest length* L_0 (the *proper length*) while all observers relative to whom O' is in motion measure a shorter length *only in the direction of motion*. A moving body appears to be shorter than one at rest.

It should be noted that length contraction occurs only in the direction of movement, meaning that any dimensions perpendicular to that of movement are unaffected.

Relativistic Velocity Addition

A source emits particles traveling at speed v' according to observer O' , who is at rest with respect to the source. A bulb is triggered to flash when the particle reaches it. This flash makes a return trip to the source. The time interval Δt_0 measured by O' is comprised of the time it takes for the particle to travel distance L_0 at speed v' and the time it takes the flash to travel the same distance at c :

$$\Delta t_0 = \frac{L_0}{v'} + \frac{L_0}{c}$$

O' is moving at speed u relative to observer O . For O , the particle traveling at speed v according to O reaches the bulb after Δt_1 , traveling a distance of $v\Delta t_1$ equal to the (contracted) length L plus the distance $u\Delta t_1$ moved by O' :

$$v\Delta t_1 = L + u\Delta t_1$$

In the interval Δt_2 , the light beam traverses distance $c\Delta t_2$ equal to length L minus the distance $u\Delta t_2$ traveled by O' :

$$c\Delta t_2 = L - u\Delta t_2$$

Solving the above two equations for Δt_1 and Δt_2 and adding them yields the total interval Δt between the emission of the particle and the arrival of the beam at the detector according to O . Applying the time dilation formula to relate this result to Δt_0 and using the length contraction formula to relate L to L_0 yields the result

$$v = \frac{v' + u}{1 + v'u/c^2} \quad (\text{relativistic velocity addition})$$

Note that when $v' = c$ (a beam of light being observed),

$$v = \frac{c + u}{1 + cu/c^2} = \frac{c + u}{1 + u/c} = \frac{c + u}{\frac{c + u}{c}} = c$$

regardless of the value of u . All observers measure the same speed of light c regardless of their reference frame.

The Relativistic Doppler Effect

The classical Doppler effect states that an observer O moving relative to the source of waves detects a frequency f' different from the true frequency f being emitted by the source S :

$$f' = f \frac{v \pm v_0}{v \mp v_S} \quad (\text{classical Doppler effect})$$

where v is the speed of the waves through some medium, v_S is the speed of the source *relative to the medium*, and O is that of the observer *relative to the medium*. The upper signs are chosen when S and O are moving towards each other while the lower ones are chosen when they are moving away from each other.

Consider a source of electromagnetic waves at rest in the reference frame of observer O . An observer O' moving relative to the source at speed u will observe a different frequency. Suppose O observes N waves emitted at frequency f . It takes an interval $\Delta t_0 = N/f$ for these N waves to be emitted from the point of view of O . This is the proper time interval in the frame of reference of O . The corresponding interval to O' is $\Delta t'$, during which O moves a distance $u\Delta t'$. The wavelength λ' according to O' is the total length occupied by the waves divided by the number of waves:

$$\lambda' = \frac{c\Delta t' + u\Delta t'}{N} = \frac{c\Delta t' + u\Delta t'}{f\Delta t_0}$$

The frequency from the reference frame of O' is $f' = c/\lambda'$, so

$$f' = \frac{c}{\lambda'} = f \frac{\Delta t_0}{\Delta t'} \frac{1}{1 + u/c}$$

Applying the time dilation formula,

$$f' = f \frac{\sqrt{1 - u^2/c^2}}{1 + u/c} = f \sqrt{\frac{1 - u/c}{1 + u/c}} \quad (\text{relativistic Doppler shift})$$

This is the formula for the *relativistic Doppler shift* where the waves are observed parallel to \vec{u} . It should be noted that unlike the classical formula, this formula does *not* distinguish between the source and observer motion; it is only dependent on the relative speed u between the source and observer.

2.5 The Lorentz Transformation

While the Galilean transformation holds for the “common-sense” experience observed at low speeds, it does not agree with experiments performed at high speeds. A new set of transformations is therefore required that is capable of reproducing relativistic effects such as time dilation, length contraction, velocity addition, and the Doppler shift.

A transformation that enables observers O and O' moving relative to each other to compare their measurements in space and time coordinates of the same event. The transformation equations relate the measurements x, y, z, t in O to x', y', z, t' in O' . This transformation must be linear (depending only on the first powers of each coordinate, following the homogeneity of space and time), consistent with Einstein’s postulates, and it must reduce to the Galilean transformation

when the relative speed between the observers is small. It is again assumed that the relative speed is only in the positive xx' direction.

The new transformation is called the *Lorentz transformation*:

$$x' = \frac{x - ut}{1 - u^2/c^2} \quad y' = y \quad z' = z \quad t' = \frac{t - (u/c^2)x}{\sqrt{1 - u^2/c^2}} \quad (\text{Lorentz transformation})$$

This is sometimes rewritten as

$$x' = \gamma(x - \beta ct) \quad y' = y \quad z' = z \quad ct' = \gamma(ct - \beta x)$$

or

$$x' = \gamma(x - vt) \quad y' = y \quad z' = z \quad t' = \gamma\left(t - x\frac{v}{c^2}\right)$$

or in matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ ct' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ct \end{bmatrix} \quad \begin{bmatrix} x' \\ ct' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix} \quad \begin{bmatrix} x \\ ct \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} x' \\ ct' \end{bmatrix}$$

where

$$\beta = \frac{u}{c} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

It is often useful to write these equation in terms of intervals by replacing each coordinate by their corresponding interval.

The equations for y' and z' are identical to their Galilean counterparts while the first and last equations reduce to theirs when $u \ll c$.

The Lorentz transformation can then be used to derive some predictions of special relativity.

Length Contraction

A rod of length L_0 is at rest as observed by O' . The rod extends from x'_1 to x'_2 ; that is, $L_0 = x'_2 - x'_1$. Observer O , relative to whom the rod is in motion, measures the ends of the rod at x_1 and x_2 . For the length of the rod to be measured by O , x_1 and x_2 must be *simultaneously* determined.

Suppose O' flashes a bulb at x'_1 and t'_1 and then flashes a bulb at x'_2 and t'_2 . The Lorentz transformation gives

$$x'_1 = \frac{x_1 - ut_1}{\sqrt{1 - u^2/c^2}} \quad x'_2 = \frac{x_2 - ut_2}{\sqrt{1 - u^2/c^2}}$$

Subtracting the first equation from the second yields

$$x'_2 - x'_1 = \frac{x_2 - x_1}{\sqrt{1 - u^2/c^2}} - \frac{u(t_2 - t_1)}{\sqrt{1 - u^2/c^2}}$$

The bulbs must be set off such that the flashes appear to be simultaneous to O . This enables a simultaneous determination to be made of the rod's endpoints. if the flashes are observed simultaneously, $t_1 = t_2$, so

$$x'_2 - x'_1 = \frac{x_2 - x_1}{\sqrt{1 - u^2/c^2}}$$

Substituting L_0 for $x'_2 - x'_1$ and L for $x_2 - x_1$ yields

$$L = L_0 \sqrt{1 - u^2/c^2}$$

Velocity Transformation

Suppose O observes a particle traveling with velocity \vec{v} . The relationship between \vec{v} and the velocity \vec{v}' observed by O' is given by the *Lorentz velocity transformation*

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} \quad v'_y = \frac{v_y \sqrt{1 - u^2/c^2}}{1 - v_x u/c^2} \quad v'_z = \frac{v_z \sqrt{1 - u^2/c^2}}{1 - v_x u/c^2} \quad (\text{Lorentz velocity transformation})$$

Solving the above equations for v_x , it can be shown that this is identical to the equation for relativistic velocity addition. It should be noted that for $u \ll c$, the Lorentz velocity transformation reduces to its Galilean counterpart. Note also that despite $y' = y$, $v'_y \neq v_y$. This is due to the Lorentz transformation's handling of t .

These transformation can be derived from the coordinate transformation. Differentiating the velocity transformation for x' yields

$$dx' = \frac{dx - u dt}{1 - u^2/c^2}$$

Differentiating the time coordinate transformation yields

$$dt' = \frac{dt - (u/c^2) dx}{1 - u^2/c^2}$$

v'_x is therefore

$$v'_x = \frac{dx'}{dt'} = \frac{\frac{dx - u dt}{1 - u^2/c^2}}{\frac{dt - (u/c^2) dx}{1 - u^2/c^2}} = \frac{dx - u dt}{dt - (u/c^2) dx} = \frac{dx/dt - u}{1 - (u/c^2) dx/dt} = \frac{v_x - u}{1 - v_x u/c^2}$$

Differentiating $y' = y$ yields $dy' = dy$, so

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{\frac{dt - (u/c^2) dx}{\sqrt{1 - u^2/c^2}}} = \frac{dy \sqrt{1 - u^2/c^2}}{dt - (u/c^2) dx} = \frac{\sqrt{1 - u^2/c^2} dy/dt}{1 - (u/c^2) dx/dt} = \frac{v_y \sqrt{1 - u^2/c^2}}{1 - u v_x/c^2}$$

Differentiating for $z' = z$ yields $dz' = dz$, so

$$v'_z = \frac{dz'}{dt'} = \frac{dz}{\frac{dt - (u/c^2) dx}{\sqrt{1 - u^2/c^2}}} = \frac{dz \sqrt{1 - u^2/c^2}}{dt - (u/c^2) dx} = \frac{\sqrt{1 - u^2/c^2} dz/dt}{1 - (u/c^2) dx/dt} = \frac{v_z \sqrt{1 - u^2/c^2}}{1 - u v_x/c^2}$$

2.6 The Twin Paradox

Suppose there are a pair of twins on earth. Twin A remains on Earth while twin B sets off on a rocket ship. A believes that as a consequence of time dilation, B should be younger than them by the time they return. For two observers in relative motion, though, *each* believes the *other's* clock to be slow. From the point of view of B, A make a round-trip away from them and back again. Under such circumstances, it should be A's clock that is running slow, so A they expect A to be younger when they return.

The resolution of this paradox comes from considering the asymmetric roles of the twins. Special

relativity applies only to inertial frames. In order to leave and reenter Earth, B must accelerate and correspondingly decelerate. Although this may be over a brief time interval, B's return journey occurs in a different inertial frame to their departure. This change in B's inertial frame is what results in the asymmetry in the twins' ages. Only B is required to change inertial frames to return, so *all observers will agree* that B is the one that is "really" in motion, so it is therefore their clock that is "really" running slow, meaning that B will indeed be the younger twin upon their return.

Spacetime Diagrams

A *spacetime* diagram plots time on the y -axis and space on the x -axis. The slope of a straight line is velocity as a proportion of c .

2.7 Relativistic Dynamics

Dynamical quantities such as momentum and kinetic energy are dependent on length, time, and velocity. As such, they are also affected by Einstein's postulates.

Consider a collision of two particles that is elastic from the reference frame of O' . Particle 1 of mass $m_1 = 2m$ is initially at rest while particle 2 of mass $m_2 = m$ is moving in the negative x direction with initial velocity $v'_{2,i} = -0.75c$. The final velocities of particles 1 and 2 are respectively $v'_{1,f} = -0.5c$ and $v'_{2,f} = 0.25c$. Analyzing this collision classically,

$$p'_i = m_1 v'_{1,i} + m_2 v'_{2,i} = -0.75mc \qquad p'_i = m_1 v'_{2,f} + m_2 v'_{2,f} = -0.75mc = p'_i$$

As the initial and final momenta are equal according to O' , energy is conserved.

Suppose the reference frame of O' is moving in the positive x direction at a speed of $u = 0.55c$ relative to O . The velocity transformation yields the velocities of $v_{1,i} = 0.55c$, $v_{2,i} = -0.34c$, $v_{1,f} = 0.069c$, and $v_{2,f} = 0.703c$. Analyzing the collision,

$$p_i = m_1 v_{1,i} + m_2 v_{2,i} = 0.76mc \qquad p_f = m_1 v_{1,f} + m_2 v_{2,f} = 0.841mc \neq p_i$$

As $p_i \neq p_f$, momentum is not conserved according to O . This means that the law of conservation of linear momentum does not satisfy Einstein's first postulate. To retain the conservation of momentum, then, a new definition of momentum is required. This definition must yield a conservation law that holds under relativity (momentum being conserved regardless of the reference frame, so long as it is inertial) and reduces to $\vec{p} = m\vec{v}$ at low speeds.

These requirements are satisfied by defining relativistic momentum to be

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \qquad \text{(relativistic momentum)}$$

The velocity v is the velocity of the particle as measured from a particular inertial frame; it is not the velocity of said frame.

Using this relativistic definition of momentum to analyze the above collision,

$$p'_i = \frac{m_1 v'_{1,i}}{\sqrt{1 - v'^2_{1,i}/c^2}} + \frac{m_2 v'_{2,i}}{\sqrt{1 - v'^2_{2,i}/c^2}} \approx 1.134mc \quad p'_f = \frac{m_1 v'_{1,f}}{\sqrt{1 - v'^2_{1,f}/c^2}} + \frac{m_2 v'_{2,f}}{\sqrt{1 - v'^2_{2,f}/c^2}} \approx 1.134mc = p'_i$$

As $p'_i = p'_f$, O' is able to conclude that momentum is conserved. To O ,

$$p_i = \frac{m_1 v_{1,i}}{\sqrt{1 - v_{1,i}^2/c^2}} + \frac{m v_{2,i}}{\sqrt{1 - v_{2,i}^2/c^2}} \approx 0.956mc \quad p_f = \frac{m_1 v_{1,f}}{\sqrt{1 - v_{1,f}^2/c^2}} + \frac{m_2 v_{2,f}}{\sqrt{1 - v_{2,f}^2/c^2}} \approx 0.946mc = p_i$$

so O' can also conclude that momentum is conserved.

Relativistic Kinetic Energy

Much like classical momentum, classical kinetic energy also fails under relativity. According to O' , the aforementioned collision has kinetic energies

$$K'_i = \frac{1}{2}m_1 v_{1,i}'^2 + \frac{1}{2}m_2 v_{2,i}'^2 \approx 0.281mc^2 \quad K'_f = \frac{1}{2}m_1 v_{1,f}'^2 + \frac{1}{2}m_2 v_{2,f}'^2 \approx 0.281mc^2 = K'_i$$

but according to O is has

$$K_i = \frac{1}{2}m_1 v_{1,i}^2 + \frac{1}{2}m_2 v_{2,i}^2 \approx 0.36mc^2 \quad K_f = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2 \approx 0.252mc^2 \neq K_i$$

so O concludes that energy is not conserved.

The relativistic definition of kinetic energy is

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \quad (\text{relativistic kinetic energy})$$

There is no limit to the amount of kinetic energy that can be given to a particle. Under the classical definition of kinetic energy, this means that speed must also lack a limit, which violates Einstein's second postulate.

Relativistic Total Energy and Rest Energy

Relativistic kinetic energy can also be expressed as

$$K = E - E_0 \quad (\text{relativistic kinetic energy})$$

where the *relativistic total energy* E is defined as

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \quad (\text{relativistic total energy})$$

and the *rest energy* E_0 is defined is

$$E_0 = mc^2 \quad (\text{rest energy})$$

This suggests that mass can be expressed in units of MeV/c^2 . The m in this equation is sometimes called the *rest mass* m_0 , which is distinguished from the “relativistic mass” m' , defined as

$$m' = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad (\text{relativistic mass})$$

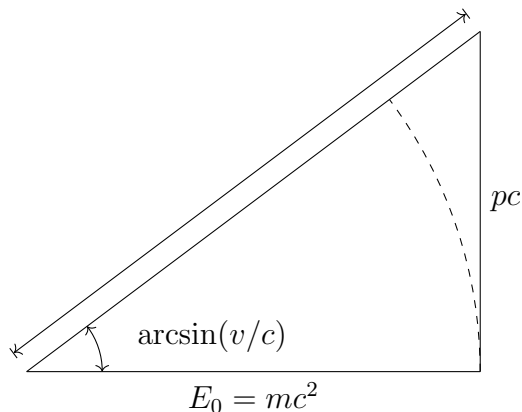
The total energy can be found as the sum of the kinetic and rest energies:

$$E = E_0 + K$$

Manipulating the equations for relativistic momentum and total energy yields the relationship

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

This can be remembered by



A particle traveling close to the speed of light ($v > 0.99c$), $K \gg E_0$. IN this case,

$$E \cong pc$$

This is the *extreme relativistic approximation* and can often simplify calculations. For massless particles (such as photons),

$$E = pc$$

All massless particles travel at c ; otherwise, their kinetic and total energies would be zero.

2.8 Conservation Laws in Relativistic Decays and Collisions

2.9 Extra

The **Epstein Cosmic Speedometer** is a plot of proper time (on the vertical axis) against space (on the horizontal axis). A body in motion will move through both space and time, making it an angle line. A stationary body will move only in time, making it a vertical line. The only thing able to move through space but not time is light, making it a horizontal line.

A spacetime diagram maintains the length of all vectors, plotting space on the horizontal axis and time on the vertical axis. The vertical component of a vector is the “proper” while its horizontal component is the speed through space.

Moving through space means moving slower through time; that is, moving objects effectively have slower clocks.