Calculus III

Arnav Patri

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Contents

Chapter 12

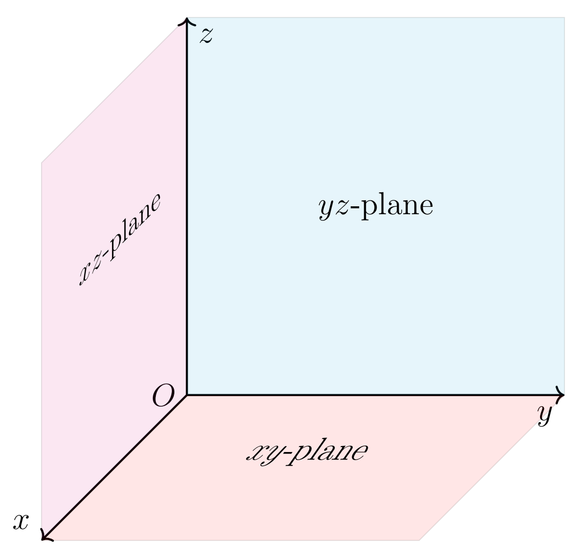
Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

Any point in a plane can be represented as an ordered pair of real numbers. Because this uses two numbers, a plane is called two-dimensional. To locate a point in space, a triplet of real-numbers is required.

3D Space

Before points can be represented in 3D space, a fixed point O (the origin) and three perpendicular lines that pass through it, called the **coordinate axes**. These axes are labeled the x-, y-, and z-axes. In general, the former two are horizontal while the third is vertical. The direction of the z-axis is determined by the **right-hand rule**. Curling the fingers of the right hand from the positive x-axis to the positive y-axis, the thumb will point in the direction of the positive z-axis.
The three coordinate axes determine the three **coordinate planes**.



Three planes divided space into eight **octants**. Illustrated above are the positive xz-, yz-, and xy-planes, constituting the **first octant**.
A point's **coordinates** are an ordered triple of real numbers. A point's **projection** onto a plane is the point with two of the same coordinates, the third becoming 0.

Plane	xy	yz	xz
(a, b, c)	$(a, b, 0)$	$(0, b, c)$	$(a, 0, c)$

The set of all ordered triples is the cartesian product of three sets of all real numbers, denoted appropriately by $\mathbb{R}^3$ and defined as

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

A one-to-one correspondence between points in space and ordered triples in $\mathbb{R}^3$ is a **three-dimensional coordinate system**. It should be noted that the first octant can be described as the set of points for which all coordinates are positive.

Distances and Spheres

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere The equation of a sphere with center (h, k, l) and radius r is

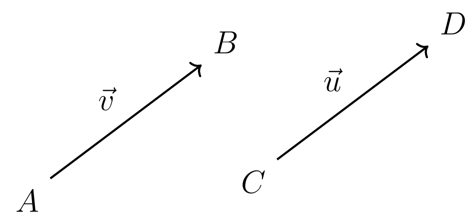
$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

12.2 Vectors

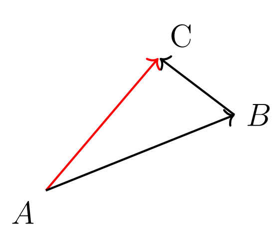
The term **vector** is used to indicate a quantity with both magnitude and direction.

Geometric Description of Vectors

A vector is often represented by an arrow, the length of which represents its magnitude. A vector is denoted by a letter in boldface ($\mathbf{v}$) or with an arrow above a letter ($\vec{v}$).
A **displacement vector** is a vector representing how something is displaced from its **initial point** (tail) to its **terminal point** (tip).



In the above figure, vector $\vec{v}$ has initial point A and terminal point B. This can be indicated by writing $\vec{v} = \overrightarrow{AB}$. If $AB = CD$ and the angles relative to the same axis are equal, then the $\vec{v}$ and $\vec{u}$ are **equivalent** (or **equal**).
The only vector without a direction is the **zero vector**, denoted by $\vec{0}$, which has length 0.
The sum of two vectors can be denoted with the initial point of the first and the terminal point of the second.

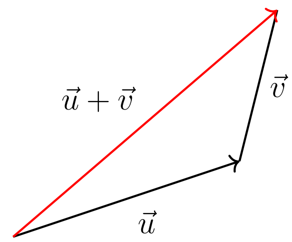


$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

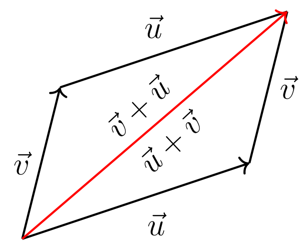
To add vectors, the one's tail can be moved to the other's tip and the resultant terminal point found.

Definition of Vector Addition The sum of two vectors position such that the initial point of one is the terminal point of the other is the vector from the initial point of the first to the terminal point of the second.

The definition of vector addition is sometimes referred to as the **Triangle Law**.



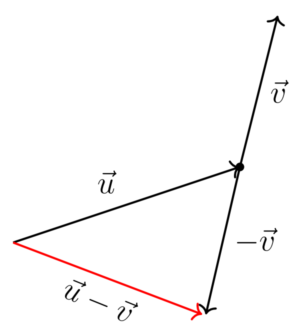
Doing the opposite addition results in the same resultant vector. This is made visible b y the **Parallelogram Law**.



A **scalar** is a number that is not a vector.

Definition of Scalar Multiplication The **scalar multiple** $c\vec{v}$ of a scalar c and a vector $\vec{v}$ is a vector of length $|c||\vec{v}|$ and whose direction is the same as $\vec{v}$ is $c > 0$ and in the opposite direction if $c < 0$. If $c = 0$ or $\vec{v} = \vec{0}$, then $c\vec{v} = \vec{0}$.

The **negative** of a vector $\vec{v}$, denoted by $-\vec{v}$, is the scalar multiple of the $\vec{v}$ and -1.
The **difference** $\vec{u} - \vec{v}$ is the sum of $\vec{u}$ and $-\vec{v}$.



Components of a Vector

A vector can be treated algebraically using **components**, which are the difference between the initial and terminal points in a dimension. The components of a vector $\vec{u}$ in 2 dimension and a vector $\vec{v}$ in 3 can be denoted as such:

$$\vec{u} = \langle u_1, u_2 \rangle \qquad\qquad \vec{v} = \langle v_1, v_2, v_3 \rangle$$

Geometric vectors can be though of as **representation** of algebraic vectors.

The representation of a vector with initial and terminal points is

$$\langle \Delta x, \Delta y, \Delta z \rangle$$

The **magnitude/length** of a vector, denoted $|\vec{v}|$ or $||\vec{v}||$ for a vector $\vec{v}$, is the length of any of its representations. This can be computed in n dimensions using the distance formula:

$$|\vec{v}| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Vectors can be added or subtracted algebraically by performing the desired operation their corresponding components.

$$\vec{u} \pm \vec{v} = \langle u_1 \pm v_1, u_2 \pm v_2, \ldots, u_n \pm v_n \rangle$$

The scalar multiple of a vector can be found by multiplying each of its components by the scalar.

$$c\vec{v} = \langle cv_1, cv_2, \ldots, cv_n \rangle$$

The set of all n-dimensional vectors is denoted by V_n. An n-dimensional vector is an n-tuple of real numbers, which are the vector's components.

Properties of Vectors If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in V_n and c and d are scalars, then

$$\begin{aligned} \vec{a} + \vec{b} &= \vec{b} + \vec{a} & \vec{a} + (\vec{b} + \vec{c}) &= (\vec{a} + \vec{b}) + \vec{c} \\ \vec{a} + \vec{0} &= \vec{a} & \vec{a} + (-\vec{a}) &= \vec{0} \\ c(\vec{a} + \vec{b}) &= c\vec{a} + c\vec{b} & (c + d)\vec{a} &= c\vec{a} + d\vec{a} \\ (cd)\vec{a} &= c(d\vec{a}) & 1\vec{a} &= \vec{a} \end{aligned}$$

A vector can be denoted in **unit-vector notation** as the sum of scalar multiples of **standard basis vectors** defined as such:

$$\hat{\imath} := \langle 0, 0, 1 \rangle \qquad \hat{\jmath} := \langle 0, 1, 0 \rangle \qquad \hat{k} := \langle 0, 0, 1 \rangle$$

A vector can therefore be rewritten using its components as such:

$$\vec{v} = \langle v_x, v_y, v_z \rangle = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

A **unit vector** is a vector of length 1. The standard basis vectors are unit vectors in the directions of the axes. A unit vector $\vec{u}$ in the same direction as a vector $\vec{v}$ is

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$$

The angle θ between a two-dimensional vector $\vec{v}$ and the positive x-axis can be calculated using inverse tangent:

$$\theta = \arctan\left(\frac{v_y}{v_x}\right)$$

This can be used to find the components of a vector of a given magnitude r and direction θ.

12.3 The Dot Product

Definition of the Dot Product The dot product of two vectors in V_n is the sum of the products of their corresponding components.

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

The dot product of two vectors is a real number. As such, the dot product is sometimes referred to as the **scalar/inner product**.

Properties of the Dot Product If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in V_n and c is a scalar, then

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2 \qquad \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} \qquad (c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$$

$$\vec{0}\cdot\vec{a} = 0$$

If θ is the angle between vectors, $\vec{u}$ and $\vec{v}$, then

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

Rewritten for θ, it can be said that

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\vec{v}|}$$

Two vectors are **orthogonal** (perpendicular) if and only if their dot product is 0.

$$\vec{u}\perp\vec{v} \iff \vec{u}\cdot\vec{v} = 0$$

Two vectors are **parallel** if their dot product is equal to the the product of their magnitudes or its negation, as θ is 0 or π.

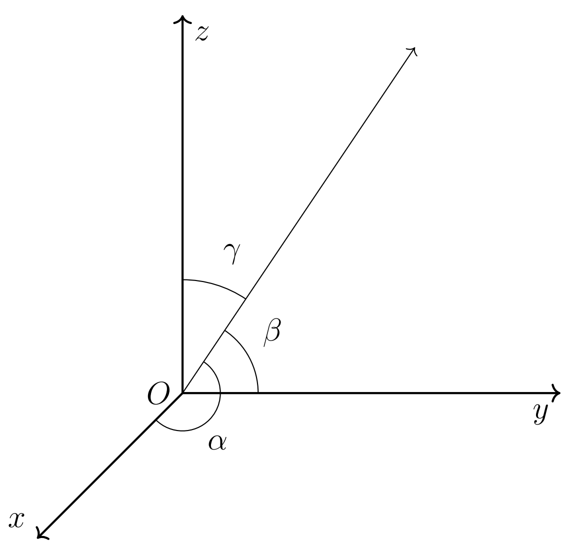
$$\vec{u} \parallel \vec{v} \iff \vec{u}\cdot\vec{v} = \pm|\vec{u}||\vec{v}|$$

Two vectors point in opposite directions if their dot product is equal to the negative of the product of their magnitudes, as θ is π.

$$\frac{\vec{u}}{|\vec{u}|} = -\frac{\vec{v}}{|\vec{v}|} \iff \vec{u}\cdot\vec{v} = -|\vec{u}||\vec{v}|$$

Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector are α, β, and γ. These are the angles made with the positive x-, y-, and z-axes respectively.



The cosines of these direction angles are called the **direction cosines**. These direction cosines can be derived using the dot products of the vector and the standard basis vectors.

$$\cos\alpha = \frac{\vec{v}\cdot\hat{\imath}}{|\vec{v}||\hat{\imath}|} = \frac{v_x}{|\vec{v}|} \qquad \cos\beta = \frac{\vec{v}\cdot\hat{\jmath}}{|\vec{v}||\hat{\jmath}|} = \frac{v_y}{|\vec{v}|} \qquad \cos\gamma = \frac{\vec{v}\cdot\hat{\mathrm{k}}}{|\vec{v}||\hat{\mathrm{k}}|} = \frac{v_z}{|\vec{v}|}$$

Squaring both sides and adding, the sum of the direction cosines can be shown to be equal to 1.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{v_x^2 + v_y^2 + v_z^2}{v_x^2 + v_y^2 + v_z^2} = 1$$

Rewriting for the components, it can be seen that the direction cosines are the components of the unit vector in the direction of the vector.

$$\begin{aligned}\vec{v} &= \langle v_x, v_y, v_z\rangle = \langle |\vec{v}|\cos\alpha, |\vec{v}|\cos\beta, |\vec{v}|\cos\gamma\rangle = |\vec{v}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle \\ \frac{\vec{v}}{|\vec{v}|} &= \langle\cos\alpha, \cos\beta, \cos\gamma\rangle\end{aligned}$$

Projections

The **scalar projection** of a vector $\vec{u}$ onto a vector $\vec{v}$ is (the **component of $\vec{u}$ along $\vec{v}$**) is the component of $\vec{u}$ in the direction of $\vec{v}$. This is denoted and found as

$$\operatorname{comp}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

The **vector projection** of this is simply equal to this scalar multiplied by a unit vector in the direction of $\vec{v}$.

$$\operatorname{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{u}}{|\vec{v}|^2}\vec{v}$$

The **orthogonal projection** of this is the the projection of $\vec{u}$ onto the vector perpendicular to $\vec{v}$, denoted and found as

$$\operatorname{orth}_{\vec{v}}\vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}}\vec{u}$$

The reason for this equation is that the projection and orthogonal projection of $\vec{u}$ onto $\vec{v}$ must logically sum to $\vec{u}$.

12.4 The Cross Product

The Cross Product of Two Vectors

A **determinant of order 2** is defined as the difference between the products of the diagonal terms and is denoted as such:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** can be defined in terms of second-order determinants.

$$\begin{aligned}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_2 + a_3b_1c_2 - a_3b_2c_1\end{aligned}$$

Definition of the Cross Product The **cross product** of two three-dimensional vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$$

The cross product of two vectors is orthogonal to both.
The magnitude of the cross product of two vectors $\vec{u}$ and $\vec{v}$ with angle θ between them is $\vec{0}$.

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

Two vectors are parallel if and only if their cross product is equal to zero.

$$\vec{u} \parallel \vec{v} \iff \vec{u} \times \vec{v} = \vec{0}$$

The cross products of the standard basis vectors are as follows:

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

Properties of the Cross Product If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and c is a scalar, then

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \qquad (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \qquad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Triple Products

The **scalar triple product** of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The volume of a parallelpiped determined by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product.

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

The **vector triple product** of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is

$$\vec{a} \times (\vec{b} \times \vec{c})$$

12.5 Equations of Lines and Planes

Lines

A line in three dimensional space is defined by a known point and a direction. The direction can be described by a vector parallel to the line. A **vector equation** of a line has a **parameter** of some scalar which is multiplied by the parallel vector. The result is then added to the initial point.

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

Rewritten in component form, this provides the **parametric equations** of the line.

The parametric equations of a line that posses through the point $P_0(x_0, y_0, z_0)$ and is parallel to the direction vector $\vec{v}$ are

$$x = x_0 + v_x t \qquad y = y_0 + v_y t \qquad z = z_0 + v_z t$$

If a vector is used to describe a line's direction, its components are called the line's **direction numbers**. As any parallel vector can be used, any set of numbers proportional to a given set of direction numbers can also be used as a set of direction numbers for the line.
A line can also be described by eliminating the parameter, solving each component's equation for the parameter.

$$t = \frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}$$

These equations are the **symmetric equations** of the line. It should be noted that the denominators are direction numbers. If one of the direction number is 0, the symmetric equations can still be found, simply equating the variable whose corresponding number is 0 to its known value. If v_x is 0,

$$x = x_0 \qquad \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}$$

The line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by the vector equation

$$\vec{r}(t) = (1 - t)\vec{r}_0 + t\vec{r}_1 \mid 0 \leq t \leq 1$$

Planes

While a line in space is determined by a point and its direction, a plane is more difficult to define. A single vector parallel to the plane is not able to properly convey its "direction", but one perpendicular to it does not completely specify it either. A plane is instead determined by a point in the plane and a vector $\vec{n}$, called the **normal vector**, that is orthogonal to the plane.
For a point $P_0(x_0, y_0, z_0)$ determining the plane with position vector $\vec{r}_0$, an arbitrary point $P(x, y, z)$ on the plane with position vector $\vec{r}$, and a normal vector $\vec{n}$, the vector $\vec{r} - \vec{r}_0$ is represented by $\overrightarrow{P_0P}$. As $\vec{n}$ is orthogonal to every vector in the plane, it is also perpendicular to $\vec{r} - \vec{r}_0$, so

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \implies \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

These equations are both **vector equations of the plane**.
A scalar equation for the plane can be derived by rewriting the vectors in terms of their components and evaluating the cross product.

A **scalar equation of the plane** through the point $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n}$ is

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$$

This can be rewritten as a **linear equation** in x, y, and z by expanding.

$$n_x x + n_y y + n_z z + d = 0 \mid d = -(n_x x_0 + n_y y_0 + n_z z_0)$$

Two planes are **parallel** if their normal vectors are parallel.

Distances

The distance from a point $P(x, y, z)$ to a plane $ax + by + cz + d = 0$ is equal to the absolute value of the scalar projection of the vector $\vec{b}$ from an arbitrary point $P_0(x_0, y_0, z_0)$ in the plane to P and the plane's normal vector.

$$D = |\operatorname{comp}_{\vec{n}} \vec{b}|$$

Expanding,

$$\begin{aligned} D &= \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} = \frac{|b_x n_x + b_y n_y + b_z n_z|}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = \frac{|n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0)|}{\sqrt{n_x^2 + n_y^2 + n_z^2}} \\ &= \frac{|(n_x x + n_y y + n_z z) - (z_x x_0 + n_y y_0 + n_z z_0)|}{\sqrt{n_x^2 + n_y^2 + n_z^2}} \end{aligned}$$

As P_0 lies in the plane, it satisfies the equation of the plane, allowing this to be further rewritten.

The distance D from a point $P(x, y, z)$ to a a plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Chapter 13

Vector Functions

13.1 Vector Functions and Space Curves

Vector-Valued Functions

A function is a rule that assigns each element in its domain to an element in its range. A **vector(-valued) function** is one with a domain of real numbers and a range of vectors.
A three-dimensional vector function can be written in terms of the sum of real valued functions corresponding to each component, called its **component functions**. If the components of $\vec{r}$ are real-valued functions f, g, and h, it can be written that

$$\vec{r} = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

Limits and Continuity

The **limit** of a vector function is defined by the limits of its component functions.

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t), \lim_{t\to c} g(t), \lim_{t\to c} h(t) \right\rangle$$

so long as the limits of the component functions exist.

A vector function $\vec{r}$ is **continuous** at c if

$$\lim_{t\to c} \vec{r}(t) = \vec{r}(c)$$

By extension, its component functions must also be continuous at c.

Space Curves

A **space curve** is a set C of all points (x, y, z) in space where x, y, and x, y, and z are determined by continuous real-valued functions on an interval I. These functions' equations are the **parametric equations of C**, and t is its **parameter**.
The collision of two space curves (being equal at the same parameter value) can be determined by equating each corresponding component and seeing if there are any values of the component for which all three equations are satisfied.
The intersection of two space curves can be determined by writing the equations in terms of different parameters, creating a system of equations equating each corresponding component and solving for one of the parameters to plug back into its corresponding vector equation to find the point.

13.2 Derivatives and Integrals of Vector Functions

Derivatives

The derivative $\vec{r}'$ of a vector function $\vec{r}$ is defined by the limit of the difference quotient (if it exists), just like the derivative of a real-valued function.

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{r}'(t) = \lim_{h\to 0}\left[\frac{\vec{r}(t+h)-\vec{r}(t)}{h}\right]$$

$\vec{r}'(t)$ is the **tangent vector** to the curve defined by $\vec{r}$ at t, provided $\vec{r}'(t)$ exists and is not $\vec{0}$. The **tangent line** to C at $\vec{r}(t)$ is the line through $\vec{r}(t)$ parallel to $\vec{r}'(t)$.

The derivative of a vector function can be found as that of each of its components.

If $\vec{r}(t) = \langle f(t), g(t), h(t)$ where each component function is differentiable,

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

A unit vector in the direction of the tangent vector is the **unit tangent vector $\vec{T}$**, defined by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

The **second derivative** of a vector function $\vec{r}$ is the derivative of $\vec{r}'$.

Differentiation Rules

If $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{u}(t)+\vec{v}(t)] = \vec{u}'(t)+\vec{v}'(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[c\vec{u}(t)] = c\vec{u}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}'(t)\cdot\vec{v}(t)+\vec{v}(t)\cdot\vec{u}(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t)+\vec{v}'(t)\times\vec{u}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t)+f(t)\vec{u}'(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$$

Integrals

The **definite integral** of a continuous vector function can be defined similarly to that of a continuous real-valued function.

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \lim_{n\to\infty}\sum_{i=1}^{\infty}\vec{r}(t_i^*)\Delta t$$

This definition can be rewritten in terms of components.

$$\begin{aligned}\int_a^b \vec{r}(t)\,\mathrm{d}t &= \left\langle \int_a^b f(t)\,\mathrm{d}t, \int_a^b g(t)\,\mathrm{d}t, \int_a^b h(t)\,\mathrm{d}t\right\rangle \\ &= \lim_{n\to\infty}\left\langle \sum_{i=1}^{n} f(t_i^*)\Delta t, \sum_{i=1}^{n} g(t_i^*)\Delta t, \sum_{i=1}^{n} h(t_i^*)\Delta t\right\rangle\end{aligned}$$

The fundamental theorem of calculus can be extended to vector functions.

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \left[\vec{R}(t)\right]_a^b = \vec{R}(b)-\vec{R}(a)$$

The constant of integration for the indefinite integral of a vector function is itself a vector $\vec{C}$.

$$\int \vec{r}(t)\,\mathrm{d}t = \vec{R}(t)+\vec{C}$$

13.3 Arc Length and Curvature

Arc Length

The arc length of a vector function is simply the integral of its derivative's magnitude.

$$L = \int_a^b |\vec{r}'(t)|\mathrm{d}t = \int_a^b \left[\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\right] \mathrm{d}t$$

A single curve can be represented by multiple vector functions, each of which is a **parametrization** of the original curve. Regardless of which parametrization is used, the arc length will be the same (so long as parameters are converted between), as arc length is a geometric property, making it independent of the parametrization used.

The Arc Length Function

For a curve C given by a continuous vector function $\vec{r}$ parametrized using $t \in [a, b]$, the **arc length function** is defined as

$$s(t) = \int_a^t |\vec{r}'(u)| \,\mathrm{d}u$$

Differentiating both sides (applying the fundamental theorem of calculus for the right side), it can be seen that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\vec{r}'(t)|$$

This **Parametrization of a curve with respect to arc length** can be used to analyze the curve, as arc length is not dependent on a particular coordinate system of parametrization, instead being an inherent geometric property of the curve's.

Curvature

A parametrization is called **smooth** on an interval if its derivative is continuous and not equal to $\vec{0}$ at any point in the interval.

The curvature of a curve C at a given point is a measure of how quickly it is changing direction. More specifically, it is the magnitude of the rate of change of the unit tangent vector with respect to arc length. (Arc length is used due to its parametrization-independence.)

The **curvature** of a curve is

$$\kappa = \left|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|$$

The curvature can be more easily computed by expressing it in terms of parameter t instead of s. Using chain rule,

$$\frac{\mathrm{d}\vec{T}}{\mathrm{d}s} = \frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \implies \kappa = \left|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\vec{T}/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t}\right|$$

Rewriting by differentiating the formula for arc length,

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

It may sometimes be more convenient to re-express curvature as

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

The Normal and Binormal Vectors

As the magnitude of the unit tangent vector is always 1 (by definition), the dot product of it and its derivative is 0 (as this is true of any vector function of constant magnitude). The **(principle) unit normal vector $\vec{N}$** is the unit vector of the derivative of the unit tangent vector.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

This vector can be thought of as indicating the direction that a curve is turning at a point.
The **binormal vector $\vec{B}$** is the cross product of the unit tangent and normal vectors.

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The plane determined by the normal and binormal vectors at point P on curve C is the **normal plane**. It consists of all lines that are orthogonal to the tangent vector.
The plane determined by the tangent and normal vectors is the **osculating plane**.
The **circle of curvature** or **osculating circle** at point P is the circle in the osculating plane with radius $1/\kappa$ whose edge contains P. The center of this circle is the **center of curvature**.

Torsion

The curvature at a point indicates how tightly a curve "bends". As the tangent vector is orthogonal to the normal plane, $\mathrm{d}\vec{T}/\mathrm{d}s$ shows how the normal plane changes moving along the curve. As the binormal vector is orthogonal to the osculating plane, $\mathrm{d}\vec{B}/\mathrm{d}s$ shows how the osculating plane changes.
As $\mathrm{d}\vec{B}/\mathrm{d}s$ is parallel to the normal vector, there is a scalar τ such that

$$\frac{\mathrm{d}\vec{B}}{\mathrm{d}s} = -\tau\vec{N}$$

Taking the dot product with $\vec{N}$ on both sides,

$$\tau(t) = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}s} \cdot \vec{N}$$

Torsion is easier to compute using parameter t, so using chain rule,

$$\tau(t) = -\frac{\vec{B}'(t) \cdot \vec{N}(t)}{|\vec{r}'(t)|}$$

Torsion can also be computed as

$$\tau(t) = \frac{(\vec{r}'(t) \times \vec{r}''(t)) \cdot \vec{r}'''(t)}{|\vec{r}'(t) \times \vec{r}''(t)|^2}$$

13.4 Motion in Space: Velocity and Acceleration

Velocity, Speed, and Acceleration

Velocity is the derivative of position with respect to time.

$$\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$$

Speed is the magnitude of velocity. It can also be found as the derivative of arc length (distance traveled) with respect to time.

$$|\vec{v}(t)| = \frac{\mathrm{d}s}{\mathrm{d}t}$$

Acceleration is the derivative of velocity.

$$\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}$$

Newton's Second Law of Motion states that for a force $\vec{F}$ acting on an object of mass m that produces an acceleration $\vec{a}$,

$$\vec{F}(t) = m\vec{a}(t)$$

Projectile Motion

The magnitude of acceleration due to gravity is

$$g \approx 9.8\,\mathrm{m/s^2}$$

Tangential and Normal Components of Acceleration

The tangential and normal components of acceleration can be found as

$$a_T = \frac{\mathrm{d}|\vec{v}|}{\mathrm{d}t} \qquad\qquad a_N = \kappa|\vec{v}|^2$$

Kepler's Laws of Planetary Motion

Kepler's laws are as follows:

1. A planet orbits in an elliptical orbit with the sun as one of its foci.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution is proportional to the cube of the semimajor axis of the orbit.

Newton's law of gravitation states that

$$\vec{F}_g = -\frac{GMm}{r^3}\vec{r} = -\frac{GMm}{r^2}\vec{u}$$

Chapter 14

Partial Derivatives

14.1 Functions of Several Variables

Functions of Two Variables

> A **function of two variables** is a rule that assigns to each ordered pair of real numbers in its domain to a real number in its range.

For a function $z = f(x, y)$, x and y are the **independent variables** and z is the **dependent variable**. A function of two variables is a function with its domain as a subset of $\mathbb{R}^2$ and its range is a subset of $\mathbb{R}$.

Graphs

> The **graph** of two-variable function f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in a subset of $\mathbb{R}^2$.

Level Curves and Contour Maps

> The **level curves** of a function F of two variables are the curves with equations $f(x, y) = k$ where k is a constant within the range of f.

A collection of level curves is a **contour map**.

14.2 Limits and Continuity

Limits of Functions of Two Variables

The notation

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

is used to denote that the values of $f(x, y)$ approaches L as x approaches a and y approaches b.

> For a function of two variables f, the **limit of $\boldsymbol{f(x, y)}$ as $\boldsymbol{(x, y)}$ approaches $\boldsymbol{a, b}$** is denoted as
>
> $$\lim_{(x,y)\to(a,b)} f(x, y)$$
>
> This is equal to L if, for every $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that
>
> $$(x, y) \in D \land 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon$$

Showing that a Limit Does Not Exist

The limit does not exist at (a, b) if f approaches two different values when approaching from different paths.

Properties of Limits

The following are true of limits:

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x\ toa} f(x) \pm \lim_{x\ toa} g(x) \qquad \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$$

$$\lim_{x \to a} [f(x) \times g(x)] = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x) \qquad \lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

A **polynomial** function is a sum of terms in the form $cx^m y^n$ where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of two polynomials.

Continuity

A function $f(x, y)$ is **continuous** at (a, b) if

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

It is continuous on an interval if this it is continuous at every point within.

Polynomials and rational functions are continuous on their entire domains.

Functions of Three or More Variables

Everything thus far can be extended to functions of more than two variables.

14.3 Partial Derivatives

Partial Derivatives of Functions of Two Variables

A **partial derivative** of a two-variable function assumes treats all but the variable being differentiated with respect to as constants. It is the instantaneous rate of change in the direction of the variable.

The partial derivatives of a function f are the functions f_x and f_y, defined as

$$f_x(x, y) = \lim_{h \to 0} \left[\frac{f(x + h, y) - f(x, y)}{h} \right] \qquad f_y(x, y) = \lim_{h \to 0} \left[\frac{f(x, y + h) - f(x, y)}{h} \right]$$

The partial derivative of a function $z = f(x, y)$ with respect to variable x can be denoted

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

When evaluating a partial derivative with respect to one variable, all other dependent variables can be regarded as constant.

Interpretations of Partial Derivatives

The partial derivative of $f(x, y)$ with respect to x is the slope of the tangent line parallel to the x-axis while that with respect to y is the slope of the tangent line parallel to the y-axis.

Higher Derivatives

The partial derivatives of a function of two variables f are themselves functions of two variables with can be differentiated. These are the **second partial derivatives** of f. If $z = f(x, y)$, two of these can be denoted

$$\begin{array}{cccccc} (f_x)_x & f_{xx} & f_{11} & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 z}{\partial x^2} \\ (f_x)_y & f_{xy} & f_{12} & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 z}{\partial x \partial y} \end{array}$$

Using Clairaut's theorem, it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if all are continuous.

14.4 Tangent Planes and Linear Approximations

Tangent Planes

Equation of a Tangent Plane If function f has continuous partial derivatives, an equation to the tangent plane to the surface $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear Approximations

The **linearization** L of f at (a, b) is its tangent plane, which can be used to approximate $f(x, y)$ at points close to (a, b).

$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

If $z = f(x)$, f can be said to be **differentiable** if Δz can be expressed as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy that approach 0 as $(\Delta x, \Delta y)$ approaches $(0, 0)$.

Differentials

For a differentiable function of a single variable $y = f(x)$, the differential $\mathrm{d}x$ is defined to be an independent variable. The differential of y is then defined as

$$\mathrm{d}y = f'(x)\mathrm{d}x$$

Δy is the change in the curve $f(x)$ while $\mathrm{d}y$ is the change in the tangent line for a change in x $\mathrm{d}x = \Delta x$. For a differentiable function of two variables $z = f(x, y)$, the **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are defined to be independent variables. The **(total) differential** $\mathrm{d}z$ is defined by

$$\mathrm{d}z = f_x(x, y)\mathrm{d}x + f_y(x, y)\mathrm{d}y = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

14.5 The Chain Rule

The chain rule for functions of a single variable provides a rule for differentiating a composite function; if $y = f(u)$ and $u = g(x)$, y is an indirectly differentiable function of x with derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

The Chain Rule: Case 1

For a function of more than 1 variable, there are multiple versions of the chain rule. The first version deals with the case where $z = f(x, y)$ and both x and y are functions of variable t, making z an indirect function of t.

> If $z = f(x, y)$ is a differentiable function of x and y where x and y are functions of t, then z is a differentiable function of t such that
>
> $$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}x}{\mathrm{d}t}$$

The Chain Rule: Case 2

> If $z = f(x, y)$ is a differentiable function of x and y where x and y are functions of t and u,
>
> $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \qquad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

In this case, t and u are **independent variables**, x and y are **intermediate variables**, and z is the **dependent** variable.

The Chain Rule: General Version

> If u is a differentiable function of n variables $x_1, x_2, \ldots, x_n$, each of which is a differentiable function of m variables $t_1, t_2, \ldots, t_m$,
>
> $$\frac{\partial u}{\partial t_i} = \sum_{j=1}^{i} \frac{\partial u}{\partial x_m}\frac{\partial x_i}{\partial t_i}$$
>
> for each i in $1, 2, \ldots, m$.

Implicit Differentiation

Suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a function of x. If F is differentiable, Case 1 of the Chain Rule can be applied to differentiate both sides of the equation.

$$\begin{aligned} 0 &= F(x, y) \\ 0 &= \frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

As $\mathrm{d}x/\mathrm{d}x$ is equal to 1,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

The **Implicit Function Theorem** states that if F is defined on a disk containing (a,b) where $F(a,b) = 0$, $F_y(a,b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x,y) = 0$ defined y as a function of x near the point (a,b).

14.6 Directional Derivatives and the Gradient Vector

Directional Derivatives

The partial derivatives f_x and f_y represent the rates of change in the x and y directions. To find the rate of change in the direction of an arbitrary unit vector $\vec{u} = \langle a, b \rangle$, the slope of the line tangent to the curve C formed by the intersection of the surface S formed by the equation $z = f(x,y)$ and the vertical plane that passes through the point of differentiation $P(x_0, y_0, z_0)$ in the direction of $\vec{u}$ can be evaluated.
If $Q(x,y,z)$ is another point on C and P' and Q' are the projections of P and Q onto the xy-plane, the vector $\overrightarrow{P'Q'}$ is parallel to $\vec{u}$, so

$$\overrightarrow{P'Q'} = h\vec{u}$$

for some scalar h. From this, it can be seen that

$$x - x_0 = ha \qquad\qquad y - y_0 = hb$$
$$x = x_0 + ha \qquad\qquad y = y_0 + hb$$

Applying the difference quotient using z,

$$\frac{\Delta z}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Taking the limit as h approaches 0 provides the slope of the tangent line in the direction of $\vec{u}$.

> The **directional derivative** of f at (x_0, y_0) in the direction of unit vector $\vec{u} = \langle a, b \rangle$ is
>
> $$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \left[\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \right]$$

The directional derivative can be computed without using limits as well.

> If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b, \rangle$ and
>
> $$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$$

The Gradient Vector

The equation for the directional derivative of a differentiable function can be written as the dot product

$$D_{\vec{u}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$$

The first vector in this dot product is called the **gradient** of f.

> If f is a function of two variables x and y, the **gradient** of f is the vector function ∇f defined by
>
> $$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

Rewriting the equation for the directional derivative using the gradient vector,

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

Functions of Three Variables

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of unit vector $\vec{u} = \langle a, b, c \rangle$ is

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \left[\frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h} \right]$$

This can be rewritten using vectors, defining $\vec{r}_0$ to be $\langle x_0, y_0$ if $n = 2$ and $\langle x_0, y_0, z_0$ is $n = 3$.

$$D_{\vec{u}} f(\vec{r}_0) = \lim_{x \to 0} \left[\frac{f(\vec{r}_0 + h\vec{u}) - f(\vec{r}_0)}{h} \right]$$

The **gradient vector** is

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \hat{\imath} + \frac{\partial f}{\partial y} \hat{\jmath} + \frac{\partial f}{\partial z} \hat{\mathrm{k}}$$

Rewriting the directional derivative using the gradient vector,

$$D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}$$

Maximizing the Directional Derivative

The way that the directional derivative is defined, it is clear that its maximum magnitude is simply that of gradient.

$$\max\{D_{\vec{u}} f(\vec{r})\} = |\nabla f(\vec{r})|$$

As the dot product is maximized for parallel vectors, this must occur when $\vec{u}$ is in the same direction as the gradient.

Tangent Planes to Level Surfaces

Suppose the surface S has equation $F(x, y, z) = k$ (making it a level surface of a function F for all 3 variables). Let point $P(x_0, y_0, z_0)$ be a point on S and C be a curve on S that passes through P. C is described by a continuous vector function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. Let $\vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. As P is on C, it must satisfy the equation of S.

$$F(x(t), y(t), z(t)) = k$$

Using the chain rule to differentiate,

$$\frac{\partial F}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z} \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

This can be rewritten as

$$\nabla F \cdot \vec{r}'(t) = 0$$

Plugging in the values for P,

$$\nabla F(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0$$

This means that the gradient vector at P is perpendicular to the tangent vector to any curve on S that passes through P. If $\nabla F(x_0, y_0, z_0 \neq \vec{0}$, the **tangent plane to the level surface** $F(x, y, z) = k$ at P can be defined as $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane passing through P with normal vector $\nabla F(x_0, y_0, z_0)$. As an equation,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The **normal line** to S at P is that passing through P perpendicular to the tangent plane. This is given by the gradient vector, making its symmetric equations

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Significance of the Gradient Vector

Properties of the Gradient Vector Let f be a differentiable function of 2 or 3 variables and that $\lambda f \neq \vec{0}$.

1. The directional derivative of f at $\vec{r}$ in the direction of unit vector $\vec{u}$ is given by

$$D_{\vec{u}} f(\vec{r}) = \nabla f(\vec{r}) \cdot \vec{u}$$

2. $\nabla f(\vec{r})$ is in the direction of the maximum rate of increase at $\vec{r}$. This maximum rate of change is $|\nabla f(\vec{r})|$

3. $\nabla f(\vec{r})$ is perpendicular to the level curve/surface of f through $\vec{r}$.

14.7 Maximum and Minimum Values

Local Maximum and Minimum Values

The local extrema of a multivariable function $f(x, y)$ are the points at which $f(x, y)$ is maximized or minimized relative to all points around it.
In order for f to have a local extremum at (a, b), its first derivatives with respect to x and y must be 0 at that point (should they exist).
A **critical point** is a point at which the first derivatives are both equal to 0. A critical point that is not an extremum is a **saddle point**.

Second Derivatives Test If the second partial derivatives of f are continuous on a disk with center (a, b) and (a, b) is a critical point,

- If $D > 0$
  - and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum
  - and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum
- If $D < 0$, $f(a, b)$ is a saddle point

where

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(ab))^2$$

D can be remembered as the determinant

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

14.8 Lagrange Multipliers

Lagrange's method enables the optimization of a function $f(x, y, z)$ subject to a constraint of the form $g(x, y, z) = k$.

Lagrange Multipliers: One Constraint

To maximize $f(x, y)$ with constraint $g(x, y) = k$ is to find the maximum value of c for which the level curve $f(x, y) = c$ intersects $g(x, y) = k$. This occurs when the curves share a common tangent line. The normal lines where they touch are identical, meaning that the gradient vectors are parallel, so

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

for some scalar λ, called a **Lagrange multiplier**.

Method of Lagrange Multipliers To find the extrema of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$,

1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all points (x, y, z) that result from step 1, comparing each result to find the extrema.

Expanding into components creates a system of four equations that can be used to solve for the four unknowns x, y, z, and λ.

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

Lagrange Multipliers: Two Constraints

The extreme values of $f(x, y, z)$ can also be found under the two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$. These are the extrema of f on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. As ∇g and ∇h are orthogonal to g and h, they are also orthogonal to C. This means that $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. As such, there are two scalars (Lagrange multipliers) λ and μ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

Writing this in terms of components creates a system of five equations that can be used to solve for the five unknowns x, y, z, λ, and μ.

$$f_x = \lambda g_x + \mu h_x \qquad f_y = \lambda g_y + \mu h_y \qquad f_z = \lambda g_z + \mu h_z \qquad g(x, y, z) = k \qquad h(x, y, z) = c$$

Chapter 15

Multiple Integrals

15.1 Double Integrals over Rectangles

Just as the area problem leads to the definite integral, the definition of the double integral can be found by attempting to find the volume of a solid.

Review of the Definite Integral

To find the area of a curve f over the interval $[a, b]$, the interval can be split into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$. Sample points x_i^* can then be chosen in each subinterval. The Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

can be evaluated as n approaches infinity to obtain the definite integral of f from a to b.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Volumes and Double Integrals

Consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \big\{(x, y) \in \mathbb{R}^2 \mid a \le x \le b \wedge c \le y \le d\big\}$$

Suppose that $f(x, y) \ge 0$. The graph of f is the surface $z = f(x, y)$. Let S be a solid that lies above R and under the graph of f.

$$S = \big\{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le f(x, y) \wedge (x, y) \in R\big\}$$

To find the volume of S, R can first be divided into sub-rectangles, dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and the interval $[c, d]$ into n subintervals $[y_{i-1}, y_i]$ of equal width $\Delta y = (d - c)/n$. Drawing lines parallel to the axes through the endpoints of these subintervals forms the sub-rectangles

$$R_{i,j} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] = \{(x, y) \mid x_{i-1} \le x \le x_i \wedge y_{j-1} \le y \le y_j\}$$

of equal area

$$\Delta A = \Delta x \Delta y$$

Selecting a **sample point** $\left(x_{i,j}^*, y_{i,j}^*\right)$ in each $R_{i,j}$ enables the approximation of the part of S that lies above $R_{i,j}$, creating boxes of volume

$$f\left(x_{i,j}^*, y_{i,j}^*\right)\Delta A$$

The sum of these boxes can be used to approximate the total volume of S.

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{i,j}^*, y_{i,j}^*\right)\Delta A$$

This is referred to as the **double Riemann sum**.

The **double integral** of f over the rectangle R is

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{i,j}^*, y_{i,j}^*\right)\Delta A$$

if this limit exists.

This definition of the double integral means that for every number $\varepsilon > 0$, there is an integer N such that

$$\left|\iint\limits_R f(x,y)\,\mathrm{d}A - \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{i,j}^*, y_{i,j}^*\right)\Delta A\right| < \varepsilon$$

for all integers m and n greater than N and for any choice of sample points $\left(x_{i,j}^*, y_{i,j}^*\right)$ in $R_{i,j}$.
A function is **integrable** if the limit definition of its definite integral converges. If f is bounded on R and f is continuous over R (except possibly on a finite number of smooth curves), then f is integrable over R.
The sample point $\left(x_{i,j}^*, y_{i,j}^*\right)$ can be chosen to be any point in sub-rectangle $R_{i,j}$, but if the upper-right corner is chosen, the expression for a double integral can be simplified.

$$\iint f(x,y)\,\mathrm{d}a = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

If $f(x,y) \geq 0$, the volume V of a solid that lies above rectangle R and below surface $z = f(x,y)$ is

$$V = \iint\limits_R f(x,y)\,\mathrm{d}A$$

The Midpoint Rule

Midpoint Rule for Double Integrals

$$\iint\limits_R f(x,y)\,\mathrm{d}A \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_i)\Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}$ is that of $[y_{i-1}, y_i]$.

Iterated Integrals

Single definite integrals are generally far easier to solve using the fundamental theorem of calculus rather than their definition. The same is true for double integrals, simply using two single integrals.

Suppose f is a function of two variables that is integrable over rectangle $R = [a, b] \times [c, d]$. The notation

$$\int_c^d f(x, y)\,\mathrm{d}y$$

is used to denote that x is held constant and $f(x, y)$ is integrated with respect to y. This is called *partial integration with respect to y.*
As the value of $\int_c^d f(x, y)\,\mathrm{d}y$ is dependent on x, it defines a function in terms of x. Integrating this function with respect to x over $[a, b]$ results in an **iterated integral**

$$\int_a^b \int_c^d f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

This integral is evaluated *from the inside out.*

Fubini's Theorem If f is continuous over rectangle

$$R = \{(x, y) \mid a \leq x \leq b \wedge c \leq y \leq d\}$$

then

$$\iint\limits_R f(x, y)\,\mathrm{d}A = \int_a^b \int_c^d f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \int_a^b f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

so long as f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

In the special case that $f(x, y)$ is factorable into the product of functions g and h of only x and only y,

$$\iint\limits_R g(x)h(y)\,\mathrm{d}A = \int_a^b g(x)\,\mathrm{d}x \int_c^d h(y)\,\mathrm{d}y$$

where $R = [a, b] \times [c, d]$.

Average Value

The **average value** of a single-variable function f on the interval a, b is

$$f_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x)\,\mathrm{d}x$$

That of a function of two variables defined on rectangle R is similarly

$$f_{\text{avg}} = \frac{1}{A(R)} \iint\limits_R f(x, y)\,\mathrm{d}A$$

where $A(R)$ is the area of R.

15.2 Double Integrals over General Regions

General Regions

Suppose a general region D is bounded, meaning that it can be enclosed by a rectangular region R. In order to integrate a function f over D, a new function F is defined with domain R as

$$F(x, y) = \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \notin D \wedge (x, y) \in R \end{cases}$$

If F is integrable over R, the **double integral of $\boldsymbol{f}$ over $\boldsymbol{d}$** is defined as

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A$$

A plane region D is **type I** if it lies between the graphs of two continuous functions of x.

$$D = \{(x,y) \mid a \le x \le b \wedge g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$. (It should be noted that g_1 and g_2 need only be continuous, meaning that they may be piecewise.)

> If D is a continuous type I region described by
>
> $$D = \{(x,y) \mid a \le x \le b \wedge g_1(x) \le y \le g_2(x)\}$$
>
> then
>
> $$\iint_D f(x,y)\,\mathrm{d}A \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

A plane region of **type II** can be expressed as

$$D = \{(x,y) \mid c \le y \le d \wedge h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous.

> If D is a continuous type II region described by
>
> $$D = \{(x,y) \mid c \le y \le d \wedge h_1(y) \le x \le h_2(y)$$
>
> then
>
> $$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Changing the Order of Integration

The order of integration can be changed to make an integral easier to evaluate.

15.3 Double Integrals in Polar Coordinates

Review of Polar Coordinates

The polar coordinates (r,θ) are related to their corresponding rectangular coordinates (x,y) by the following formulas:

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

Double Integrals in Polar Coordinates

A **polar rectangle** R is described by

$$R = \{(r, \theta) \mid a \leq r \leq b \wedge \alpha \leq \theta \leq \beta\}$$

> If f is continuous on polar rectangle R given by
>
> $$R = \{(r, \theta) \mid a \leq r \leq b \wedge \alpha \leq \theta \leq \beta\}$$
>
> where
>
> $$0 \leq \beta - \alpha \leq 2\pi$$
>
> then
>
> $$\iint\limits_R f(x, y)\,\mathrm{d}A = \int_\alpha^\theta \int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\mathrm{d}\theta$$

> If f is continuous on a polar region D of the form
>
> $$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta \wedge h_1(\theta) \leq r \leq h_2(\theta)$$
>
> then
>
> $$\iint\limits_D f(x, y)\,\mathrm{d}A = \int_\alpha^\theta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

15.4 Applications of Double Integrals

Density and Mass

If a lamina occupies a region D in the xy-plane and its **density** at a point (x, y) in D is given by $\rho(x, y)$, a continuous function on D,

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of small rectangles and the limit is taken as the dimensions of the rectangle approach 0. The total mass of the lamina is determined by

$$m = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho\big(x_{i,j}^*, y_{i,j}^*\big)\Delta A = \iint\limits_D \rho(x, y)\,\mathrm{d}A$$

Other types of density are similar. If an electric charge is distributed over region D and the charge density is given by $\sigma(x, y)$ at point (x, y) in D, then the total **electric charge** Q is given by

$$Q = \iint\limits_D \sigma(x, y)\,\mathrm{d}A$$

Density and Mass

The **moment** of a lamina **about the x-axis** is given by

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{i,j}^* \rho\big(x_{i,j}^*, y_{i,j}^*\big)\Delta A = \iint\limits_D y\rho(x, y)\,\mathrm{d}A$$

The **moment about the y-axis** is similarly

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}^{*}\rho\big(x_{i,j}^{*}, y_{i,j}^{*}\big)\Delta A = \iint\limits_{D} x\rho(x,y)\,\mathrm{d}A$$

The center of mass $(\bar{x}, \bar{y})$ is defined such that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. The lamina behaves as though its entire mass is concentrated at this point.

> The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying region D with density function $\rho(x,y)$ are
>
> $$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_{D} x\rho(x,y)\,\mathrm{d}A \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint\limits_{D} y\rho(x,y)\,\mathrm{d}A$$
>
> where mass m is given by
>
> $$m = \iint\limits_{D} \rho(x,y)\,\mathrm{d}A$$

Moment of Inertia

The **moment of inertia** or **second moment** of a particle of mass m about an axis is defined as mr^2, where m is the particle's mass and r is its distance from the axis. Extending this concept to lamina, the **moment of inertia** of a lamina **about the x-axis** is

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \big(y_{i,j}^{*}\big)^2\rho\big(x_{i,j}^{*}, y_{i,j}^{*}\big)\Delta A = \iint\limits_{D} y^2\rho(x,y)\,\mathrm{d}A$$

and the **moment of inertia about the y-axis** is

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \big(x_{i,j}^{*}\big)^2\rho\big(x_{i,j}^{*}, y_{i,j}^{*}\big)\Delta A = \iint\limits_{D} x^{*}\rho(x,y)\,\mathrm{d}A$$

The **moment of inertia about the origin** or **polar moment of inertia** is

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Big(\big(x_{i,j}^{*}\big)^2 + \big(y_{i,j}^{*}\big)^2\Big)\rho\big(x_{i,j}^{*}, y_{i,j}^{*}\big)\Delta A = \iint\limits_{D} \big(x^2+y^2\big)\rho(x,y)\,\mathrm{d}A = I_x + I_y$$

The **radius of gyration of a lamina about an axis** is the number R such that

$$mR^2 = I$$

The radii of gyration with respect to the x- and y-axes are given by

$$m\bar{\bar{y}}^2 = I_x \qquad m\bar{\bar{x}}^2 = I_y$$

$(\bar{\bar{x}}, \bar{\bar{y}})$ is the point at which tyhe mass could be concentrated without affecting the moment of inertia with respect to the coordinate axes.

Probability

The **probability density function** f of a continuous random variable X defines the probability of X falling between a and b as

$$P(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x$$

The **joint density function** of two continuous random variables X and Y is a function f of two variables such that the probability that (X, Y) lies in a region D is given by

$$P((X,Y) \in D) = \iint_D f(x,y)\,\mathrm{d}A$$

If this region is a rectangle,

$$P((x_1 \leq X \leq x_2) \wedge (y_1 \leq Y \leq y_2)) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

As probabilities can't be negative and are measured from 0 to 1

$$f(x,y) \geq 0 \qquad\qquad \iint_{\mathbb{R}^2} f(x,y)\,\mathrm{d}A = 1$$

If X and Y are random variables with respective probability density functions f_X and f_Y, X and Y are **independent random variables** if their joint density function is the product of their individual density functions.

$$f(x,y) = f_X(x) f_Y(y) \implies X \text{ and } Y \text{ are independent}$$

Expected Values

The **mean** of a continuous random variable X with probability density function f is

$$\mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

If X and Y are independent random variables with joint density function f, then the ***X*-** and ***Y*-means** or **expected values of *X* and *Y*** are

$$\mu_X = \iint_{\mathbb{R}^2} x f(x,y)\,\mathrm{d}A \qquad\qquad \mu_Y = \iint_{\mathbb{R}^2} y f(x,y)\,\mathrm{d}A$$

These are the coordinates of the center of mass of the probability density function.

15.5 Surface Area

The surface area of a surface with equation $z = f(x,y)$ with $(x,y) \in D$ where f_x and f_y are continuous is

$$A(S) = \iint_D \left[\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\right] \mathrm{d}A$$

15.6 Triple Integrals

Triple Integrals over Rectangular Boxes

A rectangular box B is defined as

$$B = \{(x, y, z) \mid a \le x \le b \land c \le y \le d \land r \le z \le s\}$$

The **triple Riemann sum** can be used to define the **triple integral** of f over B.

$$\iiint\limits_B f(x, y, z)\,\mathrm{d}V = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x^*_{i,j,k}, y^*_{i,j,k}, z^*_{i,j,k}\right)\Delta V$$

Fubini's Theorem for Triple Integrals If f is continuous over the rectangular box $R = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint\limits_D f(x, y, z)\,\mathrm{d}V = \int_a^b \int_c^d \int_r^s f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Triple Integrals over General Regions

A solid region E is **type 1** if it lies between the graphs of two continuous functions of x and y; that is to say,

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection of E onto the xy-plane.
The integral over a type 1 solid region E with xy-projection D is

$$\iiint\limits_E f(x, y, z)\,\mathrm{d}V = \iint\limits_D \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}A$$

If D is itself a type I plane region, then

$$E = \{(x, y, z) \mid a \le x \le b \land g_1(x) \le y \le g_2(x) \land u_1(x, y) \le z \le u_2(x, y)\}$$

and

$$\iiint\limits_E f(x, y, z)\,\mathrm{d}V = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

If it is a type II plane region, on the other hand, then

$$E = \{(x, y, z) \mid c \le y \le d \land h_1(y) \le x \le h_2(y) \land u_1(x, y) \le z \le u_2(x, y)\}$$

and

$$\iiint\limits_E f(x, y, z)\,\mathrm{d}V = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

A solid region E is **type 2** if it lies between the graphs of two continuous functions of y and z; that is,

$$E = \{(x, y, z) \mid (y, z) \in D \land u_1(y, z) \le x \le u_2(y, z)\}$$

where D is the projection of E onto the yz-plane.
The triple integral over a type 1 solid region E with yz-projection D is

$$\iiint\limits_E f(x,y,z)\,\mathrm{d}V = \iint\limits_D \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}A$$

A solid region E is **type 3** if it lies between the graphs of two continuous functions of x and z; that is,

$$E = \{(x,y,z) \mid (x,z) \in D \wedge u_1(x,z) \leq y \leq u_2(x,z)\}$$

where D is the projection of E onto the xz-plane. The integral over a type 3 solid region is D with xz-projection D is

$$\iiint\limits_E f(x,y,z)\,\mathrm{d}V = \iint\limits_D \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}A$$

Applications of Triple Integrals

If $f(x,y,z) = 1$ for all $(x,y,z) \in E$, then the triple integral represents the volume of E.

$$V(E) = \iiint\limits_E f(x,y,z)\,\mathrm{d}V$$

If a solid region E has density function $\rho(x,y,z)$, the mass of the region can be found as a triple integral.

$$m = \iiint\limits_E \rho(x,y,z)\,\mathrm{d}V$$

The three **moments** of E about each coordinate plane are

$$M_{yz} = \iiint\limits_E x\rho(x,y,z)\,\mathrm{d}V \qquad M_{xz} = \iiint\limits_E y\rho(x,y,z)\,\mathrm{d}V \qquad M_{yz} = \iiint\limits_E z\rho(x,y,z)\,\mathrm{d}V$$

The **center of mass** is at the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{x} \qquad \bar{y} = \frac{M_{xz}}{y} \qquad \bar{z} = \frac{M_{xy}}{z}$$

If the density is constant, the center of mass is called the **centroid** of E.
The **moments of inertia** about the coordinate axes are

$$I_x = \iiint\limits_E \left(y^2 + z^2\right)\rho(x,y.,z)\,\mathrm{d}V \quad I_y = \iiint\limits_E \left(x^2 + z^2\right)\rho(x,y,z)\,\mathrm{d}V \quad I_z = \iiint\limits_E \left(x^2 + y^2\right)\rho(x,y,z)\,\mathrm{d}V$$

The total **electric charge** on a solid occupying region E with charge density $\sigma(x,y,z)$.

$$Q = \iiint\limits_E \sigma(x,y,z)\,\mathrm{d}V$$

The **joint density function** of 3 random variables X, Y, and Z is a function of 3 variables such that the probability that (X,Y,Z) lies in E is

$$P((X,Y,Z) \in E) = \iiint\limits_E f(x,y,z)\,\mathrm{d}V$$

In particular,

$$P(x_1 \leq X \leq x_2 \wedge y_1 \leq Y \leq y_2 \wedge z_1 \leq Z \leq z_2) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z) \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x$$

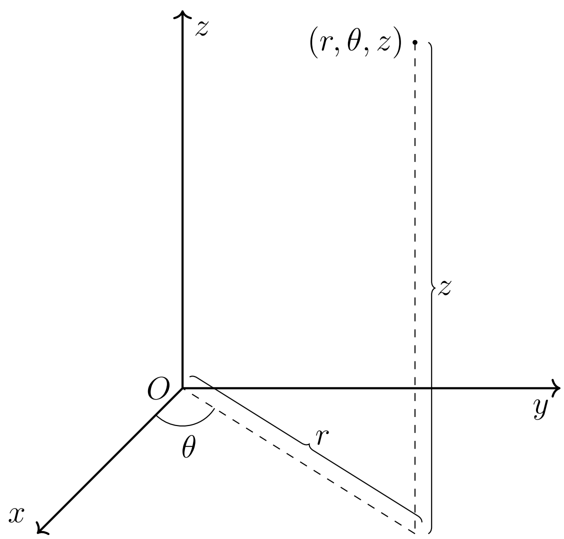
The joint density functions satisfies

$$f(x, y, z) \geq 0 \qquad\qquad \iiint\limits_{\mathbb{R}^3} f(x, y, z) \,\mathrm{d}V = 1$$

15.7 Triple Integrals in Cylindrical Coordinates

Cylindrical Coordinates

The **cylindrical coordinate system** represents a point as an ordered triple (r, θ, z), where (r, θ) is the projection of the point onto the polar plane and z is the displacement perpendicular to the plane from that projection.



Converting from cylindrical to rectangular coordinates,

$$x = r\cos\theta \qquad\qquad y = r\sin\theta \qquad\qquad z = z$$

Performing the converse operation,

$$\theta = \arctan\left(\frac{y}{x}\right) \qquad\qquad r = \sqrt{x^2 + y^2} \qquad\qquad z = z$$

Triple Integrals in Cylindrical Coordinates

If f is continuous and E is a type 1 region with projection in the xy-plane D defined as

$$E = \{(x, y, z) \mid (x, y) \in D \wedge u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta \wedge h_1(\theta) \leq r \leq h_2(\theta)\}$$

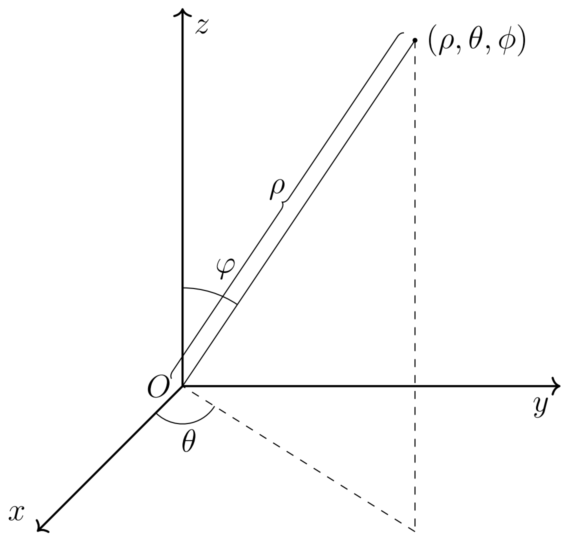
then

$$\iiint\limits_E f(x, y, z) \,\mathrm{d}V = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta$$

15.8 Triple Integrals in Spherical Coordinates

Spherical Coordinates

The **spherical coordinate system** represents a point as an ordered triple (ρ, θ, φ) where ρ is the distance to the origin, θ is the angle in the projection of the point onto the xy-plane, and φ is the angle from the positive z-axis to the line segment connecting the point to the origin.



Converting from spherical to rectangular coordinates,

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Performing the converse operation,

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \arctan\left(\frac{y}{x}\right) \qquad \phi = \arccos\left(\frac{z}{\rho}\right)$$

Triple Integrals in Spherical Coordinates

A **spherical wedge** E is defined as

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b \wedge \alpha \le \theta \le \beta \wedge \delta \le \phi \le \gamma\}$$

For a spherical wedge E given by

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b \wedge \alpha \le \theta \le \beta \wedge \delta \le \phi \le \gamma\}$$

the integral of $f(x, y, z)$ over E is

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \int_\delta^\gamma \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$

15.9 Change of Variables in Multiple Integrals

In one-dimensional calculus, the substitution rule can be written as

$$\int f(x)\,\mathrm{d}x = \int f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}x$$

Change of Variables in Double Integrals

A change of variables can generally be described by a **transformation** T to from the uv-plane to the xy-plane.

$$T(u,v) = (x,y)$$

x and y are related to u and v by

$$x = g(u,v) \qquad\qquad y = h(u,v)$$

It is typically assumed that T is a $\boldsymbol{C^1}$ **transformation**, meaning that T is a two-variable function with real inputs and outputs.
The output of an input is its **image**. If no two points share an image, the transformation is **one-to-one**.
If T is a one-to-one transformation, it has an inverse **inverse transformation** T^{-1} from the xy-plane to the uv-plane.

$$u = G(x,y) \qquad\qquad v = H(x,y)$$

The **Jacobian** of transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of Variables in a Double Integral If T is a one-to-one C^1 transformation with nonzero Jacobian that maps region S in the uv-plane to region R in the xy-plane, f is continuous on R , and R and S are type I or II plane regions, then

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \iint\limits_D f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

Change of Variables in Triple Integrals

Let T be a one-to-one transformation that maps region S in uvw-space to region R in the xyz-space.

$$x = g(u,v,w) \qquad\qquad y = h(u,v,w) \qquad\qquad z = k(u,v,w)$$

The **Jacobian** of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint\limits_R f(x,y,z) = \iiint\limits_S f(x(u,v,w),y(u,v,w),z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial u,v,w}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Chapter 16

Vector Calculus

16.1 Vector Fields

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$

A **vector field** on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns each point (x, y) in its domain D (some subset of $\mathbb{R}^2$) a two-dimensional vector $\vec{F}(x, y)$.
That on $\mathbb{R}^3$ follows the same definition, simply with 3 dimensions.

Gradient Fields

The gradient of a two-variable function f of (x, y) is

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

This is a vector field on $\mathbb{R}^2$. A vector field $\vec{F}$ is **conservative** if there is some scalar function f for which $\vec{F} = \nabla f$. If this is the case, f is a **potential function** for $\vec{F}$.

16.2 Line Integrals

Line Integrals in the Plane

If f is defined along a smooth curve C given by

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

then the **line integral of f along C** is

$$\int_C f(x, y)\,\mathrm{d}s = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

The formula for line integral of $f(x, y)$ along curve C can be rewritten as

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b \left[f(x(t), y(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\right] \mathrm{d}t$$

The value of the line integral is independent of the parametrization of the curve so long as the curve is only traversed once from a to b.

Line Integrals with Respect to x or y

The **line integrals of f along C with respect to x and y** are

$$\int_C f(x,y)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i \qquad \int_C f(x,y)\,\mathrm{d}y = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i$$

To distinguish, the line integral with respect to s is referred to as the **line integral with respect to arc length**. The parametrization of a curve determines its **orientation**, the direction of the path traces as t increases. If $-C$ denotes a curve with the same points as C but the opposite orientation,

$$\int_{-C} f(x,y)\,\mathrm{d}x = -\int_C f(x,y)\,\mathrm{d}x \qquad \int_{-C} f(x,y)\,\mathrm{d}y = -\int_C f(x,y)\,\mathrm{d}y$$

Integrating with respect to arc length, though, orientation has no impact on the value, as s is always positive.

$$\int_{-C} f(x,y)\,\mathrm{d}s = \int_C f(x,y)\,\mathrm{d}s$$

Line Integrals in Space

Let C be a smooth space curve given by

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{\mathrm{k}}$$

If f is a function of 3 variables that is continuous on some region containing C, the **line integral of f along C** (with respect to arc length) is defined as

$$\int_C f(x,y,z)\,\mathrm{d}s = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

This can be evaluated by expanding s.

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b \left[f(x,y,z)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\right]\mathrm{d}t$$

This can in turn be more compactly rewritten as

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(\vec{r}(t))|\vec{r}\,'|\,\mathrm{d}t$$

Line integrals with respect to x, y, and z are defined as

$$\int_C f(x,y,z)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta x_i = \int_a^b f(x(t),y(t),z(t))x'(t)\,\mathrm{d}t$$

$$\int_C f(x,y,z)\,\mathrm{d}y = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta y_i = \int_a^b f(x(t),y(t),z(t))y'(t)\,\mathrm{d}t$$

$$\int_C f(x,y,z)\,\mathrm{d}z = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta y_i = \int_a^b f(x(t),y(t),z(t))z'(t)\,\mathrm{d}t$$

Line Integrals of Vector Fields: Work

Work done as a particle moves along C experiencing variable force F is

$$W = \int_C \left[\vec{F} \cdot \vec{T}\right] \mathrm{d}s$$

often abbreviated as

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}$$

This can be rewritten using the definition of $\vec{T}$.

$$W = \int_a^b \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\right] \mathrm{d}t$$

If $\vec{F}$ is a continuous vector field given by vector function defined on smooth curve C given by $\vec{r}(t)$ for $a \le t \le b$, the **line integral of $\vec{F}$ along C** is

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_a^b \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\right] \mathrm{d}t = \int_C \vec{F} \cdot \vec{T} \,\mathrm{d}s$$

A continuous vector field's integral along curve C given by $\vec{r}(t)$ for $a \le t \le b$ can be split into its components as

$$\int_C F \cdot \mathrm{d}\vec{r} = \int_C [P \,\mathrm{d}x + Q \,\mathrm{d}y + R \,\mathrm{d}z]$$

where

$$\vec{F} = \langle P, Q, R \rangle$$

16.3 The Fundamental Theorem for Line Integrals

The Fundamental Theorem of Calculus states that

$$\int_a^b f'(x) \,\mathrm{d}x = f(b) - f(a)$$

where f' is continuous on $[a, b]$.

The Fundamental Theorem for Line Integrals

If C is a smooth curve given by $\vec{r}(t)$ for $a \le t \le b$ and f is a differentiable function of two or three variables whose gradient vector is continuous on C, then

$$\int_C \nabla f(x) \cdot \,\mathrm{d}\vec{r} = f(\vec{r}(b)) = f(\vec{r}(a))$$

Independence of Path

If C_1 and C_2 are two smooth curves, called **paths**, with the same initial and terminal points a and b, then in general,

$$\int_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} \ne \int_{C_2} \vec{F} \cdot \mathrm{d}\vec{r}$$

but

$$\int_C \nabla f \cdot \mathrm{d}\vec{r} = \int_{C_2} \nabla f \cdot \mathrm{d}\vec{r}$$

whenever ∇f is continuous.
Line integrals of conservative vector fields are **independent of path**.
A path is **closed** if its initial and terminal points are the same.

$$\int_C F \cdot \mathrm{d}\vec{r}$$

is independent of path in D if and only if

$$\int_C F \cdot \mathrm{d}\vec{r} = 0$$

for every closed path C in D.

If F is a vector field that is continuous on open region D, then if

$$\int_C F \cdot \mathrm{d}\vec{r}$$

is independent of path in D, then F is a conservative vector field on D.

Conservative Vector Fields and Potential Functions

If $\vec{F}(x, y)$ is a conservative vector field given by

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$$

where P and Q have continuous first-order partial derivatives on domain D, then throughout D,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

A **simple curve** is one that does not intersect itself between its endpoints.
A **simply connected region** is a contiguous region without any holes.

If

$$\vec{F} = \langle P, Q \rangle$$

is a vector field on a simply-connected region D and P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D, then, $\vec{F}$ is conservative.

16.4 Green's Theorem