BC Project

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1 Sources

2 Problems

Use the trigonometric substitution $x = \arctan\theta$ to solve each of the following:

1. $\displaystyle\int \frac{\mathrm{d}x}{1+x^2}$

2. $\displaystyle\int x\sqrt{x^2+1}\,\mathrm{d}x$

3. $\displaystyle\int \frac{\mathrm{d}x}{1+x^2}$

4. $\displaystyle\int \frac{\mathrm{d}x}{1+9x^2}$

5. Find the average value of $y = \dfrac{x^2}{x^2+4}$ from $x = 1$ to $x = 1$.

6. Find the volume of the solid generated by revolving the regions bounded between the graph of $\dfrac{1}{1+x^2}$ and the x-axis over the interval $(0, 1)$.

7. Find the arc length of the parametric function given by $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\sqrt{2}t^2}{\sqrt{t^2+1}} = \dfrac{\mathrm{d}x}{\mathrm{d}t}$ from $t = 0$ to $t = 1$.

8. $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,\mathrm{d}x$

9. Solve the IVP $\dfrac{\sqrt{1+x^2}}{x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1, y(0) = 69$ for y as a function of x.

10. Find the area under $\dfrac{y^2}{9} - x^2 = 1$ from $x = 0$ to $x = 1$.

11. During each cycle, the velocity v (in ft/s) of a robotic welding device is given by $v = \dfrac{1}{t^2-6t+13}$, where t is the time (in s). Find the expression for the displacement s (in ft) as a function of t if $s = 0$ when $t = 0$.

12. Use the integral test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{x^2\sqrt{x^2+1}}}$.

13. Find the second-degree Taylor polynomial of f centered about $x = 1$, where $f(0) = 0, f'(0) = 0$, and $f''(x) = \dfrac{x^3+8x}{(x^2+4)^{3/2}}$.

14. $\displaystyle\int \frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}\,\mathrm{d}x$

15. $\displaystyle\int \frac{1}{1+\sqrt{x^2+1}}\,\mathrm{d}x$

3 Solutions

$$\begin{aligned}
\int \sec^3\theta \mathrm{d}\theta &\implies u = \sec\theta, \mathrm{d}v = \sec^2\theta\,\mathrm{d}x \\
&\implies \mathrm{d}u = \sec\theta\tan\theta\,\mathrm{d}\theta\,, v = \tan\theta \\
&= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,\mathrm{d}\theta \\
&= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\,\mathrm{d}\theta \\
&= \sec\theta\tan\theta - \int \sec^3\theta\,\mathrm{d}\theta + \int \sec\theta\,\mathrm{d}\theta \\
&= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta\,\mathrm{d}\theta \\
2\int \sec^3\theta\,\mathrm{d}\theta &= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta \\
\int \sec^3\theta\,\mathrm{d}\theta &= \frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2}
\end{aligned}$$

1.

$$\begin{aligned}
\int \frac{1}{1+x^2}\,\mathrm{d}x &\implies x = \tan\theta \implies \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{1+\tan^2\theta}\sec^2\theta\,\mathrm{d}\theta = \int \frac{\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta = \int \mathrm{d}\theta = \theta + C = \arctan x + C
\end{aligned}$$

2.

$$\begin{aligned}
\int x\sqrt{x^2+1}\,\mathrm{d}x &\implies x = \tan\theta \implies \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta \\
&= \int \tan\theta\sec^3\theta\,\mathrm{d}\theta \implies u = \sec\theta \implies \mathrm{d}u = \sec\theta\tan\theta\,\mathrm{d}\theta \\
&= \int u^2\,\mathrm{d}u = \frac{u^3}{3} + C = \frac{\sec^3\theta}{3} + C = \frac{(x^2+1)^{3/2}}{3} + C
\end{aligned}$$

3.

4.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+9x^2}} &\implies 3x = \tan\theta \implies 3\,\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{\sqrt{1+\tan^2\theta}}\frac{\sec^2\theta}{3}\,\mathrm{d}\theta = \int \frac{\sec^2\theta}{3\sec\theta}\,\mathrm{d}\theta = \int \frac{\sec\theta}{3}\,\mathrm{d}\theta \\
&= \frac{\ln|\sec\theta + \tan\theta|}{3} + C = \frac{\ln|\sqrt{1+9x^2} + 3x|}{3} + C
\end{aligned}$$

5.

$$
\begin{aligned}
y_{\text{avg}} &= \frac{1}{10-1}\int_1^{10} \frac{x^2}{x^2+4}\,\mathrm{d}x \implies x = 2\tan\theta \implies \theta = \arctan\left(\frac{x}{2}\right), \mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{9}\int_{\arctan(1/2)}^{\arctan(10/2)} \frac{4\tan^2\theta \times 2\sec^2\theta}{4\sec^2\theta}\,\mathrm{d}\theta = \frac{1}{9}\int_{\arctan(1/2)}^{\arctan(5)} 2\tan^2\theta\,\mathrm{d}\theta = \frac{1}{9}\int_{\arctan(1/2)}^{\arctan(5)} 2\tan^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{9}\int_{\arctan(1/2)}^{\arctan(5)} 2(\sec^2\theta - 1)\,\mathrm{d}\theta = \frac{1}{9}[2\tan\theta - 2\theta]_{\arctan(1/2)}^{\arctan(5)} = \frac{2}{9}\left[5 - \arctan(5) - \left(\frac{1}{2} - \arctan\left(\frac{1}{2}\right)\right)\right]
\end{aligned}
$$

6.

$$
\begin{aligned}
V &= \pi\int_0^1 \left(\frac{1}{1+x^2}\right)^2 \mathrm{d}x \implies x = \tan\theta \implies \theta = \arctan x, \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta \\
&= \pi\int_{\arctan(0)}^{\arctan(1)} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,\mathrm{d}\theta = \pi\int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\,\mathrm{d}\theta = \pi\int_0^{\pi/4} \cos^2\theta\,\mathrm{d}\theta = \pi\int_0^{\pi/4} \frac{\cos(2\theta)+1}{2}\,\mathrm{d}\theta \\
&= \frac{\pi}{2}\left[\frac{\sin(2\theta)}{2} + \theta\right]_0^{\pi/4} = \frac{\pi}{2}\left[\frac{1}{2} + \frac{\pi}{4} - (0)\right] = \frac{2\pi + \pi^2}{8}
\end{aligned}
$$

7.

$$
s =
$$