Homework Set 3

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7 Discrete Probability

1. Boys and girls being equally likely implies that the probability of each is $1/2 = 0.5$ (assuming them to be the only 2 possible outcomes). The number of trials (the number of children) and the probability of each child being a boy are then fixed at 3 and 0.5 respectively, meaning that this situation follows a binomial setting with probability of success $p = 0.5$ and number of trials $n = 3$. As such,

$$\mu_X = np = 0.5 \times 3 = 1.5$$
$$\sigma_X = \sqrt{np(1-p)} = \sqrt{0.5(3)(1-0.5)} = \sqrt{0.75} = 0.5\sqrt{3} \approx 0.866$$

The mean is 1.5 boys while the standard deviation is about 0.866 boys.

2. X is a discrete random variable that follows the given probability distribution

x	0	1	2	3	4
$P(x)$	0.3	0.4	0.2	0.06	0.04

As such,

$$\mu_X = \sum x_i P(x_i) = 1.14$$
$$\sigma_X = \sqrt{\sum (x_i - \mu_X)^2 P(x_i)} \approx 1.039$$

The mean is 1.14 toppings while the standard deviation is about 1.039 toppings.

3. One ticked being randomly selected from $1,000$ means that the probability of any one ticket being selected is $1/1,000 = 0.001$. This describes a discrete random variable X with probability distribution

x	0	387
$P(x)$	$1 - 0.001 = 0.999$	0.001

The expected value of this random variable is then

$$\mathbb{E}[X] = \sum x_i P(x_i) = 0.387$$

X represents the money gained from the drawing rather than the profit, though. The amount of money spent on the ticket is a constant 2, so 2 can simply be subtracted from this result to yield the expected profit of entering the drawing, which is $-\$1.613$

4. The number of defective computers is a discrete random variable X with probability distribution

x	0	1	2	3	4
$P(x)$	0.4305	0.4039	0.1421	0.0222	0.0013

   As such,

$$\mu_X = \sum x_i P(x_i) = 0.7599$$

   The mean number of defective computers in a batch of 4 is 0.7599.

5. The number of defective computers is a discrete random variable X with probability distribution

x	0	1	2	3	4
$P(x)$	0.5220	0.3685	0.0975	0.0115	0.0005

   As such,

$$\sigma_X = \sqrt{\sum (x_i - \mu_X)^2 P(x_i)} \approx 0.714$$

6. The number of trials (days) and the probability of success (rain) are fixed at 3 and 0.4 respectively. This situation therefore follows a binomial setting with probability of success $p = 0.4$ and number of trials $n = 3$. Let X be a random variable following this binomial distribution. The only possible values of X are 0, 1, 2, and 3. As such, the complement of $X = 0$ is equivalent to $X \geq 1$. Therefore,

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{3}{0} 0.4^0 (1 - 0.4)^{3-0} = 1 - 0.216 = 0.784$$

   The probability of it raining on at least 1 of the 3 days is 0.784.

7. The probability of a given student owning a credit card given that they are a freshman is the relative frequency of owning one among freshman. This is simply the proportion of freshman that carry credit cards:

$$\begin{aligned} P(\text{carries a credit card} \,|\, \text{freshman}) &= \frac{\text{\# of freshman credit card carriers}}{\text{\# of freshman}} \\ &= \frac{40}{60} = \frac{2}{3} \approx 0.667 \end{aligned}$$

   The probability of a randomly selected freshman carrying a credit card is about 0.667.

8. The probability of a given student being a freshman given that they carry a credit card is simply the relative frequency of being a freshman among those that own one. This is the proportion of credit card carriers that are freshman:

$$\begin{aligned} P(\text{freshman} \,|\, \text{carries a credit card}) &= \frac{\text{\# of freshman credit card carriers}}{\text{\# of credit card carriers}} \\ &= \frac{49}{61} \approx 0.803 \end{aligned}$$

   The probability of a randomly selected credit card owner being a freshman is about 0.803.

9. The probability of a given student being over 40 and drinks cola is simply the relative frequency of people over 40 that drink cola:

$$\begin{aligned} P(\text{over 40 years of age} \cap \text{cola}) &= \frac{\#\text{ of cola drinkers over 40}}{\text{sample size}} \\ &= \frac{20}{255} = \frac{4}{51} \approx 0.078 \end{aligned}$$

he probability of a given student being over 40 and drinks cola is about 0.078.

10. The probability of a given subject drinking root beer given that they are over 40 is simply the relative frequency of root beer drinkers among those that are over 40:

$$\begin{aligned} P(\text{root beer} \mid \text{over 40 years of age}) &= \frac{\#\text{ of root beer drinkers over 40}}{\#\text{ of people over 40}} \\ &= \frac{30}{20+30+35} = \frac{30}{85} = \frac{6}{17} \approx 0.353 \end{aligned}$$

The probability of a given subject drinking root beer given that they are over 40 is about 0.353.

1 The Foundations: Logic and Proofs

1.1 Propositional Logic

1.
   a) True proposition
   b) False proposition
   c) True proposition
   d) False proposition
   e) Not a proposition
   f) Not a proposition

3.
   a) Linda is not younger than Sanjay.
   b) Mei does not make more money than Isabella.
   c) Moshe is not taller than Monica.
   d) The moon is not made of green cheese.
   e) $2^n < 100$.

5.
   a) Mei does not have an MP3 player.
   b) There is pollution in New Jersey.
   c) $2 + 1 \neq 3$.
   d) The summer in Maine is neither hot nor sunny.

11.
   a) Sharks have not been spotted near the shore.
   b) Swimming at New Jersey shore is allowed and sharks have been spotted near the shore.
   c) Swimming at the New Jersey shore is not allowed or sharks have been spotted at the shore.
   d) Swimming at the New Jersey shore being allowed implies that sharks have not been spotted near the shore.
   e) Sharks not being spotted near the shore implies that Swimming at the New Jersey shore is allowed.
   f) Swimming at the New Jersey shore not being allowed implies that sharks have not been spotted near the shore.
   g) Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore.
   h) Swimming at the New Jersey shore is not allowed or swimming at the New Jersey shore is allowed and sharks have not been spotted near the shore.

13.
a) $p \wedge q$
b) $p \wedge \neg q$
c) $\neg p \wedge \neg q$
d) $q \vee p$
e) $p \rightarrow q$
f) $p \leftrightarrow q$

15.
a) $\neg p$
b) $p \wedge \neg q$
c) $q \leftarrow p$
d) $\neg p \rightarrow \neg q$
e) $p \rightarrow q$
f) $q \rightarrow p$

19.
a) False
b) False
c) False
d) False

25.
a) If the wind blows from the northeast, then it snows.
b) If it stays warm for a week, then the apple trees will bloom.
c) If the Pistons win the championships, then they beat the Lakers.
d) If one has gotten to the top of Long's peak, they must have walked eight miles.
e) If you are famous, then you can get tenure as a professor.
f) If you drive more than 400 miles, then you will need to buy gasoline.
g) If you bought your CD player less than 90 days ago, your guarantee is good.
h) If the water is not too cold, Jan will go swimming.
i) If people believe in science, we will have a future.

27.
a) You can buy an ice cream cone if and only if it is hot outside.
b) You can win the contest if and only if you have the only winning ticket.
c) You can get promoted if and only if you have connections.
d) Your mind will decay if and only if you watch television.
e) The trains run late if and only if I take it.

31.
a) 2
b) 16
c) 64
d) 16

33.
a)

p	$\neg p$	$p \wedge \neg p$
0	1	0
1	0	0

b)

p	$\neg p$	$p \vee \neg p$
0	1	1
1	0	1

c)

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \rightarrow q$
0	0	1	1	0
0	1	0	0	1
1	0	1	1	0
1	1	0	1	1

d)

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
0	0	0	0	1
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

e)

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
0	0	1	1	1	1	1
0	1	1	1	0	1	1
1	0	0	0	1	0	1
1	1	1	0	0	1	1

f)

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \rightarrow (q \rightarrow p)$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	1
1	1	1	1	1

35. a)

p	q	$p \vee q$	$p \oplus q$	$(p \vee q) \rightarrow (p \oplus q)$
0	0	0	0	1
0	1	1	1	1
1	0	1	1	1
1	1	1	0	0

b)

p	q	$p \oplus q$	$p \wedge q$	$(p \oplus q) \rightarrow (p \wedge q)$
0	0	0	0	1
0	1	1	0	0
1	0	1	0	0
1	1	0	1	1

c)

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \oplus (p \wedge q)$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

d)

p	q	$p \leftrightarrow q$	$\neg p$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
0	0	1	1	0	1
0	1	0	1	1	1
1	0	0	0	1	1
1	1	1	0	0	1

e)

p	q	r	$p \leftrightarrow q$	$\neg p$	$\neg r$	$\neg p \leftrightarrow \neg r$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
0	0	0	1	1	1	1	0
0	0	1	1	0	1	0	1
0	1	0	0	1	1	1	1
0	1	1	0	1	0	0	0
1	0	0	0	0	1	0	0
1	0	1	0	0	0	1	1
1	1	0	1	0	1	0	1
1	1	1	1	0	0	1	0

f)

p	q	$p \oplus q$	$\neg q$	$p \oplus \neg q$	$(p \oplus q) \rightarrow (p \oplus \neg q)$
0	0	0	1	1	1
0	1	1	0	0	0
1	0	1	1	0	0
1	1	0	0	1	1

37. a)

p	q	$\neg q$	$p \rightarrow \neg q$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	0	0

b)

p	q	$\neg p$	$\neg p \leftrightarrow q$
0	0	1	0
0	1	1	1
1	0	0	1
1	1	0	0

c)

p	q	$p \to q$	$\neg p$	$\neg p \to q$	$(p \to q) \vee (\neg p \to q)$
0	0	1	1	0	1
0	1	1	1	1	1
1	0	0	0	1	1
1	1	1	0	1	1

d)

p	q	$p \to q$	$\neg p$	$\neg p \to q$	$(p \to q) \wedge (\neg p \to q)$
0	0	1	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	1	1	0	1	1

e)

p	q	$p \leftrightarrow q$	$\neg p$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \vee (\neg p \leftrightarrow q)$
0	0	1	1	0	1
0	1	0	1	1	1
1	0	0	0	1	1
1	1	1	0	0	1

f)

p	q	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$p \leftrightarrow q$	$(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$
0	0	1	1	1	1	1
0	1	1	0	0	0	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

39.

a)

p	q	r	$\neg q$	$\neg q \vee r$	$p \to (\neg q \vee r)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	1	1

b)

p	q	r	$\neg p$	$q \to r$	$\neg p \to (q \to r)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	0	0	1
1	1	1	0	1	1

c)

p	q	r	$p \to q$	$\neg p$	$\neg p \to r$	$(p \to q) \vee (\neg p \to r)$
0	0	0	1	1	0	1
0	0	1	1	1	1	1
0	1	0	1	1	0	1
0	1	1	1	1	1	1
1	0	0	0	0	1	1
1	0	1	0	0	1	1
1	1	0	1	0	1	1
1	1	1	1	0	1	1

d)

p	q	r	$p \to q$	$\neg p$	$\neg p \to r$	$(p \to q) \wedge (\neg p \to r)$
0	0	0	1	1	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	0	0	1	0
1	0	1	0	0	1	0
1	1	0	1	0	1	1
1	1	1	1	0	1	1

e)

p	q	r	$p \leftrightarrow q$	$\neg q$	$\neg q \leftrightarrow r$	$(p \leftrightarrow q) \vee (\neg p \leftrightarrow r)$
0	0	0	1	1	0	1
0	0	1	1	1	1	1
0	1	0	0	0	1	1
0	1	1	0	0	0	0
1	0	0	0	1	0	0
1	0	1	0	1	1	1
1	1	0	1	0	1	1
1	1	1	1	0	0	1

f)

p	q	r	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$q \leftrightarrow r$	$(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	0	0
0	1	0	1	0	0	0	1
0	1	1	1	0	0	1	0
1	0	0	0	1	0	1	0
1	0	1	0	1	0	0	1
1	1	0	0	0	1	0	0
1	1	1	0	0	1	1	1

Test 3

47. True

48. True

49. True

50. True

51. True

52. False

53. False

54. True

19. $p(\{\varnothing, a, \{a\}, \{\{a\}\}\})$

22. $p(\{a, \varnothing\})$

16. $(A \cup (B \cap C)) \supseteq ((A \cup B) \cap C)$

17. $((A - B) \cup (A - C)) = (A - (B \cap C))$

Test 4

9.

p	q	r	$\neg q$	$\neg r$	$r \to \neg q$	$\neg(r \to \neg q)$	$p \land \neg r$	$\neg(r \to \neg p) \lor (p \land \neg r)$
0	0	0	1	1	1	0	0	0
0	0	1	1	0	1	0	0	0
0	1	0	0	1	1	0	0	0
0	1	1	0	0	0	1	0	1
1	0	0	1	1	1	0	1	1
1	0	1	1	0	1	0	0	0
1	1	0	0	1	1	0	1	1
1	1	1	0	0	0	1	0	1

10. a) $q \land \neg p$ b) $\neg(\neg q \lor p)$

11. $p \land r \land \neg q$

15.

p	q	r	$q \to r$	$p \to (q \to r)$	$q \land r$	$p \to (q \land r)$
0	0	0	1	1	0	1
0	0	1	1	1	0	1
0	1	0	0	1	0	1
0	1	1	1	1	1	1
1	0	0	1	1	0	0
1	0	1	1	1	0	0
1	1	0	0	0	0	0
1	1	1	1	1	1	1

The 5th and and 7th columns are not equivalent, so $p \to (q \to r) \not\equiv p \to (q \land r)$.

16.

p	q	r	$q \to r$	$p \to (q \to r)$	$p \to q$	$(p \to q) \to r$
0	0	0	1	1	1	0
0	0	1	1	1	1	1
0	1	0	0	1	1	0
0	1	1	1	1	1	1
1	0	0	1	1	0	1
1	0	1	1	1	0	1
1	1	0	0	0	1	0
1	1	1	1	1	1	1

17.

$$
\begin{aligned}
(p \to q) \land (\neg p \to q) &\equiv (\neg p \lor q) \land (\neg\neg p \lor q) && \text{(definition of } \to) \\
&\equiv (\neg p \lor q) \land (p \lor q) && \text{(double negation)} \\
&\equiv ((\neg p \lor q) \land p) \lor ((\neg p \lor q) \land q) && \text{(distributive property)} \\
&\equiv ((\neg p \land p) \lor (q \land p)) \lor ((\neg p \land q) \lor (q \land q)) && \text{(distributive property)} \\
&\equiv (0 \lor (q \land p)) \lor ((\neg p \land q) \lor q) && \text{(contradiction/identity)} \\
&\equiv (p \land q) \lor ((\neg p \land q) \lor q) && \text{(identity)} \\
&\equiv (p \land q) \lor (\neg p \land q) \lor q && \text{(associativity)} \\
&\equiv q \lor q \\
&\equiv q
\end{aligned}
$$

18. $\neg(\neg p \land q)$

26.

p	q	r	$\neg q$	$\neg q \land r$	$p \to (\neg q \land r)$	$\neg p$	$r \to q$	$\neg(r \to q)$	$\neg p \lor \neg(r \to q)$
0	0	0	1	0	1	1	1	0	1
0	0	1	1	1	1	1	0	1	1
0	1	0	0	0	1	1	1	0	1
0	1	1	0	1	1	1	1	0	1
1	0	0	1	0	0	0	1	0	0
1	0	1	1	1	1	0	0	1	1
1	1	0	0	0	0	0	1	0	0
1	1	1	0	0	0	0	1	0	0

$p \to (\neg q \land r) \equiv \neg p \lor \neg(r \to q)$

29.

$$
\begin{aligned}
((p \to \neg q) \land q) \to \neg p &\equiv \neg((\neg p \lor \neg q) \land q) \lor \neg p && \text{(definition of } \to) \\
&\equiv (\neg(\neg p \lor \neg q) \lor \neg q) \lor \neg p && \text{(DeMorgan's law)} \\
&\equiv ((p \land q) \lor \neg q) \lor \neg p && \text{(DeMorgan's law)} \\
&\equiv ((p \lor \neg q) \land (q \lor \neg q)) \lor \neg p && \text{(distributive property)} \\
&\equiv ((p \lor \neg q) \land 1) \lor \neg p && \text{(tautology)} \\
&\equiv (p \lor \neg q) \lor \neg p && \text{(identity)} \\
&\equiv p \lor \neg q \lor \neg p && \text{(associativity)} \\
&\equiv p \lor \neg p \lor \neg q && \text{(commutativity)} \\
&\equiv 1 \lor q && \text{(tautology)} \\
&\equiv 1 && \text{(domination)}
\end{aligned}
$$