

Homework Set 2

Arnav Patri

10 Graphs

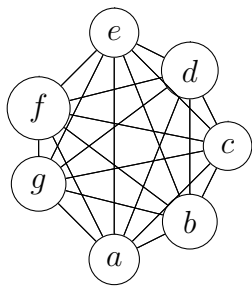
10.1 Graphs and Graph Models

3. The graph has undirected edges and no loops, making it a simple graph.
4. The graph has multiple undirected edges and no loops, making it a multigraph.
5. The graph has multiple undirected edges and loops, making it a pseudograph.
6. The graph has multiple undirected edges and no loops, making it a multigraph.
7. The graph has directed edges and loops, making it a digraph.
8. The graph has multiple directed edges and loops, making it a directed multigraph.
9. The graph has multiple directed edges and loops, making it a directed multigraph.

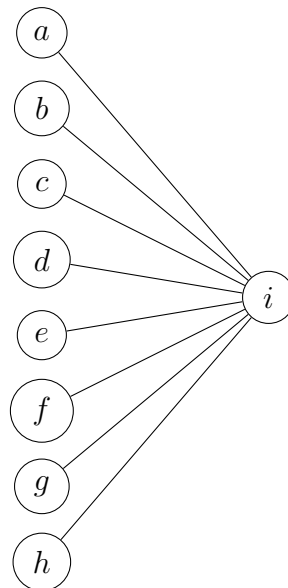
10.2 Graph Terminology and Special Types of Graphs

1. $|V| = 6$, $|E| = 6$, $\deg a = 2$, $\deg b = 4$, $\deg c = 1$ (pendant), $\deg d = 0$ (isolated), $\deg e = 2$, $\deg f = 3$
2. $|V| = 5$, $|E| = 13$, $\deg a = 6$, $\deg b = 6$, $\deg c = 6$, $\deg d = 5$, $\deg e = 3$
3. $|V| = 9$, $|E| = 12$, $\deg a = 3$, $\deg b = 2$, $\deg c = 4$, $\deg d = 0$ (isolated), $\deg e = 6$, $\deg f = 0$ (isolated), $\deg g = 4$, $\deg h = 2$, $\deg i = 3$
5. A simple graph with 15 vertices each of degree 5 cannot exist, as all graphs must have an even number of vertices of odd degree.
7. $|V| = 4$, $|E| = 7$, $\deg^- a = 3$, $\deg^+ a = 1$, $\deg^- b = 1$, $\deg^+ b = 2$, $\deg^- c = 2$, $\deg^+ c = 1$, $\deg^- d = 1$, $\deg^+ d = 3$
8. $|V| = 4$, $|E| = 8$, $\deg^- a = 1$, $\deg^- b = 3$, $\deg^- c = 2$, $\deg^- d = 1$, $\deg^+ a = 2$, $\deg^+ b = 4$, $\deg^+ c = 1$, $\deg^+ d = 1$
9. $|V| = 5$, $|E| = 13$, $\deg^- a = 6$, $\deg^+ a = 1$, $\deg^- b = 1$, $\deg^- b = 5$, $\deg^- c = 2$, $\deg^+ c = 5$, $\deg^- d = 4$, $\deg^+ d = 2$, $\deg^- e = 0$, $\deg^+ e = 0$
- 20.

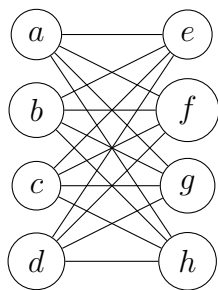
a)



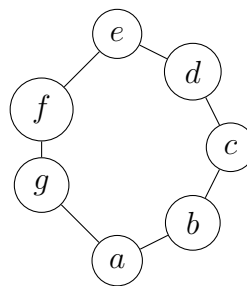
b)



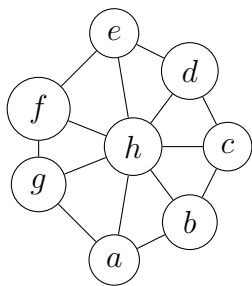
c)



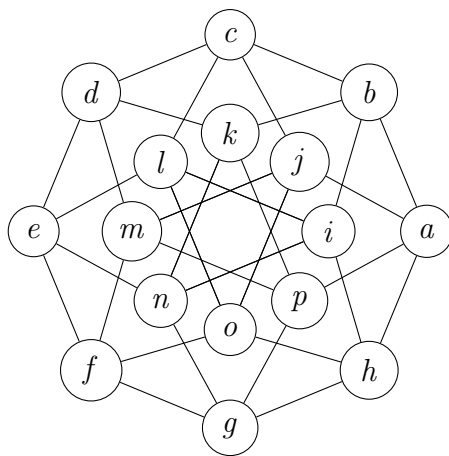
d)



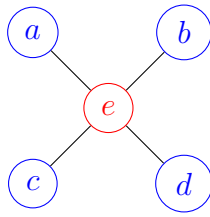
e)



f)

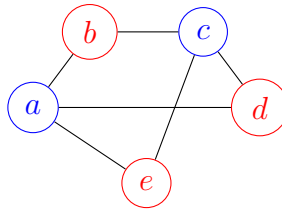


21.



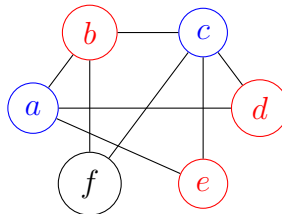
The graph is bipartite.

22.



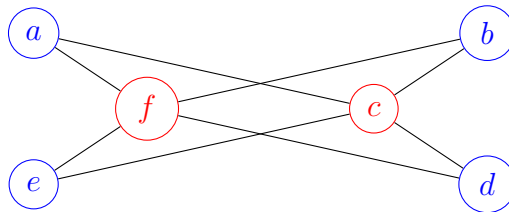
The graph is bipartite.

23.



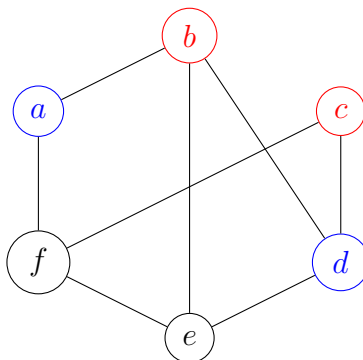
This graph is not bipartite, due to f .

24.



This graph is bipartite.

25.

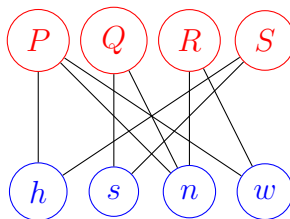


This graph is not bipartite, due to e and f .

26.

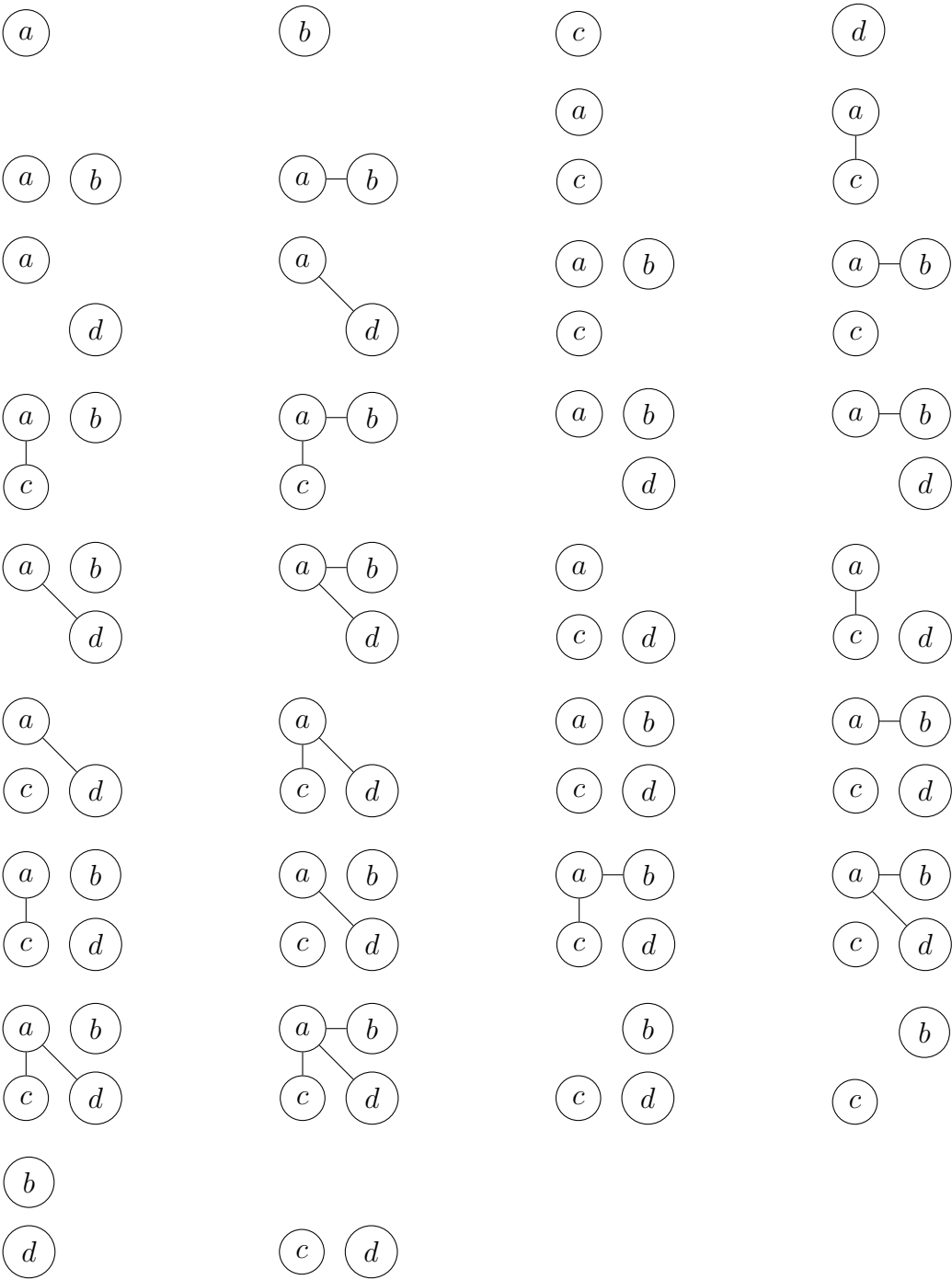
- a) K_1 and K_2 are bipartite, but K_n for $n \geq 3$ is not bipartite, as any 3 vertices are connected pairwise, so there is no way to partition them into 2 disjoint sets.
- b) C_n is bipartite whenever n is even, as the vertices can simply alternate.
- c) W_n is never bipartite, as every vertex is connected to the center of the wheel.
- d) Q_n is always bipartite.

27. a)



37. a) $|V| = n, |E| = \binom{n}{2}$ b) $|V| = n, |E| = n$ c) $|V| = n + 1, |E| = 2n$
d) $|V| = m + n, |E| = mn$ e) $|V| = 2^n, |E| = n2^{n-1}$

53.



10.3 Representing Graphs and Graph Isomorphism

1.

Vertex	Adjacent Vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

3.

Vertex	Terminal Vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

5.

$$\begin{matrix} & a & b & c & d \\ a & \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \\ b & \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \\ c & \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \\ d & \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

7.

$$\begin{matrix} & a & b & c & d \\ a & \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \\ b & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \\ c & \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \\ d & \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

9.

a)

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

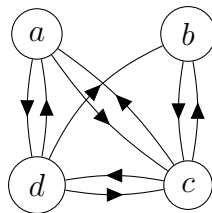
e)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

f)

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

11.



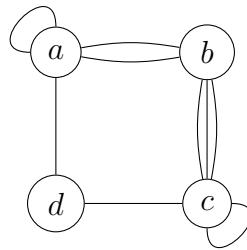
13.

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{array}{c} a \quad b \quad c \quad d \\ \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{array} \right] \end{array}$$

15.

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{array}{c} a \quad b \quad c \quad d \\ \left[\begin{array}{cccc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right] \end{array}$$

17.



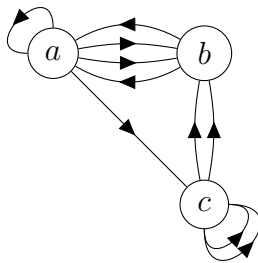
19.

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{array}{c} a \quad b \quad c \quad d \\ \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right] \end{array}$$

21.

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{array}{c} a \quad b \quad c \quad d \\ \left[\begin{array}{cccc} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{array} \right] \end{array}$$

23.



31.

$$\mathbf{M}_{13} = \begin{matrix} a \\ b \\ c \\ d \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M}_{14} = \begin{matrix} a \\ b \\ c \\ d \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_{15} = \begin{matrix} a \\ b \\ c \\ d \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

33. For an undirected graph, the sum of the entries in a column of the adjacency matrix is the number of edges that are connected to that column's vertex (loops only being counted once). For a digraph, it is the in-degree of the vertex.

35. For an undirected graph, the sum of the values of a column in the incidence matrix is equal to the number of nodes that the column's edge is incident to. This can only be 1 (if the edge is a loop) or 2.

36. a)

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

37. a)

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

d)

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$

39.

v	u_1	u_2	u_3	u_4	u_5
$f(v)$	v_1	v_3	v_2	v_5	v_3

41.

v	u_1	u_2	u_3	u_4	u_5	u_6	u_7
$f(v)$	v_1	v_3	v_5	v_7	v_2	v_4	v_6

43.

v	u_1	u_2	u_3	u_4	u_5	u_6
$f(v)$	v_5	v_2	v_3	v_6	v_4	v_1

45. u_5 is connected to exactly 2 other nodes, both of which are of degree 3. There is no node in the second graph that has this property. The graphs are therefore nonisomorphic.

47.

v	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
$f(v)$	v_6	v_5	v_8	v_{10}	v_7	v_3	v_9	v_2	v_4	v_1

63.

a)

b) No, as there is no row in the first matrix with only 1 1.

$$\begin{matrix} & v_1 & v_2 & v_3 \\ v_1 & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ v_2 & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ v_3 & \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \end{matrix} \implies \begin{matrix} & v_3 & v_1 & v_2 \\ v_3 & \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \\ v_1 & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ v_2 & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

Yes

c) No, as there is no row in the first matrix with only 1 1.

67.

v	u_1	u_2	u_3	u_4
$f(v)$	v_3	v_4	v_2	v_1

69.

v	u_1	u_2	u_3	u_4
$f(v)$	v_3	v_4	v_2	v_1

10.5 Euler and Hamilton Paths

1. a , b and c are all of degree 3, which means that 3 nodes are of odd degree, so the graph has neither an Euler path or circuit.
3. a and d are of odd degree while every other is of even degree, so an Euler path may exist. Such a path is $(a, e, c, e, b, e, d, b, a, c, d)$.
5. Every vertex is of even degree, so the graph may have an Euler circuit. Such a circuit is $(a, e, a, e, c, d, c, b, e, d, b, a)$.
7. Every vertex is of even degree, so the graph may have an Euler circuit. Such a circuit is $(a, ih, g, d, e, f, g, c, e, h, d, c, a, b, i, c, b, h, a)$.
13. This image can be drawn in a single stroke by treating the intersections of edges as nodes.
15. This image cannot be drawn in a single stroke.

19.

v	a	b	c	d
$\deg^- v$	1	3	1	2
$\deg^+ v$	2	1	2	2

$\deg^- v$ is not always equal to $\deg^+ v$, so an Euler circuit may not exist. $\deg^+ a - \deg^- a = \deg^+ c - \deg^- c = 1$, so an Euler path may not exist either.

21. An Euler path exists: $a, d, e, d, b, a, e, c, e, b, c, b, e$.

23.

v	a	b	c	d	e	f	g	h	i	j	k	l
$\deg^- v$	1	2	1	1	2	1	2	2	2	1	1	1
$\deg^+ v$	1	1	1	2	2	2	1	2	1	1	2	1

$\deg^- v$ is not always equal to $\deg^+ v$, so an Euler circuit may not exist. $\deg^+ d - \deg^- d = \deg^+ f - \deg^- f = 1$, so an Euler path may not exist either.

26.
 - a) An Euler circuit exists only if all edges are of even degree. For a complete graph K_n , all edges are of degree $n - 1$, so an Euler circuit exists whenever n is odd.
 - b) An Euler circuit always exists for C_n , as every node is of degree 2.
 - c) An Euler circuit never exists for W_n , as every node but 1 is of degree 3.
 - d) An Euler circuit only exists for Q_n if n is even, as the degree of each node is n .

31. a, b, c, d, e, a is a Hamilton circuit.

33. A Hamilton circuit does not exist, as e , f , and g are all of degree 1.

35. All edges are incident to a node of degree 2 and must therefore be in the circuit.

37. a, b, c, d, f, d, e is a Hamilton path.
39. f, e, c, b, a, d is a Hamilton path.
41. A Hamilton path does not exist, as there are 8 vertices of degree 2, only 2 of which may be the endpoints of the path. The incident edges of the other 6 nodes must be in the path. For a Hamilton path to exist, then, exactly 1 of the inside corner vertices must be an end. As this is not the case, such a path does not exist.
43. $a, b, c, f, e, d, g, h, i$ is a Hamilton path.
45. A Hamilton circuit can exist only when $m = n$ and both are at least 2.

10.6 Shortest-Path Problems

3. The shortest path is a, c, d, e, g, z , which has length 16.
5. 2: a, b, e, d, z ; 3: a, c, d, e, g, z ; 4: $a, b, e, h, l, m, p, s, z$.
7. a) a, c, d b) a, c, d, f c) c, d, f d) b, d, e, g, z
11. a) Boston, Chicago, Los Angeles b) New York, Chicago, San Francisco
 c) Dallas, Los Angeles, San Francisco d) Denver, Chicago, New York
17. a) Newark, Woodbridge, Camden; Newark, Woodbridge, Camden, Cape May.
 b) Newark, Woodbridge, Camden; Newark, Woodbridge, Camden, Cape May.
19. When the most distance possible is desired to be covered; for instance, creating a route for sightseeing.
27. Detroit, Denver, San Francisco, Los Angeles, New York, Detroit.