Assignment 1

1. a) The smallest prime number can either be 2 or not 2, making "The smallest prime number is 2" a valid statement.

   b) The sum of $\cos^2\theta$ and $\sin^2\theta$ could be 1 or not 1, so "$\cos^2\theta + \sin^2\theta = 1$" is a valid statement.

   c) It is either possible for every integer to be of the form $2k$ or $2k+1$ or there exists at least one exception, so "Every integer x is of the form $2k$ or $2k+1$" is a valid statement.

   d) 0 can either be even or odd or it could not be either, making "The number 0 is neither even nor odd" a valid statement.

   e) A question is not true or false; therefore, "Is $3 > 2$ true?" is not a valid statement.

2. a) $\forall x \in \mathbb{Z}, x^2 > 0$ b) $\forall x \in \mathbb{R}, x^3 \in \mathbb{R}$

3. a) $\exists x \in Z, \forall y \in \mathbb{R}, x + y \geq 3\sqrt{2}$ b) $\forall a \in \mathbb{N}, \exists b \in \mathbb{Q}, \forall c \in \mathbb{Z}, a = b - c$

4. a) $P(2,4) = \exists y \in \mathbb{Z}, 2(2) + 4y = 4 \implies y = 0 \in \mathbb{Z} \implies \text{true}$
   $P(2,5) = \exists y \in \mathbb{Z}, 2(2) + 4y = 5 \implies y = 0.25 \notin \mathbb{Z} \implies \text{false}$

   b) There is no condition given for n, meaning that the truth value of $P(x,n)$ cannot be determined, so "$\exists x \in \mathbb{Z}, P(x,n)$" is an open sentence depending on n.

   c) As all variables are specified, "$\forall n \in \mathbb{Z}, \exists x \in \mathbb{Z}, P(x,n)$" is a mathematical statement. As the $2(x + 2y)$ must be even given that x and y are integers, this statement is false for all odd values of n, meaning that the statement as a whole is false.

5. a)

$$\left(8^{k^2}\right)\left(4^k\right) = \left(2^{3k^2}\right)\left(2^{2k}\right) = 2^{3k^2+2k} = 2 \implies 3k^2 + 2k = 1 \implies 3k^2 + 2k - 1 = 0$$
$$\implies 0 = (3k-1)(k+1) \implies k = -1 \in \mathbb{Z} \implies \text{true}$$

   b)

$$x^2 - x + \frac{1}{4} > 0 \implies \neg\exists x = \frac{1 \pm \sqrt{1-1}}{2} = \frac{1}{2} \in \mathbb{R} \implies \text{false}$$

   c)

$$\forall x \in \{0,1,2,3\}, \forall y \in \{0,1,2,3\}, (x+y) \in \mathbb{Z}, (x^2+y^2) \in \mathbb{Z} \implies \frac{x+y}{x^2+y^2} \in \mathbb{Q} \implies \text{true}$$

d)

$$4^x + (\ln x)^2 \geq 2x \ln\left(x^2\right) = 4x \ln x$$

As 4^x grows faster than $x \ln x$, for all $x \in \mathbb{N}$ and $4 \geq 0$, the the statement is true.

e)

$$\begin{aligned} x + 2xy &= 4 \\ 1 + 2y &= \frac{2}{x} \\ y &= \frac{4}{x} - \frac{1}{2} \\ x \in \mathbb{Q} &\implies \frac{4}{x} - \frac{1}{2} \in \mathbb{Q} \implies \text{true} \end{aligned}$$

6.

a) $\forall x \in \{1, 2, 3\}, \forall y \in \{1, 2, 3\}, \dfrac{4680}{x^2 + y^2} \in \mathbb{Z}$

b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^3 = 135$

c) $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x \leq y$

d) $\exists x \in \mathbb{Q}, \exists y \in \mathbb{Q}, x^3 y^2 = 108$