Homework Set 2

Arnav Patri

10 Graphs

10.1 Graphs and Graph Models

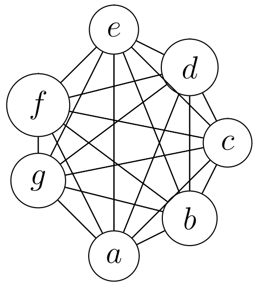
3. The graph has undirected edges and no loops, making it a simple graph.
4. The graph has multiple undirected edges and no loops, making it a multigraph.
5. The graph has multiple undirected edges and loops, making it a psuedograph.
6. The graph has multiple undirected edges and no loops, making it a multigraph.
7. The graph has directed edges and loops, making it a digraph.
8. The graph has multiple directed edges and loops, making it a directed multigraph.
9. The graph has multiple directed edges and loops, making it a directed multigraph.

10.2 Graph Terminology and Special Types of Graphs

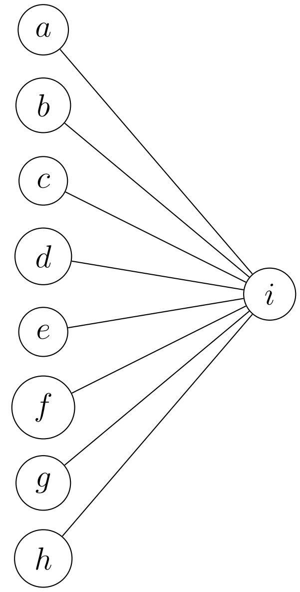
1. $|V| = 6$, $|E| = 6$, $\deg a = 2$, $\deg b = 4$, $\deg c = 1$ (pendant), $\deg d = 0$ (isolated), $\deg e = 2$, $\deg f = 3$
2. $|V| = 5$, $|E| = 13$, $\deg a = 6$, $\deg b = 6$, $\deg c = 6$, $\deg d = 5$, $\deg e = 3$
3. $|V| = 9$, $|E| = 12$, $\deg a = 3$, $\deg b = 2$, $\deg c = 4$, $\deg d = 0$ (isolated), $\deg e = 6$, $\deg f = 0$ (isolated), $\deg g = 4$, $\deg h = 2$, $\deg i = 3$
5. A simple graph with 15 vertices each of degree 5 cannot exist, as all graphs must have an even number of vertices of odd degree.
7. $|V| = 4$, $E = 7$, $\deg^- a = 3$, $\deg^+ a = 1$, $\deg^- b = 1$, $\deg^+ b = 2$, $\deg^- c = 2$, $\deg^+ c = 1$, $\deg^- d = 1$, $\deg^+ d = 3$
8. $|V| = 4$, $|E| = 8$, $\deg^- a = 1$, $\deg^- b = 3$, $\deg^- c = 2$, $\deg^- d = 1$, $\deg^+ a = 2$, $\deg^+ b = 4$, $\deg^+ c = 1$, $\deg^+ d = 1$
9. $|V| = 5$, $E = 13$, $\deg^- a = 6$, $\deg^+ a = 1$, $\deg^- b = 1$, $\deg^= b = 5$, $\deg^- c = 2$, $\deg^+ c = 5$, $\deg^- d = 4$, $\deg^= d = 2$, $\deg^- e = 0$, $\deg^= e = 0$

20.

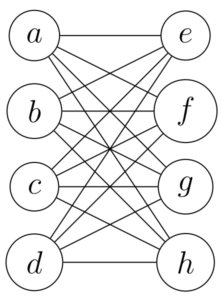
a)


e
d
f
c
g
b
a

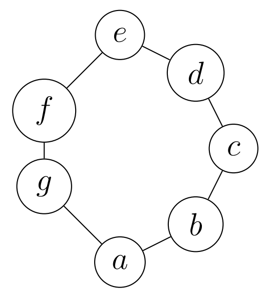
b)


a
b
c
d
i
e
f
g
h

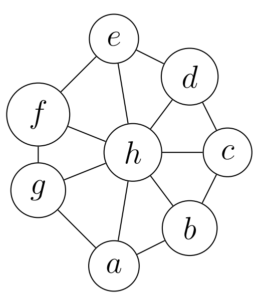
c)


a
e
b
f
c
g
d
h

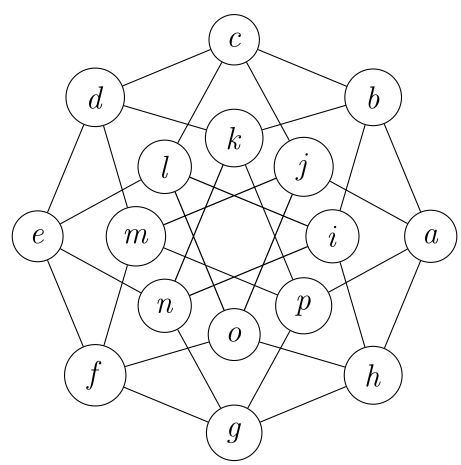
d)


e
d
f
c
g
b
a

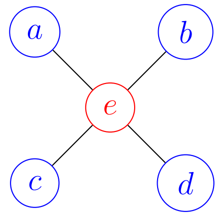
e)


e
d
f
h
c
g
b
a

f)

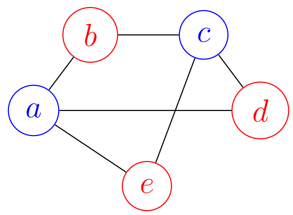

c
d
b
k
l
j
e
m
i
a
n
p
o
f
h
g

21.



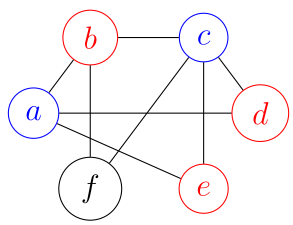
The graph is bipartite.

22.



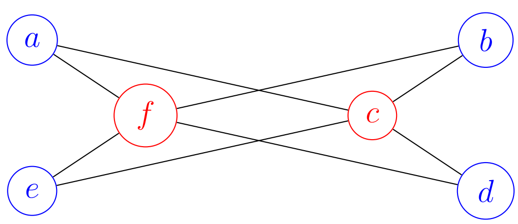
The graph is bipartite.

23.



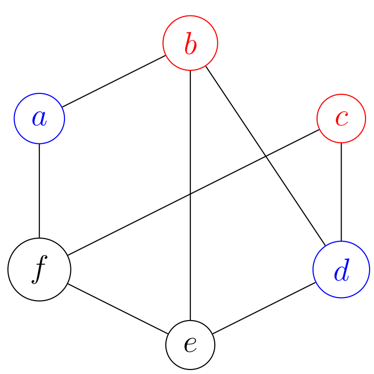
This graph is not bipartite, due to f.

24.



This graph is bipartite.

25.



This graph is not bipartite, due to e and f.

26.

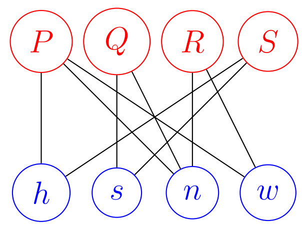
a) K_1 and K_2 are bipartite, but K_n for $n \geq 3$ is not bipartite, as any 3 vertices are connected pairwise, so there is no way to partition them into 2 disjoint sets.

b) C_n is bipartite whenever n is even, as the vertices can simply alternate.

c) W_n is never bipartite, as every vertex is connected to the center of the wheel.

d) Q_n is always bipartite.

27. a)



37. a) $|V| = n, |E| = \binom{n}{2}$

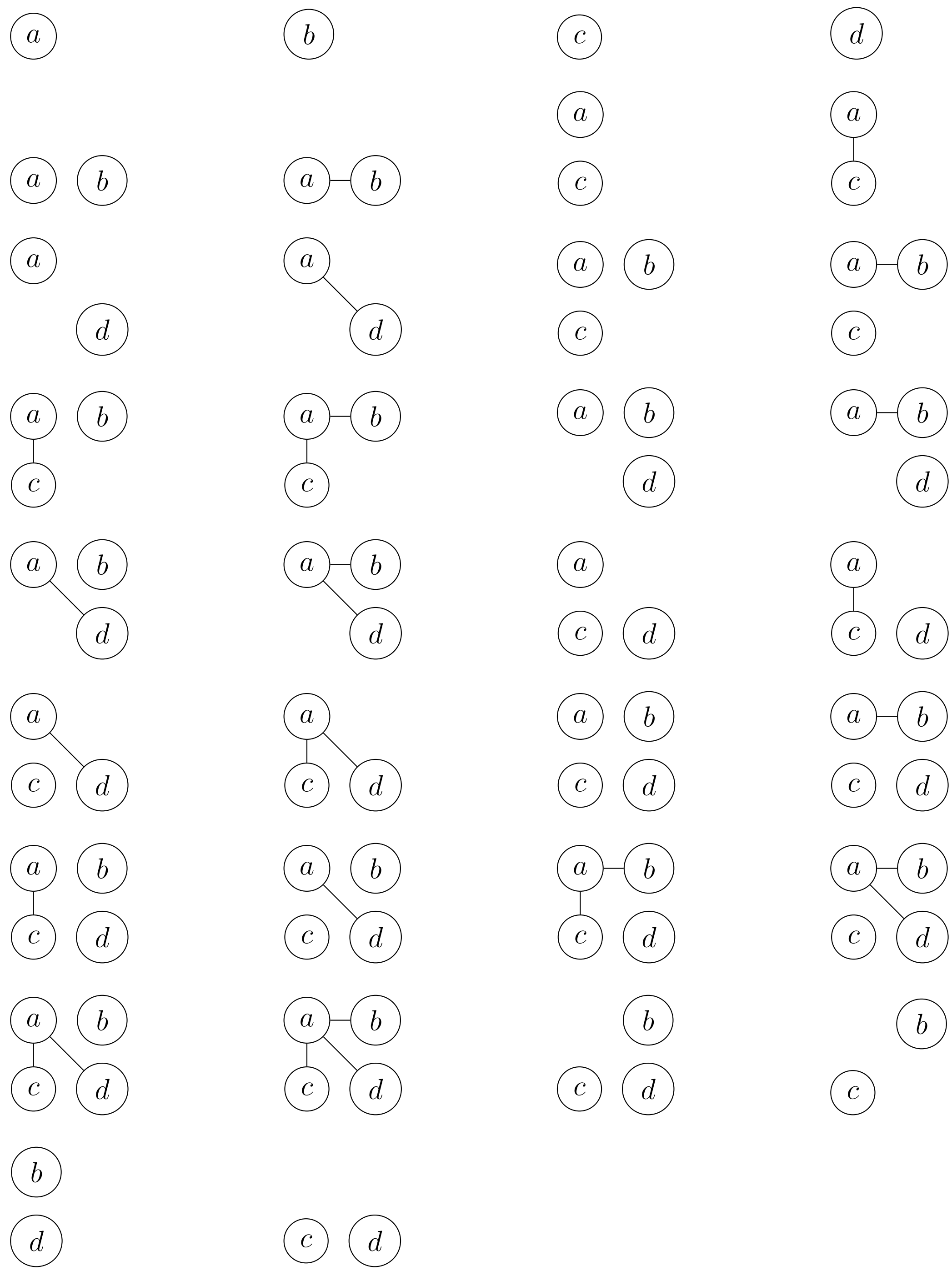
b) $|V| = n, |E| = n$

c) $|V| = n + 1, |E| = 2n$

d) $|V| = m + n, |E| = mn$

e) $|V| = 2^n, |E| = n2^{n-1}$

53.



10.3 Representing Graphs and Graph Isomorphism

1.

Vertex	Adjacent Vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

3.

Vertex	Terminal Vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

5.

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

7.

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

9. a)

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

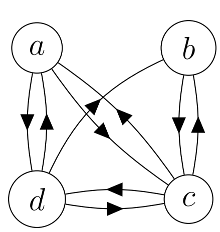
e)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

f)

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

11.



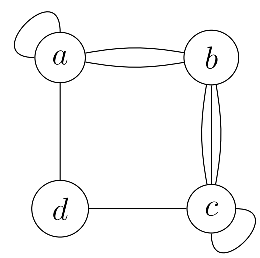
13.

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

15.

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$

17.



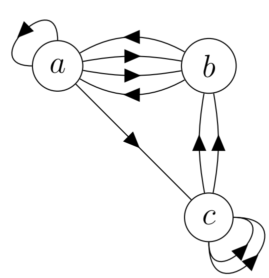
19.

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

21.

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

23.



31.

$$\mathbf{M}_{13} = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_{14} = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_{15} = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

33. For an undirected graph, the sum of the entries in a column of the adjacency matrix is the number of edges that are connected to that column's vertex (loops only being counted once). For a digraph, it is the in-degree of the vertex.

35. For an undirected graph, the sum of the values of a column in the incidence matrix is equal to the number of nodes that the column's edge is incident to. This can only be 1 (if the edge is a loop) or 2.

36. a)

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 1 & 0 & 0 & \cdots & 1 & 0 & \end{bmatrix}$$

d)

$$\begin{bmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

37. a)

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

d)

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$

39.

v	u_1	u_2	u_3	u_4	u_5
$f(v)$	v_1	v_3	v_2	v_5	v_3

41.

v	u_1	u_2	u_3	u_4	u_5	u_6	u_7
$f(v)$	v_1	v_3	v_5	v_7	v_2	v_4	v_6

43.

v	u_1	u_2	u_3	u_4	u_5	u_6
$f(v)$	v_5	v_2	v_3	v_6	v_4	v_1

45. u_5 is connected to exactly 2 other nodes, both of which are of degree 3. There is no node in the second graph that has this property. The graphs are therefore nonisomorphic.

47.

v	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
$f(v)$	v_6	v_5	v_8	v_{10}	v_7	v_3	v_9	v_2	v_4	v_1

63. a)

$$\begin{array}{c|c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \Longrightarrow \begin{array}{c|c} & \begin{array}{ccc} v_3 & v_1 & v_2 \end{array} \\ \begin{array}{c} v_3 \\ v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Yes

b) No, as there is no row in the first matrix with only 1 1.

c) No, as there is no row in the first matrix with only 1 1.

67.

v	u_1	u_2	u_3	u_4
$f(v)$	v_3	v_4	v_2	v_1

69.

v	u_1	u_2	u_3	u_4
$f(v)$	v_3	v_4	v_2	v_1

10.5 Euler and Hamilton Paths

1. a, b and c are all of degree 3, which means that 3 nodes are of odd degree, so the graph has neither an Euler path or circuit.

3. a and d are of odd degree while every other is of even degree, so an Euler path may exist. Such a path is $(a, e, c, e, b, e, d, b, a, c, d)$.

5. Every vertex is of even degree, so the graph may have an Euler circuit. Such a circuit is $(a, e, a, e, c, d, c, b, e, d, b, a)$.

7. Every vertex is of even degree, so the graph may have an Euler circuit. Such a circuit is $(a, ih, g, d, e, f, g, c, e, h, d, c, a, b, i, c, b, h, a)$.

13. This image can be drawn in a single stroke by treating the intersections of edges as nodes.

15. This image cannot be drawn in a single stroke.

19.

v	a	b	c	d
$\deg^- v$	1	3	1	2
$\deg^+ v$	2	1	2	2

$\deg^- v$ is not always equal to $\deg^+ v$, so an Euler circuit may not exist. $\deg^+ a - \deg^- a = \deg^+ c - \deg^- c = 1$, so an Euler path may not exist either.

21. An Euler path exists: $a, d, e, d, b, a, e, c, e, b, c, b, e$.

23.

v	a	b	c	d	e	f	g	h	i	j	k	l
$\deg^- v$	1	2	1	1	2	1	2	2	2	1	1	1
$\deg^+ v$	1	1	1	2	2	2	1	2	1	1	2	1

$\deg^- v$ is not always equal to $\deg^+ v$, so an Euler circuit may not exist. $\deg^+ d - \deg^- d = \deg^+ f - \deg^- f = 1$, so an Euler path may not exist either.

26.
   a) An Euler circuit exists only if all edges are of even degree. For a complete graph K_n, all edges are of degree $n - 1$, so an Euler circuit exists whenever n is odd.

   b) An Euler circuit always exists for C_n, as every node is of degree 2.

   c) An Euler circuit never exists for W_n, as every node but 1 is of degree 3.

   d) An Euler circuit only exists for Q_n if n is even, as the degree of each node is n.

31. a, b, c, d, e, a is a Hamilton circuit.

33. A Hamilton circuit does not exist, as e, f, and g are all of degree 1.

35. All edges are incident to a node of degree 2 and must therefore be in the circuit.

37. a, b, c, d, f, d, e is a Hamilton path.

39. f, e, c, b, a, d is a Hamilton path.

41. A Hamilton path does not exist, as there are 8 vertices of degree 2, only 2 of which may be the endpoints of the path. The incident edges of the other 6 nodes must be in the path. For a Hamilton path to exist, then, exactly 1 of the inside corner vertices must be an end. As this is not the case, such a path does not exist.

43. $a, b, c, f, e, d, g, h, i$ is a Hamilton path.

45. A Hamilton circuit can exist only when $m = n$ and both are at least 2.

10.6 Shortest-Path Problems

3. The shortest path is a, c, d, e, g, z, which has length 16.

5. 2: a, b, e, d, z; 3: a, c, d, e, g, z; 4: $a, b, e, h, l, m, p, s, z$.

7. a) a, c, d b) a, c, d, f c) c, d, f d) b, d, e, g, z

11. a) Boston, Chicago, Los Angeles b) New York, Chicago, San Francisco c) Dallas, Los Angeles, San Francisco d) Denver, Chicago, New York

17. a) Newark, Woodbridge, Camden; Newark, Woodbridge, Camden, Cape May.
    b) Newark, Woodbridge, Camden; Newark, Woodbridge, Camden, Cape May.

19. When the most distance possible is desired to be covered; for instance, creating a route for sightseeing.

27. Detroit, Denver, San Francisco, Los Angeles, New York, Detroit.