Assignment 0 F22

1. **Claim 1.** $S = \{x \in \mathbb{R} \mid x^3 + 2x < 4\}$ *is bounded above.*
   Proof. It is a fundamental property of the definition of S that none of its elements exceed or equal 4. Therefore, if $\alpha = 4 \in \mathbb{R}$, it is true that $x \leq \alpha$ for all $x \in S$, so S is bounded above.
   Claim 2. *S is not bounded below.*
   Proof. The definition of S provides no limitation regarding a lower bound, and as $x \to -\infty$, x^3 and $x \to -\infty$, so $x^3 + x \to -\infty$. This means that there does not exist any real number β such that $x \geq \beta$ for all $x \in S$, as x decreases infinitely, so S is unbounded below. $\square$.

2. **Claim.** *There is no order relation "$<$" on $\mathbb{C}$.*
   Proof. Assume that $i > 0$. By axiom 4,
   $$\begin{aligned} i^2 &> i(0) \\ -1 &> 0 \end{aligned}$$
   This is clearly false, so $i < 0$ by axiom 1. But by axiom 5,
   $$\begin{aligned} i^4 &< i^3(0) \\ 1 &< 0 \end{aligned}$$
   which is also false. Therefore, axiom 1 is violated, as i cannot be less than or greater than 0, meaning that there is no order relation satisfying the 5 axioms on $\mathbb{C}$. $\square$.

3. a)
   $$f(x, y) = f(x, -y) = \frac{x+1}{x^2 + y^2 + 2}$$
   as $\forall y \in \mathbb{Z}, y^2 = (-y)^2$. Therefore f is not injective.
   All $q \in \mathbb{Q}$ can be written as the ratio of $p, q \in \mathbb{Z}$, and all $p \in \mathbb{Z}$ can be written as $r + 1$ where $r \in \mathbb{Z}$, as every integer is exactly 1 more than the prior integer. The denominator goes to ∞ as $x, y \to \pm\infty$, so possible $q \in \mathbb{Q}$ can be output by f. Therefore f is surjective.

   b)
   $$f(x, y) = f(-x, -y) = xy$$
   as the negatives cancel. Therefore f is not injective.
   Every $r \in \mathbb{R}$ can be written as $r \times 1$ and $1 \in \mathbb{R}$, so f can output every $r \in \mathbb{R}$, making f surjective.

   c)
   $$f(x) = f(-x) = \frac{x^2}{1 + x^2}$$
   as for all $x \in \mathbb{R}$, $x^2 = (-x)^2$, so f is not injective.
   $f(x)$ is a rational function with a denominator never equal to 0, meaning that it is continuous for all $x \in \mathbb{R}$. $f(0) = 0$ and
   $$\lim_{x \to \pm\infty} \frac{x^2}{1 + x^2} = \frac{1}{1} = 1$$
   so by the intermediate value theorem, f must yield all outputs in $[0, 1)$, making f surjective.

4. **Claim.** *There does not exist a surjective function from X onto its power set.*
Proof. Each element $x \in X$ can be either present or not present in a given subset of X. The cardinality of the $P(X)$ is therefore $2^{|X|}$. As $|X| \in \mathbb{N}$ and $2^n > n$ for all $n \in \mathbb{N}$, there are more elements in $P(X)$ than there are in X, making a surjective function impossible. $\square$

5. a)

$$
\begin{aligned}
f \cdot g(\alpha) &= f(\alpha) * g(\alpha) = \alpha f(\alpha) g(\alpha) = g(\alpha) \\
f(\alpha) &= \frac{1}{\alpha}
\end{aligned}
$$

b)

$$
\begin{aligned}
f \cdot g(\alpha) &= f(\alpha) * g(\alpha) = \alpha f(\alpha) g(\alpha) = \alpha(2 + \alpha^2 i) g(\alpha) = (2\alpha + \alpha^3 i) g(\alpha) = 1 \\
g(\alpha) &= \frac{1}{2\alpha + \alpha^3 i}
\end{aligned}
$$

c)

$$
\begin{aligned}
f \cdot g(\alpha) &= f(\alpha) * g(\alpha) = \alpha f(\alpha) g(\alpha) = \alpha(2 + \alpha^2 i)(\alpha - i) = \alpha(2\alpha - 2i + \alpha^3 i + \alpha^2) \\
&= (\alpha^3 + 2\alpha^2) + (\alpha^4 - 2)i
\end{aligned}
$$

6. a) In order for X to contain $x + y$ for all unique $x, y \in X$, it can be defined as $X = \{0, n\}$. Then $0 + n = n \in X$ and $0 \times n = 0 \in X$, making X a sticky subset containing n.

   b) For X to to meet the criteria that for all $x, y \in X$, $x+y \in X$, it can be $X = \{kn \mid k < \mathbb{Z}\}$. Only one such set exists per number.

7. The induction is not true going from $n = 1$ to $n = 2$. For $n = 1$,

$$P(1) \implies x_1 = x_1$$

while

$$P(2) \implies x_1 = x_2$$

Removing x_2 from this yields simply x_1, which is not a statement but rather a number. The transitive property can therefore not be applied, nor can induction.

8. a)

$$a_n = n \implies \lim_{n \to \infty} a_n = \infty$$

b)

$$a_n = \frac{1}{2^n} \implies \alpha = \frac{1}{1 - 1/2} = 2$$