9.3

Integral Test

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^2+1}\right] \qquad \text{(is always positive, continuous, and decreases as } n \text{ grows)}$$

$$\int_1^{\infty}\left[\frac{1}{x^2=1}\right]dx = \lim_{a\to\infty}[\arctan x]_1^a = \lim_{a\to\infty}[\arctan a - \arctan 1] = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4} \therefore \sum_{n=1}^{\infty}\left[\frac{1}{n^2+1}\right]$$

p-series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \text{ diverges} \qquad (p = \tfrac{1}{2} \ \leq 1 \therefore \text{diverges})$$

$$\sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges} \qquad (p = 1 \leq 1 \therefore \text{diverges})$$

9.4 Comparison Tests

Direct Comparison Test

$$\sum_{n=1}^{\infty}\left[\frac{1}{2+3^n}\right] \qquad \sum_{n=1}^{\infty}\left[\frac{1}{3^n}\right] = \sum_{n=1}^{\infty}\left[\frac{1}{3}\right]^n$$

(converges)

$$\frac{1}{2+3^n} \leq \frac{1}{3^n} \qquad \text{(is always true} \wedge \text{larger series diverges} \therefore \text{original converges)}$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{10+\sqrt{n}}\right] \qquad \sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{n}}\right]$$

(diverges)

$$\frac{1}{\sqrt{n}} \leq \frac{1}{10+\sqrt{n}} \qquad \text{(false)}$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{n}\right] \qquad \text{(diverges)}$$

$$\frac{1}{n} \leq \frac{1}{10+\sqrt{n}}$$

n	1	9	16	25
$\frac{1}{n}$	1	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$
$\frac{1}{10+\sqrt{n}}$	$\frac{1}{11}$	$\frac{1}{13}$	$\frac{1}{14}$	$\frac{1}{15}$
$\frac{1}{n} \leq \frac{1}{10+\sqrt{n}}$	False	False	True	True

$$\frac{1}{n} \leq \frac{1}{10+\sqrt{n}} \text{ as } n \text{ grows larger} \wedge \frac{1}{n} \text{ diverges} \therefore \frac{1}{10+\sqrt{n}} \text{ diverges}$$

9.5

Alternating Series Test

$$\sum_{n=1}^{\infty}\left[\frac{n}{(-2)^{n-1}}\right] = \sum_{n=1}^{\infty}\left[\frac{n}{(-1\times 2)^{n-1}}\right]$$
$$= \sum_{n=1}^{\infty}\left[\frac{1}{(-1)^{n-1}}\times\frac{n}{2^{n-1}}\right]$$
$$\lim_{n\to\infty}\left[\frac{n}{2^{n-1}}\right] = \frac{\text{slow}}{\text{fast}} = 0$$
$$a_{n+1} \leq a_n$$
$$\frac{n+1}{2^n} \leq \frac{n}{2^{n-1}} \qquad \text{(larger denominator} \therefore \text{true} \therefore \text{converges)}$$

9.6

Ratio Test

$$\sum_{n=1}^{\infty}\left[\frac{2^n}{n!}\right]$$
$$\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\times\frac{n!}{2^n}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0 < 1 \therefore \text{converges}$$

Factorials

$$(n+1)! = n!(n+1)$$
$$(3n+4)! = (3n)!(3n+4)(3n+3)(3n+2)(3n+1)$$
$$(an+b)! = (an)!(an+b)(an+b-1)(an+b-2)\cdots = (an)!\prod_{i=0}^{b-1}(an+b-i) = (an)!\prod_{i=1}^{b}(an+i)$$
$$(0+1)! = 0!(0+1)$$
$$1! = 0!(1)$$
$$1 = 0!$$

Root Test

$$\sum_{n=1}^{\infty}\left[\frac{e^{2n}}{n^n}\right]$$
$$\lim_{n\to\infty}\left(\frac{e^{2n}}{n^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{e^2}{n}\right) = 0 < 1 \therefore \text{converges}$$

9.7 Power Series

$$\sum_{n=1}^{\infty}\left[\frac{(x-2)^n}{n}\right]$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n+1}\times\frac{n}{(x-2)^n}\right| < 1$$

$$\lim_{n\to\infty}|(x-2)\times 1| < 1$$

$$|x-2| < 1$$

$$x-2 < 1 \qquad\qquad x-2 > -1$$

$$x < 3 \qquad\qquad x > 1$$

$$1 < x < 3$$

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}\right]$$

9.8 Taylor and Maclaurin Polynomials

$f(x) = \sqrt{x+1}$

$f(x)=\sqrt{x+1}$	$f'(x)=\frac{1}{2}(x+1)^{-1/2}$	$f''(x)=\frac{-1}{4}(x+1)^{-3/2}$	$f^{(3)}(x)=\frac{3}{8}(x+1)^{-5/2}$	$f^{(4)}(x)=\frac{-15}{16}(x+1)^{-7/2}$
$f(0)=1$	$f'(1)=\frac{1}{2}$	$f''(0)=\frac{-1}{4}$	$f^{(3)}(0)=\frac{3}{8}$	$f^{(4)}(0)=\frac{-15}{16}$

$$P_4 = 1+\frac{1}{2}x-\frac{\frac{1}{4}x^2}{2!}+\frac{\frac{3}{8}x^3}{3!}-\frac{\frac{15}{16}x^4}{4!} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4$$

$$y = \ln(2+x)$$

$$\begin{array}{r|c} f(x)=\ln(2+x) & \ln(1)=0 \\ f'(x)=(2+x)^{-1} & (1)^{-1}=1 \\ f''(x)=-1(2+x)^{-2} & -1(-1)^2=-1 \\ f^{(3)}(x)=2(2+x)^{-3} & 2(-1)^{-3}=2 \\ f^{(4)}(x)=-6(2+x)^{-4} & -6(-1)^{-4}=-6 \end{array}$$

$$\begin{aligned} P_4 &= 0+(1)(x+1)+\frac{-1(x+1)^2}{2!}+\frac{2(x+1)^3}{3!}+frac{-}6(x+1)^4 4! \\ &= x+1-\frac{(x+1)^2}{2}+\frac{x+1}{3}-\frac{(x+1)^4}{4} \\ &= \sum_{i=1}^{4}\left[\frac{(-1)^{n+1}(x+1)^n}{n}\right] \\ y &= \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}(x+1)^n}{n}\right] \end{aligned}$$