Volume: Disk/Washer

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Sources

Calculus: Early Transcendentals 9th Edition

2. 6.2 Exercise 12
4. 6.2 Exercise 15
5. 6.2 Exercise 1
7. 6.2 Exercise 17
10. 6.2 Exercise 24

Calculus 2: Washer Method (HARD Problems) by Ludus

https://www.youtube.com/watch?v=BcFEcRimurc

6. Example 1
8. Example 2

AP Calculus Exams

11. 2021 AB FRQ 3(c)

Problems

If no instructions are given, evaluate the volume of the solid generated by revolving the region bounded by the given equations about the specified line using the disk/washer method.

1. $$\left.\begin{array}{ll} y = r & y = 0 \\ x = 0 & x = h \end{array}\right| \; y = 0$$

2. $$\left.\begin{array}{ll} y = 0 & y = \dfrac{1}{x} \\ x = 1 & x = 4 \end{array}\right| \; y = 0$$

3. $$\left.\begin{array}{ll} y = 0 & y = \dfrac{1}{\sqrt{1+x^2}} \\ x = 0 & \end{array}\right| \; y = 0$$

4. $$\left.\begin{array}{ll} y = \dfrac{x^2}{4} & y = 9 \\ x = 0 & \end{array}\right| \; x = 0$$

5. $$\left.\begin{array}{ll} y = 0 & y = x^2 + 5 \\ x = 0 & x = 3 \end{array}\right| \; y = 0$$

6. $$\left.\begin{array}{cc} y = 1 + \sec x & y = 3 \\ 1 + \sec x \leq 3 & -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \end{array}\right| \; y = 1$$

7. $$y = x^2 \quad y = 2x \;\big|\; x = 0$$

8. $$x = y^2 \quad x = 1 - y^2 \;\big|\; x = 3$$

9. $$y = x^2 - 2x \quad y = x \;\big|\; y = 4$$

10. $$\left.\begin{array}{ll} y = \sin x & y = \cos x \\ x \geq 0 & x \leq \dfrac{\pi}{4} \end{array}\right| \; y = -1$$

11. $$f(x) = cx\sqrt{4 - x^2}$$

    The solid of revolution generated by rotating the area bounded by f and the x-axis in the first quadrant about the x-axis is equal to 2π. Solve for c, given that it is a positive constant.

12.
$$x = (y-k)^2 \quad y = x + k \mid k \le 0 \text{ or } k \ge 1 \,\Big|\, x = k$$

13.
$$\left.\begin{matrix} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \mid a, b \neq 0 & x = \left|\dfrac{ay}{b}\right| \\ x = a^2 & x = \sqrt{a^2+b^2} \end{matrix}\right| x = 0$$

14.
$$\left.\begin{matrix} \dfrac{x^4}{a^2} - \dfrac{y^2}{b^2} = 1 \mid a, b \neq 0 & x^2 = \dfrac{ay}{b} \\ y = 0 & y = b \end{matrix}\right| x = 0$$

15.
$$\left.\begin{matrix} x^2 + y^2 = r^2 & \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \mid a, b > r > 0 \\ \text{(a) } y = 0 & \end{matrix}\right| \begin{matrix} \text{(a) } y = 0 \\ \text{(b) } y = b \end{matrix}$$

Solutions

1.
$$V = \pi \int_0^h \left[r^2\right] \mathrm{d}x = \pi \left[r^2 x\right]_0^h = \pi \left[r^2 h - (0)\right] = \pi r^2 h$$

2.
$$V = \pi \int_1^4 \left(\frac{1}{x}\right)^2 \mathrm{d}x = \pi \left[-\frac{1}{x}\right]_1^4 = \pi \left[-\frac{1}{4} - \left(-\frac{1}{1}\right)\right] = \frac{3\pi}{4}$$

3.
$$\frac{1}{\sqrt{1+x^2}} = 0 \implies x \to \infty$$
$$V = \pi \int_0^\infty \left(\frac{1}{\sqrt{1+x^2}}\right)^2 \mathrm{d}x = \pi \int_0^\infty \left[\frac{1}{1+x^2}\right] \mathrm{d}x = \pi[\arctan x]_0^\infty = \pi \left[\frac{\pi}{2} - (0)\right] = \frac{\pi^2}{2}$$

4.
$$\begin{aligned} y &= \frac{x^2}{4} \implies x = 2\sqrt{y} \\ y_1 &= 2\sqrt{0} = 0 \\ V &= \pi \int_0^9 (2\sqrt{y})^2 \,\mathrm{d}y = \pi \left[2y^2\right]_0^9 = 2(81)\pi = 162\pi \end{aligned}$$

5.
$$\begin{aligned} V &= \pi \int_0^3 \left(x^2+5\right)^2 \mathrm{d}x = \pi \int_0^3 \left[x^4 + 10x^2 + 25\right] \mathrm{d}x = \pi \left[\frac{x^5}{5} + \frac{10x^3}{3} + 25x\right]_0^3 \\ &= \pi \left[\frac{3^5}{5} + \frac{10(3)^3}{3} + 25(3) - (0)\right] = \pi \left[\frac{243}{5} + 90 + 75\right] = \frac{\pi(243+825)}{5} = \frac{1068\pi}{5} \end{aligned}$$

6.
$$1 + \sec x = 3 \implies \sec x = 2 \implies x = \pm\frac{\pi}{3}$$
$$\begin{aligned} V &= \pi \int_{-\pi/3}^{\pi/3} \left[(3-1)^2 - (1+\sec x - 1)^2\right] \mathrm{d}x = 2\pi \int_0^{\pi/3} \left[4 - \sec^2 x\right] \mathrm{d}x \\ &= 2\pi[4x - \tan x]_0^{\pi/3} = 2\pi \left[\frac{4\pi}{3} - \sqrt{3} - (0)\right] = \frac{8\pi^2}{3} - 2\pi\sqrt{3} \end{aligned}$$

7.
$$y = x^2 \implies x = \sqrt{y} \qquad y = 2x \implies x = \frac{y}{2}$$
$$\sqrt{y} = \frac{y}{2} \implies 4y = y^2 \implies 0 = y(y-4) \implies y_1 = 0, y_2 = 4$$
$$\begin{aligned} V &= \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right] \mathrm{d}y = \pi \int_0^4 \left[y - \frac{y^2}{4}\right] \mathrm{d}y = \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 \\ &= \pi \left[\frac{y^2(6-y)}{12}\right]_0^4 = \pi \left[\frac{4^2(6-4)}{12} - (0)\right] = \pi \left[\frac{16(2)}{12}\right] = \frac{8\pi}{3} \end{aligned}$$

8.

$$
\begin{aligned}
&y^2 = 1 - y^2 \implies 2y^2 = 1 \implies y = \pm\frac{\sqrt{2}}{2} \\
&V = \pi \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left[\left(y^2-3\right)^2 - \left(1-y^2-3\right)^2\right] \mathrm{d}y = 2\pi \int_0^{\sqrt{2}/2} \left[\left(y^2-3\right)^2 - \left(-y^2-2\right)^2\right] \mathrm{d}y \\
&\quad = 2\pi \int_0^{\sqrt{2}/2} \left[y^4 - 6y^2 + 9 - \left(y^4 + 4y^2 + 4\right)\right] \mathrm{d}y = 2\pi \int_0^{\sqrt{2}/2} \left[5 - 10y^2\right] \mathrm{d}y \\
&\quad = 2\pi \left[5y - \frac{10y^3}{3}\right]_0^{\sqrt{2}/2} = 2\pi \left[\frac{5\sqrt{2}}{2} - \frac{10}{3}\left(\frac{\sqrt{8}}{8}\right)\right] = \pi \left[5\sqrt{2} - \frac{40\sqrt{2}}{24}\right] = \frac{10\pi\sqrt{2}}{3}
\end{aligned}
$$

9.

$$
\begin{aligned}
&x = x^2 - 2x \implies 0 = x^2 - 3x = x(x-3) \implies x_1 = 0, x_2, = 3 \\
&V = \pi \int_0^3 \left[\left(x^2 - 2x - 4\right)^2 - (x-4)^2\right] \mathrm{d}x \\
&\quad = \pi \int_0^3 \left[x^4 - 2x^3 - 4x^2 - 2x^3 + 4x^2 + 8x - 4x^2 + 8x + 16 - \left(x^2 - 8x + 16\right)\right] \mathrm{d}x \\
&\quad = \pi \int_0^3 \left[x^4 - 4x^3 - 4x^2 + 16x + 16 - \left(x^2 - 8x + 16\right)\right] \mathrm{d}x \\
&\quad = \pi \int_0^3 \left[x^4 - 4x^3 - 5x^2 + 24x\right] \mathrm{d}x = \pi \left[\frac{x^5}{5} - x^4 - \frac{5x^3}{3} + 12x^2\right]_0^3 \\
&\quad = \pi \left[\frac{x^2\left(3x^3 - 15x^2 - 25x + 180\right)}{15}\right]_0^3 = \pi \left[\frac{3(81 - 135 - 75 + 180)}{5} - (0)\right] \\
&\quad = \frac{3\pi(51)}{5} = \frac{153\pi}{5}
\end{aligned}
$$

10.

$$
\begin{aligned}
&\sin x = \cos x \implies x = \frac{\pi}{4} \\
&V = \pi \int_0^{\pi/4} \left[(\cos x + 1)^2 - (\sin x + 1)^2\right] \mathrm{d}x \\
&\quad = \pi \int_0^{\pi/4} \left[\cos^2 x + 2\cos x + 1 - \sin^2 x - 2\sin x - 1\right] \mathrm{d}x \\
&\quad = \pi \int_0^{\pi/4} \left[\cos(2x) + 2\cos x - 2\sin x\right] \mathrm{d}x = \pi \left[\frac{\sin(2x)}{2} + 2\sin x + 2\cos x\right]_0^{\pi/4} \\
&\quad = \pi \left[\frac{1}{2} + \sqrt{2} + \sqrt{2} - (0 + 0 + 2)\right] = \frac{(4\sqrt{2} - 3)\pi}{2}
\end{aligned}
$$

11.

$$0 = cx\sqrt{4-x^2} \implies \begin{cases} x = 0 \\ \sqrt{4-x^2} = 0 \end{cases} \implies 4 - x^2 = 0 \implies 4 = x^2 \implies x = 2$$

$$2\pi = \pi \int_0^2 \left(cx\sqrt{4-x^2}\right)^2 \mathrm{d}x = \pi \int_0^2 \left[c^2x^2(4-x^2)\right] \mathrm{d}x = \pi \int_0^4 \left[4c^2x^2 - c^2x^4\right] \mathrm{d}x$$

$$= \pi \left[\frac{4c^2x^3}{3} - \frac{c^2x^5}{5}\right]_0^2 = \pi \left[\frac{4c^2(2)^3}{3} - \frac{c^2(2)^5}{5} - (0)\right] = \pi \left[\frac{32c^2}{3} - \frac{32c^2}{5}\right]$$

$$2 = 32c^2 \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{64c^2}{15}$$

$$c = \sqrt{\frac{30}{64}} = \sqrt{\frac{15}{32}}$$

12.

$$y = x + k \implies x = y - k$$

For both equations, k only controls a vertical shift. As this shift is the same for both equations, the area between the curves is constant regardless of the value of k. The volume when revolved about the same axis will therefore also be the same.

$$y = y^2 \implies 0 = y(y-1) \implies y_1 = 0, y_2 = 1 \qquad (k = 0)$$

$$V = \pi \int_0^1 \left[(y-k)^2 - (y^2-k)^2\right] \mathrm{d}y = \pi \int_0^1 \left[y^2 - 2ky + k^2 - y^4 + 2ky^2 - k^2\right] \mathrm{d}y$$

$$= \pi \int_0^1 \left[-y^4 + y^2(2k+1) - 2ky\right] \mathrm{d}y = \pi \left[-\frac{y^5}{5} + \frac{y^3(2k+1)}{3} - ky^2\right]_0^1$$

$$= \pi \left[\frac{-3y^5 + 5y^3(2k+1) - 15ky^2}{15}\right]_0^1 = \pi \left[\frac{-3 + 5(2k+1) - 15k}{15} - (0)\right]$$

$$= \frac{\pi(-3 + 10k + 5 - 15k)}{15} = \frac{\pi(2-5k)}{15}$$

13.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \implies x = a\sqrt{1 + \frac{y^2}{b^2}} = \frac{a\sqrt{b^2+y^2}}{b}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \implies y = \frac{b\sqrt{x^2-a^2}}{a}$$

$$y_1 = \frac{b\sqrt{a^4-a^2}}{a} = b\sqrt{a^2-1} \qquad y_2 = \frac{b\sqrt{a^2+b^2-a^2}}{a} = \frac{b^2}{a}$$

$$V = 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} \left[\left(\frac{a\sqrt{b^2+y^2}}{b}\right)^2 - \left(\frac{ay}{b}\right)^2\right] \mathrm{d}y \right| = 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} \left[\frac{a^2(b^2+y^2-y^2)}{b^2}\right] \mathrm{d}y \right|$$

$$= 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} \left[a^2\right] \mathrm{d}y \right| = 2\pi \left| \left[a^2y\right]_{b\sqrt{a^2-1}}^{b^2/a} \right| = 2\pi \left| \frac{a^2b^2}{a} - \left(a^2b\sqrt{a^2-1}\right) \right|$$

$$= \left| 2\pi ab\left(b - a\sqrt{a^2-1}\right) \right|$$

14.

$$1 = \frac{x^4}{a^2} - \frac{y^2}{b^2} \implies x = \sqrt{\frac{a\sqrt{b^2+y^2}}{b}}$$

$$V = 2\pi \left| \int_0^b \left[\frac{a\sqrt{b^2+y^2}}{b} - \frac{ay}{b} \right] \mathrm{d}y \right| = 2\pi \left| \int_0^b \left[\frac{a\left(\sqrt{b^2+y^2}-y\right)}{b} \right] \mathrm{d}y \right|$$

$$I(\alpha) = \int \left[\sqrt{\alpha^2+y^2} \right] \mathrm{d}y$$

$$y = \alpha \tan\theta \implies \mathrm{d}y = \alpha \sec^2\theta \mathrm{d}\theta$$

$$I(\alpha) = \int \left[\alpha \sec^2\theta \sqrt{\alpha^2 + \alpha^2 \tan^2\theta} \right] \mathrm{d}\theta = \int \left[\alpha^2 \sec^3\theta \right] \mathrm{d}\theta$$

$$u = \sec\theta \implies \mathrm{d}u = \sec\theta\tan\theta \qquad \mathrm{d}v = \sec^2\theta \implies v = \tan\theta$$

$$\int \left[\sec^3\theta\right] \mathrm{d}\theta = uv - \int v\,\mathrm{d}u = \sec\theta\tan\theta - \int \left[\sec\theta\tan^2\right] \mathrm{d}\theta$$

$$\int \left[\sec\theta\tan^2\theta\right] \mathrm{d}\theta = \int \left[\sec\theta\left(\sec^2\theta - 1\right)\right] \mathrm{d}\theta = \int \left[\sec^3\theta - \sec\theta\right]$$

$$= \int \left[\sec^3\theta\right] \mathrm{d}\theta - \ln|\sec\theta + \tan\theta|$$

$$\int \left[\sec^3\theta\right] \mathrm{d}\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| - \int \left[\sec^3\theta\right] \mathrm{d}\theta$$

$$2\int \left[\sec^3\theta\right] \mathrm{d}\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|$$

$$\int \left[\sec^3\theta\right] \mathrm{d}\theta = \frac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|}{2} + C$$

$$I(\alpha) = \alpha^2 \int \left[\sec^3\theta\right] \mathrm{d}\theta = \frac{\alpha^2(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)}{2}$$

$$\tan\theta = \frac{y}{\alpha} \qquad \sec\theta = \frac{\sqrt{\alpha^2+y^2}}{\alpha}$$

$$I(\alpha) = \frac{\alpha^2}{2}\left(\frac{y\sqrt{\alpha^2+y^2}}{\alpha^2} + \ln\left| \frac{y+\sqrt{\alpha^2+y^2}}{\alpha} \right| \right)$$

$$= \frac{1}{2}\left(y\sqrt{\alpha^2+y^2} + \alpha^2 \ln\left| \frac{y+\sqrt{\alpha^2+y^2}}{\alpha} \right| \right) + C$$

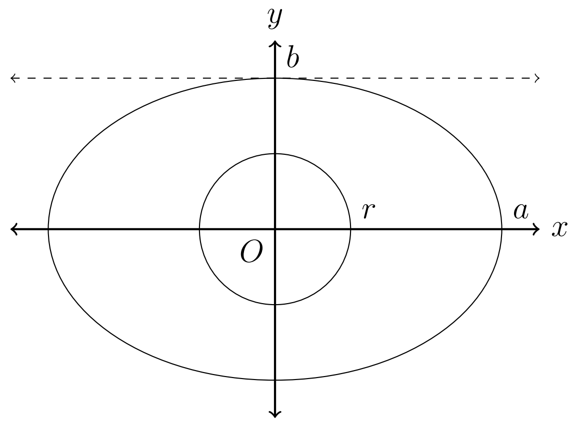
$$V = \frac{2a\pi}{b} \left| \left[I(b) - \frac{y^2}{2} \right]_0^b \right|$$

$$= \frac{2a\pi}{b} \left| \left[\frac{1}{2}\left(y\sqrt{b^2+y^2} + b^2 \ln\left| \frac{y+\sqrt{b^2+y^2}}{b} \right| - y^2 \right) \right]_0^b \right|$$

$$= \frac{a\pi}{b} \left| \left[b\sqrt{2b^2} + b^2 \ln\left| \frac{b+\sqrt{2b^2}}{b} \right| - b^2 - (0) \right] \right|$$

$$= \left| ab\pi\left(\sqrt{2} - 1 + \ln\left(1+\sqrt{2}\right)\right) \right|$$

15.



This graph assumes $a > b$, which need not be the case.

$$x^2 + y^2 = r^2 \implies y = \sqrt{r^2 - x^2} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b\sqrt{a^2 - x^2}}{a}$$

(a)

$$\begin{aligned}
V_1 &= \pi \int_0^r \left[\left(\frac{b\sqrt{a^2 - x^2}}{a}\right)^2 - \left(\sqrt{r^2 - x^2}\right)^2\right] \mathrm{d}x = \pi \int_0^r \left[\frac{b^2(a^2 - x^2)}{a^2} - (r^2 - x^2)\right] \mathrm{d}x \\
&= \pi \int_0^r \left[\frac{a^2(b^2 - r^2) + x^2(a^2 - b^2)}{a^2}\right] \mathrm{d}x = \pi \left[\frac{a^2x(b^2 - r^2)}{a^2} + \frac{x^3(a^2 - b^2)}{3a^2}\right]_0^4 \\
&= \pi \left[\frac{3a^2x(b^2 - r^2) + x^3(a^2 - b^2)}{3a^2}\right]_0^r = \pi \left[\frac{3a^2r(b^2 - r^2) + r^3(a^2 - b^2)}{3a^2}\right] \\
&= \pi \left[\frac{3a^2b^2r - 3a^2r^3 + a^2r^3 - b^2r^3}{3a^2}\right] = \pi \left(\frac{3a^2b^2r - 2a^2r^3 - b^2r^3}{3a^2}\right) \\
V_2 &= \pi \int_r^a \left(\frac{b\sqrt{a^2 - x^2}}{a}\right)^2 \mathrm{d}x = \pi \int_r^a \left[\frac{b^2(a^2 - x^2)}{a^2}\right] \mathrm{d}x = \pi \int_r^a \left[\frac{a^2b^2 - b^2x^2}{a^2}\right] \mathrm{d}x \\
&= \pi \left[\frac{a^2b^2x}{a^2} - \frac{b^2x^3}{3a^2}\right]_r^a = \pi \left[\frac{3a^2b^2x - b^2x^3}{3a^2}\right]_r^a = \left[\frac{3a^3b^2 - b^2a^3}{3a^2} - \left(\frac{3a^2b^2r - b^2r^3}{3a^2}\right)\right] \\
&= \pi \left(\frac{2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2}\right) \\
V &= 2(V_1 + V_2) = 2\pi \left(\frac{3a^2b^2r - 2a^2r^3 - b^2r^3 + 2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2}\right) \\
&= 2\pi \left(\frac{2a^3b^2 - 2a^2r^3}{3a^2}\right) = 4\pi \left(\frac{ab^2 - r^3}{3}\right)
\end{aligned}$$

(b)

$$
\begin{aligned}
V_1 &= \pi \int_0^r \left[\left(\sqrt{r^2 - x^2} - b \right)^2 - \left(\frac{b\sqrt{a^2 - x^2}}{a} - b \right)^2 \right] \mathrm{d}x \\
&= \pi \int_0^r \left[(r^2 - x^2) - 2b\sqrt{r^2 - x^2} + b^2 - \left(\frac{b^2(a^2 - x^2)}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right) \right] \mathrm{d}x \\
&= \pi \int_0^r \left[r^2 - x^2 - 2b\sqrt{r^2 - x^2} + b^2 - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2 - x^2}}{a} - b^2 \right] \mathrm{d}x \\
&= \pi \int_0^r \left[r^2 - x^2 - 2b\sqrt{r^2 - x^2} - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2 - x^2}}{a} \right] \mathrm{d}x \\
&= \pi \left(\left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right]_0^r + 2b \int_0^r \left[\frac{b\sqrt{a^2 - x^2}}{a} - \sqrt{r^2 - x^2} \right] \mathrm{d}x \right) \\
I(\alpha) &= \int \left[\sqrt{\alpha^2 - x^2} \right] \mathrm{d}x \implies x = \alpha \sin\theta \implies \mathrm{d}x = \alpha\cos\theta \mathrm{d}\theta \\
&= \int \left[\cos\theta \sqrt{\alpha^2 - \alpha^2 \sin^2\theta} \right] \mathrm{d}\theta = \int \left[\alpha^2 \cos^2\theta \right] \mathrm{d}\theta = \alpha^2 \int \left[\frac{\cos(2\theta) + 1}{2} \right] \mathrm{d}\theta \\
&= \alpha^2 \left(\frac{\sin(2\theta)}{4} + \frac{\theta}{2} \right) + C = \alpha^2 \left(\frac{2\sin\theta\cos\theta}{4} + \frac{\theta}{2} \right) + C \\
&= \alpha^2 \left(\frac{\left(\frac{x}{\alpha}\right)\left(\frac{\sqrt{\alpha^2 - x^2}}{\alpha}\right)}{2} + \frac{\arcsin(x/\alpha)}{2} \right) + C = \frac{x\sqrt{\alpha^2 - x^2} + \alpha^2 \arcsin(x/\alpha)}{2} + C \\
V_1 &= \pi \left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - a \right) \frac{x^3}{3} + 2b \left(\frac{bI(a)}{a} - I(r) \right) \right]_0^r \\
&= \pi \left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right. \\
&\quad \left. + 2b \left(\frac{bx\sqrt{a^2 - x^2} + a^2 b \arcsin(x/a)}{2a} - \frac{x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r)}{2} \right) \right]_0^r \\
&= \pi \left[(r^2 - b^2)x + \left(\frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right. \\
&\quad \left. + 2b \left(\frac{xb\sqrt{a^2 - x^2} + a^2 b \arcsin(x/a)}{2a} - \frac{x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r)}{2} \right) \right]_0^r \\
&= \pi \left[x \left(r^2 - b^2 + \frac{x^2}{3} \left(\frac{b^2}{a^2} - 1 \right) \right) \right. \\
&\quad \left. + \frac{b}{a} \left(x \left(b\sqrt{a^2 - x^2} - a\sqrt{r^2 - x^2} \right) + a^2 b \arcsin\left(\frac{x}{a} \right) - r^2 \arcsin\left(\frac{x}{r} \right) \right) \right]_0^r \\
&= \pi \left[r \left(r^2 - b^2 + \frac{r^2}{3} \left(\frac{b^2}{a^2} - 1 \right) \right) \right. \\
&\quad \left. + \frac{b}{a} \left(r \left(b\sqrt{a^2 - r^2} - 0 \right) + a^2 b \arcsin\left(\frac{r}{a} \right) - \frac{a\pi r^2}{2} \right) - (0) \right] \\
&= \pi \left(r^3 - b^2 r + \frac{b^2 r^3}{3a^2} - \frac{r^3}{3} + \frac{b^2 r\sqrt{a^2 - r^2}}{a} + ab^2 \arcsin\left(\frac{r}{a} \right) - \frac{b\pi r^2}{2} \right) \\
&= \pi \left(\frac{2r^3(2a^2 + b^2) - 3a^2 b\pi r^2 + 6ab^2 r \left(\sqrt{a^2 - r^2} - a \right) + 6a^3 b^2 \arcsin(r/a)}{6a^2} \right)
\end{aligned}
$$

(b) (cont.)

$$
\begin{aligned}
V_2 &= \pi \int_r^a \left(\frac{b\sqrt{a^2-x^2}}{a} - b\right)^2 \mathrm{d}x = \pi \int_r^a \left[\frac{b^2(a^2-x^2)}{a^2} - \frac{2b^2\sqrt{a^2-x^2}}{a} + b^2\right] \mathrm{d}x \\
&= \pi \int_r^a \left[b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2-x^2}}{a} + b^2\right] \mathrm{d}x = \pi \int_r^a \left[2b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2-x^2}}{a}\right] \mathrm{d}x \\
&= \pi \left[2b^2x - \frac{b^2x^3}{3a^2} - \frac{2b^2I(a)}{a}\right]_r^a = \pi \left[2b^2x - \frac{b^2x^3}{3a^2} - \frac{b^2x\sqrt{a^2-x^2} + a^2b^2\arcsin(x/a)}{a}\right]_r^a \\
&= \pi \left[2b^2a - \frac{b^2a^3}{3a^2} - \frac{0 + a^2b^2(\pi/2)}{a}\right. \\
&\qquad \left. - \left(2b^2r - \frac{b^2r^3}{3a^2} - \frac{b^2r\sqrt{a^2-r^2} + a^2b^2\arcsin(r/a)}{a}\right)\right] \\
&= \pi \left[2ab^2 - 2b^2r + \frac{b^2r^3 - a^3b^2}{3a^2} + \frac{b^2r\sqrt{a^2-r^2} + a^2b^2\arcsin(r/a) - 0.5a^2b^2\pi}{a}\right] \\
&= \frac{b^2\pi\left(6a^3 - 6a^2r + r^3 - a^3 + 3ar\sqrt{a^2-r^2} + 3a^3\arcsin(r/a) - 1.5a^3\pi\right)}{3a^2} \\
&= \frac{b^2\pi\left(2r^3 + 6ar\left(\sqrt{a^2-r^2} - 2a\right) + a^3(10 + 6\arcsin(r/a) - 3\pi)\right)}{6a^2} \\
V &= 4\left(V_1 + V_2\right) \\
&= \frac{2\pi}{3a^2}\left(4r^3(2a^2+b^2) - 3a^2b\pi r^2 + 6ab^2r\left(2\sqrt{a^2-r^2} - 3a\right)\right. \\
&\qquad \left. + a^3b^2\left(12\arcsin\left(\frac{r}{a}\right) + 10 - 3\pi\right)\right)
\end{aligned}
$$

Indeterminate Exponents (Type 3)

Arnav Patri

Sources

Indeterminate Forms (Type 3) by Faruk Özger

https://www.youtube.com/watch?v=2t-E1d9_cRM

1. Example 2
6. Example 4

Calculus: Early Transcendentals 9th Edition

3. 4.4 Exercise 61
5. 4.4 Exercise 57
8. 4.4 Exercise 60
9. 4.4 Exercise 65
10. 4.4 Exercise 66

Problems

Evaluate the following limits. Use comparative growth rates when necessary.

1. $$\lim_{x\to 0^+}\left[x^x\right]$$

2. $$\lim_{x\to\infty}\left(\frac{1}{x}\right)^{1/x}$$

3. $$\lim_{x\to 1^+}\left[x^{1/(1-x)}\right]$$

4. $$\lim_{x\to\infty}(\ln x)^{1/x}$$

5. $$\lim_{x\to\infty}\left(\frac{1}{e^x}\right)^{\arctan x-\pi/2}$$

6. $$\lim_{x\to 0^+}\left[x^{\sqrt{x}}\right]$$

7. $$\lim_{x\to 0^+}(1+\sin(7x))^{\cot(5x)}$$

8. $$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$$

9. $$\lim_{x\to 0^+}(4x+1)^{\cot x}$$

10. $$\lim_{x\to 0^+}(1-\cos x)^{\sin x}$$

11. $$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\cos x}{x}}$$

12. $$\lim_{x\to\infty}\left(\sum_{n=0}^{\infty}\left[\frac{1}{x^n}\right]\right)^{x}$$

13. $$\lim_{n\to 1^+}\left(\int_n^{\infty}\left[\frac{1}{x^n}\right]\mathrm{d}x\right)^{n-1}$$

14.
$$\lim_{n\to\infty}\left(\int_n^\infty \left[\frac{1}{x^n}\right]\mathrm{d}x\right)^{\int_{n+1}^\infty \left[\frac{1}{x^{n+1}}\right]\mathrm{d}x}$$

15.
$$\lim_{b\to\infty}\left(\int_0^b \left[\sum_{n=0}^\infty \left[\frac{1}{x^n}\right]\right]\mathrm{d}x\right)^{e^{-b}}$$

Solutions

1.

$$\begin{aligned}
L &= \lim_{x\to 0^+}\left[x^x\right] && \implies 0^0 \\
\ln L &= \lim_{x\to 0^+}\left[x\ln x\right] && \implies 0\times -\infty \\
&= \lim_{x\to 0^+}\left[\frac{\ln x}{1/x}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{x\to 0^+}\left[\frac{1/x}{-1/x^2}\right] = \lim_{x\to 0^+}[-x] = 0 \\
L &= e^0 = 0
\end{aligned}$$

2.

$$\begin{aligned}
L &= \lim_{x\to\infty}\left(\frac{1}{x}\right)^{1/x} && \implies 0^0 \\
\ln L &= \lim_{x\to\infty}\left[\frac{\ln(1/x)}{x}\right] = \lim_{x\to\infty}\left[-\frac{\ln x}{x}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{x\to\infty}\left[-\frac{1/x}{1}\right] = 0 \\
L &= e^0 = 0
\end{aligned}$$

3.

$$\begin{aligned}
L &= \lim_{x\to 1^+}\left[x^{1/(1-x)}\right] && \implies 1^\infty \\
\ln L &= \lim_{x\to 1^+}\left[\frac{\ln x}{1-x}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to 1^+}\left[-\frac{1/x}{1}\right] = -\frac{1/1}{1} = -1 \\
L &= e^{-1} = \frac{1}{e}
\end{aligned}$$

4.

$$\begin{aligned}
L &= \lim_{x\to\infty}(\ln x)^{1/x} && \implies \infty^0 \\
\ln L &= \lim_{x\to\infty}\left[\frac{\ln x}{x}\right] && \implies \frac{\infty}{\infty} \\
&= \lim_{x\to\infty}\left[\frac{1/x}{1}\right] = 0 \\
L &= e^0 = 1
\end{aligned}$$

5.

$$\begin{aligned}
L &= \lim_{x\to0^+}\left[x^{\sqrt{x}}\right] && \implies 0^0 \\
\ln L &= \lim_{x\to0^+}\left[\sqrt{x}\ln x\right] && \implies 0\times(-\infty) \\
&= \lim_{x\to0^+}\left[\frac{\ln x}{x^{-1/2}}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{x\to0^+}\left[-\frac{1/x}{0.5x^{-3/2}}\right] = \lim_{x\to0^+}\left[-2\sqrt{x}\right] = 0 \\
L &= e^0 = 1
\end{aligned}$$

6.

$$\begin{aligned}
L &= \lim_{x\to\infty}\left(\frac{1}{e^x}\right)^{\arctan x-\pi/2} && \implies 0^0 \\
\ln L &= \lim_{x\to\infty}\left[-x\left(\arctan x-\frac{\pi}{2}\right)\right] && \implies -\infty\times0 \\
&= \lim_{x\to\infty}\left[-\frac{\arctan x-\pi/2}{1/x}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to\infty}\left[\frac{\frac{1}{x^2+1}}{\frac{1}{x^2}}\right] = \lim_{x\to\infty}\left[\frac{x^2}{x^2+1}\right] = 1 \\
L &= e^1 = e
\end{aligned}$$

7.

$$\begin{aligned}
L &= \lim_{x\to0^+}(1+\sin(7x))^{\cot(5x)} && \implies 1^\infty \\
\ln L &= \lim_{x\to0^+}\left[\cot(5x)\ln(1+\sin(7x))\right] && \implies \infty\times0 \\
&= \lim_{x\to0^+}\left[\frac{\ln(1+\sin(7x))}{\tan(5x)}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to0^+}\left[\frac{\frac{7\cos(7x)}{1+\sin(7x)}}{5\sec^2(5x)}\right] = \lim_{x\to0^+}\left[\frac{7\cos(7x)}{5\sec^2(5x)(1+\sin(7x))}\right] = \frac{7}{5} \\
L &= e^{7/5}
\end{aligned}$$

8.

$$\begin{aligned}
L &= \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} && \implies 1^\infty \\
\ln L &= \lim_{x\to\infty}\left[bx\ln\left(1+\frac{a}{x}\right)\right] && \implies \infty\times0 \\
&= \lim_{x\to\infty}\left[\frac{b\ln\left(1+\frac{a}{x}\right)}{x^{-1}}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to\infty}\left[\frac{\frac{-bax^{-2}}{1+\frac{a}{x}}}{-x^{-2}}\right] = \lim_{x\to\infty}\left[\frac{ab}{1+\frac{a}{x}}\right] = \frac{ab}{1+0} = ab \\
L &= e^{ab}
\end{aligned}$$

9.

$$\begin{aligned}
L &= \lim_{x\to 0^+} (4x+1)^{\cot x} && \implies 1^\infty \\
\ln L &= \lim_{x\to 0^+} [(\cot x)\ln(4x+1)] && \implies \infty \times 0 \\
&= \lim_{x\to 0^+} \left[\frac{\ln(4x+1)}{\tan x}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to 0^+} \left[\frac{4/(x+1)}{\sec^2 x}\right] = \frac{4/1}{1} = 4 \\
L &= e^4
\end{aligned}$$

10.

$$\begin{aligned}
L &= \lim_{x\to 0^+} (1-\cos x)^{\sin x} && \implies 0^0 \\
\ln L &= \lim_{x\to 0^+} [(\sin x)\ln(1-\cos x)] && \implies 0 \times (-\infty) \\
&= \lim_{x\to 0^+} \left[\frac{\ln(1-\cos x)}{\csc x}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{x\to 0^+} \left[-\frac{\sin x/(1-\cos x)}{\csc x \cot x}\right] = \lim_{x\to 0^+} \left[-\frac{\sin^2 x \tan x}{1-\cos x}\right] && \implies -\frac{0}{0} \\
&= \lim_{x\to 0^+} \left[\frac{2\sin x\cos x\tan x + \sin^2 x \sec^2 x}{\sin x}\right] \\
&= \lim_{x\to 0^+} [2\cos x\tan x + \sin x \sec x] \\
&= 2(1)(0) + (0)(1) = 0 \\
L &= e^0 = 1
\end{aligned}$$

11.

$$
\begin{aligned}
&\lim_{x\to0^+}\left[\frac{\sin x}{x}\right]=\lim_{x\to0}\left[\cos x\right]=1 && \implies \frac{0}{0}\\
&\lim_{x\to0^+}\left[\frac{\cos x}{x}\right]=\frac{1}{0}=\infty\\
L&=\lim_{x\to0^+}\left(\frac{\sin x}{x}\right)^{\frac{\cos x}{x}}=\left(\lim_{x\to0^+}\left[\frac{\sin x}{x}\right]\right)^{\lim_{x\to0^+}\left[\frac{\cos x}{x}\right]} && \implies 0^\infty\\
\ln L&=\lim_{x\to0^+}\left[\frac{\cos x\ln\left(\frac{\sin x}{x}\right)}{x}\right] && \implies \frac{0}{0}\\
&=\lim_{x\to0^+}\left[\frac{\cos x(\ln(\sin x)-\ln x)}{x}\right]\\
&=\lim_{x\to0^+}\left[-\sin x\ln\left(\frac{\sin x}{x}\right)+\cos x\left(\frac{x}{\sin x}\right)\left(\frac{x\cos x-\sin x}{x^2}\right)\right]\\
&=(0)(0)+(1)\left(\frac{1}{1}\right)\lim_{x\to0^+}\left[\frac{x\cos x-\sin x}{x^2}\right] && \implies \frac{0}{0}\\
&=\lim_{x\to0^+}\left[\frac{-x\sin x+\cos x-\cos x}{2x}\right]=\lim_{x\to0^+}\left[-\frac{\sin x}{2}\right]=0\\
L&=e^0=1
\end{aligned}
$$

12.

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left[\frac{1}{x^n}\right]=\frac{1}{1-1/x}=\frac{1}{(x-1)/x}=\frac{x}{x-1}\\
L&=\lim_{x\to\infty}\left(\sum_{n=0}^{\infty}\left[\frac{1}{x^n}\right]\right)^x=\lim_{x\to\infty}\left(\frac{x}{x-1}\right)^x && \implies 1^\infty\\
\ln L&=\lim_{x\to\infty}\left[x\ln\left(\frac{x}{x-1}\right)\right] && \implies \infty\times0\\
&=\lim_{x\to\infty}\left[\frac{\ln(x/(x-1))}{1/x}\right] && \implies \frac{0}{0}\\
&=\lim_{x\to\infty}\left[\frac{1}{-1/x^2}\times\frac{1}{x/(x-1)}\times\frac{1(x-1)-1(x)}{(x-1)^2}\right]\\
&=\lim_{x\to\infty}\left[-x^2\times\frac{x-1}{x}\times\frac{x-1-x}{(x-1)^2}\right]=\lim_{x\to\infty}\left[\frac{-x^2(x-1)(-1)}{x(x-1)^2}\right]\\
&=\lim_{x\to\infty}\left[\frac{x}{x-1}\right]=1\\
L&=e^1=e
\end{aligned}
$$

13.

$$
\begin{aligned}
\int_n^\infty \left[\frac{1}{x^n}\right] \mathrm{d}x &= \left[-\frac{1}{(n-1)x^{n-1}}\right]_n^\infty && \Longleftarrow\ n \neq 1 \\
&= \lim_{b\to\infty} \left[-\frac{1}{(n-1)b^{n-1}} - \left(-\frac{1}{(n-1)n^{n-1}}\right)\right] \\
&= \frac{1}{(n-1)n^{n-1}} \\
L &= \lim_{n\to 1^+} \left(\int_n^\infty \left[\frac{1}{x^n}\right] \mathrm{d}x\right)^{n-1} = \lim_{n\to 1^+} \left(\frac{1}{(n-1)n^{n-1}}\right)^{n-1} && \Longrightarrow\ \infty^0 \\
\ln L &= \lim_{n\to 1^+} \left[(n-1)\ln\left(\frac{1}{(n-1)n^{n-1}}\right)\right] \\
&= \lim_{n\to 1^+} \left[(n-1)\left(\ln(1) - \ln\left((n-1)n^{n-1}\right)\right)\right] \\
&= \lim_{n\to 1^+} \left[-(n-1)\ln\left((n-1)n^{n-1}\right)\right] && \Longrightarrow\ 0\times(-\infty) \\
&= \lim_{n\to 1^+} \left[-\frac{\ln\left((n-1)n^{n-1}\right)}{1/(n-1)}\right] = \lim_{n\to 1^+} \left[-\frac{\ln(n-1) + (n-1)\ln n}{1/(n-1)}\right] && \Longrightarrow\ -\frac{\infty}{\infty} \\
&= \lim_{n\to 1^+} \left[\frac{\frac{1}{n-1} + \left(\ln n + \frac{n-1}{n}\right)}{1/(n-1)^2}\right] \\
&= \lim_{n\to 1^+} \left[(n-1)(1) + (n-1)^2\ln n + \frac{(n-1)^3}{n}\right] \\
&= \left[(0)(1) + (0)^2(0) + \frac{0^3}{1}\right] = 0 \\
L &= e^0 = 1
\end{aligned}
$$

14.

$$\begin{aligned}
\int_n^\infty \left[\frac{1}{x^n}\right] \mathrm{d}x &= \left[-\frac{1}{(n-1)x^{n-1}}\right]_n^\infty && \Longleftarrow n \neq 1 \\
&= \lim_{b\to\infty}\left[-\frac{1}{(n-1)b^{n-1}} - \left(-\frac{1}{(n-1)n^{n-1}}\right)\right] \\
&= \frac{1}{(n-1)n^{n-1}}
\end{aligned}$$

$$\int_{n+1}^\infty \left[\frac{1}{x^{n+1}}\right] \mathrm{d}x = \left[-\frac{1}{nx^n}\right]_{n+1}^\infty = \lim_{b\to\infty}\left[-\frac{1}{nb^n} - \left(-\frac{1}{n(n+1)^n}\right)\right] = \frac{1}{n(n+1)^n} \qquad | \; n \neq 0$$

$$\begin{aligned}
L &= \lim_{n\to\infty}\left(\int_n^\infty \left[\frac{1}{x^n}\right] \mathrm{d}x\right)^{\int_{n+1}^\infty \left[\frac{1}{x^{n+1}}\right]\mathrm{d}x} = \lim_{n\to\infty}\left(\frac{1}{(n-1)n^{n-1}}\right)^{\frac{1}{n(n+1)^n}} && \Longrightarrow 0^0 \\
\ln L &= \lim_{n\to\infty}\left[\frac{1}{n(n+1)^n}\ln\left(\frac{1}{(n-1)n^{n-1}}\right)\right] \\
&= \lim_{n\to\infty}\left[\frac{\ln(1)-\ln((n-1)n^{n-1})}{n(n+1)^n}\right] = \lim_{n\to\infty}\left[-\frac{\ln((n-1)n^{n-1})}{n(n+1)^n}\right] \\
&= \lim_{n\to\infty}\left[-\frac{\ln(n-1)+(n-1)\ln n}{n(n+1)^n}\right] \\
&= 0, \text{ as } (n+1)^n > n^n > \ln n > \ln(n-1) \\
L &= e^0 = 1
\end{aligned}$$

15.

$$\sum_{n=0}^\infty \left[\frac{1}{x^n}\right] = \frac{1}{1-\frac{1}{x}} = \frac{1}{\frac{x-1}{x}} = \frac{x}{x-1}$$

$$u = x-1 \implies \mathrm{d}u = \mathrm{d}x$$

$$\begin{aligned}
\int \left[\frac{x}{x-1}\right] \mathrm{d}x &= \int \left[\frac{u-1}{u}\right] \mathrm{d}u = \int \left[1-\frac{1}{u}\right] \mathrm{d}u = u - \ln u = x - 1 - \ln|x-1| \\
&= x - \ln|x-1| + C
\end{aligned}$$

$$\begin{aligned}
L &= \lim_{b\to\infty}\left(\int_2^b \left[\sum_{n=0}^\infty \left[\frac{1}{x^n}\right]\right] \mathrm{d}x\right)^{e^{-b}} = \lim_{b\to\infty}\left(\left[x-\ln|x-1|\right]_2^b\right)^{e^{-b}} \\
&= \lim_{b\to\infty}\left(b-\ln|b-1|-(2-0)\right)^{e^{-b}} = \lim_{b\to\infty}\left(b-2-\ln|b-1|\right)^{e^{-b}} \\
&\Longrightarrow \infty^0, \text{ as } b > \ln b > \ln|b-1| \\
\ln L &= \lim_{b\to\infty}\left[\frac{\ln|b-2-\ln|b-1||}{e^b}\right] && \Longrightarrow \frac{\infty}{\infty} \\
&= \lim_{b\to\infty}\left[\frac{1-\frac{1}{b-1}}{e^b(b-2-\ln|b-1|)}\right] \\
&= \lim_{b\to\infty}\left[\frac{b-2}{e^b(b-1)(b-2-\ln|b-1|)}\right] = 0, \text{ as } e^b b^2 > b \\
L &= e^0 = 1
\end{aligned}$$