Differential Equations Practice Final

Arnav Patri

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Q1

1. $$xy'' + x^2y' + y = 0$$

2. $$xy'' + x^2y' + y = x$$

3. $$y'' = y$$

4. $$y'' = e^x$$

5. $$y'' - y = 0, \quad y_1 = e^x$$

6. $$xy' - y = \sin(x)$$

7. $$(y'')^2 + y' = 0$$

8. $$\frac{\mathrm{d}x}{\mathrm{d}t} = x + y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - y$$

9. $$y = \mathrm{e}^x(C_1 + C_2x + C_3x^2)$$

10. $$y = \mathrm{e}^x(C_1 + C_2x + C_3x^2 + C_4x^3) + \sin(x)$$

Q2

1. $$\begin{aligned} \mathrm{d}y &= xy\,\mathrm{d}x \\ \frac{\mathrm{d}y}{y} &= x\,\mathrm{d}x \\ \ln|y| &= \frac{x^2}{2} + C \\ y &= \mathrm{e}^{x^2/2+C} = C_1\mathrm{e}^{x^2/2} \end{aligned}$$

2. (and 3.)

$$\begin{aligned}
0 &= xy\,\mathrm{d}x - \mathrm{d}y \\
M(x,y) &= xy & N(x,y) &= -1 \\
M_y &= x & N_x &= 0 \\
\int \frac{N_x - M_y}{M}\,\mathrm{d}y &= -\int \frac{x}{xy}\,\mathrm{d}y = -\int \frac{1}{y}\,\mathrm{d}y = -\ln y \\
\mu(y) &= \mathrm{e}^{-\ln y} = \frac{1}{y} \\
0 &= x\,\mathrm{d}x - \frac{\mathrm{d}y}{y} \\
M(x,y) &= x & N(x,y) &= -\frac{1}{y} \\
M_y &= 0 & N_x &= 0 = M_y \\
f(x,y) &= \int x\,\mathrm{d}x = \frac{x^2}{2} + g(y) \\
f'(x,y) = g'(y) &= -\frac{1}{y} \\
g(y) &= -\int \frac{1}{y}\,\mathrm{d}y = -\ln y \\
f(x,y) = 0 &= \frac{x^2}{2} - \ln y + C \\
y &= \mathrm{e}^{x^2/2+C} = C_1\mathrm{e}^{x^2/2}
\end{aligned}$$

4.

$$\begin{aligned}
0 &= xy\,\mathrm{d}x - \mathrm{d}y \\
M(x,y) &= xy & N(x,y) &= -1 \\
M(tx,ty) &= t^2xy = t^2M(x,y) & N(tx,ty) &= -1 = t^0N(x,y)
\end{aligned}$$

As M and N are homogenous functions of different degrees, this problem cannot be solved via substitution. If $N = x^2$,

$$
\begin{aligned}
0 &= xy\,\mathrm{d}x + x^2\,\mathrm{d}y \\
M(x,y) &= xy & N(x,y) &= x^2 \\
M(tx,ty) &= t^2xy = t^2M(x,y) & N(tx,ty) &= t^2x^2 = t^2N(x,y) \\
u &= \frac{y}{x} \\
M(x,y) &= x^2M(1,u) = x^2u & N(x,y) &= x^2N(1,u) = x^2 \\
\mathrm{d}u &= \frac{\mathrm{d}y}{\mathrm{d}x} \\
\mathrm{d}y &= u\,\mathrm{d}x + x\,\mathrm{d}u \\
0 &= x^2u\,\mathrm{d}x + x^2(u\,\mathrm{d}x + x\,\mathrm{d}u) \\
&= 2x^2u\,\mathrm{d}x + x^3\,\mathrm{d}u \\
\frac{\mathrm{d}u}{u} &= -\frac{2\,\mathrm{d}x}{x} \\
\ln u &= -2\ln x + C \\
u &= \mathrm{e}^{-2\ln x + C} = \frac{\mathrm{e}^C}{x^2} \\
y &= \frac{C_1}{x}
\end{aligned}
$$

Q3 1. (and 3.)

$$
\begin{aligned}
\sin(3x) &= y'' + 9y, \quad y(0) = 0, \quad y'(0) = \mathrm{e} \\
0 &= m^2 + 9 \\
m &= \pm 3i \\
y_1 &= \cos(3x) \qquad y_2 = \sin(3x) \\
W(y_1, y_2) &= \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix} = 3\cos^2(3x) + 3\sin^2(3x) = 1 \\
G(x,t) &= \frac{\cos(3t)\sin(3x) - \cos(3x)\sin(3t)}{1} = \cos(3t)\sin(3x) - \cos(3x)\sin(3t) \\
G(x,t)f(t) &= \cos(3t)\sin(3t)\sin(3x) - \cos(3x)\sin^2(3t) \\
\int_0^x G(x,t)f(t)\,\mathrm{d}t &= \sin(3x)\int_0^x \cos(3t)\sin(3t)\,\mathrm{d}t - \cos(3x)\int_0^x \sin^2(3t)\,\mathrm{d}t \\
&= \sin(3x)\int_0^x \frac{\sin(6t)}{2}\,\mathrm{d}t - \cos(3x)\int_0^x \frac{1-\cos(6t)}{2}\,\mathrm{d}t \\
&= \sin(3x)\left[-\frac{\cos(6t)}{12}\right]_0^x - \cos(3x)\left[\frac{x}{2} - \frac{\sin(6t)}{12}\right]_0^x \\
&= -\sin(3x)\left[\frac{1-\cos(6x)}{12}\right] - \cos(3x)\left[\frac{1}{2} - 0 - \frac{x}{2} + \frac{\sin(6x)}{12}\right] \\
&= \frac{-\sin(3x)(1-\cos(6x)) - \cos(3x)(6 - 6x + \sin(6x))}{12} \\
&= -\frac{\sin(3x)(2\sin^2(3x)) + \cos(3x)(6 - 6x + 2\sin(3x)\cos(3x))}{12} \\
&= -\frac{\sin^3(3x) + \cos(3x)(6 - 3x + \sin(3x)\cos(3x))}{6} \\
0 &= C_1\cos(0) + C_2\sin(0) = C_1 \\
\mathrm{e} &= C_2\cos(0) = C_2 \\
y &= \mathrm{e}\sin(3x) - \frac{\sin(3x)(2\sin^2(3x)) + \cos(3x)(6 - 6x + 2\sin(3x)\cos(3x))}{12}
\end{aligned}
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2. (and 4.)

Q4 (a)

$$
\begin{aligned}
W &= 4\,\mathrm{lb}, \quad k = 2\,\frac{\mathrm{lb}}{\mathrm{ft}}, \quad \beta = 1, \quad x_0 = -1\,\mathrm{ft}, \quad v_0 = 8\,\frac{\mathrm{ft}}{\mathrm{s}} \\
m &= \frac{W}{g} = \frac{1}{8}, \quad \omega^2 = \frac{k}{m} = 16, \quad 2\lambda = \frac{\beta}{m} = 8 \\
0 &= \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 8\frac{\mathrm{d}x}{\mathrm{d}t} + 16x \\
0 &= m^2 + 8m + 16 = (m+4)^2 \\
m &= -4 \\
x &= \mathrm{e}^{-4t}(C_1 + C_2 t) \\
-1 &= C_1 \\
v &= \mathrm{e}^{-4t}(C_2 - 4(C_1 + C_2 t)) = \mathrm{e}^{-4t}(C_2 - 4C_1 - 4C_2 t) \\
8 &= C_2 + 4 \\
C_2 &= 4 \\
x &= \mathrm{e}^{-4t}(4t - 1) \\
v &= \mathrm{e}^{-4t}(4 + 4 - 16t) = \mathrm{e}^{-4t}(8 - 16t) \\
0 &= \mathrm{e}^{-4t}(8 - 16t) = 8 - 16t \\
t &= 0.5\,\mathrm{s} \\
x_{\max} &= \mathrm{e}^{-2}
\end{aligned}
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(b)