

AP Physics C

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April 25, 2022

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Part I

Front Matter

Chapter 1

Mathematical Foundations

1.1 Algebra

The **summation** of all terms in a data set can be denoted with just the index beneath the summation, though this need not be included for finite data sets (with the same indices).

$$\sum_i x_i = \sum x_i$$

1.1.1 Matrices

A matrix is arranged as a two-dimensional rectangular array of numbers, comprised of rows (R) and columns (C).

A matrix with m rows and n columns can be described as being $m \times n$ (m by n). This is its dimension. If m and n are equal, the matrix is a **square matrix**.

The i^{th} row or column can be denoted R_i or C_i respectively.

Each cell of a matrix is an element. To denote the element in row i and column j of a matrix A , $A_{i,j}$ can be used.

$$A = \begin{matrix} & \begin{matrix} C_1 & C_2 & \dots & C_n \end{matrix} \\ \begin{matrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{matrix} & \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,\dots} & A_{1,n} \\ A_{2,1} & A_{2,2} & A_{2,\dots} & A_{2,n} \\ A_{\dots,1} & A_{\dots,2} & \ddots & A_{\dots,n} \\ A_{m,1} & A_{m,2} & A_{m,\dots} & A_{m,n} \end{bmatrix} \end{matrix}$$

The **determinant** of a 2×2 matrix is the difference between the products of the diagonal terms.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant of a 3×3 matrix is quite similar that of a 2×2 one, using an anchor point and multiplying that by the determinant of the matrix constructed from the remaining 4 terms.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

This trend continues for *square* matrices of higher dimension.

Row Operations

A **row operation** manipulates the rows of a matrix.

The **interchange** of the i^{th} and j^{th} rows is denoted $R_i \leftrightarrow R_j$. This operation simply swaps the rows.

$$\begin{array}{c} C_1 \quad C_2 \\ R_1 \left[\begin{array}{cc} a & b \end{array} \right] \\ R_2 \left[\begin{array}{cc} c & d \end{array} \right] \end{array} \xrightarrow{R_1 \leftrightarrow R_2} \begin{array}{c} C_1 \quad C_2 \\ R_1 \left[\begin{array}{cc} c & d \end{array} \right] \\ R_2 \left[\begin{array}{cc} a & b \end{array} \right] \end{array}$$

Interchange alone results in an equivalent matrix.

The i^{th} row can be multiplied by a scalar k , resulting in each of that row's elements being multiplied by k . This is denoted kR_i .

$$\begin{array}{c} C_1 \quad C_2 \\ R_1 \left[\begin{array}{cc} a & b \end{array} \right] \\ R_2 \left[\begin{array}{cc} c & d \end{array} \right] \end{array} \xrightarrow{kR_1} \begin{array}{c} C_1 \quad C_2 \\ R_1 \left[\begin{array}{cc} ka & kb \end{array} \right] \\ R_2 \left[\begin{array}{cc} c & d \end{array} \right] \end{array}$$

The i^{th} and j^{th} rows can be added to replace either, elements in the same column being summed. Replacing R_i in this way is denoted $R_i + R_j \rightarrow R_i$.

$$\begin{array}{c} C_1 \quad C_2 \\ R_1 \left[\begin{array}{cc} a & b \end{array} \right] \\ R_2 \left[\begin{array}{cc} c & d \end{array} \right] \end{array} \xrightarrow{R_1 + R_2 \rightarrow R_1} \begin{array}{c} C_1 \quad C_2 \\ R_1 \left[\begin{array}{cc} a+c & b+d \end{array} \right] \\ R_2 \left[\begin{array}{cc} c & d \end{array} \right] \end{array}$$

Linear Equations

Matrices can be used to solve systems of linear equations in any number of dimensions. To represent a linear equation with an **augmented matrix**, the each element of a row should correspond to the coefficient of a variable with the exception of the last column's element corresponding to a constant.

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = A \equiv \begin{array}{ccccc} & x_1 & x_2 & \dots & x_n & c \\ \left[\begin{array}{cccc|c} a_1 & a_2 & \dots & a_n & A \end{array} \right] \end{array}$$

A **system of linear equations** with m equations and n dimensions can also be represented via an augmented matrix using row operations.

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = A_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = A_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = A_m \end{array} \equiv \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \left[\begin{array}{cccc|c} x_1 & x_2 & \dots & x_n & c \\ a_{1,1} & a_{1,2} & a_{1,\dots} & a_{1,n} & A_1 \\ a_{2,1} & a_{2,2} & a_{2,\dots} & a_{2,n} & A_2 \\ a_{\dots,1} & a_{\dots,2} & \ddots & a_{\dots,n} & \vdots \\ a_{m,1} & a_{m,2} & a_{m,\dots} & a_{m,n} & A_m \end{array} \right]$$

Matrix Operations

1.1.2 Vectors

A **scalar** is a quantity with only magnitude.

A **vector** is a quantity with both magnitude and direction. A vector is denoted by an arrow (\rightarrow) above the variable. It can be represented by a 1-row matrix.

A vector can be divided into **components**, denoting its values in a given direction.¹ A specific component

¹The interpretation of vector components denoting directions should only be used by default. This may change, especially outside of physics.

can be denoted using the vector variable (without the arrow) with the subscript of the direction (x , y , or z).²

A vector can be denoted in **component form** by angled brackets, commas separating each component. The components are always ordered x , y , and z .

$$\vec{v} = \langle v_x, v_y, v_z \rangle$$

Unit vectors are vectors of magnitude 1 that have only a single component. The unit vectors in the x , y , and z directions are \hat{i} , \hat{j} , and \hat{k} respectively.

$$\hat{i} = \langle 1, 0, 0 \rangle \qquad \hat{j} = \langle 0, 1, 0 \rangle \qquad \hat{k} = \langle 0, 0, 1 \rangle$$

Unit vector notation breaks a vector into the sum of multiples of \hat{i} , \hat{j} , and \hat{k} .

$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

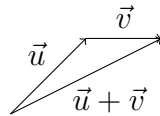
The sum of two vectors is the sum of their corresponding components and is itself a vector.

$$\vec{u} + \vec{v} = (u_x + v_x)\hat{i} + (u_y + v_y)\hat{j} + (u_z + v_z)\hat{k}$$

The **magnitude** of a vector, denoted by enclosing the vector within four vertical lines, is the square root of the sum of the squares of its components. It is a *scalar* quantity.

$$||\vec{v}|| = \sqrt{\sum_{i=1}^n v_i^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

To add two vectors graphically, the tip of one can be attached to the tail of the other, and the **resultant vector** can be drawn along the hypotenuse of the formed triangle.



The **dot product** of two vectors, denoted by a dot, is the sum of the products of their corresponding components. This is a *scalar* quantity.

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i = u_x v_x + u_y v_y + u_z v_z$$

The **cross product** of two vectors, denoted by a cross, is the

1.2 Calculus

²When the components of a vector do not denote direction, its components can be denoted by their indices in the subscript.

Part II

Mechanics

Chapter 2

Kinematics

Kinematics is the classification and comparison of motion.

A **particle** is defined as a point-like object, moving in a way such that every part of it moves at the same rate (without rotation or stretching).

The subscripts f and i (or 0) below a quantity can be used to denote initial or final values for it, while numbers can be used if more than two variations are being considered.

The subscript avg is used to denote a quantity's average over some domain.

Position must be measured relative to a reference point (typically the origin). The variables x , y , and z are used depending on the context to denote its components and the vector quantity is generally denoted \vec{r} , while *generalized* position is often denoted with q .

A change in position is a **displacement**. The displacement in a direction is denoted with a preceding Δ (meaning “change in”, though depending on the context, the variable d can instead be used in isolation.

$$\Delta q = q_f - q_i$$

The rate of change of position is **velocity**. *Average* velocity is equal to displacement divided by the change in time, while *instantaneous* velocity is the time derivative of position. Velocity is denoted using v while generalized velocity is denoted as the time derivative of generalized position, adding a dot above. (The use of dots as time derivatives of generalized position applies to any number of dots.)

a

Chapter 3

Forces

Chapter 4

Energy

Chapter 5

Systems of Particles

Chapter 6

Rotations

6.1 Rotation

A **rigid body** rotates as a unit.

The axis about which an object rotates is the **axis of rotation**. The **angular position** θ of this line is taken relative to a fixed direction, the **zero angular position**.

Although its can be changed (if specified), positive angles are conventionally **counterclockwise** from the zero angular position.

Angular dimension is measured using **radians (rad)**, which are dimensionless.

$$\theta = \frac{s}{r}$$

A **revolution** is equal to 360° , which is also equal to 2π rad.

6.2 Rolling, Torque, and Angular Momentum

For an object to **roll** is for it to move rotationally and translationally along a surface. For an object to roll **smoothly** is for it not to leave the ground while it is rolling.

Smooth rolling can be thought of as pure rotation and pure translation or as rotation about a moving contact point.

The center of mass of a rolling object moves parallel to the surface. The rest of the object rotates about the center of mass.

The **arc distance** S , the distance covered on the surface, and the velocity about the center of mass are defined as linear variables:

$$S = \theta r$$

$$v_{\text{com}} = \omega r$$

As rolling objects move both translationally and rotationally, they have both translational and rotational kinetic energy.¹

$$K = \frac{1}{2}I_{\text{com}}\omega^2 + \frac{1}{2}Mv_{\text{com}}^2$$

¹The parallel axis theorem can be used to calculate the kinetic energy about the contact point.

$$K = \frac{1}{2}I_P\omega^2 = \frac{1}{2}(I_{\text{com}} + Mr^2)\omega^2 = \frac{1}{2}I_{\text{com}}\omega^2 + \frac{1}{2}Mr^2\left(\frac{v_{\text{com}}}{r}\right)^2 = \frac{1}{2}I_{\text{com}}\omega^2 + \frac{1}{2}Mv_{\text{com}}^2$$

If no slipping occurs, then energy is conserved (even with friction). The acceleration about the center of mass follows the pattern of position and velocity.

$$a_{\text{com}} = \alpha r$$

Acceleration is only dependent on the shape of the object. Its mass and radius are irrelevant.

For an object to roll smoothly on a slope, three things are required are required:

1. The gravitational force must be vertically down.
2. The normal force must be perpendicular to the slope.
3. The force of friction must point up the slope.

The frictional and normal forces are applied to the point of contact rather than the center of mass.

A force only results in torque if it does not pass through the contact point. The torque can be calculated as the product of the perpendicular components of the force and the radius.

$$\tau = r_{\perp} F = r F_{\perp} = r F \sin \phi$$

The second law can then be used.

$$\tau_{\text{net}} = I\alpha = I \left(\frac{a_{\text{com}}}{r} \right)$$

The acceleration of a body rolling smoothly down a slope can be found as such:

$$a_{\text{com},x} = \frac{-g \sin \theta}{I + \frac{\text{com} I}{Mr^2}}$$

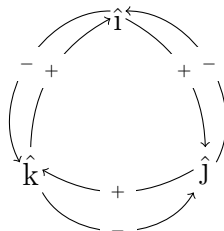
6.2.1 Torque

Torque can be defined as a vector for an individual particle moving along any path relative to a fixed point. It is the cross product of \vec{r} and \vec{F} .

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix} \times \begin{bmatrix} F_x & F_y & F_z \end{bmatrix} = \begin{bmatrix} r_y F_z - r_z F_y \\ r_x F_z - r_z F_x \\ r_x F_y - r_y F_x \end{bmatrix}$$

The resultant direction can be determined with the right hand rule, pointing the index and middle fingers of the right hand in the directions of the force and radius such that the thumb will point in the resultant direction of torque.

The angle can also be determined using unit vector notation. The product of two unit vectors is always the third. Its sign is dependent on the the “order” of the unit vectors.



6.2.2 Angular Momentum

The **angular momentum** ℓ of a particle is equal to the product of its distance and linear momentum.

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

Angular momentum has units of kgm^2/s (or Js). The magnitude of $\vec{\ell}$ is the product of the perpendicular components of \vec{r} and \vec{p} .

$$\ell = r_{\perp}p = rp_{\perp} = rp \sin \phi = mrv \sin \phi$$

Newton's second law can be rewritten to state that torque is the time derivative of angular momentum, but in order for this to be true, they must be defined about the same point.²

$$\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt}$$

The time derivative of **net angular momentum**, denoted \vec{L} , is equal to the sum of the net torques of each particle, denoted \vec{T} .

$$\frac{d\vec{L}}{dt} = \vec{T}$$

If the net external torque on a system is zero, then angular momentum is *conserved*.

Linear momentum can be written as the product of inertia and angular velocity.³

$$\ell = I\omega$$

²Using the formula for angular momentum, the fact that its time derivative is torque can be shown:

$$\frac{d}{dt}\vec{\ell} = \frac{d}{dt}m(\vec{r} \times \vec{v}) = m\vec{r} \left(\frac{d\vec{v}}{dt} \right) = m\vec{r} \times \vec{a} = \vec{F}_{\text{net}} \times \vec{r} = \vec{\tau}_{\text{net}}$$

³It can be shown that linear momentum is the product of inertia and linear momentum by rewriting its formula or by integrating its time derivative.

$$\begin{aligned} \ell &= mrv = mr^2\omega = I\omega \\ &= \int \tau \, dt = \int I\alpha \, dt = I\omega \end{aligned}$$

Chapter 7

Statics and Oscillations

7.1 Equilibrium and Elasticity

An object in **static equilibrium** have a center of mass with constant linear momentum and have constant angular momentum about any point.

$$\text{equilibrium} \implies (\vec{P} \wedge \vec{L}) \text{ are constant}$$

If a body returns to static equilibrium after a slight displacement, it is in **stable static equilibrium**. Otherwise, it is **unstable**.

The requirements for equilibrium can be rewritten using Newton's second law.

$$\text{equilibrium} \implies \vec{F}_{\text{net}}, \vec{T}_{\text{net}} = 0$$

Forces are often only considered in the xy plane, further simplifying the requirements.

$$\text{equilibrium} \implies F_{\text{net},x}, F_{\text{net},y}, \tau_{\text{net},z} = 0$$

7.1.1 Center of Gravity

The gravitational force \vec{F}_g on a body is the sum of all of the gravitational forces acting on its individual particles. This can be simplified by treating it as a single force that acts on the **center of gravity (cog)**. If the gravitational force is the same for all of a body's particles, then the center of gravity is simply the center of mass.

7.2 Simple Harmonic Motion

An oscillation's **frequency** (f) is the number of times per second that it takes for it to complete an **oscillation** (cycle).

The unit of frequency is **hertz (Hz)**, equal to one s^{-1} .

$$\text{Hz} = \frac{1}{\text{s}}$$

The **period** (T) is the inverse of the frequency, making it the amount of time that it takes for a full period to be completed.

$$T = \frac{1}{f}$$

Any process that regularly repeats is **periodic**.

Simple harmonic motion is periodic motion that is a sinusoidal function of time. It can be described in an equation with parameters of amplitude (x_m), angular frequency (ω), and phase angle (ϕ) as such:

$$x(t) = x_m \cos(\omega t + \phi)$$

The **amplitude** is the maximum displacement in either direction.

The **phase** is the argument of cosine while the phase angle adjusts for the initial conditions.

The **angular frequency** is the product of the period of cosine and the frequency.

$$\omega = 2\pi f$$

The velocity is the derivative of motion with respect to time.

$$v(t) = \frac{d}{dt}x(t) = -\omega x_m \sin(\omega t + \phi)$$

The product of the angular frequency and amplitude is the **velocity amplitude** (vm).

The acceleration is the time derivative of motion.

$$a(t) = \frac{d}{dt}v(t) = -\omega^2 x_m \cos(\omega t + \phi) = -\omega^2 x(t)$$

The product of the square of the angular frequency and the amplitude is the **acceleration amplitude** (am).

Newton's second law can be applied to the formula for acceleration to derive the force.

$$F = ma = m(-\omega^2 x) = -m\omega^2 x$$

Simple harmonic motion is the motion of a particle with a force acting on it proportional to the particle's displacement but in the opposite direction.

The formula for force can be rewritten similarly to Hooke's law.

$$F = ma = -kx$$
$$a = \frac{-k}{m}x$$

If the force is proportional to the first power of x , the motion can be described as **linear simple harmonic oscillation** can be described as such:

$$a = \frac{-k}{m}x = -\omega^2 x$$
$$\omega = \frac{k}{m}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The equations for kinetic and potential energy can be rewritten using kinematics:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$
$$K = \frac{1}{2}mv^2 = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$

Chapter 8

Gravitation

Gravitation is the tendency for bodies to attract each other.

The gravitational force is dependent on mass. It is always attractive.

Newton's **law of gravitation** defines the strength of the attractive force between particles.

$$F_G = \frac{Gm_1m_2}{r^2}$$

G is the **gravitational constant**

$$G \approx 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2$$

Newton's law of gravitation can be rewritten in vector form using \hat{r} , which is a vector of magnitude 1 that points from one particle to another.

$$\vec{F}_G = \frac{Gm_1m_2}{r^2} \hat{r}$$

As is made clear by this vector form, gravitational forces always come in *third-law pairs*.

The **shell theorem** states that a uniform spherical shell of matter attracts a particle that is outside the shell as though all of the shell's mass was concentrated at its center.

The net gravitational force can be calculated by the **principle of superposition**, which simply states that the net effect is the sum of the individual effects.

$$\vec{F}_{\text{net}} = \sum \vec{F}_i$$

For a real (extended) object, this sum becomes an integral.

$$\vec{F}_{\text{net}} = \int d\vec{F}$$

If the object in question is a uniform sphere or shell, it can be treated as having its mass at its center rather than considering the object as a whole.

Newton's law of gravitation can be combined with Newton's second law to calculate the acceleration due to gravity.

$$a_{G,1} = \frac{F_g}{m_1} = \frac{Gm_1m_2/r^2}{m_1} = \frac{Gm_2}{r^2}$$

An additional implication of the shell theorem is that no net gravitational force is exerted on a particle within the shell. (The forces still exist, but balance out.)

The **gravitational potential energy** of a two-particle system is as follows:¹

$$U_G = \frac{-Gm_1m_2}{r}$$

The gravitational potential energy of a system is the sum of all the gravitational potential energies of every possible pair of particles. Gravitational potential energy is a property of a pair of particles. The potential energy cannot be allocated to the individual particles.

When one object is far more massive than the other, the smaller object can be given all of the gravitational potential energy (for practical calculations).

For a projectile to escape a body's gravitational pulls, it must come to rest only at infinity, if at all. This means that K must both be 0, and U is 0 because r is ∞ . Their sum must therefore be at least 0 at the surface of the body.

$$K + U = 0 = \frac{1}{2}m_1v^2 + \left(\frac{-Gm_1m_2}{r}\right)$$

$$v = \sqrt{\frac{2Gm_2}{r}}$$

8.1 Kepler's Laws of Planetary Motion

8.1.1 The Law of Orbits

All planets in the solar system move in elliptical orbits with the Sun as a focus.

An orbit is defined by its **semimajor axis** a and its **eccentricity** e .

8.1.2 The Law of Areas

A line connecting a planet to the Sun sweeps out equal areas in the plane of the planet's orbit for an equal time interval; that is, the rate $\frac{dA}{dt}$ at which it sweeps out an area A is constant.

8.1.3 The Law of Periods

The square of the period of a planet's orbit is proportional to the cube of its semimajor axis.

$$T^2 \propto a^3$$

More concretely, the law of periods can be written as such:

$$T^2 = \frac{4\pi^2a^3}{GM}$$

¹The formula for gravitational potential energy is the integral of that of the force of gravity with respect to r .

$$U_G = \int F_G dr = \int \frac{Gm_1m_2}{r^2} dr = \frac{-Gm_1m_2}{r}$$

8.2 Satellites

The total mechanical energy of a circular orbit can be described as such:²

$$E = -K = \frac{-GMm}{2r}$$

For an elliptical orbit, the semimajor axis can be used in place of the radius, as the eccentricity of the orbit has of no bearing on the total kinetic energy.

$$E = \frac{-GMm}{2a}$$

²The gravitational force can be related to the centripetal acceleration of a satellite to derive its kinetic energy.

$$\begin{aligned}a_c &= \frac{v^2}{r} = \frac{F_G}{m} = \frac{GMm}{mr^2} \\v^2 &= \frac{GM}{r} \\K &= \frac{1}{2}mv^2 = \frac{1}{2} \left(\frac{GMm}{r} \right) = \frac{GMm}{2r} \\&= \frac{-U_G}{2}\end{aligned}$$

The net mechanical energy can then be found.

$$E = K + U = \frac{-U}{2} + U = \frac{U}{2} = \frac{-GMm}{2r} = -K$$

Part III

Electricity and Magnetism