15.5.13. (p. 1082)

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Find the area of the surface of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies within the cylinder $x^2 + y^2 = ax$ and above the xy-plane.

$$\begin{aligned}
ax &= x^2 + y^2 \\
ar\cos\theta &= r^2 \\
r &= a\cos\theta \\
D &= \{(r,\theta) \,|\, 0 \le r \le a\cos\theta, 0 \le \theta \le \pi\} \\
x^2 + y^2 + z^2 &= a^2 \\
z &= \sqrt{a^2 - x^2 - y^2} \\
\frac{\partial z}{\partial x} &= \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}} \\
A &= \iint\limits_D \left[\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\right] \mathrm{d}A = \iint\limits_D \left[\sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}}\right] \mathrm{d}A \\
&= \iint\limits_D \left[\sqrt{\frac{a^2}{a^2 - x^2 - y^2}}\right] \mathrm{d}A = \int_0^\pi \int_0^{a\cos\theta} \left[r\sqrt{\frac{a^2}{a^2 - r^2}}\right] \mathrm{d}r\,\mathrm{d}\theta \\
u &= a^2 - r^2 \qquad u_1 = a^2 - 0 = a^2 \qquad u_2 = a^2 - a^2\cos^2\theta = a^2\sin^2\theta \\
\mathrm{d}u &= -2r\mathrm{d}r \\
A &= -\frac{a}{2}\int_0^\pi \int_{a^2}^{a^2\sin^2\theta} \left[\sqrt{\frac{1}{u}}\right] \mathrm{d}u\,\mathrm{d}\theta = \frac{a}{2}\int_\pi^0 \left[2\sqrt{u}\right]_{a^2}^{a^2\sin^2\theta} \mathrm{d}\theta \\
&= a\int_\pi^0 [a\sin\theta - a]\,\mathrm{d}\theta = a[-a\cos\theta - a\theta]_\pi^0 = a[-a - 0 - a + a\pi] = a^2(\pi - 2)
\end{aligned}$$