Volume: Disk/Washer

Sources

Calculus: Early Transcendentals 9$^{\text{th}}$ Edition

1. 6.2 Exercise 12
2. 6.2 Exercise 15
3. 6.2 Exercise 1
4. 6.2 Exercise 17
5. 6.2 Exercise 24

AP Calculus Exams

6. 2021 AB FRQ 3(c)

Problems

If no instructions are given, evaluate the volume of the solid generated by revolving the region bounded by the given equations about the specified line using the disk/washer method.

1. $$\left.\begin{array}{ll} y = 0 & y = \dfrac{1}{x} \\ x = 1 & x = 4 \end{array}\right| \; y = 0$$

2. $$\left.\begin{array}{ll} y = \dfrac{x^2}{4} & y = 9 \\ x = 0 & \end{array}\right| \; x = 0$$

3. $$\left.\begin{array}{ll} y = 0 & y = x^2 + 5 \\ x = 0 & x = 3 \end{array}\right| \; y = 0$$

4. $$\left.\begin{array}{l} y = x^2 \\ y = 2x \end{array}\right| \; x = 0$$

5. $$\left.\begin{array}{ll} y = \sin x & y = \cos x \\ x \geq 0 & x \leq \dfrac{\pi}{4} \end{array}\right| \; y = -1$$

6. $$f(x) = cx\sqrt{4 - x^2}$$

   The solid of revolution generated by rotating the area bounded by f and the x-axis in the first quadrant about the x-axis is equal to 2π. Solve for c, given that it is a positive constant.

7.

8. $$\left.\begin{array}{ll} x^2 + y^2 = r^2 & \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \mid a, b > r > 0 \\ \text{(a) } y = 0 & \end{array}\right| \begin{array}{l} \text{(a) } y = 0 \\ \text{(b) } y = b \end{array}$$

Solutions

1.

$$V = \pi \int_1^4 \left(\frac{1}{x}\right)^2 \mathrm{d}x = \pi \left[-\frac{1}{x}\right]_1^4 = \pi \left[-\frac{1}{4} - \left(-\frac{1}{1}\right)\right] = \frac{3\pi}{4}$$

2.

$$\begin{aligned}
y &= \frac{x^2}{4} \implies x = 2\sqrt{y} \\
y_1 &= 2\sqrt{0} = 0 \\
V &= \pi \int_0^9 (2\sqrt{y})^2 \,\mathrm{d}y = \pi \left[2y^2\right]_0^9 = 2(81)\pi = 162\pi
\end{aligned}$$

3.

$$\begin{aligned}
V &= \pi \int_0^3 \left(x^2+5\right)^2 \mathrm{d}x = \pi \int_0^3 \left[x^4 + 10x^2 + 25\right] \mathrm{d}x = \pi \left[\frac{x^5}{5} + \frac{10x^3}{3} + 25x\right]_0^3 \\
&= \pi \left[\frac{3^5}{5} + \frac{10(3)^3}{3} + 25(3) - (0)\right] = \pi \left[\frac{243}{5} + 90 + 75\right] = \frac{\pi(243+825)}{5} = \frac{1068\pi}{5}
\end{aligned}$$

4.

$$\begin{aligned}
y &= x^2 \implies x = \sqrt{y} \qquad y = 2x \implies x = \frac{y}{2} \\
\sqrt{y} &= \frac{y}{2} \implies 4y = y^2 \implies 0 = y(y-4) \implies y_1 = 0, y_2 = 4 \\
V &= \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right] \mathrm{d}y = \pi \int_0^4 \left[y - \frac{y^2}{4}\right] \mathrm{d}y = \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 \\
&= \pi \left[\frac{y^2(6-y)}{12}\right]_0^4 = \pi \left[\frac{4^2(6-4)}{12} - (0)\right] = \pi \left[\frac{16(2)}{12}\right] = \frac{8\pi}{3}
\end{aligned}$$

5.

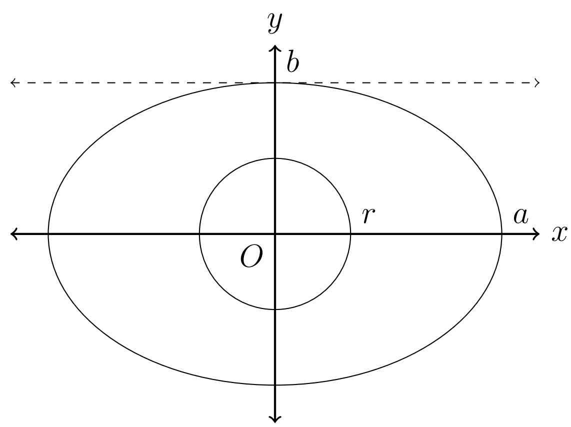
$$\begin{aligned}
\sin x = \cos x &\implies x = \frac{\pi}{4} \\
V &= \pi \int_0^{\pi/4} \left[(\cos x + 1)^2 - (\sin x + 1)^2\right] \mathrm{d}x \\
&= \pi \int_0^{\pi/4} \left[\cos^2 x + 2\cos x + 1 - \sin^2 x - 2\sin x - 1\right] \mathrm{d}x \\
&= \pi \int_0^{\pi/4} \left[\cos(2x) + 2\cos x - 2\sin x\right] \mathrm{d}x = \pi \left[\frac{\sin(2x)}{2} + 2\sin x + 2\cos x\right]_0^{\pi/4} \\
&= \pi \left[\frac{1}{2} + \sqrt{2} + \sqrt{2} - (0+0+2)\right] = \frac{(4\sqrt{2}-3)\pi}{2}
\end{aligned}$$

6.

$$0 = cx\sqrt{4-x^2} \implies \begin{cases} x = 0 \\ \sqrt{4-x^2} = 0 \quad \implies 4 - x^2 = 0 \implies 4 = x^2 \implies x = 2 \end{cases}$$

$$\begin{aligned}
2\pi &= \pi \int_0^2 \left(cx\sqrt{4-x^2}\right)^2 \mathrm{d}x = \pi \int_0^2 \left[c^2x^2(4-x^2)\right] \mathrm{d}x = \pi \int_0^4 \left[4c^2x^2 - c^2x^4\right] \mathrm{d}x \\
&= \pi \left[\frac{4c^2x^3}{3} - \frac{c^2x^5}{5}\right]_0^2 = \pi \left[\frac{4c^2(2)^3}{3} - \frac{c^2(2)^5}{5} - (0)\right] = \pi \left[\frac{32c^2}{3} - \frac{32c^2}{5}\right] \\
2 &= 32c^2 \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{64c^2}{15} \\
c &= \sqrt{\frac{30}{64}} = \sqrt{\frac{15}{32}}
\end{aligned}$$

7.



This graph assumes $a > b$, which need not be the case.

$$x^2 + y^2 = r^2 \implies y = \sqrt{r^2 - x^2} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b\sqrt{a^2 - x^2}}{a}$$

(a)

$$\begin{aligned}
V_1 &= \pi \int_0^r \left[\left(\frac{b\sqrt{a^2 - x^2}}{a} \right)^2 - \left(\sqrt{r^2 - x^2} \right)^2 \right] \mathrm{d}x = \pi \int_0^r \left[\frac{b^2(a^2 - x^2)}{a^2} - (r^2 - x^2) \right] \mathrm{d}x \\
&= \pi \int_0^r \left[\frac{a^2(b^2 - r^2) + x^2(a^2 - b^2)}{a^2} \right] \mathrm{d}x = \pi \left[\frac{a^2x(b^2 - r^2)}{a^2} + \frac{x^3(a^2 - b^2)}{3a^2} \right]_0^4 \\
&= \pi \left[\frac{3a^2x(b^2 - r^2) + x^3(a^2 - b^2)}{3a^2} \right]_0^r = \pi \left[\frac{3a^2r(b^2 - r^2) + r^3(a^2 - b^2)}{3a^2} \right] \\
&= \pi \left[\frac{3a^2b^2r - 3a^2r^3 + a^2r^3 - b^2r^3}{3a^2} \right] = \pi \left(\frac{3a^2b^2r - 2a^2r^3 - b^2r^3}{3a^2} \right) \\
V_2 &= \pi \int_r^a \left(\frac{b\sqrt{a^2 - x^2}}{a} \right)^2 \mathrm{d}x = \pi \int_r^a \left[\frac{b^2(a^2 - x^2)}{a^2} \right] \mathrm{d}x = \pi \int_r^a \left[\frac{a^2b^2 - b^2x^2}{a^2} \right] \mathrm{d}x \\
&= \pi \left[\frac{a^2b^2x}{a^2} - \frac{b^2x^3}{3a^2} \right]_r^a = \pi \left[\frac{3a^2b^2x - b^2x^3}{3a^2} \right]_r^a = \left[\frac{3a^3b^2 - b^2a^3}{3a^2} - \left(\frac{3a^2b^2r - b^2r^3}{3a^2} \right) \right] \\
&= \pi \left(\frac{2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2} \right) \\
V &= 2V_1 + 2V_2 = 2\pi \left(\frac{3a^2b^2r - 2a^2r^3 - b^2r^3 + 2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2} \right) \\
&= 2\pi \left(\frac{2a^3b^2 - 2a^2r^3}{3a^2} \right) = 4\pi \left(\frac{ab^2 - r^3}{3} \right)
\end{aligned}$$

(b)

$$
\begin{aligned}
V_1 &= \pi \int_0^r \left[\left(\sqrt{r^2-x^2}-b\right)^2 - \left(\frac{b\sqrt{a^2-x^2}}{a}-b\right)^2\right] \mathrm{d}x \\
&= \pi \int_0^r \left[(r^2-x^2) - 2b\sqrt{r^2-x^2} + b^2 - \left(\frac{b^2(a^2-x^2)}{a^2} - \frac{2b^2\sqrt{a^2-x^2}}{a} + b^2\right)\right] \mathrm{d}x \\
&= \pi \int_0^r \left[r^2 - x^2 - 2b\sqrt{r^2-x^2} + b^2 - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2-x^2}}{a} - b^2\right] \mathrm{d}x \\
&= \pi \int_0^r \left[r^2 - x^2 - 2b\sqrt{r^2-x^2} - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2-x^2}}{a}\right] \mathrm{d}x \\
&= \pi \left(\left[(r^2-b^2)x + \left(\frac{b^2}{a^2}-1\right)\frac{x^3}{3}\right]_0^r + 2b\int_0^r \left[\frac{b\sqrt{a^2-x^2}}{a} - \sqrt{r^2-x^2}\right] \mathrm{d}x\right) \\
I(\alpha) &= \int \left[\sqrt{\alpha^2-x^2}\right] \mathrm{d}x \implies x = \alpha\sin\theta \implies \mathrm{d}x = \alpha\cos\theta\mathrm{d}\theta \\
&= \int \left[\cos\theta\sqrt{\alpha^2-\alpha^2\sin^2\theta}\right] \mathrm{d}\theta = \int \left[\alpha^2\cos^2\theta\right] \mathrm{d}\theta = \alpha^2 \int \left[\frac{\cos(2\theta)+1}{2}\right] \mathrm{d}\theta \\
&= \alpha\left(\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right) + C = \alpha\left(\frac{2\sin\theta\cos\theta}{4} + \frac{\theta}{2}\right) + C \\
&= \alpha^2\left(\frac{\left(\frac{x}{\alpha}\right)\left(\frac{\sqrt{\alpha^2-x^2}}{\alpha}\right)}{2} + \frac{\arcsin(x/\alpha)}{2}\right) + C = \frac{x\sqrt{\alpha^2-x^2} + \alpha^2\arcsin(x/\alpha)}{2} + C \\
V_1 &= \pi\left[(r^2-b^2)x + \left(\frac{b^2}{a^2}-a\right)\frac{x^3}{3} + 2b\left(\frac{bI(a)}{a} - I(r)\right)\right]_0^r \\
&= \pi\Bigg[(r^2-b^2)x + \left(\frac{b^2}{a^2}-1\right)\frac{x^3}{3} \\
&\quad +2b\left(\frac{bx\sqrt{a^2-x^2} + a^2b\arcsin(x/a)}{2a} - \frac{x\sqrt{r^2-x^2} + r^2\arcsin(x/r)}{2}\right)\Bigg]_0^r \\
&= \pi\Bigg[(r^2-b^2)x + \left(\frac{b^2}{a^2}-1\right)\frac{x^3}{3} \\
&\quad +2b\left(\frac{xb\sqrt{a^2-x^2} + a^2b\arcsin(x/a)}{2a} - \frac{x\sqrt{r^2-x^2} + r^2\arcsin(x/r)}{2}\right)\Bigg]_0^r \\
&= \pi\Bigg[x\left(r^2 - b^2 + \frac{x^2}{3}\left(\frac{b^2}{a^2}-1\right)\right) \\
&\quad +\frac{b}{a}\left(x\left(b\sqrt{a^2-x^2} - a\sqrt{r^2-x^2}\right) + a^2b\arcsin\left(\frac{x}{a}\right) - r^2\arcsin\left(\frac{x}{r}\right)\right)\Bigg]_0^r \\
&= \pi\Bigg[r\left(r^2 - b^2 + \frac{r^2}{3}\left(\frac{b^2}{a^2}-1\right)\right) \\
&\quad +\frac{b}{a}\left(r\left(b\sqrt{a^2-r^2} - 0\right) + a^2b\arcsin\left(\frac{r}{a}\right) - \frac{a\pi r^2}{2}\right) - (0)\Bigg] \\
&= \pi\left(r^3 - b^2r + \frac{b^2r^3}{3a^2} - \frac{r^3}{3} + \frac{b^2r\sqrt{a^2-r^2}}{a} + ab^2\arcsin\left(\frac{r}{a}\right) - \frac{b\pi r^2}{2}\right) \\
&= \pi\left(\frac{2r^3(2a^2+b^2) - 3a^2b\pi r^2 + 6ab^2r\left(\sqrt{a^2-r^2}-a\right) + 6a^3b^2\arcsin(r/a)}{6a^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= \pi \int_r^a \left(\frac{b\sqrt{a^2-x^2}}{a} - b\right)^2 \mathrm{d}x = \pi \int_r^a \left[\frac{b^2(a^2-x^2)}{a^2} - \frac{2b^2\sqrt{a^2-x^2}}{a} + b^2\right] \mathrm{d}x \\
&= \pi \int_r^a \left[b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2-x^2}}{a} + b^2\right] \mathrm{d}x = \pi \int_r^a \left[2b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2-x^2}}{a}\right] \mathrm{d}x \\
&= \pi \left[2b^2x - \frac{b^2x^3}{3a^2} - \frac{2b^2I(a)}{a}\right]_r^a = \pi \left[2b^2x - \frac{b^2x^3}{3a^2} - \frac{b^2x\sqrt{a^2-x^2} + a^2b^2\arcsin(x/a)}{a}\right]_r^a \\
&= \pi \left[2b^2a - \frac{b^2a^3}{3a^2} - \frac{0 + a^2b^2(\pi/2)}{a}\right. \\
&\qquad \left. - \left(2b^2r - \frac{b^2r^3}{3a^2} - \frac{b^2r\sqrt{a^2-r^2} + a^2b^2\arcsin(r/a)}{a}\right)\right] \\
&= \pi \left[2ab^2 - 2b^2r + \frac{b^2r^3 - a^3b^2}{3a^2} + \frac{b^2r\sqrt{a^2-r^2} + a^2b^2\arcsin(r/a) - 0.5a^2b^2\pi}{a}\right] \\
&= \frac{b^2\pi\left(6a^3 - 6a^2r + r^3 - a^3 + 3ar\sqrt{a^2-r^2} + 3a^3\arcsin(r/a) - 1.5a^3\pi\right)}{3a^2} \\
&= \frac{b^2\pi\left(2r^3 + 6ar\left(\sqrt{a^2-r^2} - 2a\right) + a^3(10 + 6\arcsin(r/a) - 3\pi)\right)}{6a^2} \\
V &= 4\left(V_1 + V_2\right) \\
&= \frac{2\pi}{3a^2}\left(4r^3(2a^2+b^2) - 3a^2b\pi r^2 + 6ab^2r\left(2\sqrt{a^2-r^2} - 3a\right)\right. \\
&\qquad \left. + a^3b^2\left(12\arcsin\left(\frac{r}{a}\right) + 10 - 3\pi\right)\right)
\end{aligned}
$$

Indeterminate Exponents (Type 3)

Sources

Calculus: Early Transcendentals 9$^{\text{th}}$ Edition

1. 4.4 Exercise 61
2. 4.4 Exercise 57
3. 4.4 Exercise 60
4. 4.4 Exercise 65
5. 4.4 Exercise 66

Problems

Evaluate the following limits.

1. $$\lim_{x\to 1^+}\left[x^{1/(1-x)}\right]$$

2. $$\lim_{x\to 0^+}\left[x^{\sqrt{x}}\right]$$

3. $$\lim_{x\to \infty}\left(1+\frac{a}{x}\right)^{bx}$$

4. $$\lim_{x\to 0^+}(4x+1)^{\cot x}$$

5. $$\lim_{x\to 0^+}(1-\cos x)^{\sin x}$$

Solutions

1.

$$\begin{aligned}
L &= \lim_{x\to 1^+}\left[x^{1/(1-x)}\right] && \implies 1^\infty \\
\ln L &= \lim_{x\to 1^+}\left[\frac{\ln x}{1-x}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to 1^+}\left[-\frac{1/x}{1}\right] = -\frac{1/1}{1} = -1 \\
L &= \frac{1}{e}
\end{aligned}$$

2.

$$\begin{aligned}
L &= \lim_{x\to 0^+}\left[x^{\sqrt{x}}\right] && \implies 0^0 \\
\ln L &= \lim_{x\to 0^+}\left[\sqrt{x}\ln x\right] && \implies 0\times(-\infty) \\
&= \lim_{x\to 0^+}\left[\frac{\ln x}{x^{-1/2}}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{x\to 0^+}\left[-\frac{1/x}{0.5x^{-3/2}}\right] = \lim_{x\to 0^+}\left[-2\sqrt{x}\right] = 0 \\
L &= e^0 = 1
\end{aligned}$$

3.

$$\begin{aligned}
L &= \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} && \implies 1^\infty \\
\ln L &= \lim_{x\to\infty}\left[bx\ln\left(1+\frac{a}{x}\right)\right] && \implies \infty\times 0 \\
&= \lim_{x\to\infty}\left[\frac{b\ln\left(1+\frac{a}{x}\right)}{x^{-1}}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to\infty}\left[\frac{\frac{-bax^{-2}}{1+\frac{a}{x}}}{-x^{-2}}\right] = \lim_{x\to\infty}\left[\frac{ab}{1+\frac{a}{x}}\right] = \frac{ab}{1+0} = ab \\
L &= e^{ab}
\end{aligned}$$

4.

$$\begin{aligned}
L &= \lim_{x\to 0^+}(4x+1)^{\cot x} && \implies 1^\infty \\
\ln L &= \lim_{x\to 0^+}\left[(\cot x)\ln(4x+1)\right] && \implies \infty\times 0 \\
&= \lim_{x\to 0^+}\left[\frac{\ln(4x+1)}{\tan x}\right] && \implies \frac{0}{0} \\
&= \lim_{x\to 0^+}\left[\frac{4/(x+1)}{\sec^2 x}\right] = \frac{4/1}{1} = 4 \\
L &= e^4
\end{aligned}$$

5.

$$
\begin{aligned}
L &= \lim_{x\to 0^+} (1-\cos x)^{\sin x} && \implies 0^0 \\
\ln L &= \lim_{x\to 0^+} \left[(\sin x)\ln(1-\cos x)\right] && \implies 0\times(-\infty) \\
&= \lim_{x\to 0^+} \left[\frac{\ln(1-\cos x)}{\csc x}\right] && \implies -\frac{\infty}{\infty} \\
&= \lim_{x\to 0^+} \left[-\frac{\sin x/(1-\cos x)}{\csc x\cot x}\right] = \lim_{x\to 0^+} \left[-\frac{\sin^2 x\tan x}{1-\cos x}\right] && \implies -\frac{0}{0} \\
&= \lim_{x\to 0^+} \left[\frac{2\sin x\cos x\tan x+\sin^2 x\sec^2 x}{\sin x}\right] = \lim_{x\to 0^+} \left[2\cos x\tan x+\sin x\sec x\right] \\
&= 2(1)(0)+(0)(1) = 0 \\
L &= 1
\end{aligned}
$$