Differential Equations

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Contents

Chapter 1

Introduction to Differential Equations

1.1   Definitions and Terminology

A Definition

> **Differential Equation**   An equation containing the derivatives of one or more unknown functions (or dependent variables) with respect to one or more independent variables is a **differential equation (DE)**.

Classification by Type

A differential containing only ordinary derivatives with respect to a *single* independent variables is an **ordinary differential equation (ODE)**. One involving partial derivatives is a **partial differential equation (PDE)**.

Notation

Leibniz notation denotes derivatives as ratios of differentials with the operators and variables raised to the n for the nth derivative.   **Prime notation** denotes the nth derivative with either n primes or (n) in superscript of the dependent variable or the function.   The nth derivative of $y = f(x)$ can thusly be denoted as

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = y^{(n)} = f^{(n)}(x)$$

Newton's **dot notation** is sometimes used to denote derivatives with respect to time, placing n dots above the dependent variable to denote its nth derivative with respect to t. The second derivative of x with respect to t can be denoted as

$$\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

Subscript Notation is often used for partial derivatives, indicating the independent variable in the subscript. The second partial x derivative with respect to z can be denoted as

$$z_{xx} = \frac{\partial^2 z}{\partial x^2}$$

Classification by Order

The **order of a differential equation** is the order of the highest derivartive in the equation.
A first-order ODE is sometimes written in the **differential form**

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0$$

Symbolically, an nth-order ODE in one dependent variable can be expressed generally as

$$F\left(x, y, y', \dots, y^{(n)}\right) = 0$$

where F is a real-valued function of $n + 2$ variables.
It is assumed that it is possible to solve an ODE in the form above uniquely for the highest derivative
$y^{(n)}$ in terms of the remaining $n + 1$ variables.
The **normal form** of the above expression is

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, y', \dots, y^{(n-1)}\right)$$

where f is a real-values continuous function.
The following normal forms can be used to represent general first- and second-order ODEs:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \qquad\qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x, y, y')$$

Classification by Linearity

An general n^{th} order ODE is **linear** if F is linear in $y, y', y^{(n)}$. This means that it is linear when

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Two important special cases of the above are linear first-[1] and second-order DEs:

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \qquad\qquad a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

The characteristic properties of linear ODEs are that the dependent variable and all of its derivatives
are of first degree and that the coefficients of those terms are dependent at most on the independent
variable.
A **nonlinear** ODE is one that is not linear.
Nonlinear functions of the dependent variable cannot appear in linear ODEs.
A DE can not be classified as linear or nonlinear if both differentials are in the numerator.

Solutions

> **Solution of an ODE** Any function φ defined on an interval I with at least n derivatives
> that are continuous on I which when substituted into an nth-order ODE reduce the equation
> to an identity is a **solution** of the equation on the interval.

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[1] A first order ODE written in differential form as $M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0$ may be linear or nonlinear, as there
is no indication of which symbol is the dependent variable.

A DE need not have a solutions. A solution of a DE may involve **integral-defined function**[2].
A solution of a general n^{th}-order ODE is a function φ with at least n derivatives for which

$$F\big(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(n)}(x)\big) = 0 \qquad \forall\, x \in I$$

φ is said to *satisfy* the differential equation on I. It is assumed that a solution φ is a real-valued function.
A solution is occasionally denoted alternatively by $y(x)$.

Interval of Definition

The interval I over which φ satisfies the ODE is referred to as the **interval of definition/existence/validity** or the **domain of the solution**.
A solution of a DE that is identically 0 on an interval I is said to be a **trivial solution**.

Solution Curve

The graph of φ is called a **solution curve**.
The domain of φ need not be the same as I.

Explicit and Implicit Solutions

A function that expresses the dependent variable solely in terms of the independent variable and constants is said to be *explicit*. An **explicit solution** is a solution with an explicit function. It can be thought of as an explicit formula $y = \varphi(x)$ that can be manipulated.
An explicit solution is generally not needed over an implicit one.
An implicit solution $G(x, y) = 0$ may define a differentiable function that is a solution of a DE despite $G(x, y) = 0$ potentially not being solvable analytically. The solution curve may be a segment of the graph of $G(x, y) = 0$.

Families of Solutions

When solving a first-order DE, the solution *usually* contains a single constant or parameter C, similar to the constant of integration obtained from the indefinite integral. A solution of $F(x, y, y') = 0$ containing constant C is a set of solutions $G(x, y, C) = 0$ called a **one-parameter family of solutions**.

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[2] A function F of a single variable x can be defined as

$$F(x) = \int_a^x g(t)\, \mathrm{d}t$$

If the integrand g is continuous over $[a, b]$ and x falls within the interval, then F is differentiable on the open interval and

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x g(t)\, \mathrm{d}t = g(x)$$

The integral is often **nonelementary**, meaning that it is not composed of elementary functions.
Elementary functions include constant, polynomial, rational, exponential, logarithmic, trigonometric, and inverse trigonometric functions, as well as rational powers and finite combinations using the four basic arithmetic operations and compositions of these functions.

An nth-order DE[3] often yields an **n-parameter family of solutions**[4] $G(x, y, C_1, C_2, \ldots, C_n) = 0$. The parameters in a family of solutions are *arbitrary* up to a point, but they should always take on values that make sense in the real-number system.

A **singular solution** is one that cannot be obtained by specializing *any* of the parameters in the family of solutions.

Systems of Differential Equations

A **system of ODEs** is comprised of multiple unknown functions of a single independent variable. A **solution** of a system is a pair of differentiable functions defined on common interval I that satisfy each equation of the system on the interval.

1.2   Initial-Value Problems

Often, a solution to a DE must meet other conditions imposed on it and its derivatives.

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\big(x, y, y', \ldots, y^{(n-1)}\big) \mid y(x_0) = y_0, y'(x) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

If these desired values are constants, this is an **n^{th}-order initial-value problem (IVP)**. The desired values are called **initial conditions (IC)**.[5]

Solving an nth-order IVP often requires that an n-parameter family of solutions be found that can then be used in tandem with the constraints to find the constants. The resulting particular solution is defined on some interval that contains x_0.

Geometric Interpretation

A first-order IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \mid y(x_0) = y_0$$

can be interpreted as finding a solution $y(x)$ with a graph that passes through the point (x_0, y_0).

A second-order IVP

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x, y, y') \mid y(x_0) = y_0, y'(x_0) = y_1$$

can be interpreted as finding a solution $y(x)$ with a graph that passes through (x_0, y_0) with slope y_1.

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[3] $F\big(x, y, y', \ldots, y^{(n)}\big) = 0$ may not always be solvable for $y^{(n)}$.

[4] If *every* solution of an nth-order ODE on an interval can be found by manipulating the parameters of an n-parameter family of solutions, then the family is said to be the **general solution** of the DE.

Nonlinear ODEs are often difficult of impossible to solve in terms of elementary functions, so if a family of solutions is found for one, it is unclear whether it is a general solution. Practically, the designation of "general solution" is only given for solutions to linear ODEs.

[5] If conditions are prescribed at multiple points, called **boundary conditions**, the problem is called a **boundary-value problem (BVP)**.

Existence and Uniqueness

It can be assumed that *most* DEs will hav solutions and that the solutions of IVPs will *generally* be unique.

Theorem 1.2.1 Let R be a rectangular region in the xy-plane defined by

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

that contains point (x_0, y_0). If $f(x, y)$ and $\partial f/\partial y$ are continuous over R^a, then there exists some interval $I_0 : (x_0 - h, x_0 + h)$ (where $h > 0$) contained in $[a, b]$ and a unique function $y(x)$ defined on this interval that is a solution of the IVP.

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[a] These conditions are sufficient but not necessary. When $f(x, y)$ and $\partial f/\partial y$ are continuous on R, a solution of the IVP exists and is unique so long as (x_0, y_0) is contained within R. If these conditions are not met, though, the IVP *may* still have a solution that *may* be unique, or it may have multiple or no solutions.

Interval of Existence/Uniqueness

The domain of the function that represents a solution to an IVP, the interval I over which the solution is defined or exists, and the interval I_0 of existence *and* uniqueness.

Suppose (x_0, y_0) is a point in the interior of rectangle R. The continuity of function $f(x, y)$ on R is sufficient to guarantee the existence of at least on solution of $dy/dx = f(x, y)$, $y(x_0) = y_0$), defined on some interval I. The interval of definition I for this IVP is generally taken to be the largest interval containing x_0 over which the solution $y(x)$ is both defined and differentiable. The interval depends both on the DE and the initial condition.

The condition of continuity of $\partial f/\partial y$ on R means that the solution on I_0 containing x_0 is the *only* solution satisfying the initial condition.

It should be noted that the interval of definition I may not be as wide as R and the interval of existence I_0 and uniqueness may not be as large as I. The number $h > 0$ that defines I_0 may be very small, so the solution $y(x)$ should bed thought of as *unique locally*: a solution defined near (x_0, y_0).

# 1.3 Differential Equations as Mathematical Models

Mathematical Models

A **mathematical model** is a mathematical description of some system or phenomenon.

To construct a mathematical model, one must first identify the independent variables of the system. The model's **level of resolution** is determined by which variables are chosen to be included From there, a set of hypotheses about the system can be made. These will include any empirical laws that may apply.

Assumptions made regarding a system often involve *rates of change*, so their models often include *derivatives*; that is to say, mathematical models often take the form of DEs or system of them.

If the DE or system is solvable, then the model can be considered reasonable if the solution is in line with data or known facts. Otherwise, the level of resolution can be increased or alternative assumptions can be made.

Steps in the modeling process with differential equations
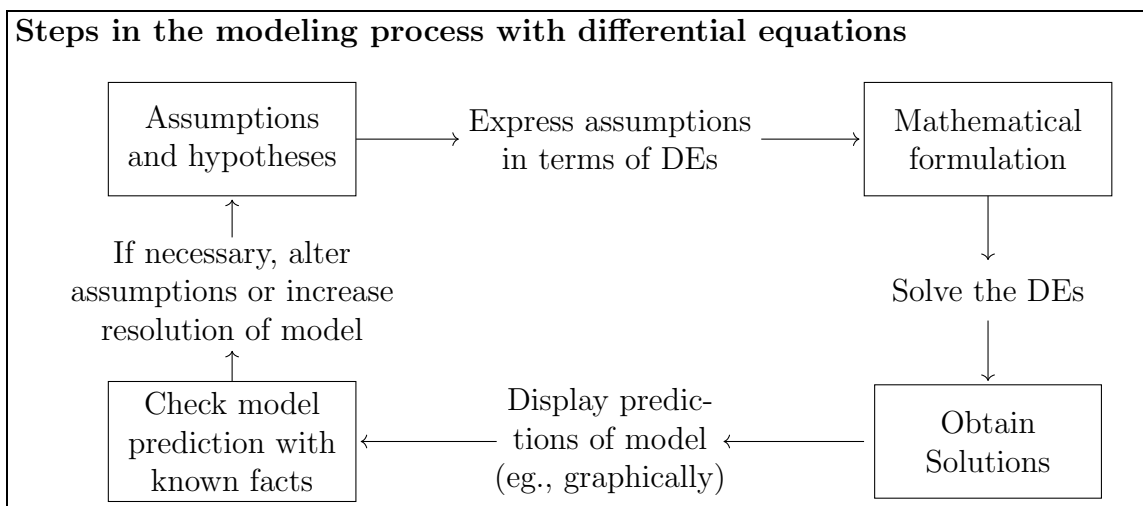
Increasing the resolution also increases the complexity of the model, meaning that an explicit solution becomes less likely.

Mathematical models of physical systems often involve time as a variable t. A solution gives the **state of the system**.

Population Dynamics

The **Malthusian model for population growth** assumes that the the growth rate over a certain time is proportional to the total population at that time.

$$\frac{\mathrm{d}P}{\mathrm{d}t} \propto P \qquad \text{or} \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

Due to its simplicity, it is only used to model the *growth of small populations over short intervals*. This model is also used for the model of continuous compound interest $\mathrm{d}S/\mathrm{d}t = rS$ (where S is capital and r is the annual interest rate).

Radioactive Decay

Radioactive decay can be modeled under the assumption that the rate $\mathrm{d}A/\mathrm{d}t$ at which a substance's nuclei decay is proportional to the number of nuclei $A(t)$ of the substance remaining at time t.

$$\frac{\mathrm{d}A}{\mathrm{d}t} \propto A \qquad \text{or} \qquad \frac{\mathrm{d}A}{\mathrm{d}t} = kA$$

This model is also used to determine a drug's half-life and in the model of a first-order chemical reaction.

A single differential equation may serve as a mathematical model for many phenomena.

Mathematical models often have side conditions, meaning that they may either be IVPs or BVPs.

Newton's Law of Warming/Cooling

Newton's law of cooling/warming can be expressed as

$$\frac{\mathrm{d}T}{\mathrm{d}t} \propto T - T_m \qquad \text{or} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_m)$$

where T is temperature of the body, T_m is the temperature of the surrounding medium, and t is time.

Spread of a Disease

The spread of a disease can be modeled as

$$dvxt = kxy$$

where $x(t)$ is the number of people that have contracted the disease and $y(t)$ is the number of people that have not been exposed. The product of these two can be used to approximate the number of interactions between the two groups.

Chemical Reactions

Radioactive decay is a **first-order reaction**. Such a reaction can be modeled as

$$\frac{dX}{dt} = kX$$

where $X(t)$ is the amount of substance A remaining.
Suppose one molecule each of substances A and B is used to form a single molecule of substance C. If X is the amount of C formed and α and β are the initial amounts of A and B, then the instantaneous amounts of A and B that have not yet been converted are $\alpha - X$ and $\beta - X$ respectively. The rate of formation of C is therefore given by

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$$

A reaction modeled by this is said to be a **second-order reaction**

Mixtures

If $A(t)$ denotes the amount of salt at time t, then the rate at which this changes is

$$\frac{dA}{dt} = R_{\text{in}} - R_{\text{out}}$$

Draining a Tank

Torricelli's law states that the speed v of efflux of water through a sharp-edged hole at the bottom of a container filled to depth h is equal to the speed that a body would acquire in free fall from the same height ($v = \sqrt{2gh}$ where g is acceleration due to gravity). If the are of the hole is A_h and the speed of water leaving is $v = \sqrt{2gh}$, then the volume of water leaving per second is $A_h\sqrt{2gh}$. If $V(t)$ denotes the volume of water remaining at time t, then

$$\frac{dV}{dt} = -A_h\sqrt{2gh}$$

Series Circuits

Consider the *LCR*-series circuit
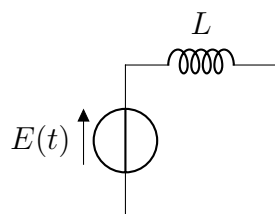
Chapter 2

First-Order Differential Equations

2.1 Solution Curves without a Solution

If a DE is not explicitly or analytically solvable, it still provides information regarding its solution curve.

2.1.1 Direction Fields

Slope

As the solution $y = y(x)$ of the first order DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

must be differentiable on its interval of definition I, it must also be continuous on said interval. The function f in normal form, as presented above, is called the **slope/rate function**. The slope of the tangent line at point $(x, y(x)$ is equal to $f(x, y(x))$.
Let (x, y) represent any point in the xy-plane for which f is defined. The value of the function at that point represents the slope of a line segment called a **lineal element**, which is some segment of the tangent line at that point.

Direction Field

A **direction/slope field** is a plot of small lineal elements over some region in the xy-plane. Visually, it provides information regarding the shape of a family of solution curves, allowing qualitative aspects of said family to be discerned. A solution curve passing through the field must follow the flow of the grid, being tangent to the lineal element at any point it intersects one.

Increasing/Decreasing

The sign of the first derivative provides information regarding whether the function is increasing or decreasing.

$\mathrm{d}y/\mathrm{d}x$	< 0	> 0
function behavior	decreasing	increasing

2.1.2 Autonomous First-Order DEs

Autonomous First-Order DEs

An ODE that does not explicitly contain the independent variable is said to be **autonomous**. If x is independent and y is dependent, an autonomous DE would take the form

$$f(y, y') = 0 \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$$

It is assumed that f isa continuous and differentiable function of y on some interval I.

Critical Points

A real number c is a **critical/equilibrium/stationary point** of an autonomous DE if $f(c) = 0$. If the constant function $y(x) = c$ is substituted into the normal form of an autonomous de, both sides equate to 0.

> If c is a critical point of an autonomous DE, then $y(x) = c$ is a constant solution.

A constant solution $y(x) = c$ is also referred to as an **equilibrium solution**, equilibria being the *only* constant solutions. A **(one dimensional) phase portrait** consists of a vertical **phase line** (the P-axis) displaying the intervals on which a function is increasing or decreasing between equilibria.

Solution Curves

As f is independent of x, it may be considered defined for $-\infty < x < \infty$ or $0 \le x < \infty$. As f is continuous and differentiable for y on some interval I of the y-axis, some horizontal region R can be formed in the xy-plane using I. Through any point x_0, y_0 in R passes only a single solution curve. Suppose R is split into subregions R_i by equilibria $y(x) = c_i$.

- If a solution curve passes through point x_i, y_i in R_i, it must stay within R_i for all x, as the curve is continuous and cannot cross equilibria.

- As f is continuous over R, $f(y)$ must be either entirely positive or entirely negative for all x in R_i.

- As $\mathrm{d}y/\mathrm{d}x = f(y(x))$, all solution curves must be monotonic.

- If $y(x)$ is *bounded* by 1 or 2 critical points, then the graph must asymptotically approach them.

Attractors and Repellers

A critical point c may be an **attractor** of $y(x)$, an initial point sufficiently close resulting in $\lim_{x \to \infty} y(x) = c$, a **repeller**, an initial point sufficiently close resulting in movement away from c, or neither, attracting on one side and repelling on the other. Respectively, these are referred to as

asymptotically stable, **unstable**, and **semi-stable**.
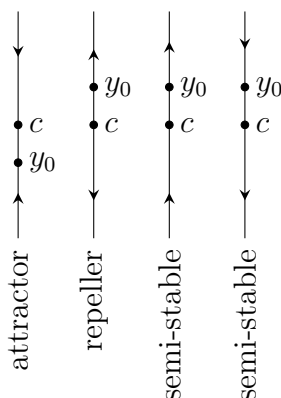
Autonomous DEs and Direction Fields

As dy/dx is solely dependent on y, the slopes of the lineal elements displayed in a direction field will depend solely on the points' y-coordinates.

Lineal elements that pass through the same *horizontal* line have the same slopes while those that pass through the same *vertical* line may vary.

Translation Property

> If $y(x)$ is a solution to an autonomous DE, then $y_1(x) = y(x - k)$ where k is a constant is as well.

This is due to the fact that an autonomous DE is not dependent on x while the solution curve is.

2.2 Separable Equations

The simplest of all DEs fall under the category of first-order separable ODEs.

Solution by Integration

When dy/dx is solely dependent on x ($f(x, y) = g(x)$), integration can be used to solve the DE. If both sides are continuous,

$$y = \int g(x)\, dx = G(x) + C$$

A Definition

> **Separable Equation**   A first-order DE of the form
>
> $$\frac{dy}{dx} = g(x)h(y)$$
>
> is said to be **separable**, having **separable variables**.

From this form, it can be seen that

$$p(y)\frac{dy}{dx} = g(x)$$

where $p(y) = 1/h(y)$. The derivative can then be split so that both differentials are in the numerator.

$$\int p(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$$

Integrating,

$$P(y) = G(x) + C$$

Method of Solution

A on-parameter family of solutions can be obtained by integrating.[1]

Losing a Solution

It should be noted that variable divisors may be zero at a point. Constant solutions are often lost through division.

Use of Computers

A computer algebra system (CAS) can be used to produce level curves defined by equating the implicit solution $G(x, y)$ to various values of c.
It should be noted that an IVP may have nontrivial solutions that are part of the same family. In fact, all member of a family may be solutions.

An Integral-Defined Function

The solution of an IVP $\mathrm{d}y/\mathrm{d}x = g(x)$, $y(x_0) = y_0$ defined on interval I containing x_0 and x over which g is continuous is given by

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,\mathrm{d}t$$

When the integral is nonelementary, this is acceptable as a final solution.

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[1]It should be noted that there is no need for multiple constants of integration for a separable equation, as their difference can simply be replaced by a single constant.