AP Calc BC Project: Integration by Parts

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1st Hour

Problems

1. $\int e^x \cos x \,\mathrm{d}x$
2. $\int x^2 \sin x \,\mathrm{d}x$
3. $\int \sin^{-1}(3x) \,\mathrm{d}x$
4. $\int x\sqrt{x+3} \,\mathrm{d}x$
5. $\int \frac{1}{x(\ln x)^3} \,\mathrm{d}x$
6. $\int x^2 \ln x \,\mathrm{d}x$
7. $\int x^7 \sin\left(2x^4\right) \mathrm{d}x$
8. $\int x^3 e^{-x^2} \,\mathrm{d}x$
9. $\int_1^2 \frac{\ln x}{x^2} \,\mathrm{d}x$
10. $\int_1^4 (2-x)^2 \ln(4x) \,\mathrm{d}x$
11. $\int \sin(\ln x) \,\mathrm{d}x$
12. $e^{3x} \sin(e^x) \,\mathrm{d}x$
13. $\int (x+1)^2 \ln(3x) \,\mathrm{d}x$
14. $\int x \ln(1+x) \,\mathrm{d}x$
15. $\int (4x^3 - 9x^2 + 7x + 3)e^{-x} \,\mathrm{d}x$

Solutions

1.

$$\int e^x \cos x \, \mathrm{d}x \implies \begin{array}{|c|ccc|}\hline \pm & + & - & + \\ \hline u & \cos x & -\sin x & -\cos x \\ \hline \mathrm{d}v & e^x & e^x & e^x \\ \hline \end{array}$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x \, \mathrm{d}x$$

$$2 \int e^x \cos x = e^x(\cos x + \sin x)$$

$$\int e^x \cos x = \frac{e^x(\cos x + \sin x)}{2} + C$$

2.

$$\int x^2 \sin x \, \mathrm{d}x \implies \begin{array}{|c|cccc|}\hline \pm & + & - & + & - \\ \hline u & x^2 & 2x & 2 & 0 \\ \mathrm{d}v & \sin x & -\cos x & -\sin x & \cos x \\ \hline \end{array}$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

3.

$$\int \sin^{-1}(3x) \implies \begin{array}{|c|cc|}\hline \pm & + & - \\ \hline u & \sin^{-1}(3x) & \frac{3}{\sqrt{1-9x^2}} \\ \hline \mathrm{d}v & 1 & x \\ \hline \end{array}$$

$$= x \sin^{-1}(3x) - \int \frac{3x}{\sqrt{1-9x^2}} \, \mathrm{d}x \implies u = 1 - 9x^2 \implies \mathrm{d}u = -18x \, \mathrm{d}x$$

$$= x \sin^{-1}(3x) + \frac{1}{6} \int \frac{\mathrm{d}u}{\sqrt{u}} = x \sin^{-1}(3x) + \frac{1}{6} \frac{u^{1/2}}{1/2} + C = x \sin^{-1}(3x) + \frac{1}{3}\sqrt{1-9x^2} + C$$

4.

$$\int x\sqrt{x+3} \, \mathrm{d}x \implies \begin{array}{|c|ccc|}\hline \pm & + & - & + \\ \hline u & x & 1 & 0 \\ \hline \mathrm{d}v & \sqrt{x+3} & \frac{2(x+3)^{3/2}}{3} & \frac{4(x+3)^{5/2}}{15} \\ \hline \end{array}$$

$$= \frac{2}{3}x(x+3)^{3/2} - \frac{4}{15}(x+3)^{5/2} + C$$

5.

$$\int \frac{1}{x(\ln x)^3} \implies \begin{array}{|c|cc|} \hline \pm & + & - \\ \hline u & (\ln x)^{-3} & -3x^{-1}(\ln x)^{-4} \\ \hline \mathrm{d}v & x^{-1} & \ln x \\ \hline \end{array}$$

$$= \frac{\ln x}{(\ln x)^3} + 3\int \frac{\ln x}{x(\ln x)^4}\,\mathrm{d}x = \frac{1}{(\ln x)^2} + 3\int \frac{1}{x(\ln x)^3}$$

$$2\int \frac{1}{x(\ln x)^3} = -\frac{1}{(\ln x)^2}$$

$$\int \frac{1}{x(\ln x)^3} = -\frac{1}{2(\ln x)^2} + C$$

6.

$$\int x^2 \ln x\,\mathrm{d}x \implies \begin{array}{|c|cc|} \hline \pm & + & - \\ \hline u & \ln x & x^{-1} \\ \hline \mathrm{d}v & x^2 & x^3/3 \\ \hline \end{array}$$

$$= \frac{1}{3}x^3\ln x - \int \frac{x^3}{3x}\,\mathrm{d}x = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2\,\mathrm{d}x = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$$

7.

$$\int x^7\sin(2x^4)\,\mathrm{d}x \implies \begin{array}{|c|cc|} \hline \pm & + & - \\ \hline u & x^4 & 4x^3 \\ \hline \mathrm{d}v & x^3\sin(2x^4) & -\cos(2x^4)/8 \\ \hline \end{array}$$

$$= -\frac{1}{8}x^4\cos(2x^4) + \int \frac{4x^3\cos(2x^4)}{8}\,\mathrm{d}x = -\frac{1}{8}x^4\cos(2x^4) + \frac{1}{16}\sin(2x^4) + C$$

8.

$$\int x^3 e^{-x^2}\,\mathrm{d}x \implies w = x^2 \implies \mathrm{d}w = 2x\,\mathrm{d}x$$

$$= \frac{1}{2}\int we^{-w}\,\mathrm{d}w \implies \begin{array}{|c|ccc|} \hline \pm & + & - & + \\ \hline u & w & 1 & 0 \\ \hline \mathrm{d}v & e^{-w} & -e^{-w} & e^{-w} \\ \hline \end{array}$$

$$= -\frac{e^{-w}}{2}(w+1) = -\frac{e^{-x^2}}{2}(x^2+1)$$

9.

$$\int_1^2 \frac{\ln(x)}{x^2}\,\mathrm{d}x \implies \begin{array}{|c|cc|} \hline \pm & + & - \\ \hline u & \ln x & x^{-1} \\ \hline \mathrm{d}v & x^{-2} & -x^{-1} \\ \hline \end{array}$$

$$= -\frac{\ln x}{x}\bigg|_1^2 + \int_1^2 \frac{1}{x^2}\,\mathrm{d}x = -\frac{\ln(2)}{2} + \frac{\ln(1)}{(1)} - \frac{1}{x}\bigg|_1^2 = -\frac{\ln(2)}{2} - \frac{1}{2} + \frac{1}{1} = -\frac{\ln(2)-1}{2}$$

10.

$$\int_1^4 (2-x)^2 \ln(4x)\,\mathrm{d}x \implies \begin{array}{|c|cc|}\hline \pm & + & - \\ \hline u & \ln(4x) & x^{-1} \\ \hline \mathrm{d}v & (2-x)^2 & x^3/3 - 2x^2 + 4x \\ \hline \end{array}$$

$$= \left(\frac{x^3}{3} - 2x^2 + 4x\right)\ln(4x)\Big|_1^4 - \int_1^4 \left(\frac{x^2}{2} - 2x + 4\right)\mathrm{d}x$$

$$= \left(\frac{x^3}{3} - 2x^2 + 4x\right)\ln(4x) - \frac{x^3}{9} + x^2 - 4x\Big|_1^4$$

$$= \left(\frac{64}{3} - 32 + 16\right)\ln(16) - \frac{64}{9} + 16 - 16 - \left(\left(\frac{1}{3} - 2 + 4\right)\ln(4) - \frac{1}{9} + 1 - 4\right)$$

$$= \left(\frac{64}{3} - 16\right)\ln(16) - \left(\frac{1}{3} + 2\right)\ln(4) - \frac{63}{9} + 3 = \frac{16\ln(16) - 7\ln(4)}{3} - 4$$

11.

$$\int \sin(\ln x)\,\mathrm{d}x \implies w = \ln x \implies \mathrm{d}w = \frac{1}{x}\,\mathrm{d}x \implies \mathrm{d}x = x\,\mathrm{d}w = e^w\,\mathrm{d}w$$

$$= \int e^w \sin w\,\mathrm{d}w \implies \begin{array}{|c|ccc|}\hline \pm & + & - & + \\ \hline u & \sin w & \cos w & -\sin w \\ \hline \mathrm{d}v & e^w & e^w & e^w \\ \hline \end{array}$$

$$= e^w(\sin w - \cos w) - \int e^w \sin w\,\mathrm{d}w$$

$$2\int e^w \sin w\,\mathrm{d}w = e^w(\sin w - \cos w)$$

$$\int e^w \sin w = \frac{e^w(\sin w - \cos w)}{2} + C = \frac{x(\sin(\ln x) - \cos(\ln x))}{2} + C$$

12.

$$\int e^{3x}\sin(e^x) \implies w = e^x \implies \mathrm{d}w = e^x\,\mathrm{d}x$$

$$= \int w^2 \sin w\,\mathrm{d}w \implies \begin{array}{|c|cccc|}\hline \pm & + & - & + & - \\ \hline u & w^2 & 2w & 2 & 0 \\ \mathrm{d}v & \sin w & -\cos w & -\sin w & \cos w \\ \hline \end{array}$$

$$= -w^2\cos w + 2w\sin w + 2\cos w + C = -e^{2x}\cos(e^x) + 2e^x\sin(e^x) + 2\cos(e^x) + C$$

13.

$$\int (x+1)^2 \ln(3x)\,\mathrm{d}x \implies \begin{array}{|c|cc|}\hline \pm & + & - \\ \hline u & \ln(3x) & x^{-1} \\ \hline \mathrm{d}v & (x+1)^2 & x^3/3 + x^2 + x \\ \hline \end{array}$$

$$= \left(\frac{x^3}{3} + x^2 + x\right)\ln(3x) - \int \left(\frac{x^2}{3} + x + 1\right)\mathrm{d}x$$

$$= \left(\frac{x^3}{3} + x^2 + x\right)\ln(3x) - \frac{x^3}{9} - \frac{x^2}{2} - x + C$$

14.

$$\int x\ln(1+x)\,\mathrm{d}x \implies \begin{array}{|c|cc|}\hline \pm & + & - \\ \hline u & \ln(1+x) & \frac{1}{1+x} \\ \hline \mathrm{d}v & x & x^2/2 \\ \hline \end{array}$$

$$\begin{aligned} &= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2}{2(1+x)}\,\mathrm{d}x = \frac{x^2\ln(1+x)}{2} - \frac{1}{2}\int \frac{x^2}{x+1}\,\mathrm{d}x \\ &= \frac{x^2\ln(1+x)}{2} - \frac{1}{2}\int\left(\frac{1}{x+1} + x - 1\right)\mathrm{d}x = \frac{x^2\ln(1+x)}{2} - \frac{1}{2}\left(\ln(x+1) + \frac{x^2}{2} - x\right) \\ &= \frac{x^2\ln(x+1)}{2} - \frac{\ln(x+1)}{2} - \frac{x^2}{4} + \frac{x}{2} + C \end{aligned}$$

15.

$$\int (4x^3 - 9x^2 + 7x + 3)e^{-x}\,\mathrm{d}x \implies \begin{array}{|c|ccccc|}\hline \pm & + & - & + & - & + \\ \hline u & 4x^3 - 9x^2 + 7x + 3 & 12x^2 - 18x + 7 & 24x - 18 & 24 & 0 \\ \hline \mathrm{d}v & e^{-x} & -e^{-x} & e^{-x} & -e^{-x} & e^{-x} \\ \hline \end{array}$$

$$\begin{aligned} &= -e^{-x}(4x^3 - 9x^2 + 7x + 3 + 12x^2 - 18x + 7 + 24x - 18 + 24) \\ &= -e^{-x}(4x^3 + 3x^3 + 13x + 16) + C \end{aligned}$$