Differential Equations

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Contents

Chapter 1

Introduction to Differential Equations

1.1 Definitions and Terminology

A Definition

Differential Equation An equation containing the derivatives of one or more unknown functions (or dependent variables) with respect to one or more independent variables is a **differential equation (DE)**.

Classification by Type

A differential containing only ordinary derivatives with respect to a *single* independent variables is an **ordinary differential equation (ODE)**. One involving partial derivatives is a **partial differential equation (PDE)**.

Notation

Leibniz notation denotes derivatives as ratios of differentials with the operators and variables raised to the n for the n^{th} derivative. **Prime notation** denotes the n^{th} derivative with either n primes or (n) in superscript of the dependent variable or the function. The n^{th} derivative of $y = f(x)$ can thusly be denoted as

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = y^{(n)} = f^{(n)}(x)$$

Newton's **dot notation** is sometimes used to denote derivatives with respect to time, placing n dots above the dependent variable to denote its n^{th} derivative with respect to t. The second derivative of x with respect to t can be denoted as

$$\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

Subscript Notation is often used for partial derivatives, indicating the independent variable in the subscript. The second partial x derivative with respect to z can be denoted as

$$z_{xx} = \frac{\partial^2 z}{\partial x^2}$$

Classification by Order

The **order of a differential equation** is the order of the highest derivartive in the equation.
A first-order ODE is sometimes written in the **differential form**

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

Symbolically, an n^{th}-order ODE in one dependent variable can be expressed generally as

$$F\big(x, y, y', \dots, y^{(n)}\big) = 0$$

where F is a real-valued function of $n+2$ variables.
It is assumed that it is possible to solve an ODE in the form above uniquely for the highest derivative $y^(n)$ in terms of the remaining $n+1$ variables.
The **normal form** of the above expression is

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\big(x, y, y', \dots, y^{(n-1)}\big)$$

where f is a real-values continuous function.
The following normal forms can be used to represent general first- and second-order ODEs:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \qquad\qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x,y,y')$$

Classification by Linearity

An general n^{th} order ODE is **linear** if F is linear in $y, y', y^{(n)}$. This means that it is linear when

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Two important special cases of the above are linear first-[1] and second-order DEs:

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \qquad\qquad a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

The characteristic properties of linear ODEs are that the dependent variable and all of its derivatives are of first degree and that the coefficients of those terms are dependent at most on the independent variable.
A **nonlinear** ODE is one that is not linear.
Nonlinear functions of the dependent variable cannot appear in linear ODEs.
A DE can not be classified as linear or nonlinear if both differentials are in the numerator.

Solutions

Solution of an ODE Any function φ defined on an interval I with at least n derivatives that are continuous on I which when substituted into an n^{th}-order ODE reduce the equation to an identity is a **solution** of the equation on the interval.

[1] A first order ODE written in differential form as $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ may be linear or nonlinear, as there is no indication of which symbol is the dependent variable.

A DE need not have a solutions. A solution of a DE may involve **integral-defined function**[2].
A solution of a general n^{th}-order ODE is a function φ with at least n derivatives for which

$$F\big(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(n)}(x)\big) = 0 \qquad \forall\, x \in I$$

φ is said to *satisfy* the differential equation on I. It is assumed that a solution φ is a real-valued function.
A solution is occasionally denoted alternatively by $y(x)$.

Interval of Definition

The interval I over which φ satisfies the ODE is referred to as the **interval of definition/existence/validity** or the **domain of the solution**.
A solution of a DE that is identically 0 on an interval I is said to be a **trivial solution**.

Solution Curve

The graph of φ is called a **solution curve**.
The domain of φ need not be the same as I.

Explicit and Implicit Solutions

A function that expresses the dependent variable solely in terms of the independent variable and constants is said to be *explicit*. An **explicit solution** is a solution with an explicit function. It can be thought of as an explicit formula $y = \varphi(x)$ that can be manipulated.
An explicit solution is generally not needed over an implicit one.
An implicit solution $G(x, y) = 0$ may define a differentiable function that is a solution of a DE despite $G(x, y) = 0$ potentially not being solvable analytically. The solution curve may be a segment of the graph of $G(x, y) = 0$.

Families of Solutions

When solving a first-order DE, the solution *usually* contains a single constant or parameter C, similar to the constant of integration obtained from the indefinite integral. A solution of $F(x, y, y') = 0$ containing constant C is a set of solutions $G(x, y, C) = 0$ called a **one-parameter family of solutions**.

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[2] A function F of a single variable x can be defined as

$$F(x) = \int_a^x g(t)\,\mathrm{d}t$$

If the integrand g is continuous over $[a, b]$ and x falls within the interval, then F is differentiable on the open interval and

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x g(t)\,\mathrm{d}t = g(x)$$

The integral is often **nonelementary**, meaning that it is not composed of elementary functions.
Elementary functions include constant, polynomial, rational, exponential, logarithmic, trigonometric, and inverse trigonometric functions, as well as rational powers and finite combinations using the four basic arithmetic operations and compositions of these functions.

An n^{th}-order DE[3] often yields an ***n*-parameter family of solutions**[4] $G(x, y, C_1, C_2, \ldots, C_n) = 0$. The parameters in a family of solutions are *arbitrary* up to a point, but they should always take on values that make sense in the real-number system.
A **singular solution** is one that cannot be obtained by specializing *any* of the parameters in the family of solutions.

Systems of Differential Equations

A **system of ODEs** is comprised of multiple unknown functions of a single independent variable.
A **solution** of a system is a pair of differentiable functions defined on common interval I that satisfy each equation of the system on the interval.

1.2 Initial-Value Problems

Often, a solution to a DE must meet other conditions imposed on it and its derivatives.

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\big(x, y, y', \ldots, y^{(n-1)}\big) \qquad \text{subject to} \qquad y(x_0) = y_0, y'(x) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

If these desired values are constants, this is an **n^{th}-order initial-value problem (IVP)**. The desired values are called **initial conditions (IC)**.[5]
Solving an n^{th}-order IVP often requires that an n-parameter family of solutions be found that can then be used in tandem with the constraints to find the constants. The resulting particular solution is defined on some interval that contains x_0.

Geometric Interpretation

A first-order IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \mid y(x_0) = y_0$$

can be interpreted as finding a solution $y(x)$ with a graph that passes through the point (x_0, y_0).
A second-order IVP

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x, y, y') \mid y(x_0) = y_0, y'(x_0) = y_1$$

can be interpreted as finding a solution $y(x)$ with a graph that passes through (x_0, y_0) with slope y_1.

[3] $F\big(x, y, y', \ldots, y^{(n)}\big) = 0$ may not always be solvable for $y^{(n)}$.
[4] If *every* solution of an n^{th}-order ODE on an interval can be found by manipulating the parameters of an n-parameter family of solutions, then the family is said to be the **general solution** of the DE.
Nonlinear ODEs are often difficult of impossible to solve in terms of elementary functions, so if a family of solutions is found for one, it is unclear whether it is a general solution. Practically, the designation of "general solution" is only given for solutions to linear ODEs.
[5] If conditions are prescribed at multiple points, called **boundary conditions**, the problem is called a **boundary-value problem (BVP)**.

Existence and Uniqueness

It can be assumed that *most* DEs will hav solutions and that the solutions of IVPs will *generally* be unique.

Theorem 1.2.1 Let R be a rectangular region in the xy-plane defined by

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

that contains point (x_0, y_0). If $f(x, y)$ and $\partial f/\partial y$ are continuous over R[a], then there exists some interval $I_0 : (x_0 - h, x_0 + h)$ (where $h > 0$) contained in $[a, b]$ and a unique function $y(x)$ defined on this interval that is a solution of the IVP.

[a] These conditions are sufficient but not necessary. When $f(x, y)$ and $\partial f/\partial y$ are continuous on R, a solution of the IVP exists and is unique so long as (x_0, y_0) is contained within R. If these conditions are not met, though, the IVP *may* still have a solution that *may* be unique, or it may have multiple or no solutions.

Interval of Existence/Uniqueness

The domain of the function that represents a solution to an IVP, the interval I over which the solution is defined or exists, and the interval I_0 of existence *and* uniqueness.

Suppose (x_0, y_0) is a point in the interior of rectangle R. The continuity of function $f(x, y)$ on R is sufficient to guarantee the existence of at least on solution of $\mathrm{d}y/\mathrm{d}x = f(x, y)$, $y(x_0) = y_0$), defined on some interval I. The interval of definition I for this IVP is generally taken to be the largest interval containing x_0 over which the solution $y(x)$ is both defined and differentiable. The interval depends both on the DE and the initial condition.

The condition of continuity of $\partial f/\partial y$ on R means that the solution on I_0 containing x_0 is the *only* solution satisfying the initial condition.

It should be noted that the interval of definition I may not be as wide as R and the interval of existence I_0 and uniqueness may not be as large as I. The number $h > 0$ that defines I_0 may be very small, so the solution $y(x)$ should bed thought of as *unique locally*: a solution defined near (x_0, y_0).

1.3 Differential Equations as Mathematical Models

Mathematical Models

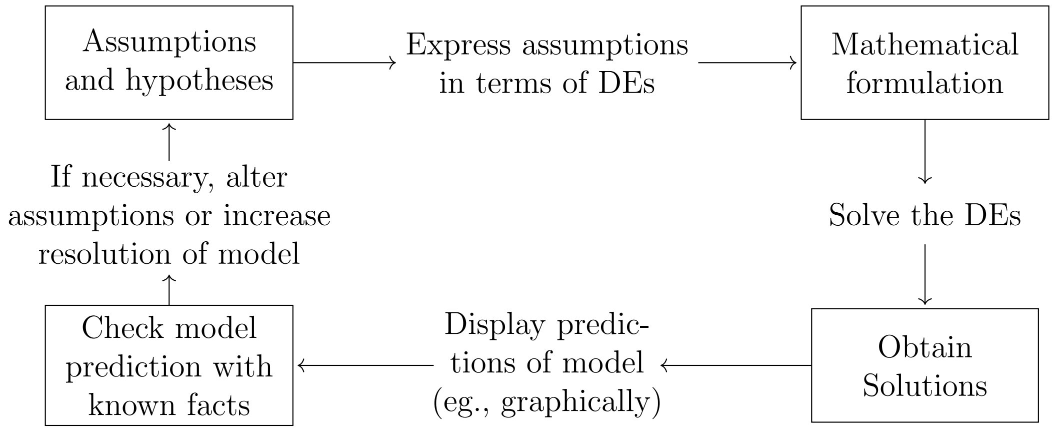
A **mathematical model** is a mathematical description of some system or phenomenon.

To construct a mathematical model, one must first identify the independent variables of the system. The model's **level of resolution** is determined by which variables are chosen to be included From there, a set of hypotheses about the system can be made. These will include any empirical laws that may apply.

Assumptions made regarding a system often involve *rates of change*, so their models often include *derivatives*; that is to say, mathematical models often take the form of DEs or system of them.

If the DE or system is solvable, then the model can be considered reasonable if the solution is in line with data or known facts. Otherwise, the level of resolution can be increased or alternative assumptions can be made.

Steps in the modeling process with differential equations



Increasing the resolution also increases the complexity of the model, meaning that an explicit solution becomes less likely.
Mathematical models of physical systems often involve time as a variable t. A solution gives the **state of the system**.

Population Dynamics

The **Malthusian model for population growth** assumes that the the growth rate over a certain time is proportional to the total population at that time.

$$\frac{\mathrm{d}P}{\mathrm{d}t} \propto P \qquad \text{or} \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

Due to its simplicity, it is only used to model the *growth of small populations over short intervals*. This model is also used for the model of continuous compound interest $\mathrm{d}S/\mathrm{d}t = rS$ (where S is capital and r is the annual interest rate).

Radioactive Decay

Radioactive decay can be modeled under the assumption that the rate $\mathrm{d}A/\mathrm{d}t$ at which a substance's nuclei decay is proportional to the number of nuclei $A(t)$ of the substance remaining at time t.

$$\frac{\mathrm{d}A}{\mathrm{d}t} \propto A \qquad \text{or} \qquad \frac{\mathrm{d}A}{\mathrm{d}t} = kA$$

This model is also used to determine a drug's half-life and in the model of a first-order chemical reaction.

A single differential equation may serve as a mathematical model for many phenomena.

Mathematical models often have side conditions, meaning that they may either be IVPs or BVPs.

Newton's Law of Warming/Cooling

Newton's law of cooling/warming can be expressed as

$$\frac{\mathrm{d}T}{\mathrm{d}t} \propto T - T_m \qquad \text{or} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_m)$$

where T is temperature of the body, T_m is the temperature of the surrounding medium, and t is time.

Spread of a Disease

The spread of a disease can be modeled as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kxy$$

where $x(t)$ is the number of people that have contracted the disease and $y(t)$ is the number of people that have not been exposed. The product of these two can be used to approximate the number of interactions between the two groups.

Chemical Reactions

Radioactive decay is a **first-order reaction**. Such a reaction can be modeled as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = kX$$

where $X(t)$ is the amount of substance A remaining.
Suppose one molecule each of substances A and B is used to form a single molecule of substance C. If X is the amount of C formed and α and β are the initial amounts of A and B, then the instantaneous amounts of A and B that have not yet been converted are $\alpha - X$ and $\beta - X$ respectively. The rate of formation of C is therefore given by

$$\frac{\mathrm{d}X}{\mathrm{d}t} = k(\alpha - X)(\beta - X)$$

A reaction modeled by this is said to be a **second-order reaction**

Mixtures

If $A(t)$ denotes the amount of salt at time t, then the rate at which this changes is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = R_{\text{in}} - R_{\text{out}}$$

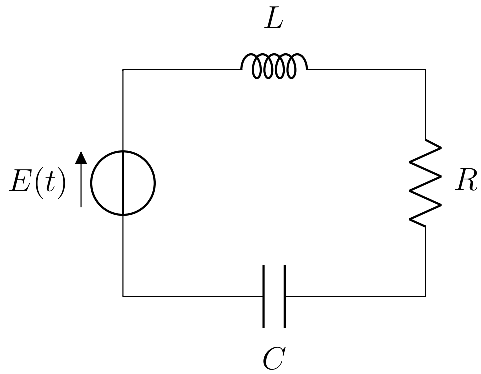
Draining a Tank

Torricelli's law states that the speed v of efflux of water through a sharp-edged hole at the bottom of a container filled to depth h is equal to the speed that a body would acquire in free fall from the same height ($v = \sqrt{2gh}$ where g is acceleration due to gravity). If the are of the hole is A_h and the speed of water leaving is $v = \sqrt{2gh}$, then the volume of water leaving per second is $A_h\sqrt{2gh}$. If $V(t)$ denotes the volume of water remaining at time t, then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -A_h\sqrt{2gh}$$

Series Circuits

Consider the single-loop LRC-series circuit



containing an inductor, a resistor, and a capacitor. The current remaining in a circuit after a switch is closed is denoted by $i(t)$ while the charge on a capacitor is denoted by $q(t)$. L, R, and C denote inductance, resistance, and capacitance respectively and are generally constants.
According to **Kirchhoff's second law**, the impressed voltage $E(t)$ on a closed loop must be equal to the sum of the voltage drops in the loop. As current $i(t)$ is related to charge $q(t)$ by $i = \mathrm{d}q/\mathrm{d}t$, equating the sum of the 3 voltages

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} \qquad iR = R\frac{\mathrm{d}q}{\mathrm{d}t} \qquad \frac{1}{C}q$$

to the impressed voltage yields a second-order DE

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$

Falling Bodies

Newton's first law of motion states that a body in motion will stay in motion and one at rest will stay at rest unless acted upon by an external force. Both statements are equivalent to stating that the sum of the forces (the *net*/resultant force) acting on the body is 0, then so is its acceleration.
Newton's second law of motion states that $F = ma$, where F is force, m is mass, and a is acceleration.

Falling Bodies and Air Resistance

A falling body of mass m may encounter air resistance proportion to its velocity v. The net force acting on such a mass is given by $F = mg - kv$, (k being a positive constant of proportionality) the former term being the force of gravity and the latter being that of air resistance, called **viscous damping**. As $a = \mathrm{d}v/\mathrm{d}t$, Newton's second law can be rewritten a

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

Equating the net force to this form of Newton's second law yields a first-order DE for the velocity $v(t)$:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv$$

In terms of position s,

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = mg - k\frac{\mathrm{d}s}{\mathrm{d}t}$$

Suspended Cables

Suppose a flexible cable is suspended between two vertical supports. Let P_1 denote its lowers hanging point and P_2 some arbitrary point. The portion of the cable connecting these two points is a curve in the xy-plane, the y-axis passing through P_1 and the x-axis being a units below P_1. There are 3 forces acting on the cable: the tensions $\vec{T}_1$ and $\vec{T}_2$ tangent to the cable at P_1 and P_2 respectively and the weight of the cable $\vec{W}$. Let $T_1 = |\vec{T}_1|$, $T_2 = |\vec{T}_2|$, and $W = |\vec{W}|$. The tension $\vec{T}_2$ is the only force with both vertical and horizontal components. As the system, is in static equilibrium,

$$T_1 = T_2 \cos\theta \qquad\qquad W = T_2 \sin\theta$$

Dividing the former equation by the latter, T_2,

$$\tan\theta = \frac{W}{T_1}$$

As $\mathrm{d}y/\mathrm{d}x = \tan\theta$,

$$\mathrm{d}y/\mathrm{d}x = \frac{W}{T_1}$$

Chapter 2

First-Order Differential Equations

2.1 Solution Curves without a Solution

If a DE is not explicitly or analytically solvable, it still provides information regarding its solution curve.

2.1.1 Direction Fields

Slope

As the solution $y = y(x)$ of the first order DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

must be differentiable on its interval of definition I, it must also be continuous on said interval. The function f in normal form, as presented above, is called the **slope/rate function**. The slope of the tangent line at point $(x, y(x))$ is equal to $f(x, y(x))$.
Let (x, y) represent any point in the xy-plane for which f is defined. The value of the function at that point represents the slope of a line segment called a **lineal element**, which is some segment of the tangent line at that point.

Direction Field

A **direction/slope field** is a plot of small lineal elements over some region in the xy-plane. Visually, it provides information regarding the shape of a family of solution curves, allowing qualitative aspects of said family to be discerned. A solution curve passing through the field must follow the flow of the grid, being tangent to the lineal element at any point it intersects one.

Increasing/Decreasing

The sign of the first derivative provides information regarding whether the function is increasing or decreasing.

$\mathrm{d}y/\mathrm{d}x$	< 0	> 0
function behavior	decreasing	increasing

2.1.2 Autonomous First-Order DEs

Autonomous First-Order DEs

An ODE that does not explicitly contain the independent variable is said to be **autonomous**. If x is independent and y is dependent, an autonomous DE would take the form

$$f(y, y') = 0 \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$$

It is assumed that f is a continuous and differentiable function of y on some interval I.

Critical Points

A real number c is a **critical/equilibrium/stationary point** of an autonomous DE if $f(c) = 0$. If the constant function $y(x) = c$ is substituted into the normal form of an autonomous de, both sides equate to 0.

If c is a critical point of an autonomous DE, then $y(x) = c$ is a constant solution.

A constant solution $y(x) = c$ is also referred to as an **equilibrium solution**, equilibria being the *only* constant solutions. A **(one dimensional) phase portrait** consists of a vertical **phase line** (the P-axis) displaying the intervals on which a function is increasing or decreasing between equilibria.

Solution Curves

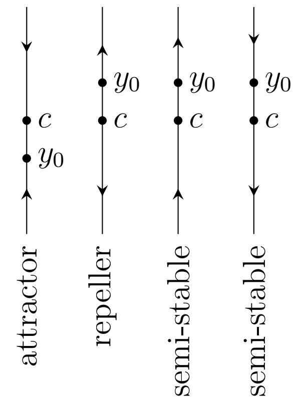
As f is independent of x, it may be considered defined for $-\infty < x < \infty$ or $0 \leq x < \infty$. As f is continuous and differentiable for y on some interval I of the y-axis, some horizontal region R can be formed in the xy-plane using I. Through any point x_0, y_0 in R passes only a single solution curve. Suppose R is split into subregions R_i by equilibria $y(x) = c_i$.

- If a solution curve passes through point x_i, y_i in R_i, it must stay within R_i for all x, as the curve is continuous and cannot cross equilibria.
- As f is continuous over R, $f(y)$ must be either entirely positive or entirely negative for all x in R_i.
- As $\mathrm{d}y/\mathrm{d}x = f(y(x))$, all solution curves must be monotonic.
- If $y(x)$ is *bounded* by 1 or 2 critical points, then the graph must asymptotically approach them.

Attractors and Repellers

A critical point c may be an **attractor** of $y(x)$, an initial point sufficiently close resulting in $\lim_{x\to\infty} y(x) = c$, a **repeller**, an initial point sufficiently close resulting in movement away from c, or neither, attracting on one side and repelling on the other. Respectively, these are referred to as

asymptotically stable, **unstable**, and **semi-stable**.



Autonomous DEs and Direction Fields

As $\mathrm{d}y/\mathrm{d}x$ is solely dependent on y, the slopes of the lineal elements displayed in a direction field will depend solely on the points' y-coordinates.
Lineal elements that pass through the same *horizontal* line have the same slopes while those that pass through the same *vertical* line may vary.

Translation Property

If $y(x)$ is a solution to an autonomous DE, then $y_1(x) = y(x-k)$ where k is a constant is as well.

This is due to the fact that an autonomous DE is not dependent on x while the solution curve is.

2.2 Separable Equations

The simplest of all DEs fall under the category of first-order separable ODEs.

Solution by Integration

When $\mathrm{d}y/\mathrm{d}x$ is solely dependent on x ($f(x,y) = g(x)$, integration can be used to solve the DE. If both sides are continuous,

$$y = \int g(x)\,\mathrm{d}x = G(x) + C$$

A Definition

Separable Equation A first-order DE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$$

is said to be **separable**, having **separable variables**.

From this form, it can be seen that

$$p(y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$

where $p(y) = 1/h(y)$. The derivative can then be split so that both differentials are in the numerator.

$$\int p(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$$

Integrating,

$$P(y) = G(x) + C$$

Method of Solution

A on-parameter family of solutions can be obtained by integrating.[1]

Losing a Solution

It should be noted that variable divisors may be zero at a point. Constant solutions are often lost through division.

Use of Computers

A computer algebra system (CAS) can be used to produce level curves defined by equating the implicit solution $G(x, y)$ to various values of c.
It should be noted that an IVP may have nontrivial solutions that are part of the same family. In fact, all member of a family may be solutions.

An Integral-Defined Function

The solution of an IVP $\mathrm{d}y/\mathrm{d}x = g(x)$, $y(x_0) = y_0$ defined on interval I containing x_0 and x over which g is continuous is given by

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,\mathrm{d}t$$

When the integral is nonelementary, this is acceptable as a final solution.

2.3 Linear Equations

A Definition

Definition 2.3.1. Linear Equation A first-order DE of the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

is a **linear equation** in y.[a]

[a]A first-order DE may occasionally be linear in one variable but not the other.

[1]It should be noted that there is no need for multiple constants of integration for a separable equation, as their difference can simply be replaced by a single constant.

Standard Form

By dividing by $a_1(x)$, the **standard form** of a first-order linear equation is obtained:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x)$$

A solution of this should be on an interval I over which both P and f are continuous.

Method of Solution

The left hand side of the standard form of a first-order linear DE can be rewritten as the derivative of a product by multiplying by $\mu(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)] = \mu\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}\mu}{\mathrm{d}x}y = \mu\frac{\mathrm{d}y}{\mathrm{d}x} + \mu P y$$

Evidently,

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \mu P \implies \frac{\mathrm{d}\mu}{\mu} = P\,\mathrm{d}x \implies \ln|\mu| = \int P(x)\,\mathrm{d}x \implies \mu = \mathrm{e}^{\int P(x)\mathrm{d}x}$$

The constant of integration is chosen to be 1 for simplicity. This result is called the **integrating factor**. Substituting in,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\mathrm{e}^{\int P(x)\mathrm{d}x}\right] = \int P(x)\,\mathrm{d}x\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\mathrm{e}^{\int P(x)\mathrm{d}x}y = \mathrm{e}^{\int P(x)\mathrm{d}x}f(x)$$

Integrating both sides,

$$y\mathrm{e}^{\int P(x)\mathrm{d}x} = \int \mathrm{e}^{\int P(x)\mathrm{d}x}f(x)\,\mathrm{d}x$$

Solving for y produces a one-parameter family of solutions

$$y = \mathrm{e}^{-\int P(x)\mathrm{d}x}\left(\int \mathrm{e}^{\int P(x)\mathrm{d}x}f(x)\,\mathrm{d}x + C\right)$$

Solving a First-Order Linear DE

1. Put the equation into standard form.

2. Identify $P(x)$ to find the integrating factor $\mathrm{e}^{\int P(x)\mathrm{d}x}$. Evaluate the integral, forgoing the integration constant.

3. Multiply both sides by the integrating factor:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left[y\mathrm{e}^{\int P(x)\mathrm{d}x}\right] = \mathrm{e}^{\int P(x)\mathrm{d}x}f(x)$$

4. Integrate both sides and solve for y.

General Solutions

Suppose P and f, found in the standard form of a first-order linear DE, are continuous on I. *If* a solution of the DE exists on I, it must be of the form found via the integrating factor. Conversely, any function in this form is a solution on I. In other words, the family of solutions found via the integrating factor contains *every* solution defined on I. This family is therefore referred to as the **general solution** on I.

A **transient term** in a solution os one that becomes negligible for increasing values of the dependent variable.

Piecewise-Linear Differential Equation

When $P(x)$ or $f(x)$ are piecewise functions, the equation is referred to as a **piecewise-linear differential equation**.

Error Function

Many important functions are defined in terms of nonelementary integrals. Two such functions are the **error** and **complementary error functions**, denoted erf and erfc respectively and defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t \qquad\qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t$$

It is known that the definite integral from 0 to ∞ of e^{-t^2} is equal to $\sqrt{2\pi}/2$, so

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t = 1$$

Using the additive property of definite integrals, this can be rewritten as

$$\frac{2}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t + \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t = 1$$

It is then apparent why that

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$$

When the solution of an IVP involves a nonelementary integral, it is often beneficial to use a *definite* integral from x_0 to x.

2.4 Exact Equations

Differential of a Function of Two Variables

Recall that for a function $z = f(x, y)$ with continuous partial derivatives in region R, its **differential** is defined as

$$\mathrm{d}z = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$$

If z is constant,

$$\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = 0$$

So given a one parameter family of solutions $f(x, y) = c$, f first-order DE can be generated by computing the differential of both sides of the equality.

A Definition

Exact Equation A differential expression $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y$ is an **exact differential** in region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$ on R. A first-order DE of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

is said to be an **exact equation** if the expression on the left is an exact differential.

Theorem 2.4.1 Criterion for an Exact Differential If $M(x,y)$ and $N(x,y)$ are continuous and first-differentiable in a rectangular region R, then $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y$ is said to be an exact differential if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Method of Solution

If the differential is exact, there exists a function f for which

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x,y)$$

Integration and partial differentiation can be used to find this function.

$$f(x,y) = \int M(x,y)\,\mathrm{d}x + g(y)$$
$$\frac{\partial f}{\partial y} = N(x,y) = \frac{\partial}{\partial y}\int M(x,y)\,\mathrm{d}x + g'(y)$$

2.5 Solutions by Substitutions

Substitutions

Solving a DE often involves first transforming it into another DE via **substitution**.
To transform the first-order DE $\mathrm{d}y/\mathrm{d}x = f(x,y)$ using the substitution $y = g(x,u)$, where u is regarded as a function of x, the chain rule can be used (so long as g has first-partial derivatives)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial g}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{\partial g}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g_x(x,u) + g_u(x,u)\frac{\mathrm{d}u}{\mathrm{d}x}$$

The DE then becomes

$$g_x(x,u) + g_u(x,u)\frac{\mathrm{d}u}{\mathrm{d}x} = f(x, g(x,u))$$

Solving for $\mathrm{d}u/\mathrm{d}x$ yields the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = F(x,u)$$

If a solution $u = \varphi(x)$ can be found, then a solution of the original DE is

$$y = g(x, \varphi(x))$$

Homogenous Equations

A function f is said to be a **homogenous function** of degree α if $f(tx, ty) = t^\alpha f(x, y)$ for some real number α.
A first-order DE in differential form

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

is **homogenous** if both M and N are homogenous functions of the *same* degree; that is, if

$$M(tx, ty) = t^\alpha M(x, y) \qquad \text{and} \qquad N(tx, ty) = t^\alpha N(x, y)$$

If this is true, it can also be written that

$$M(x, y) = x^\alpha M(1, u) \qquad \text{and} \qquad N(x, y) = x^\alpha N(1, u) \qquad \text{where } u = \frac{y}{x}$$

and

$$M(x, y) = y^\alpha M(v, 1) \qquad \text{and} \qquad N(x, y) = y^\alpha N(v, 1) \qquad \text{where } v = \frac{x}{y}$$

Either $y = ux$ or $x = vy$ can be used to reduce a homogenous DE to a *separable* first-order De.

Bernoulli's Equation

The DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x)y^n$$

where n is any real number is called **Bernoulli's equation**. For $n \neq 0, 1$, the substitution $u = y^{1-n}$ can be used.

Reduction to Separation of Variables

A DE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(Ax + By + C)$$

can always be reduced to a separable equation by the substitution $Ax + By + C$ so long as $B \neq 0$.

Chapter 3

Modeling with First-Order Differential Equations

3.1 Linear Models

Growth and Decay

The IVP

$$\frac{\mathrm{d}x}{\mathrm{d}t} \propto x, \quad x(t_0) = x_0$$

can model growth or decay.

Half-Life

A substance's **half-life** is the amount of time that it takes for half of the atoms in the initial amount A_0 to disintegrate or transmute into those of another element. The longer a substance's half-life, the more stable it is.

Newton's Law of Cooling/Warming

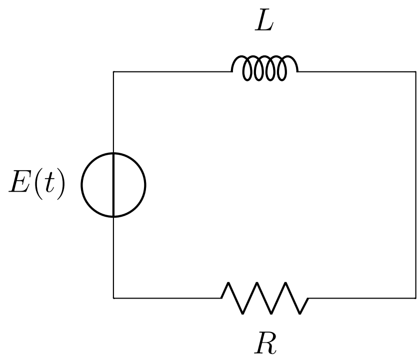
Newton's law of cooling/warming is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_m)$$

where $T(t)$ is the temperature of the object, T_m is the ambient temperature, and k is a proportionality constant.

Series Circuits

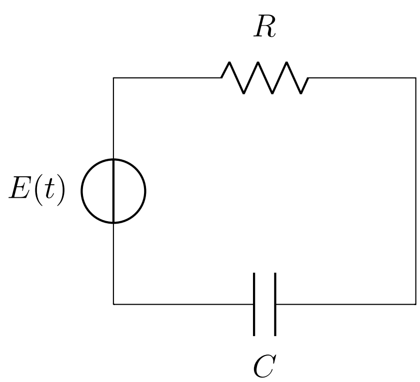
For an LR-series circuit



containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor ($L(\mathrm{d}i/\mathrm{d}t)$) and that across the resistor (iR) is equal to the impressed voltage ($E(t)$) on the circuit. This provides the first-order linear DE

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = E(t)$$

for the current $i(t)$, where L and R are the inductance and resistance respectively. The current is also called the **response** os the system.
The voltage drop across a capacitor with capacitance C is given by $q(t)/C$, q being the charge on the capacitor. For an RC-series circuit



Kirchhoff's second law then gives

$$Ri + \frac{1}{C}q = E(t)$$

As i and q are related by $i = \mathrm{d}q/\mathrm{d}t$, though, this can be rewritten as the linear DE

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$

3.2 Nonlinear Models

Population Dynamics

If $P(t)$ denotes a population's size at time t, the model for exponential growth begins by assuming that $\mathrm{d}P/\mathrm{d}t + kP$ for some $k > 0$. In this model, the **relative/specific growth rate**, as defined by

$$\frac{\mathrm{d}P/\mathrm{d}t}{P}$$

is a constant k.
As resources are finite, true exponential growth over long periods of time is essentially unheard of. The assumption that a population's growth is dependent only on the size of the population can be states as

$$\frac{\mathrm{d}P/\mathrm{d}t}{P} = f(P) \qquad \text{or} \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = Pf(P)$$

This DE is called the **density-dependent hypothesis**.

Logistic Equation

Suppose that the maximum number of individuals sustainable in a population is K. This quantity K is called the environment's **carrying capacity**. For the DE, then,

$$f(K) = 0 \qquad \text{and} \qquad f(0) = r$$

The simplest assumption to satisfy these conditions is that $f(P)$ is linear:

$$f(P) = C_1 P + C_2$$

Using the conditions, we find that $c_2 = r$ and $c_1 = -r/K$, so

$$f(P) = r - \frac{r}{K}P$$

The DE then becomes

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P\Big(r - \frac{r}{K}P\Big)$$

Relabeling the constants,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(a - bP)$$

Solution of the Logistic Equation

The logistic model can be solved via separation of variables:

$$a$$

$$\begin{aligned}
\left(\frac{1/a}{P} + \frac{b/a}{a - bP}\right)\mathrm{d}P &= \mathrm{d}t \\
\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a - bP| &= t + C \\
\ln\left|\frac{P}{a - bP}\right| &= at + aC \\
\frac{P}{a - bP} &= C_1 \mathrm{e}^{at}
\end{aligned}$$

It then follows that

$$P(t) = \frac{aC_1\mathrm{e}^{at}}{1 + bC_1\mathrm{e}^{at}} = \frac{aC_1}{bC_1 + \mathrm{e}^{-at}}$$

If $P(0) = P_0 \neq a/b$, then $C_1 = P_0/(a - bP_0)$, so

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)\mathrm{e}^{-at}}$$

Chemical Reactions

Suppose a grams of chemical A and b grams of chemical B are combined. If there are M parts of A and N parts of B formed in the compound, then the number of grams of chemicals A and B remaining at time t are respectively

$$a - \frac{M}{M+N}X \qquad \text{and} \qquad b - \frac{N}{M+N}X$$

where $X(t)$ is the number of grams of chemical C formed.
The law of mass action states that when temperature is constant, the rate at which two substances react is proportional to the amounts of each that are untransformed at time t:

$$\frac{\mathrm{d}X}{\mathrm{d}t} \propto \left(a - \frac{M}{M+N}X\right)\left(b - \frac{N}{M+N}X\right)$$

Factoring out $M/(M+N)$ from the first factor and $N/(M+N)$ from the second and introducing constant of proportionality $k > 0$, this can be rewritten as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = k(\alpha - X)(\beta - X)$$

where

$$\alpha = a\frac{M+N}{M} \qquad \text{and} \qquad \beta = b\frac{M+N}{N}$$

A chemical reaction governed by this nonlinear De is said to be a **second-order reaction**.

3.3 Modeling with Systems of First-Order Differential Equations

Linear/Nonlinear Systems

A system of two related first-order DEs may be

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g_1(t, x, y) \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g_2(t, x, y)$$

When g_1 and g_2 are linear in x and y, being of the forms

$$g_1(t, x, y) = c_1x + c_2y + f_1(t) \qquad\qquad g_2(t, x, y) = c_3x + c_4y + f_2(t)$$

where the coefficients c_i may depend on t, the system is said to be **linear**. It is otherwise **nonlinear**.

Radioactive Series

When a substance decays via radioactivity, it generally doesn't simply transmute in a single step into a stable substance; a **radioactive decay series** is the process of continuous decaying into gradually more stable elements.
Schematically, a radioactive series may be described as

$$X \xrightarrow{-\lambda_1} Y \xrightarrow{-\lambda_2} Z$$

where

$$k_1 = -\lambda_1 < 0 \qquad \text{and} \qquad k_2 = -\lambda_2 < 0$$

are the decay constants of substances X and Y respectively and Z is a stable element. Suppose as well that $x(t)$, $y(t)$, and $z(t)$ denote the amounts os substances X, Y, and Z remaining at time t. The decay of X is described by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\lambda_1 x$$

while that of Y is the net rate

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda_1 x - \lambda_2 y$$

as Y is *gaining* atoms from the decay of X while *losing* them due to its own decay.
As Z is a stable element, its aggregation is simply from the decay of Y:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \lambda_2 y$$

These three first-order DEs compose a linear system that models the radioactive decay series of 3 elements.

A Predator-Prey Model

Suppose that two species interact within the same ecosystem and that one eats only vegetation while the other preys only on the former; the latter is the predator, the former the prey. Let $x(t)$ denote the predator population and $y(t)$ denote the prey population. When there are no prey, the decline in predators would decline corresponding to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax, \quad a > 0$$

When there are prey in the environment it seems reasonable that the number of interactions between the species would be jointly proportional to their populations. So when prey are present, the predator population would increase at rate $bxy, b > 0$. Adding this to the last equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax + bxy$$

If there are no predators, the prey thrive, growing proportionally to the population:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = dy, \quad d > 0$$

When predators are present, though, the prey population would decline at a rate modeled by cxy, decreasing by the rate at which they are eaten in their encounters:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = dy - cxy$$

These formula constitute the **Lotka-Volterra predator-prey model**:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax + bxy = x(-a + by) \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = dy - cxy = y(d - cx)$$

where a, b, c, and d are positive constant.
Apart from two constant solutions, $x(t) = 0$, $y(t) = 0$ and $x(t) = d/c$, $y(t) = a/b$, this system cannot be solved in terms of elementary functions.

Competition Models

Suppose two different species occupy the same ecosystem, competing for resources. In isolation, assume that the rate at which each population grows is respectively

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cy$$

As they are in direct competition, it may be assumed that each rate is diminished by the mere existence of the other population, providing the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - by \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cy - dx$$

where a, b, c, and d are positive constants.
If the growth rate is instead affected proportionally to the number of interactions, the resulting system would be

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cy - dxy$$

which is quite similar to the Lotka-Volterra predator-prey model.
It may be more realistic to replace the isolated rates with logistic models rather than exponential ones, making them

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_1x - b_1x^2 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = a_2y - b_2y^2$$

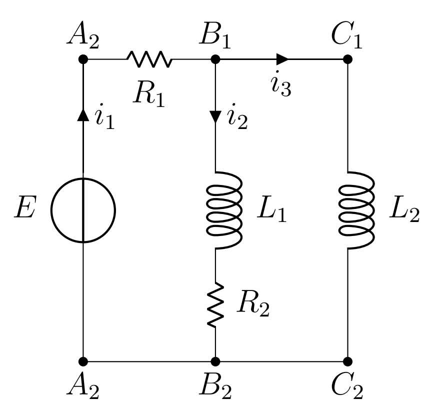
which results in another nonlinear model

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= a_1x - b_1x^2 - c_1xy = x(a_1 - b_1x - c_1y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= a_2y - b_2y^2 - c_2xy = y(a_2 - b_2 - c_2x) \end{aligned}$$

in which all coefficients are positive.
All of these models are called **competition models**.

Networks

An electrical network with more than one loop such as



results in simultaneous DEs. In this case, the current $i_1(t)$ splits in two at one of the network's *branch point* B_1. **Kirchhoff's first law** shows that

$$i_1(t) = i_2(t) + i_3(t)$$

Kirchhoff's second law can also be applied to each loop. Summing the voltage drops across each part of loop $A_1B_1B_2A_2A_2A_1$ gives

$$E(t) = i_1R_1 + L_1\frac{\mathrm{d}i_2}{\mathrm{d}t} + i_2R_2$$

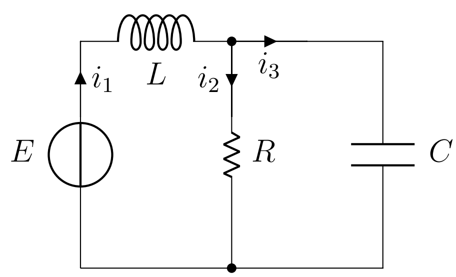
Doing the same for loop $A_1B_1C_1C_2B_2A_2A_1$ results in

$$E(t) = i_1R_1 + L_2\frac{\mathrm{d}i_3}{\mathrm{d}t}$$

Using the first derived equation to eliminate i_1 in the sums yields a linear system of two first order equations:

$$\begin{aligned} L_1\frac{\mathrm{d}i_2}{\mathrm{d}t} + (R_1+R_2)i_2 + R_1i_3 &= E(t) \\ L_3\frac{\mathrm{d}i_3}{\mathrm{d}t} + R_1i_2 + R_1I_3 &= E(T) \end{aligned}$$

Similarly,



results in the system

$$\begin{aligned} L\frac{\mathrm{d}i_1}{\mathrm{d}t} + Ri_2 &= E(t) \\ RC\frac{\mathrm{d}i_2}{\mathrm{d}t} + i_2 - i_1 &= 0 \end{aligned}$$

Chapter 4

Higher-Order Differential Equations

4.1 Preliminary Theory—Linear Equations

4.1.1 Initial-Value and Boundary-Value Problems

Initial-Value Problem

For a linear DE, an n^{th}**-order initial-value problem (IVP)** is

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = g(x)$$
$$\text{subject to} \quad y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

Existence and Uniqueness

Theorem 4.1.1 Existence of a Unique Solution Let $a_n(x)$, $a_{n-1}(x)$, ..., $a_1(x)$, $a_0(x)$, and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in the interval. If $x = x_0$ at any point within I, then a solution $y(x)$ of the IVP both exists on the interval and is unique.

Boundary-Value Problem

A linear DE of order two or greater in which the dependent variable or its derivatives are specified at *different points*, such as

$$a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(q) \qquad \text{subject to} \qquad y(a) = y_0, y(b) = y_1$$

is called a **boundary-value problem (BVP)**. The specified values are called **boundary conditions (BC)**. A solution of the above problem is a function that satisfies the DE on some interval I containing both a and b that pases through the points a, y_0 and b, y_2.

4.1.2 Homogenous Equations

A linear n^{th}-order DE of the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0$$

is said to be **homogenous** while one of the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

where $g(x)$ is not identically 0 is said to be **nonhomogenous**.
It should be noted that word *homogenous* as used here does not refer to coefficients that are homogenous functions.

When stating definitions or theorems regarding linear equations, it shall always be assumed that on some common interval I, the coefficient functions $a_i(x)$ and $g(x)$ are continuous and that $a_n(x) \neq 0$ for every x in the interval.

Differential Operators

The symbol D is called a **differential operator**, as it transforms a differentiable function into another function. In general,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = D^n y$$

where y is a sufficiently differentiable function. Polynomial expressions that involve D are also differential operators. In general, an $\boldsymbol{n}^{\mathbf{th}}$**-order differential/polynomial operator** to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

As $D(c(f(x)) = cDf(x)$ (where c is a constant) and $D\{f(x) + g(x)\} = Df(x) + Dg(x)$, the differential operator L also posses the property that acting on a linear combination of differentiable functions is the same as the linear combination of L operating on the individual functions. Symbolically,

$$L\{\alpha f(x) + \beta g(x)\} = \alpha L(f(x)) + \beta(g(x))$$

where α and β are constant. Because of this property, L can be said to be a **linear operator**.

Differential Equations

Any linear DE can be expressed in terms of D notation.

Superposition Principle

Theorem 4.1.2 Superposition Principle—Homogenous Equations Let $y_1, y_2, \ldots, y_k$ be solutions of the homogenous n^{th}-order DE $L(y) = 0$ on an interval I. The linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

where the c_i are arbitrary constants, is also a solution of the DE on I.

Linear Dependence and Linear Independence

Linear Dependence/Independence A set of functions $f_1(x), f_x(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants c_i (that are not all 0) such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. If a set of functions are not linearly dependent on an interval, they are **linearly independent**.

If a set of two functions are linearly dependent, then they are constant multiples of each other.
A set of functions is linearly dependent on a interval if at least one of the functions can be expressed as a linear combination of the others.

Solutions of Differential Equations

Wronskian Let each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is the **Wronskian** of the functions.

Theorem 4.1.3 Criterion for Linearly Independent Solutions Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogenous linear n^{th}-order DE $L(y) = 0$ on interval I. The set of solutions is **linearly independent** on I if an only if $W(y_1, y_2, \ldots, y_n) \not\equiv 0$ for every x in the interval.

Fundamental Set of Solutions Any set of n linearly independent solutions of the homogenous linear n^{th}-order linear DE $L(y) = 0$ on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem 4.1.4 Existence of a Fundamental Set A fundamental set of solutions for the homogenous linear n^{th}-order DE $L(y) = 0$ exists on any interval I.

General Solution—Homogenous Equations Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogenous linear n^{th}-order DE $L(y) = 0$ on an interval I. The **general solution** of the equation on the interval is given by

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

where c_i are arbitrary constants.

4.1.3 Nonhomogenous Equations

Any function y_p with no parameters that satisfies a nonhomogenous equation is said to be a **particular solution**. If $y_{1\cdots k}$ corresponds to solutions of a homogenous equation on interval I and y_p is a particular solution of a nonhomgenous equation on the same interval, the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y(k) + y_p(x)$$

Theorem 4.1.6 General Solution—Nonhomogenous Equation Let y_p be any particular solution of the nonhomogenous linear n^{th}-order DE $L(y) = g(x)$ on interval I and let $y_{1\cdots}$ be a fundamental set of solutions of the corresponding homogenous DE $L(y) = 0$ on the same interval. The **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where $c_{i\cdots n}$ are arbitrary constants.

Complementary Functions

The general solution of a nonhomegenous DE can be rewritten as the sum of two functions

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x) = y_c(x) + y_p(x)$$

$y_c(x)$, which is a general solution of the corresponding homogenous DE, is called the **complementary function** of the nonhomogenous DE. The general solution of a nonhomgenous DE can then be rewritten as

$$y = \text{complementary function} + \text{any particular solution} = y_c + y_p$$

Another Superposition Principle

Superposition Principle – Nonhomogenous Equations Let $y_{p,i\cdots k}$ be k particular solutions to a nonhomogenous linear n^{th}-order DE of the form $L(y) - g(x)$ on an interval I that correspond to k distinct functions $g_{1\cdots k}$; that is, suppose y_p, i denotes a particular solution to the DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x)$$

Then

$$y_p(x) = y_{p,1}(x) + y_{p,2}(x) + \cdots + y_{i,k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

4.2 Reduction of Order

Reduction of Order

Suppose that y_1 denotes a nontrivial solution of the second-order homogenous linear DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

that is defined on I. If a second solution y_2 is linearly independent on I, then their quotient y_2/y_1 is nonconstant on I. Knowing this, a function $u(x)$ can be found by substituting $y_2(x) = u(x)y_1(x)$ into the DE. This method is called **reduction of order**, as a linear first-order DE must be solved to find u.

General Case

Putting the second-order homogenous linear DE into **standard form**,

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x)$ and $Q(x)$ are continuous on I. Let y_1 be a known solution to this DE on I that is never equal to 0 on the interval. Defining $y = u(x)y_1(x)$,

$$\begin{aligned} y' &= u'y_1 + y_1'u \\ y'' &= u''y_1 + y_1'u' + y_1''u + u'y_1' = uy_1'' + 2u'y_1' + u''y_1 \\ y'' + Py' + Qy &= u(y_1'' + Py_1' + Qy_1) + y_1u'' + (2y' + Py_1)u' \\ 0 &= y_1u'' + (2y_1' + Py_1)u' \end{aligned}$$

Letting $w = u'$,

$$0 = y_1 w' + (2y' + Py_1)w$$

This equation is both linear and separable.

$$\begin{aligned} \frac{\mathrm{d}w}{w} + 2\frac{y_1'}{y_1}\,\mathrm{d}x + P\,\mathrm{d}x = 0 \\ \ln|wy_1^2| = -\int P\,\mathrm{d}x \\ wy_1^2 = C\mathrm{e}^{-\int P\mathrm{d}x} \end{aligned}$$

Substituting u back in,

$$u = C\int \frac{\mathrm{e}^{-\int P\mathrm{d}x}}{y_1^2}\,\mathrm{d}x$$

Letting $C = 1$ and knowing that $y(x) = u(x)y_1(x)$,

$$y_2(x) = y_1(x)\int \frac{\mathrm{e}^{-\int P(x)\mathrm{d}x}}{y_1^2(x)}\,\mathrm{d}x$$

In the case that this integral is nonelementary, $y_2(x)$ can be defined as an integral function:

$$y_2(x) = y_1(x)\int_{x_0}^{x} \frac{\mathrm{e}^{-\int P(t)\mathrm{d}t}}{y_1^2(t)}\,\mathrm{d}t$$

4.3 Homogenous Linear Equations with Constant Coefficients

Substituting $y = \mathrm{e}^{mx}$ into the first-order homogenous DE

$$ay' + by = 0$$

yields

$$am\mathrm{e}^{mx} + b\mathrm{e}^{mx} = \mathrm{e}^{mx}(am + b) = 0$$

As $\mathrm{e}^{x} = 0$ has no real solutions, this is only satisfied when $am + b = 0$. For this value of m, $y = \mathrm{e}^{mx}$ is a solution of the DE.

Auxiliary Equation

Consider the second-order equation

$$ay'' + by' + cy = 0$$

where a, b, and c are constants. Substituting $y = \mathrm{e}^{mx}$, this becomes

$$am^2\mathrm{e}^{mx} + bm\mathrm{e}^{mx} + c\mathrm{e}^{mx} = \mathrm{e}^{mx}(am^2 + bm + c) = 0$$

$\mathrm{e}^{x} = 0$ has no real solutions,

$$am^2 + bm + c = 0$$

This is called the **auxiliary equation** of the DE. Using the quadratic formula to solve for m,

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $b^2 - 4ac$ can be used to determine the number of real solutions:

$b^2 - 4ac$	> 0	0	< 0
real solutions	2	1	1

Case I: Distinct Real Roots

If the auxiliary equation has two distinct real roots m_1 and m_2, two solutions $y_1 = \mathrm{e}^{m_1 x}$ and $y_2 = \mathrm{e}^{m_2 x}$ are found. These two functions are linearly independent on $\mathbb{R}$, making the general solution on the interval

$$y = C_1 \mathrm{e}^{m_1 x} + C_2 \mathrm{e}^{m_2 x}$$

Case II: Repeated Real Roots

When only a single real root m_1 of the auxiliary equation exists, only a single exponential solution $y_1 = \mathrm{e}^{m_1 x}$ is found. As $m_1 = m_2$ only if $b^2 - 4ac = 0$, $m_1 = -b/2a$. It then follows from reduction of order that a second solution is

$$y_2 = \mathrm{e}^{m_1 x} \int \frac{\mathrm{e}^{-\int -\frac{b}{a} \mathrm{d}x}}{\mathrm{e}^{2mx}} \,\mathrm{d}x = \mathrm{e}^{mx} \int \frac{\mathrm{e}^{\int 2m \mathrm{d}x}}{\mathrm{e}^{2mx}} \,\mathrm{d}x = \mathrm{e}^{mx} \int \frac{\mathrm{e}^{2mx}}{\mathrm{e}^{2mx}} \,\mathrm{d}x = \mathrm{e}^{mx} \int \mathrm{d}x = x\mathrm{e}^{mx}$$

The general solution is then

$$y = C_1 \mathrm{e}^{mx} + C_2 x \mathrm{e}^{mx}$$

Case III: Complex Conjugate Roots

If m_1 and m_2 are both complex, it can be written that

$$m_1 = \alpha + i\beta \qquad \text{and} \qquad m_2 = \alpha - i\beta$$

where $\alpha, \beta \in \mathbb{R}^+$ ($\mathbb{R}^+$ being the set of positive real numbers).
This is formally identical to having two distinct real roots, so the general solution can therefore be written as

$$y = C_1 \mathrm{e}^{(\alpha + i\beta)x} + C_2 \mathrm{e}^{(\alpha - i\beta)x}$$

In practice, though, real functions are preferred to complex exponentials. As such, **Euler's formula**

$$\mathrm{e}^{i\theta} = \cos\theta + i \sin\theta$$

(where θ is any real number) is used. It then follows that

$$\mathrm{e}^{i\beta x} = \cos\beta x + i \sin\beta x \qquad \text{and} \qquad \mathrm{e}^{-i\beta x} = \cos\beta x - i \sin\beta x$$

The addition and subtraction of these two formulas results in

$$\mathrm{e}^{i\beta x} + \mathrm{e}^{-i\beta x} = 2\cos\beta x \qquad \text{and} \qquad \mathrm{e}^{i\beta x} - \mathrm{e}^{-i\beta x} = 2i \sin\beta x$$

Letting $C_1 = C_2 = 1$ results in

$$y_1 = \mathrm{e}^{(\alpha+i\beta)x} + \mathrm{e}^{(\alpha-i\beta)x}$$

while letting $C_1 = 1$ and $C_2 = -1$ gives

$$y_2 = \mathrm{e}^{(\alpha+i\beta)x} - \mathrm{e}^{(\alpha-i\beta)x}$$

Refactoring,

$$y_1 = \mathrm{e}^{\alpha x}\left(\mathrm{e}^{i\beta x} + \mathrm{e}^{-i\beta x}\right) = 2\mathrm{e}^{\alpha x}\cos\beta x \qquad \text{and} \qquad y_2 = \mathrm{e}^{\alpha x}\left(\mathrm{e}^{i\beta x} - \mathrm{e}^{-\beta x}\right) = 2i\mathrm{e}^{\alpha x}\sin\beta x$$

As 2 and i are constants, $\mathrm{e}^{\alpha x}\cos\beta x$ and $\mathrm{e}^{\alpha x}\sin\beta x$ are *real* solutions. These two solutions form a fundamental set on $\mathbb{R}$. The general solution is consequently

$$y = C_1\mathrm{e}^{\alpha x}\cos\beta x + C_2\mathrm{e}^{\alpha x}\sin\beta x = \mathrm{e}^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$$

Two Equations Worth Knowing

The DEs

$$y'' \pm k^2 y = 0$$

where $k \in \mathbb{R}^+$ are quite useful in applied mathematics.
The auxiliary equation of $y'' + k^2 y = 0$ is

$$m^2 + k^2 = 0$$

which has imaginary roots $m_1 = ki$ and $m_2 = -ki$. With $\alpha = 0$ and $\beta = k$, the general solution can be seen to be

$$y = C_1\cos(kx) + C_2\sin(kx)$$

The auxiliary equation of $y'' - k^2 y = 0$ is

$$m^2 - k^2 = 0$$

which has real roots $m_1 = k$ and $m_2 = -k$, making the general solution of the DE

$$y = C_1\mathrm{e}^{kx} + C_2\mathrm{e}^{kx}$$

Letting $C_1 = C_2 = 1/2$, and $C_1 = 1/2$ and $C_2 = -1/2$ yields the particular solutions

$$y_1 = \frac{1}{2}\left(\mathrm{e}^{kx} + \mathrm{e}^{-kx}\right) = \cosh(kx) \qquad \text{and} \qquad y_2 = \frac{1}{2}\left(\mathrm{e}^{kx} - \mathrm{e}^{-kx}\right) = \sinh(kx)$$

meaning that another form of the general solution is

$$y = C_1\cosh(kx) + C_2\sinh(kx)$$

Higher-Order Equations

In general, in order to solve an n^{th}-order homogenous DE $L(y) = 0$, the n^{th} degree polynomial

$$a_n m^n + a_{n-1}m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0$$

must be solved. If all roots of this equation are distinct, the general solution is

$$y = C_1\mathrm{e}^{m_1 x} + C_2\mathrm{e}^{m_2} + \cdots + C_n\mathrm{e}^{m_n x}$$

4.4 Undetermined Coefficients—Superposition Approach

To solve a nonhomogenous DE $L(y) = g(x)$, the complementary function y_c and *any* particular solution must be found. The general solution is then $y = y_c + y_p$.
The complementary function y_c is the general solution of the associated homogenous DE $L(y) = 0$.

Method of Undetermined Coefficients

The **method of undetermined coefficients** is a technique of obtaining a particular solution y_p of a nonhomogenous DE. This method is limited to linear DEs with constant coefficients and $g(x)$ comprised of finite products and sums of constants, polynomials, exponentials, or sines or cosines. The reason for this limitation is that these types of functions have the property that the derivatives of their sums or products are comprised of sums or products of themselves. Because the linear combination of derivatives y_p must be identical to $g(x)$ it is reasonable to assume that y_p must also *have the same form* as $g(x)$.

Trial Particular Solutions

$g(x)$	Form of y_p
k	A
$ax^n + bx^{n-1} + \cdots$	$Ax^n + Bx^{n-1} + \cdots$
$\sin(\alpha x)$ $\cos(\alpha x)$	$A\cos(\alpha x) + B\sin(\alpha x)$
e^{kx}	$A\mathrm{e}^{kx}$

The form of y_p when $g(x)$ is the product of two functions is that of one function with the second function's substituted in for the constants. If $g(x) = (ax + b)\sin x$, for example, then $y_p = (Ax + B)\cos x + (Cx + E)\sin x$. When $g(x)$ is a sum, the form is simply the sum of the forms of the individual terms of $g(x)$.

Case I

Form Rule for Case I *The form of y_p is a linear combination of all linearly independent functions generated by repeated differentiations of $g(x)$.*

Case II

Multiplication Rule for Case II *If any $y_{p,i}$ contains duplicate terms from y_c, then that $y_{p,i}$* must be multiplied by x^n where n is the smallest positive integer that eliminates the duplication.

4.5 Undetermined Coefficients—Annihilator Approach

An n^{th}-order DE can be written

$$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 Dy + a_0 y = g(x)$$

where $D^k = \mathrm{d}^k y/\mathrm{d}x^k$ for $k = 0, \ldots n$. This can also be written $L(y) = g(x)$, where L denotes the n^{th}-order differential operator

$$L(y) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

This operator notation is not only useful as shorthand, but also serves to justify the rules for determining the form of y_p.

Factoring Operators

When the coefficients a_i are real constants, a linear differential operator L can be factored if the characteristic polynomial

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0$$

can be factored; that is to say, if r_1 is a root of the auxiliary equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

then $L = (D - r_1)P(D)$, the polynomial expression $P(D)$ being a linear differential operator of order $n - 1$.

Factors of a linear differential operator with constant coefficients commute.

Annihilator Operator

If L is a linear differential operator with constant coefficients and f is a sufficiently differential function such that $L(f(x)) = 0$, then L is said to be an **annihilator** of f.
The functions annihilated by L are simply those that can be obtained from the general solution of the homogenous DE $L(y) = 0$.
A polynomial can be annihilated by an operator that annihilates the highest power x.
The differential operator $(D - \alpha)^n$ annihilates any $x^k \mathrm{e}^{\alpha x}$ for k in $0, \ldots, n - 1$.
The differential opeartor $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^n$ annihilates $x^k \mathrm{e}^{\alpha x} \cos \beta x$ and $x^k \mathrm{e}^{\alpha x} \sin \beta x$ for k in $0, \ldots, n - 1$.
It should be noted that multiple differential operators may be able to annihilate a function, so when finding one, that of the *lowest possible order* should be sought.

Undetermined Coefficients

Let $L(y) = g(x)$ be a linear DE with constant coefficients such that $g(x)$ meets the condition for undetermined coefficients to be used, being that it is a linear combination of functions of the forms

$$k, \quad x^m, \quad x^m \mathrm{e}^{\alpha x}, \quad x^m \mathrm{e}^{\alpha x} \cos \beta x, \quad \text{and} \quad x^m \mathrm{e}^{\alpha x} \sin \beta x$$

where $m \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. Such a function can be annihilated by a differential operator L_1 of lowest order, which consists of a product of

$$D^n, \quad (D - \alpha)^n \quad \text{and} \quad (D - 2\alpha D + \alpha^2 + \beta^2)^n$$

Applying L_1 to both sides of the DE yields

$$L_1 L(y) = L_1(g(x)) = 0$$

By solving the *homogenous higher-order* equation $L_1 L(y) = 0$, the *form* of y_p can be found. Substituting this into $L(y) = g(x)$, an explicit particular solution can be found. This procedure is called the **method of undetermined coefficients**.

Summary of the Method

Undetermined Coefficients—Annihilator Approach The DE $L(y) = g(x)$ has constant coefficients and $g(x)$ is comprised of the finite sums and products of constants, polynomials, exponentials, sines, and cosines.

1. Find the complementary solution y_c to the homogenous equation $L(y) = 0$.
2. Operate on both sides of the nonhomogenous equation $L(y) = g(x)$ using differential operator L_1 that annihilates $g(x)$.
3. Find the general solution of the higher-order homogenous DE $L_1 L(y) = 0$.
4. Delete all terms that are duplicated in y_c. Form a linear combination of those that remain. This is the form of a particular solution of $L(y) = g(x)$.
5. Substitute y_p into $L(y) = g(x)$. Match coefficients of the various functions on each side of the equality and solve the resulting system of equations for the unknown coefficients of y_p.
6. Form the general solution $y = y_c + y_p$.

4.6 Variation of Parameters

Linear First-Order DEs Revisited

The general solution of the linear first-order DE

$$a_1(x)y' + a_0(x)y = g(x)$$

is found by first writing it in standard form as

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x)$$

If $P(x)$ and $f(x)$ are continuous on common interval I, the integrating factor can be used to identify the solution as

$$y = C\mathrm{e}^{-\int P(x)\mathrm{d}x} + \mathrm{e}^{-\int P(x)\mathrm{d}x} \int \mathrm{e}^{-\int P(x)\mathrm{d}x} f(x)\,\mathrm{d}x$$

This is of the form $y_c + y_p$, where

$$y_p = C\mathrm{e}^{-\int P(x)\mathrm{d}x} \qquad \text{and} \qquad y_c = \mathrm{e}^{-\int P(x)\mathrm{d}x} \int \mathrm{e}^{-\int P(x)\mathrm{d}x} f(x)\,\mathrm{d}x$$

This particular solution can be derived via **variation of parameters**.
Suppose that y_1 is a particular solution of a homogenous first-order linear DE; that is,

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} + P(x)y_1 = 0$$

The solution

$$y = \mathrm{e}^{-\int P(x)\mathrm{d}x}$$

is also known, and as the equation is linear, $C_1y_1(x)$ is its general solution. Variation of parameters consists of finding a particular solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = 0$$

that is of the form

$$y_p = u_1(x)g_1(x)$$

The *parameter* C_1 has been replaced by the *function* u_1.
Substituting $y_p = u_1g_1$ into the equation,

$$\begin{aligned} f(x) &= \frac{\mathrm{d}y}{\mathrm{d}x}[u_1y_1] + P(x)u_1y_1 \\ &= u_1\frac{\mathrm{d}y_1}{\mathrm{d}x} + y_1\frac{\mathrm{d}u_1}{\mathrm{d}x} + P(x)u_1y_1 \\ &= u_1\overbrace{\left(\frac{\mathrm{d}y_1}{\mathrm{d}x} + P(x)y_1\right)}^{0} + y_1\frac{\mathrm{d}u_1}{\mathrm{d}x} \\ &= y_1\frac{\mathrm{d}u_1}{\mathrm{d}x} \end{aligned}$$

Separating variables and integrating,

$$\begin{aligned} \mathrm{d}u_1 &= \frac{f(x)}{y_1(x)}\,\mathrm{d}x \\ u_1 &= \int \frac{f(x)}{y_1(x)}\,\mathrm{d}x \end{aligned}$$

The particular solution is therefore

$$y = u_1y_1 = y_1\int \frac{f(x)}{y_1(x)}\,\mathrm{d}x$$

Linear Second-Order DEs

Consider the linear second-order DE

$$a_2(x)y'' + a_1(x)y' + a_0y = g(x)$$

In standard form,

$$y'' + P(x)y' + Q(x)y = f(x)$$

Suppose that $P(x)$, $Q(x)$, and $f(x)$ are continuous on common interval I. The complementary solution

$$y_c = C_1y(x) + C_2y(x)$$

Replacing the parameters with functions,

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Using product rule,

$$\begin{aligned}
y_p' &= u_1'y_1 + y_1'u_1 + u_2'y_2 + y_2'u_2 \\
y_p'' &= u_1''y_1 + y_1'u_1' + y_1''u_1 + u_1'y_1' + u_2''y_2 + y_2'u_2 + y_2''u_2 + u_2'y_2'
\end{aligned}$$

Substituting,

$$\begin{aligned}
y_p'' + P(x)y_p' + Q(x)y_p &= u_1\overbrace{(y_1'' + Py_1' + Qy_1)}^{0} + u_2(y) + u_2\overbrace{(y_2'' + Py_2' + Qy_2)}^{0} \\
&\quad + y_1u_1'' + u_1'y_1' + y_2u_2'' + u_2'y_2' + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' \\
&= \frac{\mathrm{d}}{\mathrm{d}x}[y_1u_1'] + \frac{\mathrm{d}}{\mathrm{d}x}[y_2u_2'] + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' \\
&= \frac{\mathrm{d}}{\mathrm{d}x}[y_1u_1' + y_2u_2'] + P(y_1u_1' + y_2u_2') + y_1u_1' + y_2u_2' = f(x)
\end{aligned}$$

As there are two unknowns, two equations are needed. These are obtained by further assuming that

$$y_1u_1' + y_2u_2' = 0$$

as this results in DE reducing to

$$y_1u_1 + y_2u_2 = f(x)$$

This provides the desired two equations. By Cramer's Rule, the solution of this system can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2f(x)}{W} \qquad \text{and} \qquad u_2' = \frac{W_2}{W} = \frac{y_1f(x)}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad \text{and} \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

The functions u_1 and u_2 can then be found simply by integrating the results.
The determinant W is the Wronskian of y_1 and y_2. As they are linearly independent, $W(y_1(x), y_2(x))$ is never 0 for any x in I.

Summary of the Method

Due to the long-winded nature of the procedure, it is more efficient here to simply memorize the formulas for first- and second-order equations.
Solving

$$a_2y'' + a_1y' + a_0y = g(x)$$

the complementary function can be found as

$$y_c = C_1y_1 + C_2y_2$$

The Wronskian ($W(y_1(x), y_2(x))$ can then be computed. Dividing by a_1, the DE can be put into standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

to determine $f(x)$. u_1 and u_2 can be as

$$u_1 = -\int \frac{y_1 f(x)}{W}\,\mathrm{d}x \qquad \text{and} \qquad u_2 = \int \frac{y_1 f(x)}{W}\,\mathrm{d}x$$

A particular solution is then

$$y_p = u_1 y_1 + u_2 y_2$$

The general solution is then

$$y = y_c + y_p = C_1 y_2 + C_2 y_2 + u_1 y_2 + u_2 y_2$$

Higher-Order Equations

The method used for second-order DEs can be generalized to those of order n in standard form

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \cdots + P_1(x) y' + P_0(x) y = f(x)$$

If

$$y_c = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

then a particular solution is

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$$

where u_k' is determined by the equations

$$\begin{array}{ccccccccc} y_1 u_1' & + & y_2 u_2' & + & \cdots & + & y_n u_n' & = & 0 \\ y_1' u_1' & + & y_2' u_2' & + & \cdots & + & y_n' u_n' & = & 0 \\ \vdots & & & & & & & & \vdots \\ y_1^{(n-1)} u_1 & + & y_2^{(n-1)} u_2' & + & \cdots & + & y_n^{(n-1)} u_n' & = & 0 \end{array}$$

The first $n-1$ equations in this system are assumptions made to simplify the resultant equation of substituting y_p into the DE. Cramer's Rule gives

$$u_k' = \frac{W_k}{W}$$

where W is the Wronskian of $y_{1\cdots n}$ and W_k is the determinant obtained by replacing the k^{th} column of the Wronskian with a column comprised of $(0, 0, \ldots, f(x))$.

4.7 Cauchy-Euler Equations

Cauchy-Euler Equation

A linear DE of the form

$$a_n x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1} x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1 x \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = g(x)$$

(where the coefficients $a_{0\cdots n}$ are constants) is a **Cauchy-Euler equation**. The defining characteristic of this type of equation is the fact that the degree of the monomial coefficients x^k match the orders k of differentiation.
It should be noted that the lead coefficient $a_n x^n$ of any Cauchy-Euler equatino is zero at $x = 0$, so to ensure the existence of a unique solution, the interval $(0, \infty)$ is generally considered.

Method of Solution

Substituting x^m into a Cauchy-Euler equation yields a polynomial in m times x^m, as

$$a_k x^k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} = a_k x^k m(m-1)(m-2)\cdots(m-k+1)x^{m-k} = a_k m(m-1)(m-2)\cdots(m-k+1)x^m$$

Substituting into a second-order equation,

$$ax^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + bx\frac{\mathrm{d}y}{\mathrm{d}x} + cy = am(m-1)x^m + bmx^m + cx^m = (am(m-1) + bm + c)x^m$$

Thus $y = x^m$ is a solution of the DE if m is a solution of the **auxiliary equation**

$$am(m-1) + bm + c = 0 \qquad \text{or} \qquad am^2 + (b-a)m + c = 0$$

Case I: Distinct Real Roots

Let m_1 and m_2 denote distinct real roots of the auxiliary equation. This implies that $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions, making the general solution

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

Case II: Repeated Real Roots

If the roots of the auxiliary equation are repeated, then only a single solution $y = x^m$ is obtained. The roots of a quadratic being zero necessitates that the discriminant be 0, meaning that

$$m = -\frac{b-a}{2a}$$

Rewriting the Cauchy-Euler Equation in standard form yields

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{b}{ax}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{ax^2}y = 0$$

meaning that

$$P(x) = \frac{b}{ax} \qquad \text{and} \qquad \int P(x)\,\mathrm{d}x = \frac{b}{a}\ln x$$

Thus

$$\begin{aligned} y_2 &= x^m \int \frac{\mathrm{e}^{-\frac{b}{a}\ln x}}{x^{2m}}\,\mathrm{d}x = x^m \int \left[x^{-b/a} \times x^{-2m}\right]\mathrm{d}x = x^m \int \left[x^{-b/a} \times x^{\frac{b-a}{a}}\right]\mathrm{d}x \\ &= x^m \int \frac{\mathrm{d}x}{x} = x^m \ln x \end{aligned}$$

The general solution is therefore

$$y = C_1 x^m + C_2 x^m \ln x$$

Case III: Conjugate Complex Roots

If the roots of the auxiliary equation are the conjugate pair $m = \alpha \pm i\beta$ (where α and $\beta > 0$ are real, then a solution is

$$y = C_1 x^{\alpha+i\beta} + C_2 x^{\alpha+i\beta}$$

The positive complex term can be rewritten as

$$x^{i\beta} = \mathrm{e}^{i\beta \ln x} = \cos(\beta \ln x) + i \sin(\beta \ln x)$$

The negative complex term can similarly be rewritten as

$$x^{-i\beta} = \cos(\beta \ln x) - i \sin(\beta \ln x)$$

Adding and subtracting these results yield

$$x^{i\beta} + x^{-i\beta} = 2\cos(\beta \ln x) \qquad \text{and} \qquad x^{i\beta} - x^{-i\beta} = 2i \sin(\beta \ln x)$$

Letting $C_1 = C_2 = 1$ gives

$$y_1 = x^\alpha \left(x^{i\beta} + x^{-i\beta}\right) = 2x^\alpha \cos(\beta \ln x)$$

while letting $C_1 = 1$ and $C_1 = -1$ yields

$$y_2 = x^\alpha \left(x^{i\beta} - x^{-i\beta}\right) = 2ix^\alpha \sin(\beta \ln x)$$

As

$$W\left(x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)\right) = \beta x^{2n-1} \not\equiv 0$$

for $\beta, x \in \mathbb{R}^+$, it can be concluded that

$$y_1 = x^\alpha \cos(\beta \ln x) \qquad \text{and} \qquad y_2 = x^\alpha \sin(\beta \ln x)$$

constitute a fundamental set of real solutions. The general solution is therefore

$$y = C_1 x^\alpha \cos(\beta \ln x) + C_2 x^\alpha \sin(\beta \ln x)$$

4.8 Green's Function

4.8.1 Initial-Value Problems

Three Initial-Value Problems

Consider the second-order IVP

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

The solution y can be expressed as the superposition of two solutions

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the solution of the associated homogenous DE with nonhomogenous initial conditions

$$y'' + P(x)y' + Q(x)y = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

and y_p is the solution of the nonhomogenous DE with homogenous (0) initial conditions

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0$$

It is assumed that at least one of the numbers y_0 or y_1 is not 0. Otherwise, the $y_h = 0$.

If the coefficients P and Q are constants, y_h can be found without issue using its auxiliary equation. Due the initial conditions being 0, y_p can describe a physical system that is initially at rest, giving it the moniker **rest solution**.

Green's Function

If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions on I of the associated homogenous form of the IVP, then a particular solution of the nonhomogenous equation on the interval of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

can be found on by variation of parameters.
The variable coefficients are defined by

$$u_1'(x) = -\frac{y_1(x)f(x)}{W}, \quad u_2'(x) = \frac{y_1(x)f(x)}{W}$$

The linear independence of y_1 and y_2 on I guarantees that $W(y_1(x), y_2(x)) \neq 0$ for all x in I. If x and x_0 are both in I, the integrating the derivatives on the interval $[x_0, x]$ and substituting the results into y_p yields

$$y_p(x) = -y_1(x)\int_{x_0}^{x} \frac{y_2(t)f(t)}{W(t)}\,\mathrm{d}t + y_2(x)\int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,\mathrm{d}t$$

As $y_1(x)$ and $y_2(x)$ are constant with respect to t,

$$y_p(x) = -\int_{x_0}^{x} \frac{y_1(x)y_2(t)}{W(t)} f(t)\,\mathrm{d}t + \int_{x_0}^{x} \frac{y_1(t)y_2(x)}{W(t)} f(t)\,\mathrm{d}t$$

These two integrals can be rewritten as the single integral

$$y_p(x) = \int_{x_0}^{x} G(x,t)f(t)\,\mathrm{d}t$$

where

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

is the **Green's function** for the DE.
Note that a Green's function is dependent only on the fundamental solutions y_1 and y_2 of the associated homogenous DE of the IVP and *not* on the forcing function f. Therefore all linear second-order DEs with the same left-hand side but with different forcing functions will have the same Green's function. An alternative title for the Green's function is the **Green's function for the second-order differential operator $\boldsymbol{D^2 + P(x)D + Q(x)}$**.

4.8.2 Boundary Value Problems

A BVP for a second-order DE specifies y and y' at two different points. Conditions for BVPs are special cases of the more general homogenous boundary conditions

$$A_1y(a) + B_1y'(a) = 0 \qquad\qquad A_2y(b) + B_2(b) = 0$$

where A_1, A_2, B_1, and B_2 are constants. The goal is to find the integral solution $y_p(x)$ for the nonhomogenous BVP of the form

$$\begin{gathered} y'' + P(x)y' + Q(x)y = f(x) \\ A_1y(a) + B_1y'(a) = 0 \\ A_2y(b) + B_2y'(b) = 0 \end{gathered}$$

It is assumed that $P(x)$, $Q(x)$, and $f(x)$ are all continuous on $[a,b]$ as well as that the corresponding homogenous BVP has only the trivial solution $y=0$. This latter assumption is sufficient to guarantee that a unique solution of the nonhomogenous BVP exists and is given by

$$y_p(x) = \int_a^b G(x,t)\,\mathrm{d}t$$

Another Green's Function

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions on $[a,b]$ of the associated of the associated homogenous BVP at x be within the interval. These can be integrated over two different intervals:

$$u_1(x) = -\int_b^x \frac{y_2(t)f(t)}{W(t)}\,\mathrm{d}t \qquad\qquad u_2(x) = \int_a^x \frac{y_1(t)f(t)}{W(t)}\,\mathrm{d}t$$

A particular solution is then

$$y_p(x) = \int_a^x \frac{y_1(x)y_2(t)}{W(t)}f(t)\,\mathrm{d}t + \int_x^b \frac{y_1(t)y_2(x)}{W(t)}f(t)\,\mathrm{d}t$$

This can be compactly written as

$$y_p(x) = \int_a^b G(x,t)f(t)\,\mathrm{d}t$$

where

$$G(x,t) = \begin{cases} \frac{y_1(t)y_2(x)}{W(t)} & a \le t \le x \\ \frac{y_1(x)y_2(t)}{W(t)} & x \le t \le b \end{cases}$$

This piecewise-defined function is called a **Green's function** for the BVP. It can be proven that this function is continuous on $[a,b]$.
If the solutions y_1 and y_2 are chosen such that at, $x=a$

$$A_1y_1(a) + B_1y_1'(a) = 0 \qquad \text{and} \qquad A_2y_2(b) + B_2y_2'(b) = 0$$

then y_p satisfies both homogenous boundary conditions.

4.9 Solving Systems of DEs by Elimination

Systematic Elimination

A linear equation

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = g(t)$$

where a_i are constants can be written as

$$\left(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0\right)y = g(t)$$

If the n^{th}-order differential operator factors into several of lower order, the factors commute

Solution of a System

A **solution** of a system of DEs is a set of sufficiently differentiable functions $x = \varphi_1(t)$, $y = \varphi_2(t)$, $z = \varphi_3(t)$, and so on, that satisfies each equation in the system on some interval I.

Method of Solution

Linear DEs can be rewritten using differential operators. Differential operators can then be used on each equation to get them to eliminate each other through addition or subtraction.

4.10 Nonlinear Differential Equations

Some Differences

Nonlinear DEs do not possess the property of superposability.
Nonlinear DEs may possess singular solutions, unlike linear DEs.

Reduction of Order

The nonlinear second-order DEs $F(x, y', y'') = 0$, where the dependent variable y is absent, and $F(y, y', y'') = 0$, where the independent variable x is missing, can be reduced to first-order equations by the substitution $u = y'$.

Dependent Variable Missing

If $u = y'$, then

$$F(x, y', y'') = 0 \implies F(x, u, u') = 0$$

If this equation is solved for u, then y can be found by means of integration. As a second-order equation is being solved, the solution will contain two constants.

Independent Variable Missing

If x is missing, then the DE must be transformed so that x becomes the dependent variable and y the independent. Letting $u = y'$,

$$y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}u}{\mathrm{d}y}$$

so

$$F(y, y', y'') = 0 \implies F\left(y, u, u\frac{\mathrm{d}u}{\mathrm{d}y}\right) = 0$$

Use of a Numerical Solver

To analyze an n^{th}-order IVP numerically, the n^{th}-order DE if first expressed as a system of n first-order equations. A second-order IVP must first be put into normal form $y'' = f(x, y, y')$. Letting $u = y'$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = u_0$$

then $y'' = u'$ and $y'(x_0) = u(x_0)$, so the IVP becomes

$$\begin{cases} y' = u \\ u' = f(x, y, u) \end{cases} \qquad \text{subject to} \qquad y(x_0) = y_0, \quad u(x_0) = y_0'$$

Chapter 5

Modeling with Higher-Order Differential Equations

5.1 Linear Models: Initial-Value Problems

5.1.1 Spring/Mass Systems: Free Undampened Motion

Hooke's Law

Suppose a rigid body of is attached to a flexible spring. The spring force F_s is proportional to the displacement s of the body from its equilibrium position and is in the direction of equilibrium; that is

$$F_s = -ks$$

Newton's Second Law

Newton's second law of motion states that

$$F_{\text{net}} = ma$$

where m is mass and a is acceleration or $\mathrm{d}^2x/\mathrm{d}t^2$.
If the mass on a spring vibrates without regard to any external forces, having **free motion**, and the mass is hanging vertically from the spring, then Newton's second law gives

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \overbrace{-(x+s)+mg}^{F_{\text{net}}} = -kx + \overbrace{mg-ks}^{0} = -kx$$

where s is the equilibrium position (where $ks = mg$) and x is the displacement from equilibrium.

DE of Free Undampened Motion

Dividing by m yields the second-order DE

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$$

where $\omega^2 = k/m$. This equation is said to describe **simple harmonic motion (SHM)** or **free undampened motion**.

Equation of Motion

The auxiliary equation of the SHM DE is

$$m^2 + \omega^2 = 0$$

so the solutions are

$$m = \pm i\omega$$

making the general solution

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

The **period** of motion $\boldsymbol{T}$ (in s), the amount of time it takes for a full oscillation to occur, is

$$T = \frac{2\pi}{\omega}$$

The **frequency** of motion $\boldsymbol{f}$ (in s^{-1} or hz) is

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

The number

$$\omega = \sqrt{\frac{k}{x}}$$

(in rad/s) and f are both sometimes referred to as the **natural frequency** of the system.
The equation derived by solving for the constants of the general solution is the **equation of motion** of the system.

Alternative Forms of $\boldsymbol{x(t)}$

When $C_1, C_2 \neq 0$, the **actual amplitude** $\boldsymbol{A}$ of free vibrations is not immediately obvious, so it is often convenient to convert the equation of SHM to the simpler form

$$x(t) = A \sin(\omega t + \varphi)$$

where

$$A = \sqrt{C_1^2 + C_2^2}$$

and φ is a **phase angle** defined by

$$\left.\begin{aligned} \sin\varphi &= \frac{C_1}{A} \\ \cos\varphi &= \frac{C_2}{A} \end{aligned}\right\} \tan\varphi = \frac{C_1}{C_2}$$

A cosine function is sometimes preferred, making the solution

$$x(t) = A \cos(\omega t + \varphi)$$

where φ is defined by

$$\left.\begin{aligned} \sin\varphi &= \frac{C_2}{A} \\ \cos\varphi &= \frac{C_1}{A} \end{aligned}\right\} \tan\varphi = \frac{C_2}{C_1}$$

Double Spring Systems

The **effective spring constant k_{eff}** of a system with two *parallel* springs with spring constants k_1 and k_2 is

$$k_{\text{eff}} = k_1 + k_2$$

That of a system with two *series* springs is

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2}$$

Systems with Variable Spring Constants

In reality, it is reasonable to expect the spring constant to decay over time. One model for the **aging spring** replaces the spring constant k with the decreasing function

$$K(t) = k\mathrm{e}^{-\alpha t}$$

where k and α are positive constants. The linear DE

$$mx'' + k\mathrm{e}^{-\alpha t}x = 0$$

cannot be solved with the methods discussed thus far.
When a spring/mass system is subject to a rapidly decreasing temperature, k may be replaced with $K(t) = kt$ where k is a positive constant, a function that increases with time. The resulting model

$$mx'' + ktx = 0$$

is a form of **Airy's differential equation**.

5.1.2 Spring/Mass Systems: Free Dampened Motion

DE of Free Dampened Motion

Damping forces are proportional to a power of the instantaneous velocity. It is assumed here that it is a constant multiple of $\mathrm{d}x/\mathrm{d}t$. When no external forces are present, it follows from Newton's second law that

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t}$$

where β is a positive *damping constant*. Dividing by m yields

$$0 = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\beta}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x$$

where

$$2\lambda = \frac{\beta}{m}$$

2λ is used for convenience, as the auxiliary equation then becomes

$$0 = m^2 + 2\lambda m + \omega^2$$

which has roots

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

The discriminant is then $\lambda^2 - \omega^2$, providing 3 possible cases.
As each solution contains the *damping factor* $\mathrm{e}^{-\lambda t}$ and λ is positive, the displacements become negligible as t increases.

Case I: $\boldsymbol{\lambda^2 - \omega^2 > 0}$

The system is said to be **overdamped** when $\lambda^2 > \omega^2$, as β is large compared to k. The corresponding solution is

$$x(t) = \mathrm{e}^{-\lambda t}\left(C_1\mathrm{e}^{t\sqrt{\lambda^2-\omega^2}} + C_2\mathrm{e}^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

This equation describes smooth nonoscillatory motion.

Case II: $\boldsymbol{\lambda^2 - \omega^2 = 0}$

The system is said to be **critically dampened** when $\lambda^2 = \omega^2$, as any slight decrease in the damping force would yield oscillatory motion. The general solution is

$$x(t) = \mathrm{e}^{-\lambda t}\left(C_1 + C_2 t\right)$$

The motion described by this equation is quite similar to that of an overdampened system. Note that the mass may pass through the equilibrium position at most once.

Case III: $\boldsymbol{\lambda^2 - \omega^2 < 0}$

The system is said to be **underdamped** when $\lambda^2 < \omega^2$, as β is small compared to k. The roots are now complex, being

$$m = -\lambda \pm i\sqrt{\omega^2 - \lambda^2}$$

The general solution is

$$x(t) = \mathrm{e}^{-\lambda t}\left(C_1\cos\left(t\sqrt{\omega^2-\lambda^2}\right) + C_2\sin\left(t\sqrt{\omega^2-\lambda^2}\right)\right)$$

The motion described by this equation is oscillatory, but due to the $\mathrm{e}^{-\lambda t}$ term, the amplitude approaches 0 as t increases.

Alternative Form of $\boldsymbol{x(t)}$

Any solution

$$x(t) = \mathrm{e}^{-\lambda t}\left(C_1\cos\left(t\sqrt{\omega^2-\lambda^2}\right) + C_2\sin\left(t\sqrt{\omega^2-\lambda^2}\right)\right)$$

can be rewritten as

$$x(t) = A\mathrm{e}^{-\lambda t}\sin\left(t\sqrt{\omega^2-\lambda^2} + \varphi\right)$$

where

$$A = \sqrt{C_1^2 + C_2^2} \qquad \text{and} \qquad \tan\varphi = \frac{C_1}{C_2}$$

$A\mathrm{e}^{-\lambda t}$ is sometimes referred to as the **damped amplitude** of vibrations.
As the solution is not periodic, the number

$$\frac{2\pi}{\sqrt{\omega^2-\lambda^2}}$$

is called the **quasi period** and

$$\frac{\sqrt{\omega^2-\lambda^2}}{2\pi}$$

the **quasi frequency**.
The quasi period is the interval between two successive maxima.

5.1.3 Spring/Mass Systems: Driven Motion

DE of Driven Motion with Damping

Consider an external force $f(t)$. Including it in the formulation of Newton's second law yields the DE of **driven/forced motion**:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t}$$

Dividing by m yields

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t)$$

where

$$F(t) = \frac{f(t)}{m}$$

Transient and Steady-State Terms

When F is a periodic function, such as

$$F(t) = F_0\sin(\gamma t) \qquad \text{or} \qquad F(t) = F_0\cos(\gamma t)$$

5.2 Linear Models: Boundary-Value Problems

Deflection of a Beam

Many structures employ girders or beams to keep stable. Such beams distort or deflect under their own weight or under the influence of some external force. This deflection $y(x)$ is governed by a relatively simple 4^{th}-order DE.

Assume that a beam of length L is homogenous with uniform cross-sections along its length. Without any load (including the beam's own weight, a curve that joins the centroids of each cross section in a straight line is called the **axis of symmetry**. If a load is then applied in a vertical plane containing the axis of symmetry, the beam undergoes a distortion. The curve that now connects all centroids is the **deflection/elastic curve**. Let the axis of symmetry be the x-axis and that the deflection $y(x)$ is measured relative to it pointing downwards. The bending moment $M(x)$ at a point x along the beam can be found from the load per unit length $w(x)$ by

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2} = w(x)$$

It is also proportional to the curvature κ as

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2} = EI\kappa$$

where E and I are constants, being Young's modulus of the beam's material and the moment of inertia of a cross section (about an axis referred to as the neutral axis) respectively. The product is called the **flexural rigidity** of the beam.

Curvature is given by

$$\kappa = \frac{y''}{\left(1 + (y')^2\right)^{3/2}}$$

When the deflection is small, the slope $y' \approx 0$, so $(1 + (y')^2)^{3/2} \approx 1$. Letting $\kappa \approx y''$, $M = EIy''$, so

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + EI\frac{\mathrm{d}^2}{\mathrm{d}x^2}y'' = EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4}$$

It can then be seen that

$$EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = w(x)$$

The boundary conditions are dependent on how the beam is supported. A cantilever beam is **embedded** or **clamped** at one end and **free** at the other. For a cantilever, $y(0) = y'(0) = y''(L) = y'''(L) = 0$

The function $F(x) = \mathrm{d}M/\mathrm{d}x = EI\,\mathrm{d}^3y/\mathrm{d}x^3$ is called the shear force. If the end of a beam is **simply/pin/fulcrum supported** or **hinged**, then $y = y' = 0$ at that end. In summary,

Ends	Boundary	Conditions
Embedded	$y = 0$,	$y' = 0$
Free	$y'' = 0$,	$y''' = 0$
Simply Supported or HInged	$y = 0$,	$y'' = 0$

Eigenvalues and Eigenfunctions

Many problems require the solving of a 2-point BVP involving a linear DE containing a parameter λ. *Nontrivial (nonzero)* solutions are sought.

Consider the BVP

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

This presents 3 cases:

Case I: $\boldsymbol{\lambda = 0}$

For $\lambda = 0$, the solution of $y'' = 0$ is $y = C_1 x + C_2$. The conditions imply that $C_1 = C_2 = 0$, so the only solution is the trivial solution $y = 0$.

Case II: $\boldsymbol{\lambda < 0}$

For $\lambda < 0$, it is convenient to make the substitution $\lambda = \alpha^2$ (where $\alpha \in \mathbb{R}^+$). The roots of the auxiliary equation $m^2 - \alpha^2 = 0$ are then $m = \pm\alpha$. As the interval is finite, the general solution of

$$y'' - \alpha^2 y = 0$$

is written as

$$y = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x)$$

making $y(0)$

$$y(0) = C_1 \cosh(0) + C_2 \sinh(0) = C_1$$

$y(0) = 0$ then implies that $C_1 = 0$, so y becomes

$$y = C_2 \sinh(\alpha x)$$

The second condition, $y(L) = 0$, means that

$$y(L) = C_2 \sinh(\alpha L) = 0$$

which, for $\alpha \neq 0$ necessitates that $C_2 = 0$. The only solution for this BVP is then the trivial solution $y = 0$.

Case III: $\boldsymbol{\lambda > 0}$

For $\lambda > 0$, it is also convenient to write $\lambda = -\alpha^2$ (where $\alpha \in \mathbb{R}^+$). The roots of the auxiliary equation $m^2 + \alpha^2 = 0$ are then $m = \pm i\alpha$, so the general solution of

$$y'' + \alpha^2 y = 0$$

is

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$$

$y(0)$ again implies that $C_1 0$ and the condition $y(L) = 0$ or

$$C_2 \sin(\alpha L) = 0$$

is satisfied by $C_2 = 0$. This again yields the trivial solution $y = 0$. If it is required that $C_2 \neq 0$, though, then $\sin(\alpha L) = 0$ is satisfied so long as αL is an integer multiple of π:

$$\alpha L = n\pi \quad \text{or} \quad \alpha = \frac{n\pi}{L} \quad \text{or} \quad \lambda_n = \alpha_n^2 = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{Z}^+$$

For any real nonzero C_2,

$$y_n(x) = C_2 \sin\left(\frac{n\pi x}{L}\right)$$

is a solution for $n \in \mathbb{Z}^+$. As the DE is homogenous, any constant multiple of a solution is itself also a solution, so C_2 can be set to 1. In other words, for each number in the sequence

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{for} \quad n \in \mathbb{Z}^+$$

the corresponding function

$$y_n = \sin\left(\frac{n\pi}{L}\right)$$

is a nontrivial solution of the problem

$$y'' + \lambda_n y = 0, \quad y(0) = 0, \quad y(L) = 0$$

The numbers λ_n are known as **eigenvalues**. The nontrivial solutions are called **eigenfunctions**.

Chapter 6

Series Solutions of Linear Equations

6.1 Review of Power Series

Power Series

A **power series centered at a** is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Important Facts

Convergence

A power series is **convergent** at a value of x if its sequence of partial sums $\{\{S_N(x)\}\}$ converges; that is,

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$$

must exist. If this limit does not exist, the series is said to be **divergent**.
The **interval of convergence** is the set of *all* real numbers x for which the series converges. Every power series has one.
The radius R of the interval of convergence is the **radius of convergence**. If $R > 0$, then a power series converges for $|x-a| < R$ (equivalently $a - R < x < a$ and diverges for $|x-a| > R$. If the series is only convergent at its center, $R = 0$. If it converges for all $x \in \mathbb{R}$, then $R = \infty$. It may or may not converge at the endpoints of the interval.
The power series **converges absolutely** within its interval of convergence (not inclusive), meaning that

$$\sum_{n=0}^{\infty} |c_n (x-a)^n|$$

converges.
The convergence of a power series can often be determined by the **ratio test**. If $c_n \neq 0$ for all $n \in \mathbb{N}$, let

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L$$

If $L < 1$, the series converges absolutely. If $L > 1$, it diverges. If $L = 1$, the test is inconclusive. This test is always inconclusive at the endpoints of the interval of convergence.

A Power Series Defines a Function

A power series defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

whose domain is the the series' interval of convergence. If the radius of convergence is $R > 0$, the f is continuous, differentiable, and integrable on $a \pm R$. If it is ∞, f is continuous, differentiable, and integrable on $\mathbb{R}$. $f'(x)$ and $\int f(x)\,\mathrm{d}x$ can be found term-by-term via differentiation or integration. Convergence at the endpoints may be gained through integration or lost through differentiation.
If

$$y = \sum_{n=0}^{\infty} c_n x^n$$

is a power series, then

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$$

It is then clear that the first term of y' and the first 2 of y'' are 0. Omitting these, they become

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

Note in particular the changed lower bound of the summation in the derivatives.

Properties

The **identity property** states that if

$$\sum_{n=0}^{\infty} c_n (x-a)^n = 0$$

and $R > 0$, then $c_n = 0$ for all $n \in \mathbb{N}$.
A function f is said to be **analytic at a point** if it can be represented at that point with a power series with a radius of convergence that is either positive or infinite.
Power series may be combined through addition, multiplication, and division.

Common Maclaurin Series

$f(x)$	Maclaurin Series	Interval of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$\mathbb{R}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$\mathbb{R}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\mathbb{R}$
$\arctan x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$[-1, 1]$
$\cosh x$	$\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$\mathbb{R}$
$\sinh x$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$\mathbb{R}$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$	$(-1, 1]$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1, 1)$

A Preview

To find a power series solution to DE, the desired derivatives must first be calculated. These can then be substituted back into the DE. The indices can be shifted to combine the summations. For a homogenous DE, identity can be used to solve for the coefficients.

6.2 Solutions About Ordinary Points

A Definition

Dividing the homogenous linear second-order DE

Definition 6.2.1 Ordinary and Singular Points A point $x = x_0$ is said to be an **ordinary point** of the DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

if both $P(x)$ and $Q(x)$ are analytic at x_0, where

$$y'' + P(x)y' + Q(x)y = 0$$

is the standard form of the DE. A point that is not an ordinary point of this DE is a **singular point** of it.

Polynomial Coefficients

A polynomial is analytic at any value of x, and a rational function is whenever its denominator is not zero. Both coefficients

$$P(x) = \frac{a_1(x)}{a_2(x)} \qquad \text{and} \qquad Q(x) = \frac{a_0(x)}{a_2(x)}$$

are analytic wherever $a_2(x) \neq 0$. It then follows that *a number $x = x_0$ is an ordinary point of a linear second-order homogenous DE if $a_2(x_0) \neq 0$, and $x = x_0$ is a singular point if $a_2(x_0) = 0$.*

Theorem 6.2.1 Existence of Power Series Solutions Let $x = x_0$ be an ordinary point of a linear second-order homogenous DE. Two linearly independent solutions in the form of the power series

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

can always be found. A power series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

A solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

is said to be a **solution about the ordinary point $\boldsymbol{x_0}$**. The distance R is the *minimum value* or *lower bound* of the radius of convergence.

6.3 Solutions about Singular Points

6.3.1 A Definition

A singular point x_0 of the standard-form second-order homogenous linear DE

$$y'' + P(x)y' + Q(x)y = 0$$

can be further classified as either regular or irregular.

Definition 6.3.1 Regular and Irregular Singular Points A singular point $x = x_0$ is said to be **regular** if the functions

$$p(x) = (x - x_0)P(x) \qquad \text{and} \qquad q(x) = (x - x_0)^2 Q(x)$$

are both analytic at x_0. One that is not regular is **irregular**.

Polynomial Coefficients

In order for $x = x_0$ to be a singular point, either $P(x)$ or $Q(x)$ is not analytic at x_0. As $a_2(x)$ is a polynomial and x_0 is one of its zeros, it follows that $x - x_0$ is one of its factors. After simplifying the rational functions, then, the factor $x - x_0$ must remain to some positive integer power in at least one of the denominators.

Suppose $x = x_0$ is a singular point and that $p(x)$ and $q(x)$ are analytic at x_0. Multiplying $P(x)$ by

$x - x_0$ and $Q(x)$ by $(x - x_0)^2$ must then result in cancellation with the denominator, as $x - x_0$ no longer appears in either. The regularity of x_0 can then be determined by checking the denominators.

If $x - x_0$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$, then $x = x_0$ is a regular singular point.

Observe also that if $x = x_0$ is a regular singular point and the DE is multiplied by $(x - x_0)^2$, then

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$$

where p and q are analytic at $x = x_0$.

Method of Frobenius

Theorem 6.3.1 Frobenius' Theorem If $x = x_0$ is a regular singular point of a second-order homogenous linear DE, then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where r is a constant. The series converges at least on some interval $0 < x - x_0 < R$.

Indicial Equation

The **indicial equation** of a problem is the quadratic equation in r found by equating the *total coefficient of the lower power of x to 0* after substituting the form of the solution into the DE. The values of r are the **indicial roots/exponents** of the singularity $x = x_0$.. These values can then be substituted into a recurrence relation, relating c_k to c_{k+1}. Frobenius' theorem guarantees that at least one solution of the assumed series form can be found.
The indicial equation can be obtained without substituting. If $x = 0$ is a regular singular point, then both

$$p(x) = xP(x) \qquad \text{and} \qquad q(x) = x^2 Q(x)$$

are analytic at $x = 0$; that is, their power series expansions

$$p(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad q(x) = \sum_{n=0}^{\infty} b_n x^n$$

are valid on intervals with positive radii of convergence. Multiplying the standard form by x^2 yields

$$x^2 y'' + x[xP(x)]y' + [x^2 Q(x)]y = 0$$

Substituting

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

and the series expansions of $p(x)$ and $q(x)$ into this yields the general indicial equation

$$r(r-1) + a_0 r + b_0 = 0$$

where a_0 and b_0 are the constant terms of the power series expansions of $p(x)$ and $q(x)$ respectively.

Three Cases

Let $x = 0$ be a regular singular point of a linear second-order homogenous DE and r_1 and r_2 be real. When employing the method of Frobenius, three cases can be distinguished that correspond to the nature of the indicial roots.

Case I

If $r_1 > r_2$ and $r_1 - r_2$ is not a positive integer, then there exist 2 linearly independent solutions of the DE of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0 \qquad \text{and} \qquad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0$$

Case II

If $r_1 - r_2 = N$ where $N \in \mathbb{Z}^+$, then there exist 2 linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0 \qquad \text{and} \qquad y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0$$

where C is a constant (that may be 0).

Case III

If $r_1 = r_2$, then there exist 2 linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0 \qquad \text{and} \qquad y_2(x) = y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}$$

Finding a Second Solution

In case II, there may be 2 solutions of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

This is cannot be known in advance, instead being determined after finding the indicial roots and examining the recurrence relation that defines the coefficients. It is possible that $C = 0$; that is,

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \qquad \text{and} \qquad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

6.4 Special Functions

Solution of Bessel's Equation

Bessel's equation of order ν is the DE

$$x^2y'' + xy' + \left(x^2 - \nu^2\right) y = 0$$

As $x = 0$ is a singular point of this equation, there exists at least one solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \implies y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \implies y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

Substituting this into the DE,

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} \left[(n+r)(n+r-1)c_n x^{n+r}\right] + \sum_{n=0}^{\infty} \left[(n+r)c_n x^{n+r}\right] + \sum_{n=0}^{\infty} \left[\left(x^2 - \nu^2\right) c_n x^{n+r}\right] \\
&= x^r \sum_{n=1}^{\infty} \left[\left((n+r)^2 - \nu^2\right) c_n x^n\right] + x^r \sum_{n=0}^{\infty} \left[c_n x^{n+2}\right] + c_0 \left(r(r-1) + r - \nu^2\right) x^r \\
&= x^r \sum_{n=1}^{\infty} \left[\left((n+r)^2 - \nu^2\right) c_n x^n\right] + x^r \sum_{n=0}^{\infty} \left[c_n x^{n+2}\right] + c_0 \left(r^2 - \nu^2\right) x^r
\end{aligned}$$

Rewriting the equation in standard form, The indicial equation is then

$$0 = r^2 - \nu^2$$

which has roots $r = \pm\nu$, so

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+\nu} \qquad \text{and} \qquad y_2 = \sum_{n=0}^{\infty} c_n x^{n-\nu}$$

Substituting y_1 into the DE,

$$\begin{aligned}
0 &= x^\nu \sum_{n=1}^{\infty} \left[\left(n^2 + 2n\nu\right) c_n x^n\right] + x^\nu \sum_{n=0}^{\infty} \left[c_n x^{n+2}\right] \\
&= x^\nu \left((1+2\nu) + \underbrace{\sum_{n=2}^{\infty} \left[c_n (n+2\nu) x^n\right]}_{k=n-2} + \underbrace{\sum_{n=0}^{\infty} \left[c_n x^{n+2}\right]}_{k=n} \right) \\
&= x^\nu \left((1+2\nu)c_1 x + \sum_{k=0}^{\infty} \left[(k+2)(k+2+2\nu)c_{k+2} + c_k\right] x^{k+2} \right)
\end{aligned}$$

The recurrence relation is then

$$c_{k+2} = -\frac{c_k}{(k+2)(k+2+2\nu)}$$

Letting $c_1 = 0$ implies that $c_3 = c_5 = \cdots = 0$, so for $k = 0, 2, \cdots$, letting $k + 2 = 2n$ for $n \in \mathbb{Z}^+$ means that

$$c_{2n} = -\frac{c_{2n-2}}{2^2 n(n+\nu)}$$

This can be rewritten in terms of c_0 as

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n! \prod_{i=1}^{n} [i+\nu]}$$

It is standard practice to let c_0 be the specific value

$$c_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

where $\Gamma(1+\nu)$ is the gamma function. This posses the property that

$$\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$$

so the denominator of c_{2n} can be further reduced to a single term:

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(1+\nu+n)}$$

Bessel Functions of the First Kind

Using the coefficients c_{2n} and $r = \nu$, a series solution of Bessel's equation is

$$y = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$$

This solution is typically denoted by $J_\nu(v)$:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

If $\nu \geq 0$, the series converges at least on $[0, \infty)$. For the second root $r_2 = -nu$,

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

The above two functions are called **Bessel functions of the first kind** of orders ν and $-\nu$ respectively. The value of ν may result in negative power of x, making the interval of convergence $(0, \infty)$.
When $\nu = 0$, $J_\nu = J_{-\nu}$.
If $\nu > 0$ and $r_1 - r_2 = 2\nu$ is not a positive integer, then it follows from Case I that J_ν and $J_{-\nu}$ are linearly independent solutions on $(0, \infty)$, making the general solution on the interval

$$y = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$$

When 2ν is a positive integer, then from Case II it follows that a second series solution *may* exist. If $\nu = m \in \mathbb{Z}^+$, J_m and J_{-m} are not linearly independent. Additionally, 2ν can be a positive integer when ν is half of an odd positive integer. In this case, J_ν and $J_{-\nu}$ are linearly independent. In summary, The general solution of the Bessel equation on $(0, \infty)$ is

$$y = C_1 J_\nu(x) + C_2 J_{-\nu}(x), \quad \nu \notin \mathbb{Z}$$

Bessel Functions of the Second Kind

If $v \notin \mathbb{Z}$, the function defined by the linear combination

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

and the function $J_\nu(x)$ are linearly independent solutions of Bessel's equation. Another form for the general solution of this equation is then

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x), \quad \nu \notin \mathbb{Z}$$

As $\nu \to m \in \mathbb{Z}$, $Y_\nu(x)$ has the indeterminate form $0/0$. It can be shown by L'Hôpital's Rule that this limit exists. Moreover, the function

$$Y_m = \lim_{\nu \to m} Y_\nu(x)$$

and $J_m(x)$ are linearly independent solutions of Bessel's equation. For *any* value of ν, then, the general solution of Bessel's equation on $(0, \infty)$ can be written as

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

$Y_\nu(x)$ is called the **Bessel function of the second kind of order $\boldsymbol{\nu}$**.

DEs Solvable in Terms of Bessel Functions

Some DEs can be transformed via change change of variable into Bessel's equation. Letting $t = \alpha x$ where $\alpha > 0$, for example, yields the **parametric Bessel equation of order $\boldsymbol{\nu}$**:

$$x^2 y'' + x y' + \left(\alpha^2 x^2 - \nu^2\right) y = 0$$

then by chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} tx = \alpha \frac{\mathrm{d}y}{\mathrm{d}t}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x} = \alpha^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$$

Substituting these back in,

$$\begin{aligned} 0 &= \left(\frac{t}{\alpha}\right)^2 \alpha^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \left(\frac{t}{\alpha}\right) \alpha \frac{\mathrm{d}y}{\mathrm{d}t} + \left(t^2 - \nu^2\right) y \\ &= t^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + t \frac{\mathrm{d}y}{\mathrm{d}t} + \left(t^2 - \nu^2\right) y \end{aligned}$$

This is Bessel's equation of order ν with solution

$$y = C_1 J_\nu(t) + C_2 Y_\nu(t)$$

Resubstituting $t = \alpha x$ yields

$$y = C_1 J_\nu(\alpha x) + C_2 Y_\nu(\alpha x)$$

Chapter 7

The Laplace Transform

7.1 Definition of the Laplace Transform

Differentiation and integration are *transforms*, meaning that, roughly speaking, these operations transform a function into another function. Moreover, these two transforms possess the **linearity property** that a transform of linear combinations of functions is a linear combination of their transforms. For two constants α and β,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\alpha f(x)+\beta g(x)] = \alpha f'(x)+\beta g'(x) \qquad \text{and} \qquad \int [\alpha f(x)+\beta g(x)]\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x+\beta \int g(x)\,\mathrm{d}x$$

provided that each derivative and integral exists. The **Laplace transform** is a special type of integral transform that has several interesting properties in addition to linearity.

Integral Transform

If $f(x, y)$ is a function of two variables, a definite integral of f with respect to one of those variables results in a function of the other variable:

$$\int_a^b f(x, y)\,\mathrm{d}x = g(y)$$

A definite integral such as

$$\int_a^b K(s, t) f(t)\,\mathrm{d}t$$

similarly transforms a function f of variable t into a function F of variable s. An **integral transform**, where the interval of integration is $[0, \infty)$, is of particular interest. If $f(t)$ is defined for $t \geq 0$, then

$$\int_0^\infty K(s, t) f(t)\,\mathrm{d}t = \lim_{b\to\infty} \int_0^b K(s, t) f(t)\,\mathrm{d}t$$

If this limit exists, the integral exists or is **convergent**; otherwise, it does not exist or is **divergent**. The above limit generally exists only for specific values of s (which is henceforth assumed to be a real variable).

A Definition

The function $K(s,t)$ is referred to as the **kernel** of the transform. The choice $K(s,t) = \mathrm{e}^{-st}$ yields an especially important integral transform.

Definition 7.1.1 Laplace Transform Let f be a function defined for $t \geq 0$. The integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

is said to be the **Laplace transform** of f, provided it converges.

When the Laplace transform converges, the resultant function s is generally denoted as the uppercase form of the lowercase letter used to denote the function of t on which the transform was performed.
The domain of $F(s)$ is dependent on $f(t)$.
The notation

$$]_0^\infty$$

is used as shorthand for

$$\lim_{b\to\infty} [\,]_0^b$$

$\mathscr{L}$ Is a Linear Transform

For a linear combination of functions,

$$\int_0^\infty \mathrm{e}^{-st}[\alpha f(t) + \beta g(t)]\,\mathrm{d}t = \alpha \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t + \beta \int_0^\infty \mathrm{e}^{-st} g(t)\,\mathrm{d}t$$

whenever both integrals converge for $s > c$. It then follows that

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

Due to this property, $\mathscr{L}$ is said to be a **linear transform**.

Theorem 7.1.1 Transforms of Some Basic Functions

$f(t)$	1	t^n	$\mathrm{e}^{\alpha t}$	$\sin(kt)$	$\cos(kt)$	$\sinh(kt)$	$\cosh(kt)$
$\mathscr{L}\{f(t)\}$	$\frac{1}{s}$	$\frac{n!}{s^{n+1}}, n \in \mathbb{Z}^+$	$\frac{1}{s-a}$	$\frac{k}{s^2+n^2}$	$\frac{s}{s^2+k^2}$	$\frac{k}{s^2-k^2}$	$\frac{s}{s^2-k^2}$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

The Laplace transform's integral does not necessarily converge. Sufficient conditions to guarantee the existence of $\mathscr{L}\{f(t)\}$ are that f be piecewise continuous on $[0,\infty)$ and that f be of exponential order for $t > T$. A function is **piecewise continuous** on $[0,\infty)$ if for any interval $0 \leq a \leq t \leq b$, there are at most a finite number of points t_k at which f has finite discontinuities and is continuous on each open interval (t_{k-1}, t_k).

Definition 7.1.2 Exponential Order A function f is said to be of **exponential order** if their eists constants c, $M > 0$ and $T > 0$ such that $|f(t)| \leq M\mathrm{e}^{ct}$ for all $t > T$.

If f is *increasing*, this simply means that on the interval (T, ∞), f must not increase faster than $M\mathrm{e}^{ct}$, where $c > 0$.
A positive integral power of t. is always of exponential power, as for $c > 0$,

$$|t^n| \leq M\mathrm{e}^{ct} \qquad \text{or} \qquad \left|\frac{t^n}{\mathrm{e}^{ct}}\right| \leq M \quad \text{for} \quad t > T$$

is equivalent to showing that

$$\lim_{t\to\infty} \frac{t^n}{\mathrm{e}^{ct}}$$

is finite for $n \in \mathbb{Z}+$. The result follows from n applications of Lôhpital's rule.

Theorem 7.1.2 Sufficient Conditions for Existence If f is piecewise continuous on $[0, \infty)$ and of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

By the additive property of definite integrals,

$$\mathscr{L}\{f(t)\} = \int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t + \int_T^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t = I_1 + I_2$$

The integral I_1 exists, as it can be written as a sum of integrals over intervals over which $\mathrm{e}^{-st}f(t)$ is continuous. As f is of exponential order, there exist constants c, $M > 0$, $T > 0$ such that $|f(t)| \leq M\mathrm{e}^{ct}$ for $t > T$. It can then be written that

$$|I_2| \leq \int_T^\infty |\mathrm{e}^{-st}f(t)|\,\mathrm{d}t \leq M\int_T^\infty \mathrm{e}^{-st}\mathrm{e}^{ct}\,\mathrm{d}t = M\int_T^\infty \mathrm{e}^{(-s-c)t}\,\mathrm{d}t = M\frac{\mathrm{e}^{(-s-c)T}}{s-c}$$

for $s > c$. As $\int_T^\infty M\mathrm{e}^{-(s-c)t}\,\mathrm{d}t$ converges, so must $\int_T^\infty \mathrm{e}^{-st}f(t)\,\mathrm{d}t$ by the comparison test for improper integrals. As both I_1 and I_2 exist, so does $\mathscr{L}\{f(t)\}$.

Theorem 7.1.3 Behavior of $F(s)$ as $s \to \infty$
If $F(s) = \mathscr{L}\{f(t)\}$ exists, then $\lim_{s\to\infty} F(s) = 0$.

As f is of exponential order, there exist constants γ and $M_1, T \in \mathbb{R}^+$ such that $|f(t)| \leq M_1\mathrm{e}^{\gamma t}$ for $t > T$. As f is piecewise continuous over $[0, T]$, it must be bounded on that interval; that is, $|f(t)| \leq M_2 = m_2\mathrm{e}^{0t}$. If $M = \max\{M_1, M_2\}$ and $c = \max\{0, \gamma\}$, then

$$|F(s)| \leq \int_0^\infty \mathrm{e}^{-st}|f(t)|\,\mathrm{d}t \leq M\int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{ct} = M\int_0^\infty \mathrm{e}^{-(s-c)t}\,\mathrm{d}t = \frac{M}{s-c}$$

for $s > c$. As $s \to \infty$, the denominator also goes to ∞, so $|F(s)| \to 0$, so $F(s) = \mathscr{L}\{f(t)\} \to 0$.

7.2 Inverse Transforms and Transforms of Derivatives

7.2.1 Inverse Transforms

The Inverse Problem

If $F(s) = \mathscr{L}\{f(t)\}$, is can be said that $f(t)$ is the **inverse Laplace transform** of $F(s)$, denoted

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

Theorem 7.2.1 Some Inverse Transforms

$F(s)$	$\dfrac{1}{s}$	$\dfrac{n!}{s^{n+1}}, n \in \mathbb{N}$	$\dfrac{1}{s-a}$	$\dfrac{k}{s^2+k^2}$	$\dfrac{s}{s^2+k^2}$	$\dfrac{k}{s^2-k^2}$	$\dfrac{s}{s^2-k^2}$
$\mathscr{L}^{-1}\{F(s)\}$	1	t^n	e^{at}	$\sin(kt)$	$\cos(kt)$	$\sinh(kt)$	$\cosh(kt)$

$\mathscr{L}^{-1}$ is a Linear Transform

The inverse Laplace transform is also linear; that is,

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\,\mathscr{L}^{-1}\{F(s)\} + \beta\,\mathscr{L}^{-1}\{G(s)\}$$

for some constants α and β.

Partial Fractions

Decomposing a function into its partial fractions is often useful in evaluating its inverse Laplace transform.

7.2.2 Transforms of Derivatives

Transform a Derivative

If f' is continuous, integration by parts gives

$$\mathscr{L}\{f'(t)\} = \int_0^\infty \mathrm{e}^{-st} f'(t)\,\mathrm{d}t = \left[\mathrm{e}^{-st} f(t)\right]_0^\infty + s\int_0^\infty \mathrm{e}^{-st} f(t) = -f(0) + s\,\mathscr{L}\{f(t)\}$$

or

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

Using this result

$$\begin{aligned}\mathscr{L}\{f''(t)\} &= \int_0^\infty \mathrm{e}^{-st} f''(t)\,\mathrm{d}t = \left[\mathrm{e}^{-st} f'(t)\right]_0^\infty + s\int_0^\infty \mathrm{e}^{-st} f'(t)\,\mathrm{d}t \\ &= -f'(0) + s\,\mathscr{L}\{f'(t)\} = s[sF(s) - f(0)] - f'(0)\end{aligned}$$

or

$$\mathscr{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

It can similarly be shown that

$$\mathscr{L}\{f''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

Theorem 7.2.2 Transform of a Derivative If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty$ and of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

where $F(s) = \mathscr{L}\{f(t)\}$.

Solving Linear ODEs

$\mathscr{L}\{\mathrm{d}^n y/\mathrm{d}t^n\}$ is evidently dependent on $Y(s) = \mathscr{L}\{y(t)\}$ and on the $n-1$ derivatives of $y(t)$ at $t = 0$. This property makes the Laplace transform well-suited to solving linear IVPs in which the DE has *constant coefficients*. Such a DE is simply a linear combination of y and its n derivatives:

$$a_0\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_0 y = g(t), \qquad y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$$

where $a_{0\cdots n}$ and $y_{0\cdots n-1}$ are constants. By linearity, the Laplace transform of this linear combination is itself a linear combination of Laplace transforms:

$$a_n\,\mathscr{L}\left\{\frac{\mathrm{d}^n y}{\mathrm{d}t^n}\right\} + a_{n-1}\,\mathscr{L}\left\{\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}\right\} + \cdots + a_0\,\mathscr{L}\{y\} = \mathscr{L}\{g(t)\}$$

This becomes

$$a_n\left[s^n Y(s)\sum_{i=1}^{n} s^{n-i} y^{(i-1)}(0)\right] + a_{n-1}\left[s^{n-1}Y(s) - \sum_{i=2}^{n} s^{(n-i)} y^{(i-2)}\right] + \cdots + a_0 Y(s) = G(s)$$

where $Y(s) = \mathscr{L}\{y(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$.

The Laplace transform of a linear DE with constant coefficients becomes an algebraic equation in $Y(s)$.

It can be written that

$$P(s)Y(s) = Q(s) + G(s)$$

where

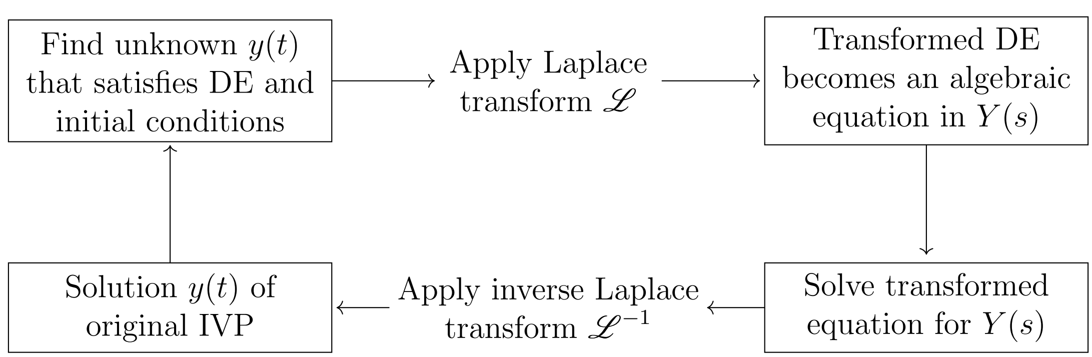
$$P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

and $Q(s)$ is a polynomial in s of degree $\leq n-1$ comprised of the various products of the coefficients a_i and the prescribed initial conditions y_i. Rewriting,

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

These two terms are typically combined over the least common denominator before being composed into partial fractions. The solution $y(t)$ is then simply $\mathscr{L}^{-1}\{Y(s)\}$.

Figure 7.2.1 Steps in solving an IVP by the Laplace transform



7.3 Operational Properties I

7.3.1 Translation on the s-Axis

A Translation

In general, if $\mathscr{L}\{f(t)\} = F(s)$ is known, then $\mathscr{L}\{e^{at} f(t)\}$ can be computed.

Theorem 7.3.1 First Translation Theorem If $\mathscr{L}\{f(t)\} = F(s)$ and a is a real number, then

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a)$$

By the definition of the Laplace transform,

$$\mathscr{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$$

If s is regarded as a real variable, the graph of $F(s-a)$ is simply the graph of $F(s)$ shifted along the s-axis by $|a|$. If $a > 0$, the shift is to the right, while it is to the left if $a < 0$.
For emphasis, it is sometimes useful to use the notation

$$\mathscr{L}\{e^{at} f(t)\} = \mathscr{L}\{f(t)\}|_{s \to s-a}$$

where $s \to s - a$ means that the Laplace transform $F(s)$ replaces s with $s - a$ wherever it appears.

Inverse Form of a Translation

To compute the inverse of $F(s-a)$, $F(s)$ must be recognized. $\mathscr{L}^{-1}\{F(s-a)\}$ is then simply the product of $f(t) = \mathscr{L}^{-1} F(s)$ and e^{at}. Symbolically, this can be summarized as

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at} f(t)$$

7.3.2 Translation on the t-Axis

Unit Step Function

Definition 7.3.1 Unit Step Function The **unit step function** (or **Heaveside function**) $\mathscr{U}(t-a)$ is defined to be

$$\mathscr{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$$

Note that $\mathscr{U}(t-a)$ is defined only on the nonnegative t-axis, as this is all that must be considered when working with the Laplace transform. In a broader sense, $\mathscr{U}(t-a) = 0$ for all $t < a$.
When a function f defined for $t \ge 0$ is multiplied by $\mathscr{U}(t-a)$, the unit step function "turns off" the portion of the graph that is before $t = a$.
The unit step function can be used to compactly write piecewise functions:

$$f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases} = g(t) - g(t)\,\mathscr{U}(t-a) + h(t)\,\mathscr{U}(t-a)$$

$$f(t) = \begin{cases} 0 & 0 \le t < a \\ g(t) & a \le t < b \\ 0 & t \ge b \end{cases} = g(t)[\mathscr{U}(t-a) - \mathscr{U}(t-b)]$$

Theorem 7.3.2 Second Translation Theorem If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\,\mathscr{U}(t-a)\} = \mathrm{e}^{-as}F(s)$$

By the additive property of integrals,

$$\begin{aligned}
\mathscr{L}\{f(t-a)\,\mathscr{U}(t-a)\} &= \int_0^\infty \mathrm{e}^{-st} f(t-a)\,\mathscr{U}(t-a)\,\mathrm{d}t \\
&= \int_0^a \mathrm{e}^{-st} f(t-a)\underbrace{\mathscr{U}(t-a)}_{0 \text{ for } 0\le t<a}\,\mathrm{d}t + \int_a^\infty \mathrm{e}^{-st} f(t-a)\underbrace{\mathscr{U}(t-a)}_{1 for t\ge a}\,\mathrm{d}t \\
&= \int_a^\infty \mathrm{e}^{-st} f(t-a)\,\mathrm{d}t
\end{aligned}$$

Letting $v = t - a$ and $\mathrm{d}v = \mathrm{d}t$,

$$\begin{aligned}
\mathscr{L}\{f(t-a)\,\mathscr{U}(t-a)\} &= \int_a^\infty \mathrm{e}^{-st} f(t-a)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-s(v+a)} f(v)\,\mathrm{d}v \\
&= \mathrm{e}^{-as}\int_0^\infty \mathrm{e}^{-sv} f(v)\,\mathrm{d}v = \mathrm{e}^{-as}\,\mathscr{L}\{f(t)\}
\end{aligned}$$

The Laplace transform of just a unit step function is

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{\mathrm{e}^{-as}}{s}$$

Inverse Transform to a Step Function

When $a > 0$,

$$\mathscr{L}^{-1}\{\mathrm{e}^{-as}F(s)\} = f(t-a)\,\mathscr{U}(t-a)$$

Alternative Form

Using the definition of $\mathscr{U}(t-a)$ and the substitution $u = t - a$, the transform of the product of a function and a step function can be rewritten as

$$\mathscr{L}\{g(t)\,\mathscr{U}(t-a)\} = \int_a^\infty \mathrm{e}^{-st} g(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-s(u+a)} g(u+a)\,\mathrm{d}u$$

That is,

$$\boxed{\mathscr{L}\{g(t)\,\mathscr{U}(t-a)\} = \mathrm{e}^{-as}\,\mathscr{L}\{g(t+a)\}}$$