16.2

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15. (p. 1141) Evaluate the line integral

$$\int_C \left[z \, \mathrm{d}x + xy \, \mathrm{d}y + y^2 \, \mathrm{d}z\right]$$

where C is the given space curve

$$C : x = \sin t, y = \cos t, z = \tan t, -\frac{\pi}{4} \leq t \leq \frac{\pi}{4}$$

$$\begin{aligned} x &= \sin t & y &= \cos t & z &= \tan t \\ \mathrm{d}x &= \cos t \, \mathrm{d}t & \mathrm{d}y &= -\sin t \, \mathrm{d}t & \mathrm{d}z &= \sec^2 t \, \mathrm{d}t \end{aligned}$$

$$\int_C \left[z \, \mathrm{d}x + xy \, \mathrm{d}y + y^2 \, \mathrm{d}z\right] = \int_{-\pi/4}^{\pi/4} \left[\sin t - \sin^2 t \cos t + 1\right] \mathrm{d}t = [t - \cos t]_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4} \left[\sin^2 t \cos t\right] \mathrm{d}t$$

$$u = \sin t \implies u_1 = -\frac{\sqrt{2}}{2}, u_2 = \frac{\sqrt{2}}{2}, \mathrm{d}u = \cos t \, \mathrm{d}t$$

$$\int_{-\pi/4}^{\pi/4} \left[\sin^2 t \cos t\right] \mathrm{d}t = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left[u^2\right] \mathrm{d}u = \left[\frac{u^3}{3}\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = \frac{2^{3/2}}{24} - \left(\frac{-2^{3/2}}{24}\right) = \frac{2\left(2\sqrt{2}\right)}{24} = \frac{\sqrt{2}}{6}$$

$$[t - \cos t]_{-\pi/4}^{\pi/4} = \frac{\pi}{4} - \frac{\sqrt{2}}{2} - \left(-\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right) = \frac{\pi}{2}$$

$$\int_C \left[z \, \mathrm{d}x + xy \, \mathrm{d}y + y^2 \, \mathrm{d}z\right] = \frac{\pi}{2} - \frac{\sqrt{2}}{6} = \frac{3\pi - \sqrt{2}}{6}$$

23. (p. 1142) Evaluate the line integral $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$, where C is given by the vector function $\vec{r}(t)$.

$$\vec{F}(x, y, z) = \sin x \, \hat{\imath} + \cos y \, \hat{\jmath} + xz \, \hat{\mathrm{k}} \qquad \vec{r}(t) = t^3 \, \hat{\imath} - t^2 \, \hat{\jmath} + t \, \hat{\mathrm{k}}, 0 \leq t \leq 1$$

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= \sin\left(t^3\right) \hat{\imath} + \cos\left(-t\right)^2 \hat{\jmath} + t^4 \, \hat{\mathrm{k}} = \sin\left(t^3\right) \hat{\imath} + \cos\left(t^2\right) \hat{\jmath} + t^4 \, \hat{\mathrm{k}} \\ \vec{r}'(t) &= 3t^2 \, \hat{\imath} - 2t \, \hat{\jmath} + \hat{\mathrm{k}} \\ \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= 3t^2 \sin\left(t^3\right) - 2t \cos\left(t^2\right) + t^4 \\ \int_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, \mathrm{d}t = \int_0^1 \left[3t^2 \sin\left(t^3\right) - 2t \cos\left(t^2\right) + t^4\right] \mathrm{d}t \\ &= \left[-\cos\left(t^3\right) - \sin\left(t^2\right) + 0.2t^5\right]_0^1 = -\cos(1) - \sin(1) + 0.2 - (-1 - 0 + 0) \\ &= 1.2 - \sin(1) - \cos(1) \end{aligned}$$