Contemporary Physics

Arnav Patri

February 5, 2023

Contents

Chapter 1

The Failures of Classical Physics

Under classical physics,

$$\vec{v} = \frac{\Delta \vec{r}}{t} \qquad K = \frac{1}{2}mv^2 \qquad \vec{p} = m\vec{v} \qquad \vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$$

where $\vec{v}$ is velocity, $\vec{r}$ is position, m is mass, $\vec{p}$ is momentum, $\vec{L}$ is angular momentum, and I is rotational inertia.

Working on very small scales, the unit of charge used is often the magnitude of that of an electron (or proton)

$$1\,e \approx 1.602 \times 10^{-19}\,\mathrm{C}$$

and that of energy is the **electron-volt (eV)**, which is the energy of an electron subjected to a $1\,\mathrm{V}$ potential difference:

$$1\,\mathrm{eV} \approx 1.602 \times 10^{-19}\,\mathrm{J}$$

An **atomic mass unit (amu)** is defined such that

$$12\,\mathrm{amu} = \text{carbon atom}$$

The speed of light c and Planck's constant h are

$$c \approx 3 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}} \qquad\qquad h \approx 6.626 \times 10^{-34}\,\mathrm{J\,s} \approx 4.14 \times 10^{-15}\,\mathrm{eV\,s}$$

The product of these two constants is

$$hc \approx 1240\,\mathrm{eV\,nm}$$

Failures of space and time require relativity while those of particle theories require quantum mechanics.

A **theory** is an organized body of facts. It provides a model of explanation.

A **hypothesis** is a prediction as to how something works. (With data to back up a hypothesis, it can become a theory.)

A **law** is a correlation between variables.

Chapter 2

The Special Theory of Relativity

2.1 Classical Relativity

A "theory of relativity" is simply a way for observers in different reference frames to compare their observations.

The mathematical basis for converting between reference frames is called a *transformation*.

Consider two observers O and O' observing the same event, O' moving relative O with constant velocity $\vec{u}$. In their own reference frames O and O' are both at rest, but they move relative to one another with constant velocity $\vec{u}$. According to O, the space and time coordinates are x, y, z, and t, while according to O', those of the *same event* are x', y', z', and t'. If the relative velocity is only in the common xx' direction, $\vec{u}$ can represent the velocity of O' as measured by O.

It is assumed that each observer is able to test and verify Newton's laws in their reference frames. A reference frame that follows Newton's first law (the law of inertia) is said to be an *inertial frame*. In order for a frame to be inertial, it must not be accelerating.

The classical *Galilean* transformation relating the coordinates assumes as a postulate of classical physics that $t = t'$; that is, time is the same for all observers. It is also assumed for simplicity that the coordinate systems are chosen such that their origins overlap at $t = 0$. Consider an object in O' at coordinates (x', y', z'). According to O, the y and z coordinates are the same as those observed in O'. Along the x direction, though, O would observe the object at $x = x' + ut$. This yields the *Galilean coordinate transformation*

$$x' = x - ut \qquad y' = y \qquad z' = z \qquad t' = t \qquad \text{(Galilean coordinate transformation)}$$

The velocities as observed by O can be found simply by differentiating the prior results:

$$v'_x = v_x - u \qquad v'_y = v_y \qquad v'_z = v_z \qquad \text{(Galilean velocity transformation)}$$

Again differentiating yields the relationships between the accelerations:

$$a'_x = a_x \qquad\qquad a'_y = a_y \qquad\qquad a'_z = a_z$$

This shows that Newton's laws hold for both observes so long as u is constant, as the observers measure identical accelerations, meaning that the results of applying $\vec{F} = m\vec{a}$ are also identical.

2.3 Einstein's Postulates

1. **Principle of Relativity** The laws of physics are identical for all inertial reference frames.

2. The speed of light is constant for all inertial reference frames.

2.4 Consequences of Einstein's Postulates

As a consequence of these two postulates, the motion of the observer determines their measurements.

The Relativity of Time

Consider a beam of light traveling up to a mirror and back down (in the y direction) in a reference frame moving at speed u in the x direction. An observer. An observer O at rest and an observer O' moving in the same reference frame as the mirror.

From the perspective of O', the time interval between the light leaving and returning from its source is

$$\Delta t_0 = \frac{2L_0}{c}$$

where L_0 is the distance between the origin of the light and the mirror.

From the perspective of O, the light beam travels a distance $2L$, where

$$L = \sqrt{L_0^2 + \left(\frac{u\Delta t}{2}\right)^2}$$

as derived by Pythagorean theorem. As O observes the beam of light to travel at c, the time interval measured is

$$\Delta t = \frac{2L}{c} = \frac{2\sqrt{L_0^2 + (u\Delta t/2)^2}}{c}$$

Substituting for L_0 and solving for Δt yields

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} \qquad \text{(time dilation)}$$

As the $u \leq c$, the denominator of this expression is at most 1, meaning that $\Delta t \leq \Delta t_0$; that is, O measures a longer time interval than O'. For observer O, the beginning and ending of the time interval occur at the same location, so the interval Δt_0 observed by them is known as *proper time*.

The Relativity of Length

Suppose the beam of light instead travels in the same direction as that of the motion of O'. According to O, the distance traveled is L, while from the perspective of O' (from which the clock is at rest), it is L_0.

The light takes time interval Δt_1 to reach the mirror, in which it travels distance $c\Delta t_1$, which is equal to the length L plus the distance $u\Delta t_1$ traveled by by O':

$$c\Delta t_1 = L + u\Delta t_1$$

The light then takes time interval Δt_2 to return to its source, in which it travels distance $c\Delta t_2$, equal to the length L minus the distance $u\Delta t_2$:

$$c\Delta t_2 = L - u\Delta t_2$$

Adding these times yields Δt:

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{L}{c-u} + \frac{L}{c+u} = \frac{2L}{c}\frac{1}{1-u^2/c^2}$$

Applying the formula for time dilation,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-u^2/c^2}} = \frac{2L_0}{c}\frac{1}{\sqrt{1-u^2/c^2}}$$

Setting the above two equations equal to each other yields

$$L = L_0\sqrt{1-\frac{u^2}{c^2}} \qquad \text{(length contraction)}$$

Observer O', who is at rest relative to the object, measures the *rest length* L_0 (the *proper length*) while all observers relative to whom O' is in motion measure a shorter length *only in the direction of motion*. A moving body appears to be shorter than one at rest.

It should be noted that length contraction occurs only in the direction of movement, meaning that any dimensions perpendicular to that of movement are unaffected.

Relativistic Velocity Addition

A source emits particles traveling at speed v' according to observer O', who is at rest with respect with respect to the source. A bulb is triggered to flash when the particle reaches it. This flash makes a return trip to the source. The time interval Δt_0 measured by O' is comprised of the time it takes for the particle to travel distance L_0 at speed v' and the time it takes the flash to travel the same distance at c:

$$\Delta t_0 = \frac{L_0}{v'} + \frac{L_0}{c}$$

O' is moving at speed u relative to observer O. For O, the particle traveling at speed v according to O reaches the bulb after Δt_1, traveling a distance of $v\Delta t_1$ equal to the (contracted) length L plus the distance $u\Delta t_1$ moved by O':

$$v\Delta t_1 = L + u\Delta t_1$$

In the interval Δt_2, the light beam traverses distance $c\Delta t_2$ equal to length L minus the distance $u\Delta t_2$ traveled by O':

$$c\Delta t_2 = L - u\Delta t_2$$

Solving the above two equation for Δt_1 and Δt_2 and adding them yields the total interval Δt between the emission of the particle and the arrival of the beam at the detector according to O. Applying the time dilation formula to relate this result to Δt_0 and using the length contraction formula to relate L to L_0 yields the result

$$v = \frac{v'+u}{1+v'/uc^2} \qquad \text{(relativistic velocity addition)}$$

Note that when $v' = c$ (a beam of light being observed),

$$v = \frac{c+u}{1+cu/c^2} = \frac{c+u}{1+u/c} = \frac{c+u}{\dfrac{c+u}{c}} = c$$

regardless of the value of u. All observers measure the same speed of light c regardless of their reference frame.

The Relativistic Doppler Effect

The classical Doppler effect states that an observer O moving relative to the source of waves detects a frequency f' different from the true frequency f being emitted by the source S:

$$f' = f\frac{v \pm v_O}{v \mp v_S} \qquad \text{(classical Doppler effect)}$$

where v is the speed of the waves through some medium, v_S is the speed of the source *relative to the medium*, and O is that of the observer *relative to the medium*. The upper signs are chosen when S and O are moving towards each other while the lower ones are chosen when they are moving away from each other.

Consider a source of electromagnetic waves at rest in the reference frame of observer O. An observer O' moving relative to the source at speed u will observe a different frequency. Suppose O observes N waves emitted at frequency f. It takes an interval $\Delta t_0 = N/f$ for these N waves to be emitted from the point of view of O. This is the proper time interval in the frame of reference of O. The corresponding interval to O' is $\Delta t'$, during which O moves a distance $u\Delta t'$. The wavelength λ' according to O' is the total length occupied by the waves divided by the number of waves:

$$\lambda' = \frac{c\Delta t' + u\Delta t'}{N} = \frac{c\Delta t' + u\Delta t'}{f\Delta t_0}$$

The frequency from the reference frame of O' is $f' = c/\lambda'$, so

$$f' = \frac{c}{\lambda'} = f\frac{\Delta t_0}{\Delta t'}\frac{1}{1 + u/c}$$

Applying the time dilation formula,

$$f' = f\frac{\sqrt{1 - u^2/c^2}}{1 + u/c} = f\sqrt{\frac{1 - u/c}{1 + u/c}} \qquad \text{(relativistic Doppler shift)}$$

This is the formula for the *relativistic Doppler shift* where the waves are observed parallel to $\vec{u}$. It should be noted that unlike the classical formula, this formula does *not* distinguish between the source and observer motion; it is only dependent on the relative speed u between the source and observer.

Restated,

$$\lambda' = \lambda\sqrt{\frac{1 + u/c}{1 - u/c}} \qquad\qquad \frac{u}{c} = \frac{(\lambda'/\lambda)^2 - 1}{(\lambda'/\lambda)^2 + 1}$$

2.5 The Lorentz Transformation

While the Galilean transformation holds for the "common-sense" experience observed at low speeds, it does not agree with experiments performed at high speeds. A new set of transformations is therefore required that is capable of reproducing relativistic effects such as time dilation, length contraction, velocity addition, and the Doppler shift.

A transformation that enables observers O and O' moving relative to each other to compare their measurements in space and time coordinates of the same event. Teh transformation equations

relate the measurements x, y, z, t in O to x', y', z, and t' in O'. This transformation must be linear (depending only on the first powers of each coordinate, following the homogeneity of space and time), consistent with Einstein's postulates, and it must reduce to the Galilean transformation when the relative speed between the observers is small. It is again assumed that the relative speed is only in the positive xx' direction.

The new transformation is called the *Lorentz transformation*:

$$x' = \frac{x - ut}{1 - u^2/c^2} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - (u/c^2)x}{\sqrt{1 - u^2/c^2}} \qquad \text{(Lorentz transformation)}$$

This is sometimes rewritten as

$$x' = \gamma(x - \beta ct) \qquad y' = y \qquad z' = z \qquad ct' = \gamma(ct - \beta x)$$

or

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma\left(t - x\frac{v}{c^2}\right)$$

or in matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ ct' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ct \end{bmatrix} \qquad \begin{bmatrix} x' \\ ct' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix} \qquad \begin{bmatrix} x \\ ct \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} x' \\ ct'' \end{bmatrix}$$

where

$$\beta = \frac{u}{c} \qquad \text{and} \qquad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

It is often useful to write these equation in terms of intervals by replacing each coordinate by their corresponding interval.

The equations for y' and z' are identical to their Galilean counterparts while the first and last equations reduce to theirs when $u \ll c$.

The Lorentz transformation can then be used to derive some predictions of special relativity.

Length Contraction

A rod of length L_0 is at rest as observed by O'. The rod extends from x_1' to x_2'; that is, $L_0 = x_2' - x_1'$. Observer O, relative to whom the rod is in motion, measures the ends of the rod at x_1 and x_2. For the length of the rod to be measured by O, x_1 and x_2 must be *simultaneously* determined.

Suppose O' flashes a bulb at x_1' and t_1' and then flashes a bulb at x_2' and t_2'. The Lorentz transformation gives

$$x_1' = \frac{x_1 - ut_1}{\sqrt{1 - u^2/c^2}} \qquad\qquad x_2' = \frac{x_2 - ut_2}{\sqrt{1 - u^2/c^2}}$$

Subtracting the first equation from the second yields

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - u^2/c^2}} - \frac{u(t_2 - t_2)}{\sqrt{1 - u^2/c^2}}$$

The bulbs must be set off such that the flashes appear to be simultaneous to O. This enables a simultaneous determination to be made of the rod's endpoints. if the flashes are observed simultaneously, $t_1 = t_2$, so

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - u^2/c^2}}$$

Substituting L_0 for $x_2' - x_1'$ and L for $x_2 - x_1$ yields

$$L = L_0\sqrt{1 - u^2/c^2}$$

Velocity Transformation

Suppose O observes a particle traveling with velocity $\vec{v}$. The relationship between $\vec{v}$ and the velocity $\vec{v}'$ observed by O' is given by the *Lorentz velocity transformation*

$$v_x' = \frac{v_x - u}{1 - v_x u/c^2} \quad v_y' = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2} \quad v_z' = \frac{v_z\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2} \quad \text{(Lorentz velocity transformation)}$$

Solving the above equations for v_x, it can be shown that this is identical to the equation for relativistic velocity addition. It should be noted that for $u \ll c$, the Lorentz velocity transformation reduces to its Galilean counterpart. Note also that despite $y' = y$, $v_y' \neq v_y$. This is due to the Lorentz transformation's handling of t.

These transformation can be derived from the coordinate transformation. Differentiating the coordinate transformation for x' yields

$$\mathrm{d}x' = \frac{\mathrm{d}x - u\,\mathrm{d}t}{1 - u^2/c^2}$$

Differentiating the time coordinate transformation yields

$$\mathrm{d}t' = \frac{\mathrm{d}t - (u/c^2)\,\mathrm{d}x}{1 - u^2/c^2}$$

v_x' is therefore

$$v_x' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\dfrac{\mathrm{d}x - u\,\mathrm{d}t}{1 - u^2/c^2}}{\dfrac{\mathrm{d}t - (u/c^2)\,\mathrm{d}x}{1 - u^2/c^2}} = \frac{\mathrm{d}x - u\,\mathrm{d}t}{\mathrm{d}t - (u/c^2)\,\mathrm{d}x} = \frac{\mathrm{d}x/\mathrm{d}t - u}{1 - (u/c^2)\,\mathrm{d}x/\mathrm{d}t} = \frac{v_x - u}{1 - v_x u/c^2}$$

Differentiating $y' = y$ yields $\mathrm{d}y' = \mathrm{d}y$, so

$$v_y' = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\dfrac{\mathrm{d}t - (u/c^2)\,\mathrm{d}x}{\sqrt{1 - u^2/c^2}}} = \frac{\mathrm{d}y\sqrt{1 - u^2/c^2}}{\mathrm{d}t - (u/c^2)\,\mathrm{d}x} = \frac{\sqrt{1 - u^2/c^2}\,\mathrm{d}y/\mathrm{d}t}{1 - (u/c^2)\,\mathrm{d}x/\mathrm{d}t} = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - uv_x/c^2}$$

Differentiating for $z' = z$ yields $\mathrm{d}z' = \mathrm{d}z$, so

$$v_z' = \frac{\mathrm{d}z'}{\mathrm{d}t'} = \frac{\mathrm{d}z}{\dfrac{\mathrm{d}t - (u/c^2)\,\mathrm{d}x}{\sqrt{1 - u^2/c^2}}} = \frac{\mathrm{d}z\sqrt{1 - u^2/c^2}}{\mathrm{d}t - (u/c^2)\,\mathrm{d}x} = \frac{\sqrt{1 - u^2/c^2}\,\mathrm{d}z/\mathrm{d}t}{1 - (u/c^2)\,\mathrm{d}x/\mathrm{d}t} = \frac{v_z\sqrt{1 - u^2/c^2}}{1 - uv_x/c^2}$$

2.6 The Twin Paradox

Suppose there are a pair of twins on earth. Twin A remains on Earth while twin B sets off on a rocket ship. A believes that as a consequence of time dilation, B should be younger than them by the time they return. For two observers in relative motion, though, *each* believes the *other's* clock to be slow. From the point of view of B, A make a round-trip away from them and back again. Under such circumstances, it should be A's clock that is running slow, so A they expect A to be younger when they return.

The resolution of this paradox comes from considering the asymmetric roles of the twins. Special relativity applies only to inertial frames. In order to leave and reenter Earth, B must accelerate and correspondingly decelerate. Although this may be over a brief time interval, B's return journey occurs in a different inertial frame to their departure. This change in B's inertial frame is what results in the asymmetry in the twins' ages. Only B is required to change inertial frames to return, so *all observers will agree* that B is the one that is "really" in motion, so it is therefore their clock that is "really" running slow, meaning that B will indeed be the younger twin upon their return.

Spacetime Diagrams

A *spacetime* diagram plots time on the y-axis and space on the x-axis. The slope of a straight line is velocity as a proportion of c.

2.7 Relativistic Dynamics

Dynamical quantities such as momentum and kinetic energy are dependent on length, time, and velocity. As such, they are also affected by Einstein's postulates.

Consider a collision of two particles that is elastic from the reference frame of O'. Particle 1 of mass $m_1 = 2m$ is initially at rest while particle 2 of mass $m_2 = m$ is moving in the negative x direction with initial velocity $v'_{2,i} = -0.75c$. The final velocities of particles 1 and 2 are respectively $v'_{1,f} = -0.5c$ and $v'_{2,f} = 0.25c$. Analyzing this collision classically,

$$p'_i = m_1 v'_{1,i} + m_2 v'_{2,i} = -0.75mc \qquad\qquad p'_i = m_1 v'_{2,f} + m_2 v'_{2,f} = -0.75mc = p'_i$$

As the initial and final momenta are equal according to O', energy is conserved.

Suppose the reference frame of O' is moving in the positive x direction at a speed of $u = 0.55c$ relative to O. The velocity transformation yields the velocities of $v_{1,i} = 0.55c$, $v_{2,i} = -0.34c$, $v_{1,f} = 0.069c$, and $v_{2,f} = 0.703c$. Analyzing the collision,

$$p_i = m_1 v_{1,i} + m_2 v_{2,i} = 0.76mc \qquad\qquad p_f = m_1 v_{1,f} + m_2 v_{2,f} = 0.841mc \neq p_i$$

As $p_i \neq p_f$, momentum is not conserved according to O. This means that the law of conservation of linear momentum does not satisfy Einstein's first postulate. To retain the conservation of momentum, then, a new definition of momentum is required. This definition must yield a conservation law that holds under relativity (momentum being conserved regardless of the reference frame, so long as it is inertial) and reduces to $\vec{p} = m\vec{v}$ at low speeds.

These requirements are satisfied by defining relativistic momentum to be

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \qquad\qquad \text{(relativistic momentum)}$$

The velocity v is the velocity of the particle as measured from a particular inertial frame; it is not the velocity of said frame.

Using this relativistic definition of momentum to analyze the above collision,

$$p_i' = \frac{m_1 v_{1,i}'}{\sqrt{1 - v_{1,i}'^2/c^2}} + \frac{m v_{2,i}'}{\sqrt{1 - v_{2,i}'^2/c^2}} \approx 1.134 mc \quad p_f' = \frac{m_2 v_{1,f}'}{\sqrt{1 - v_{1,f}'^2/c^2}} + \frac{m_2 v_{2,f}'}{\sqrt{1 - v_{2,f}'^2/c^2}} \approx 1.134 mc = p_i'$$

As $p_i' = p_f'$, O' is able to conclude that momentum is conserved. To O,

$$p_i = \frac{m_1 v_{1,i}}{\sqrt{1 - v_{1,i}^2/c^2}} + \frac{m v_{2,i}}{\sqrt{1 - v_{2,i}^2/c^2}} \approx 0.956 mc \quad p_f = \frac{m_1 v_{1,f}}{\sqrt{1 - v_{1,f}^2/c^2}} + \frac{m_2 v_{2,f}}{\sqrt{1 - v_{2,f}^2/c^2}} \approx 0.946 mc = p_i$$

so O' can also conclude that momentum is conserved.

Relativistic Kinetic Energy

Much like classical momentum, classical kinetic energy also fails under relativity. According to O', the aforementioned collision has kinetic energies

$$K_i' = \frac{1}{2} m_1 v_{1,i}'^2 + \frac{1}{2} m_2 v_{2,i}'^2 \approx 0.281 mc^2 \qquad K_f' = \frac{1}{2} m_1 v_{1,f}'^2 + \frac{1}{2} m_2 v_{2,f}'^2 \approx 0.281 mc^2 = K_i'$$

but according to O is has

$$K_i = \frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2} m_2 v_{2,i}^2 \approx 0.36 mc^2 \qquad K_f = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 \approx 0.252 mc^2 \neq K_i$$

so O concludes that energy is not conserved.

The relativistic definition of kinetic energy is

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \qquad \text{(relativistic kinetic energy)}$$

There is no limit to the amount of kinetic energy that can be given to a particle. Under the classical definition of kinetic energy, this means that speed must also lack a limit, which violates Einstein's second postulate.

Relativistic Total Energy and Rest Energy

Relativistic kinetic energy can also be expressed as

$$K = E - E_0 \qquad \text{(relativistic kinetic energy)}$$

where the *relativistic total energy* E is defined as

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \qquad \text{(relativistic total energy)}$$

and the *rest energy* E_0 is defined is

$$E_0 = mc^2 \qquad \text{(rest energy)}$$

That of en electron is

$$E_e \approx 0.511 \, \text{MeV}$$

This suggests that mass can be expressed in units of MeV/c^2. The m in this equation is sometimes called the *rest mass* m_0, which is distinguished from the "relativistic mass" m', defined as

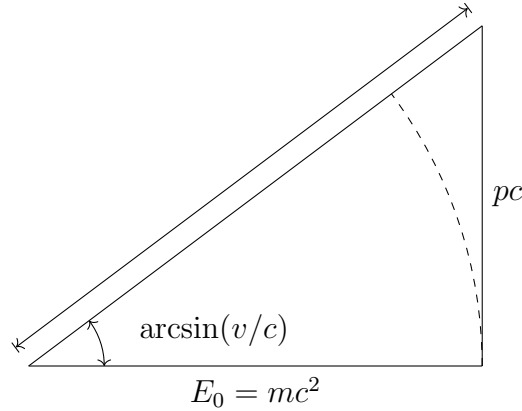$$m' = \frac{m_0}{\sqrt{1 - v^2/c^2}} \qquad \text{(relativistic mass)}$$

The total energy can be found as the sum of the kinetic and rest energies:

$$E = E_0 + K$$

Manipulating the equations for relativistic momentum and total energy yields the relationship

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

This can be remembered by

A particle traveling close to the speed of light ($v > 0.99c$), $K \gg E_0$. IN this case,

$$E \cong pc$$

This is the *extreme relativistic approximation* and can often simplify calculations. For massless particles (such as photons),

$$E = pc$$

All massless particles travel at c; otherwise, their kinetic and total energies would be zero.

2.8 Conservation Laws in Relativistic Decays and Collisions

The law of conservation of momentum can be applied to all collisions. The only difference for high speeds is that the relativistic definition of momentum must be used. It can be stated the same:

> *In an isolated system of particles, the total linear momentum remains constant.*

Classically, the only form of energy present in an elastic collision is kinetic, so the conservation of energy is simply that of kinetic energy. In inelastic collisions or decay processes, kinetic energy is not conserved. The total energy does indeed remain constant, though; the other forms of energy are simply unaccounted for. This missing energy is usually stored in particles as atomic or nuclear energy.

Relativistically, the internal stored energy is accounted for by the rest energy. In atomic and nuclear processes, the only forms of energy generally accounted for are kinetic and rest (though radiation will be considered later). A loss of kinetic energy then implies an increase in rest energy, as the total relativistic energy does not change.

Consider a reaction in which new particles are produced. The loss of kinetic energy in the original particles is what provides the rest energy of the new particle. In a nuclear decay process, on the other hand, the initial nucleus loses some rest energy in the form of the kinetic energy given to the decay products.

The relativistic law of conservation of energy is as follows:

> *In an isolated system of particles, the relativistic total energy (kinetic energy plus rest energy) remains constant.*

When applying this law to relativistic collisions, it does not matter whether a collision is elastic or inelastic, as the inclusion of rest energy accounts for any potential loss of kinetic energy.

2.9   Extra

The **Epstein Cosmic Speedometer** is a plot of proper time (on the vertical axis) against space (on the horizontal axis). A body in motion will move through both space and time, making it an angle line. A stationary body will move only in time, making it a vertical line. The only thing able to move through space but not time is light, making it a horizontal line.

A spacetime diagram maintains the length of all vectors, plotting space on the horizontal axis and time on the vertical axis. The vertical component of a vector is the "proper" while its horizontal component is the speed through space.

Moving through space means moving slower through time; that is, moving objects effectively have slower clocks.

Graphs can be made in Lorentz/Minkowski Space. The axes are x and ct. The new axes can be found as lines with direct variation of slopes x' and ct'. Note that ct' and x' are not perpendicular. To graph with them, then, one must first go along one axis and then parallel along the other. The greater β, is, the closer together the ct' and x' axes; that is, the slower a reference frame is traveling, the less it skews spacetime. The graph can be used to plot a point in one reference frame and see its coordinates in the other.

It should be noted that
$$(ct)^2 - x^2 = (ct')^2 - x'^2 \qquad \text{(invariant)}$$

Chapter 3

The Particlelike Properties of Electromagnetic Radiation

Wave mechanics is the second theory on which modern physics is based. One of its consequences is the breakdown of the classically made distinction between particles and waves. Light, which is usually treated as a wave, can have properties associated with particles. Rather than spreading energy smoothly over a wave, it is delivered in discrete packets (*quantum*) known as *photons*.

3.1   Review of Electromagnetic Waves

An electromagnetic field is characterized by its electric field $\vec{E}$ and magnetic field $\vec{B}$. The electric field at distance r from a point charge q is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \qquad\qquad \text{(electric field)}$$

where $\hat{r}$ is a unit vector in the radial direction. The magnetic field at distance r from a long, straight wire carrying a current i along the z-axis is

$$\vec{B} = \frac{\mu_0 i}{2\pi r}\hat{\varphi} \qquad\qquad \text{(magnetic field)}$$

where $\hat{\varphi}$ is a unit vector in the azimuthal direction (in the xy-plane) in cylindrical coordinates. (Note this definition is different than its mathematical definition, where the azimuthal angle is in the yz-plane.)

If the charges are accelerated or the current is varied with time, an electromagnetic wave is produced, in which the electric and magnetic fields vary not only with $\vec{r}$ but also with t. The mathematical expression used to describe this can take several forms depending on the properties of the source wave and the medium that is being traversed. A special form is the *plane wave*, for which the wave fronts are planes. (A point source produces spherical waves, having spherical wave fronts.) A plane electromagnetic wave traveling in the positive z direction is described by

$$\vec{E} = \vec{E}_0 \sin(kz - \omega t) \qquad\qquad \vec{B} = \vec{B}_0 \sin(kz - \omega t) \qquad\qquad \text{(planar wave)}$$

where the *wave number* k is found as

$$k = \frac{2\pi}{\lambda} \qquad\qquad \text{(wave number)}$$

where λ is the wavelength and the *angular frequency* ω is found as

$$\omega = 2\pi f \qquad \text{(angular frequency)}$$

where f is the frequency. As $c = \lambda f$,

$$c = \frac{\omega}{k}$$

The wave's polarization is represented by $\vec{E}_0$, the plane of polarization being determined by the direction of $\vec{E}_0$ and the direction of propagation (in this case $+z$). The direction of $\vec{B}_0$ is then fixed by the requirement that $\vec{B}$ be perpendicular to $\vec{E}$ and the direction of travel and that $\vec{E} \times \vec{B}$ must point in the direction of travel.

The magnitude of $\vec{B}_0$ is determined by

$$B_0 = \frac{E_0}{c}$$

An electromagnetic wave transfers energy, the flux of which is determined by the *Poynting vector* $\vec{S}$:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \qquad \text{(Poynting vector)}$$

For a plane wave, this reduces to

$$\vec{S} = \frac{1}{\mu_0} E_0 B_0 \sin^2(kz - \omega t) \ \hat{k}$$

The units of the Poynting vector are that of power per unit area, such as W/m^2.

An electromagnetic wave with $\vec{E}$ oscillating about the x-axis and $\vec{B}$ about the y, $\vec{P}$ points in the $+z$ direction.

Consider a detector of electromagnetic radiation with area A placed perpendicular to the z axis. The power entering the receiver is

$$P = SA = \frac{1}{\mu_0} E_0 B_0 A \sin^2(kz - \omega t)$$

which can be rewritten as

$$P = \frac{1}{\mu_0 c} E_0^2 A \sin^2(kz - \omega t)$$

Note that the intensity (the average power per unit area) is proportional to E_0^2. In general, *the intensity of a wave is proportional to the square of its amplitude.*

It should also be noted that the intensity fluctuates with time with frequency

$$f = \frac{\omega}{2\pi}$$

This rapid fluctuation is typically not observed. The average power can be found from the observation time T as

$$P_{\text{avg}} = \frac{1}{T} \int_0^T P \, dt$$

The intensity I can be obtained as

$$I = \frac{P_{\text{avg}}}{A} = \frac{1}{2\mu_0 c} E_0^2$$

as the average value of $\sin^2 \theta$ is 0.5.

Interference and Diffraction

The most unique property of waves as physical phenomena is the *principle of superposition*, which allows two waves meeting at a point to have a combined disturbance on that point that may be greater or less than that provided each wave individually.

This property leads to *interference* and *diffraction*.

The simplest example of the former is *Young's double-slit experiment*. A monochromatic plane wave is given two narrow slits. The slits *diffract* the plane wave, resulting in the area covered by the light passing through each slit being far greater than the areas of the slits. This causes the light from each slit to overlap, resulting in an interference pattern.

A bright region in the interference pattern is from *constructive interference*, where the peaks of each beam align, resulting in their amplitudes being added. It occurs at point where the distance from one slit is exactly one wavelength greater than that from the other.

If X_1 and X_2 are distances from the point to the two slits, then in order for maximum interference to occur

$$|X_1 - X_2| = n\lambda, \quad n \in \mathbb{Z}^+ \qquad \text{(constructive interference)}$$

When the two waves cancel, there is a dark region, created by *destructive interference*. (The existence of destructive interference shows that the powers are not added, as power is always positive, but rather the electric fields.) This occurs when the distances are such that the phases of the waves differ by an integer number of half-cycles:

$$|X_1 - X_2| = \frac{2n+1}{2}, \quad n \in \mathbb{N} \qquad \text{(destructive interference)}$$

Crystal Diffraction of X Rays

3.2 The Photoelectric Effect

When a light is shone on a metal surface, electrons may be emitted. This is know as the *photoelectric effect*. The emitted electrons are called *photoelectrons*.

The rate of electron emission from the surface can be measured as an electric current by an ammeter. The maximum kinetic energy of the electrons can then be measured by applying a negative potential V_S (called the *stopping potential* to a collector for the electrons that is just enough to repel the most energetic electrons (the ammeter dropping to 0). This means that

$$K_{\mathrm{max}} = -\Delta U = eV_{\mathrm{s}}$$

Classically, the metal's surface is illuminated by en electromagnetic wave of intensity I. The surface absorbs the energy until that energy exceeds the binding energy of the electron, at which point the electron is released. This minimum energy is called the metal's *work function* φ.

The Classical Theory of the Photoelectric Effect

Classical wave theory makes 3 key predictions regarding the emitted photoelectrons:

1. *The maximum kinetic energy of the electrons should be proportional to the intensity of the radiation.* As the brightness increases, so does the energy carried by the radiation, meaning that more energy is transferred to the surface (the electric field is greater), so the kinetic energy of the electrons should also be greater.

2. *The photoelectric effect should occur for light of any frequency or wavelength.* So long as the light is intense enough to release electrons, the photoelectric effect should occur regardless of the frequency or wavelength.

3. *The first electrons should be emitted on the order of a few seconds from the radiation beginning to strike the surface.* The energy of the wave is uniformly distributed over the wave front. If the energy is absorbed directly from the wave, the amount of energy emitted to an electron is dependent on how much radiant energy is incident on the area in which the electron is confined. Assuming this area to be about the size of an atom, the time lag should be on the order of seconds.

The Quantum Theory of the Photoelectric Effect

Einstein proposed that the energy of electromagnetic radiation is not continuously distributed but rather in discrete packets called *quanta* (or *photons*. A photon's energy is associated with the wavelength and frequency as

$$E = hf = \frac{hc}{\lambda} \qquad \text{(energy of a photon)}$$

Photons are often treated as particles. Like electromagnetic waves, they travel at the speed of light. They must also obey the relativistic relationship $p = E/c$, which means that

$$p = \frac{h}{\lambda}$$

In addition to energy, photons carry linear momentum.
Because a proton travels at the speed of light, its mass must be 0; otherwise, it would have infinite energy and momentum. Its rest energy is also 0.
According to Einstein, a photoelectron is released as a result of an encounter with a *single photon*, the entire energy of which is delivered instantaneously to a *single photoelectron*. If the photon energy is greater than the work function of the material, the photoelectron will be released. Otherwise, the photoelectric effect will not occur. This explanation accounts for two failures of classical wave theory: the existence of a cutoff frequency and the lack of a measurable time delay.
If the photon's energy exceeds the work function, the excess energy becomes kinetic energy for the electron:

$$K_{\mathrm{max}} = hf - \varphi$$

The intensity of the light is not factored into this equation. Increasing the intensity while maintaining the frequency and wavelength simply means that the number of photoelectrons released is increased; their maximum kinetic energies are unchanged.
A photon with energy equal to the work function corresponds with light of frequency f_{c}, called the cutoff frequency:

$$f_{\mathrm{c}} = \frac{\varphi}{h} \qquad \text{(cutoff frequency)}$$

The corresponding cutoff wavelength is

$$\lambda_{\mathrm{c}} = \frac{hc}{\varphi} \qquad \text{(cutoff wavelength)}$$

This is the *largest* wavelength for which the cutoff frequency can be observed.

3.3　Thermal Radiation

Thermal radiation, the radiation released by all objects due to their temperature, is not explained by classical wave theory.

Experiments yield two key characteristics regarding radiation:

1. The total intensity radiated over all wavelengths (the area under each curve) increases with temperature as

$$I = \sigma T^4 \qquad \text{(Stefan's law)}$$

   where proportionality constant σ is

$$\sigma \approx 5.67 \times 10^{-8} \, \frac{\text{W}}{\text{m}^2} \, \text{K}^4 \qquad \text{(Stefan-Boltzmann constant)}$$

   as determined experimentally.

2. The wavelength λ_{max} at which the intensity peaks decreases as the temperature increases:

$$\lambda_{\text{max}} T \approx 2.9 \times 10^{-3} \, \text{m K} \qquad \text{(Wien's displacement law)}$$

To simplify analysis of thermal radiation, a *blackbody* is considered, which absorbs all radiation incident on it, reflecting none of it. To further simplify, a special type is considered: a hole in a hollow metal box with walls that are in thermal equilibrium at temperature T. The box is filled with electromagnetic radiation that is emitted and reflected by the walls. A small hole in one of the walls allows some radiation to escape. *It is the hole, not the box itself, that is the blackbody.* Radiation from outside that is incident on the hole enters the box, having a negligible change of reemerging from the hole; hence no reflections occur from the blackbody. The radiation that emerges from the hold is a sample of the radiation inside the box, so understanding the radiation inside the box enables an understanding of the radiation leaving through the hole.

Consider the radiation inside the box. It has energy density per unit wavelength interval $u(\lambda)$; that is, the energy density of the electromagnetic radiation with wavelengths between λ and $\lambda + d\lambda$ in a small volume element is $u(\lambda) \, d\lambda$. At a given instant, half of the radiation within the box is moving away from the hole. The other half is moving towards it with velocity of magnitude c directed in a range of angles. Averaging over this range to evaluate the energy flowing perpendicular to the hole's surface introduces another factor of $1/2$, making the contribution of the radiation in this wavelength interval to the intensity passing through the hole

$$I(\lambda) = \frac{c}{4} u(\lambda)$$

The quantity $I(\lambda) \, d\lambda$ is the radiant intensity over the interval $d\lambda$. To find the total intensity emitted, one must integrate this quantity over all wavelengths:

$$I = \int_0^\infty I(\lambda) \, d\lambda$$

Classical Theory of Thermal Radiation

The following predictions are made by classical theories of electromagnetism and thermodynamics regarding the relationship between I and λ.

1. *The box is filled with electromagnetic standing waves.* As the box's walls are metal, the radiation is reflected back and forth with a node of the electric field at each wall (the electric field within a conductor being 0). This same condition applies to other standing waves.

2. *The number of standing waves with wavelengths between λ and $\lambda + \mathrm{d}\lambda$ is*

$$N(\lambda)\,\mathrm{d}\lambda = \frac{8\pi V}{\lambda^4}\,\mathrm{d}\lambda$$

   where V is the volume of the box. For one-dimensional standing waves, as on a string of length L, the allowed wavelengths are

$$\lambda = \frac{2L}{n} \quad \text{for} \quad n \in \mathbb{Z}^+$$

   The number of possible standing waves with wavelengths between λ_1 and λ_2 is

$$n_2 - n_1 = 2L\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$$

   In the interval from λ to $\mathrm{d}\lambda$, the number of standing waves is

$$N(\lambda)\,\mathrm{d}\lambda = \left|\frac{\mathrm{d}n}{\mathrm{d}\lambda}\right|\mathrm{d}\lambda = \frac{2L}{\lambda^2}\,\mathrm{d}\lambda$$

   Extending this approach to three dimensions yields the number of standing waves between λ and $\lambda + \mathrm{d}\lambda$ as stated above.

3. *Each individual waves contributes an average energy kT to the radiation in the box.*

Quantum Theory of Thermal Radiation

3.4   The Compton Effect

The Compton effect describes the phenomenon of radiation scattering from loosely bound, nearly free electrons. Part of the energy of the radiation is given to the electron while the remainder is reradiated as electromagnetic radiation. Under classical wave theory, the scattered radiation is less energetic than the incident radiation (as some energy must go into the kinetic energy of the electron). The concept of photons, however, gives rise to a much different prediction.

The scattering process is analyzed as an interaction between a single photon and electron, which is assumed to be at rest. Initially, the photon has energy E and linear momentum p given by

$$E = hf = \frac{hc}{\lambda} \qquad \text{and} \qquad p = \frac{E}{c}$$

The electron has rest energy $m_e c^2$. After the scattering, the photon has energy and momentum

$$E' = \frac{hc}{\lambda'} \qquad \text{and} p' = \frac{E'}{c}$$

and is moving in a direction at angle θ with respect to that of the incident photon. The electron has total final energy E_e and momentum p_e and moves in a direction at angle φ with respect to the

initial photon. (To account for high-energy incident photons given energetic scattering electrons, relativistic kinematics are used for the electron.) Applying the relativistic conservation laws,

$$E_i = E_f : \qquad\qquad E + m_e c^2 = E' + E_e$$

$$p_{x,i} = p_{x,f} : \qquad\qquad p = p_e \cos\varphi + p' \cos\theta$$

$$p_{y,i} = p_{y,f} : \qquad\qquad 0 = p_e \sin\varphi + p' \sin\theta$$

This provides three equations with four unknowns (θ, φ, E_e, E'; p_e and p' are not independent unknowns) that cannot be solved uniquely. Any two of them can be eliminated by solving the equations simultaneously. Choosing to measure the energy and direction of the scattered photons, eliminates E_e and φ. The latter is eliminated by first rewriting the momentum equations as

$$p_e \cos\varphi = p - p' \cos\theta \qquad \text{and} \qquad p_e \sin\varphi = p' \sin\theta$$

Squaring and adding yields

$$p_e^2 = p^2 - 2pp' \cos\theta + p'^2$$

The relativistic relationship between energy and momentum is

$$E_e^2 = c^2 p_e^2 + m_e^2 c^4$$

Substituting this into the equation for E_e obtained by the conservation of energy and for p_e^2 from the above equation yields

$$(E + m_e c^2 - E')^2 = c^2(p^2 - 2pp' \cos\theta + p'^2) + m_e^2 c^4$$

Rewriting,

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2}(1 - \cos\theta)$$

In terms of wavelengths,

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

where λ is the wavelength of the incident photon and λ' is that of the scattered photon, and the quantity

$$\lambda_c = \frac{h}{m_e c} \approx 0.002\,42\,\text{nm} \qquad\qquad \text{(Compton wavelength of the electron)}$$

is the *Compton wavelength of the electron*. It should be noted that this is a *change* of wavelength rather than a wavelength in and of itself.

The above equations give the change in energy or wavelength as a function of the *scattering angle* θ. As the quantity on the right side is always positive, E' is always less than E, so the scattered photon has less energy than the original incident photon; the difference is simply the kinetic energy given to the electron, $E_e - m_e c^2$. Similarly, λ' is greater than λ, meaning that the wavelength of the scattered photon is always longer than that of the incident photon, the change in wavelength ranging from 0 at $\theta = 0°$ to $2\lambda_c$ at $\theta = 180°$. These two descriptions are of course equivalent, the choice of which to use being simply a matter of convenience.

Using

$$E_e = K_e + m_e c^2$$

conservation of energy can also be written as

$$E = m_e c^2 = E' + K_e + m_e c^2$$

19

Solving for K_e yields

$$K_e = E - E'$$

That is, the kinetic energy of the electron is equal to the difference in the energies of the initial and final photon energies.

The direction of the electron's motion can be found by dividing the momentum relationships:

$$\tan \varphi = \frac{p_e \sin \varphi}{p_e \cos \varphi} = \frac{p' \sin \theta}{p - p' \cos \theta} = \frac{E' \sin \theta}{E - E' \cos \theta}$$

the final result coming from

$$p = \frac{E}{c} \qquad \text{and} \qquad p' = \frac{E'}{c}$$

3.5   Other Photon Processes