Assignment 5

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June 27, 2022

1. Find the Directional derivative of the function $f(x, y) = x^3 y - 2x^2 y^2$ in the direction of the vector $\vec{v} = \hat{\imath} + 2\hat{\jmath}$ at the point $P(1, 1)$.

   Solution

   $$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 3x^2 y - 4xy^2, x^3 - 4x^2 y \rangle$$

   $$\nabla f(1, 1) = \langle -1, -3 \rangle$$

   $$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{\sqrt{1^2 + 2^2}} = \frac{\vec{v}}{\sqrt{5}}$$

   $$D_{\vec{u}} f(1, 1) = \nabla f(1, 1) \cdot \vec{u} = \frac{(-1)(1) + (-3)(2)}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$

2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2 - y^2 + z^2 - 2z = 4$

   Solution

   Let $F(x, y, z) = x^2 - y^2 + z^2 - 2z$. If $z \neq 1$,

   $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z - 2} = -\frac{x}{z - 1} \qquad\qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-2y}{2z - 1} = \frac{y}{z - 1}$$

3. Find the local maximum and local minimum values and saddle points of $f(x, y) = xy - 2x - 2y - x^2 - y^2$.

   Solution

   $$f_x(x, y) = y - 2 - 2x = 0 \qquad\qquad f_y(x, y) = x - 2 - 2y = 0$$

   $$y = 2x + 2 \qquad\qquad\qquad\qquad y = \frac{x}{2} - 1$$

   $$2x + 2 = \frac{x}{2} - 1$$

   $$4x + 4 = x - 2$$

   $$3x = -6$$

   $$x = -2 \qquad\qquad\qquad\qquad\qquad y = -2$$

   $$f(-2, 2) = 4$$

   $$f_{xx}(x, y) = -2 \qquad\qquad\qquad\qquad f_{yy}(x, y) = -2$$

   $$f_{xx}(-2, -2) = -2 \qquad\qquad\qquad\quad f_{yy}(-2, -2) = -2$$

   $$f_{xy}(x, y) = 1$$

   $$f_{xy}(-2, -2) = 1$$

   $$D(-2, -2) = (-2)(-2) - 1^2$$

   $$= 3$$

   As $D(-2, -2) > 0$ and $f_{xx}(-2, -2) < 0$, the point $(-2, -2, 4)$ is a relative maximum of f. As f lacks any other critical values, it lacks relative minima and saddle points.