

# Volume: Disk/Washer

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## Sources

### Calculus: Early Transcendentals 9<sup>th</sup> Edition

- 2. 6.2 Exercise 12
- 4. 6.2 Exercise 15
- 5. 6.2 Exercise 1
- 7. 6.2 Exercise 17
- 10. 6.2 Exercise 24

### Calculus 2: Washer Method (HARD Problems) by Ludus

<https://www.youtube.com/watch?v=BcFEcRimurc>

- 6. Example 1
- 8. Example 2

### AP Calculus Exams

- 11. 2021 AB FRQ 3(c)

## Problems

If no instructions are given, evaluate the volume of the solid generated by revolving the region bounded by the given equations about the specified line using the disk/washer method.

1.

$$\left. \begin{array}{l} y = r \quad y = 0 \\ x = 0 \quad x = h \end{array} \right| y = 0$$

2.

$$\left. \begin{array}{l} y = 0 \quad y = \frac{1}{x} \\ x = 1 \quad x = 4 \end{array} \right| y = 0$$

3.

$$\left. \begin{array}{l} y = 0 \quad y = \frac{1}{\sqrt{1+x^2}} \\ x = 0 \end{array} \right| y = 0$$

4.

$$\left. \begin{array}{l} y = \frac{x^2}{4} \quad y = 9 \\ x = 0 \end{array} \right| x = 0$$

5.

$$\left. \begin{array}{l} y = 0 \quad y = x^2 + 5 \\ x = 0 \quad x = 3 \end{array} \right| y = 0$$

6.

$$\left. \begin{array}{l} y = 1 + \sec x \quad y = 3 \\ 1 + \sec x \leq 3 \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \end{array} \right| y = 1$$

7.

$$\left. \begin{array}{l} y = x^2 \quad y = 2x \end{array} \right| x = 0$$

8.

$$\left. \begin{array}{l} x = y^2 \quad x = 1 - y^2 \end{array} \right| x = 3$$

9.

$$\left. \begin{array}{l} y = x^2 - 2x \quad y = x \end{array} \right| y = 4$$

10.

$$\left. \begin{array}{l} y = \sin x \quad y = \cos x \\ x \geq 0 \quad x \leq \frac{\pi}{4} \end{array} \right| y = -1$$

11.

$$f(x) = cx\sqrt{4-x^2}$$

The solid of revolution generated by rotating the area bounded by  $f$  and the  $x$ -axis in the first quadrant about the  $x$ -axis is equal to  $2\pi$ . Solve for  $c$ , given that it is a positive constant.

12.

$$x = (y - k)^2 \quad y = x + k \mid k \leq 0 \text{ or } k \geq 1 \mid x = k$$

13.

$$\left. \begin{array}{l} \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \mid a, b \neq 0 \quad x = \left| \frac{ay}{b} \right| \\ x = a^2 \quad x = \sqrt{a^2 + b^2} \end{array} \right| x = 0$$

14.

$$\left. \begin{array}{l} \frac{x^4}{a^2} - \frac{y^2}{b^2} = 1 \mid a, b \neq 0 \quad x^2 = \frac{ay}{b} \\ y = 0 \quad y = b \end{array} \right| x = 0$$

15.

$$\left. \begin{array}{l} x^2 + y^2 = r^2 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \mid a, b > r > 0 \\ \text{(a) } y = 0 \quad \text{(b) } y = b \end{array} \right|$$

## Solutions

1.

$$V = \pi \int_0^h [r^2] dx = \pi [r^2 x]_0^h = \pi [r^2 h - (0)] = \pi r^2 h$$

2.

$$V = \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx = \pi \left[-\frac{1}{x}\right]_1^4 = \pi \left[-\frac{1}{4} - \left(-\frac{1}{1}\right)\right] = \frac{3\pi}{4}$$

3.

$$\frac{1}{\sqrt{1+x^2}} = 0 \implies x \rightarrow \infty$$

$$V = \pi \int_0^\infty \left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx = \pi \int_0^\infty \left[\frac{1}{1+x^2}\right] dx = \pi [\arctan x]_0^\infty = \pi \left[\frac{\pi}{2} - (0)\right] = \frac{\pi^2}{2}$$

4.

$$y = \frac{x^2}{4} \implies x = 2\sqrt{y}$$

$$y_1 = 2\sqrt{0} = 0$$

$$V = \pi \int_0^9 (2\sqrt{y})^2 dy = \pi [2y^2]_0^9 = 2(81)\pi = 162\pi$$

5.

$$\begin{aligned} V &= \pi \int_0^3 (x^2 + 5)^2 dx = \pi \int_0^3 [x^4 + 10x^2 + 25] dx = \pi \left[\frac{x^5}{5} + \frac{10x^3}{3} + 25x\right]_0^3 \\ &= \pi \left[\frac{3^5}{5} + \frac{10(3)^3}{3} + 25(3) - (0)\right] = \pi \left[\frac{243}{5} + 90 + 75\right] = \frac{\pi(243 + 825)}{5} = \frac{1068\pi}{5} \end{aligned}$$

6.

$$1 + \sec x = 3 \implies \sec x = 2 \implies x = \pm \frac{\pi}{3}$$

$$\begin{aligned} V &= \pi \int_{-\pi/3}^{\pi/3} [(3-1)^2 - (1+\sec x-1)^2] dx = 2\pi \int_0^{\pi/3} [4 - \sec^2 x] dx \\ &= 2\pi [4x - \tan x]_0^{\pi/3} = 2\pi \left[\frac{4\pi}{3} - \sqrt{3} - (0)\right] = \frac{8\pi^2}{3} - 2\pi\sqrt{3} \end{aligned}$$

7.

$$y = x^2 \implies x = \sqrt{y} \quad y = 2x \implies x = \frac{y}{2}$$

$$\sqrt{y} = \frac{y}{2} \implies 4y = y^2 \implies 0 = y(y-4) \implies y_1 = 0, y_2 = 4$$

$$\begin{aligned} V &= \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right] dy = \pi \int_0^4 \left[y - \frac{y^2}{4}\right] dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 \\ &= \pi \left[\frac{y^2(6-y)}{12}\right]_0^4 = \pi \left[\frac{4^2(6-4)}{12} - (0)\right] = \pi \left[\frac{16(2)}{12}\right] = \frac{8\pi}{3} \end{aligned}$$

8.

$$\begin{aligned}
y^2 = 1 - y^2 &\implies 2y^2 = 1 \implies y = \pm \frac{\sqrt{2}}{2} \\
V &= \pi \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left[ (y^2 - 3)^2 - (1 - y^2 - 3)^2 \right] dy = 2\pi \int_0^{\sqrt{2}/2} \left[ (y^2 - 3)^2 - (-y^2 - 2)^2 \right] dy \\
&= 2\pi \int_0^{\sqrt{2}/2} [y^4 - 6y^2 + 9 - (y^4 + 4y^2 + 4)] dy = 2\pi \int_0^{\sqrt{2}/2} [5 - 10y^2] dy \\
&= 2\pi \left[ 5y - \frac{10y^3}{3} \right]_0^{\sqrt{2}/2} = 2\pi \left[ \frac{5\sqrt{2}}{2} - \frac{10}{3} \left( \frac{\sqrt{8}}{8} \right) \right] = \pi \left[ 5\sqrt{2} - \frac{40\sqrt{2}}{24} \right] = \frac{10\pi\sqrt{2}}{3}
\end{aligned}$$

9.

$$\begin{aligned}
x = x^2 - 2x &\implies 0 = x^2 - 3x = x(x - 3) \implies x_1 = 0, x_2 = 3 \\
V &= \pi \int_0^3 \left[ (x^2 - 2x - 4)^2 - (x - 4)^2 \right] dx \\
&= \pi \int_0^3 [x^4 - 2x^3 - 4x^2 - 2x^3 + 4x^2 + 8x - 4x^2 + 8x + 16 - (x^2 - 8x + 16)] dx \\
&= \pi \int_0^3 [x^4 - 4x^3 - 4x^2 + 16x + 16 - (x^2 - 8x + 16)] dx \\
&= \pi \int_0^3 [x^4 - 4x^3 - 5x^2 + 24x] dx = \pi \left[ \frac{x^5}{5} - x^4 - \frac{5x^3}{3} + 12x^2 \right]_0^3 \\
&= \pi \left[ \frac{x^2(3x^3 - 15x^2 - 25x + 180)}{15} \right]_0^3 = \pi \left[ \frac{3(81 - 135 - 75 + 180)}{5} - (0) \right] \\
&= \frac{3\pi(51)}{5} = \frac{153\pi}{5}
\end{aligned}$$

10.

$$\begin{aligned}
\sin x = \cos x &\implies x = \frac{\pi}{4} \\
V &= \pi \int_0^{\pi/4} [(\cos x + 1)^2 - (\sin x + 1)^2] dx \\
&= \pi \int_0^{\pi/4} [\cos^2 x + 2\cos x + 1 - \sin^2 x - 2\sin x - 1] dx \\
&= \pi \int_0^{\pi/4} [\cos(2x) + 2\cos x - 2\sin x] dx = \pi \left[ \frac{\sin(2x)}{2} + 2\sin x + 2\cos x \right]_0^{\pi/4} \\
&= \pi \left[ \frac{1}{2} + \sqrt{2} + \sqrt{2} - (0 + 0 + 2) \right] = \frac{(4\sqrt{2} - 3)\pi}{2}
\end{aligned}$$

11.

$$\begin{aligned}
0 = cx\sqrt{4-x^2} &\implies \begin{cases} x=0 \\ \sqrt{4-x^2}=0 \end{cases} \implies 4-x^2=0 \implies 4=x^2 \implies x=2 \\
2\pi = \pi \int_0^2 (cx\sqrt{4-x^2})^2 dx &= \pi \int_0^2 [c^2x^2(4-x^2)] dx = \pi \int_0^4 [4c^2x^2 - c^2x^4] dx \\
&= \pi \left[ \frac{4c^2x^3}{3} - \frac{c^2x^5}{5} \right]_0^2 = \pi \left[ \frac{4c^2(2)^3}{3} - \frac{c^2(2)^5}{5} - (0) \right] = \pi \left[ \frac{32c^2}{3} - \frac{32c^2}{5} \right] \\
2 &= 32c^2 \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{64c^2}{15} \\
c &= \sqrt{\frac{30}{64}} = \sqrt{\frac{15}{32}}
\end{aligned}$$

12.

$$y = x + k \implies x = y - k$$

For both equations,  $k$  only controls a vertical shift. As this shift is the same for both equations, the area between the curves is constant regardless of the value of  $k$ . The volume when revolved about the same axis will therefore also be the same.

$$\begin{aligned}
y = y^2 &\implies 0 = y(y-1) \implies y_1 = 0, y_2 = 1 \quad (k=0) \\
V = \pi \int_0^1 [(y-k)^2 - (y^2-k)^2] dy &= \pi \int_0^1 [y^2 - 2ky + k^2 - y^4 + 2ky^2 - k^2] dy \\
&= \pi \int_0^1 [-y^4 + y^2(2k+1) - 2ky] dy = \pi \left[ -\frac{y^5}{5} + \frac{y^3(2k+1)}{3} - ky^2 \right]_0^1 \\
&= \pi \left[ \frac{-3y^5 + 5y^3(2k+1) - 15ky^2}{15} \right]_0^1 = \pi \left[ \frac{-3 + 5(2k+1) - 15k}{15} - (0) \right] \\
&= \frac{\pi(-3 + 10k + 5 - 15k)}{15} = \frac{\pi(2 - 5k)}{15}
\end{aligned}$$

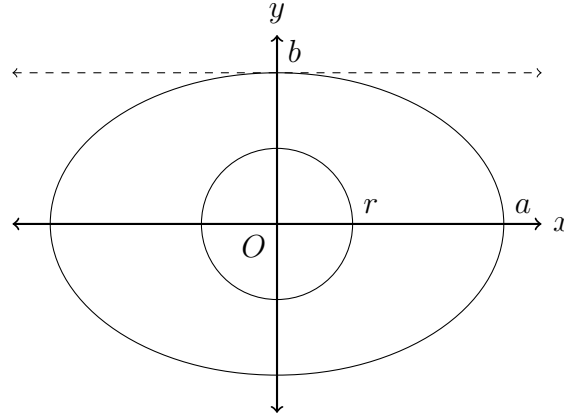
13.

$$\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 &\implies x = a\sqrt{1 + \frac{y^2}{b^2}} = \frac{a\sqrt{b^2 + y^2}}{b} \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 &\implies y = \frac{b\sqrt{x^2 - a^2}}{a} \\
y_1 = \frac{b\sqrt{a^4 - a^2}}{a} &= b\sqrt{a^2 - 1} \quad y_2 = \frac{b\sqrt{a^2 + b^2 - a^2}}{a} = \frac{b^2}{a} \\
V = 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} \left[ \left( \frac{a\sqrt{b^2 + y^2}}{b} \right)^2 - \left( \frac{ay}{b} \right)^2 \right] dy \right| &= 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} \left[ \frac{a^2(b^2 + y^2 - y^2)}{b^2} \right] dy \right| \\
&= 2\pi \left| \int_{b\sqrt{a^2-1}}^{b^2/a} [a^2] dy \right| = 2\pi \left| [a^2y]_{b\sqrt{a^2-1}}^{b^2/a} \right| = 2\pi \left| \frac{a^2b^2}{a} - (a^2b\sqrt{a^2-1}) \right| \\
&= \left| 2\pi ab(b - a\sqrt{a^2-1}) \right|
\end{aligned}$$

14.

$$\begin{aligned}
1 &= \frac{x^4}{a^2} - \frac{y^2}{b^2} \implies x = \sqrt{\frac{a\sqrt{b^2+y^2}}{b}} \\
V &= 2\pi \left| \int_0^b \left[ \frac{a\sqrt{b^2+y^2}}{b} - \frac{ay}{b} \right] dy \right| = 2\pi \left| \int_0^b \left[ \frac{a(\sqrt{b^2+y^2}-y)}{b} \right] dy \right| \\
I(\alpha) &= \int \left[ \sqrt{\alpha^2+y^2} \right] dy \\
y &= \alpha \tan \theta \implies dy = \alpha \sec^2 \theta d\theta \\
I(\alpha) &= \int \left[ \alpha \sec^2 \theta \sqrt{\alpha^2 + \alpha^2 \tan^2 \theta} \right] d\theta = \int \left[ \alpha^2 \sec^3 \theta \right] d\theta \\
u &= \sec \theta \implies du = \sec \theta \tan \theta \quad dv = \sec^2 \theta \implies v = \tan \theta \\
\int \left[ \sec^3 \theta \right] d\theta &= uv - \int v du = \sec \theta \tan \theta - \int \left[ \sec \theta \tan^2 \theta \right] d\theta \\
\int \left[ \sec \theta \tan^2 \theta \right] d\theta &= \int \left[ \sec \theta (\sec^2 \theta - 1) \right] d\theta = \int \left[ \sec^3 \theta - \sec \theta \right] \\
&= \int \left[ \sec^3 \theta \right] d\theta - \ln |\sec \theta + \tan \theta| \\
\int \left[ \sec^3 \theta \right] d\theta &= \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| - \int \left[ \sec^3 \theta \right] d\theta \\
2 \int \left[ \sec^3 \theta \right] d\theta &= \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \\
\int \left[ \sec^3 \theta \right] d\theta &= \frac{\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|}{2} + C \\
I(\alpha) &= \alpha^2 \int \left[ \sec^3 \theta \right] d\theta = \frac{\alpha^2 (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)}{2} \\
\tan \theta &= \frac{y}{\alpha} \quad \sec \theta = \frac{\sqrt{\alpha^2+y^2}}{\alpha} \\
I(\alpha) &= \frac{\alpha^2}{2} \left( \frac{y\sqrt{\alpha^2+y^2}}{\alpha^2} + \ln \left| \frac{y + \sqrt{\alpha^2+y^2}}{\alpha} \right| \right) \\
&= \frac{1}{2} \left( y\sqrt{\alpha^2+y^2} + \alpha^2 \ln \left| \frac{y + \sqrt{\alpha^2+y^2}}{\alpha} \right| \right) + C \\
V &= \frac{2a\pi}{b} \left| \left[ I(b) - \frac{y^2}{2} \right]_0^b \right| \\
&= \frac{2a\pi}{b} \left| \left[ \frac{1}{2} \left( y\sqrt{b^2+y^2} + b^2 \ln \left| \frac{y + \sqrt{b^2+y^2}}{b} \right| - y^2 \right) \right]_0^b \right| \\
&= \frac{a\pi}{b} \left| \left[ b\sqrt{2b^2} + b^2 \ln \left| \frac{b + \sqrt{2b^2}}{b} \right| - b^2 - (0) \right] \right| \\
&= \left| ab\pi \left( \sqrt{2} - 1 + \ln(1 + \sqrt{2}) \right) \right|
\end{aligned}$$

15.



This graph assumes  $a > b$ , which need not be the case.

$$x^2 + y^2 = r^2 \implies y = \sqrt{r^2 - x^2} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b\sqrt{a^2 - x^2}}{a}$$

(a)

$$\begin{aligned}
 V_1 &= \pi \int_0^r \left[ \left( \frac{b\sqrt{a^2 - x^2}}{a} \right)^2 - \left( \sqrt{r^2 - x^2} \right)^2 \right] dx = \pi \int_0^r \left[ \frac{b^2(a^2 - x^2)}{a^2} - (r^2 - x^2) \right] dx \\
 &= \pi \int_0^r \left[ \frac{a^2(b^2 - r^2) + x^2(a^2 - b^2)}{a^2} \right] dx = \pi \left[ \frac{a^2x(b^2 - r^2)}{a^2} + \frac{x^3(a^2 - b^2)}{3a^2} \right]_0^r \\
 &= \pi \left[ \frac{3a^2x(b^2 - r^2) + x^3(a^2 - b^2)}{3a^2} \right]_0^r = \pi \left[ \frac{3a^2r(b^2 - r^2) + r^3(a^2 - b^2)}{3a^2} \right] \\
 &= \pi \left[ \frac{3a^2b^2r - 3a^2r^3 + a^2r^3 - b^2r^3}{3a^2} \right] = \pi \left( \frac{3a^2b^2r - 2a^2r^3 - b^2r^3}{3a^2} \right) \\
 V_2 &= \pi \int_r^a \left( \frac{b\sqrt{a^2 - x^2}}{a} \right)^2 dx = \pi \int_r^a \left[ \frac{b^2(a^2 - x^2)}{a^2} \right] dx = \pi \int_r^a \left[ \frac{a^2b^2 - b^2x^2}{a^2} \right] dx \\
 &= \pi \left[ \frac{a^2b^2x}{a^2} - \frac{b^2x^3}{3a^2} \right]_r^a = \pi \left[ \frac{3a^2b^2x - b^2x^3}{3a^2} \right]_r^a = \left[ \frac{3a^3b^2 - b^2a^3}{3a^2} - \left( \frac{3a^2b^2r - b^2r^3}{3a^2} \right) \right] \\
 &= \pi \left( \frac{2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2} \right) \\
 V &= 2(V_1 + V_2) = 2\pi \left( \frac{3a^2b^2r - 2a^2r^3 - b^2r^3 + 2a^3b^2 - 3a^2b^2r + b^2r^3}{3a^2} \right) \\
 &= 2\pi \left( \frac{2a^3b^2 - 2a^2r^3}{3a^2} \right) = 4\pi \left( \frac{ab^2 - r^3}{3} \right)
 \end{aligned}$$





(b)

$$\begin{aligned}
V_1 &= \pi \int_0^r \left[ \left( \sqrt{r^2 - x^2} - b \right)^2 - \left( \frac{b\sqrt{a^2 - x^2}}{a} - b \right)^2 \right] dx \\
&= \pi \int_0^r \left[ (r^2 - x^2) - 2b\sqrt{r^2 - x^2} + b^2 - \left( \frac{b^2(a^2 - x^2)}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right) \right] dx \\
&= \pi \int_0^r \left[ r^2 - x^2 - 2b\sqrt{r^2 - x^2} + b^2 - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2 - x^2}}{a} - b^2 \right] dx \\
&= \pi \int_0^r \left[ r^2 - x^2 - 2b\sqrt{r^2 - x^2} - b^2 + \frac{b^2x^2}{a^2} + \frac{2b^2\sqrt{a^2 - x^2}}{a} \right] dx \\
&= \pi \left( \left[ (r^2 - b^2)x + \left( \frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right]_0^r + 2b \int_0^r \left[ \frac{b\sqrt{a^2 - x^2}}{a} - \sqrt{r^2 - x^2} \right] dx \right) \\
I(\alpha) &= \int \left[ \sqrt{\alpha^2 - x^2} \right] dx \implies x = \alpha \sin \theta \implies dx = \alpha \cos \theta d\theta \\
&= \int \left[ \cos \theta \sqrt{\alpha^2 - \alpha^2 \sin^2 \theta} \right] d\theta = \int \left[ \alpha^2 \cos^2 \theta \right] d\theta = \alpha^2 \int \left[ \frac{\cos(2\theta) + 1}{2} \right] d\theta \\
&= \alpha^2 \left( \frac{\sin(2\theta)}{4} + \frac{\theta}{2} \right) + C = \alpha^2 \left( \frac{2 \sin \theta \cos \theta}{4} + \frac{\theta}{2} \right) + C \\
&= \alpha^2 \left( \frac{\left( \frac{x}{\alpha} \right) \left( \frac{\sqrt{\alpha^2 - x^2}}{\alpha} \right)}{2} + \frac{\arcsin(x/\alpha)}{2} \right) + C = \frac{x\sqrt{\alpha^2 - x^2} + \alpha^2 \arcsin(x/\alpha)}{2} + C \\
V_1 &= \pi \left[ (r^2 - b^2)x + \left( \frac{b^2}{a^2} - a \right) \frac{x^3}{3} + 2b \left( \frac{bI(a)}{a} - I(r) \right) \right]_0^r \\
&= \pi \left[ (r^2 - b^2)x + \left( \frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right. \\
&\quad \left. + 2b \left( \frac{bx\sqrt{a^2 - x^2} + a^2b \arcsin(x/a)}{2a} - \frac{x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r)}{2} \right) \right]_0^r \\
&= \pi \left[ (r^2 - b^2)x + \left( \frac{b^2}{a^2} - 1 \right) \frac{x^3}{3} \right. \\
&\quad \left. + 2b \left( \frac{xb\sqrt{a^2 - x^2} + a^2b \arcsin(x/a)}{2a} - \frac{x\sqrt{r^2 - x^2} + r^2 \arcsin(x/r)}{2} \right) \right]_0^r \\
&= \pi \left[ x \left( r^2 - b^2 + \frac{x^2}{3} \left( \frac{b^2}{a^2} - 1 \right) \right) \right. \\
&\quad \left. + \frac{b}{a} \left( x \left( b\sqrt{a^2 - x^2} - a\sqrt{r^2 - x^2} \right) + a^2b \arcsin \left( \frac{x}{a} \right) - r^2 \arcsin \left( \frac{x}{r} \right) \right) \right]_0^r \\
&= \pi \left[ r \left( r^2 - b^2 + \frac{r^2}{3} \left( \frac{b^2}{a^2} - 1 \right) \right) \right. \\
&\quad \left. + \frac{b}{a} \left( r \left( b\sqrt{a^2 - r^2} - 0 \right) + a^2b \arcsin \left( \frac{r}{a} \right) - \frac{a\pi r^2}{2} \right) - (0) \right] \\
&= \pi \left( r^3 - b^2r + \frac{b^2r^3}{3a^2} - \frac{r^3}{3} + \frac{b^2r\sqrt{a^2 - r^2}}{a} + ab^2 \arcsin \left( \frac{r}{a} \right) - \frac{b\pi r^2}{2} \right) \\
&= \pi \left( \frac{2r^3(2a^2 + b^2) - 3a^2b\pi r^2 + 6ab^2r(\sqrt{a^2 - r^2} - a) + 6a^3b^2 \arcsin(r/a)}{6a^2} \right)
\end{aligned}$$

(b) (cont.)

$$\begin{aligned}
V_2 &= \pi \int_r^a \left( \frac{b\sqrt{a^2 - x^2}}{a} - b \right)^2 dx = \pi \int_r^a \left[ \frac{b^2(a^2 - x^2)}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right] dx \\
&= \pi \int_r^a \left[ b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} + b^2 \right] dx = \pi \int_r^a \left[ 2b^2 - \frac{b^2x^2}{a^2} - \frac{2b^2\sqrt{a^2 - x^2}}{a} \right] dx \\
&= \pi \left[ 2b^2x - \frac{b^2x^3}{3a^2} - \frac{2b^2I(a)}{a} \right]_r^a = \pi \left[ 2b^2x - \frac{b^2x^3}{3a^2} - \frac{b^2x\sqrt{a^2 - x^2} + a^2b^2 \arcsin(x/a)}{a} \right]_r^a \\
&= \pi \left[ 2b^2a - \frac{b^2a^3}{3a^2} - \frac{0 + a^2b^2(\pi/2)}{a} \right. \\
&\quad \left. - \left( 2b^2r - \frac{b^2r^3}{3a^2} - \frac{b^2r\sqrt{a^2 - r^2} + a^2b^2 \arcsin(r/a)}{a} \right) \right] \\
&= \pi \left[ 2ab^2 - 2b^2r + \frac{b^2r^3 - a^3b^2}{3a^2} + \frac{b^2r\sqrt{a^2 - r^2} + a^2b^2 \arcsin(r/a) - 0.5a^2b^2\pi}{a} \right] \\
&= \frac{b^2\pi (6a^3 - 6a^2r + r^3 - a^3 + 3ar\sqrt{a^2 - r^2} + 3a^3 \arcsin(r/a) - 1.5a^3\pi)}{3a^2} \\
&= \frac{b^2\pi (2r^3 + 6ar(\sqrt{a^2 - r^2} - 2a) + a^3(10 + 6 \arcsin(r/a) - 3\pi))}{6a^2}
\end{aligned}$$

$$\begin{aligned}
V &= 4(V_1 + V_2) \\
&= \frac{2\pi}{3a^2} \left( 4r^3(2a^2 + b^2) - 3a^2b\pi r^2 + 6ab^2r \left( 2\sqrt{a^2 - r^2} - 3a \right) \right. \\
&\quad \left. + a^3b^2 \left( 12 \arcsin \left( \frac{r}{a} \right) + 10 - 3\pi \right) \right)
\end{aligned}$$

# Indeterminate Exponents (Type 3)

Arnav Patri

## Sources

**Indeterminate Forms (Type 3) by Faruk Özger**

[https://www.youtube.com/watch?v=2t-E1d9\\_cRM](https://www.youtube.com/watch?v=2t-E1d9_cRM)

1. Example 2
6. Example 4

**Calculus: Early Transcendentals 9<sup>th</sup> Edition**

3. 4.4 Exercise 61
5. 4.4 Exercise 57
8. 4.4 Exercise 60
9. 4.4 Exercise 65
10. 4.4 Exercise 66

## Problems

Evaluate the following limits. Use comparative growth rates when necessary.

1.

$$\lim_{x \rightarrow 0^+} [x^x]$$

2.

$$\lim_{x \rightarrow \infty} \left( \frac{1}{x} \right)^{1/x}$$

3.

$$\lim_{x \rightarrow 1^+} [x^{1/(1-x)}]$$

4.

$$\lim_{x \rightarrow \infty} (\ln x)^{1/x}$$

5.

$$\lim_{x \rightarrow \infty} \left( \frac{1}{e^x} \right)^{\arctan x - \pi/2}$$

6.

$$\lim_{x \rightarrow 0^+} [x^{\sqrt{x}}]$$

7.

$$\lim_{x \rightarrow 0^+} (1 + \sin(7x))^{\cot(5x)}$$

8.

$$\lim_{x \rightarrow \infty} \left( 1 + \frac{a}{x} \right)^{bx}$$

9.

$$\lim_{x \rightarrow 0^+} (4x + 1)^{\cot x}$$

10.

$$\lim_{x \rightarrow 0^+} (1 - \cos x)^{\sin x}$$

11.

$$\lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^{\frac{\cos x}{x}}$$

12.

$$\lim_{x \rightarrow \infty} \left( \sum_{n=0}^{\infty} \left[ \frac{1}{x^n} \right] \right)^x$$

13.

$$\lim_{n \rightarrow 1^+} \left( \int_n^{\infty} \left[ \frac{1}{x^n} \right] dx \right)^{n-1}$$

14.

$$\lim_{n \rightarrow \infty} \left( \int_n^\infty \left[ \frac{1}{x^n} \right] dx \right)^{\int_{n+1}^\infty \left[ \frac{1}{x^{n+1}} \right] dx}$$

15.

$$\lim_{b \rightarrow \infty} \left( \int_0^b \left[ \sum_{n=0}^\infty \left[ \frac{1}{x^n} \right] \right] dx \right)^{e^{-b}}$$

## Solutions

1.

$$\begin{aligned}
 L &= \lim_{x \rightarrow 0^+} [x^x] && \implies 0^0 \\
 \ln L &= \lim_{x \rightarrow 0^+} [x \ln x] && \implies 0 \times -\infty \\
 &= \lim_{x \rightarrow 0^+} \left[ \frac{\ln x}{1/x} \right] && \implies -\frac{\infty}{\infty} \\
 &= \lim_{x \rightarrow 0^+} \left[ \frac{1/x}{-1/x^2} \right] = \lim_{x \rightarrow 0^+} [-x] = 0 \\
 L &= e^0 = 1
 \end{aligned}$$

2.

$$\begin{aligned}
 L &= \lim_{x \rightarrow \infty} \left( \frac{1}{x} \right)^{1/x} && \implies 0^0 \\
 \ln L &= \lim_{x \rightarrow \infty} \left[ \frac{\ln(1/x)}{x} \right] = \lim_{x \rightarrow \infty} \left[ -\frac{\ln x}{x} \right] && \implies -\frac{\infty}{\infty} \\
 &= \lim_{x \rightarrow \infty} \left[ -\frac{1/x}{1} \right] = 0 \\
 L &= e^0 = 1
 \end{aligned}$$

3.

$$\begin{aligned}
 L &= \lim_{x \rightarrow 1^+} [x^{1/(1-x)}] && \implies 1^\infty \\
 \ln L &= \lim_{x \rightarrow 1^+} \left[ \frac{\ln x}{1-x} \right] && \implies \frac{0}{0} \\
 &= \lim_{x \rightarrow 1^+} \left[ -\frac{1/x}{1} \right] = -\frac{1/1}{1} = -1 \\
 L &= e^{-1} = \frac{1}{e}
 \end{aligned}$$

4.

$$\begin{aligned}
 L &= \lim_{x \rightarrow \infty} (\ln x)^{1/x} && \implies \infty^0 \\
 \ln L &= \lim_{x \rightarrow \infty} \left[ \frac{\ln x}{x} \right] && \implies \frac{\infty}{\infty} \\
 &= \lim_{x \rightarrow \infty} \left[ \frac{1/x}{1} \right] = 0 \\
 L &= e^0 = 1
 \end{aligned}$$

5.

$$\begin{aligned}
L &= \lim_{x \rightarrow 0^+} \left[ x^{\sqrt{x}} \right] && \Rightarrow 0^0 \\
\ln L &= \lim_{x \rightarrow 0^+} \left[ \sqrt{x} \ln x \right] && \Rightarrow 0 \times (-\infty) \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{\ln x}{x^{-1/2}} \right] && \Rightarrow -\frac{\infty}{\infty} \\
&= \lim_{x \rightarrow 0^+} \left[ -\frac{1/x}{0.5x^{-3/2}} \right] = \lim_{x \rightarrow 0^+} [-2\sqrt{x}] = 0 \\
L &= e^0 = 1
\end{aligned}$$

6.

$$\begin{aligned}
L &= \lim_{x \rightarrow \infty} \left( \frac{1}{e^x} \right)^{\arctan x - \pi/2} && \Rightarrow 0^0 \\
\ln L &= \lim_{x \rightarrow \infty} \left[ -x \left( \arctan x - \frac{\pi}{2} \right) \right] && \Rightarrow -\infty \times 0 \\
&= \lim_{x \rightarrow \infty} \left[ -\frac{\arctan x - \pi/2}{1/x} \right] && \Rightarrow \frac{0}{0} \\
&= \lim_{x \rightarrow \infty} \left[ \frac{\frac{1}{x^2+1}}{\frac{1}{x^2}} \right] = \lim_{x \rightarrow \infty} \left[ \frac{x^2}{x^2+1} \right] = 1 \\
L &= e^1 = e
\end{aligned}$$

7.

$$\begin{aligned}
L &= \lim_{x \rightarrow 0^+} (1 + \sin(7x))^{\cot(5x)} && \Rightarrow 1^\infty \\
\ln L &= \lim_{x \rightarrow 0^+} [\cot(5x) \ln(1 + \sin(7x))] && \Rightarrow \infty \times 0 \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{\ln(1 + \sin(7x))}{\tan(5x)} \right] && \Rightarrow \frac{0}{0} \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{\frac{7 \cos(7x)}{1 + \sin(7x)}}{5 \sec^2(5x)} \right] = \lim_{x \rightarrow 0^+} \left[ \frac{7 \cos(7x)}{5 \sec^2(5x)(1 + \sin(7x))} \right] = \frac{7}{5} \\
L &= e^{7/5}
\end{aligned}$$

8.

$$\begin{aligned}
L &= \lim_{x \rightarrow \infty} \left( 1 + \frac{a}{x} \right)^{bx} && \Rightarrow 1^\infty \\
\ln L &= \lim_{x \rightarrow \infty} \left[ bx \ln \left( 1 + \frac{a}{x} \right) \right] && \Rightarrow \infty \times 0 \\
&= \lim_{x \rightarrow \infty} \left[ \frac{b \ln \left( 1 + \frac{a}{x} \right)}{x^{-1}} \right] && \Rightarrow \frac{0}{0} \\
&= \lim_{x \rightarrow \infty} \left[ \frac{\frac{-ba x^{-2}}{1 + \frac{a}{x}}}{-x^{-2}} \right] = \lim_{x \rightarrow \infty} \left[ \frac{ab}{1 + \frac{a}{x}} \right] = \frac{ab}{1 + 0} = ab \\
L &= e^{ab}
\end{aligned}$$



9.

$$\begin{aligned}
L &= \lim_{x \rightarrow 0^+} (4x + 1)^{\cot x} && \Rightarrow 1^\infty \\
\ln L &= \lim_{x \rightarrow 0^+} [(\cot x) \ln(4x + 1)] && \Rightarrow \infty \times 0 \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{\ln(4x + 1)}{\tan x} \right] && \Rightarrow \frac{0}{0} \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{4/(x + 1)}{\sec^2 x} \right] = \frac{4/1}{1} = 4 \\
L &= e^4
\end{aligned}$$

10.

$$\begin{aligned}
L &= \lim_{x \rightarrow 0^+} (1 - \cos x)^{\sin x} && \Rightarrow 0^0 \\
\ln L &= \lim_{x \rightarrow 0^+} [(\sin x) \ln(1 - \cos x)] && \Rightarrow 0 \times (-\infty) \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{\ln(1 - \cos x)}{\csc x} \right] && \Rightarrow -\frac{\infty}{\infty} \\
&= \lim_{x \rightarrow 0^+} \left[ -\frac{\sin x / (1 - \cos x)}{\csc x \cot x} \right] = \lim_{x \rightarrow 0^+} \left[ -\frac{\sin^2 x \tan x}{1 - \cos x} \right] && \Rightarrow -\frac{0}{0} \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{2 \sin x \cos x \tan x + \sin^2 x \sec^2 x}{\sin x} \right] \\
&= \lim_{x \rightarrow 0^+} [2 \cos x \tan x + \sin x \sec x] \\
&= 2(1)(0) + (0)(1) = 0 \\
L &= e^0 = 1
\end{aligned}$$

11.

$$\begin{aligned}
\lim_{x \rightarrow 0^+} \left[ \frac{\sin x}{x} \right] &= \lim_{x \rightarrow 0} [\cos x] = 1 && \Rightarrow \frac{0}{0} \\
\lim_{x \rightarrow 0^+} \left[ \frac{\cos x}{x} \right] &= \frac{1}{0} = \infty \\
L &= \lim_{x \rightarrow 0^+} \left( \frac{\sin x}{x} \right)^{\frac{\cos x}{x}} = \left( \lim_{x \rightarrow 0^+} \left[ \frac{\sin x}{x} \right] \right)^{\lim_{x \rightarrow 0^+} \left[ \frac{\cos x}{x} \right]} && \Rightarrow 0^\infty \\
\ln L &= \lim_{x \rightarrow 0^+} \left[ \frac{\cos x \ln \left( \frac{\sin x}{x} \right)}{x} \right] && \Rightarrow \frac{0}{0} \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{\cos x (\ln(\sin x) - \ln x)}{x} \right] \\
&= \lim_{x \rightarrow 0^+} \left[ -\sin x \ln \left( \frac{\sin x}{x} \right) + \cos x \left( \frac{x}{\sin x} \right) \left( \frac{x \cos x - \sin x}{x^2} \right) \right] \\
&= (0)(0) + (1) \left( \frac{1}{1} \right) \lim_{x \rightarrow 0^+} \left[ \frac{x \cos x - \sin x}{x^2} \right] && \Rightarrow \frac{0}{0} \\
&= \lim_{x \rightarrow 0^+} \left[ \frac{-x \sin x + \cos x - \cos x}{2x} \right] = \lim_{x \rightarrow 0^+} \left[ -\frac{\sin x}{2} \right] = 0 \\
L &= e^0 = 1
\end{aligned}$$

12.

$$\begin{aligned}
\sum_{n=0}^{\infty} \left[ \frac{1}{x^n} \right] &= \frac{1}{1 - 1/x} = \frac{1}{(x-1)/x} = \frac{x}{x-1} \\
L &= \lim_{x \rightarrow \infty} \left( \sum_{n=0}^{\infty} \left[ \frac{1}{x^n} \right] \right)^x = \lim_{x \rightarrow \infty} \left( \frac{x}{x-1} \right)^x && \Rightarrow 1^\infty \\
\ln L &= \lim_{x \rightarrow \infty} \left[ x \ln \left( \frac{x}{x-1} \right) \right] && \Rightarrow \infty \times 0 \\
&= \lim_{x \rightarrow \infty} \left[ \frac{\ln(x/(x-1))}{1/x} \right] && \Rightarrow \frac{0}{0} \\
&= \lim_{x \rightarrow \infty} \left[ \frac{1}{-1/x^2} \times \frac{1}{x/(x-1)} \times \frac{1(x-1) - 1(x)}{(x-1)^2} \right] \\
&= \lim_{x \rightarrow \infty} \left[ -x^2 \times \frac{x-1}{x} \times \frac{x-1-x}{(x-1)^2} \right] = \lim_{x \rightarrow \infty} \left[ \frac{-x^2(x-1)(-1)}{x(x-1)^2} \right] \\
&= \lim_{x \rightarrow \infty} \left[ \frac{x}{x-1} \right] = 1 \\
L &= e^1 = e
\end{aligned}$$

13.

$$\begin{aligned}
\int_n^\infty \left[ \frac{1}{x^n} \right] dx &= \left[ -\frac{1}{(n-1)x^{n-1}} \right]_n^\infty && \Leftarrow n \neq 1 \\
&= \lim_{b \rightarrow \infty} \left[ -\frac{1}{(n-1)b^{n-1}} - \left( -\frac{1}{(n-1)n^{n-1}} \right) \right] \\
&= \frac{1}{(n-1)n^{n-1}} \\
L &= \lim_{n \rightarrow 1^+} \left( \int_n^\infty \left[ \frac{1}{x^n} \right] dx \right)^{n-1} = \lim_{n \rightarrow 1^+} \left( \frac{1}{(n-1)n^{n-1}} \right)^{n-1} && \Rightarrow \infty^0 \\
\ln L &= \lim_{n \rightarrow 1^+} \left[ (n-1) \ln \left( \frac{1}{(n-1)n^{n-1}} \right) \right] \\
&= \lim_{n \rightarrow 1^+} \left[ (n-1) (\ln(1) - \ln((n-1)n^{n-1})) \right] \\
&= \lim_{n \rightarrow 1^+} \left[ -(n-1) \ln((n-1)n^{n-1}) \right] && \Rightarrow 0 \times (-\infty) \\
&= \lim_{n \rightarrow 1^+} \left[ -\frac{\ln((n-1)n^{n-1})}{1/(n-1)} \right] = \lim_{n \rightarrow 1^+} \left[ -\frac{\ln(n-1) + (n-1) \ln n}{1/(n-1)} \right] && \Rightarrow -\frac{\infty}{\infty} \\
&= \lim_{n \rightarrow 1^+} \left[ \frac{\frac{1}{n-1} + (\ln n + \frac{n-1}{n})}{1/(n-1)^2} \right] \\
&= \lim_{n \rightarrow 1^+} \left[ (n-1)(1) + (n-1)^2 \ln n + \frac{(n-1)^3}{n} \right] \\
&= \left[ (0)(1) + (0)^2(0) + \frac{0^3}{1} \right] = 0 \\
L &= e^0 = 1
\end{aligned}$$

14.

$$\begin{aligned}
\int_n^\infty \left[ \frac{1}{x^n} \right] dx &= \left[ -\frac{1}{(n-1)x^{n-1}} \right]_n^\infty && \Longleftarrow n \neq 1 \\
&= \lim_{b \rightarrow \infty} \left[ -\frac{1}{(n-1)b^{n-1}} - \left( -\frac{1}{(n-1)n^{n-1}} \right) \right] \\
&= \frac{1}{(n-1)n^{n-1}} \\
\int_{n+1}^\infty \left[ \frac{1}{x^{n+1}} \right] dx &= \left[ -\frac{1}{nx^n} \right]_{n+1}^\infty = \lim_{b \rightarrow \infty} \left[ -\frac{1}{nb^n} - \left( -\frac{1}{n(n+1)^n} \right) \right] = \frac{1}{n(n+1)^n} && | n \neq 0 \\
L &= \lim_{n \rightarrow \infty} \left( \int_n^\infty \left[ \frac{1}{x^n} \right] dx \right)^{\int_{n+1}^\infty \left[ \frac{1}{x^{n+1}} \right] dx} = \lim_{n \rightarrow \infty} \left( \frac{1}{(n-1)n^{n-1}} \right)^{\frac{1}{n(n+1)^n}} && \Rightarrow 0^0 \\
\ln L &= \lim_{n \rightarrow \infty} \left[ \frac{1}{n(n+1)^n} \ln \left( \frac{1}{(n-1)n^{n-1}} \right) \right] \\
&= \lim_{n \rightarrow \infty} \left[ \frac{\ln(1) - \ln((n-1)n^{n-1})}{n(n+1)^n} \right] = \lim_{n \rightarrow \infty} \left[ -\frac{\ln((n-1)n^{n-1})}{n(n+1)^n} \right] \\
&= \lim_{n \rightarrow \infty} \left[ -\frac{\ln(n-1) + (n-1)\ln n}{n(n+1)^n} \right] \\
&= 0, \text{ as } (n+1)^n > n^n > \ln n > \ln(n-1) \\
L &= e^0 = 1
\end{aligned}$$

15.

$$\begin{aligned}
\sum_{n=0}^\infty \left[ \frac{1}{x^n} \right] &= \frac{1}{1 - \frac{1}{x}} = \frac{1}{\frac{x-1}{x}} = \frac{x}{x-1} \\
u &= x-1 \Rightarrow du = dx \\
\int \left[ \frac{x}{x-1} \right] dx &= \int \left[ \frac{u+1}{u} \right] du = \int \left[ 1 + \frac{1}{u} \right] du = u + \ln u = x-1 + \ln |x-1| \\
&= x - \ln |x-1| + C \\
L &= \lim_{b \rightarrow \infty} \left( \int_2^b \left[ \sum_{n=0}^\infty \left[ \frac{1}{x^n} \right] \right] dx \right)^{e^{-b}} = \lim_{b \rightarrow \infty} \left( [x - \ln |x-1|]_2^b \right)^{e^{-b}} \\
&= \lim_{b \rightarrow \infty} (b - \ln |b-1| - (2-0))^{e^{-b}} = \lim_{b \rightarrow \infty} (b-2 - \ln |b-1|)^{e^{-b}} \\
&\Rightarrow \infty^0, \text{ as } b > \ln b > \ln |b-1| \\
\ln L &= \lim_{b \rightarrow \infty} \left[ \frac{\ln |b-2 - \ln |b-1||}{e^b} \right] && \Rightarrow \frac{\infty}{\infty} \\
&= \lim_{b \rightarrow \infty} \left[ \frac{1 - \frac{1}{b-1}}{e^b(b-2 - \ln |b-1|)} \right] \\
&= \lim_{b \rightarrow \infty} \left[ \frac{b-2}{e^b(b-1)(b-2 - \ln |b-1|)} \right] = 0, \text{ as } e^b b^2 > b \\
L &= e^0 = 1
\end{aligned}$$