AP Physics C

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Contents

I Front Matter 2

1 Mathematical Foundations 3
- 1.1 Algebra . . . 3
  - 1.1.1 Matrices . . . 3
  - 1.1.2 Vectors . . . 4
- 1.2 Calculus . . . 5

II Mechanics 6

2 Kinematics 7

3 Forces 8

4 Energy 9

5 Systems of Particles 10

6 Rotations 11
- 6.1 Rotation . . . 11
- 6.2 Rolling, Torque, and Angular Momentum . . . 11
  - 6.2.1 Torque . . . 12
  - 6.2.2 Angular Momentum . . . 13

7 Statics and Oscillations 14
- 7.1 Equilibrium and Elasticity . . . 14
  - 7.1.1 Center of Gravity . . . 14

8 Gravitation 15

III Electricity and Magnetism 16

Part I

Front Matter

Chapter 1

Mathematical Foundations

1.1 Algebra

The **summation** of all terms in a data set can be denoted with just the index beneath the summation, though this need not be included for finite data sets (with the same indices).

$$\sum_i x_i = \sum x_i$$

1.1.1 Matrices

A matrix is arranged as a two-dimensional rectangular array or numbers, comprised of rows (R) and columns (C).
A matrix with m rows and n columns can be described as being $m \times n$ (m by n). This is its dimension. If m and n are equal, the matrix is a **square matrix**.
The i^{th} row or column can be denoted R_i or C_i respectively.
Each cell of a matrix is an element. To denote the element in row i and column j of a matrix A, $A_{i,j}$ can be used.

$$A = \begin{array}{c c} & \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_m \end{array} & \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,\ldots} & A_{1,n} \\ A_{2,1} & A_{2,2} & A_{2,\ldots} & A_{2,n} \\ A_{\ldots,1} & A_{\ldots,2} & \ddots & A_{\ldots,n} \\ A_{m,1} & A_{m,2} & A_{m,\ldots} & A_{m,n} \end{bmatrix} \end{array}$$

The **determinant** of a 2×2 matrix is the difference between the products of the diagonal terms.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

The determinant of a 3×3 matrix is quite similar that of a 2×2 one, using an anchor point and multiplying that by the determinant of the matrix constructed from the remaining 4 terms.

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) + b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & i \end{bmatrix}\right)$$

This trend continues for *square* matrices of higher dimension.

Row Operations

A **row operation** manipulates the rows of a matrix.
The **interchange** of the i^{th} and j^{th} rows is denoted $R_i \leftrightarrow J$. This operation simply swaps the rows.

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \end{array} \xrightarrow{R_1 \leftrightarrow R_2} \begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \left[\begin{array}{cc} c & d \\ a & b \end{array}\right] \end{array}$$

$$\boxed{\text{Interchange alone results in an equivalent matrix.}}$$

The i^{th} row can be multiplied by a scalar k, resulting in each of that row's elements being multiplied by k. This is denoted kR_i.

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \end{array} \xrightarrow{kR_1} \begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \left[\begin{array}{cc} ka & kb \\ c & d \end{array}\right] \end{array}$$

The i^{th} and j^{th} rows can be added to replace either, elements in the same column being summed. Replacing R_i in this way is denoted $R_i + R_j \to R_i$.

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \end{array} \xrightarrow{R_1 + R_2 \to R_1} \begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \left[\begin{array}{cc} a+c & b+d \\ c & d \end{array}\right] \end{array}$$

Linear Equations

Matrices can be used to solve systems of linear equations in any number of dimensions. To represent a linear equation with an **augmented matrix**, the each element of a row should correspond to the coefficient of a variable with the exception of the last column's element corresponding to a constant.

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = A \equiv \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & \cdots & x_n & c \end{array} \\ \left[\begin{array}{cccc|c} a_1 & a_2 & \cdots & a_n & A \end{array}\right] \end{array}$$

A **system of linear equations** with m equations and n dimensions can also be represented via an augmented matrix using row operations.

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}nx_n &= A_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}nx_n &= A_2 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}nx_n &= A_m \end{aligned} \equiv \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & \cdots & x_n & c \end{array} \\ \left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & a_{1,\ldots} & a_{1,n} & A_1 \\ a_{2,1} & a_{2,2} & a_{2,\ldots} & a_{2,n} & A_2 \\ a_{\ldots,1} & a_{\ldots,2} & \ddots & a_{\ldots,n} & \vdots \\ a_{m,1} & a_{m,2} & a_{m,\ldots} & a_{m,n} & A_m \end{array}\right] \end{array}$$

Matrix Operations

1.1.2 Vectors

A **scalar** is a quantity with only magnitude.
A **vector** is a quantity with both magnitude and direction. A vector is denoted by an arrow ($\rightarrow$) above the variable. It can be represented by a 1-row matrix.
A vector can be divided into **components**, denoting its values in a given direction.[1] A specific component

[1]The interpretation of vector components denoting directions should only be used by default. This may change, especially outside of physics.

can be denoted using the vector variable (without the arrow) with the subscript of the direction (x, y, or z).[2]

A vector can be denoted in **component form** by angled brackets, commas separating each component. The components are always ordered x, y, and z.

$$\vec{v} = \langle v_x, v_y, v_z \rangle$$

Unit vectors are vectors of magnitude 1 that have only a single component. The unit vectors in the x, y, and z directions are $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ respectively.

$$\hat{\imath} = \langle 1, 0, 0 \rangle \qquad \hat{\jmath} = \langle 0, 1, 0 \rangle \qquad \hat{k} = \langle 0, 0, 1 \rangle$$

Unit vector notation breaks a vector into the sum of multiples of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

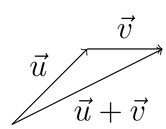
The sum of two vectors is the sum of their corresponding components and is itself a vector.

$$\vec{u} + \vec{v} = (u_x + v_x)\hat{\imath} + (u_y + v_y)\hat{\jmath} + (u_z + v_z)\hat{k}$$

The **magnitude** of a vector, denoted by enclosing the vector within four vertical lines, is the square root of the sum of the squares of its components. It is a *scalar* quantity.

$$||\vec{v}|| = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

To add two vectors graphically, the tip of one can be attached to the tail of the other, and the **resultant vector** can be drawn along the hypotenuse of the formed triangle.



The **dot product** of two vectors, denoted by a dot, is the sum of the products of their corresponding components. This is a *scalar* quantity.

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_x v_x + u_y v_y + u_z v_z$$

The **cross product** of two vectors, denoted by a cross, is the

1.2 Calculus

[2]When the components of a vector do not denote direction, its components can be denoted by their indices in the subscript.

Part II

Mechanics

Chapter 2

Kinematics

Kinematics is the classification and comparison of motion.
A **particle** is defined as a point-like object, moving in a way such that every part of it moves at the same rate (without rotation or stretching).
The subscripts f and i (or 0) below a quantity can be used to denote initial or final values for it, while numbers can be used if more than two variations are being considered.
The subscript avg is used to denote a quantity's average over some domain.
Position must be measured relative to a reference point (typically the origin). The variables x, y, and z are used depending on the context to denote it, while *generalized* position is often denoted with q.
A change in position is **displacement**. The displacement in a direction is denoted with a preceding Δ (meaning "change in", though depending on the context, the variable d can instead be used in isolation.

$$\Delta q = q_f - q_i$$

The rate of change of position is **velocity**. *Average* velocity is equal to displacement divided by the change in time, while *instantaneous* velocity is the time derivative of position. Velocity is denoted using v in

Chapter 3

Forces

Chapter 4

Energy

Chapter 5

Systems of Particles

Chapter 6

Rotations

6.1 Rotation

A **rigid body** rotates as a unit.
The axis about which an object rotates is the **axis of rotation**. The **angular position $\boldsymbol{\theta}$** of this line is taken relative to a fixed direction, the **zero angular position**.

> Although its can be changed (if specified), positive angles are conventionally **counterclockwise** from the zero angular position.

Angular dimension is measured using **radians** (**rad**), which are dimensionless.

$$\theta = \frac{s}{r}$$

A **revolution** is equal to 360°, which is also equal to 2π rad.

6.2 Rolling, Torque, and Angular Momentum

For an object to **roll** is for it to move rotationally and translationally along a surface. For an object to roll **smoothly** is for it not to leave the ground while it is rolling.

> Smooth rolling can be thought of as pure rotation and pure translation or as rotation about a moving contact point.

The center of mass of a rolling object moves parallel to the surface. The rest of the object rotates about the center of mass.
The **arc distance $\boldsymbol{S}$**, the distance covered on the surface, and the velocity about the center of mass are defined as linear variables:

$$S = \theta r \qquad\qquad \mathrm{com}\, v = \omega r$$

As rolling objects move both translationally and rotationally, they have both translational and rotational kinetic energy.[1]

$$K = \frac{1}{2} I_{\mathrm{com}} \omega^2 + \frac{1}{2} M v_{com}^2$$

[1] The parallel axis theorem can be used to calculate the kinetic energy about the contact point.

$$K = \frac{1}{2} I_P \omega^2 = \frac{1}{2}(I_{\mathrm{com}} + Mr^2)\omega^2 = \frac{1}{2} I_{\mathrm{com}} \omega^2 + \frac{1}{2} Mr^2 \left(\frac{v_{\mathrm{com}}}{r}\right)^2 = \frac{1}{2} I_{\mathrm{com}} \omega^2 + \frac{1}{2} M v_{\mathrm{com}}^2$$

If no slipping occurs, then energy is conserved (even with friction). The acceleration about the center of mass follows the pattern of position and velocity.

$$a_{\text{com}} = \alpha r$$

$$\boxed{\text{Acceleration is only dependent on the shape of the object. Its mass and radius are irrelevant.}}$$

For an object to roll smoothly on a slope, three things are required are required:

1. The gravitational force must be vertically down.
2. The normal force must be perpendicular to the slope.
3. The force of friction must point up the slope.

$$\boxed{\text{The frictional and normal forces are applied to the point of contact rather than the center of mass.}}$$

A force only results in torque if it does not pass through the contact point. The torque can be calculated as the product of the perpendicular components of the force and the radius.

$$\tau = r_\perp F = rF_\perp = rF \sin\phi$$

The second law can then be used.

$$\text{net}\,\tau = I\alpha = I\left(\frac{a_{\text{com}}}{r}\right)$$

The acceleration of a body rolling smoothly down a slope can be found as such:

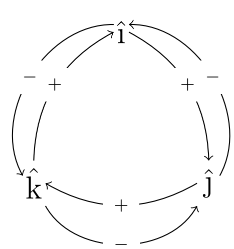
$$a_{\text{com},x} = \frac{-g\sin\theta}{I + \frac{\text{com}\,I}{Mr^2}}$$

6.2.1 Torque

Torque can be defined as a vector for an individual particle moving along any path relative to a fixed point. It is the cross product of $\vec{r}$ and $\vec{F}$.

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix} \times \begin{bmatrix} F_x & F_y & F_z \end{bmatrix} = \begin{bmatrix} r_yF_z - r_zF_y \\ r_xF_z - r_zF_x \\ r_xF_z - r_yF_z \end{bmatrix}$$

The resultant direction can be determined with the right hand rule, pointing the index and middle fingers of the right hand in the directions of the force and radius such that the thumb will point in the resultant direction of torque.
The angle can also be determined using unit vector notation. The product of two unit vectors is always the third. Its sign is dependent on the the “order” of the unit vectors.



6.2.2 Angular Momentum

The **angular momentum** $\boldsymbol{\ell}$ of a particle is equal to the product of its distance and linear momentum.

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

Angular momentum has units of kgm^2/s (or Js). The magnitude of $\vec{\ell}$ is the product of the perpendicular components of $\vec{r}$ and $\vec{p}$.

$$\ell = r_{\perp} p = r p_{\perp} = rp \sin\phi = mrv \sin\phi$$

Newton's second law can be rewritten to state that torque is the time derivative of angular momentum, but in order for this to be true, they must be defined about the same point.[2]

$$\vec{\tau}_{\text{net}} = \frac{\mathrm{d}\vec{\ell}}{\mathrm{d}t}$$

The time derivative of **net angular momentum**, denoted $\vec{L}$, is equal to the sum of the net torques of each particle, denoted $\vec{\mathrm{T}}$.

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\mathrm{T}}$$

If the net external torque on a system is zero, then angular momentum is *conserved*.
Linear momentum can be written as the product of inertia and angular velocity.[3]

$$\ell = I\omega$$

[2]Using the formula for angular momentum, the fact that its time derivative is torque can be shown:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\ell} = \frac{\mathrm{d}}{\mathrm{d}t} m(\vec{r} \times \vec{v}) = m\vec{r}\left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right) = m\vec{r} \times \vec{a} = \vec{F}_{\text{net}} \times \vec{r} = \vec{\tau}_{\text{net}}$$

[3]It can be shown that linear momentum is the product of inertia and linear momentum by rewriting its formula or by integrating its time derivative.

$$\begin{aligned} \ell &= mrv = mr^2\omega = I\omega \\ &= \int \tau \,\mathrm{d}t = \int I\alpha \,\mathrm{d}t = I\omega \end{aligned}$$

Chapter 7

Statics and Oscillations

7.1 Equilibrium and Elasticity

An object in **static equilibrium** have a center of mass with constant linear momentum and have constant angular momentum about any point.

$$\text{equilibrium} \implies (\vec{P} \wedge \vec{L}) \text{ are constant}$$

If a body returns to static equilibrium after a slight displacement, it is in **stable static equilibrium**. Otherwise, it is **unstable**.
The requirements for equilibrium can be rewritten using Newton's second law.

$$\text{equilibrium} \implies \vec{F}_{\text{net}}, \vec{\mathrm{T}}_{\text{net}} = 0$$

Forces are often only considered in the xy plane, further simplifying the requirements.

$$\text{equilibrium} \implies F_{\text{net},x}, F_{\text{net},y}, \tau_{\text{net},z} = 0$$

7.1.1 Center of Gravity

The gravitational force $\vec{F}_g$ on a body is the sum of all of the gravitational forces acting on its individual particles. This can be simplified by treating it as as single force that acts on the **center of gravity** (**cog**). If the gravitational force is the same for all of a body's particles, then the center of gravity is simply the center of mass.

Chapter 8

Gravitation

Gravitation is the tendency for bodies to attract each other.
The gravitational force is dependent on mass. It is always attractive.
Newton's **law of gravitation** defines the strength of the attractive force between particles.

$$F_G = G\frac{m_1 m_2}{r^2}$$

G is the **gravitational constant**

$$G \approx 6.67 \times 10^{-11}\ \mathrm{N} \cdot \mathrm{m}^2 \,/\, \mathrm{kg}^2$$

Newton's law of gravitation can be rewritten in vector form using $\hat{r}$, which is a vector of magnitude 1 that points from one particle to another.

$$\vec{F}_G = G\frac{m_1 m_2}{r^2}\hat{r}$$

As is made clear by this vector form, gravitational forces always come in *third-law pairs*.
The **shell theorem** states that a uniform spherical shell of matter attracts a particle that is outside the shell as though all of the shell's mass was concentrated at its center.
The net gravitational force can be calculated by the **principle of superposition**, which simply states that the net effect is the sum of the individual effects.

$$\vec{F}_{\text{net}} = \sum \vec{F}_i$$

For a real (extended) object, this sum becomes an integral.

$$\vec{F}_{\text{net}} = \int \mathrm{d}\vec{F}$$

If the object in question is a uniform sphere or shell, it can be treated as having its mass at its center rather than considering the object as a whole.
Newton's law of gravitation can be combined with Newton's second law to calculate the acceleration due to gravity.

$$a_{G,1} = \frac{F_g}{m_1} = \frac{G m_1 m_2 / r^2}{m_1} = \frac{G m_2}{r^2}$$

Part III

Electricity and Magnetism