Calculus III

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June 4, 2022

Contents

Chapter 12

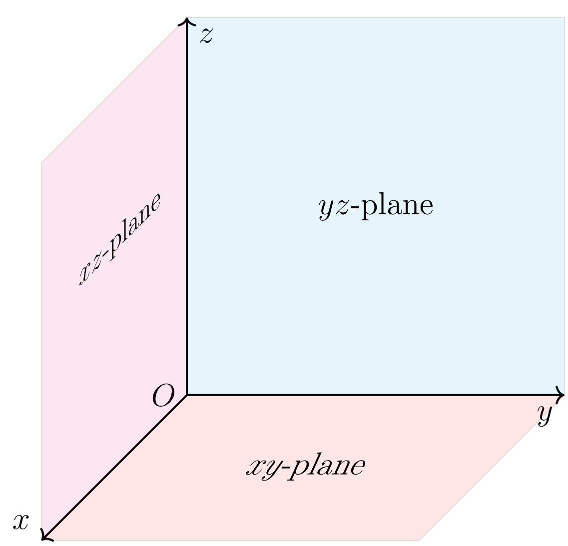
Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

Any point in a plane can be represented as an ordered pair of real numbers. Because this uses two numbers, a plane is called two-dimensional. To locate a point in space, a triplet of real-numbers is required.

3D Space

Before points can be represented in 3D space, a fixed point O (the origin) and three perpendicular lines that pass through it, called the **coordinate axes**. These axes are labeled the x-, y-, and z-axes. In general, the former two are horizontal while the third is vertical. The direction of the z-axis is determined by the **right-hand rule**. Curling the fingers of the right hand from the positive x-axis to the positive y-axis, the thumb will point in the direction of the positive z-axis.
The three coordinate axes determine the three **coordinate planes**.



Three planes divided space into eight **octants**. Illustrated above are the positive xz-, yz-, and xy-planes, constituting the **first octant**.
A point's **coordinates** are an ordered triple of real numbers. A point's **projection** onto a plane is the point with two of the same coordinates, the third becoming 0.

Plane	xy	yz	xz
(a, b, c)	$(a, b, 0)$	$(0, b, c)$	$(a, 0, c)$

The set of all ordered triples is the cartesian product of three sets of all real numbers, denoted appropriately by $\mathbb{R}^3$ and defined as

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

A one-to-one correspondence between points in space and ordered triples in $\mathbb{R}^3$ is a **three-dimensional coordinate system**. It should be noted that the first octant can be described as the set of points for which all coordinates are positive.

Distances and Spheres

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere The equation of a sphere with center (h, k, l) and radius r is

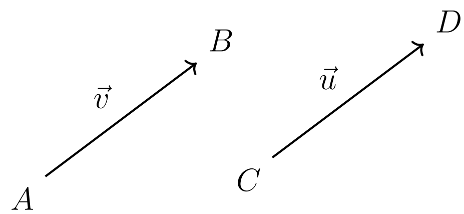
$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

12.2 Vectors

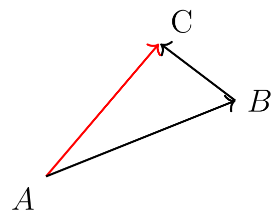
The term **vector** is used to indicate a quantity with both magnitude and direction.

Geometric Description of Vectors

A vector is often represented by an arrow, the length of which represents its magnitude. A vector is denoted by a letter in boldface ($\mathbf{v}$) or with an arrow above a letter ($\vec{v}$).
A **displacement vector** is a vector representing how something is displaced from its initial point (tail) to its **terminal point** (tip).



In the above figure, vector $\vec{v}$ has initial point A and terminal point B. This can be indicated by writing $\vec{v} = \overrightarrow{AB}$. If $AB = CD$ and the angles relative to the same axis are equal, then the $\vec{v}$ and $\vec{u}$ are **equivalent** (or **equal**).
The only vector without a direction is the **zero vector**, denoted by $\vec{0}$, which has length 0.
The sum of two vectors can be denoted with the initial point of the first and the terminal point of the second.

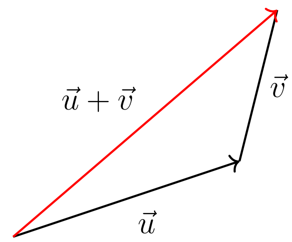


$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

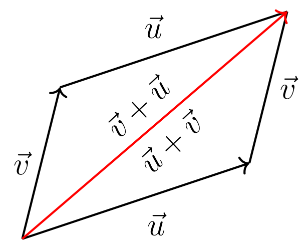
To add vectors, the one's tail can be moved to the other's tip and the resultant terminal point found.

> **Definition of Vector Addition** The sum of two vectors position such that the initial point of one is the terminal point of the other is the vector from the initial point of the first to the terminal point of the second.

The definition of vector addition is sometimes referred to as the **Triangle Law**.



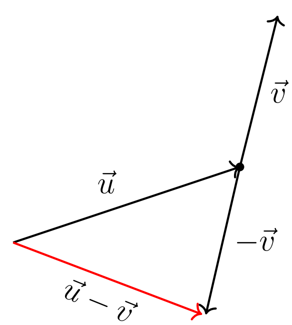
Doing the opposite addition results in the same resultant vector. This is made visible b y the **Parallelogram Law**.



A **scalar** is a number that is not a vector.

> **Definition of Scalar Multiplication** The **scalar multiple** $c\vec{v}$ of a scalar c and a vector $\vec{v}$ is a vector of length $|c||\vec{v}|$ and whose direction is the same as $\vec{v}$ is $c > 0$ and in the opposite direction if $c < 0$. If $c = 0$ or $\vec{v} = \vec{0}$, then $c\vec{v} = \vec{0}$.

The **negative** of a vector $\vec{v}$, denoted by $-\vec{v}$, is the scalar multiple of the $\vec{v}$ and -1.
The **difference** $\vec{u} - \vec{v}$ is the sum of $\vec{u}$ and $-\vec{v}$.



Components of a Vector

A vector can be treated algebraically using **components**, which are the difference between the initial and terminal points in a dimension. The components of a vector $\vec{u}$ in 2 dimension and a vector $\vec{v}$ in 3 can be denoted as such:

$$\vec{u} = \langle u_1, u_2 \rangle \qquad \vec{v} = \langle v_1, v_2, v_3 \rangle$$

Geometric vectors can be though of as **representation** of algebraic vectors.

> The representation of a vector with initial and terminal points is
>
> $$\langle \Delta x, \Delta y, \Delta z \rangle$$

The **magnitude/length** of a vector, denoted $|\vec{v}|$ or $||\vec{v}||$ for a vector $\vec{v}$, is the length of any of its representations. This can be computed in n dimensions using the distance formula:

$$|\vec{v}| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Vectors can be added or subtracted algebraically by performing the desired operation their corresponding components.

$$\vec{u} \pm \vec{v} = \langle u_1 \pm v_1, u_2 \pm v_2, \ldots, u_n \pm v_n \rangle$$

The scalar multiple of a vector can be found by multiplying each of its components by the scalar.

$$c\vec{v} = \langle cv_1, cv_2, \ldots, cv_n \rangle$$

The set of all n-dimensional vectors is denoted by V_n. An n-dimensional vector is an n-tuple of real numbers, which are the vector's components.

Properties of Vectors If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in V_n and c and d are scalars, then

$$\begin{aligned} \vec{a} + \vec{b} &= \vec{b} + \vec{a} & \vec{a} + (\vec{b} + \vec{c}) &= (\vec{a} + \vec{b}) + \vec{c} \\ \vec{a} + \vec{0} &= \vec{a} & \vec{a} + (-\vec{a}) &= \vec{0} \\ c(\vec{a} + \vec{b}) &= c\vec{a} + c\vec{b} & (c + d)\vec{a} &= c\vec{a} + d\vec{a} \\ (cd)\vec{a} &= c(d\vec{a}) & 1\vec{a} &= \vec{a} \end{aligned}$$

A vector can be denoted in **unit-vector notation** as the sum of scalar multiples of **standard basis vectors** defined as such:

$$\hat{\imath} := \langle 0, 0, 1 \rangle \qquad \hat{\jmath} := \langle 0, 1, 0 \rangle \qquad \hat{\mathrm{k}} := \langle 0, 0, 1 \rangle$$

A vector can therefore be rewritten using its components as such:

$$\vec{v} = \langle v_x, v_y, v_z \rangle = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{\mathrm{k}}$$

A **unit vector** is a vector of length 1. The standard basis vectors are unit vectors in the directions of the axes. A unit vector $\vec{u}$ in the same direction as a vector $\vec{v}$ is

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$$

The angle θ between a two-dimensional vector $\vec{v}$ and the positive x-axis can be calculated using inverse tangent:

$$\theta = \arctan\left(\frac{v_y}{v_x}\right)$$

This can be used to find the components of a vector of a given magnitude r and direction θ.

12.3 The Dot Product

Definition of the Dot Product The dot product of two vectors in V_n is the sum of the products of their corresponding components.

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

The dot product of two vectors is a real number. As such, the dot product is sometimes referred to as the **scalar/inner product**.

Properties of the Dot Product If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in V_n and c is a scalar, then

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2 \qquad \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$
$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} \qquad (c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$$
$$\vec{0}\cdot\vec{a} = 0$$

If θ is the angle between vectors, $\vec{u}$ and $\vec{v}$, then

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

Rewritten for θ, it can be said that

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

Two vectors are **orthogonal** (perpendicular) if and only if their dot product is 0.

$$\vec{u}\perp\vec{v} \iff \vec{u}\cdot\vec{v} = 0$$

Two vectors are **parallel** if their dot product is equal to the the product of their magnitudes or its negation, as θ is 0 or π.

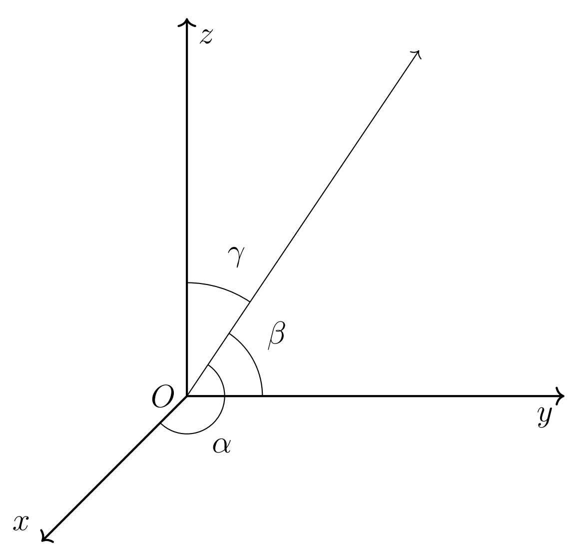
$$\vec{u} \parallel \vec{v} \iff \vec{u}\cdot\vec{v} = \pm|\vec{u}||\vec{v}|$$

Two vectors point in opposite directions if their dot product is equal to the negative of the product of their magnitudes, as θ is π.

$$\frac{\vec{u}}{|\vec{u}|} = -\frac{\vec{v}}{|\vec{v}|} \iff \vec{u}\cdot\vec{v} = -|\vec{u}||\vec{v}|$$

Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector are α, β, and γ. These are the angles made with the positive x-, y-, and z-axes respectively.



The cosines of these direction angles are called the **direction cosines**. These direction cosines can be derived using the dot products of the vector and the standard basis vectors.

$$\cos\alpha = \frac{\vec{v}\cdot\hat{\imath}}{|\vec{v}||\hat{\imath}|} = \frac{v_x}{|\vec{v}|} \qquad \cos\beta = \frac{\vec{v}\cdot\hat{\jmath}}{|\vec{v}||\hat{\jmath}|} = \frac{v_y}{|\vec{v}|} \qquad \cos\gamma = \frac{\vec{v}\cdot\hat{\mathrm{k}}}{|\vec{v}||\hat{\mathrm{k}}|} = \frac{v_z}{|\vec{v}|}$$

Squaring both sides and adding, the sum of the direction cosines can be shown to be equal to 1.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{v_x^2 + v_y^2 + v_z^2}{v_x^2 + v_y^2 + v_z^2} = 1$$

Rewriting for the components, it can be seen that the direction cosines are the components of the unit vector in the direction of the vector.

$$\vec{v} = \langle v_x, v_y, v_z\rangle = \langle |\vec{v}|\cos\alpha, |\vec{v}|\cos\beta, |\vec{v}|\cos\gamma\rangle = |\vec{v}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$
$$\frac{\vec{v}}{|\vec{v}|} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$

Projections

The **scalar projection** of a vector $\vec{u}$ onto a vector $\vec{v}$ is (the **component of $\vec{u}$ along $\vec{v}$**) is the component of $\vec{u}$ in the direction of $\vec{v}$. This is denoted and found as

$$\operatorname{comp}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

The **vector projection** of this is simply equal to this scalar multiplied by a unit vector in the direction of $\vec{v}$.

$$\operatorname{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{u}}{|\vec{v}|^2}\vec{v}$$

The **orthogonal projection** of this is the the projection of $\vec{u}$ onto the vector perpendicular to $\vec{v}$, denoted and found as

$$\operatorname{orth}_{\vec{v}}\vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}}\vec{u}$$

The reason for this equation is that the projection and orthogonal projection of $\vec{u}$ onto $\vec{v}$ must logically sum to $\vec{u}$.

12.4 The Cross Product

The Cross Product of Two Vectors

A **determinant of order 2** is defined as the difference between the products of the diagonal terms and is denoted as such:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** can be defined in terms of second-order determinants.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_2 + a_3b_1c_2 - a_3b_2c_1$$

Definition of the Cross Product The **cross product** of two three-dimensional vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$$

The cross product of two vectors is orthogonal to both.
The magnitude of the cross product of two vectors $\vec{u}$ and $\vec{v}$ with angle θ between them is $\vec{0}$.

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

Two vectors are parallel if and only if their cross product is equal to zero.

$$\vec{u} \parallel \vec{v} \iff \vec{u} \times \vec{v} = \vec{0}$$

The cross products of the standard basis vectors are as follows:

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

Properties of the Cross Product If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and c is a scalar, then

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \qquad (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \qquad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Triple Products

The **scalar triple product** of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The volume of a parallelpiped determined by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product.

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

The **vector triple product** of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is

$$\vec{a} \times (\vec{b} \times \vec{c})$$

12.5 Equations of Lines and Planes

Lines

A line in three dimensional space is defined by a known point and a direction. The direction can be described by a vector parallel to the line. A **vector equation** of a line has a **parameter** of some scalar which is multiplied by the parallel vector. The result is then added to the initial point.

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

Rewritten in component form, this provides the **parametric equations** of the line.

> The parametric equations of a line that posses through the point $P_0(x_0, y_0, z_0)$ and is parallel to the direction vector $\vec{v}$ are
>
> $$x = x_0 + v_x t \qquad y = y_0 + v_y t \qquad z = z_0 + v_z t$$

If a vector is used to describe a line's direction, its components are called the line's **direction numbers**. As any parallel vector can be used, any set of numbers proportional to a given set of direction numbers can also be used as a set of direction numbers for the line.
A line can also be described by eliminating the parameter, solving each component's equation for the parameter.

$$t = \frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}$$

These equations are the **symmetric equations** of the line. It should be noted that the denominators are direction numbers. If one of the direction number is 0, the symmetric equations can still be found, simply equating the variable whose corresponding number is 0 to its known value. If v_x is 0,

$$x = x_0 \qquad \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}$$

The line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by the vector equation

$$\vec{r}(t) = (1 - t)\vec{r}_0 + t\vec{r}_1 \mid 0 \leq t \leq 1$$

Planes

While a line in space is determined by a point and its direction, a plane is more difficult to define. A single vector parallel to the plane is not able to properly convey its "direction", but one perpendicular to it does not completely specify it either. A plane is instead determined by a point in the plane and a vector $\vec{n}$, called the **normal vector**, that is orthogonal to the plane.
For a point $P_0(x_0, y_0, z_0)$ determining the plane with position vector $\vec{r}_0$, an arbitrary point $P(x, y, z)$ on the plane with position vector $\vec{r}$, and a normal vector $\vec{n}$, the vector $\vec{r} - \vec{r}_0$ is represented by $\overrightarrow{P_0P}$. As $\vec{n}$ is orthogonal to every vector in the plane, it is also perpendicular to $\vec{r} - \vec{r}_0$, so

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \implies \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

These equations are both **vector equations of the plane**.
A scalar equation for the plane can be derived by rewriting the vectors in terms of their components and evaluating the cross product.

> A **scalar equation of the plane** through the point $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n}$ is
>
> $$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$$
>
> This can be rewritten as a **linear equation** in x, y, and z by expanding.
>
> $$n_x x + n_y y + n_z z + d = 0 \mid d = -(n_x x_0 + n_y y_0 + n_z z_0)$$

Two planes are **parallel** if their normal vectors are parallel.

Distances

The distance from a point $P(x, y, z)$ to a plane $ax + by + cz + d = 0$ is equal to the absolute value of the scalar projection of the vector $\vec{b}$ from an arbitrary point $P_0(x_0, y_0, z_0)$ in the plane to P and the plane's normal vector.

$$D = |\operatorname{comp}_{\vec{n}} \vec{b}|$$

Expanding,

$$\begin{aligned} D &= \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} = \frac{|b_x n_x + b_y n_y + b_z n_z|}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = \frac{|n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0)|}{\sqrt{n_x^2 + n_y^2 + n_z^2}} \\ &= \frac{|(n_x x + n_y y + n_z z) - (z_x x_0 + n_y y_0 + n_z z_0)|}{\sqrt{n_x^2 + n_y^2 + n_z^2}} \end{aligned}$$

As P_0 lies in the plane, it satisfies the equation of the plane, allowing this to be further rewritten.

> The distance D from a point $P(x, y, z)$ to a a plane $ax + by + cz + d = 0$ is
>
> $$D = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Chapter 13

Vector Functions

13.1 Vector Functions and Space Curves

Vector-Valued Functions

A function is a rule that assigns each element in its domain to an element in its range. A **vector(-valued) function** is one with a domain of real numbers and a range of vectors.
A three-dimensional vector function can be written in terms of the sum of real valued functions corresponding to each component, called its **component functions**. If the components of $\vec{r}$ are real-valued functions f, g, and h, it can be written that

$$\vec{r} = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{\mathrm{k}}$$

Limits and Continuity

The **limit** of a vector function is defined by the limits of its component functions.

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t), \lim_{t\to c} g(t), \lim_{t\to c} h(t) \right\rangle$$

so long as the limits of the component functions exist.

A vector function $\vec{r}$ is **continuous** at c if

$$\lim_{t\to c} \vec{r}(t) = \vec{r}(c)$$

By extension, its component functions must also be continuous at c.

Space Curves

A **space curve** is a set C of all points (x, y, z) in space where x, y, and x, y, and z are determined by continuous real-valued functions on an interval I. These functions' equations are the **parametric equations of $\boldsymbol{C}$**, and t is its **parameter**.

13.2 Derivatives and Integrals of Vector Functions

Derivatives

The derivative $\vec{r}'$ of a vector function $\vec{r}$ is defined by the limit of the difference quotient (if it exists), just like the derivative of a real-valued function.

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{r}'(t) = \lim_{h\to 0}\left[\frac{\vec{r}(t+h)-\vec{r}(t)}{h}\right]$$

$\vec{r}'(t)$ is the **tangent vector** to the curve defined by $\vec{r}$ at t, provided $\vec{r}'(t)$ exists and is not $\vec{0}$. The **tangent line** to C at $\vec{r}(t)$ is the line through $\vec{r}(t)$ parallel to $\vec{r}'(t)$.

The derivative of a vector function can be found as that of each of its components.

> If $\vec{r}(t) = \langle f(t), g(t), h(t)$ where each component function is differentiable,
>
> $$\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

A unit vector in the direction of the tangent vector is the **unit tangent vector $\vec{T}$**, defined by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

The **second derivative** of a vector function $\vec{r}$ is the derivative of $\vec{r}'$.

Differentiation Rules

> If $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function,
>
> $$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{u}(t)+\vec{v}(t)] = \vec{u}'(t)+\vec{v}'(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[c\vec{u}(t)] = c\vec{u}'(t)$$
> $$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}'(t)\cdot\vec{v}(t)+\vec{v}(t)\cdot\vec{u}(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t)+\vec{v}'(t)\times\vec{u}$$
> $$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t)+f(t)\vec{u}'(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$$

Integrals

The **definite integral** of a continuous vector function can be defined similarly to that of a continuous real-valued function.

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \lim_{n\to\infty}\sum_{i=1}^{\infty}\vec{r}(t_i^*)\Delta t$$

This definition can be rewritten in terms of components.

$$\begin{aligned}\int_a^b \vec{r}(t)\,\mathrm{d}t &= \left\langle \int_a^b f(t)\,\mathrm{d}t, \int_a^b g(t)\,\mathrm{d}t, \int_a^b h(t)\,\mathrm{d}t\right\rangle \\ &= \lim_{n\to\infty}\left\langle \sum_{i=1}^{n} f(t_i^*)\Delta t, \sum_{i=1}^{n} g(t_i^*)\Delta t, \sum_{i=1}^{n} h(t_i^*)\Delta t\right\rangle\end{aligned}$$

The fundamental theorem of calculus can be extended to vector functions.

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \left[\vec{R}(t)\right]_a^b = \vec{R}(b)-\vec{R}(a)$$

The constant of integration for the indefinite integral of a vector function is itself a vector $\vec{C}$.

$$\int \vec{r}(t)\,\mathrm{d}t = \vec{R}(t)+\vec{C}$$

13.3 Arc Length and Curvature

Arc Length

The arc length of a vector function is simply the integral of its derivative's magnitude.

$$L = \int_a^b |\vec{r}'(t)|\mathrm{d}t = \int_a^b \left[\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\right] \mathrm{d}t$$

A single curve can be represented by multiple vector functions, each of which is a **parametrization** of the original curve. Regardless of which parametrization is used, the arc length will be the same (so long as parameters are converted between), as arc length is a geometric property, making it independent of the parametrization used.

The Arc Length Function

For a curve C given by a continuous vector function $\vec{r}$ parametrized using $t \in [a, b]$, the **arc length function** is defined as

$$s(t) = \int_a^t |\vec{r}'(u)|\mathrm{d}u$$

Differentiating both sides (applying the fundamental theorem of calculus for the right side), it can be seen that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\vec{r}'(t)|$$

This **Parametrization of a curve with respect to arc length** can be used to analyze the curve, as arc length is not dependent on a particular coordinate system of parametrization, instead being an inherent geometric property of the curve's.

Curvature

A parametrization is called **smooth** on an interval if its derivative is continuous and not equal to $\vec{0}$ at any point in the interval.

The curvature of a curve C at a given point is a measure of how quickly it is changing direction. More specifically, it is the magnitude of the rate of change of the unit tangent vector with respect to arc length. (Arc length is used due to its parametrization-independence.)

The **curvature** of a curve is

$$\kappa = \left|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|$$

The curvature can be more easily computed by expressing it in terms of parameter t instead of s. Using chain rule,

$$\frac{\mathrm{d}\vec{T}}{\mathrm{d}s} = \frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \implies \kappa = \left|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\vec{T}/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t}\right|$$

Rewriting by differentiating the formula for arc length,

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

It may sometimes be more convenient to re-express curvature as

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$$

The Normal and Binormal Vectors