

Assignment 0 F22

1. **Claim.** $S = \{x \in \mathbb{R} \mid x^3 + 2x < 4\}$ bounded above.

Proof. It is a fundamental property of the definition of S that none of its elements exceed or equal 4. Therefore, if $\alpha = 4 \in \mathbb{R}$, it is true that $x \leq \alpha$ for all $x \in S$, so S is bounded above.

Claim. S is not bounded below.

Proof. The definition of S provides no limitation regarding a lower bound, and as $x \rightarrow -\infty$, x^3 and $x \rightarrow -\infty$, so $x^3 + x \rightarrow -\infty$. This means that there does not exist any real number β such that $x \geq \beta$ for all $x \in S$, as x decreases infinitely, so S is unbounded below. \square .

2. **Claim.** There is no order relation “ $<$ ” on \mathbb{C} .

Proof. Assume that $i > 0$. By axiom 4,

$$\begin{aligned} i^2 &> i(0) \\ -1 &> 0 \end{aligned}$$

This is clearly false, so $i < 0$ by axiom 1. But by axiom 5,

$$\begin{aligned} i^4 &< i^3(0) \\ 1 &< 0 \end{aligned}$$

which is also false. Therefore, axiom 1 is violated, as i cannot be less than or greater than 0, meaning that there is no order relation satisfying the 5 axioms on \mathbb{C} . \square .

3. a)

$$f(x, y) = f(x, -y) = \frac{x+1}{x^2 + y^2 + 2}$$

as $\forall y \in \mathbb{Z}, y^2 = (-y)^2$. Therefore f is not injective.

All $q \in \mathbb{Q}$ can be written as the ratio of $p, q \in \mathbb{Z}$, and all $p \in \mathbb{Z}$ can be written as $r + 1$ where $r \in \mathbb{Z}$, as every integer is exactly 1 more than the prior integer. The denominator goes to ∞ as $x, y \rightarrow \pm\infty$, so possible $q \in \mathbb{Q}$ can be output by f . Therefore f is surjective.

- b)

$$f(x, y) = f(-x, -y) = xy$$

as the negatives cancel. Therefore f is not injective.

Every $r \in \mathbb{R}$ can be written as $r \times 1$ and $1 \in \mathbb{R}$, so f can output every $r \in \mathbb{R}$, making f surjective.

- c)

$$f(x) = f(-x) = \frac{x^2}{1+x^2}$$

as for all $x \in \mathbb{R}, x^2 = (-x)^2$, so f is not injective.

$f(x)$ is a rational function with a denominator never equal to 0, meaning that it is continuous for all $x \in \mathbb{R}$. $f = 0$ and

$$\lim_{x \rightarrow \pm\infty} \frac{x^2}{1+x^2} = \frac{1}{1} = 1$$

so by the intermediate value theorem, f must yield all outputs in $[0, 1)$, making f surjective.

4. **Claim.** *There does not exist a surjective function from X onto its power set.*

Proof. Each element $x \in X$ can be either present or not present in a given subset of X . The cardinality of the $P(X)$ is therefore $2^{|X|}$. As $|X| \in \mathbb{N}$ and $2^n > n$ for all $n \in \mathbb{N}$, there are more elements in $P(X)$ than there are in X , making a surjective function impossible. \square

5. a)

$$f \cdot g(\alpha) = f(\alpha) * g(\alpha) = \alpha f(\alpha)g(\alpha) = g(\alpha)$$

$$f(\alpha) = \frac{1}{\alpha}$$

- b)

$$f \cdot g(\alpha) = f(\alpha) * g(\alpha) = \alpha f(\alpha)g(\alpha) = \alpha(2 + \alpha^2 i)g(\alpha) = (2\alpha + \alpha^3 i)g(\alpha) = 1$$

$$g(\alpha) = \frac{1}{2\alpha + \alpha^3 i}$$

- c)

$$f \cdot g(\alpha) = f(\alpha) * g(\alpha) = \alpha f(\alpha)g(\alpha) = \alpha(2 + \alpha^2 i)(\alpha - i) = \alpha(2\alpha - 2i + \alpha^3 i + \alpha^2)$$

$$= (\alpha^3 + 2\alpha^2) + (\alpha^4 - 2)i$$

6. a) In order for X to contain $x + y$ for all unique $x, y \in X$, it can be defined as $X = \{0, n\}$. Then $0 + n = n \in X$ and $0 \times n = 0 \in X$, making X a sticky subset containing n .
- b) For X to meet the criteria that for all $x, y \in X$, $x + y \in X$, it can be $X = \{kn \mid k < \mathbb{Z}\}$. Only one such set exists per number.

7. The induction is not true going from $n = 1$ to $n = 2$. For $n = 1$,

$$P(1) \implies x_1 = x_1$$

while

$$P(2) \implies x_1 = x_2$$

Removing x_2 from this yields simply x_1 , which is not a statement but rather a number. The transitive property can therefore not be applied, nor can induction.

8. a)

$$a_n = n \implies \lim_{n \rightarrow \infty} a_n = \infty$$

- b)

$$a_n = \frac{1}{2^n} \implies \alpha = \frac{1}{1 - 1/2} = 2$$