Calculus III

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Contents

Chapter 12

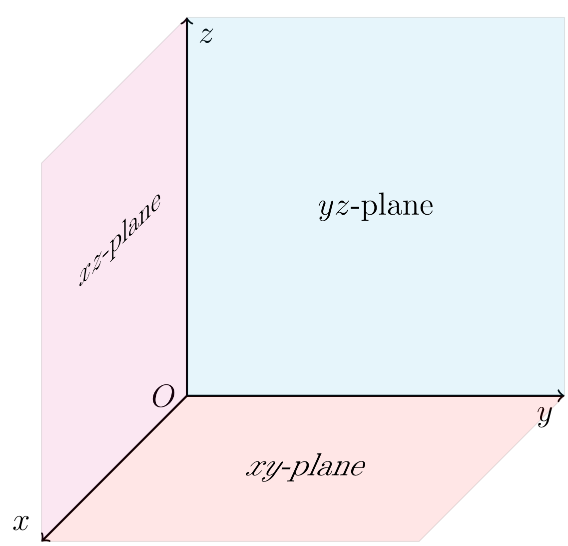
Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

Any point in a plane can be represented as an ordered pair of real numbers. Because this uses two numbers, a plane is called two-dimensional. To locate a point in space, a triplet of real-numbers is required.

3D Space

Before points can be represented in 3D space, a fixed point O (the origin) and three perpendicular lines that pass through it, called the **coordinate axes**. These axes are labeled the x-, y-, and z-axes. In general, the former two are horizontal while the third is vertical. The direction of the z-axis is determined by the **right-hand rule**. Curling the fingers of the right hand from the positive x-axis to the positive y-axis, the thumb will point in the direction of the positive z-axis.
The three coordinate axes determine the three **coordinate planes**.



Three planes divided space into eight **octants**. Illustrated above are the positive xz-, yz-, and xy-planes, constituting the **first octant**.
A point's **coordinates** are an ordered triple of real numbers. A point's **projection** onto a plane is the point with two of the same coordinates, the third becoming 0.

Plane	xy	yz	xz
(a, b, c)	$(a, b, 0)$	$(0, b, c)$	$(a, 0, c)$

The set of all ordered triples is the cartesian product of three sets of all real numbers, denoted appropriately by $\mathbb{R}^3$ and defined as

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

A one-to-one correspondence between points in space and ordered triples in $\mathbb{R}^3$ is a **three-dimensional coordinate system**. It should be noted that the first octant can be described as the set of points for which all coordinates are positive.

Distances and Spheres

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere The equation of a sphere with center (h, k, l) and radius r is

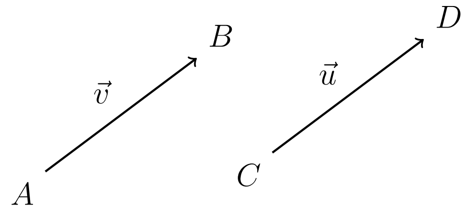
$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

12.2 Vectors

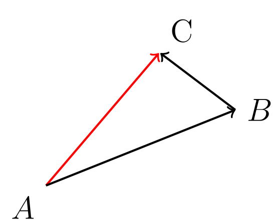
The term **vector** is used to indicate a quantity with both magnitude and direction.

Geometric Description of Vectors

A vector is often represented by an arrow, the length of which represents its magnitude. A vector is denoted by a letter in boldface ($\mathbf{v}$) or with an arrow above a letter ($\vec{v}$).
A **displacement vector** is a vector representing how something is displaced from its **initial point** (tail) to its **terminal point** (tip).



In the above figure, vector $\vec{v}$ has initial point A and terminal point B. This can be indicated by writing $\vec{v} = \overrightarrow{AB}$. If $AB = CD$ and the angles relative to the same axis are equal, then the $\vec{v}$ and $\vec{u}$ are **equivalent** (or **equal**).
The only vector without a direction is the **zero vector**, denoted by $\mathbf{0}$, which has length 0.
The sum of two vectors can be denoted with the initial point of the first and the terminal point of the second.

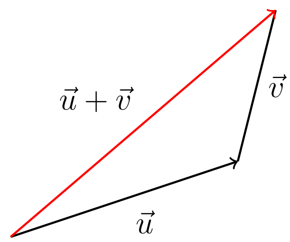


$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

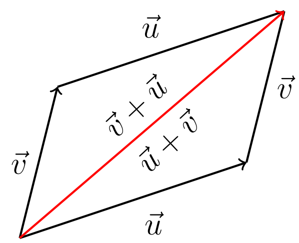
To add vectors, the one's tail can be moved to the other's tip and the resultant terminal point found.

Definition of vector addition The sum of two vectors position such that the initial point of one is the terminal point of the other is the vector from the initial point of the first to the terminal point of the second.

The definition of vector addition is sometimes referred to as the **Triangle Law**.



Doing the opposite addition results in the same resultant vector. This is made visible b y the **Parallelogram Law**.



A **scalar** is a number that is not a vector.

Definition of Scalar Multiplication The **scalar multiple** $c\vec{v}$ of a scalar c and a vector $\vec{v}$