

MATH 135 Assignments

Arnav Patri

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Contents

Fall 2022	2
Assignment 1	2
Assignment 2	4
Assignment 4	7

Fall 2022

Assignment 1

1.
 - a) The smallest prime number can either be 2 or not 2, making “The smallest prime number is 2” a valid statement.
 - b) The sum of $\cos^2 \theta$ and $\sin^2 \theta$ could be 1 or not 1, so “ $\cos^2 \theta + \sin^2 \theta = 1$ ” is a valid statement.
 - c) It is either possible for every integer to be of the form $2k$ or $2k + 1$ or there exists at least one exception, so “Every integer x is of the form $2k$ or $2k + 1$ ” is a valid statement.
 - d) 0 can either be even or odd or it could not be either, making “The number 0 is neither even nor odd” a valid statement.
 - e) A question is not true or false; therefore, “Is $3 > 2$ true?” is not a valid statement.
2.
 - a) $\forall x \in \mathbb{Z}, x^2 > 0$
 - b) $\forall x \in \mathbb{R}, x^3 \in \mathbb{R}$
3.
 - a) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{R}, x + y \geq 3\sqrt{2}$
 - b) $\forall a \in \mathbb{N}, \exists b \in \mathbb{Q}, \forall c \in \mathbb{Z}, a = b - c$
4.
 - a) $P(2, 4) = \exists y \in \mathbb{Z}, 2(2) + 4y = 4 \implies y = 0 \in \mathbb{Z} \implies \text{true}$
 $P(2, 5) = \exists y \in \mathbb{Z}, 2(2) + 4y = 5 \implies y = 0.25 \notin \mathbb{Z} \implies \text{false}$
 - b) There is no condition given for n , meaning that the truth value of $P(x, n)$ cannot be determined, so “ $\exists x \in \mathbb{Z}, P(x, n)$ ” is an open sentence depending on n .
 - c) As all variables are specified, “ $\forall n \in \mathbb{Z}, \exists x \in \mathbb{Z}, P(x, n)$ ” is a mathematical statement. As the $2(x + 2y)$ must be even given that x and y are integers, this statement is false for all odd values of n , meaning that the statement as a whole is false.
5.
 - a)

$$\begin{aligned} \binom{8^{k^2}}{(4^k)} &= \binom{2^{3k^2}}{(2^{2k})} = 2^{3k^2+2k} = 2 \implies 3k^2 + 2k = 1 \implies 3k^2 + 2k - 1 = 0 \\ &\implies 0 = (3k - 1)(k + 1) \implies k = -1 \in \mathbb{Z} \implies \text{true} \end{aligned}$$

b)

$$x^2 - x + \frac{1}{4} > 0 \implies \neg \exists x = \frac{1 \pm \sqrt{1-1}}{2} = \frac{1}{2} \in \mathbb{R} \implies \text{false}$$

c)

$$\forall x \in \{0, 1, 2, 3\}, \forall y \in \{0, 1, 2, 3\}, (x+y) \in \mathbb{Z}, (x^2+y^2) \in \mathbb{Z} \implies \frac{x+y}{x^2+y^2} \in \mathbb{Q} \implies \text{true}$$

d)

$$4^x + (\ln x)^2 \geq 2x \ln(x^2) = 4x \ln x$$

As 4^x grows faster than $x \ln x$, for all $x \in \mathbb{N}$ and $4 \geq 0$, the the statement is true.

e)

$$x + 2xy = 4$$

$$1 + 2y = \frac{2}{x}$$

$$y = \frac{4}{x} - \frac{1}{2}$$

$$x \in \mathbb{Q} \implies \frac{4}{x} - \frac{1}{2} \in \mathbb{Q} \implies \text{true}$$

6.

$$\text{a) } \forall x \in \{1, 2, 3\}, \forall y \in \{1, 2, 3\}, \frac{4680}{x^2+y^2} \in \mathbb{Z} \quad \text{b) } \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^3 = 135$$

$$\text{c) } \exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x \leq y$$

$$\text{d) } \exists x \in \mathbb{Q}, \exists y \in \mathbb{Q}, x^3 y^2 = 108$$

Assignment 2

1.
 - a) The hypothesis is “ $xy < 0$ ”.
 - b) The conclusion is “ $x > 0$ and $y < 0$ ”.
 - c) The converse is “If $x > 0$ and $y < 0$, then $xy > 0$ ”.
 - d) The contrapositive is “If $x \leq 0$ or $y \geq 0$, then $xy \leq 0$ ”.
 - e) The negation is “ $xy < 0$ or $x \leq 0$ or $y \geq 0$ ”.
 - f) In order for the product of two numbers to be negative, one number must be positive and the other negative. If x is negative and y is positive, though, their product is still negative. Therefore, $\forall x, y \in \mathbb{R}, S(x, y)$ is false.
2.
 - a) This fails to restrict the domain of x .
 - b) This fails to consider the case $n = 0$.
3.
 - a) The truth table for this is

P	Q	R	$P \implies Q$	$\neg R$	$P \wedge (\neg R)$	$(P \implies Q) \vee (P \wedge (\neg R))$
T	T	T	T	F	F	T
T	T	F	T	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	T	F	F	T
F	T	F	T	T	F	T
F	F	T	T	F	F	T
F	F	F	T	T	F	T

The only cases where the statement is true are when P and Q are both true, P is true and R is false, or when P is false.

- b) The truth table these statements is

A	B	C	$A \implies B$	$(A \implies B) \implies C$	$C \vee A$	$B \implies C$	$(C \vee A) \wedge (B \implies C)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	F	F	T	F

As the columns of $(A \implies B) \implies C$ and $(C \vee A) \wedge (B \implies C)$ are identical, they are logically equivalent.

- 4.

$$\begin{aligned}
 P \vee \neg((\neg Q) \vee R) &\equiv P \vee (Q \wedge (\neg R)) && \text{(DeM/def. of } \neg) \\
 &\equiv (P \vee Q) \wedge (P \vee (\neg R)) && \text{(dist. law)} \\
 &\equiv (R \implies P) \wedge (P \vee Q) && \text{(comm. law/def. of } \implies)
 \end{aligned}$$

5. **Claim.** $\forall a \in \mathbb{Z}, 2 \nmid (a^3 - 6a^2 + 5a + 1)$.

Proof. Let $a \in \mathbb{Z}$. a must be either even or odd.

Case 1. $2 \mid a$. This means that for some $k \in \mathbb{Z}$, $a = 2k$. Substituting with this yields

$$\begin{aligned} a^3 - 6a^2 + 5a + 1 &= (2k)^3 - 6(2k)^2 + 5(2k) + 1 \\ &= 8k^3 - 24k^2 + 10k + 1 \\ &= 2(4k^3 - 12k^2 + 5k) + 1 \end{aligned}$$

As $4k^3 - 12k^2 + 5k \in \mathbb{Z}$, by the definition of divisibility $2 \nmid (a^3 - 6a^2 + 5a + 1)$.

Case 2. $2 \nmid a$. This means that for some $k \in \mathbb{Z}$, $a = 2k + 1$. Substituting yields

$$\begin{aligned} a^3 - 6a^2 + 5a + 1 &= (2k + 1)^3 - 6(2k + 1)^2 + 5(2k + 1) + 1 \\ &= ((2k)^3 + 3(2k)^2 + 3(2k) + 1) - 6((2k)^2 + 2(2k) + 1) + 10k + 5 + 1 \\ &= 8k^3 + 12k^2 + 6k + 1 - 24k^2 - 24k - 6 + 10k + 6 \\ &= 8k^3 - 12k^2 - 8k + 1 \\ &= 2(4k^3 - 6k^2 - 4k) + 1 \end{aligned}$$

As $4k^3 - 6k^2 - 4k \in \mathbb{Z}$, by the definition of divisibility $2 \nmid (a^3 - 6a^2 + 5a + 1)$.

Regardless of whether a is even or odd, $2 \nmid (a^3 - 6a^2 + 5a + 1)$. \square

6. **Claim.** $\forall x \in \mathbb{R}, 1 + 99 \sin^2 x \geq 10 \sin(2x)$.

Proof. Let $x \in \mathbb{R}$. Suppose that both sides of the inequality are equal for this x :

$$1 + 99 \sin^2 x = 10 \sin(2x)$$

The double angle formula and Pythagorean identity yield

$$\begin{aligned} \sin^2 x + \cos^2 x + 99 \sin^2 x &= 20 \sin x \cos x \\ 0 &= 100 \sin^2 x - 20 \sin x \cos x + \cos^2 x \end{aligned}$$

This formula may only have solutions if for some $x, y \leq 1$,

$$0 = 100x^2 - 20xy + y^2$$

Applying the quadratic formula yields

$$\begin{aligned} x &= \frac{20y \pm \sqrt{400y^2 - 400y^2}}{200} \\ &= \frac{y}{10} \end{aligned}$$

This means that the equality is only valid when

$$\begin{aligned} \sin x &= \frac{\cos x}{10} \\ x &= \arctan\left(\frac{1}{10}\right) \end{aligned}$$

Substituting this into the equality with the double angle formula applied,

$$\begin{aligned}
 1 + 99 \sin^2 \left(\arctan \left(\frac{1}{10} \right) \right) &= 20 \sin \left(\arctan \left(\frac{1}{10} \right) \right) \cos \left(\arctan \left(\frac{1}{10} \right) \right) \\
 1 + \frac{99}{101} &= 20 \frac{100}{101} \\
 \frac{200}{101} &= \frac{200}{101}
 \end{aligned}$$

Pythagorean theorem was used to obtain the values of $\sin x$ and $\cos x$. As this is the only solution in the first period (note that the period of $\sin(2x)$ is π , meaning that $x = \arctan(1/10) + \pi$ is in the second period), the sign of

$$y = 1 + 99 \sin^2 x - 10 \sin(2x)$$

must not change at any other point in that period. At $x = 0$,

$$y = 1 + 0 - 0 = 1$$

At $x = \pi$,

$$y = 1 + 0 - 0 = 1$$

This means that $y \geq 0$ for $x \in [0, \pi]$; that is,

$$\begin{aligned}
 1 + 99 \sin^2 x - 10 \sin(2x) &\geq 0 \\
 1 + 99 \sin^2 x &\geq 10 \sin(2x)
 \end{aligned}$$

As sine is a periodic function, this inequality is true for all $x \in \mathbb{R}$. \square .

Assignment 4

1.
 - a) If $m \geq n$, $(m - n)! = 0$, meaning that the denominator of $\binom{n}{m}$ would be 0.
 - b) The bounds of summation were changed so that in the inductive step, the sums from $P(k)$ and $P(k + 1)$ could be combined, enabling PI to be applied before changing the bounds again to prove $P(k + 1)$.
 - c) The upper bound of the summation for the case with $a = 0$ was $m = n$, making $a^{m-n} = 0^0$.

d) **Claim.**

$$\forall a \in \mathbb{Z}, \forall n \in \mathbb{N}, a \mid ((2 + a)^n - 2^n)$$

Proof. This is a proof by induction on n , where $P(x)$ is

$$\forall a \in \mathbb{Z}, a \mid ((2 + a)^n - 2^n)$$

Base Case: Let $n = 1$. Then

$$2 + a - 2 = a$$

By the definition of divisibility, for all $a \in \mathbb{Z}$, $a \mid a$, so $P(1)$ is true.

Inductive Step: Assume that for some $k \in \mathbb{N}$, $P(k)$ is true; that is,

$$\forall a \in \mathbb{Z}, a \mid ((2 + a)^k - 2^k)$$

By BT2 and the definition of divisibility, the expression becomes

$$\begin{aligned} (2 + a)^k - 2^k &= \sum_{i=0}^k \left[\binom{k}{i} 2^{k-i} a^i \right] - 2^k \\ &= xa \end{aligned}$$

for some $x \in \mathbb{Z}$.

The expression for $P(k + 1)$ is

$$(2 + a)^{k+1} - 2^{k+1} = \sum_{i=0}^{k+1} \left[\binom{k+1}{i} 2^{k+1-i} a^i \right] - 2^{k+1}$$

by BT2. By Pascal's identity and the induction hypothesis,

$$\begin{aligned} (2 + a)^{k+1} - 2^{k+1} &= \binom{k+1}{0} 2^{k+1} a^0 + \sum_{i=1}^{k+1} \left[\left(\binom{k}{i-1} + \binom{k}{i} \right) 2^{k+1-i} a^i \right] - 2^{k+1} \\ &= 2^{k+1} + a \sum_{i=0}^k \left[\binom{k}{i} 2^{k-i} a^i \right] \\ &\quad + 2 \sum_{i=0}^k \left[\binom{k}{i} 2^{k-i} a^i \right] - \binom{k}{0} 2^{k+1} a^0 + \binom{k}{k+1} 2^0 a^{k+1} - 2^{k+1} \\ &= a(xa + 2^k) + 2(xa + 2^k) - 2^{k+1} + 0 \\ &= a(xa + 2^k) + 2xa + 2^{k+1} - 2^{k+1} \\ &= a(xa + 2^k + 2x) \end{aligned}$$

As $xa + 2^k + 2x \in \mathbb{Z}$, $(2 + a)^{k+1} - 2^{k+1}$ is by definition divisible by a , making $P(k + 1)$ true. By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

Alternate Proof. Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. By BT2,

$$\begin{aligned}(2+a)^n - 2^n &= \sum_{i=0}^n \left[\binom{n}{i} 2^{n-i} a^i \right] - 2^n \\ &= 2^n + \sum_{i=1}^n \left[\binom{n}{i} 2^{n-i} a^i \right] - 2^n \\ &= a \sum_{i=1}^n \binom{n}{i} 2^{n-i} a^{i-1}\end{aligned}$$

As $\sum_{i=1}^n \binom{n}{i} 2^{n-i} a^i \in \mathbb{Z}$, $a \mid ((2+a)^n - 2^n)$.

2. Claim.

$$\forall n \in \mathbb{Z}, (4 \nmid (n^4 + 3)) \iff (n \mid 2)$$

Proof. Let $n \in \mathbb{Z}$.

Case 1. $n \nmid 2$.

This means that for some $k \in \mathbb{Z}$, $n = 2k + 1$, so

$$\begin{aligned}n^4 + 3 &= (2k + 1)^4 + 3 \\ &= (2k)^4 + 4(2k)^3 + 6(2k)^2 + 4(2k) + 1 + 3 \\ &= 16k^4 + 24k^3 + 24k^2 + 8k + 4 \\ &= 4(4k^4 + 6k^3 + 6k^2 + 2k + 1)\end{aligned}$$

$4k^4 + 6k^3 + 6k^2 + 2k + 1 \in \mathbb{Z}$, so by the definition of divisibility, $(n \nmid 2) \implies (4 \nmid (n^4 + 3))$.

Case 2. $n \mid 2$.

In this case, for some $k \in \mathbb{Z}$, $n = 2k$, so

$$\begin{aligned}n^4 + 3 &= (2k)^4 + 3 \\ &= 16k^4 + 3 \\ &= 4(4k^4) + 3\end{aligned}$$

As $4k^4 \in \mathbb{Z}$ and $4 \nmid 3$, $(n \mid 2) \implies (4 \nmid (n^4 + 3))$.

This means that $(4 \nmid (n^4 + 3)) \iff (n \mid 2)$. \square

3. Claim. $\forall a \in \mathbb{N}, \forall n \in \mathbb{Z}, (a^2 - 10) \neq n^2$.

Proof. Let $a \in \mathbb{N}$, and assume that for some $n \in \mathbb{Z}$,

$$a^2 - 10 = n^2$$

This means that

$$a^2 - n^2 = 10$$

This implies that there are two perfect squares separated by distance 10. The sequence of the difference of the perfect squares is given by

$$\begin{aligned}c_i &= (i+1)^2 - i^2 \\ &= i^2 + 2i + 1 - i^2 \\ &= 2i + 1\end{aligned}$$

It is clear that this difference is always odd, meaning that it can never be equal to 10. There is therefore no $a \in \mathbb{N}$ that is 10 more than a perfect square. \square

4. **Claim.**

$$\forall (w, x, y, z \in \mathbb{Z}), ((w \neq y) \wedge (wz - xy \neq 0)) \implies \exists! r \in \mathbb{Q}, \frac{wr + x}{yr + z} = 1$$

Proof. Let $w, x, y, z \in \mathbb{Z}$ with $w \neq y$ and $wz \neq xy$. Assume that for some $r \in \mathbb{Q}$,

$$\begin{aligned} \frac{wr + x}{yr + z} &= 1 \\ wr + x &= yr + z \\ wr - yr &= z - x \\ r &= \frac{z - x}{w - y} \end{aligned}$$

As $w \neq y$, the denominator is not 0. Substituting this back into the equation,

$$\begin{aligned} 1 &= \frac{wr + x}{yr + z} \\ &= \frac{w \frac{z-x}{w-y} + x}{y \frac{z-x}{w-y} + z} \\ &= \frac{w(z-x) + x(w-y)}{y(z-x) + z(w-y)} \\ &= \frac{wz - wx + wx - xy}{yz - xy + wz - yz} \\ &= \frac{wz - xy}{wz - xy} \\ &= 1 \end{aligned}$$

meaning that this solution is not extraneous. ($wz \neq xy$, meaning that the second to last expression is not indeterminate.) The expression for r is dependent on the values of the parameters alone, meaning that it is simply the ratio of two fixed differences. There is therefore only one such r that may exist. \square

5. **Claim**

$$\forall n \in \mathbb{N}, \sum_{i=1}^n (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

Proof. This is a proof by induction on n , where $P(n)$ is the statement

$$\sum_{i=1}^n (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

Base Case. For $n = 1$,

$$\begin{aligned} \sum_{i=1}^1 (-1)^i i^2 &= (-1)^1 1^2 \\ &= -1 \\ &= -\frac{2}{2} \\ &= \frac{(-1)^1 1(1+2)}{2} \end{aligned}$$

so $P(1)$ is true.

Inductive Step Assume that for some $k \in \mathbb{N}$, $P(k)$ is true; that is,

$$\sum_{i=1}^k (-1)^i i^2 = \frac{(-1)^k k(k+1)}{2}$$

Adding the next term to the sum yields

$$\begin{aligned} \sum_{i=1}^{k+1} (-1)^i i^2 &= (-1)^{k+1} (k+1)^2 + \sum_{i=1}^k (-1)^i i^2 \\ &= (-1)^{k+1} (k+1)^2 + \frac{(-1)^k k(k+1)}{2} \\ &= \frac{2(-1)^{k+1} (k+1)^2 + (-1)^k k(k+1)}{2} \\ &= \frac{(-1)^{k+1} (2(k^2 + 2k + 1) - (k^2 + k))}{2} \\ &= \frac{(-1)^{k+1} (2k^2 + 4k + 2 - k^2 - k)}{2} \\ &= \frac{(-1)^{k+1} (k^2 + 3k + 2)}{2} \\ &= \frac{(-1)^{k+1} (k+1)(k+2)}{2} \\ &= \frac{(-1)^{k+1} (k+1)((k+1) + 1)}{2} \end{aligned}$$

so $P(k+1)$ is also true.

By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

6. Claim.

$$\forall x, y \in \mathbb{R}, (0 < y < x) \implies \forall n \in \mathbb{N}, x^n - y^n \leq nx^{n-1}(x - y)$$

Proof. Let $x, y \in \mathbb{R}$ with $0 < y < x$. This is a proof by induction on n , where $P(n)$ is

$$x^n - y^n \leq nx^{n-1}(x - y)$$

Base Case. Let $n = 1$. Then

$$\begin{aligned} x^1 - y^1 &\leq (1)x^{1-1}(x - y) \\ x - y &\leq x - y \end{aligned}$$

so $P(1)$ is true.

Inductive Step. Assume that for some $k \in \mathbb{N}$, $P(k)$ is true; that is,

$$x^k - y^k \leq kx^{k-1}(x - y)$$

Multiplying by x on both sides and expanding

$$x^{k+1} - xy^k \leq kx^{k+1} - kx^k y$$

Adding $xy^k - y^{k+1}$ to both sides and recalling that $0 < y < x$,

$$\begin{aligned}x^{k+1} - y^{k+1} &\leq kx^{k+1} - kx^ky + xy^k - y^{k+1} \\&= kx^{k+1} - kx^ky + xy^k\left(1 - \frac{y}{x}\right) \\&< kx^{k+1} - kx^ky + x^{k+1}\left(1 - \frac{y}{x}\right) \\&= kx^{k+1} - kx^ky + x^{k+1} - x^ky \\&= (k+1)x^k(x-y)\end{aligned}$$

making $P(k+1)$ true as well.

By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square