Topology – Munkres

Arnav Patri

Chapter 2 – Topological Spaces and Continuous Functions

Section 13 – Basis for a Topology

1. Let (X, τ) be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is some open $U \in \tau \cap \mathcal{P}(A)$ with $x \in U$. Show that $A \in \tau$.

   Solution: For each $x \in A$, let $U_x \in \tau \cap \mathcal{P}(A)$ such that $x \in U_x$, and let $U = \bigcup_{x \in A} U_x$. Clearly, $U = A$, making the latter a union of open sets and thus itself open. $\square$

3. Show that

$$\tau_{cf} = \{U \subseteq X \mid |X \setminus U| < \aleph_0\} \cup \varnothing$$

   is a topology. Is

$$\tau_\infty = \{U \subseteq X \mid |X \setminus U| \geq \aleph_0\} \cup \{\varnothing, X\}$$

   a topology?

   Solution:
   If $|X| < \aleph_0$, then $\tau_{cf} = \mathcal{P}(X)$, yielding the trivial topology. Suppose instead that X is infinite, and let $\{U_\lambda\}_{\lambda \in \Lambda}, \{V_i\}_{i=1}^n \subseteq \tau_{cf}$. $\varnothing \in \tau_{cf}$ by construction, and

$$|X \setminus X| = |\varnothing| = 0 < \aleph_0$$

   so $X \in \tau_{cf}$ as well. Furthermore, for any $\alpha \in \Lambda$,

$$\left| X \setminus \bigcup_{\lambda \in \Lambda} U_\lambda \right| \leq |X \setminus U_\alpha| < \aleph_0$$

   making τ_{cf} closed under arbitrary unions. If $V_i = \varnothing$ for some i, $\bigcap_{i=1}^n V_i = \varnothing \in \tau_{cf}$, and otherwise, $|X \setminus V_i| < \aleph_0$, so

$$\left| X \setminus \bigcap_{i=1}^n V_i \right| = \left| \bigcup_{i=1}^n [X \setminus V_i] \right| \leq \sum_{i=1}^n |X \setminus V_i| < \aleph_0$$

   so τ_{cf} is also closed under finite intersections and is thus a valid topology. $\square$

   If $|X| < \aleph_0$, τ_∞ is the trivial topology. Otherwise, for each $x \in X$,

$$|X \setminus \{x\}| = |X| - |\{x\}| = \aleph_0 - 1 = \aleph_0$$

   so each singleton is in τ_∞. Despite this,

$$\left| X \setminus \bigcup_{y \in X \setminus \{x\}} \{y\} \right| = |X \setminus (X \setminus \{y\})| = |\{y\}| = 1$$

   meaning that τ_∞ is not closed under arbitrary unions and is thus not a topology.

4. (a) Let $\{\tau_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{P}(\mathcal{P}(X))$ be a family of topologies on X. Show that $\tau = \bigcap_{\lambda \in \Lambda} \tau_\lambda$ is also a topology. Is $\tau' = \bigcup_{\lambda \in \Lambda} \tau_\lambda$?

   Solution: Let $\left\{U_\gamma\right\}_{\gamma \in \Gamma}, \{V_i\}_{i=1}^n \subseteq \tau$. Then each U_γ and V_i is open in each τ_λ, meaning that the union of the former collection and the intersection of the latter are also in each τ_λ and thus in the intersection τ as well, making the latter a topology. $\square$

   Let $X = \{1, 2, 3\}$, let $\tau_1 = \{\varnothing, \{1\}, X\}$, and let $\tau_2 = \{\varnothing, \{2\}, X\}$. These are clearly both valid topologies, so consider $\tau' = \tau_1 \cup \tau_2$. This includes both $\{1\}$ and $\{2\}$ but not their union and is thus not a topology.

(b) Let $\{\tau_\lambda\}_{\lambda\in\Lambda}$ be a family of topologies on X. Show that there are a unique smallest topology τ_Λ finer than and largest topology τ_0 coarser than every τ_λ.

Solution: Let τ_Λ be the topology generated by $\mathcal{B} = \bigcup_{\lambda\in\Lambda} \tau_\lambda$. By Lemma 13.1, this means that

$$\tau_\Lambda = \left\{ \bigcup B \,\middle|\, B \subseteq \bigcup_{\lambda\in\Lambda} \tau_\lambda \right\}$$

Clearly, each $\tau_\lambda \subseteq \tau_\Lambda$. Let τ' be a topology such that each $\tau_\lambda \subseteq \tau'$, and let $U \in \tau_\Lambda$. Let $B \subseteq \mathcal{B}$ such that $\bigcup B = U$. Each element of B is open in some τ_λ and is thus open in τ', so its union U is also in τ'; that is, $\tau_\Lambda \subseteq \tau'$.

Let $\tau_0 = \bigcap_{\lambda\in\Lambda} \tau_\lambda$. By 4.(a), this is a valid topology. Let τ'' be a topology contained in each τ_λ. Then for $U \in \tau''$, $U \in \tau_\lambda$ for each λ, so $\tau'' \subseteq \tau_0$. □

5. Show that if $\mathcal{A}$ is a basis or subbasis generating (X, τ), then τ is the intersection of all topologies on X containing $\mathcal{A}$.

Solution: Let $\mathcal{A} \subseteq X$ be a basis generating topology $\tau_\mathcal{A}$, and let τ' be a topology on containing $\mathcal{A}$. By Lemma 13.1,

$$\tau_\mathcal{A} = \left\{ \bigcup A \,\middle|\, A \subseteq \mathcal{A} \right\}$$

As $\mathcal{A} \subseteq \tau'$, $\tau_\mathcal{A} \subseteq \tau'$ by closure under unions.

Let $\mathcal{S} \subseteq X$ be a subbasis generating topology $\tau_\mathcal{S}$ and let τ'' be a topology containing $\mathcal{S}$. By closure under arbitrary finite intersections and countable unions, $\tau_\mathcal{S} \subseteq \tau''$. □

8. (a) Show from Lemma 13.2 that

$$\mathcal{B} = \{(a, b) \mid a < b \in \mathbb{Q}\}$$

is a basis that generates the standard topology τ on $\mathbb{R}$.

Solution: Let $a < b \in \mathbb{R}$ and let $x \in (a, b)$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, let $c \in [a, x) \cap \mathbb{Q}$ and $d \in (x, b] \cap \mathbb{Q}$. Then $x \in (d, e) \in \mathcal{B}$. τ is generated by bounded open intervals, so for any $U \in \tau$ and $x \in U$, there is some open interval and thus some element of $\mathbb{Q}$ within U containing x, so by Lemma 13.2, $\mathcal{B}$ generates τ. □

(b) Show that

$$\mathcal{C} = \{[a, b) \mid a < b \in \mathbb{Q}\}$$

does not generate the lower limit topology on $\mathbb{R}$.

Solution: Let $a < b \in \mathbb{R} \setminus \mathbb{Q}$. Then for all $x \in [a, b) \setminus \mathbb{Q}$, $a < x$, meaning that there is no element of $\mathcal{C} \cap \mathcal{P}([a, b))$ containing a. By Lemma 13.3, the topology generated by $\mathcal{C}$ is not finer than and thus not equal to the lower limit topology. □

Section 16 – The Subspace Topology

1. Show that a subset of a subspace inherits the same topology from both the subspace and the parent space.

   Solution: Let (Y, τ_Y) be a subspace of (X, τ) and let $A \subset Y$ inherit topologies τ_A from Y and $\tau_{A'}$ from X. Then

$$\begin{aligned} \tau_A &= \{Z \cap U \mid U \in \tau_Y\} = \{Z \cap (U \cap V) \mid V \in \tau\} \\ &= \{Z \cap U \mid U \in \tau\} \end{aligned}$$

   by closure under finite unions. □

2. Let τ and τ' be topologies on X with the latter strictly finer than the former. How do their subspace topologies on $Y \subseteq X$ compare?

   Solution: Let τ_Y and τ'_Y respectively denote the subspace topologies induced by τ and τ'. Then

$$\tau_Y = \{Z \cap U \mid U \in \tau\} \subseteq \{Z \cap U \mid U \in \tau'\}$$

   the former is coarser than the latter. If there is some $U \in \mathcal{P}(Y) \cap (\tau' \setminus \tau)$, then clearly this relation is strict.

   Consider $X = \{1, 2, 3\}$, $\tau = \{\varnothing, \{1\}, X\}$, $\tau' = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, X\}$, and $Y = \{1\}$. Then despite τ being strictly coarser than τ', $\tau_Y = \tau'_Y$.

4. A map $f : X \to Y$ between topological spaces (X, τ_X) and (Y, τ_Y) is an **open map** when it carries open sets to open sets. Show that the projection maps from $X \times Y$ with the product topology $\tau_{X\times Y}$ are open maps.

   Solution: Let $U \in \tau_{X\times Y}$. By Lemma 13.1, let $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda} \subseteq \tau_X \times \tau_Y$ such that

$$U = \bigcup_{\lambda\in\Lambda} [A_\lambda \times B_\lambda]$$

   Then

$$\pi_1(U) = \pi_1\left(\bigcup_{\lambda\in\Lambda} [A_\lambda \times B_\lambda]\right) = \bigcup_{\lambda\in\Lambda} \pi_1(A_\lambda \times B_\lambda) = \bigcup_{\lambda\in\Lambda} A_\lambda \in \tau_X$$

   making π_1 an open map. Similarly, π_2 is as well. □