Topology – Munkres

Selected Exercise Solutions by Arnav Patri

Contents

Notation

This document uses the following conventions:

- $A \triangleq B$ denotes that B is the definition of A.
- $A \subset B$ denotes that A is a *proper* subset of B. Similarly, $A \supset B$ denotes that A is a proper superset of B.
- The power set of A is denoted $\mathcal{P}(A)$.
- $0 \in \mathbb{N} = \omega$.
- χ_A denotes the indicator function equal to 1 on A and zero on $A^\complement$.
- id denotes the identity function and may be subscripted with its domain.
- Let (X, τ) be a topological space, let $A \subseteq X$, and let $x \in X$.
  - The closure of A is denoted A or $\mathrm{Cl}(A)$.
  - The interior is denoted by A° or $\mathrm{Int}(A)$.
  - The set of limit points is denoted A' or $\mathrm{Lim}(A)$.
  - The boundary of A is denoted ∂A and is defined as $\overline{X} \setminus X^\circ$.
  - A neighbourhood of x is some $U \in \mathcal{P}(X)$ such that there is some $V \in \mathcal{P}(U) \cap \tau$ such that $x \in V$.
  - The neighbourhood system about x, the set of neighbourhoods of x, is denoted $\mathcal{N}(x)$,
  - The open neighbourhood system about x, the set of open neighbourhoods of x, is denoted $\mathcal{N}^\circ(x)$.
  - A punctured neighbourhood of x is some $U \setminus \{x\}$, where $U \in \mathcal{N}(x)$, and the punctured neighbourhood system is denoted $\mathcal{N}^*(x)$.
  - The system of punctured open neighbourhoods is denoted $\mathcal{N}^{\circledast}(x)$.
  - Cl, Int, Lim, ∂, and $\mathcal{N}$ maybe subscripted with either X or τ for clarity.
- Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda}$ be a collection of topological spaces. Their product with the product topology is denoted $\prod_{\lambda \in \Lambda} X_\lambda$, while that with the box topology is denoted $\square_{\lambda \in \Lambda} X_\lambda$.
- Let (X, d) be a metric space with topology τ, let $x \in X$, and let $\varepsilon > 0$.
  - The open ball centred at x with radius ε is

$$B(x, \varepsilon) \triangleq \{y \in X \mid d(x, y) < \varepsilon\}$$

  - Its closed counterpart is

$$\overline{B}(x, \varepsilon) \triangleq \{y \in X \mid d(x, y) \leq \varepsilon\}$$

  - The punctured open ball is given by

$$B^*(x, \varepsilon) \triangleq \{y \in X \mid d(x, y) \in (0, \varepsilon)\}$$

  - B may be subscripted by X, d, or τ for clarity.

Part I – General Topology

Chapter 2 – Topological Spaces and Continuous Functions

Section 13 – Basis for a Topology

1. Let (X, τ) be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is some open $U \in \tau \cap \mathcal{P}(A)$ with $x \in U$. Show that $A \in \tau$.

   Solution: For each $x \in A$, let $U_x \in \tau \cap \mathcal{P}(A)$ such that $x \in U_x$, and let $U = \bigcup_{x \in A} U_x$. Clearly, $U = A$, making the latter a union of open sets and thus itself open. $\square$

3. Show that

$$\tau_{cf} = \{U \subseteq X \,|\, |X \setminus U| < \aleph_0\} \cup \varnothing$$

   is a topology. Is

$$\tau_\infty = \{U \subseteq X \,|\, |X \setminus U| \geq \aleph_0\} \cup \{\varnothing, X\}$$

   a topology?

   Solution: If $|X| < \aleph_0$, then $\tau_{cf} = \mathcal{P}(X)$, yielding the discrete topology. Suppose instead that X is infinite, and let $\{U_\lambda\}_{\lambda \in \Lambda}, \{V_i\}_{i=1}^n \subseteq \tau_{cf}$. $\varnothing \in \tau_{cf}$ by construction, and

$$|X \setminus X| = |\varnothing| = 0 < \aleph_0$$

   so $X \in \tau_{cf}$ as well. Furthermore, for any $\alpha \in \Lambda$,

$$\left| X \setminus \bigcup_{\lambda \in \Lambda} U_\lambda \right| \leq |X \setminus U_\alpha| < \aleph_0$$

   making τ_{cf} closed under arbitrary unions. If $V_i = \varnothing$ for some i, $\bigcap_{i=1}^n V_i = \varnothing \in \tau_{cf}$, and otherwise, $|X \setminus V_i| < \aleph_0$, so

$$\left| X \setminus \bigcap_{i=1}^n V_i \right| = \left| \bigcup_{i=1}^n [X \setminus V_i] \right| \leq \sum_{i=1}^n |X \setminus V_i| < \aleph_0$$

   so τ_{cf} is also closed under finite intersections and is thus a valid topology. $\square$

   If $|X| < \aleph_0$, τ_∞ is the trivial topology. Otherwise, for each $x \in X$,

$$|X \setminus \{x\}| = |X| - |\{x\}| = \aleph_0 - 1 = \aleph_0$$

   so each singleton is in τ_∞. Despite this,

$$\left| X \setminus \bigcup_{y \in X \setminus \{x\}} \{y\} \right| = |X \setminus (X \setminus \{y\})| = |\{y\}| = 1$$

   meaning that τ_∞ is not closed under arbitrary unions and is thus not a topology.

4. (a) Let $\{\tau_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{P}(\mathcal{P}(X))$ be a family of topologies on X. Show that $\tau = \bigcap_{\lambda \in \Lambda} \tau_\lambda$ is also a topology. Is $\tau' = \bigcup_{\lambda \in \Lambda} \tau_\lambda$?

   Solution: Let $\{U_\gamma\}_{\gamma \in \Gamma}, \{V_i\}_{i=1}^n \subseteq \tau$. Then each U_γ and V_i is open in each τ_λ, meaning that the union of the former collection and the intersection of the latter are also in each τ_λ and thus in the intersection τ as well, making the latter a topology. $\square$

   Let $X = \{1, 2, 3\}$, let $\tau_1 = \{\varnothing, \{1\}, X\}$, and let $\tau_2 = \{\varnothing, \{2\}, X\}$. These are clearly both valid topologies, so consider $\tau' = \tau_1 \cup \tau_2$. This includes both $\{1\}$ and $\{2\}$ but not their union and is thus not a topology.

   (b) Let $\{\tau_\lambda\}_{\lambda \in \Lambda}$ be a family of topologies on X. Show that there are a unique smallest topology τ_Λ finer than and largest topology τ_0 coarser than every τ_λ.

Solution: Let τ_Λ be the topology generated by $\mathcal{B} = \bigcup_{\lambda\in\Lambda} \tau_\lambda$. By Lemma 13.1, this means that

$$\tau_\Lambda = \left\{ \bigcup B \,\middle|\, B \subseteq \bigcup_{\lambda\in\Lambda} \tau_\lambda \right\}$$

Clearly, each $\tau_\lambda \subseteq \tau_\Lambda$. Let τ' be a topology such that each $\tau_\lambda \subseteq \tau'$, and let $U \in \tau_\Lambda$. Let $B \subseteq \mathcal{B}$ such that $\bigcup B = U$. Each element of B is open in some τ_λ and is thus open in τ', so its union U is also in τ'; that is, $\tau_\Lambda \subseteq \tau'$.

Let $\tau_0 = \bigcap_{\lambda\in\Lambda} \tau_\lambda$. By 4.(a), this is a valid topology. Let τ'' be a topology contained in each τ_λ. Then for $U \in \tau''$, $U \in \tau_\lambda$ for each λ, so $\tau'' \subseteq \tau_0$. ☐

5. Show that if $\mathcal{A}$ is a basis or subbasis generating (X, τ), then τ is the intersection of all topologies on X containing $\mathcal{A}$.

   Solution: Let $\mathcal{A} \subseteq X$ be a basis generating topology $\tau_{\mathcal{A}}$, and let τ' be a topology on containing $\mathcal{A}$. By Lemma 13.1,

   $$\tau_{\mathcal{A}} = \left\{ \bigcup A \,\middle|\, A \subseteq \mathcal{A} \right\}$$

   As $\mathcal{A} \subseteq \tau'$, $\tau_{\mathcal{A}} \subseteq \tau'$ by closure under unions.

   Let $\mathcal{S} \subseteq X$ be a subbasis generating topology $\tau_{\mathcal{S}}$ and let τ'' be a topology containing $\mathcal{S}$. By closure under arbitrary finite intersections and countable unions, $\tau_{\mathcal{S}} \subseteq \tau''$. ☐

8. (a) Show from Lemma 13.2 that

   $$\mathcal{B} = \{(a, b) \mid a < b \in \mathbb{Q}\}$$

   is a basis that generates the standard topology τ on $\mathbb{R}$.

   Solution: Let $a < b \in \mathbb{R}$ and let $x \in (a, b)$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, let $c \in [a, x) \cap \mathbb{Q}$ and $d \in (x, b] \cap \mathbb{Q}$. Then $x \in (d, e) \in \mathcal{B}$. τ is generated by bounded open intervals, so for any $U \in \tau$ and $x \in U$, there is some open interval and thus some element of $\mathbb{Q}$ within U containing x, so by Lemma 13.2, $\mathcal{B}$ generates τ. ☐

   (b) Show that

   $$\mathcal{C} = \{[a, b) \mid a < b \in \mathbb{Q}\}$$

   does not generate the lower limit topology on $\mathbb{R}$.

   Solution: Let $a < b \in \mathbb{R} \setminus \mathbb{Q}$. Then for all $x \in [a, b) \setminus \mathbb{Q}$, $a < x$, meaning that there is no element of $\mathcal{C} \cap \mathcal{P}([a, b))$ containing a. By Lemma 13.3, the topology generated by $\mathcal{C}$ is not finer than and thus not equal to the lower limit topology. ☐

Section 16 – The Subspace Topology

1. Show that a subset of a subspace inherits the same topology from both the subspace and the parent space.

   Solution: Let (Y, τ_Y) be a subspace of (X, τ) and let $A \subset Y$ inherit topologies τ_A from Y and $\tau_{A'}$ from X. Then

$$\begin{aligned} \tau_A &= \{Z \cap U \mid U \in \tau_Y\} = \{Z \cap (U \cap V) \mid V \in \tau\} \\ &= \{Z \cap U \mid U \in \tau\} \end{aligned}$$

   by closure under finite unions. □

2. Let τ and τ' be topologies on X with the latter strictly finer than the former. How do their subspace topologies on $Y \subseteq X$ compare?

   Solution: Let τ_Y and τ'_Y respectively denote the subspace topologies induced by τ and τ'. Then

$$\tau_Y = \{Z \cap U(|)U \in \tau\} \subseteq \{Z \cap U(|)U \in \tau'\}$$

   the former is coarser than the latter. If there is some $U \in \mathcal{P}(Y) \cap (\tau' \setminus \tau)$, then clearly this relation is strict.

   Consider $X = \{1, 2, 3\}$, $\tau = \{\varnothing, \{1\}, X\}$, $\tau' = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, X\}$, and $Y = \{1\}$. Then despite τ being strictly coarser than τ', $\tau_Y = \tau'_Y$.

4. A map $f : X \to Y$ between topological spaces (X, τ_X) and (Y, τ_Y) is an **open map** when it carries open sets to open sets. Show that the projection maps from $X \times Y$ with the product topology $\tau_{X\times Y}$ are open maps.

   Solution: Let $U \in \tau_{X\times Y}$. By Lemma 13.1, let $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda} \subseteq \tau_X \times \tau_Y$ such that

$$U = \bigcup_{\lambda\in\Lambda} [A_\lambda \times B_\lambda]$$

   Then

$$\pi_1(U) = \pi_1\left(\bigcup_{\lambda\in\Lambda} [A_\lambda \times B_\lambda]\right) = \bigcup_{\lambda\in\Lambda} \pi_1(A_\lambda \times B_\lambda) = \bigcup_{\lambda\in\Lambda} A_\lambda \in \tau_X$$

   making π_1 an open map. Similarly, π_2 is as well. □

5. Let τ_X and τ'_X be topologies on X, let τ_Y and τ'_Y be topologies on Y, and let $\tau_{X\times Y}$ and $\tau'_{X\times Y}$ denote respectively the product topologies induced by the paris τ_X and τ_Y and τ'_X and τ'_Y.

   (a) Show that if τ'_X and τ'_Y are respectively finer than τ_X and τ_Y, $\tau_{X\times Y} \subseteq \tau'_{X\times Y}$

   Solution: Let $U \in \tau_{X\times Y}$. Then by Lemma 13.1, some $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda} \subseteq \tau_X \times \tau_Y$,

$$U = \bigcup_{\lambda\in\Lambda} [A_\lambda \times B_\lambda]$$

   U is a union of basis elements of $\tau'_{X\times Y}$ and thus one of its elements by closure under unions. As such, $\tau_{X\times Y} \subseteq \tau'_{X\times Y}$. □

   (b) Suppose conversely that $\tau_{X\times Y} \subseteq \tau'_{X\times Y}$, and determine whether it must be the case that $\tau_X \subseteq \tau'_X$ and $\tau_Y \subseteq \tau'_Y$.

   Solution: Let $U \in \tau_X$ and $V \in \tau_Y$. Then $U \times V \in \tau'_{X\times Y}$, so by Lemma 13.1, let $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda} \subseteq \tau'_X \times \tau'_Y$ such that

$$U \times V = \bigcup_{\lambda \in \Lambda} [A_\lambda \times B_\lambda]$$

Clearly, $U = \bigcup_{\lambda \in \Lambda} A_\lambda \in \tau'_X$ and $V = \bigcup_{\lambda \in \Lambda} B_\lambda \in \tau'_Y$, making them respectively finer than τ_X and τ_Y. □

6. Show that

$$\mathcal{B} = \{(a, b) \times (c, d) \mid a < b \in \mathbb{Q}, c < d \in \mathbb{Q}\}$$

is a basis for the standard topology τ_2 on $\mathbb{R}^2$.

Solution: Let $U \in \tau_2$ and let $x \in U$. By Lemma 13.1, let $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda} \subseteq \tau \times \tau$ such that

$$U = \bigcup_{\lambda \in \Lambda} [A_\lambda \times B_\lambda]$$

and let $\alpha \in \Lambda$ such that $a \in A_\alpha \times B_\alpha$. By 13.8.(a) ,

$$\mathcal{C} = \{(a, b) \mid a < b \in \mathbb{Q}\}$$

is a basis for the standard topology τ on $\mathbb{R}$, so by Lemma 13.1, let $\{C_n\}_{n \in N}, \{D_n\}_{n \in N} \subseteq \mathcal{C}$ such that

$$A_\alpha = \bigcup_{n \in \mathbb{N}} C_n \quad \text{and} \quad B_\alpha = \bigcup_{n \in \mathbb{N}} D_n$$

For some $n \in \mathbb{N}$, $x \in C_n \times D_n \in \mathcal{B} \cap \mathcal{P}(U)$, making $\mathcal{B}$ a basis for $(\mathbb{R}_2, \tau_2)$ by Lemma 13.2. □

7. Are all convex strict subsets of an ordered set necessarily either intervals or rays?

Solution: Consider

$$Y = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

with the usual order. This is cleary convex, but as it lacks a supremum in $\mathbb{Q}$, it is neither an interval nor a ray.

Section 17 – Closed Sets and Limit Points

1. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ with $\varnothing, X \in \mathcal{C}$ be closed under finite unions and arbitrary intersections. Show that
$$\tau = \{X \setminus C \mid C \in \mathcal{C}\}$$
is a topology.

   Solution: $X \setminus \varnothing = X$ while $X \setminus X = \varnothing$, so both are in τ. For $\{U_\lambda\}_{\lambda \in \Lambda}, \{V_i\}_{i=1}^n \subseteq \tau$,
$$\bigcup_{\lambda \in \Lambda} U_\lambda = X \setminus \bigcap_{\lambda \in \Lambda} [X \setminus U_\lambda] \in \tau$$
by closure under intersections and
$$\bigcap_{i=1}^n V_i = X \setminus \bigcup_{i=1}^n [X \setminus V_i] \in \tau$$
by closure under finite intersections, making τ a topology. □

2. Show that if A is closed in a subspace (Y, τ_Y) which is itself closed in (X, τ), then A is closed in X.

   Solution: By Theorem 17.2, $A = C \cap Y$ for some closed $C \subseteq X$, making A the intersection of two closed sets and hence itself closed. □

3. Show that if A is closed in (X, τ_X) and B is closed in (Y, τ_Y), then $A \times B$ is closed in the product space $(X \times Y, \tau_{X \times Y})$.

   Solution: Since A and B are closed, $X \setminus A$ and $Y \setminus B$ are both open, so their product is a basis element of $\tau_{X \times Y}$, making
$$A \times B = (X \times Y) \setminus ((X \setminus A) \times (X \setminus B))$$
closed. □

4. Show that if U and A are respectively open and closed in (X, τ), then $U \setminus A$ is open while $A \setminus U$ is closed.

   Solution: By closure under finite intersections,
$$U \setminus A = U \cap (X \setminus A) \in \tau$$
while by Theorem 17.1,
$$A \setminus U = A \cap (X \setminus U)$$
is closed. □

5. Show that in the order topology τ on $(X, <)$, $\overline{(a, b)} \subseteq [a, b]$ for $a < b \in X$, and determine when equality holds.

   Solution: For $x < a$, $(-\infty, a)$ is a neighbourhood that does not intersect (a, b), while for $y > b$, (b, ∞) suffices. As such, neither may be limit points of (a, b), so $\overline{(a, b)} \subseteq [a, b]$. Equality holds when a and b are themselves limit points of X. □

6. Let (X, τ) be a topological space and let $\{A, B\} \cup \{A_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{P}(X)$, and prove the following:
   (a) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$

   Solution: Let $x \in A'$. Then for all $U \in \mathcal{N}^*(x)$, $U \cap A \subseteq U \cap B$ is nonempty, so $x \in B'$. Since $A \subseteq B \subseteq \overline{A}$,

$$\overline{A} = A \cup A' \subseteq B \cup B' = \overline{B}$$

by Theorem 17.6. $\square$

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Solution: By Theorem 17.6, $\overline{A \cup B} = (A \cup B) \cup (A \cup B)'$, $\overline{A} = A \cup A'$, and $\overline{B} = B \cup B'$. If $x \in A \cup B$, then clearly $x \in \overline{A} \cup \overline{B}$. Let $x \in (A \cup B)'$. Then for all $U \in \mathcal{N}^*(x)$, $U \cap (A \cup B)$ is nonempty, so either $U \cap A$ or $U \cap B$ is nonempty, making x a limit point of either A or B and thus an element of $\overline{A} \cup \overline{B}$. By proxy, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

If $x \in A$ or $x \in B$, then $x \in A \cup B$ by definition. If $x \in A'$, then for each $U \in \mathcal{N}^*(x)$, $U \cap A \neq \varnothing$, so $U \cap (A \cup B) \neq \varnothing$; that is, $x \in (A \cup B)'$, so $\overline{A \cup B} = \overline{A} \cup \overline{B}$ $\square$

(c) $\overline{\bigcup_{\lambda \in \Lambda} A_\lambda} \supseteq \bigcup_{\lambda \in \Lambda} \overline{A_\lambda}$, and equality may not hold.

Solution: By Theorem 17.6, $\overline{A_\alpha} = A_\alpha \cup A'_\alpha$ for each $\alpha \in \Lambda$ and

$$\overline{\bigcup_{\lambda \in \Lambda} A_\lambda} = \bigcup_{\lambda \in \Lambda} A_\lambda \cup \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right)'$$

Let $\alpha \in \Lambda$. If $x \in A_\alpha$ then clearly $x \in \overline{\bigcup_{\lambda \in \Lambda} A_\lambda}$. If $x \in A'_\alpha$, then for each $U \in \mathcal{N}^*(x)$, $U \cap A_\alpha \neq \varnothing$, so $U \cap \bigcup_{\lambda \in \Lambda} A_\lambda$ is also nonempty; that is, $\bigcup_{\lambda \in \Lambda} \overline{A_\lambda} \subseteq \overline{\bigcup_{\lambda \in \Lambda} A_\lambda}$.

Let $B_n = [2^{-n}, 1]$ for $n \in \mathbb{N}$. Then in the standard topology on $\mathbb{R}$, each B_n is closed and thus contains all of its limit points, but

$$B = \bigcup_{n \in \mathbb{N}} B_n = (0, 1]$$

is not, having 0 as a limit point, meaning that $\overline{B} \supset \bigcup_{n \in \mathbb{N}} \overline{B_n}$. $\square$

9. Let A and B be subsets of (X, τ_X) and (Y, τ_Y) respectively. Show that in the product space $(X \times Y, \tau_{X \times Y})$, $\mathrm{Cl}_{X \times Y}(A \times B) = \mathrm{Cl}_X(A) \times \mathrm{Cl}_Y(B)$.

Solution: By Theorem 17.6,

$$\begin{aligned}
\mathrm{Cl}_{X \times Y}(A \times B) &= (A \times B) \cup \mathrm{Lim}_{X \times Y}(A \times B) \\
\mathrm{Cl}_X(A) &= A \cup \mathrm{Lim}_X(A) \\
\mathrm{Cl}_Y(B) &= B \cup \mathrm{Lim}_Y(B)
\end{aligned}$$

For $(x, y) \in \mathrm{Lim}_{X \times Y}(A \times B)$, $U \in \mathcal{N}^*_X(x)$, and $V \in \mathcal{N}^*_Y(x)$, $U \times V$ is a punctured neighbourhood of (x, y) and thus intersects $A \times B$, meaning that $A \cap U$ and $B \cap V$ must be nonempty, as if either were empty, the product $(A \cap U) \times (B \cap Y) = (A \times B) \cap (U \times Y)$ would be as well. This implies that $\mathrm{Cl}_{X \times Y}(A \times B) \subseteq \mathrm{Cl}_X(A) \times \mathrm{Cl}_Y(B)$.

If $x \in \mathrm{Lim}_X(A)$, $y \in \mathrm{Lim}_Y(B)$, and $U \in \mathcal{N}^*_{X \times Y}((x, y))$, then by Lemma 13.1, there are some $\{(C_\lambda, D_\lambda)\}_{\lambda \in \Lambda} \subseteq \tau_X \times \tau_Y$ such that

$$U = \bigcup_{\lambda \in \Lambda} [C_\lambda \times D_\lambda]$$

Each C_λ and D_λ is respectively a punctured neighbourhood of x and y and therefore respectively intersect A and B, meaning that their product intersects A and B, as does U by proxy. As such, $\mathrm{Cl}_{X \times Y}(A \times B) = \mathrm{Cl}_X(A) \times \mathrm{Cl}_Y(B)$. $\square$

11. Show that the product of two Hausdorff spaces is Hausdorff.

Solution: Let (X, τ_X) and (Y, τ_Y) be Hausdorff spaces, let $(X \times Y, \tau_{X\times Y})$ denote their product, and let $x \neq y \in X \times Y$. If $\pi_1(x) \neq \pi_1(y)$, let $U \in \mathcal{N}_X^\circ(\pi_1(x))$ and $V \in \mathcal{N}_X^\circ(\pi_1(y))$ to obtain disjoint $U \times Y \in \mathcal{N}_{X\times Y}^\circ(x)$ and $V \times Y \in \mathcal{N}_{X\times Y}^\circ(y)$; otherwise, let $U \in \mathcal{N}_X^\circ(\pi_2(x))$ and $V \in \mathcal{N}_Y^\circ(\pi_2(y))$ for disjoint $X \times U \in \mathcal{N}_{X\times Y}^\circ(x)$ and $X \times V \in \mathcal{N}_{X\times Y}^\circ(y)$. In either case, the existence of such neighbourhoods makes the product space Hausdorff. □

12. Show that a subspace of a Hausdorff space is Hausdorff.

Solution: Let (A, τ_A) be a subspace of a Hausdorff space (X, τ), and let $x \neq y \in A$. If $U \in \mathcal{N}_X^\circ(x)$ and $V \in \mathcal{N}_X^\circ(y)$ are disjoint, then so are $U \cap A \in \mathcal{N}_A^\circ(x)$ and $V \cap A \in \mathcal{N}_A^\circ(y)$, making A Hausdorff. □

13. Show that (X, τ) is Hausdorff if and only if the **diagonal**

$$\Delta \triangleq \{(x, x) \mid x \in X\}$$

is closed in the product space $(X \times X, \tau_2)$.

Solution: Suppose that (X, τ) is Hausdorff, and let $x \neq y \in X$. Let $U \in \mathcal{N}^\circ(x)$ and $V \in \mathcal{N}^\circ(y)$ be disjoint. Then since $U \times V \in \mathcal{N}_2^\circ((x, y))$ does not intersect Δ, since neither set contains a duplicate from the other, (x, y) is not a limit point of Δ, which therefore contains all of its is limit points and is thusly closed by Corollary 17.7.

Suppose conversely that Δ is closed in the product space, and let $x \neq y \in X$. Then by Corollary 17.7, it contains all of its limit points, so $(x, y) \notin \mathrm{Lim}_2(\Delta)$; that is, there is some $U \in \mathcal{N}_2^\circ((x, y))$ disjoint from Δ. By Lemma 13.1, let $\{(A_\lambda, B_\lambda)\}_{\lambda\in\Lambda} \in \tau \times \tau$ such that

$$U = \bigcup_{\lambda\in\Lambda} [A_\lambda \times B_\lambda]$$

Then

$$x \in \pi_1(U) = \pi_1\left(\bigcup_{\lambda\in\Lambda} [A_\lambda \times B_\lambda]\right) = \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathcal{N}^\circ(x)$$

and similarly, $\pi_2(U) \in \mathcal{N}^\circ(y)$, separating x and y and making X Hausdorff. □

15. Show that the T_1 axioms is equivalent to the condition that all distinct pairs of points have a neighbourhood not containing the other.

Solution: Suppose that (X, τ) is a T_1-space, and let $x \neq y \in X$. Then $X \setminus \{y\} \in \mathcal{N}^\circ(x)$ and $X \setminus \{x\} \in \mathcal{N}^\circ(y)$ are neighbourhoods not containing the other point.

Suppose conversely that for all $x \neq y \in X$, there are some $U \in \mathcal{N}^\circ(x)$ and $V \in \mathcal{N}^\circ(y)$ such that neither point is in the other's neighbourhood. Let $x \in X$ and for $y \in X \setminus \{x\}$, let $U_y \in \mathcal{N}^\circ(y)$ such that $x \notin U_y$. Then

$$\bigcup_{y\in X\setminus\{x\}} U_Y = X \setminus \{x\} \in \tau$$

so $\{x\}$ is closed; that is, singletons are closed, so X is T_1. □

Section 18 – Continuous Functions

1. Show that the topological and ε-δ definition of continuity on for $\mathbb{R}^{\mathbb{R}}$ are equivalent.

   Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous in the topological sense, and let $x \in \mathbb{R}$ and $\delta > 0$. Since $B(x, \varepsilon)$ is open, so is its preimage under f, so by Lemma 13.1, let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a collection of open intervals such that

   $$f^{-1}(B(x,\varepsilon)) = \bigcup_{\lambda\in\Lambda} I_\lambda$$

   Let $\alpha \in \Lambda$ such that $x \in I_\alpha$, and let

   $$\delta = \min\{x - \inf(I_\alpha), \sup(I_\alpha) - x\}$$

   Then $B(x,\delta) \subseteq I_\lambda \subseteq f^{-1}(B(x,\varepsilon))$, so $f(B(x,\delta)) \subseteq B(x,\varepsilon)$, making f continuous in the ε-δ sense.

   Let $g : \mathbb{R} \to \mathbb{R}$ be continuous in the ε-δ sense, and let $U \subseteq \mathbb{R}$ be open. By Lemma 13.1, let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a colletion of open sets such that

   $$U = \bigcup_{\lambda\in\Lambda} I_\lambda$$

   For each $x \in f^{-1}(U)$, there is some $\alpha_x \in \Lambda$ such that $f(x) \in I_{\alpha_x}$ and thus some $\varepsilon_x > 0$ such that $B(f(x), \varepsilon_x) \subseteq I_{\alpha_x}$ and some corresponding $\delta_x > 0$ such that $f(B(x,\delta_x)) \subseteq B(f(x),\varepsilon_x)$, so

   $$f^{-1}(U) = \bigcup_{x\in f^{-1}(U)} B(x,\delta_x)$$

   is open, making f topological continuous. $\square$

2. Let $f : X \to Y$ be a continuous function between spaces (X, τ_X) and (Y, τ_Y). If x is a limit point of $A \subseteq X$, is $f(x)$ a limit point of $f(A)$?

   Solution: Suppose that Y is T_1. Let $U \in \tau_X$, let $y \in Y$, and let

   $$\begin{aligned} f : X &\to Y \\ x &\mapsto y \end{aligned}$$

   This is trivially continuous, but $\{y\}$ is a singleton and thus closed, meaning that it contains all of its limit points by Corollary 17.7, but since no neighbourhood of y can intersect $\{y\}$ nontrivially, the set lacks limit points altogether.

3. Let X be a set in topologies τ and τ' and let $\iota : X \to X$ be the identity function from (X, τ') to (X, τ), where these are the respective subspace topologies.
   (a) Show that ι is continuous if and only if $\tau' \supseteq \tau$.

   Solution: Suppose that ι is continuous. Then for every $U \in \tau$, $\iota^{-1}(U) = U \in \tau'$, so $\tau' \supseteq \tau$.

   Suppose conversely that $\tau' \supseteq \tau$. Then for $U \in \tau$, $\iota^{-1}(U) = U \in \tau'_X$, making ι continuous. $\square$

   (b) Show that ι is a homeomorphism if and only if $\tau' = \tau$.

   Solution: Suppose that ι is a homeomorphism. Then $U \in \tau$ if and only if $\iota^{-1}(U) = U \in \tau'$, so $\tau = \tau'$.

   Suppose conversely that $\tau' = \tau$. Then $U \in \tau$ if and only if $U = \iota^{-1} \in \tau'$, so ι is a homeomorphism. $\square$

4.