# Optimization Techniques (MAT 5285)

**Module 1**: Formulation of Linear programming problem, solution of LPP by Simplex method, 2-phase method, Dual Simplex method, Duality theory. Transportation problem: VAM (Vogel's approximation method), MODI method. Assignment problem – Hungarian method.

**Module 2**: Project Management: Networks, project planning, project control using PERT and CPM. Project Crashing.

**Module 3**: Game theory: 2-Persons Zero-Sum game, Minimax principle games with Mixed strategies. Dominance theory, solution using LPP.

**Reference books:**


1. P.K. Gupta & Man Mohan, Operations Research, Sultan chand & Pub, 1980.
2. Hamdy A. Taha, operations Research, PHI, 1995.
3. J.K. Sharma, Operations Research theory and problems, Macmillan, 2007.
4. Bronson Richard, Theory and problems of operations Research, Schaums Series MGH, 1983.
5. Higher engineering mathematics, B.S. Grewal.


**Module 1**: LPP → Operations Research.

$\boxed{\text{Operations Research}}$ → World war II → to manage limited/scarce resources.

Decision making → to utilise the limited resources in an efficient manner so as to maximise the profit / to minimize the cost or to yield the maximum production etc. These types of problems are known as optimization problems.

Linear programming in particular, deals with optimization (Maximization/minimization) of linear functions subject to linear constraints. This method/techniques was propounded by George B. Dantzig in 1947 while working on a project for US Air force. He also developed the iterative process known as "Simplex method" for the solution of LPP in 1951

Application. LPP is widely used to tackle a number of industrial, economic, marketing & distribution problems etc. Product mix problems, blending problems and diet problems.

$\boxed{\text{LPP may be applied to any situation where a linear function of variables has to be optimized subject to a set of linear equations or inequalities}}$

In this module, our purpose is to present the principles of LPP and the techniques of its applications in a manner that will suit both engineers and scientists who are increasingly using this technique to solve their problems.

# Mathematical Model Formulation

Present → LPP form — define the variables, establishing the relationship between them and formulating the objective function and constraints:

The general LPP (model) with $n$ variables (decision variables) and $m$ constraints (limitations) can be stated in the form

Optimize: Max/Min $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq = \geq)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\ \text{"}\ )\ b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq = \geq)\ b_m$$

and $x_1, x_2, x_3, \ldots, x_n \geq 0$.

Max/Min $Z = \sum_{j=1}^{n} c_j x_j$ → objective function.

s.to $\sum_{j=1}^{n} a_{ij} x_j \ (\leq = \geq)\ b_i \quad i = 1, 2, 3, \ldots, m.$ → Constraints

and $x_j \geq 0.\ j = 1, 2, \ldots, n$ → non-negativity Constraints.

$c_j$: coefficients representing the per unit contribution of $x_j$

$a_{ij}$: input-output coefficients; $b_i$ – total availability of $i^{th}$ resources.

## Guidelines for LPP formulation

Step 1: Identify the decision variable $x_j$ $\quad x_1, x_2, \ldots x_n$

Step 2: Identify the problem data:

Step 3: Formulate the constraints

Step 4: Formulate the objective function

Q1) A manufacturer produces two types of models $M_1$ and $M_2$. Each $M_1$ model requires 4 hours of grinding and 2 hours of polishing, where as each $M_2$ model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works 40 hrs a week and polisher works 60 hours a week. Profit on $M_1$ model is Rs 3 and on an $M_2$ is Rs 4. what ever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week.

Soln: Let $x_1$ and $x_2$ be the number of $M_1$ models and $M_2$ models produced per week. Then the profit is $3x_1 + 4x_2$

objective function is $Z = 3x_1 + 4x_2$.

| | grinder | Polisher | Profit |
|---|---|---|---|
| $M_1$ | 4 | 2 | 3 |
| $M_2$ | 2 | 5 | 4 |

grinding hours: $4x_1 + 2x_2 \leq 80$

polishing hours: $2x_1 + 5x_2 \leq 180$

LPP model is

$$\boxed{\begin{aligned} &\text{Max } Z = 3x_1 + 4x_2 \\ &\text{Subject to } \quad 4x_1 + 2x_2 \le 80 \\ &\qquad\qquad\quad 2x_1 + 5x_2 \le 180 \\ &\qquad\qquad\quad x_1, x_2 \ge 0 \end{aligned}}$$

→ objective function

Constraints

Q2) Consider the following problem faced by a production planner in a soft drink plant. He has two bottling machines A and B. A is designed for 8-ounce bottles and B for 16-ounce bottle. However each can be used on both types with some loss of efficiency. The following is available

| Machine | 8-ounce | 16-ounce. |
|---|---|---|
| A | 100/minute | 40/minute |
| B | 60/minute | 75/minute |

The machines can be down 8-hours per day, 5 days/week. Profit in 8-ounce bottle is 15 paise and on a 16-ounce bottle is 25 paise. Weekly production of drink can't exceed 300000 ounces and the market can absorb 25000 8-ounce bottles and 7000 16-ounce bottles per week. Formulate this as a LPP to maximize his profit.

Soln: Let $x_1$: number of 8-ounce bottles produced/week

$x_2$: " 16-ounce "

Profit is $Z = 0.15x_1 + 0.25x_2$.

Since an 8-ounce bottle takes $\frac{1}{100}$ minute and 16-ounce bottle $\frac{1}{40}$ minutes on machine A and the machine can run 2400 min per week (8 hr/day, 5 days/week)

then we have $\frac{1}{100}x_1 + \frac{1}{40}x_2 \le 2400$.

Similarly for machine B: $\frac{1}{60}x_1 + \frac{1}{75}x_2 \le 2400$.

$8x_1 + 16x_2 \le 300000$.

$0 \le x_1 \le 25000$ and $0 \le x_2 \le 7000$

LPP model is

$$\boxed{\begin{aligned} &\text{Max } Z = 0.15x_1 + 0.25x_2 \\ &\text{Subject to } \quad 2x_1 + 5x_2 \le 480000, \\ &\qquad\qquad\quad 5x_1 + 4x_2 \le 720000 \\ &\qquad\qquad\quad x_1 + 2x_2 \le 37500, \quad 0 \le x_1 \le 25000, \quad 0 \le x_2 \le 7000. \end{aligned}}$$

(Q3)

**Example 2.12** A company produces two types of sauces: A and B. Both these sauces are made by blending two ingredients – X and Y. A certain level of flexibility is permitted in the formulae of these products. Indeed, the restrictions are that (i) B must contain no more than 75 per cent of X, and (ii) A must contain no less than 25 per cent of X, and no less than 50 per cent of Y. Up to 400 kg of X and 300 kg of Y could be purchased. The company can sell as much of these sauces as it produces at a price of Rs 18 for A and Rs 17 for B. The X and Y cost Rs 1.60 and 2.05 per kg, respectively.

The company wishes to maximize its net revenue from the sale of these sauces. Formulate this problem as an LP model.

Soln. Model formulation

Let $x_1, x_2$ = kg of Sauces A and B to be produced respectively

$y_1, y_2$ = kg of ingredient X used to make sauces A and B respectively

$y_3, y_4$ = kg of " Y " " " "

Maximize $Z = 18x_1 + 17x_2 - 1.6(y_1+y_2) - 2.05(y_3+y_4)$

Subject to (i) $y_1 + y_3 - x_1 = 0$

(ii) $y_2 + y_4 - x_2 = 0$

(iii) $y_1 + y_2 \le 400$, $y_3 + y_4 \le 300$ } Purchase

(iv) $y_2 - 0.75x_2 \geq 0$ (Sauce B)

(v) $y_1 - 0.25x_1 \geq 0$, $y_2 - 0.5x_2 \geq 0$ } (Sauce A)

$x_1, x_2, y_1, y_2, y_3, y_4 \geq 0$.

HW

1) **Ex. 7: A manufacturer of a line of patent medicines is preparing a production plan on medicines A and B. There are sufficient ingradients available to make 20,000 bottles of A and 40,000 bottles of B but there are onoy 45,000 bottles into which either of the medicines can be put. Further more it takes 3 hours to prepare enough materials to fill 1,000 bottles of A, it takes one hour to prepare enough materials to fill 1,000 bottles of B and there are 66 hours available for the operation. The profit is Rs. 8 per bottles for A and Rs. 7 for B. Formulate this problem as a linear programming problem.**

2) A transistor radio company manufacture models A, B and C which have profit contributions of 8, 15 and 25 respectively. The weekly minimum production requirements are 100 for model A, 150 for model B and 75 for model C.
Each type of radio requires a certain amount of time for the manufacturing of component parts, for assembling, and packing. Specially a dozen units of model A require three hours for manufacturing four hours for assembling, and one hour for packing. The corresponding figures for a dozen units of model B are 3.5, 5 and 1.5 and for a dozen unit of model C are 5, 8 and 3. During the forthcoming week the company has available 150 hours of manufacturing 200 hours of assembling and 60 hours of packing time.
Formulate the production scheduling problem as a linear programming model.

3) 5. A farmer, has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can raise. The price he can obtain is Rs. 5 per Kg of tomatoes, Rs. 2.00 a head lettuce and Rs. 8.00 per kg for radishes. The average yield per acre is 2000 kgs of tomatoes, 3000 heads of lettuce and 1000 kgs of radishes. Fertilizer is available at Rs. 5.00 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce and 50 kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man days for tomatoes and radishes and 6 man days for lettuce. A total of 400 man-days of labour are available at Rs. 100.00 per man day. Formulate the problem as a L. P. P. which maximises the total profit of the farmer.

4) 6. A firm produces an alloy having the following specifications (1) Specific granity $\leq 0.95$. (ii) chromium $\geq 9\%$ (iii) melting point $\geq 460^0$C. Raw materials A, B, C having the properties shown in the table can be used to make the alloy.

| Property | Raw Materials A | B | C |
|---|---|---|---|
| Specific gravity | .90 | .94 | 1.04 |
| Chromium | 8% | 14% | 17% |
| Melting Point | 460°C | 500°C | 490°C |

Cost of the various raw materials per unit tone are Rs. 100 for A, Rs. 300 for B and Rs. 50 for C. It is desired to find the units of raw materials which will minimise the total cost.

Solution of LPP

(i) Graphical method : LPP involving 2 variables

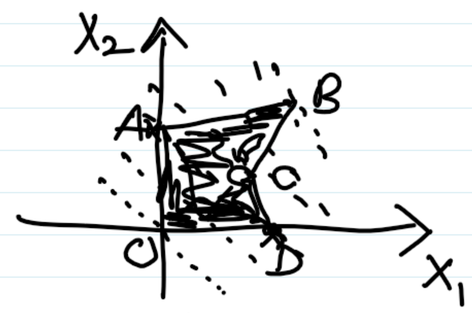
Working procedure

Step1 : Formulate the given pblm as a LPP

Step2 : plot the given Constraints as equalities on $x_1 - x_2$ plane and determine the Convex region :

fig 1 Convex | fig 2 Convex | fig 3 not Convex | fig 4. not Convex

Step3 : Determine the Vertices of the Convex region and find the value of Z at each vertex. The vertex which gives optimal (Max/Min) value of Z gives the required solution




OR

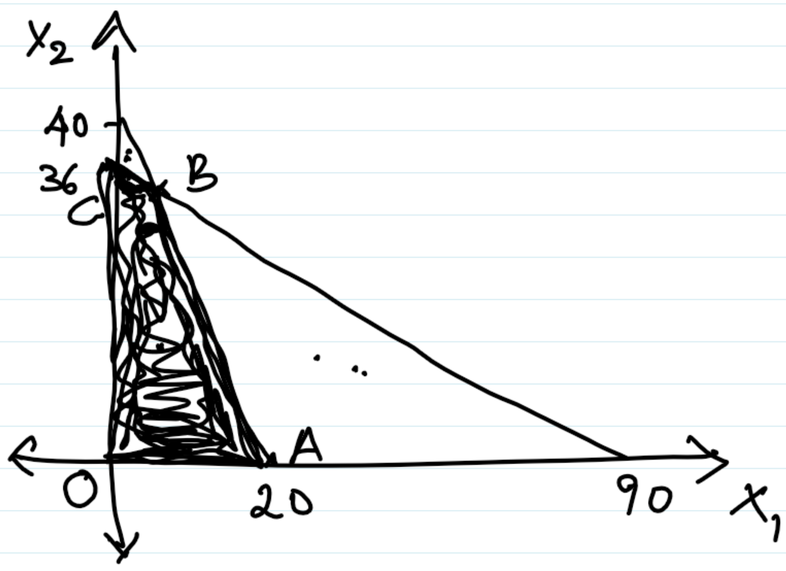
Example: Solve the LPP graphically :

$(Z=0)$

Max $Z = 3x_1 + 4x_2$

Subject to $4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$, $x_1, x_2 \geq 0$.




$3x_1 + 4x_2$

Solution : $4x_1 + 2x_2 = 80$

| $x_1$ | 20 | 0 |
|---|---|---|
| $x_2$ | 0 | 40 |

$2x_1 + 5x_2 = 180$

| $x_1$ | 0 | 90 |
|---|---|---|
| $x_2$ | 36 | 0 |

| | value of Z |
|---|---|
| O(0,0) | 0 |
| A(20,0) | 60 |
| B(2.5,35) | $3\times2.5 + 4\times35$ = Max value of Z |
| C(0,36) | $0 + 4\times36$ |

.. Solution is $x_1 = 2.5$, $x_2 = 35$ and Max $Z = 147.5$

Exceptional cases

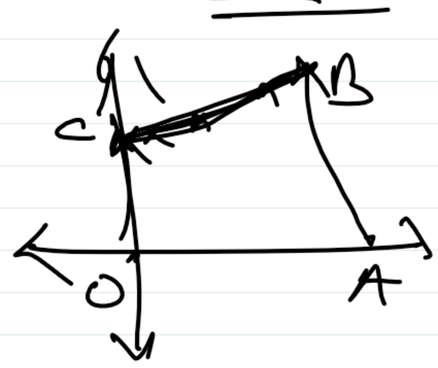
LPP may have

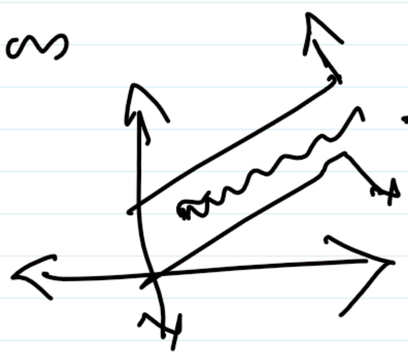
(i) unique solution (Previous eg.)

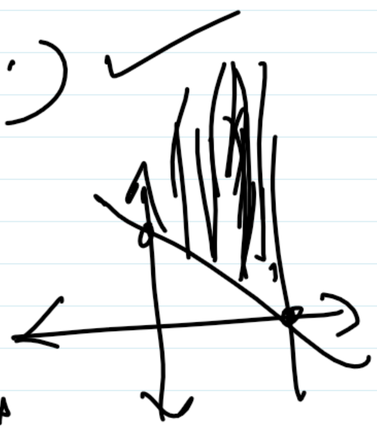
(ii) infinite solution

(iii) unbounded solutions

(iv) no solution





# General LPP

$$\text{Max}|\text{Min } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ \text{---------} \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{array}\right\} \text{Constraints}$$

$$x_1, x_2, \ldots, x_n \geq 0$$

**Defn:** A set of values $x_1, x_2, \ldots, x_n$ which satisfies the constraints of the LPP is called its <u>solution</u>

**Defn:** Any solution to a LPP which satisfies the non-negativity restriction ($x_i \geq 0$) is called a <u>feasible solution</u>

**Defn:** Any feasible solution which optimizes (Max/Min) the objective function is called its optimal solution.

**Defn:** <u>Slack variables</u>: The non-negative variable $s_i$, added to the "$\le$" type inequality to make it equality, is called <u>slack variable</u>

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b$$
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + s_1 = b$$

**Defn:** <u>Surplus variable</u>: The non-negative variable $s_i$, which is subtracted from the LHS of "$\geq$" type inequality to make it equality is called <u>surplus variable</u>.

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \geq b_1$$

then $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - s_2 = b_1$

## Canonical and standard forms of LPP

<u>Step 1</u>: Formulation

(1) <u>Canonical form</u>

$$\text{Max/Min } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$; $i = 1, 2, \ldots, m$

$$x_1, x_2, \ldots, x_n \geq 0.$$

<u>Characteristics</u>

(i) Objective function is of Max: type

(ii) All constraints are of "$\le$" type

(iii) All $x_i \geq 0$ (non-negative)

(2) <u>Standard form</u>

$$\text{Max/Min } Z = c_1x_1 + \cdots + c_nx_n$$

Subject to .. Constraints

<u>Characteristics</u> (i) Object functions is of Max type

(ii) All constraints are expressed as equations

(iii) All variables are non-negative ie $x_i \geq 0$

Note: 1) Min $Z = c_1x_1 + \cdots + c_nx_n \longrightarrow$ Max $Z' = (-Z) = -c_1x_1 - c_2x_2 - \cdots - c_nx_n$

Note 2): Max/Min $Z = c_i x_i$

s.to ... Constraints

$x_1, x_2, \ldots, x_{n-1} \geq 0$

$x_n$ "Unrestricted"

$x_n = x_n' - x_n''$ where $x_n', x_n'' \geq 0$

New Max/Min $Z = c_i x_i$

s.to Constraints

$x_1, x_2, \ldots, x_{n-1}, x_n', x_n'' \geq 0$

Q1) Convert the following LPP into standard form:

Min $Z = 3x_1 + 5x_2 + 4x_3$

subject to $2x_1 + 3x_2 \leq 8$; $2x_2 + 5x_3 \geq 10$; $3x_1 + 2x_2 + 4x_3 \leq 15$

$x_1, x_2, x_3 \geq 0$

Solution

Max $Z' = -3x_1 - 5x_2 - 4x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to $2x_1 + 3x_2 + S_1 + 0S_2 + 0S_3 = 8$

$2x_2 + 5x_2 + 0S_1 - S_2 + 0S_3 = 10$

$3x_1 + 2x_2 + 4x_3 + 0S_1 + 0S_2 + S_3 = 15$, $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

standard form.

Q2) Max $Z = 3x_1 + 5x_2 + 7x_3$

Subject to $6x_1 - 4x_2 \leq 5$; $3x_1 + 2x_2 + 5x_3 \geq 11$; $4x_1 + 3x_3 \leq 2$

$\boxed{x_1, x_2 \geq 0}$

Soln: As $x_3$ is unrestricted, let $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$.

Standard form is

Max $Z = 3x_1 + 5x_2 + 7x_3$

Subject to $6x_1 - 4x_2 + S_1 + 0S_2 + 0S_3 = 5$; $3x_1 + 2x_2 + 5(x_3' - x_3'') + 0S_1 - S_2 + 0S_3 = 11$

$4x_1 + 3(x_3' - x_3'') + 0S_1 + 0S_2 + S_3 = 2$; $x_1, x_2, x_3', x_3'', S_1, S_2, S_3 \geq 0$.

## Solution of LPP using Simplex Method (Simplex Algorithm)

Step 1: Find the basic solution

$2x_1 + x_2 + 4x_3 = 11$;

$3x_1 + x_2 + 5x_3 = 14$

$m = 2$; $m + n = 3$ $\boxed{3 - 2 = 1}$

No. of basic solution $= {}^3C_2 = 3$

| No. of basic soln | Basic variable | Non-basic variable | Basic Soln | feas/not | degenerate or not |
|---|---|---|---|---|---|
| 1 | $x_1, x_2$ | $x_3$ | $2x_1 + x_2 = 11$<br>$3x_1 + x_2 = 14$<br>$\Rightarrow x_1 = 3$; $x_2 = 5$ | Yes | No |
| 2 | $x_1, x_3$ | $x_2$ | $x_1 = 0.5$; $x_3 = 5/2$ | Yes | NO |
| 3 | $x_2, x_3$ | $x_1$ | $x_2 = 3$; $x_3 = -1$ | NO | No |

Assume that there are 'm' Constraints and m+n variables (decision + slack/surplus) $(m \leq n)$

Starting Solution: setting 'n' variables to zero and solving the remaining 'm' equations

The 'n' zero variables are called non-basic variables while the remaining 'm' variables are called basic variables and they form a basic solution

Note:

| Basis | $x_3$ | $x_4$ | - |
|---|---|---|---|
| $x_1$ | | | |
| $x_2$ | | | |

| Basis | $x_2$ | $x_4$ | - - - |
|---|---|---|---|
| $x_1$ | | | |
| $x_3$ | | | |

Optimal solution

| | |
|---|---|
| ⋮ | |

optim.

This process is repeated till no further improvement is possible.
The resulting solution $\Rightarrow$ optimal solution.

## Simplex Method

(i) Feasibility Conditions

(ii) Optimality Condition

**Example:** Max $Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$

Subject to $2x_1 + 3x_2 - x_3 + 4x_4 = 8$; $x_1 - 2x_2 + 6x_3 - 7x_4 = -3$

$x_1, x_2, x_3, x_4 \geq 0$.

**Solution** Given no. of Constraints $m = 2$; no. of Variables $n = 4$.

no. of non-basic variables $= 4 - 2 = 2$. No. of basic Solution $= {}^4C_2 = 6$

| No. | Basic Variable | Non-basic Variable | Basic solution | Is feasible? | Value of Z | Is optimal |
|---|---|---|---|---|---|---|
| 1 | $x_1, x_2$ | $x_3, x_4 = 0$ | $x_1 = 1$; $x_2 = 2$ | Yes | 8 | No. |
| 2 | $x_1, x_3$ | $x_2, x_4 = 0$ | $x_1 = -\frac{14}{13}$, $x_2 = -\frac{67}{13}$ | No | — | — |
| 3 | $x_1, x_4$ | $x_2, x_3 = 0$ | $x_1 = \frac{26}{9}$, $x_4 = \frac{7}{9}$ | Yes | 10.3 | NO |
| 4 | $x_2, x_3$ | $x_1, x_4 = 0$ | $x_2 = \frac{45}{16}$, $x_3 = \frac{7}{16}$ | Yes | 10.2 | NO |
| 5 | $x_2, x_4$ | $x_1, x_3 = 0$ | $x_2 = \frac{132}{39}$, $x_4 = \frac{-7}{13}$ | No | — | — |
| 6 | $x_3, x_4$ | $x_1, x_2 = 0$ | $x_3 = \frac{44}{17}$, $x_4 = 45/17$ | Yes | 28.9 ✓ | Yes |

Hence the optimal Solution is $x_1 = 0$; $x_2 = 0$; $x_3 = \frac{44}{17}$, $x_4 = \frac{45}{17}$

Max $Z = 28.9$

# Working Procedure of Simplex Method

**Step 1:** (i) Check whether $Z$ is to be Max or Min
If $Z$ is to be Min, then converting it into Max.

(ii) Check whether all $b_i$'s are positive.

**Step 2:** Express the problem in Standard form.

**Step 3:** Find an initial basic feasible solution.

| $C_j$ | | | $c_1$ | $c_2$ | $c_3$ | $\cdots$ | $c_n$ | $0$ | $0$ | $0$ | $0$ | $\cdots$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $b = x_B$ | $x_1$ | $x_2$ | $x_3$ | $\cdots$ | $x_n$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $\cdots$ | Min ratio |
| $c_{B_1}$ 0 | $S_1$ | $b_1$ | $a_{11}$ | $a_{12}$ | $a_{13}$ 0 | $\cdots$ | $a_{1n}$ | 1 | 0 | 0 | 0 | | $\theta = b/x_2$ $b_1/a_{12} = \frac{1}{2}$ |
| $c_{B_2}$ 0 | $S_2$ | $b_2$ | $a_{21}$ | $a_{22}$ | $a_{23}$ 0 | $\cdots$ | $a_{2n}$ | 0 | 1 | 0 | 0 | $\cdots$ | $b_2/a_{22} = 1/5$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $c_3$ | $x_3$ | | $a_{m1}$ | $a_{m2}$ | $a_{m3}$ | $\cdots$ | $a_{mn}$ | 0 | 0 | $\cdots$ | 1 | | $b_m/a_{m2} = \frac{1}{16}$ |

Body matrix — unit matrix.

**Step 4:** Apply optimality test

$C_j = c_j - z_j$

Compute $C_j = c_j - \sum c_B a_{ij}$ or $C_j = c_j - z_j$ where $z_j = \sum c_B a_{ij}$

$C_j$ is called net evaluation row.

If all $C_j$ are negative, then the initial basic feasible solution is optimal and terminate the procedure

If even one $C_j$ is +ve, then the current basic feasible solution is not optimal and proceed to next step.

**Step 5:** Identify the incoming and outgoing variable

**Step 6:** goto step 4 and repeat the computational procedure until an optimal solution (unbounded solution) is obtained.

Q1) Using Simplex method

$$\text{Max } Z = 5x_1 + 3x_2$$

s.t. $x_1 + x_2 \le 2$; $5x_1 + 2x_2 \le 10$; $3x_1 + 8x_2 \le 12$; $x_1, x_2 \ge 0$.

**Solution:** Step 1: satisfied

Step 2: Express the given pblm in standard form:

Let $S_1, S_2, S_3$ be the slack variables. Then

$$\left.\begin{aligned} \text{Max } Z &= 5x_1 + 3x_2 + 0S_1 + 0S_2 + 0S_3 \\ \text{Subject to } x_1 + x_2 + S_1 + 0S_2 + 0S_3 &= 2 \\ 5x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 &= 10 \\ 3x_1 + 8x_2 + 0S_1 + 0S_2 + S_3 &= 12 \\ x_1, x_2, S_1, S_2, S_3 &\ge 0 \end{aligned}\right\} \text{Standard form.}$$

Step 3: Find an initial basic feasible solution.

3 equations in 5 variables i.e. $n = 5$, $m = 3$, $\boxed{n - m = 2}$.

$x_1 = 0$, $x_2 = 0$. $\Rightarrow S_1 = 2$, $S_2 = 10$, $S_3 = 12$.

Basic feasible solution is $\boxed{x_1 = x_2 = 0}$ → non-basic and $\boxed{S_1 = 2,\ S_2 = 10,\ S_3 = 12}$ → basic

| $C_B$ | Basis $B$ | Solution $b (= x_B)$ | $c_j$: 5 $x_1$ | 3 $x_2$ | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | Min. ratio $\theta = b/x_i$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 2 | $\boxed{1}$ key element | 1 | 1 | 0 | 0 | $2/1 \leftarrow$ |
| 0 | $S_2$ | 10 | 5 → 0 | 2 | 0 | 1 | 0 | $10/5$ |
| 0 | $S_3$ | 12 | 3 → 0 | 8 | 0 | 0 | 1 | $12/3$ |
| | $Z_j = \Sigma C_B a_{ij}$ | | 0 | 0 | 0 | 0 | 0 | |
| | $C_j = c_j - Z_j$ | | 5 | 3 | 0 | 0 | 0 | |

| $C_B$ | Basis $B$ | Solution $b$ | $c_j$: 5 $x_1$ | 3 $x_2$ | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ |
|---|---|---|---|---|---|---|---|
| 5 | $x_1$ | 2 | 1 | 1 | 1 | 0 | 0 |
| 0 | $S_2$ | 0 | 0 | -3 | -5 | 1 | 0 |
| 0 | $S_3$ | 6 | 0 | 5 | -3 | 0 | 1 |
| | $Z_j = \Sigma C_B a_{ij}$ | | 5 | 5 | 5 | 0 | 0 |
| | $C_j = c_j - Z_j$ | | 0 | -2 | -5 | 0 | 0 |

$x_1, x_2$

$x_1 = 2$

$x_2 = 0$

Max $Z = 10 = \boxed{\Sigma C_B \cdot b}$

Since all $C_j$'s are $\leq 0$, the above table gives the optimal solution

This optimal solution is $x_1 = 2$; $x_2 = 0$; and Max $Z = 10$

Q2) Solve the following LPP by Simplex method.

Min $Z = x_1 - 3x_2 + 3x_3$

Subject to $3x_1 - x_2 + 2x_3 \leq 7$; $2x_1 + 4x_2 \geq -12$ (convert it into +ve)

$-4x_1 + 3x_2 + 8x_3 \leq 10$; $x_1, x_2, x_3 \geq 0$.

Soln: Here the objective function is of Min type. So convert it into Max type. Max $Z' = -x_1 + 3x_2 - 3x_3$ where $Z' = -Z$

As the R.H.S of second constraint is negative, we write it as

$-2x_1 - 4x_2 \leq 12$

Standard form: Max $Z' = -x_1 + 3x_2 - 3x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to $3x_1 - x_2 + 2x_3 + S_1 + 0S_2 + 0S_3 = 7$

$-2x_1 - 4x_2 + 0x_3 + 0S_1 + S_2 + 0S_3 = 12$

$-4x_1 + 3x_2 + 8x_3 + 0S_1 + 0S_2 + S_3 = 10$; $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$.

| $C_B$ | Basis $B$ | Solution $b$ | $c_j$: -1 $x_1$ | 3 $x_2$ | -3 $x_3$ | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | Min ratio $\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 7 | 3 | -1 | 2 | 1 | 0 | 0 | $7/-1$ |
| 0 | $S_2$ | 12 | -2 | -4 | 0 | 0 | 1 | 0 | $12/-4$ |
| 0 | $S_3$ | 10 | -4 | $\boxed{3}$ | 8 | 0 | 0 | 1 | $10/3 \leftarrow$ |
| | $Z_j$ | | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $C_j = c_j - Z_j$ | | -1 | 3 | -3 | 0 | 0 | 0 | |

| $C_B$ | Bases $B$ | Solution $b$ | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | $\theta$: Min ratio |
|---|---|---|---|---|---|---|---|---|---|
| | | $C_j$ | $-1$ | $3$ | $-3$ | $0$ | $0$ | $0$ | |
| 0 | $S_1$ | $31/3$ | $5/3$ | $0$ ✓ | $14/3$ | 1 | 0 | $1/3$ | $31/5$ ← |
| 0 | $S_2$ | $76/3$ | $-22/3$ | $0$ ✓ | $32/3$ | 0 | 1 | $4/3$ | $-38/11$ |
| 3 | $x_2$ | $10/3$ | $-4/3$ | $1$ ✓ | $8/3$ | 0 | 0 | $1/3$ | $-5/2$ |
| | $Z_j$ | | $-4$ | $3$ | $8$ | $0$ | $0$ | $1$ | |
| | $C_j = C_j - Z_j$ | | $3$ | $0$ | $-11$ | $0$ | $0$ | $-1$ | |

| $C_B$ | Bases $B$ | Solution $b$ | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|---|---|---|---|---|
| | | $C_j$ | $-1$ | $3$ | $-3$ | $0$ | $0$ | $0$ |
| $-1$ | $x_1$ | $31/5$ | 1 | 0 | $14/5$ | $3/5$ | 0 | $1/5$ |
| 0 | $S_2$ | $354/5$ | 0 | 0 | $156/5$ | $22/5$ | 1 | $14/5$ |
| 3 | $x_2$ | $58/5$ | 0 | 1 | $32/5$ | $4/5$ | 0 | $3/5$ |
| | $Z_j$ | | $-1$ | 3 | $82/5$ | $9/5$ | 0 | $8/5$ |
| | $C_j = C_j - Z_j$ | | 0 | 0 | $-97/5$ | $-9/5$ | 0 | $-8/5$ |

Now since all $C_j \leq 0$, it gives the optimal solution.

$x_1 = 31/5$, $x_2 = 58/5$, $x_3 = 0$ and $Z'_{max} = 143/5$.

Hence $Z_{min} = -\dfrac{143}{5}$

## Artificial Variable Techniques

" $\geq$ " " $=$ "

(i) M-method (method of Penalties)

(ii) Two-phase method.

### (i) M-method

**Step 1:** Express the LPP in standard form

**Step 2:** Add non-negative variables to LHS of all constraints which are of " $\geq$ " or " $=$ " type. These variable are called artificial variables.

**Purpose:** to obtain an initial basic feasible solution.

We assign a very large Penalty $(-M)$ to these artificial variable in the objective function.

**Step 3:** Solve the modified LPP by simplex method.

Then one of the following three cases may arise

**Case i:** There is no artificial variable in the basis and the optimality condition is satisfied. Then the solution is an optimal solution to the given problem

**Case ii:** There is atleast one artificial variable in the basis with zero value in b-column and the optimality condition is satisfied. Then the solution is a degenerate optimal solution

**Case iii:** There is atleast one artificial variable in the basis with positive value in b-column and the optimality condition is satisfied. Then the problem has no feasible solution

ie the final soln is not optimal. Such solution is called Pseudo optimal solution.

**Step 4:** Continue the simplex method until either an optimal solution or an unbounded solution is obtained.

**Note:** Artificial variables are only computational device for getting a starting solution. Once an artificial variable leaves the basis, the column for this variable is omitted from the next simplex table.

Q1) Use method of Penalty Cost (M-method) (charne's penalty method) to

$$\text{Min } Z = 2x_1 + x_2$$

Subject to $3x_1 + x_2 \boxed{=} 3$ ; $4x_1 + 3x_2 \boxed{\geq} 6$ ; $x_1 + 2x_2 \boxed{\leq} 3$ ; $x_1, x_2 \geq 0$.

Soln: Step 1: Standard form: We require $S_1, S_2, A_1, A_2$.

$$\text{Max } Z' = -2x_1 + -x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

Subject to

$$3x_1 + x_2 + 0S_1 + 0S_2 + A_1 + 0A_2 = 3$$
$$4x_1 + 3x_2 - S_1 + 0S_2 + 0A_1 + A_2 = 6$$
$$x_1 + 2x_2 + 0S_1 + S_2 + 0A_1 + 0A_2 = 3$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0.$$

| | | $c_j$ | $-2$ | $-1$ | $0$ | $0$ | $-M$ | $-M$ | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ | Ratio θ |
| $-M$ | $A_1$ | 3 | 3 | 1 | 0 | 0 | 1 | 0 | 3/3 |
| $-M$ | $A_2$ | 6 | 4 | 3 | $-1$ | 0 | 0 | 1 | 6/4 |
| 0 | $S_2$ | 3 | 1 | 2 | 0 | 1 | 0 | 0 | 3/1 |
| | $Z_j$ | | $-7M$ | $-4M$ | $M$ | 0 | $-M$ | $-M$ | |
| | $C_j = c_j - Z_j$ | | $7M-2$ | $4M-1$ | $-M$ | 0 | 0 | 0 | |

Initial Simplex table.

Since $C_j$ is +ve under $x_1$ and $x_2$ column, this is not an optimal solution.

New Simplex table is:

| | | $c_j$ | $-2$ | $-1$ | $0$ | $0$ | $-M$ | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $A_2$ | Ratio θ |
| $-2$ | $x_1$ | 1 | 1 | 1/3 | 0 | 0 | 0 | $1/1/3 = 3$ |
| $-M$ | $A_2$ | 2 | 0 | 5/3 | $-1$ | 0 | 1 | $2/5/3 = \frac{6}{5}$ |
| 0 | $S_2$ | 2 | 0 | 5/3 | 0 | 1 | 0 | $2/5/3 = 6/5$ |
| | $Z_j$ | | $-2$ | $-\frac{2}{3} - \frac{5M}{3}$ | $M$ | 0 | $-M$ | |
| | $C_j = c_j - Z_j$ | | 0 | $\frac{5M}{3} - \frac{1}{3}$ | $-M$ | 0 | 0 | |

Since $C_j$ is +ve under $x_2$-column, this is not an optimal solution

The revised Simplex table is:

| | | $c_j$ | $-2$ | $-1$ | $0$ | $0$ | |
|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $S_1$ | $S_2$ | |
| $-2$ | $x_1$ | 3/5 | 1 | 0 | 1/5 | 0 | Here all $C_j \leq 0$. This is an optimal solution and the optimal solution is $x_1 = 3/5$, $x_2 = 6/5$ |
| $-1$ | $x_2$ | 6/5 | 0 | 1 | $-3/5$ | 0 | |
| | $S_2$ | 0 | 0 | 0 | 1 | 1 | Max $Z' = -12/5$ |
| | $Z_j$ | $-12/5$ | $-2$ | $-1$ | 1/5 | 0 | |
| | $C_j = c_j - Z_j$ | | 0 | 0 | $-1/5$ | 0 | |

Here Min $Z = 12/5$

Q2) Max $Z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \le 2$ ; $3x_1 + 4x_2 \ge 12$ ; $x_1, x_2 \ge 0$

① ①①

Soln: Step 1: Standard form: We require $S_1, S_2, A$

Max $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA$.

Subject $2x_1 + x_2 + S_1 + 0S_2 + 0A = 2$.

$3x_1 + 4x_2 + 0S_1 - S_2 + A = 12$ ; $x_1, x_2, S_1, S_2, A \ge 0$.

| | $C_j$ | 3 | 2 | 0 | 0 | $-M$ | |
|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $S_1$ | $S_2$ | A | Ratio θ |
| 0 | $S_1$ | 2 | 2 | $\boxed{1}$ | 1 | 0 | 0 | 2/1 |
| $-M$ | A | 12 | 3 | 4 | 0 | $-1$ | 1 | 12/4 |
| | $Z_j$ | $-12M$ | $-3M$ | $-4M$ | 0 | M | $-M$ | |
| | $C_j = C_j - Z_j$ | | $3M+3$ | $4M+2$ | 0 | $-M$ | 0 | |

Initial Simplex table.

The new Simplex table is:

| | $C_j$ | 3 | 2 | 0 | 0 | $-M$ |
|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $S_1$ | $S_2$ | A |
| 2 | $x_2$ | 2 | 2 | 1 | 1 | 0 | 0 |
| $-M$ | A | $\boxed{4}$ | $-5$ | 0 | $-4$ | $-1$ | 1 |
| | $Z_j$ | $4-4M$ | $4+5M$ | 2 | $2+4M$ | M | $-M$ |
| | $C_j = C_j - Z_j$ | | $-1-5M$ | 0 | $-2-4M$ | $-M$ | 0 |

Here all $C_j \le 0$. So the optimality condition is satisfied. But an artificial variable appears in the basis at non-zero level (ie value = 4). Thus there exist a pseudo optimal solution to the problem.

Hw 1) Solve using M-method

Max $Z = 3x_1 + 2x_2 + 3x_3$

Subject to $2x_1 + x_2 + x_3 \le 2$ ; $3x_1 + 4x_2 + 2x_3 \ge 8$ ; $x_1, x_2, x_3 \ge 0$

2) Min $Z = 4x_1 + 3x_2 + x_3$

Subject to $x_1 + 2x_2 + 4x_3 \ge 12$ ; $3x_1 + 2x_2 + x_3 \ge 8$ ; $x_1, x_2, x_3 \ge 0$.

## II Two-phase method.

Here the LPP is solved in two phases.

### Phase-I

Step 1: Express the given problem in std form.

Step 2: Formulate an artificial objective function

$$Z^* = -A_1 - A_2 - \cdots - A_m$$ by assigning (-1) cost to each of the artificial variable

Step 3: Max. $Z^*$ subject to the constraints of the original problem using simplex method

Then three cases arise:

Case i: Max $Z^* < 0$ and atleast one artificial variable appears in optimal basis, at +ve level, then the original problem doesnot possess any feasible solution and procedure comes to an end.

Case ii: Max $Z^* = 0$ and no artificial variable appears in the optimal basis, then a basic feasible solution is obtained and we proceed to phase II for finding the optimal solution to the original problem.

Case iii: Max $Z^* = 0$ and atleast one artificial variable appears in the optimal basis, at zero level then a feasible solution to the auxillary LPP is also a feasible solution to the original problem with all artificial variable set = 0.

To obtain a basic feasible solution, we prolong phase I for pushing all the artificial variables out of the basis.

### Phase II

The basic feasible solution obtained in phase I is used as the starting solution for phase II. i.e The final simplex table of phase I is taken as the initial simplex table of phase II. The artificial objective function is replaced by the original objective function and we find the optimal solution to the given problem.

Q1) Use two-phase method to
Minimize $Z = 7.5x_1 - 3x_2$

Subject to $3x_1 - x_2 - x_3 \geq 3$; $x_1 - x_2 + x_3 \geq 2$; $x_1, x_2, x_3 \geq 0$.

Solution: Phase I : Step 1: Standard form:

$$\text{Max } Z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - A_1 - A_2$$

Subject to
$$3x_1 - x_2 - x_3 - S_1 + 0S_2 + A_1 + 0A_2 = 3$$
$$x_1 - x_2 + x_3 + 0S_1 - S_2 + 0A_1 + A_2 = 2$$
$$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0.$$

Initial Simplex table:

| | $c_j$ | 0 | 0 | 0 | 0 | 0 | -1 | -1 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ | $\theta$ |
| -1 | $A_1$ | 3 | $\boxed{3}$ | -1 | -1 | -1 | 0 | 1 | 0 | $\frac{3}{3}$ |
| -1 | $A_2$ | 2 | 1 | -1 | 1 | 0 | -1 | 0 | 1 | $\frac{2}{1}$ |
| | $Z_j^*$ | -5 | -4 | 2 | 0 | 1 | 1 | -1 | -1 | |
| | $C_j = c_j - Z_j^*$ | | 4 | -2 | 0 | -1 | -1 | 0 | 0 | |

| | $c_j$ | 0 | 0 | 0 | 0 | 0 | -1 | -1 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ | $\theta$ |
| 0 | $x_1$ | 1 | 1 | $-1/3$ | $-1/3$ | $-1/3$ | 0 | $1/3$ | 0 | -3 |
| -1 | $A_2$ | 1 | 0 | $-2/3$ | $4/3$ | $1/3$ | -1 | $-1/3$ | 1 | $\frac{3}{4}$ |
| | $Z_j^*$ | -1 | 0 | $2/3$ | $-4/3$ | $-1/3$ | 1 | $1/3$ | -1 | |
| | $C_j = c_j - Z_j^*$ | | 0 | $-2/3$ | $4/3$ | $1/3$ | -1 | $-4/3$ | 0 | |
| | | | | | (4) | (+) | | | | |

-ve ignore

| | $c_j$ | 0 | 0 | 0 | 0 | 0 | -1 | -1 |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ |
| 0 | $x_1$ | $\frac{5}{4}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| 0 | $x_3$ | $3/4$ | 0 | $-\frac{1}{2}$ | 1 | $\frac{1}{4}$ | $-\frac{3}{4}$ | $-\frac{1}{4}$ | $\frac{3}{4}$ |
| | $Z_j^*$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $C_j = c_j - Z_j^*$ | | 0 | 0 | 0 | 0 | 0 | -1 | -1 |

Since all $C_j \leq 0$, this table gives the optimal solution.
Also $Z^*_{Max} = 0$ and no artificial variables appears in the basis. Thus an optimal basic feasible solution to the auxillary LPP has been attained. Next go to phase II.

**Phase II** : Consider Max $Z' = -7.5x_1 + 3x_2 + 0x_3 + 0S_1 + 0S_2 + 0A_1 + 0A_2$

Subject to $3x_1 - x_2 - x_3 - S_1 + 0S_2 + A_1 + 0A_2 = 3$

$x_1 - x_2 + x_3 + 0S_1 - S_2 + 0A_1 + A_2 = 2$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$.

Now Consider the final Simplex table of phase I as the initial table for phase II.

| | $C_j$ | $-7.5$ | $3$ | $0$ | $0$ | $0$ |
|---|---|---|---|---|---|---|
| $C_B$ | Basis | $b$ | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ |
| $-7.5$ | $x_1$ | $5/4$ | $1$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{4}$ | $-1/4$ |
| $0$ | $x_3$ | $3/4$ | $0$ | $-\frac{1}{2}$ | $1$ | $\frac{1}{4}$ | $-3/4$ |
| | $Z_j$ | $-75/8$ | $-7.5$ | $15/4$ | $0$ | $15/8$ | $15/8$ |
| | $C_j = C_j - Z_j$ | | $0$ | $-3/4$ | $0$ | $-15/8$ | $-15/8$ |

Since all $C_j \leq 0$, this Solution is optimal.

Hence an optimal Solution is given by

$$x_1 = \frac{5}{4}\ ;\ x_2 = 0\ ;\ x_3 = \frac{3}{4},\quad \text{Max } Z' = -\frac{75}{8} \text{ i.e. Min } Z = \frac{75}{8}.$$

**Hw**

Q1) Min $Z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 4$ ; $x_1 + 7x_2 \geq 7$ ; $x_1, x_2 \geq 0$.

Q2) Max $Z = 0.5x_1 + 6x_2 + 5x_3$

Subject to $4x_1 + 6x_2 + 3x_3 \leq 24$ ; $x_1 + 1.5x_2 + 3x_3 \leq 12$ ;

$3x_1 + x_2 \leq 12$, $x_1, x_2, x_3 \geq 0$.

Exceptional Cases

(i) Tie for the incoming variable.

(ii) Tie for the outgoing variable

(iii) Degeneracy : Cycling :

| $C_j$ : | $x_1$ | $x_2$ | $x_3$ | $\theta$ |
|---|---|---|---|---|
| Basis | | | | |
| $A_1$ | — | — | — | $\frac{1}{3}$ |
| $A_2$ ← | — | — | — | $\frac{1}{3}$ ← |
| $C_j$ | 7 ↑ | 7 ↑ | $-3$ | 2 |

To avoid cycling :→ Perturbation procedure.

(i) Divide each element in the tied rows by the positive coefficients of the key column in that row.

(ii) Compare the resulting ratios first of unit matrix and then of the body matrix, column by column

(iii) The out going variable lies in that row which first contains the smallest algebraic ratio

Example: Max $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \le 2$; $5x_1 + 2x_2 \le 10$; $3x_1 + 8x_2 \le 12$; $x_1, x_2 \ge 0$.

---

## DUALITY CONCEPT

| Two LPP's → Duals | | n | m |
|---|---|---|---|
| | Primal | 2 | 5 |
| | Dual. | 5 | 2 |

### Importance of duality

(i) If the primal contains a large no. of constraints and a smaller of variables, the labour of computations can be reduced by converting it into the dual pblm. and then solve it.

(ii) The interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions in the activities being programmed.

## FORMULATON OF DUAL PROBLEM

Consider Max $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \; ; \; x_1, x_2, \ldots, x_n \ge 0. \end{aligned}\right\} \text{Primal}$$

To construct the dual, we adopt the following steps:

(i) The Max. problem in primal becomes the Min problem in the dual and vice versa.

(ii) "$\le$" type constraints in the primal become "$\ge$" type in the dual. and vice versa.

(iii) The coefficient $c_1, c_2, \ldots, c_n$ in the objective function of the primal become $b_1, b_2, \ldots, b_m$ in the objective function of dual.

(iv) The constants $b_1, b_2, \ldots, b_m$ in constraints of the primal become

$c_1, c_2, \ldots, c_n$ in the constraints of the dual.

(v) If the primal has $n$ variables and $m$ constraints, then the dual will have $m$ variables and $n$ constraints.

(vi) The variables in both primal and dual are non-negative.

Then the dual problem will be

$$\begin{array}{|cccc|}\hline a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ \hline \end{array}$$

$n$ - variable, $m$ - constraints } Primal

$m$ - variables, $n$ - constraints } dual

$$\text{Min } W = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

Subject to $a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m \geq c_1$

$a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m \geq c_2$

$\cdots\cdots\cdots\cdots$

$a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m \geq c_n$ ; $y_1, y_2, \ldots, y_m \geq 0$.

Q1) Write the dual of the following L.P.P:

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Subject to $3x_1 + 5x_2 + 4x_3 \geq 7$, $6x_1 + x_2 + 3x_3 \geq 4$.

Convert $\to \geq$ $\leftarrow$ $7x_1 - 2x_2 - x_3 \leq 10$, $x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$; $x_1, x_2, x_3 \geq 0$.

Primal: 3 - variables, 5 - constraints.

Soln: Since the problem is of Min, all constraints should be "$\geq$".
Here the third constraint is "$\leq$" type, so convert it into "$\geq$" type.
Multiply both sides by $(-1) \Rightarrow -7x_1 + 2x_2 + x_3 \geq -10$.

Let $y_1, y_2, y_3, y_4$ and $y_5$ be the dual variables associated with the above five constraints. Then the dual is given by

$$\text{Max } W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$$

Subject to $3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$

$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$

$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$; $y_1, y_2, y_3, y_4, y_5 \geq 0$

## Formulation of dual problem when the primal has equality constraints

Consider $\boxed{\text{Max}}$ $Z = c_1 x_1 + c_2 x_2$

Subject to $\boxed{a_{11} x_1 + a_{12} x_2 = b_1}$; $a_{21} x_1 + a_{22} x_2 \leq b_2$; $x_1, x_2 \geq 0$.

Method: Re write $a_{11} x_1 + a_{12} x_2 = b_1$ as

$a_{11} x_1 + a_{12} x_2 \leq b_1$ & $a_{11} x_1 + a_{12} x_2 \geq b_1$

ie $a_{11} x_1 + a_{12} x_2 \leq b_1$ & $-a_{11} x_1 - a_{12} x_2 \leq -b_1$

Let $y_1'$, $y_1''$ and $y_2$ be the dual variables.

Then the dual problem is

$$\text{Min } W = b_1\boxed{(y_1' - y_1'')} + b_2 y_2$$

Subject to $a_{11}(y_1' - y_1'') + a_{21} y_2 \geq c_1$

$a_{12}(y_1' - y_1'') + a_{22} y_2 \geq c_2$; $y_1', y_1'', y_2 \geq 0$.

This can also be written as

$$\text{Min } W = b_1 y_1 + b_2 y_2$$

Subject to $a_{11} y_1 + a_{21} y_2 \geq c_1$

$a_{12} y_1 + a_{22} y_2 \geq c_2$; $y_1$ unrestricted in sign, $y_2 \geq 0$.

Example: Construct the dual of the LPP:

$$\text{Max } Z = 4x_1 + 9x_2 + 2x_3$$

Subject to $\underbrace{\boxed{2x_1 + 3x_2 + 2x_3 \leq 7;}}_{y_1}$ $\underbrace{\boxed{3x_1 - 2x_2 + 4x_3 = 5;}}_{y_2}$ $x_1, x_2, x_3 \geq 0$.

Soln :→

$$\text{Min } W = 7y_1 + 5y_2$$

Subject to $2y_1 + 3y_2 \geq 4$;

$3y_1 - 2y_2 \geq 9$;

$2y_1 + 4y_2 \geq 2$;

$y_1 \geq 0$; $y_2$ - unrestricted in sign

$y_2 = y_2' - y_2''$

$y_2', y_2'' \geq 0$

Hw: Write the dual of the following:

1) Max $Z = 10x_1 + 13x_2 + 19x_3$

Subject to $6x_1 + 5x_2 + 3x_3 \leq 26$; $4x_1 + 2x_2 + 5x_3 \leq 7$; $x_1, x_2, x_3 \geq 0$

2) Max $Z = 3x_1 + x_2 + 2x_3$

Subject to $x_1 + x_2 + x_3 \geq 6$; $3x_1 + x_2 + 7x_3 = 3$; $x_1 + 4x_2 - 6x_3 = 4$; $x_1, x_2, x_3 \geq 0$

## DUALITY PRINCIPLE

Statement: $\underbrace{\text{Max } Z}_{\text{Primal}} = \underbrace{\text{Min } W}_{\text{dual}}$ } Fundamental theorem of duality.

## Working rule for optimal Solution

**Rule 1**: If the primal variables corresponds to a slack starting variable in the dual problem, then its optimal value is directly given by the coefficients of the slack variables with changed sign, in the $C_j$ row of the optimal dual simplex table and viceversa.

**Rule 2**: If the primal variables corresponds to an artificial starting variable the dual problem, then its optimal value is directly given by the coefficient of artificial variable in the $C_j$ row of the optimal dual simplex table after ignoring M and vice versa.

Q) Construct the dual of the following problem and solve the primal and the dual:

$$\text{Max } Z = 2x_1 + x_2,$$

Subject to $-x_1 + 2x_2 \le 2;\ x_1 + x_2 \le 4;\ x_1 \le 3;\ x_1, x_2 \ge 0.$

Soln: find the solution of the primal using simplex method (do it as HW: Soln: $x_1 = 3;\ x_2 = 1,\ \text{Max } Z = 7$)

Solution using dual problem

The dual of the given problem is

$$\text{Min } W = 2y_1 + 4y_2 + 3y_3$$

Subject to $-y_1 + y_2 + y_3 \ge 2;\ 2y_1 + y_2 + 0y_3 \ge 1;\ y_1, y_2, y_3 \ge 0.$

Step i: Standard form:

$$\text{Max } W' = -2y_1 - 4y_2 - 3y_3 + 0S_1 + 0S_2 - MA_1 - MA_2$$

Subject to $-y_1 + y_2 + y_3 - S_1 + 0S_2 + A_1 + 0A_2 = 2;$

$2y_1 + y_2 + 0y_3 + 0S_1 - S_2 + 0A_1 + A_2 = 1$

| | | $C_j$ | $-2$ | $-4$ | $-3$ | $0$ | $0$ | $-M$ | $-M$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b-value | $y_1$ | $y_2$ | $y_3$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ | $\theta$ |
| $-M$ | $A_1$ | 2 | $-1$ | 1 → zero | 1 | $-1$ | 0 | 1 | 0 | $2/1$ |
| $-M$ | $A_2$ ↑ $y_2$ | 1 | 2 | $\boxed{1}$ | 0 | 0 | $-1$ | 0 | 1 | $1/1$ ← |
| | $Z_j$ | $-3M$ | $-M$ | $-2M$ | $-M$ | $M$ | $M$ | $-M$ | $-M$ | |
| | $C_j$ | | $M-2$ | $2M-4$ | $M-3$ | $-M$ | $-M$ | 0 | 0 | |

As $C_j$ is positive under some columns, solution is not optimal.

Introduce $y_2$ and drop $A_2$, then the new Simplex table is

| | | $c_j$ | $-2$ | $-4$ | $-3$ | $0$ | $0$ | $-M$ | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $b$ | $y_1$ | $y_2$ | $y_3$ | $S_1$ | $S_2$ | $A_1$ | $\theta$-ratio |
| $-M$ | $A_1$ | $1$ | $-3$ | $0$ | $1$ | $-1$ | $1$ | $1$ | $1/1$ |
| $-4$ | $y_2$ | $1$ | $2$ | $1$ | $0$ | $0$ | $-1$ | $0$ | $1/0$ |
| | $Z_j$ | $-M-4$ | $3M-8$ | $-4$ | $-M$ | $M$ | $4-M$ | $-M$ | |
| | $C_j$ | | $6-3M$ | $0$ | $M-3$ | $-M$ | $M-4$ | $0$ | |

As some $c_j$'s are positive, this solution is not optimal.

Now drop $A_1$ and introduce $y_3$, we get the new table as follows:

| | | $c_j$ | $-2$ | $-4$ | $-3$ | $0$ | $0$ | $-M$ | $-M$ |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $b$ | $y_1$ | $y_2$ | $y_3$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ |
| $-3$ | $y_3$ | $1$ | $-3$ | $0$ | $1$ | $-1$ | $1$ | $1$ | $-1$ |
| $-4$ | $y_2$ | $1$ | $2$ | $1$ | $0$ | $0$ | $-1$ | $0$ | $1$ |
| | $Z_j$ | $-7$ | $1$ | $-4$ | $-3$ | $3$ | $1$ | $-3$ | $1$ |
| | $C_j$ | | $-3$ | $0$ | $0$ | $-3$ | $-1$ | $3-M$ | $1-M$ |

Here all $C_j \le 0$, the optimal solution is attained.

Thus the optimal solution to the dual problem is

$y_1 = 0,\ y_2 = 1,\ y_3 = 1,\ \text{Max } W' = -7 \Rightarrow \boxed{\text{Min } W = 7}$

The optimal solution to the primal problem is

$x_1 = 3,\ x_2 = 1,\ \boxed{\text{Max } Z = 7}$

Q2) Using duality theory solve the LPP:

$$\text{Min } Z = 0.7x_1 + 0.5x_2$$

Subject to $x_1 \ge 4;\ x_2 \ge 6;\ x_1 + 2x_2 \ge 20;\ 2x_1 + x_2 \ge 18;\ x_1, x_2 \ge 0.$

Soln: First write the dual.

$$\text{Max } W = 4y_1 + 6y_2 + 20y_3 + 18y_4$$

Subject to $y_1 + y_3 + 2y_4 \le 0.7;\ y_2 + 2y_3 + y_4 \le 0.5,\ y_1, y_2, y_3, y_4 \ge 0$

Std form: $\text{Max } W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0S_1 + 0S_2$

Subject to $y_1 + 0y_2 + y_3 + 2y_4 + S_1 + 0S_2 = 0.7$

$0y_1 + y_2 + 2y_3 + y_4 + 0S_1 + S_2 = 0.5,\ y_1, y_2, y_3, y_4, S_1, S_2 \ge 0.$

Initial Simplex table:

| | | $c_j$ | $4$ | $6$ | $20$ | $18$ | $0$ | $0$ | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $b$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $S_1$ | $S_2$ | $\theta$-ratio |
| $0$ | $S_1$ | $0.7$ | $1$ | $0$ | $1$ | $2$ | $1$ | $0$ | $0.7/1$ |
| $0$ | $S_2$ | $0.5$ | $0$ | $1$ | $2$ | $1$ | $0$ | $1$ | $0.5/2$ |
| | $Z_j$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | |
| | $C_j$ | | $4$ | $6$ | $20$ | $18$ | $0$ | $0$ | |

Introduce $y_3$ and drop $S_2$.

| | $C_j$ | 4 | 6 | 20 | 18 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $b$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $S_1$ | $S_2$ | Q-Ratio |
| 0 | $S_1$ | $9/20$ | 1 | $-1/2$ | 0 | $3/2$ | 1 | $-1/2$ | $9/20 / 3/2$ |
| 20 | $y_3$ | $1/4$ | 0 | $1/2$ | 1 | $1/2$ | 0 | $1/2$ | $1/4 / 1/2$ |
| | $Z_j$ | 5 | 0 | 10 | 20 | 10 | 0 | 10 | |
| | $C_j$ | | 4 | $-4$ | 0 | 8 | 0 | $-10$ | |

As $C_j$ is positive under some columns, this solution is not optimal

Now introduce $y_4$ and drop $S_1$, we get the revised table as:

| | $C_j$ | 4 | 6 | 20 | 18 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $b$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $S_1$ | $S_2$ |
| 18 | $y_4$ | $\frac{3}{10}$ | $2/3$ | $-1/3$ | 0 | 1 | $2/3$ | $-1/3$ |
| 20 | $y_3$ | $1/10$ | $-1/3$ | $2/3$ | 1 | 0 | $-1/3$ | $2/3$ |
| | $Z_j$ | $74/10$ | $16/3$ | $22/3$ | 20 | 18 | $16/3$ | $22/3$ |
| | $C_j$ | | $-4/3$ | $-4/3$ | 0 | 0 | $-16/3$ | $-22/3$ |

Since all $C_j \le 0$, this table gives the optimal solution.
Thus the optimal solution to the dual problem is

$$y_1 = 0, \quad y_2 = 0, \quad y_3 = \frac{1}{10}, \quad y_4 = \frac{3}{10}, \quad \text{Max } W = 7.4$$

Hence the optimal solution to the primal is
(Rule 1) $x_1 = 16/3$, $x_2 = 22/3$, Min $Z = 7.4$

HW Q1) Using duality solve the following:

Min $Z = 2x_1 + 9x_2 + x_3$

Subject to $x_1 + 4x_2 + 2x_3 \geq 5$; $3x_1 + x_2 + 2x_3 \geq 4$; $x_1, x_2, x_3 \geq 0$

Q2) Max $Z = -2x_1 - 2x_2 - 4x_3$

Subject to $2x_1 + 3x_2 + 5x_3 \geq 2$; $3x_1 + x_2 + 7x_3 \geq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$; $x_1, x_2, x_3 \geq 0$ } indirect type

(first write the dual and then solve)

# DUAL SIMPLEX METHOD

We starts with basic infeasible solution (but optimal) and works towards feasibility. The only difference lies in the criteria used for selecting incoming and outgoing variables. ie Here we first determine the the outgoing variable and then the incoming variable

**Working Procedure:**

**Step 1:** (i) Convert the LPP to Max: form

(ii) Convert "$\geq$" to "$\leq$" (Constraints)

(iii) Express the problem in std form.

**Step 2:** Find the initial basic solution and form the dual simplex table.

**Step 3:** Test the nature of $C_j = C_j - Z_j$:

(a) If all $C_j \leq 0$ and all $b_i \geq 0$, then the optimal solution has been attained

(b) If all $C_j \leq 0$ and atleast one $b_i < 0$, then go to Step 4

(c) If any $C_j \geq 0$, then the method fails

**Step 4:** Mark the outgoing variable: Select the row that contains the most negative $b_i$. This will be the key row and the corresponding basic variable is the outgoing variable

**Step 5:** Test the nature of key row elements:

(a) If all these elements are $\geq 0$, the problem does not have a feasible solution

(b) If atleast one element $< 0$, find the ratio of the corresponding element of $C_j$-row to these elements choose the smallest of these ratios. The corresponding column is the key column and the associated variable is the incoming variable.

**Step 6:** Iterate towards optimal solution: Make the key element unity. Perform row operations as in the regular simplex method and repeat the iterations until an optimal solution is attained.

Q1) Using dual simplex method:

$$\text{Max } Z = -3x_1 - 2x_2$$

Subject to $x_1 + x_2 \geq 1$; $x_1 + x_2 \leq 7$; $x_1 + 2x_2 \geq 10$; $x_2 \leq 3$; $x_1, x_2 \geq 0$.

**Soln:** Convert the first and third constraints into "$\leq$" type.

$$-x_1 - x_2 \leq -1; \quad -x_1 - 2x_2 \leq -10.$$

Std form: Max $Z = -3x_1 - 2x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$

Subject to $-x_1 - x_2 + S_1 = -1$;

$x_1 + x_2 + S_2 = 7$

$-x_1 - 2x_2 + S_3 = -10$; $x_2 + S_4 = 3$; $x_1, x_2, S_1, S_2, S_3 \geq 0$

Step 2: Dual simplex table:

| | | $C_j$ | $-3$ | $-2$ | $0$ | $0$ | $0$ | $0$ |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b-value | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
| 0 | $S_1$ | $-1$ | $-1$ | $-1$ | 1 | 0 | 0 | 0 |
| 0 | $S_2$ | 7 | 1 | 1 | 0 | 1 | 0 | 0 |
| 0 | $S_3$ | $-10$ | $-1$ | $-2$ | 0 | 0 | 1 | 0 |
| 0 | $S_4$ | 3 | 0 | 1 | 0 | 0 | 0 | 1 |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $C_j$ | | $-3$ | $-2$ | 0 | 0 | 0 | 0 |
| | | | $(-3/-1)$ | $(-2/-2)$ | | | | |

Since all $C_j \leq 0$ and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but not feasible.

Outgoing variable: Outgoing variable is $S_3$. Now to find incoming variable: calculate the ratio of elts in $C_j$ to the corresponding -ve elts of the key row. i.e. $\frac{-3}{-1}$ and $\frac{-2}{-2}$

Since the smaller ratio is $\frac{-2}{-2}$, the incoming variable is $x_2$.

Now drop $S_3$ and introduce $x_2$, then the revised table is

| | | $C_j$ | $-3$ | $-2$ | $0$ | $0$ | $0$ | $0$ |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
| 0 | $S_1$ | 4 | $-\frac{1}{2}$ | 0 | 1 | 0 | $-\frac{1}{2}$ | 0 |
| 0 | $S_2$ | 2 | $\frac{1}{2}$ | 0 | 0 | 1 | $\frac{1}{2}$ | 0 |
| $-2$ | $x_2$ | 5 | $\frac{1}{2}$ | 1 | 0 | 0 | $-\frac{1}{2}$ | 0 |
| 0 | $S_4$ | $-2$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | 1 |
| | $Z_j$ | $-10$ | $-1$ | $-2$ | 0 | 0 | 1 | 0 |
| | $C_j$ | | $-2$ | 0 | 0 | 0 | $-1$ | 0 |

Now introduce $x_1$ and drop $S_4$, then the next table is

| | | $C_j$ | $-3$ | $-2$ | $0$ | $0$ | $0$ | $0$ |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | b | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
| 0 | $S_1$ | 6 | 0 | 0 | 1 | 0 | $-1$ | $-1$ |
| 0 | $S_2$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| $-2$ | $x_2$ | 3 | 0 | 1 | 0 | 0 | 0 | 0 |
| $-3$ | $x_1$ | 4 | 1 | 0 | 0 | 0 | $-10$ | $-2$ |
| | $Z_j$ | $-18$ | $-3$ | $-2$ | 0 | 0 | 3 | 4 |
| | $C_j$ | | 0 | 0 | 0 | 0 | $-3$ | $-4$ |

Since all $C_j \leq 0$ and all $b_i$'s are $\geq 0$, therefore this solution is optimal and feasible. Thus the optimal solution is

$x_1 = 4$; $x_2 = 3$; $Z_{max} = -18$.

Hw 1) Using dual Simplex method, Solve:

$$\text{Min } Z = 2x_1 + 2x_2 + 4x_3$$

Subject to $2x_1 + 3x_2 + 5x_3 \geq 2$; $3x_1 + x_2 + 7x_3 \leq 3$;

$x_1 + 4x_2 + 6x_3 \leq 5$; $x_1, x_2, x_3 \geq 0$.

Soln: $x_1 = 0$; $x_2 = 2/3$, $x_3 = 0$; Max $Z' = -\frac{4}{3}$, Min $Z = \frac{4}{3}$.

## Transportation Problem

It is a Special case of LPP:

Objective: to transport a Single Commodity from various origins to different destinations at a Minimum cost.

Formulation of a transportation problem

m: (origins)
n: (destination)

Let there are m plant locations (origin) and n distribution Centre (destinations).

$a_i$ : production capacity at $i^{th}$ plant.

$b_j$ : no. of units required at $j^{th}$ destination centre.

$c_{ij}$ : transportation cost of one unit from $i^{th} \to j^{th}$.

So our objective to determine the no. of units to be transported from $i^{th} \to j^{th}$ with minimum transportation cost:

Mathematical representation: Let $x_{ij}$ : no. of units shipped from $i^{th} \to j^{th}$.

Then

$$\text{Min } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$x_{i1} + x_{i2} + \cdots + x_{in} = a_i$ ; $i = 1, 2, \ldots, m$ (for $i^{th}$ origin)

$x_{1j} + x_{2j} + \cdots + x_{mj} = b_j$ ; $j = 1, 2, \ldots, n$ (for $j^{th}$ destination).

$x_{ij} \geq 0$.

Note 1) Condition for a feasible solution $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Transportation Problem Satisfying this condition is called a balanced transportation problem.

2) A feasible Solution to a transportation problem is said to be a basic feasible solution if it contains at the most $(m+n-1)$ strictly positive allocation, otherwise the solution will degenerate. If the total no. of positive (non-zero) allocation is exactly $m+n-1$, then the basic feasible solution is said to non-degenerate.

3) A feasible solution which minimize the transportation cost is called optimal solution.

| Origins (Plant) | Destination (distribution centre) 1 | 2 | 3 | … | j | … | n | Supply |
|---|---|---|---|---|---|---|---|---|
| 1 | $c_{11}$ | $c_{12}$ | $c_{13}$ | | $c_{1j}$ | | $c_{1n}$ | $a_1$ |
| 2 | | | | | | | | $a_2$ |
| ⋮ | | | | | | | | ⋮ |
| i | $c_{i1}$ | $c_{i2}$ | $c_{i3}$ | | $c_{ij}$ | | $c_{in}$ | $a_i$ |
| ⋮ | | | | | | | | ⋮ |
| m | $c_{m1}$ | $c_{m2}$ | $c_{m3}$ | | | | $c_{mn}$ | $a_m$ |
| Demand | $b_1$ | $b_2$ | $b_3$ | … | $b_j$ | … | $b_n$ | $\Sigma a_i = \Sigma b_j$ |

Transportation table

Working procedure for transportation problem

Step 1: Construct the transportation table. Check balanced/not.

Qn: Solve the following transportation problem:

| | A | B | C | D | Availability |
|---|---|---|---|---|---|
| Source I | 21 | 16 | 25 | 13 | 11 |
| Source II | 17 | 18 | 14 | 23 | 13 |
| Source III | 32 | 27 | 18 | 41 | 19 |
| Requirement | 6 | 10 | 12 | 15 | 43 |

balanced.

Step 2: Finding initial basic feasible soln: By VAM (Vogel's approxn: method)

(i)

| | | | | | |
|---|---|---|---|---|---|
| 21 | 16 | 25 | 11 / 13 | 11 | (3) |
| 17 | 18 | 14 | 23 | 13 | (3) |
| 32 | 27 | 18 | 41 | 19 | (9) |
| 6 | 10 | 12 | 15 | | |
| (4) | (2) | (4) | (10) ✓ | | |

Display the difference b/w the least and next to least cost in each row/column

(ii) Identifying the row/column with largest difference and allocate as much as possible under the rim requirements to the lowest cost in that row/column.

| | | | | | |
|---|---|---|---|---|---|
| 17 | 18 | 14 | 4 | ~~13~~ 9 | (3) |
| 32 | 27 | 18 | 41 | 19 | (9) |
| 6 | 10 | 12 | 4 | | |
| (15) | (9) | (4) | (18) | | |

| | | | | | |
|---|---|---|---|---|---|
| 6 / 17 | 18 | 14 | 9 | (3) | |
| 32 | 27 | 18 | 19 | (9) | |
| 6 (15) | 10 (9) | 12 (4) | | | |

| | | | | | |
|---|---|---|---|---|---|
| 3 / 18 | 14 | 3 | (4) | | |
| 27 | 18 | 19 | (9) | | |
| 10 (9) | 12 (4) | | | | |

| | | | |
|---|---|---|---|
| 7 / 27 | 12 / 18 | 19 | |
| 7 | 12 | | |

Basic solution is

| | | | |
|---|---|---|---|
| 21 | 16 | 25 | 13 |
| 6 | 3 | 4 | |
| 17 | 18 | 14 | 23 |
| | 7 | 12 | |
| 32 | 27 | 18 | 41 |

$m = 3$

$n = 4$

$m+n-1 = 3+4-1$
$= 6$
$=$ no. of allocation

Initial basic feasible solution (VAM).

## Step 3: Apply optimality check. (MODI method)

(i) Note $u_i$ - along left and $v_j$ along the top of the cost matrix such that $\boxed{u_i + v_j = c_{ij}}$ to the occupied cells. Initially assume some $u_i = 0$.

$u_1 + v_4 = 13$; $u_2 + v_1 = 17$; $u_2 + v_2 = 18$;
$u_2 + v_4 = 23$; $u_3 + v_2 = 27$; $u_3 + v_3 = 18$

Let $u_2 = 0$, then $v_1 = 17$, $v_2 = 18$, ...

| $u_i$ \ $v_j$ | $v_1 = 17$ | $v_2 = 18$ | $v_3 = 9$ | $v_4 = 23$ |
|---|---|---|---|---|
| $u_1 = -10$ | (-) 21 | (-) 16 | θ (-) 25 | 11 13 |
| $u_2 = 0$ | 6 17 | 3 18 | (-) 14 | 4 23 |
| $u_3 = 9$ | (-) 32 | 7 27 | 12 18 | (-) 41 |

(ii) Compute net evaluations $\boxed{w_{ij} = u_i + v_j - c_{ij}}$ for all empty cells and enter them in upper right-hand corner of that cell.

If all $w_{ij} \le 0$, then the current basic feasible soln is optimal.

So for this example, it represents an optimal solution and the optimal solution is given by

$x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$; $x_{32} = 7$, $x_{33} = 12$.

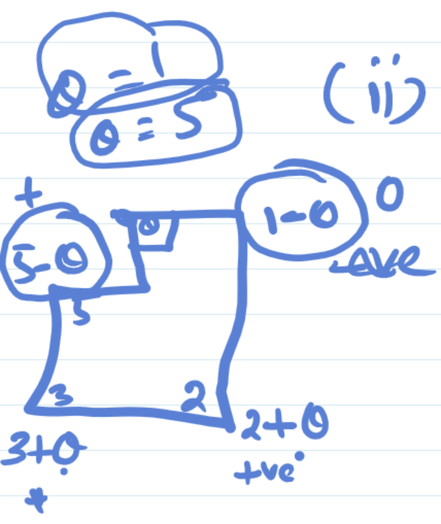
The minimum transportation cost is $11 \times 13 + 6 \times 17 + 3 \times 18 + 4 \times 23 + 7 \times 27 + 12 \times 18$

$= ₹\ 796$

If any $w_{ij} > 0$, Then this solution is not optimal and we proceed further.

## Step 4: Iterate towards optimal solution

(i) choose the unoccupied cell with largest $w_{ij}$ and mark 'θ' in it




(ii) Draw a closed path consisting of horizontal and vertical lines begin and ending at θ-cell and having its other corners at the allocated cell. Add and subtract θ alternately to and from the corners of the closed path. Assign a maximum value to θ so that one allocation (basic variable) becomes zero and other allocation remain non-negative.

Now return to Step 3 and repeat the process until an optimal soln is obtained.

**Example 2:** Solve the following transportation problem:

| Factories \ Project site | 1 | 2 | 3 | 4 | Supply |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 11 | 7 | 6 |
| 2 | 1 | 0 | 6 | 1 | 1 |
| 3 | 5 | 8 | 15 | 9 | 10 |
| Demand | 7 | 5 | 3 | 2 | 17 |

balanced.

**Step 1:** Initial basic feasible solution by VAM.

Compute $w_{ij} = u_i + v_j - c_{ij}$ for unoccupied cell

| $u_i \backslash v_j$ | | | | |
|---|---|---|---|---|
| | [1] 2 | [5] 3 | (+1) 11 | (−) 7 |
| | (−) 1 | (−) 0 | [θ] (+1) 6 | [1] −θ = 0 1 |
| | [6] 5 | (−) 8 | [3] −θ 15 | [1] +θ 9 |

This Solution is not optimal as two $w_{ij}$ values are $> 0$. Choose the unoccupied cell with maximum $w_{ij}$. In the case of tie, select the cell with lower cost. Take $\theta = 1$.

$u_1 + v_1 = 2$

$u_1 + v_2 = 3$

$u_2 + v_3 = 6$

$u_3 + v_1 = 5$

$u_3 + v_3 = 15$

$u_3 + v_4 = 9$. Take $u_3 = 0$ and solve

| $u_i \backslash v_j$ | $v_1 = 5$ | $v_2 = 6$ | $v_3 = 15$ | $v_4 = 9$ |
|---|---|---|---|---|
| $u_1$ | [1] −θ 2 | [5] 3 | [θ] (+1) 11 | (−) 7 |
| $u_1 = -3$ | (−) 1 | (−) 0 | [1] 6 | (−) 1 |
| $u_2 = -9$ | [6] +θ 5 | (−) 8 | [2] −θ 15 | [2] 9 |
| $u_3 = 0$ | | | | |

New table → Again calculate $w_{ij}$ for unoccupied cell

| $u_i \backslash v_j$ | | | | |
|---|---|---|---|---|
| | (−) 2 | [5] 3 | [1] 11 | (−) 7 |
| | (−) 1 | 0 | [1] 6 | (−) 1 |
| | [7] 5 | (−) 8 | [1] 15 | [2] 9 |

Since all $w_{ij} \le 0$ for the empty cell, this Solution is optimal.

The optimal Solution is $x_{12} = 5$, $x_{13} = 1$, $x_{23} = 1$, $x_{31} = 7$,

$x_{33} = 1$, $x_{34} = 2$.

The minimum transportation cost is $5 \times 3 + 1 \times 11 + 1 \times 6 + 7 \times 5 + 1 \times 15 + 2 \times 9$

$= ₹\ \underline{\underline{100}}$

<u>Note</u>: MODI is applicable if no. of allocation $= m+n-1$.

## Degeneracy in Transportation Problem:

5 $\quad m+n-1 = 7$

When no. of allocation is $< m+n-1$, the basic Solution degenerates. To remove degeneracy, we assign a small positive value $\epsilon$ to the unoccupied cells. Now solve the resulting problem using MODI method. At the final step (optimal solution), we let $\epsilon \to 0$.

<u>Example</u>:

| $u_i \backslash v_j$ | 4 | 3 | 7 | 0 | 0 | 5 |
|---|---|---|---|---|---|---|
| 2 | (−) 9 | (−) 12 | [5] 9 | (−) 6 | (−) 9 | (−) 10 |
| 0 | (−) 7 | [4] 3 | [ε] 7 | (−) 7 | (−) 5 | [2] 5 |
| 2 | [1] 6 | (−) 5 | [1] 9 | (−) 11 | (−) 3 | (−) 11 |
| 2 | [3] 6 | (−) 8 | (−) 11 | [2] 2 | [4] 2 | (−) 10 |

Here total allocation $= 8$

$m+n-1 = 4+6-1$

$= 9$

$\boxed{8 < 9}$

Since all $w_{ij} \le 0$ for empty cells, this solution is optimal.

The Minimum transportation cost is $5 \times 9 + 4 \times 3 + 7 \times \epsilon + 2 \times 5 + 1 \times 6 + 1 \times 9 + 3 \times 6 + 2 \times 2 + 4 \times 2 = 112 + 7\epsilon = 112$ as $\epsilon \to 0$.

HW 1)

| Origin \ Destination | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Availability |
|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | 1 | 4 | 30 |
| $O_2$ | 3 | 3 | 2 | 1 | 50 |
| $O_3$ | 4 | 2 | 5 | 9 | 20 |
| Requirement | 20 | 40 | 30 | 10 | |

2)

| Consumers / Suppliers | A | B | C | Available |
|---|---|---|---|---|
| I | 6 | 8 | 4 | 14 |
| II | 4 | 9 | 8 | 12 |
| III | 1 | 2 | 6 | 5 |
| Required | 6 | 10 | 15 | |

## The unbalanced transportation problems

$\sum a_i \neq \sum b_j$

(i) If $\sum a_i > \sum b_j$ ✓
add a dummy destination.

(ii) If $\sum a_i < \sum b_j$
add a dummy origin

| | | | | |
|---|---|---|---|---|
| | | | | 2 |
| | | | | 10 |
| | | | | 20 |
| 5 | 20 | 3 | 2 | |

$\sum a_i = 32$
$\sum b_j = 30$

$\sum a_i = \sum b_j$ (balanced transportation)

| | | | | dummy |
|---|---|---|---|---|
| | | | | 0 |
| | | | | 0 |
| | | | | 0 |
| | | | | 2 |

$\boxed{\sum a_i = \sum b_j}$

| | | | | |
|---|---|---|---|---|
| dummy | 0 | 0 | 0 | 0 |

Q1)

| | A | B | C | |
|---|---|---|---|---|
| $O_1$ | 4 | 8 | 8 | 76 |
| $O_2$ | 16 | 24 | 16 | 82 |
| $O_3$ | 8 | 16 | 24 | 77 |
| | 72 | 102 | 41 | |

$\sum a_i = 235$
$\sum b_j = 215$ } unbalanced

Soln: The given problem is unbalanced ($\sum a_i = 235$, $\sum b_j = 215$)
So add a dummy column with cost set to 0 (zero)

| | A | B | C | D(dummy) | |
|---|---|---|---|---|---|
| $O_1$ | 4 | 8 | 8 | 0 | 76 |
| $O_2$ | 16 | 24 | 16 | 0 | 82 |
| $O_3$ | 8 | 16 | 24 | 0 | 77 |
| | 72 | 102 | 41 | 20 | |

Solve using VAM →

| | $V_1$ | $V_2$ | $V_3$ | $V_4$ |
|---|---|---|---|---|
| $U_1$ | 4 | 35 | 41 8 | 0 |
| $U_2$ | 16 | 62 24 | 16 | 20 0 |
| $U_3$ | 72 8 | 5 16 | 24 | 0 |

No. of allocation $= 6 = m+n-1$.

Now apply optimality test using MODI method.

$U_1+V_2 = 35$; $U_1+V_3 = 41$; $U_2+V_2 = 62$; $U_2+V_4 = 20$; $U_3+V_1 = 72$; $U_3+V_2 = 5$.

Set $U_1 = 0$ and solve:

| | | | | |
|---|---|---|---|---|
| 4 | 76 8 | 8 | 0 |
| 21 16 | 41 24 | 16 | 20 0 |
| 72 8 | 5 16 | 24 | 0 |

The final table is

Total transportation cost
$= 8\times76 + 24\times21 + 16\times41 + 0\times20 + 8\times72 + 16\times5 = \text{Rs. } 2424$

## II. Maximization transportation problems

This may be converted to Minimization problems by subtracting all the cost values from the highest cost value in the table and then can be solved as usual.

Q1) Solve the following transportation problem for maximizing the profit.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | |
|---|---|---|---|---|---|---|
| $F_1$ | 15 | 17 | 12 | 11 | 11 | 140 |
| $F_2$ | 5 | 9 | 7 | 15 | 7 | 190 |
| $F_3$ | 14 | 15 | 16 | 20 | 10 | 115 |
| | 74 | 94 | 69 | 89 | 119 | |

Here $\Sigma a_i = \Sigma b_j$ So balanced

Soln: Here the highest value cost is 20.

| | | | | | |
|---|---|---|---|---|---|
| 5 | 3 | 8 | 9 | 9 | 140 |
| 15 | 11 | 13 | 5 | 13 | 190 |
| 6 | 5 | 4 | 0 | 10 | 115 |
| 74 | 94 | 69 | 89 | 119 | 445 |

balanced.

Next apply VAM to find the initial basic feasible solution and then apply MODI method. (Complete it).

## ASSIGNMENT PROBLEM (Special type of transportation problem)

Objective: To assign a number of origins to an equal no. of destination at a Minimum cost (Max. profit).

Formulation: There are $n$ new machines $M_i$ $(i=1,2,\dots,n)$ which are to be installed in a Machine Shop. There are $n$ vacant spaces $S_j$ $(j=1,2,\dots,n)$ available. The cost of installation is $c_{ij}$

Let $x_{ij}$ be the assignment of machine $M_i$ to space $S_j$

ie $x_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ machine is installed at } j^{th} \text{ place} \\ 0, & \text{otherwise} \end{cases}$

$$x_{i1} + x_{i2} + \dots + x_{in} = 1 \quad \text{for machine } M_i$$

$$x_{1j} + x_{2j} + \dots + x_{nj} = 1, \quad \text{for space } S_j$$

Total installation cost is $\sum\sum c_{ij}x_{ij}$

Thus the assignment problem can be stated as follows:

Determine $x_{ij} \geq 0$ so as to

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints $\sum_{i=1}^{n} x_{ij} = 1$ and $\sum_{j=1}^{n} x_{ij} = 1$.

| | $S_1$ | $S_2$ | - - - - | $S_n$ |
|---|---|---|---|---|
| $M_1$ | $c_{11}$ | $c_{12}$ | - - - | $c_{1n}$ |
| $M_2$ | $c_{21}$ | $c_{22}$ | - - - | $c_{2n}$ |
| ⋮ | ⋮ | ⋮ | | ⋮ |
| $M_n$ | $c_{n1}$ | $c_{n2}$ | - - - | $c_{nn}$ |

Solution of Assignment problem by

$\boxed{\text{Hungarian method.}}$

Working procedure

Step 1: Reduce the matrix

Qn: Four jobs are to be done on four different machines. The cost of producing ith job on the jth machine is given below:

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | 15 | 11 | 13 | 15 |
| $J_2$ | 17 | 12 | 12 | 13 |
| $J_3$ | 14 | 15 | 10 | 14 |
| $J_4$ | 16 | 13 | 11 | 17 |

Assign the job to different Machines so as to minimize the total cost.

Soln: Step 1: Reduce the matrix

(Reduced matrix)

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | – |
| 1 | 0 | 0 | 0 | – |
| 0 | 5 | 0 | 3 | – |
| 1 | 2 | 0 | 5 | – |
| | | | | |

(final solution)

Step 2: Assign the zeros

(a)

(b)

Step 3: Apply optimality check

Repeat step 2 until one of the following occurs:

(i) If no row/column is without assignment, then the current assignement is optimal

(ii) If there is some row/column without an assignement then the current assignment is not optimal and go to next step ...

(go to next page)

4. *Find the minimum number of lines crossing all zeros.* This consists of the following sub-steps:

(*a*) Mark (√) the rows that do not have assignments.

(*b*) Mark (√) the columns (not already marked) that have zeros in marked rows.

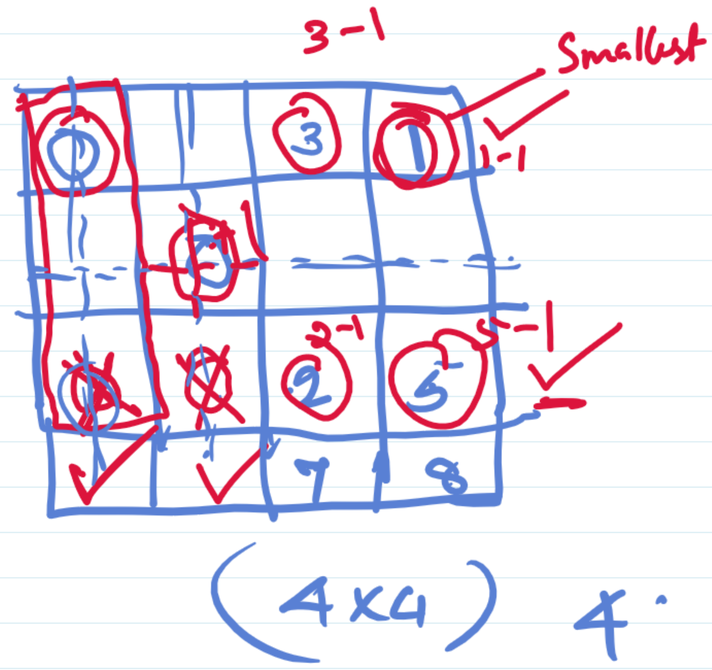
(*c*) Mark (√) the rows (not already marked) that have assignments in marked columns.

(*d*) Repeat sub-steps (*b*) and (*c*) till no more rows or columns can be marked.

(*e*) Draw straight lines through all unmarked rows and marked columns. This gives the minimum number of lines crossing all zeros. If this number is equal to the order of the matrix, then it is an optimal solution, otherwise go to step 5.

5. *Iterate towards the optimal solution.* Examine the uncovered elements. Select the smallest element and subtract it from all the uncovered elements. Add this smallest element to every element that lies at the intersection of two lines. Leave the remaining elements of the matrix as such. This yields second basic feasible solution.

6. Repeat steps 3 through 5 successively until the number of lines crossing all zeros becomes equal to the order of the matrix. In such a case every row and column will have one assignment. This indicates that an optimal solution has been obtained.




Example 2

and the minimum assignment cost = Rs. (11 + 13 + 14 + 11) = Rs. 49.

**Example 12.33.** *A marketing manager has 5 salesmen and 5 sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows :*

| | | Sales districts | | | | |
|---|---|---|---|---|---|---|
| | | A | B | C | D | E |
| Salesmen | 1 | 32 | 38 | 40 | 28 | 40 |
| | 2 | 40 | 24 | 28 | 21 | 36 |
| | 3 | 41 | 27 | 33 | 30 | 37 |
| | 4 | 22 | 38 | 41 | 36 | 36 |
| | 5 | 29 | 33 | 40 | 35 | 39 |

*Find the assignment of salesmen to districts that will result in maximum sales.*

(*Madras, B.E., 2000*)

Special Case:
(Max: problem)

Soln: First we need to convert it into a Minimization problem as follows:

**Solution** *consists of the following steps :*

*Step 1. Reduce the matrix.* Convert the given maximization problem into a minimization problem, by making all the profits negative, since max. $Z$ = min. $(-Z)$. Then subtract the smallest element of each row from the elements of that row. Now subtract the smallest element of each col. (not containing zero) from the elements of that col. This gives the *reduced matrix* (Table 1).

*Table 1*

| | | | | |
|---|---|---|---|---|
| 8 | 0 | ~~0~~ | 7 | 0 |
| 0 | 14 | 12 | 14 | 4 |
| 0 | 12 | 8 | 6 | 4 |
| 19 | 1 | 0 | ~~0~~ | 5 |
| 11 | 5 | 0 | ~~0~~ | 1 |

*Table 2*

| | | | | |
|---|---|---|---|---|
| 12 | 0 | 0 | 7 | 0 |
| 0 | 10 | 8 | 10 | 0 |
| 0 | 8 | 4 | 2 | 0 |
| 23 | 1 | 0 | 0 | 5 |
| 15 | 5 | 0 | 0 | 1 |

*Step 2. Assign the zeros.* Rows 2 and 3 have each a single unmarked zero in col. 1. Encircle these. Columns 2 and 5 have each a single unmarked zero in row 1. Encircle these and cross the zero in row 1. Columns 3 and 4 have each unmarked zeros. Encircle the zeros in each of the rows 4 and 5 as shown in Table 1 and cross other zeros.

*Step 3. Apply optimality check.* As col. 4 is without assignment, this solution is not optimal. Therefore we go to next step.

*Step 4. Find minimum number of lines crossing all zeros.* Draw the least number of horizontal and vertical (dotted) lines which cover all the zeros. Since there are four dotted lines which are less than the order of the cost matrix (= 5), we go to step 5.

*Step 5. Iterate towards optimal solution.* Select the smallest element in the table 1, not covered by the dotted lines. Such an element is 4 which lies at two different positions. Selecting the element that lies at position (3, 5) arbitrarily, subtract it from all the uncovered elements of the cost matrix (table 1) and add the same to the elements lying at the intersection of two dotted lines. Now draw more minimum number of dotted lines so as to cover the new zero. Here we draw such a line in col. 5 (Table 2).

Now, since the number of dotted lines is equal to the order of the cost matrix, the optimal solution is attained.

Finally, to determine this optimal assignment, we consider only the zero elements (Table 3) :

(*i*) Examine successively the rows with exactly one zero. There is no such row.

(*ii*) Examine successively the columns with exactly one zero. Col. 2 has one zero, encircle it and cross all zeros of row 1.

(*iii*) Encircle arbitrarily the zero in position (2, 1) and cross all zeros in row 2 and col. 1. Then encircle the unmarked zero in row 3. Now encircle arbitrarily the zero in

*Table 3*

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | 0 | ~~0~~ | | |
| 2 | 0 | | | | ~~0~~ |
| 3 | ~~0~~ | | | | 0 |
| 4 | | | 0 | ~~0~~ | |
| 5 | | | ~~0~~ | 0 | |

Soln:

| | | | | |
|---|---|---|---|---|
| -32 | -38 | -40 | -28 | -40 |
| -40 | -24 | -28 | -21 | -36 |
| -41 | -27 | -33 | -30 | -37 |
| -22 | -38 | -41 | -36 | -36 |
| -29 | -33 | -40 | -35 | -39 |

| | | | | | |
|---|---|---|---|---|---|
| 8 | 0 | ~~0~~ | 7 | ~~0~~ | |
| 0 | 14 | 12 | 14 | 4 | ✓ |
| ~~0~~ | 12 | 8 | 6 | 4 | ✓ |
| 19 | 1 | 0 | ~~0~~ | 5 | |
| 11 | 5 | ~~0~~ | 0 | 1 | 2 |

✓

| | A | B | C | D | E |
|---|---|---|---|---|---|
| $S_1$ | 12 | 0 | ~~0~~ | 7 | ~~0~~ |
| $S_2$ | 0 | 10 | 8 | 10 | ~~0~~ |
| $S_3$ | ~~0~~ | 8 | 4 | 2 | 0 |
| $S_4$ | 23 | 1 | 0 | ~~0~~ | 5 |
| $S_5$ | 15 | 5 | ~~0~~ | 0 ~~0~~ | 1 |

Here no. of assignments = Order of the matrix

Hence this gives the optimal solution

Optimal solution is $S_1 \to B$; $S_2 \to A$; $S_3 \to E$; $S_4 \to C$; $S_5 \to D$.

Max. Sale $= 38 + 36 + 37 + 41 + 35$

$= \boxed{18700}$

Note: Select the largest cost and subtract all costs from this largest cost, which gives the reduced matrix
→ Second method for maximization

## Unbalanced Assignment Problems

Hungarian method is applicable only when the given matrix is a square matrix.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| $S_1$ | | | | |
| $S_2$ | | | | |
| $S_3$ | | | | |
| Dummy | 0 | 0 | 0 | 0 |

| | $P_1$ | $P_2$ | $P_3$ | Dummy |
|---|---|---|---|---|
| $J_1$ | | | | 0 |
| $J_2$ | | | | 0 |
| $J_3$ | | | | 0 |
| $J_4$ | | | | 0 |

Q1)

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ |
|---|---|---|---|---|---|---|
| A | 12 | 10 | 15 | 22 | 18 | 8 |
| B | 10 | 18 | 25 | 15 | 16 | 12 |
| C | 11 | 10 | 3 | 8 | 5 | 9 |
| D | 6 | 14 | 10 | 13 | 13 | 12 |
| E | 8 | 12 | 11 | 7 | 13 | 10 |
| F | 0 | 0 | 0 | 0 | 0 | 0 |

Min: problem

Now apply Hungarian method to find the initial basic feasible soln:

} Complete VAM and optimality check.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ |
|---|---|---|---|---|---|---|
| A | 6 | 0 | 5 | 12 | 8 | ~~0~~ |
| B | ~~0~~ | 4 | 11 | 1 | 2 | 0 |
| C | 12 | 7 | 0 | 5 | 2 | 8 |
| D | 0 | 4 | ~~0~~ | 3 | 3 | 4 |
| E | 5 | 5 | 4 | 0 | 6 | 5 |
| Dummy F | 4 | ~~0~~ | ~~0~~ | ~~0~~ | 0 | 2 |

optimal Solution

Min. Value = 38

HW

A small garment making unit has five tailors stitching five different types of garments. All the five tailors are capable of stitching all the five types of garments. The output per day per tailor and the profit (Rs.) for each type of garment are given below:

**Table 4.91**

| Tailors | Garments 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | 7 | 9 | 4 | 8 | 6 |
| B | 4 | 9 | 5 | 7 | 8 |
| C | 8 | 5 | 2 | 9 | 8 |
| D | 6 | 5 | 8 | 10 | 10 |
| E | 7 | 8 | 10 | 9 | 9 |
| Profit (Rs.) per garment | 2 | 3 | 2 | 3 | 4 |

(i) Which type of garment should be assigned to which tailor in order to maximize profit, assuming that there are no other constraints?

(ii) If tailor D is absent for a specified period and no other substitute tailor is available, what should be the optimal assignment? [I.C.W.A. Dec., 1990]

A fast-food chain wants to build four stores. In the past the chain has used six different construction companies, and having been satisfied with each, has invited them to bid for each job. The final bids (in thousands of rupees) are shown in the following table:

**Table 4.45**

| Stores | Construction companies 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 85.3 | 90 | 87.5 | 82.4 | 89.1 | 91.3 |
| 2 | 78.9 | 84.5 | 99.4 | 80.4 | 89.3 | 88.4 |
| 3 | 82.0 | 31.3 | 28.5 | 66.5 | 80.4 | 109.7 |
| 4 | 84.3 | 34.6 | 86.2 | 83.3 | 85.0 | 85.5 |

Since the fast-food chain wants to have each of the new stores ready as quickly as possible, it will allot at most one job to a construction company. What assignment will result in the minimum total cost?

Five wagons are available at stations 1, 2, 3, 4 and 5. These are required at five stations I, II, III, IV and V. The mileages between various stations are given by the table below. How should the wagons be transported so as to minimize the total mileage covered?

**Table 4.28**

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| 1 | 10 | 5 | 9 | 18 | 11 |
| 2 | 13 | 9 | 6 | 12 | 14 |
| 3 | 3 | 2 | 4 | 4 | 5 |
| 4 | 18 | 9 | 12 | 17 | 15 |
| 5 | 11 | 6 | 14 | 19 | 10 |

Solve the following assignment problem:

**Table 4.22**

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| 1 | 11 | 17 | 8 | 16 | 20 |
| 2 | 9 | 7 | 12 | 6 | 15 |
| 3 | 13 | 16 | 15 | 12 | 16 |
| 4 | 21 | 24 | 17 | 28 | 26 |
| 5 | 14 | 10 | 12 | 11 | 13 |

# Unit - II PROJECT MANAGEMENT :

**Defn : Project :**

**Defn : Network :**

PERT (Programme Evaluation and Review Technique)

CPM (Critical Path method).

**PERT** : developed in 1958

**CPM** : Du pont Company & Remington Rand Corp.

**PERT** : 1.
2.
3. (event - oriented technique)
4.
5.

**CPM** : 1.
2.
3.

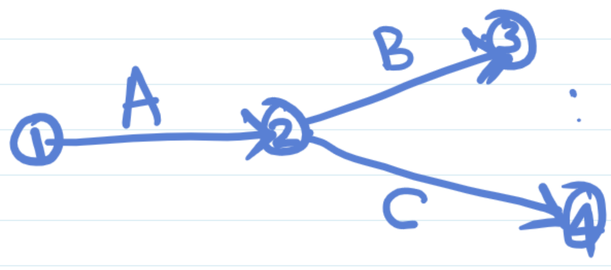
**Defn : Activity** : item of work to be done

i-j, activity A.

**Defn : Event** : Consumes no time ; represented by circles.
- also known as node

**Types of Activities** : 3 types of activities

(i) Predecessor activity :

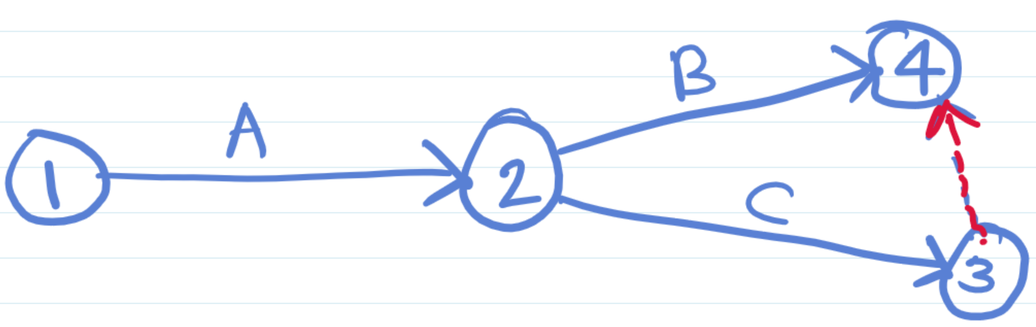



Here A is the predecessor activity of B and C

(ii) Successor Activity :

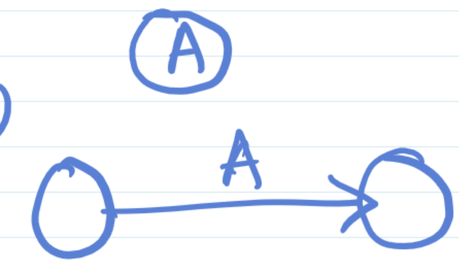
Here B is the successor of A.

(iii) Dummy Activity :



Activity-on-Node (AON) network :

Activity-on-Arrow (AOA) network :

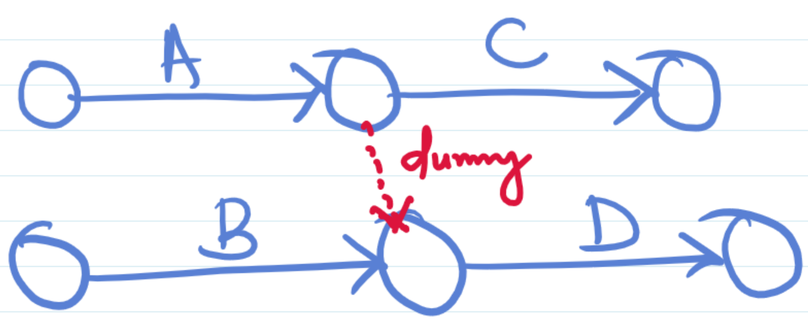
(i) 

} Activity A.

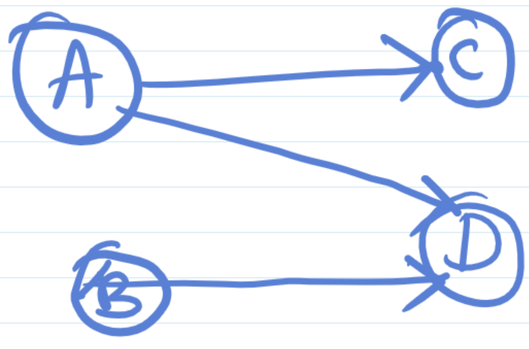
(ii) B must follow A : AOA AON

(iii) C must follow A, and D must follow A and B.

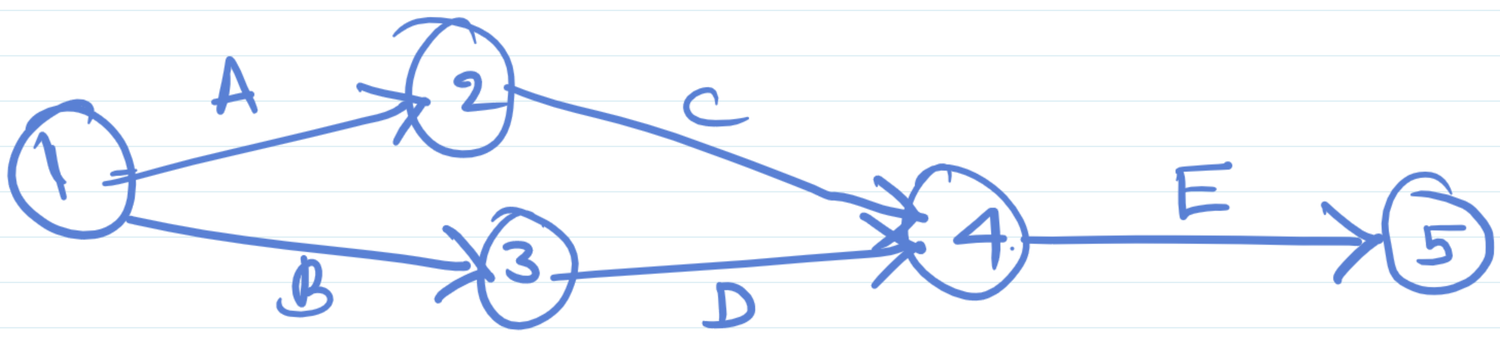
AOA :




AON :




Defn : Start and Terminal Activities




Here A, B : Start activity , E : Terminal activity

Remark : Dummy activity is introduced when

(i) Two or more parallel activities have same tail and head events.

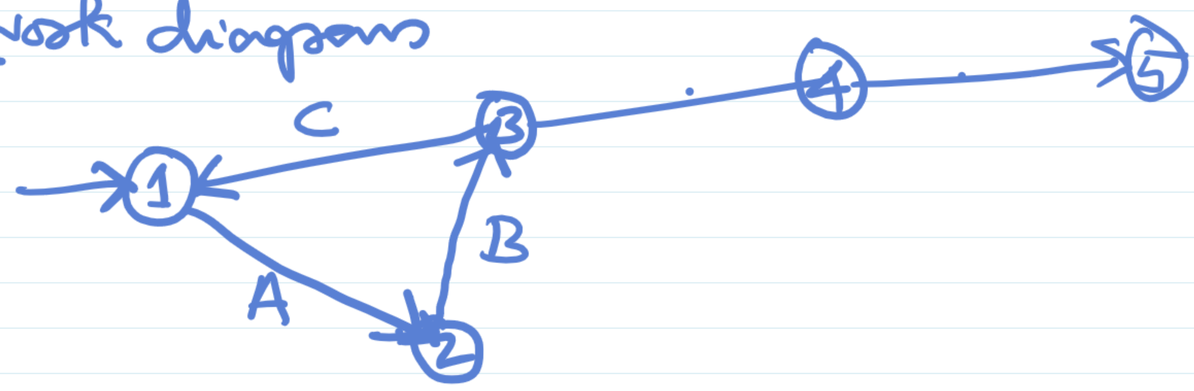
(ii) Two or more activities have some of their immediate predecessor activities in common.

Logical sequence : $A < B$ : A must be completed before the start of B.

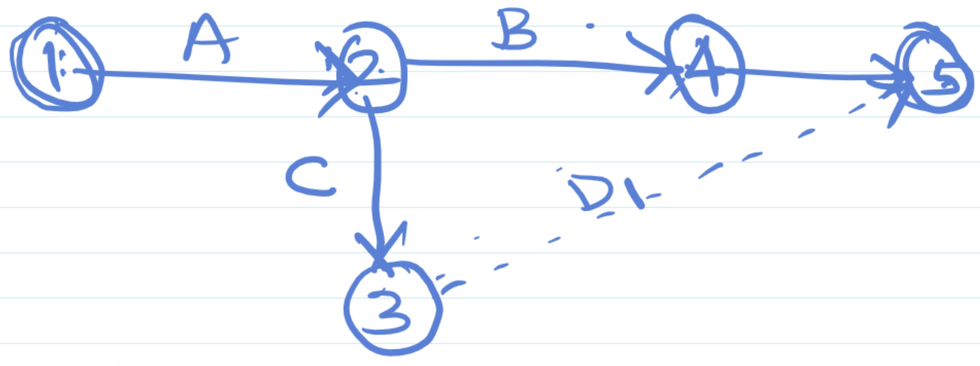
$A < B, C$

Errors in network diagrams

(i) Looping :




(ii) Dangling :




## NETWORK DIAGRAM

graph, $\rightarrow$ O

Network diagrams are of 2 types:

(i) Event oriented

(ii) Activity oriented

(i) Event oriented : PERT

(ii) Activity oriented : CPM

## Rules for network Construction

(i) each activity by one and only one arrow.

(ii) Start node and end node

(iii) events/nodes are number : tail no. < head no.

(iv) logical Sequence :

(a) $A < B, C$

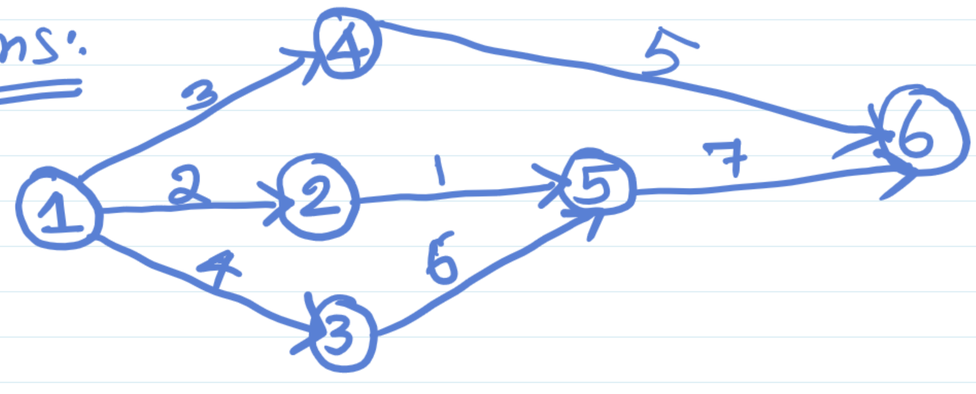
(b)

(c)

Q1) Draw a network diagram to the following activities.

| Activity (i,j) | Time duration |
|---|---|
| 1-2. ✓ | 2 |
| 1-3. | 4 |
| 1-4. | 3 |
| 2-5. | 1 |
| 3-5. | 6 |
| 4-6. | 5 |
| 5-6. | 7 |

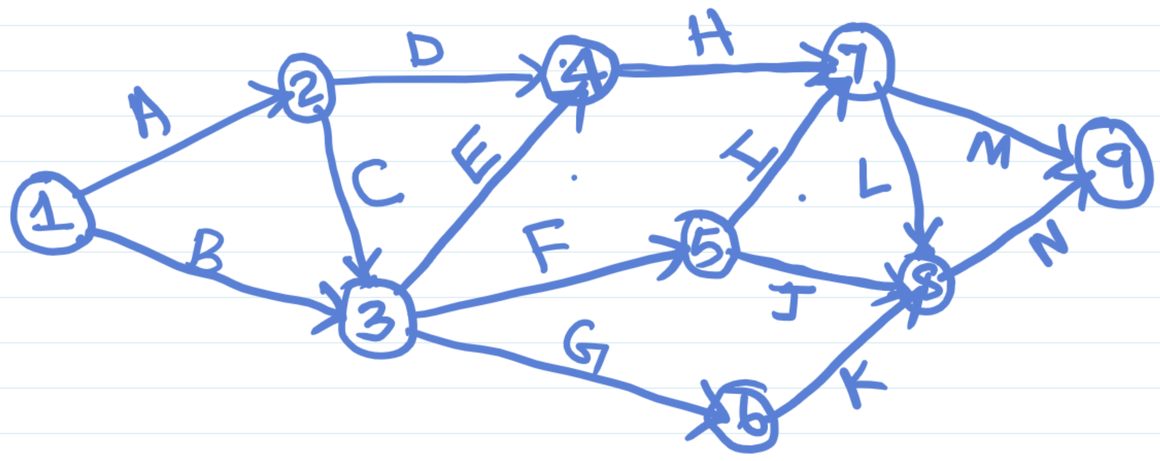
Ans:



Q2) Draw a network diagram to the following set of Activities.

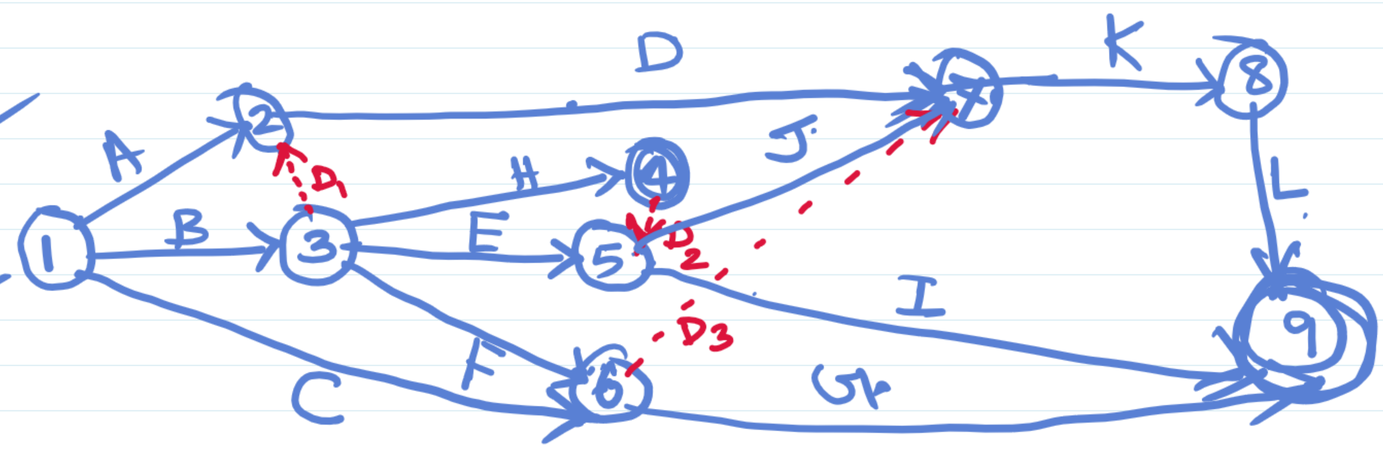
| Activity | preceding activity |
|---|---|
| A ✓ | none |
| B ✓ | none |
| C ✓ | A |
| D ✓ | A |
| E ✓ | B,C |
| F ✓ | B,C |
| G ✓ | B,C |
| H ✓ | D,E |
| I ✓ | F |
| J ✓ | F |
| K ✓ | G |
| L ✓ | H,I |
| M ✓ | H,I |
| N | J,K,L |

Soln:




Q3) Construct the network diagram comprising activities A, B, C, ..., L such that the following relationships are satisfied

1) A, B, C are the first activities of the project, can start simultaneously
2) A, B precede D ✓ ∴ Soln:
3) B precedes E, F, H ✓
4) F, C precedes G ✓
5) E, H precedes I, J ✓
6) C, D, F, J precede K ✓
7) K precedes L ✓
8) I, G, L are the terminal activities of the project.

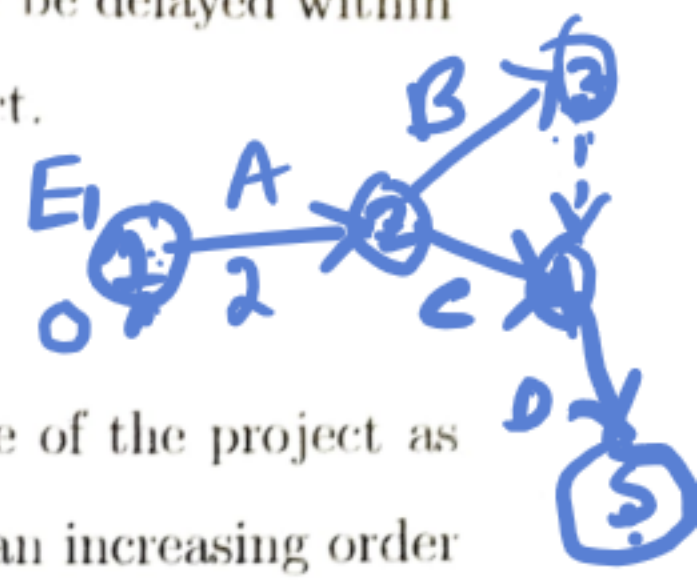



HW Construct the network diagram:

| Activity: | Immediate predecessor |
|---|---|
| A | — |
| B | — |
| C | — |
| D | A |
| E | B,C |
| F | A |
| G | C |
| H | D,E,F |
| I | D |

| Activity | predecessor |
|---|---|
| J | G |
| K | G |
| L | H,J |
| M | K |
| N | I,L |

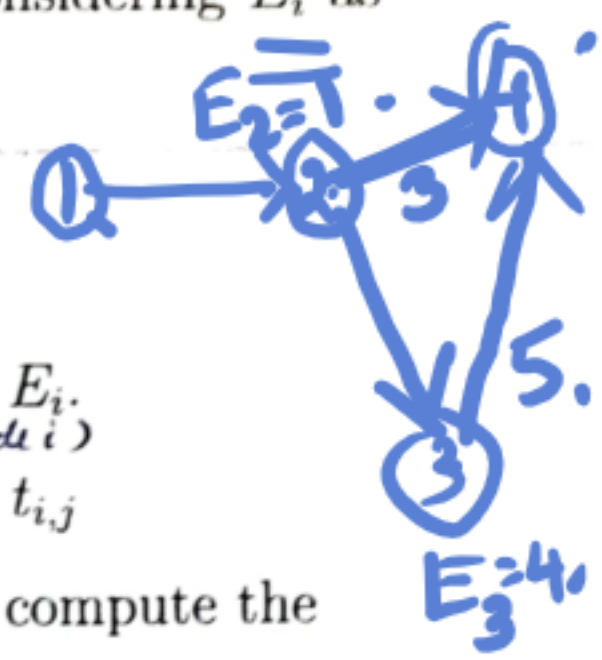
scheduling slack so that the start time of the activity may be delayed within limit without effecting the completion of the entire project.




The Critical path calculations are done in two ways;

(i)**Forward Pass Calculation**:

We will start from the initial node with starting time of the project as zero. We proceed through the network visiting events in an increasing order of the event number and ends at the final event of the network. At each node we calculate the EST and EFT for each activity by considering $E_i$ as the earliest occurrence of the event 'i'.




Step 1: Let $E_1 = 0$

Step 2: Set the EST for each activity starts with node 'i' by $E_i$.

(i.e that beginnat node i)

Step 3: Compute EFT for each activity 'i-j' by $EFT = E_i + t_{i,j}$

Step 4: Move on to to the next node say 'j' where $j > i$ and compute the earliest occurrence for node 'j' by $E_j = Max_i\{E_i + t_{i,j}\}$ for every immediate predecessor activities 'i-j'.

$\underbrace{\qquad\qquad}_{\max_i}$

$E_4 = 1+3 = 4$

$E_4 = 4+5 = 9$

(i)**Backward Pass Calculation**:

We will start from the final node with starting time of the project as zero. We proceed through the network visiting events in decreasing order of the event number and ends at the initial node of the network. At each node we calculate the LFT and LST for each activity by considering $L_j$ as the latest occurrence of the event 'j'.

Step 1: Let $L_n = E_n$ for the final node 'n'

Step 2: Set the LFT for each activity ends with node 'j' by $L_j$.

Step 3: Compute LST for each activity 'i-j' by $LST = L_j - t_{i,j}$

Step 4:

**Step 4:** Proceed backward to the node in the sequence that decrease 'j' by 1 and compute the latest occurrence time for node 'i' where '$i < j$' by

$L_i = \text{Min}_j\{L_j - t_{i,j}\}$.  $L_i = \text{Min}_j\{L_j - t_{i,j}\}$.

**Definition 2.7** *(Float)* (Float) (Slack) (Free time)

*The float of an activity is the amount of the time by which it is possible to* delay its completion time *without effecting the total project completion time.*

Floats are of different types: There are 3 types of floats:

**Definition 2.8** *(i)(Total float)*

*The total float of an activity represents the amount of time by which an activity can be delayed without delay in the projects completion date. ie; total float is the* +ve *difference between EFT and LFT of the activity.*

Total float $= L_j - (E_i + t_{i,j})$

**Definition 2.9** *(ii)(Free float)*

*Free float is a portion of* ~~a portion of~~ *positive total float that can be used by an activity without delaying any succeeding activity. ie; Free float = EST of the successor activity − EFT of the present activity.*

free float $= E_j - (E_i + t_{i,j})$

**Definition 2.10** *(iii)(Independent float)*

*The independent float of the activity is the portion of the total float within which an activity can be delayed for the start without effecting the floats of the* preceding *preceding activities.*

*ie; Independent float = EST of the subsequent activity − LFT for the* preceding activity *preceding activity − time duration of the activity.*

Independent float $= E_j - (L_i + t_{i,j})$

# Critical Path

**Defn:** The critical activities of a network that constitute an uninterrupted path which span the entire network from start to finish is called a critical path.

Also we can say that, an activity is critical if its total float is zero

Q1) The following table gives the activities in a construction project and time duration.

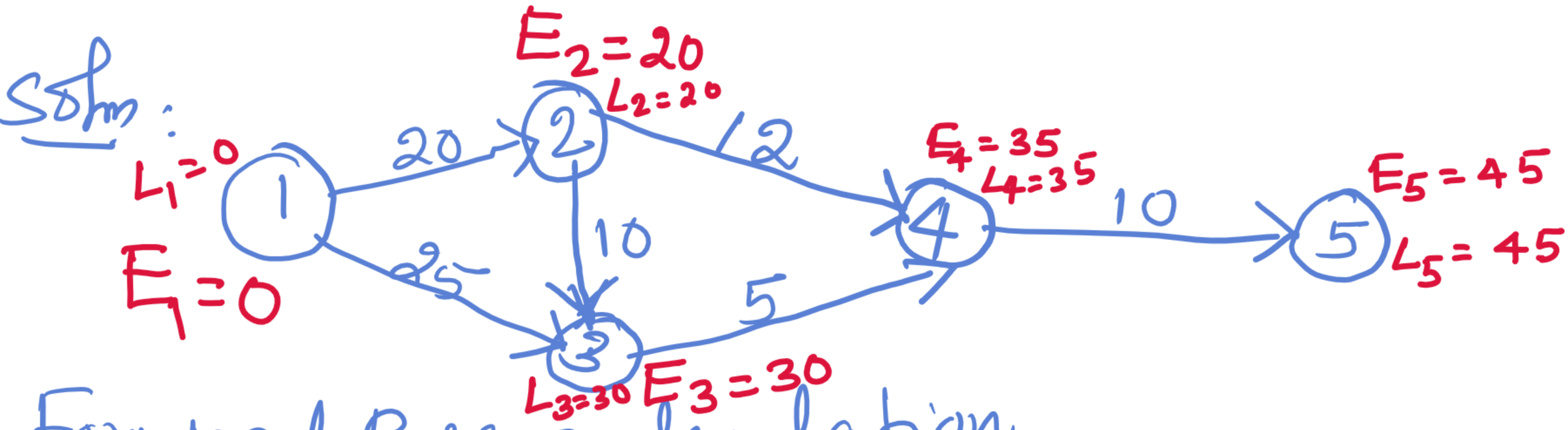
| Activity | Preceding activity | activity duration (days) |
|---|---|---|
| 1-2 | — | 20 |
| 1-3 | — | 25 |
| 2-3 | 1-2 | 10 |
| 2-4 | 1-2 | 12 |
| 3-4 | 1-3, 2-3 | 5 |
| 4-5 | 2-4, 3-4 | 10 |

(i) Draw the network diagram of the project.

(ii) Find the total float and free float for each activity

(iii) Determine the critical path and project duration.

Soln:




Forward Pass calculation

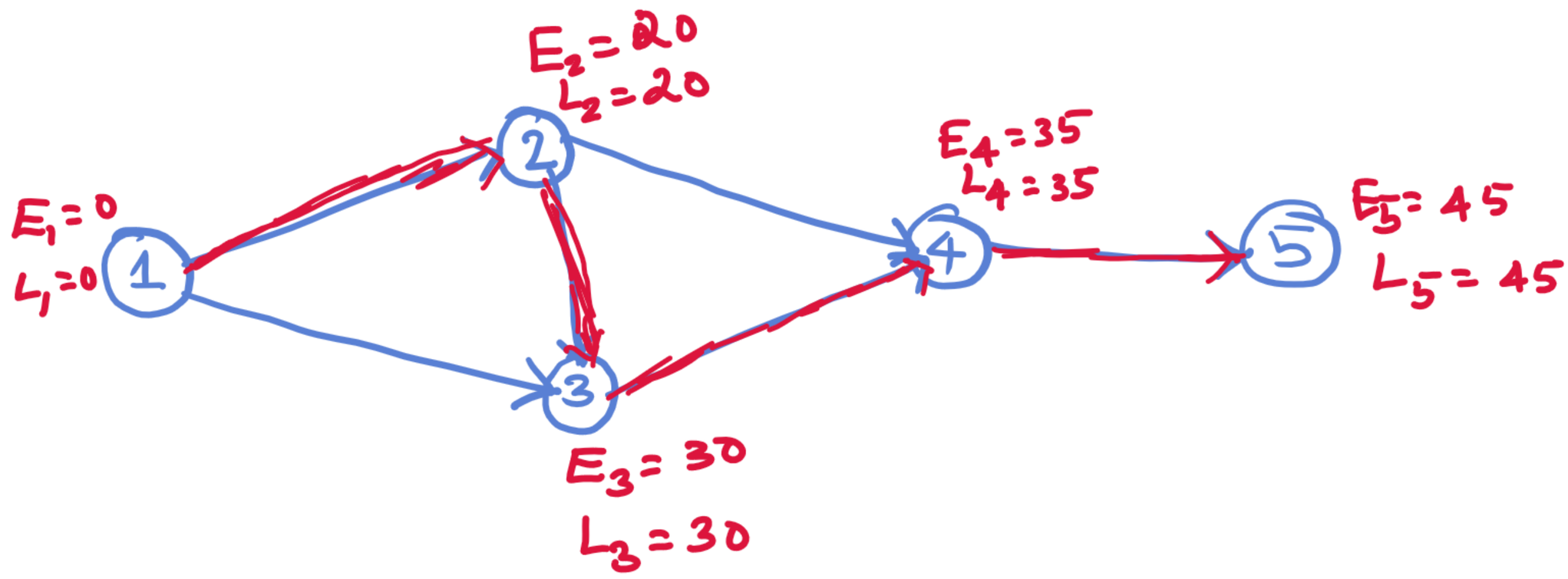
$E_1 = 0$; $E_2 = E_1 + t_{1,2} = 0 + 20 = 20$

$E_3 = \text{Max}\{E_1 + t_{1,3}, E_2 + t_{2,3}\} = \text{Max}\{25, 30\}$
$= 30$

$E_4 = \text{Max}\{E_2 + t_{2,4},\ E_3 + t_{3,4}\} = \text{Max}\{22, 35\} = 35$

$E_5 = E_4 + t_{4,5} = 35 + 10 = 45$

(do the Backward pass calculations and obtains all $L_j$'s)




| Activity | time | Earliest time | | Latest time | | Float | |
|---|---|---|---|---|---|---|---|
| | | $E_i$ | $(E_i + t_{ij})$ | $(L_j - t_{ij})$ | $L_j$ | Total $L_j - (E_i + t_{ij})$ | free float $E_j - (E_i + t_{ij})$ |
| 1-2 | 20 | 0 | 20 | 0 | 20 | 0 | (do it later) |
| 1-3 | 25 | 0 | 25 | 5 | 30 | 5 | |
| 2-3 | 10 | 20 | 30 | 20 | 30 | 0 | |
| 2-4 | 12 | 20 | 32 | 23 | 35 | 3 | |
| 3-4 | 5 | 30 | 35 | 30 | 35 | 0 | |
| 4-5 | 10 | 35 | 45 | 35 | 45 | 0 | |

The critical path is 1-2-3-4-5.

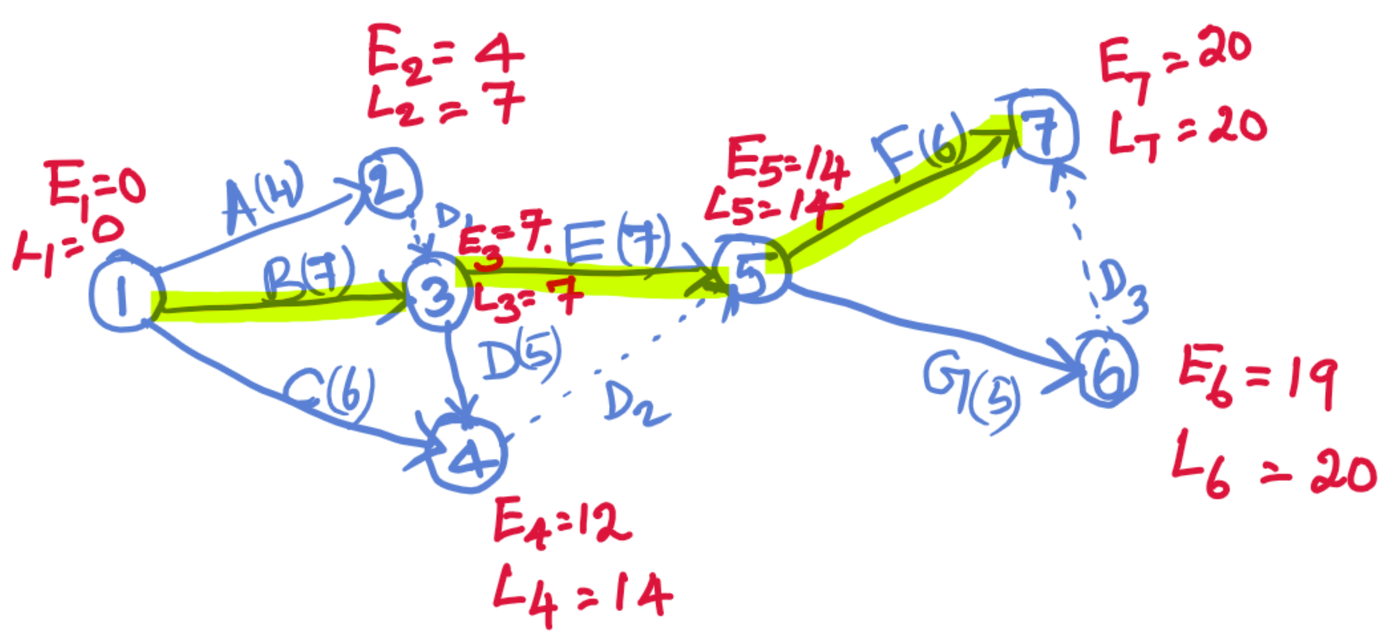
Project duration = 45 days.

Q2) A small project consists of seven activities for which the relevant data are given below:

| Activity | Preceding activity | Duration (days) |
|---|---|---|
| A | – | 4 |
| B | – | 7 |
| C | – | 6 |
| D | A, B | 5 |
| E | A, B | 7 |
| F | C, D, E | 6 |
| G | C, D, E | 5 |

(i) Draw the network and find the project completion time
(ii) Calculate the total float for each activity and hence find the CP.

Soln:




Forward calculations: Start with $E_1=0$

Backward pass calculations start with $L_7=20$.

| Activity | time $t_{ij}$ | Earliest time $E_i$ | Earliest time $E_i+t_{ij}$ | Latest time $L_j-t_{ij}$ | Latest time $L_j$ | Total float $L_j-(E_i+t_{ij})$ |
|---|---|---|---|---|---|---|
| A(1-2) | 4 | 0 | 4 | 3 | 7 | 3 |
| B(1-3) | 7 | 0 | 7 | 0 | 7 | 0 |
| C(1-4) | 6 | 0 | 6 | 8 | 14 | 8 |
| D(3-4) | 5 | 7 | 12 | 9 | 14 | 2 |
| E(3-5) | 7 | 7 | 14 | 7 | 14 | 0 |
| F(5-7) | 6 | 14 | 20 | 14 | 20 | 0 |
| G(5-6) | 5 | 14 | 19 | 15 | 20 | 1 |

The critical path is 1 – 3 – 5 – 7.

Project completion time = 20 days.

Hw Tasks A, B, C, ..., I constitute a project with the precedence relationships $A < D$, $A < E$, $B < F$, $D < F$, $C < G$, $C < H$, $F < I$, $G < I$.

Draw the network diagram and find the project completion time. Given

| Tasks: | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| Time: | 8 | 10 | 8 | 10 | 16 | 17 | 18 | 14 | 9 |

(days).

## PERT:

Objective: To find out whether a job could be finished on a given time.

(i) optimistic time ($t_o$)

(ii) Pessimistic time ($t_p$):

(iii) Most likely time ($t_m$):

## Steps involved in PERT calculation

(i) Construct the network diagram.

(ii) obtain the estimates $t_o, t_p, t_m$.

(iii) Compute the expected time ($t_e$) for each activity by

$$t_e = \frac{t_o + 4t_m + t_p}{6}.$$

(iv) Determine the critical path

(v) Find the variance of the time estimates of all activities by

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

(vi) Find the probability of finishing the project on some fixed target by using Normal distribution table. The value of $Z$ is obtained by $Z = \frac{T_s - T_e}{\sigma}$ where

$T_s$ = due date, $T_e$ = Expected date of completion,

$\sigma$ = Standard deviation of the critical path

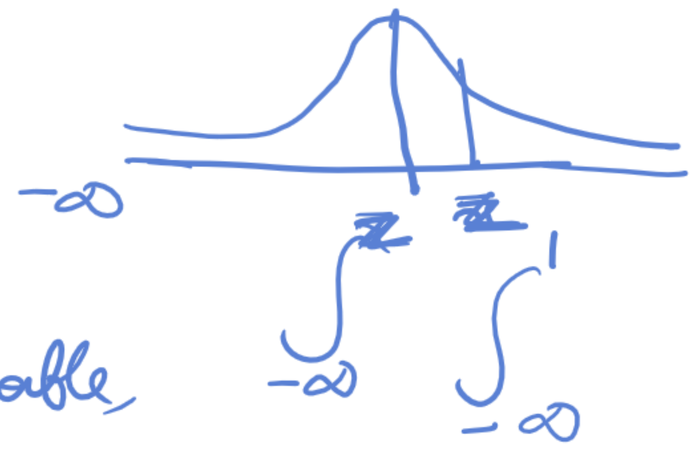
= Square root of the sum of the variances of the critical activities.

**Example:** A project is expected to take 15 months along the critical path having a S.d of 3 months. What is the probability of completing the project on the due date, if the due date is (i) 18 months (ii) 12 months.

**Soln:** Given $T_e = 15$ months, $\sigma = 3$ months.

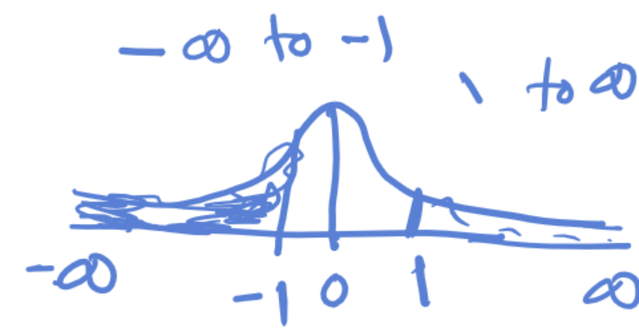
(i) due date $= T_s = 18$ months.

$$z = \frac{T_s - T_e}{\sigma} = \frac{18-15}{3} = 1.$$




when $z=1$, from the standard normal table, we get Probability $= 0.8413$

(ii) $T_s = 12$, $z = \frac{T_s - T_e}{\sigma} = \frac{12-15}{3} = -1$




from standard normal table, we get

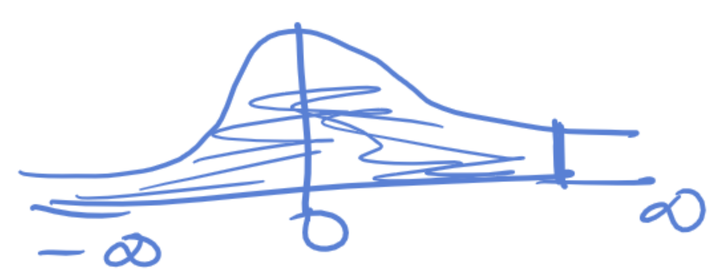
Required probability $= 1 - P(-\infty \text{ to } 1)$

$= 1 - 0.8413 = 0.1587$

Q2) PERT calculation yield a project length of 50 weeks with a variance 16. Within how many weeks would you expect the project to be completed with probability of (i) 95% (ii) 40%.

**Soln:** $T_e = 50$, $\sigma = \sqrt{16} = 4$

(i) Given prob $= 95\% = 0.95$.




From the normal distbn table, we get $z = 1.65$.

ie $\frac{T_s - T_e}{\sigma} = 1.65 \Rightarrow \frac{T_s - 50}{4} = 1.65$

$$\begin{array}{r} 1.65 \\ 4 \\ \hline 6.60 \end{array}$$

$\Rightarrow T_s = 50 + 4 \times 1.65 = 56.6 \approx 57$

due date $= 57$ weeks.

(ii) Given prob. = 40% = 0.4, from the normal distbn table, we get $z = -0.25$.

$$\int_{-\infty}^{z} = 40\%$$

60% = 0.6

$z = .25$

ie $\dfrac{T_s - T_e}{\sigma} = -0.25$

$\dfrac{T_s - 50}{4} = -0.25 \Rightarrow T_s = 50 - 4 \times 0.25$

$= 49$.

due date is $= 49$ weeks.

(Assignment 2 is scheduled on 29th Saturday)

Qn) A small project is composed of 7 activities whose time estimates are given below:

| Activity | Estimated duration $t_o$ | $t_m$ | $t_p$ |
|---|---|---|---|
| 1-2 | 1 | 1 | 7 |
| 1-3 | 1 | 4 | 7 |
| 1-4 | 2 | 2 | 8 |
| 2-5 | 1 | 1 | 1 |
| 3-5 | 2 | 5 | 14 |
| 4-6 | 2 | 5 | 8 |
| 5-6 | 3 | 6 | 15 |

1) Draw the project network.

2) Find the expected duration and variance for each activity. What is the expected project length

3) Calculate the variance and S.D of the project length

4) What is the probability that the project will be completed

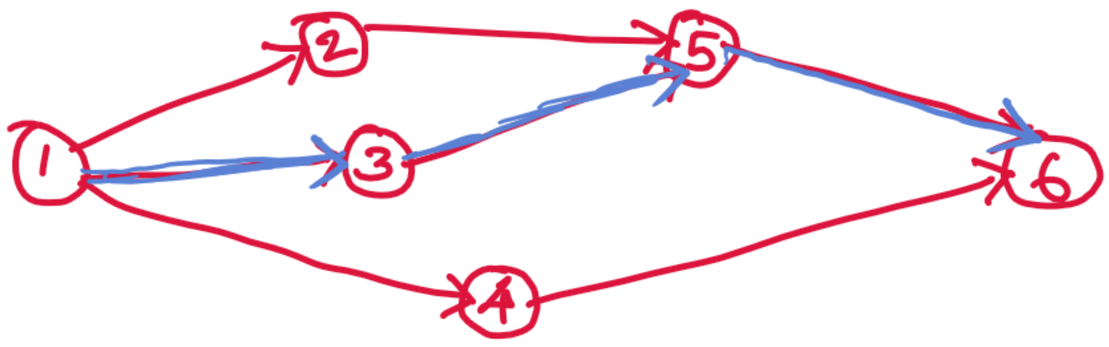
(a) atleast 4 weeks earlier than expected time.

(b) no more than 4 weeks latest than expected time

5) If the project due date is 19 weeks, then what is the probability of not meeting the due date?

Soln:

(i)




(ii)

| Activity | $t_o$ | $t_m$ | $t_p$ | $t_e = \frac{t_o + 4t_m + t_p}{6}$ | $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$ |
|---|---|---|---|---|---|
| 1-2 | 1 | 1 | 7 | 2 | 1 |
| 1-3 | 1 | 4 | 7 | 4 | 1 |
| 1-4 | 2 | 2 | 8 | 3 | 1 |
| 2-5 | 1 | 1 | 1 | 1 | 0 |
| 3-5 | 2 | 5 | 14 | 6 | 4 |
| 4-6 | 2 | 5 | 8 | 5 | 1 |
| 5-6 | 3 | 6 | 15 | 7 | 4 |

Forward calculation

$E_1 = 0,\ E_2 = 2,\ E_3 = 4,\ E_4 = 3,\ E_5 = 10$

$E_6 = 17$

Backward calculation

$L_6 = E_6 = 17,\ L_5 = 10,\ L_4 = 12,\ L_3 = 4;$

$L_2 = 9;\ L_1 = 0.$

| Activity | time($t_{ij}$) | $E_i$ | $E_i + t_{ij}$ | $L_j - t_{ij}$ | $L_j$ | Total float $(L_j - (E_i + t_{ij}))$ |
|---|---|---|---|---|---|---|
| 1-2 | 2 | 0 | 2 | 7 | 9 | 7 |
| 1-3 | 4 | 0 | 4 | 0 | 4 | 0 |
| 1-4 | 3 | 0 | 3 | 9 | 12 | 9 |
| 2-5 | 1 | 2 | 3 | 9 | 10 | 7 |
| 3-5 | 6 | 4 | 10 | 4 | 10 | 0 |
| 4-6 | 5 | 3 | 8 | 12 | 17 | 9 |
| 5-6 | 7 | 10 | 17 | 10 | 17 | 0 |

Critical Path is 1-3-5-6

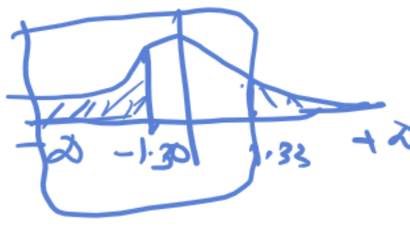
Expected project length is 17 weeks.

(3) Variance of the project length, $\sigma^2 = 1+4+4 = 9$ weeks.

Standard deviation $= \sigma = \sqrt{9} = 3$ weeks

(4) (a) $T_e = 17$ weeks, $T_s = 17 - 4 = 13$ weeks, $\sigma = 3$ weeks

$Z = \dfrac{T_s - T_e}{\sigma} = \dfrac{13-17}{3} = -1.33.$ $P(Z \le -1.33) = ?$




$P(Z \le -1.33) = 1 - P(Z \le 1.33) = 1 - 0.9082 = 0.0918$

$= 9.18\%$ Confidence

(b) $T_e = 17,\ T_s = 17 + 4 = 21,\ \sigma = 3;\ Z = \dfrac{21-17}{3} = 1.33$

$P(Z \le 1.33) = 0.9082 = 90.82\%$ confidence.

(5) Given $T_s = 19$ weeks, $Z = \dfrac{19-17}{3} = 0.67.$

$P(\text{meeting the due date}) = P(Z \le 0.67) = 0.7486$

Hence $P(\text{not meeting the due date}) = 1 - P(\text{meeting due date})$

$= 1 - 0.7486$

$= 0.2514$

$= 25.14\%$

Hw 1. A project consists of 8 activities, as below:

| Activity | Immediate predecessor | Estimated time $t_o$ | $t_m$ | $t_p$ |
|---|---|---|---|---|
| A | — | 1 | 1 | 7 |
| B | — | 1 | 4 | 7 |
| C | — | 2 | 2 | 8 |
| D | A | 1 | 1 | 1 |
| E | B | 2 | 5 | 14 |
| F | C | 2 | 5 | 8 |
| G | D,E | 3 | 6 | 15 |
| H | F,G | 1 | 2 | 3 |

(i) Draw the project network and find out the expected project completion time

(ii) What duration will have 95% confidence for project completion?

# GAME THEORY

*

**Definition 0.1** ***(GAME)***: *A competitive situation is called a* ***Game*** *if it has the following properties;*

*(i) There are finite number of participants (competitors) called players.*

*(ii) Each player has a finite number of strategies (rules or alternatives) available to him.*

*(iii) A play of the game takes place when each player employs his strategies.*

*(iv)Every game results in an outcome usually called pay off to some player.*

**Definition 0.2** ***(Two-Person Game)***: *A game between two competitors is called* ***Two-Person Game.***

*If the number persons in the game is "n" then such game are called* ***n-person game.***

**Definition 0.3** ***(Zero sum game)***: *In a game the algebraic sum of gains and losses of all players is zero, such game is called* ***Zero sum game.***

**Definition 0.4** ***(Two person zero sum game)****: A zero sum game with two players is called* ***Two person zero sum game****.*

**Remark 0.5** *:If the sum of gains and losses is not equal to zero then such game is called* ***non zero sum game****.*

**Example 0.6** *Consider a two person cion tossing game. Each player A and B tosses an unbaised coins simultaneously. Player B pays Rs. 7 to A if HH occurs and Rs. 4 if TT occurs; otherwise player A pays Rs. 3 to B.*

**Definition 0.7** ***Strategy****: A* ***strategy*** *for a player is the set of all rules or alternative course of actions available to him in advance by which player decides the course of action that he adopts.*

There are three types of strategies

**Definition 0.8** ***(Pure Strategy)****: A player select the same strategy each time then it is called* ***Pure strategy****.*

**Definition 0.9** ***(Mixed Strategy)****: When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occasion. Then this strategy is called mixed strategy.*

**Definition 0.10** ***(Optimum Strategy)****: A course of action which puts the player in the most performed position, irrespective of the strategy of his competitor is called an* ***optimum strategy****.*

**Definition 0.11** ***(Value of a Game)***: *It is the expected outcome of the player when all the players of the game follow their optimum strategy. The value of the game is denoted by "v".*

**Definition 0.12** ***(Pay off Matrix)***: *When player select their particular strategies then their outcome is called pay off, can be expressed in the form of a matrix is called* ***Pay off Matrix****.*

Let player A has "m" strategies $A_1, A_2, ...A_m$ and player B has "n" strategies $B_1, B_2, ...B_n$. Assume that each player has their choices from amongst the pure strategies. **Assume that player A is always the gainer and player B is the looser.**

Let $a_{ij}$ be the pay off , which player A gains from player B if player A chooses strategy $A_i$ and player B chooses strategy $B_i$ then the pay off matrix to player A is;

$$
\begin{array}{cc}
 & \begin{array}{cccccc} & B_1 & Player\,B & & & B_n \\ & & B_2 & - & - & - & \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ Player\,A \\ \vdots \\ \\ A_m \end{array} &
\begin{pmatrix}
a_{11} & a_{12} & . & . & . & a_{1n} \\
a_{21} & a_{22} & . & . & . & a_{2n} \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
a_{m1} & a_{m2} & . & . & . & a_{mn}
\end{pmatrix}
\end{array}
$$

If the game is zero sum game then the pay off matrix of player B is $[a_{ij}]$.

**Example 0.13** *In the example (0.6) the pay of matrix of player A is*

$$\text{Player } A \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{cc} H & T \end{array} \\ \begin{pmatrix} 7 & -3 \\ -3 & 4 \end{pmatrix} \end{array}$$

*there fore, the pay off matrix of B is*

$$\text{Player } A \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \text{Player } B \\ \begin{pmatrix} -7 & 3 \\ 3 & -4 \end{pmatrix} \end{array}$$

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Min each row} \\ \hline A_1 & 5 & 3 & 2 & 2 \\ A_2 & 1 & -2 & 0 & -2 \\ A_3 & 8 & -1 & 1 & -1 \\ \hline \text{Max column} & 8 & 3 & 2 & \end{array}$$

2 → Maximin

2 → Minimax.

**Definition 0.14** ***(Solution of a Game)****: Solving a game means finding the best strategies for both players and the value of the game.*

**Definition 0.15** ***(Maximin and Minimax)****:* ***Maximin*** *is the maximum of the minimum pay off's of each row. It is denoted by* $\underline{v}$.

***Minimax*** *is the minimum of the maximum pay off's in each column. It is denoted by* $\overline{v}$.

**Definition 0.16** ***(Saddle Point)****: If the maximin value is equal to the minimax value then the game is said to have saddle point and the corresponding strategies are called optimum strategies.*

*If the game has a saddle point then the pay off at the saddle point is the value of the game.*

**Definition 0.17** ***(Fair and Strictly determinable Game)****: A game is said to be* ***fair*** *if* $\underline{v} = 0 = \overline{v}$.

*A game is said to be* ***strictly determinable*** *if* $\underline{v} = v = \overline{v}$.

**Working rule to determine the saddle point of the game:** In the pay off matrix of the game

(i) Select the minimum element of each row of the pay off matrix and mark it by "*".

(ii)Select the maximum element of each column of the pay off matrix and mark it by "↑".

(iii) If there is an element in the pay off matrix marked both $*$ and ↑ then the game is with saddle point and the pay off at that place is the value of the game and the corresponding strategies are the optimum strategies.

# 1 GAME WITHOUT SADDLE POINT

In a two person zero sum game if there is no saddle point then we can't apply the maximin - minimax principle to solve the game. In this case we introduce the concept of chance move.

In this game both the players must determine an optimum mixture of strategies to find the saddle point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability being chosen.

**Maximin - Minimax criteria**

Consider an (mxn) game$[a_{ij}]$ without a saddle point, with mixed strategies.

Let $p_1, p_2, ...p_m$ be the probabilities with which player A will chose the moves $A_1, A_2, ...A_m$ respectively then $p_i \geq 0$ for every $1 \leq i \leq m$ and $p_1 + p_2 + ... + p_m = 1$.

Please go to Page 6

Example: Determine the following game is strictly determinable and fair. Give optimum strategies for each player in the case of strictly determinable game.

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 5^{\uparrow} & 0^{*} \\ 0^{*} & 2^{\uparrow} \end{bmatrix} \end{array}. \quad \underline{V} = 0, \ \overline{V} = 2.$$

$$\underline{V} \neq \overline{V}$$

So the game is not strictly determinable.

Example

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 2^{*} & 4^{\downarrow} & 7 \\ A_2 & 10^{\uparrow} & 7^{*\uparrow} & 11^{\uparrow} \\ A_3 & 4^{*} & 6 & 8 \end{array}$$

$\underline{V} = \overline{V} = 7$

optimum strategies are

player A : $A_2$ and for

player B : $B_2$

Example: Consider the following game:

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 1 & -2^{*} \\ 2^{\uparrow} & -1^{*\uparrow} \end{bmatrix} \end{array}$$

(i) what is the value of game?

(ii) who will be the winner of the game? why?

Maximin $= \underline{V} = -1$

Minimax $= \overline{V} = -1$

. So the game is with saddle point since $\underline{V} = \overline{V}$.

(i) Value of the game $= -1$.

(ii) Since the value of the game is negative, Y wins the game.

**Example**: For what value of $\lambda$, the game with following payoff matrix is strictly determinable?

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array}.$$

**Soln**: Take $\lambda$ as neither maximin nor minimax.

$$\text{Maximin} = \underline{v} = \text{Max}\{2, -7, -2\} = 2$$

$$\text{Minimax} = \bar{v} = \text{Min}\{-1, 6, 2\} = -1$$

$\therefore$ $\lambda$ will be a saddle point only when $\lambda \geq -1$ and $\lambda \leq 2$. ie $\boxed{-1 \leq \lambda \leq 2}$

Example (game without saddle point)

$$\begin{array}{cc} & \text{player 2} \\ \text{Player 1} & \begin{bmatrix} -2^* & -1 \uparrow \\ 2 \uparrow & -3^* \end{bmatrix} \end{array}.$$

Here $\underline{v} \neq \bar{v}$. Game without saddle point.

$$S_{\text{player 1}} = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ where } p_1 = \frac{-3-2}{(-2+-3)-(2+-1)}$$

$$= \frac{-5}{-6} = 5/6$$

$\therefore$ $p_2 = 1 - p_1 = 1 - 5/6 = 1/6$ $\therefore$ $S_{\text{player 1}} = \begin{bmatrix} A_1 & A_2 \\ 5/6 & 1/6 \end{bmatrix}$

Similarly $S_{\text{player 2}} = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ 1/3 & 2/3 \end{bmatrix}$. Value of game $V = \frac{-2 \times -3 - 2 \times -1}{-6}$

$q_1 = \frac{(-3--1)}{-6} = 1/3$

$= -4/3$ //

Let $q_1, q_2, ...q_n$ be the probabilities with which player B will chose the moves $B_1, B_2, ...B_n$respectively then $q_j \geq 0$ for every $1 \leq j \leq n$ and $q_1 + q_2 + ... + q_n = 1$.

There fore, the expected pay off function for player A is given by, $E(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$ where p and q denote the mixed strategies for player A and B.

**Theorem 1.1** *For any 2x2 two person zero sum game without any saddle point having the pay off matrix for player A is*

$$Player A \begin{pmatrix} & Player B & \\ a_{11} & a_{12} & \\ a_{21} & a_{22} & \end{pmatrix}$$

*then the optimum mixed strategies of A and B are*

$$S_A = \begin{pmatrix} A_1 & A_2 & . & . & A_m \\ p_1 & p_2 & . & . & p_m \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & . & . & B_n \\ q_1 & q_2 & . & . & q_n \end{pmatrix}$$

*where* $p_1 + p_2 = 1$ *and* $q_1 + q_2 = 1$ ;$p_1 = \dfrac{a_{22} - a_{a21}}{(a_{11} + a_{22}) - (a_{12} + a_{a_{21}})}$ ;

$q_1$

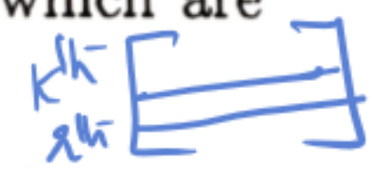
$q_1 = \dfrac{a_{22} - a_{a12}}{(a_{11} + a_{22}) - (a_{12} + a_{a_{21}})}$

$q_1$

*and value of the game* $v = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{a_{21}})}$ ;

# 2 DOMINANCE RULE

In this property we can reduce the size of the pay off matrix corresponding to the 2x2 two person zero sum game by deleting the strategies which are dominated by the others.




ie; if each element in one row[say, $k^{th}$] of the pay off matrix $[a_{ij}]$ is less than or equal to the corresponding element in the $r^{th}$ row say. Then player A will never choose $k^{th}$ strategy.

ie; the probability to choose the $k^{th}$ strategy is $p_k = 0$ if $a_{kj} \leq a_{rj}$ for every $1 \leq j \leq n$.

If we are deleting the $k^{th}$ row from the pay off matrix then the value of the game and the non zero choice of probabilities remains unchanged.

### Rules for Dominance

Step (i): If all the elements of $k^{th}$ row are less than the corresponding elements in any other row say $r^{th}$ then $k^{th}$ row is dominated by $r^{th}$.

Step(ii):If all the elements of $k^{th}$ column are greater than the corresponding elements in any other column say $r^{th}$ then $k^{th}$ column is dominated by $r^{th}$.

Step(iii): Dominated rows or columns to be deleted to reduce the size of the pay off matrix then the optimum strategies will remains unchanged.

**Note:** If the average of some rows or columns dominates some other rows or column say, $i^{th}$ then the $i^{th}$ row or column will be deleted.

Qn) Solve the following game:

$$\text{Player A} \quad \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 10 & 5 & -2^* \\ A_2 & 13 & 12^* & 15^\uparrow \\ A_3 & 16^\uparrow & 14^\uparrow & 10^* \end{array}$$

Soln: The given game is without saddle point.

Now we have to reduce the size of the matrix by applying dominance rule.

Every element in first row is less than the corresponding elements in 2nd row, 1st row is dominated by 2nd row. So delete 1st row, we get

$$\begin{array}{c} \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 13 & 12 & 15 \\ 16 & 14 & 10 \end{bmatrix} \end{array}$$

Now every element in column 1 is greater than corresponding elements in column 2. So column 2 dominates column 1 and delete column 1, we get

$$\begin{array}{c} \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{bmatrix} 12^* & 15^\uparrow \\ 14^\uparrow & 10^* \end{bmatrix} \end{array}$$

$\therefore$ Let $S_A = \begin{bmatrix} A_2 & A_3 \\ p_2 & p_3 \end{bmatrix}$, $S_B = \begin{bmatrix} B_2 & B_3 \\ q_2 & q_3 \end{bmatrix}$

Now $p_2 = \dfrac{10-14}{(12+10)-(14+15)} = \dfrac{-4}{-7} = 4/7 \Rightarrow p_3 = 1 - \dfrac{4}{7} = \dfrac{3}{7}$

$q_2 = \dfrac{(10-15)}{-7} = 5/7 \Rightarrow q_3 = 1 - \dfrac{5}{7} = \dfrac{2}{7}$.

Value of the game $v = \dfrac{12 \times 10 - 14 \times 15}{-7} = \dfrac{-90}{-7} = 90/7$

Solution of the game is

$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 4/7 & 3/7 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 5/7 & 2/7 \end{bmatrix}$

$v = \dfrac{90}{7}$

Q2) Solve

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 1^* & 7^\uparrow & 3 & 4 \\ A_2 & 5 & 6 & 4^{*\uparrow} & 5^\uparrow \\ A_3 & 7^\uparrow & 2 & 0^* & 3 \end{array}$$

. Game with saddle point

Soln: Optimum strategy for A is $A_2$ and for B is $B_3$,

Value of the game $= \underline{\underline{4}}$

Q3) Solve the game:

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 3 & 2 & 4^\uparrow & 0^* \\ A_2 & 3 & 4^\uparrow & 2^* & 4 \\ A_3 & 4^\uparrow & 2 & 4^\uparrow & 0^* \\ A_4 & 0^* & 4^\uparrow & 0^* & 8^\uparrow \end{array}$$

Soln: It is a game without saddle point. Now apply Dominance rule.

The first row is dominated by 3rd row. So delete 1st row.

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_2 & 3 & 4 & 2 & 4 \\ A_3 & 4 & 2 & 4 & 0 \\ A_4 & 0 & 4 & 0 & 8 \end{array}$$

. Now $B_1$ dominates $B_3$ and delete $B_1$, we get

$$\begin{array}{c|ccc} & B_2 & B_3 & B_4 \\ \hline A_2 & 4 & 2 & 4 \\ A_3 & 2 & 4 & 0 \\ A_4 & 4 & 0 & 8 \end{array}$$

. Now note that none of the pure strategies of player A and B is inferior to any of their other strategies. However average of payoff of $B_3$ and $B_4$ $\left\{\frac{2+4}{2}=3, \frac{4+0}{2}=2, \frac{0+8}{2}=4\right\}$ is superior to the payoff due to $B_2$ of player B.

So $B_2$ may be deleted and we get

$$\begin{array}{c|cc} & B_3 & B_4 \\ \hline A_2 & 2 & 4 \\ A_3 & 4 & 0 \\ A_4 & 0 & 8 \end{array}$$

. Average of $A_3$ and $A_4$ is same as the payoff of $A_2$. Hence player A will gain the same amount even if $A_2$ is never used. So we can delete $A_2$, and we get

$$\begin{array}{c|cc} & B_3 & B_4 \\ \hline A_3 & 4 & 0 \\ A_4 & 0 & 8 \end{array}$$

. $S_A = \begin{bmatrix} A_3 & A_4 \\ p_1 & p_2 \end{bmatrix}$, $S_B = \begin{bmatrix} B_3 & B_4 \\ q_1 & q_2 \end{bmatrix}$ where

$$p_1 = \frac{8-0}{(4+8)-(0+0)} = \frac{8}{12} = \frac{2}{3} \Rightarrow p_2 = \frac{1}{3}. \quad q_1 = \frac{8-0}{12} = 2/3, \; q_2 = 1-2/3 = 1/3$$

value of the game $v = \frac{32-0}{12} = \frac{32}{12} = \underline{\underline{\frac{8}{3}}}$. Soln is

$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$; value of the game $v = \frac{8}{3}$

Hw 1) Solve the following game:

Union Strategy

| Company Strategy | $U_1$ | $U_2$ | $U_3$ | $U_4$ |
|---|---|---|---|---|
| $C_1$ | 0.25 | 0.27 | 0.35 | 0.02 |
| $C_2$ | 0.2 | 0.06 | 0.08 | 0.08 |
| $C_3$ | 0.14 | 0.12 | 0.05 | 0.03 |
| $C_4$ | 0.3 | 0.14 | 0.19 | 0 |

2)

Player B.

| Player A | | | |
|---|---|---|---|
| | 30 | 40 | -80 |
| | 0 | 15 | -20 |
| | 90 | 20 | 50 |

Solution of game using LPP

$$\begin{array}{c|cccc} & B_1 & B_2 & \cdots & B_n \\ \hline A_1 & a_{11} & & & \\ A_2 & a_{21} & & & \\ \vdots & & & & \\ A_m & a_{m1} & & & \end{array}$$

To illustrate the transformation of a game problem to a linear programming problem, consider a payoff matrix of size $m \times n$. Let $a_{ij}$ be the element in the $i$th row and $j$th column of game payoff matrix, and letting $p_i$ be the probabilities of $m$ strategies ($i = 1, 2, \ldots, m$) for player $A$. Then, the expected gains for player $A$, for each of player $B$'s strategies will be:

$$\boxed{V = \sum_{i=1}^{m} p_i\, a_{ij},} \quad j = 1, 2, \ldots, n$$

The aim of player $A$ is to select a set of strategies with probability $p_i$ ($i = 1, 2, \ldots, m$) on any play of game such that he can maximize his minimum expected gains.

Now to obtain values of probability $p_i$, the value of the game to player $A$ for all strategies by player $B$ must be at least equal to $V$. Thus to maximize the minimum expected gains, it is necessary that:

$$\begin{aligned} a_{11}\, p_1 + a_{21}\, p_2 + \ldots + a_{m1}\, p_m &\geq V \\ a_{12}\, p_1 + a_{22}\, p_2 + \ldots + a_{m2}\, p_m &\geq V \\ \vdots \qquad\qquad\qquad\qquad & \vdots \\ a_{1n}\, p_1 + a_{2n}\, p_2 + \ldots + a_{mn}\, p_m &\geq V \end{aligned}$$

$$a_{11}\frac{p_1}{V} + a_{21}\frac{p_2}{V} + \cdots + a_{m1}\frac{p_m}{V} \geq 1$$

where $\quad p_1 + p_2 + \ldots + p_m = 1$, and $p_i \geq 0$ for all $i$.

Dividing both sides of the $m$ inequalities and equation by $V$ the division is valid as long as $V > 0$. In case $V < 0$, the direction of inequality constraints must be reversed. But if $V = 0$, the division would be meaningless. In this case a constant can be added to all entries of the matrix, ensuring that the value of the game ($V$) for the revised matrix becomes more than zero. After optimal solution is obtained, the true value of the game is obtained by substracting the same constant value. Let $p_i/V = x_i$, ($\geq 0$), we then have

$$\begin{aligned} a_{11}\frac{p_1}{V} + a_{21}\frac{p_2}{V} + \ldots + a_{m1}\frac{p_m}{V} &\geq 1 \\ a_{12}\frac{p_1}{V} + a_{22}\frac{p_2}{V} + \ldots + a_{m2}\frac{p_m}{V} &\geq 1 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \\ a_{1n}\frac{p_1}{V} + a_{2n}\frac{p_2}{V} + \ldots + a_{mn}\frac{p_m}{V} &\geq 1 \\ \frac{p_1}{V} + \frac{p_2}{V} + \ldots + \frac{p_m}{V} &= 1 \end{aligned}$$

$$\left.\begin{aligned} a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m &\geq 1 \\ \vdots \qquad\qquad\qquad & \vdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m &\geq 1 \\ x_1 + x_2 + \cdots + x_m &= 1. \end{aligned}\right\} \text{Con-straints}$$

Since the objective of player $A$ is to maximize the value of the game, $V$, which is equivalent to minimizing $1/V$, the resulting linear programming problem can be stated as:

Minimize $Z_p\ (= 1/V) = x_1 + x_2 + \ldots + x_m$

subject to the constraints

$$\begin{aligned} a_{11}\, x_1 + a_{21}\, x_2 + \ldots + a_{m1}\, x_m &\geq 1 \\ a_{12}\, x_1 + a_{22}\, x_2 + \ldots + a_{m2}\, x_m &\geq 1 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \\ a_{1n}\, x_1 + a_{2n}\, x_2 + \ldots + a_{mn}\, x_m &\geq 1 \end{aligned}$$

and $\quad x_1, x_2, \ldots, x_m \geq 0$

where $\quad x_i = p_i/V \geq 0;\ i = 1, 2, \ldots, m$

Similarly, Player $B$ has a similar problem with the inequalities of the constraints reversed, i.e. minimize the expected loss. Since minimizing $V$ is equivalent to maximizing $1/V$, therefore, the resulting linear programming problem can be stated as:

Maximize $Z_q\ (= 1/V) = y_1 + y_2 + \ldots + y_n$

subject to the constraints

$$\begin{aligned} a_{11}\, y_1 + a_{12}\, y_2 + \ldots + a_{1n}\, y_n &\leq 1 \\ a_{21}\, y_1 + a_{22}\, y_2 + \ldots + a_{2n}\, y_n &\leq 1 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \\ a_{m1}\, y_1 + a_{m2}\, y_2 + \ldots + a_{mn}\, y_n &\leq 1 \end{aligned}$$

$y_1, y_2, \ldots, y_n \geq 0$

where, $\quad y_j = q_j/V \geq 0;\quad j = 1, 2, \ldots, n$

It may be noted that the LP problem for player $B$ is the dual of LP problem for player $A$ and vice versa. Therefore, the solution of the dual problem can be obtained from the primal simplex table. Since for

Since for

both the players $Z_p = Z_q$, the expected gain to player $A$ in the game will be exactly equal to expected loss to player $B$.

Imp:

**Remark** Linear programming technique requires all variables to be non-negative and therefore to obtain a non-negative of value $V$ of the game, the data to the problem, i.e. $a_{ij}$ in the payoff table should all be non-negative. If there are some negative elements in the payoff table, a constant to every element in the payoff table must be added so as to make the smallest element zero; the solution to this new game will give an optimal mixed strategy for the original game. The value of the original game then equals the value of the new game minus the constant.

Q1) For the following payoff matrix, transform the given game into an equivalent LPP and solve it by Simplex method:

| Player A \ Player B | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $A_1$ | 1 | $-1^{*}$ | $3^{\uparrow}$ |
| $A_2$ | 3 | $5^{\uparrow}$ | $-3^{*}$ |
| $A_3$ | $6^{\uparrow}$ | 2 | $-2^{*}$ |

Given game is without saddle point.

Soln: Since some payoff elements are negative, convert it into a payoff matrix having non negative value. For this purpose add a constant 4, to all the elements, we get the new payoff matrix as,

| Player A \ Player B | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $A_1$ | 5 | 3 | 7 |
| $A_2$ | 7 | 9 | 1 |
| $A_3$ | 10 | 6 | 2 |

Let $p_i$ $(i=1,2,3)$ and $q_j$ $(j=1,2,3)$ be the probabilities of selecting strategies $A_i$ and $B_j$ respectively.

The expected gain for player A will be as follows:

$$5p_1+7p_2+10p_3 \geq V \quad (\text{If B uses the strategy } B_1)$$
$$3p_1+9p_2+6p_3 \geq V \quad (\text{" " " " } B_2)$$
$$7p_1+p_2+2p_3 \geq V \quad (\text{" " " " } B_3)$$
$$p_1+p_2+p_3 = 1, \quad p_1, p_2, p_3 \geq 0.$$

$$\Rightarrow \quad 5x_1+7x_2+10x_3 \geq 1; \quad 3x_1+9x_2+6x_3 \geq 1; \quad 7x_1+x_2+2x_3 \geq 1;$$
$$x_1, x_2, x_3 \geq 0$$

The problem for player A becomes

$$\text{Min } Z_p \left(= \frac{1}{V}\right) = x_1+x_2+x_3$$

Subject to $5x_1+7x_2+10x_3 \geq 1$; $3x_1+9x_2+6x_3 \geq 1$; $7x_1+x_2+2x_3 \geq 1$;
$x_1, x_2, x_3 \geq 0$. where $x_1 = \frac{p_1}{V}$, $x_2 = \frac{p_2}{V}$, $x_3 = \frac{p_3}{V}$.

Similarly for player B, the problem becomes

$$\text{Max } Z_q = \left(\frac{1}{V}\right) = y_1+y_2+y_3$$

Subject to $5y_1+3y_2+7y_3 \leq 1$; $7y_1+9y_2+y_3 \leq 1$; $10y_1+6y_2+2y_3 \leq 1$
$y_1, y_2, y_3 \geq 0$ where $y_1 = \frac{q_1}{V}$, $y_2 = \frac{q_2}{V}$, $y_3 = \frac{q_3}{V}$

Note that the problem of player A is the dual of problem of player B.

To solve problem of B: (dual simplex method)

$$\text{Max } Z_q \left(= \frac{1}{V}\right) = y_1 + y_2 + y_3 + 0S_1 + 0S_2 + 0S_3$$

Subject to

$$5y_1 + 3y_2 + 7y_3 + S_1 + 0S_2 + 0S_3 = 1.$$
$$7y_1 + 9y_2 + y_3 + 0S_1 + S_2 + 0S_3 = 1.$$
$$10y_1 + 6y_2 + 2y_3 + 0S_1 + 0S_2 + S_3 = 1; \quad y_1, y_2, y_3, S_1, S_2, S_3 \geq 0.$$

Standard form.

| | | $c_j$ | 1 | 1 | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis $B$ | Solution $y_B(=b)$ | $y_1$ | $y_2$ | $y_3$ | $S_1$ | $S_2$ | $S_3$ | Min ratio $\theta$ |
| 0 | $S_1$ | 1 | 5 | 3 | 7 | 1 | 0 | 0 | 1/5 |
| 0 | $S_2$ | 1 | 7 | 9 | 1 | 0 | 1 | 0 | 1/7 |
| 0 | $S_3$ | 1 | 10 | 6 | 2 | 0 | 0 | 1 | 1/10 |
| $Z=0$ | | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | |
| | | $c_j - Z_j$ | 1 | 1 | 1 | 0 | 0 | 0 | |

Proceeding with the usual simplex method, the optimal solution is shown below:

| | | $c_j$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis $B$ | $y_B(=b)$ | $y_1$ | $y_2$ | $y_3$ | $S_1$ | $S_2$ | $S_3$ |
| 1 | $y_3$ | 1/10 | 2/5 | 0 | 1 | 3/20 | -1/10 | 0 |
| 1 | $y_2$ | 1/10 | 11/15 | 1 | 0 | -1/60 | 7/60 | |
| 0 | $S_3$ | 1/5 | 24/5 | 0 | 0 | -1/5 | -3/5 | 1 |
| $Z = 1/5$ | | $Z_j$ | 17/15 | 0 | 0 | 2/15 | 1/5 | 0 |
| | | $c_j - Z_j$ | -2/15 | 0 | 0 | -2/15 | -1/15 | 0 |

The optimal solution for B is $y_1 = 0$; $y_2 = \frac{1}{10}$; $y_3 = \frac{1}{10}$ and the expected value of the game is $Z = \frac{1}{V} - \text{Constant} = 5 - 4 = 1$.

The solution values are now converted back into the original variables: If $\frac{1}{V} = \frac{1}{5}$, then $V = 5$

$y_1 = \frac{q_1}{V} \Rightarrow q_1 = y_1 * V = 0 \times V = 0$

$q_2 = y_2 \times V = \frac{1}{10} \times 5 = \frac{1}{2}$

$q_3 = y_3 \times V = \frac{1}{10} \times 5 = \frac{1}{2}$.

The optimal strategies for player A are obtained from the $c_j - z_j$ row of the optimal table.

ie $x_1 = \frac{2}{15}$ ; $x_2 = \frac{1}{15}$ ; $x_3 = 0$.

$p_1 = x_1 \times V = \frac{2}{15} \times 5 = \frac{2}{3}$, $p_2 = x_2 \times V = \frac{1}{15} \times 5 = \frac{1}{3}$, $p_3 = 0$.

Hence the solution is

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 2/3 & 1/3 & 0 \end{pmatrix} ; \quad S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Value of the game $= 5 - 4 = \underline{\underline{1}}$