Notes on Matlab 2D Boussinesq implementation

Jeffrey J. Early

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The 2D Boussinesq equations are

$$\nabla^2\psi_t + J\left(\psi, \nabla^2\psi\right) = \frac{g}{\rho_0}\rho_x + \nu Q(k)\psi_{xxxx} + \nu_z Q(m)\psi_{zzzz} \tag{1a}$$

$$\rho_t + J\left(\psi, \rho\right) + \psi_x \bar{\rho}_z = 0 \tag{1b}$$

where $J(a,b) \equiv a_x b_z - a_z b_x$. The linear solution written as a stream function is

$$\psi(x,z,t) = Uh\cos\theta G(z) \tag{2}$$

$$b(x,z,t) = -UN^2\frac{kh}{\omega}\cos\theta G(z) \tag{3}$$

Note we need to add damping on the density to match the damping that occurs on w. We should write this in terms of buoyancy, $b(x,y,z,t) \equiv -\frac{g}{\rho_0}\rho$.

$$\rho_t + \psi_x\rho_z - \psi_z\rho_x + \psi_x\bar{\rho}_z = 0 \tag{4a}$$

$$b_t + \psi_x b_z - \psi_z b_x + \psi_x N^2 = 0 \tag{4b}$$

$$b_t = \psi_z b_x - \psi_x(N^2 + b_z) \tag{4c}$$

So our equations to model are,

$$\nabla^2\psi_t = -\psi_x\nabla^2\psi_z + \psi_z\nabla^2\psi_x - b_x \tag{5a}$$

$$b_t = \psi_z b_x - \psi_x(N^2 + b_z) \tag{5b}$$

Using that,

$$\psi(x,z) = \psi_{km}e^{ikx}\sin mz \tag{6}$$

$$b(x,z) = b_{km}e^{ikx}\sin mz \tag{7}$$

$$\tag{8}$$

the derivatives of ψ are,

$$\psi_x = ik\psi_{km}e^{ikx}\sin mz \tag{9}$$
$$\psi_z = m\psi_{km}e^{ikx}\cos mz \tag{10}$$
$$\nabla^2\psi = -(k^2+m^2)\psi_{km}e^{ikx}\sin mz \tag{11}$$
$$\nabla^2\psi_x = -ik(k^2+m^2)\psi_{km}e^{ikx}\sin mz \tag{12}$$
$$\nabla^2\psi_z = -m(k^2+m^2)\psi_{km}e^{ikx}\cos mz \tag{13}$$
$$\psi_{xxxx} = k^4 e^{ikx}\sin mz \tag{14}$$
$$\psi_{zzzz} = m^4 e^{ikx}\sin mz \tag{15}$$

and the derivatives of b are,

$$b_x = ikb_{km}e^{ikx}\sin mz \tag{16}$$
$$b_z = mb_{km}e^{ikx}\cos mz \tag{17}$$

1 Damping

We have to get the damping right so that the buoyancy gets damped at a rate consistent with velocity.

$$u_t = -\frac{1}{\rho_0}p_x - \nu_x u_{xx} - \nu_z u_{zz} \tag{18a}$$
$$w_t = -\frac{1}{\rho_0}p_z - g\frac{\rho}{\rho_0} - \nu_x w_{xx} - \nu_z w_{zz} \tag{18b}$$
$$u_x + w_z = 0 \tag{18c}$$
$$\rho_t + w\bar{\rho}_z = 0 \tag{18d}$$