Damping in the Winter's Model

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1 Damping time scale

Using the terminology of Kraig Winters we need to define a reasonable coefficient for damping.

Given that,

$$\frac{\partial u}{\partial t} = \nu_n \nabla^{2n} u \tag{1}$$

is spectrally,

$$\frac{\partial u}{\partial t} = \nu_n (ik2\pi)^{2n} u \tag{2}$$

which has the solution

$$u = e^{\nu_n (ik2\pi)^{2n} t}. \tag{3}$$

We want to convert this into an e-fold time, so we want

$$e^{-t/T} = e^{\nu_n (ik2\pi)^{2n} t}. \tag{4}$$

Using that $k = \frac{1}{2\Delta}$ where Δ is the sample interval and solving for ν in terms of the other variables,

$$\nu_n = \frac{(-1)^{n+1}}{T} \left(\frac{\Delta}{\pi} \right)^{2n} \tag{5}$$

1.1 Reynolds number

The usual approach to setting a damping scale is to require that the Reynolds number is 1 at the grid scale. So, $\frac{U\Delta}{\nu_1} = 1$. Of course, this is not the hyperviscous coefficient, but the regular viscosity coefficient, i.e., ν_1 in our notation.

How does this hyperviscous ν_n compare to the usual ν_1?

$$\nu_n (k2\pi)^{2n} u = \nu_1 (k2\pi)^2 u \tag{6}$$

$$\nu_1 = \nu_n (k2\pi)^{2n-2} \tag{7}$$

$$\nu_1 = \frac{1}{T} \left(\frac{\Delta}{\pi}\right)^2 \tag{8}$$

where we used equation 5. Now use the fact that we want $Re = 1$,

$$\frac{U\Delta}{\nu_1} = 1 \tag{9}$$

$$\frac{(-1)^{n+1}}{U\Delta} \left(\frac{\Delta}{\pi}\right)^2 = T \tag{10}$$

Note that the dependence on the order of the hyperviscous operator, n, is gone. This makes sense because we're simply setting the time scale at which the highest wavenumber decays—and we want it to decay at the same rate for ALL case. It's only the other wavenumbers that differ.

1.2 Hyper-Reynolds number

An alternative way of looking at this is to create a 'hyper-Reynolds number'. So,

$$\frac{\partial u}{\partial t} + u u_x = \nu_n \nabla^{2n} u \tag{11}$$

would lead to a Reynolds number that looks like,

$$Re = \frac{U L^{2n}}{L \nu_n} \tag{12}$$

Requiring this to be viscous at the grid scale is,

$$\nu_n = U \Delta^{2n-1} \tag{13}$$

Now use this in equation 5 to find the time scale,

$$U \Delta^{2n-1} = \frac{(-1)^{n+1}}{T} \left(\frac{\Delta}{\pi}\right)^{2n} \tag{14}$$

and, other than some factors of π we get that,

$$T = \frac{\Delta}{U}. \tag{15}$$

2 With forcing

Now add some forcing,

$$\frac{\partial \hat{u}}{\partial t} = \hat{F} + \nu(ik2\pi)^{2n}\hat{u} \tag{16}$$

solution

$$\hat{u} = u_0 e^{\nu(-1)^n(k2\pi)^{2n}t} - (-1)^n \frac{F}{\nu(k2\pi)^{2n}} \tag{17}$$

Then, in steady state,

$$\hat{u}\hat{u}^* = \frac{F^2}{\nu^2}(2\pi k)^{-4n} \tag{18}$$

- We can compute the wavenumber at which damping drops the amplitude more than 50 percent during the length of the simulation.
- We can compute, given U, the cfl criteria, and then ask that the Reynolds number be one at the grid scale.

3 Decorrelation time

$$\begin{aligned}
R(\tau \geq 0) &= \int_0^\infty u(t)u(t+\tau)\,dt && (19)\\
&= \int_0^\infty e^{\nu(ik2\pi)^{2n}t} e^{\nu(ik2\pi)^{2n}(t+\tau)}\,dt && (20)\\
&= \left.\frac{e^{\nu(ik2\pi)^{2n}(2t+\tau)}}{2\nu(ik2\pi)^{2n}}\right|_0^\infty && (21)
\end{aligned}$$

So what is $\nu(k2\pi)^{2n}$?

$$\begin{aligned}
\nu(ik2\pi)^{2n} &= \frac{(-1)^{n+1}}{T}\left(\frac{\Delta}{\pi}\right)^{2n}(ik2\pi)^{2n} && (22)\\
&= -\frac{(k2\Delta)^{2n}}{T} && (23)
\end{aligned}$$

So then,

$$R(\tau) = -\frac{T}{2(k2\Delta)^{2n}} e^{-(k2\Delta)^{2n}\frac{\tau}{T}} \tag{24}$$

If we normalize by the total variance of the integrated velocity, then we have that,

$$R(\tau) = e^{-(k2\Delta)^{2n}\frac{\tau}{T}} \tag{25}$$

So how long does it take for each wavenumber to decay to $\epsilon = 0.5$?

$$\ln\epsilon = -(k2\Delta)^{2n}\frac{\tau}{T} \tag{26}$$

$$\tau = \frac{T\ln\epsilon}{-(k2\Delta)^{2n}} \tag{27}$$

I also want to know which wavenumbers take a certain amount of time to decay.

$$(k2\Delta)^{2n} = -\frac{T\ln\epsilon}{\tau} \tag{28}$$

$$k = \frac{1}{2\Delta}\left(-\frac{T\ln\epsilon}{\tau}\right)^{\frac{1}{2n}} \tag{29}$$

In terms of mode number,

$$\frac{j}{2N\Delta} = \frac{1}{2\Delta}\left(-\frac{T\ln\epsilon}{\tau}\right)^{\frac{1}{2n}} \tag{30}$$

$$j = N\left(-\frac{T\ln\epsilon}{\tau}\right)^{\frac{1}{2n}} \tag{31}$$

4 2D Decorrelation time

In two-dimensions the damping solution is,

$$u = e^{(\lambda_x+\lambda_z)t}. \tag{32}$$

So,

$$R(\tau \geq 0) = \int_0^\infty u(t)u(t+\tau)\,dt \tag{33}$$

$$= \int_0^\infty e^{(\lambda_x+\lambda_z)t}e^{(\lambda_x+\lambda_z)(t+\tau)}\,dt \tag{34}$$

$$= \frac{e^{(\lambda_x+\lambda_z)(2t+\tau)}}{2(\lambda_x+\lambda_z)}\bigg|_0^\infty \tag{35}$$

$$= \frac{e^{(\lambda_x+\lambda_z)\tau}}{2(\lambda_x+\lambda_z)} \tag{36}$$

Normalized, this is quite simply,

$$R(\tau \geq 0) = e^{(\lambda_x+\lambda_z)\tau} \tag{37}$$