Notes on Matlab 2D Boussinesq implementation

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The 2D Boussinesq equations are The equations of motion for fluid velocity $u(x,z,t)$ and $w(x,z,t)$ are given by,

$$u_t + uu_x + wu_z = -\frac{1}{\rho_0} p_x \tag{1a}$$
$$w_t + uw_x + ww_z = -\frac{1}{\rho_0} p_z - g\frac{\rho}{\rho_0} \tag{1b}$$
$$u_x + w_z = 0 \tag{1c}$$
$$\rho_t + u\rho_x + w\rho_z + w\bar{\rho}_z = 0 \tag{1d}$$

We should write this in terms of buoyancy, $b(x,y,z,t) \equiv -\frac{g}{\rho_0}\rho$,

$$u_t + uu_x + wu_z = -\frac{1}{\rho_0} p_x \tag{2a}$$
$$w_t + uw_x + ww_z = -\frac{1}{\rho_0} p_z + b \tag{2b}$$
$$u_x + w_z = 0 \tag{2c}$$
$$b_t + ub_x + wb_z + wN^2 = 0 \tag{2d}$$

For mathematical convenience we define a stream function $u = \psi_z$ and $w = -\psi_x$ that automatically enforces the continuity. With this we have that,

$$\psi_{zt} + \psi_z\psi_{xz} - \psi_x\psi_{zz} = -\frac{1}{\rho_0} p_x \tag{3a}$$
$$-\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz} = -\frac{1}{\rho_0} p_z + b \tag{3b}$$
$$b_t + ub_x + wb_z + wN^2 = 0 \tag{3c}$$

so then take the curl,

$$\begin{aligned} \psi_{zzt} + \psi_{xxt} + \psi_{zz}\psi_{xz} + \psi_z\psi_{xzz} - \psi_{xz}\psi_{zz} - \psi_x\psi_{zzz} \\ +\psi_{xz}\psi_{xx} + \psi_z\psi_{xxx} - \psi_{xx}\psi_{xz} - \psi_x\psi_{xxz} = -b_x \end{aligned} \tag{4a}$$
$$b_t + ub_x + wb_z + wN^2 = 0 \tag{4b}$$

which organizes as,

$$\begin{aligned}\nabla^2\psi_t + \psi_z\nabla^2\psi_x - \psi_x\nabla^2\psi_z &= -b_x \\ b_t + \psi_z b_x - \psi_x(b_z + N^2) &= 0 \qquad (5a)\end{aligned}$$

which is then,

$$\nabla^2\psi_t + J\left(\psi, \nabla^2\psi\right) = -\, b_x + \nu Q(k)\psi_{xxxx} + \nu_z Q(m)\psi_{zzzz} \tag{6a}$$

$$b_t + J\left(\psi, b\right) - \psi_x\bar{\rho}_z = 0 \tag{6b}$$

where $J(a, b) \equiv a_z b_x - a_x b_z$. The linear solution written as a stream function is

$$\psi(x, z, t) = Uh\cos\theta G(z) \tag{7}$$

$$b(x, z, t) = UN^2\frac{kh}{\omega}\cos\theta G(z) \tag{8}$$

Note we need to add damping on the density to match the damping that occurs on w.

So our equations to model are,

$$\nabla^2\psi_t = -\,\psi_z\nabla^2\psi_x + \psi_x\nabla^2\psi_z - b_x \tag{9a}$$

$$b_t = -\,\psi_z b_x + \psi_x(N^2 + b_z) \tag{9b}$$

Using that,

$$\psi(x, z) = \psi_{km}e^{ikx}\sin mz \tag{10}$$

$$b(x, z) = b_{km}e^{ikx}\sin mz \tag{11}$$

$$\tag{12}$$

the derivatives of ψ are,

$$\psi_x = ik\psi_{km}e^{ikx}\sin mz \tag{13}$$

$$\psi_z = m\psi_{km}e^{ikx}\cos mz \tag{14}$$

$$\nabla^2\psi = -\,(k^2 + m^2)\psi_{km}e^{ikx}\sin mz \tag{15}$$

$$\nabla^2\psi_x = -\,ik(k^2 + m^2)\psi_{km}e^{ikx}\sin mz \tag{16}$$

$$\nabla^2\psi_z = -\,m(k^2 + m^2)\psi_{km}e^{ikx}\cos mz \tag{17}$$

$$\psi_{xxxx} = k^4e^{ikx}\sin mz \tag{18}$$

$$\psi_{zzzz} = m^4e^{ikx}\sin mz \tag{19}$$

and the derivatives of b are,

$$b_x = ikb_{km}e^{ikx}\sin mz \tag{20}$$

$$b_z = mb_{km}e^{ikx}\cos mz \tag{21}$$

1 Damping

We have to get the damping right so that the buoyancy gets damped at a rate consistent with velocity.

$$u_t = -\frac{1}{\rho_0}p_x - \nu_x u_{xx} - \nu_z u_{zz} \tag{22a}$$

$$w_t = -\frac{1}{\rho_0}p_z - g\frac{\rho}{\rho_0} - \nu_x w_{xx} - \nu_z w_{zz} \tag{22b}$$

$$u_x + w_z = 0 \tag{22c}$$

$$\rho_t + w\bar{\rho}_z = 0 \tag{22d}$$