Damping in the Winter's Model

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Using the terminology of Kraig Winters we need to define a reasonable coefficient for damping.

Given that,

$$\frac{\partial u}{\partial t} = \nu \nabla^{2n} u \tag{1}$$

is spectrally,

$$\frac{\partial u}{\partial t} = \nu (ik2\pi)^{2n} u \tag{2}$$

which has the solution

$$u = e^{\nu (ik2\pi)^{2n} t}. \tag{3}$$

We want to convert this into an e-fold time, so we want

$$e^{-t/T} = e^{\nu (ik2\pi)^{2n} t}. \tag{4}$$

Using that $k = \frac{1}{2\Delta}$ where Δ is the sample interval and solving for ν in terms of the other variables,

$$\nu = \frac{(-1)^{n+1}}{T} \left(\frac{\Delta}{\pi} \right)^{2n} \tag{5}$$

Now add some forcing,

$$\frac{\partial \hat{u}}{\partial t} = \hat{F} + \nu (ik2\pi)^{2n} \hat{u} \tag{6}$$

solution

$$\hat{u} = u_0 e^{\nu (-1)^n (k2\pi)^{2n} t} - (-1)^n \frac{F}{\nu (k2\pi)^{2n}} \tag{7}$$

Then, in steady state,

$$\hat{u}\hat{u}^* = \frac{F^2}{\nu^2} (2\pi k)^{-4n} \tag{8}$$

- We can compute the wavenumber at which damping drops the amplitude more than 50 percent during the length of the simulation.

- We can compute, given U, the cfl criteria, and then ask that the Reynolds number be one at the grid scale.

How does this hyperviscous ν compare to the usual ν_0?

$$\nu(k2\pi)^{2n}u = \nu_0(k2\pi)^2 u \tag{9}$$
$$\nu_0 = \nu(k2\pi)^{2n-2} \tag{10}$$
$$\nu_0 = \frac{1}{T}\left(\frac{\Delta}{\pi}\right)^2 \tag{11}$$

$$\frac{U\Delta}{\nu_0} = 1 \tag{12}$$
$$\frac{(-1)^{n+1}}{U\Delta}\left(\frac{\Delta}{\pi}\right)^2 = T \tag{13}$$

For one simulation, we have that $U = 0.0365$ m/s and $\Delta = 6750$ m with $n = 3$. This suggests a damping time scale of $T = 18000$s. The actually simulation used twice that.

The linear wave mode propagation speed seems to be the maximum velocity in these simulations. Wait. There's no surface, so that can't matter.

1 Decorrelation time

$$R(\tau \geq 0) = \int_0^\infty u(t)u(t+\tau)\,dt \tag{14}$$
$$= \int_0^\infty e^{\nu(ik2\pi)^{2n}t} e^{\nu(ik2\pi)^{2n}(t+\tau)}\,dt \tag{15}$$
$$= \left.\frac{e^{\nu(ik2\pi)^{2n}(2t+\tau)}}{2\nu(ik2\pi)^{2n}}\right|_0^\infty \tag{16}$$

So what is $\nu(k2\pi)^{2n}$?

$$\nu(ik2\pi)^{2n} = \frac{(-1)^{n+1}}{T}\left(\frac{\Delta}{\pi}\right)^{2n}(ik2\pi)^{2n} \tag{17}$$
$$= -\frac{(k2\Delta)^{2n}}{T} \tag{18}$$

So then,

$$R(\tau) = -\frac{T}{2(k2\Delta)^{2n}} e^{-(k2\Delta)^{2n}\frac{\tau}{T}} \tag{19}$$

If we normalize by the total variance of the integrated velocity, then we have that,

$$R(\tau) = e^{-(k2\Delta)^{2n}\frac{\tau}{T}} \tag{20}$$

So how long does it take for each wavenumber to decay to $\epsilon = 0.5$?

$$\ln\epsilon = -\,(k2\Delta)^{2n}\frac{\tau}{T} \tag{21}$$

$$\tau = \frac{T\ln\epsilon}{-(k2\Delta)^{2n}} \tag{22}$$

I also want to know which wavenumbers take a certain amount of time to decay.

$$(k2\Delta)^{2n} = -\frac{T\ln\epsilon}{\tau} \tag{23}$$

$$k = \frac{1}{2\Delta}\left(-\frac{T\ln\epsilon}{\tau}\right)^{\frac{1}{2n}} \tag{24}$$

In terms of mode number,

$$\frac{j}{2N\Delta} = \frac{1}{2\Delta}\left(-\frac{T\ln\epsilon}{\tau}\right)^{\frac{1}{2n}} \tag{25}$$

$$j = N\left(-\frac{T\ln\epsilon}{\tau}\right)^{\frac{1}{2n}} \tag{26}$$

2 2D Decorrelation time

In two-dimensions the damping solution is,

$$u = e^{(\lambda_x+\lambda_z)t}. \tag{27}$$

So,

$$R(\tau \geq 0) = \int_0^\infty u(t)u(t+\tau)\,dt \tag{28}$$

$$= \int_0^\infty e^{(\lambda_x+\lambda_z)t}e^{(\lambda_x+\lambda_z)(t+\tau)}\,dt \tag{29}$$

$$= \frac{e^{(\lambda_x+\lambda_z)(2t+\tau)}}{2(\lambda_x+\lambda_z)}\bigg|_0^\infty \tag{30}$$

$$= \frac{e^{(\lambda_x+\lambda_z)\tau}}{2(\lambda_x+\lambda_z)} \tag{31}$$

Normalized, this is quite simply,

$$R(\tau \geq 0) = e^{(\lambda_x+\lambda_z)\tau} \tag{32}$$

This should just be additive.

$$\begin{aligned}
R(\tau \geq 0) &= \int_0^\infty u(t)u(t+\tau)\,dt && (33)\\
&= \int_0^\infty \left[e^{\nu_x(ik2\pi)^{2n}t} + e^{\nu_z(im2\pi)^{2n}t}\right]\left[e^{\nu_x(ik2\pi)^{2n}(t+\tau)} + e^{\nu_z(im2\pi)^{2n}(t+\tau)}\right]\,dt && (34)\\
&= \frac{e^{\nu_x(ik2\pi)^{2n}(2t+\tau)}}{2\nu_x(ik2\pi)^{2n}}\Bigg|_0^\infty + \frac{e^{\nu_z(ik2\pi)^{2n}(2t+\tau)}}{2\nu_z(ik2\pi)^{2n}}\Bigg|_0^\infty && (35)\\
&\quad + \left[e^{\nu_x(ik2\pi)^{2n}\tau} + e^{\nu_z(im2\pi)^{2n}\tau}\right]\frac{e^{(\nu_x(ik2\pi)^{2n}+\nu_z(im2\pi)^{2n})t}}{\nu_x(ik2\pi)^{2n}+\nu_z(im2\pi)^{2n}}\Bigg|_0^\infty && (36)\\
&\frac{e^{\nu_x(ik2\pi)^{2n}\tau}}{2\nu_x(ik2\pi)^{2n}} + \frac{e^{\nu_z(ik2\pi)^{2n}\tau}}{2\nu_z(im2\pi)^{2n}} + \frac{e^{\nu_x(ik2\pi)^{2n}\tau} + e^{\nu_z(im2\pi)^{2n}\tau}}{\nu_x(ik2\pi)^{2n}+\nu_z(im2\pi)^{2n}} && (37)
\end{aligned}$$

Now we want to normalize this, so that it goes to 1 at lag 0. Let's simplify notation,

$$\begin{aligned}
R(\tau \geq 0) &= \frac{e^{\lambda_x\tau}}{2\lambda_x} + \frac{e^{\lambda_z\tau}}{2\lambda_z} + \frac{e^{\lambda_x\tau}+e^{\lambda_z\tau}}{\lambda_x+\lambda_z} && (38)\\
&= \frac{\lambda_z(\lambda_x+\lambda_z)e^{\lambda_x\tau} + \lambda_x(\lambda_x+\lambda_z)e^{\lambda_z\tau} + 2\lambda_x\lambda_z\left(e^{\lambda_x\tau}+e^{\lambda_z\tau}\right)}{2\lambda_x\lambda_z(\lambda_x+\lambda_z)} && (39)\\
R(0) &= \frac{\lambda_z(\lambda_x+\lambda_z) + \lambda_x(\lambda_x+\lambda_z) + 4\lambda_x\lambda_z}{2\lambda_x\lambda_z(\lambda_x+\lambda_z)} && (40)\\
&= \frac{\lambda_x^2+\lambda_z^2+6\lambda_x\lambda_z}{2\lambda_x\lambda_z(\lambda_x+\lambda_z)} && (41)\\
&= 1 && (42)
\end{aligned}$$

So, we have that,

$$R(\tau \geq 0) = \frac{\lambda_z^2 e^{\lambda_x\tau} + \lambda_x^2 e^{\lambda_z\tau} + 3\lambda_x\lambda_z\left(e^{\lambda_x\tau}+e^{\lambda_z\tau}\right)}{\lambda_x^2+\lambda_z^2+6\lambda_x\lambda_z} \tag{43}$$