

Damping in the Winter's Model

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May 9, 2018

Using the terminology of Kraig Winters we need to define a reasonable coefficient for damping.

Given that,

$$\frac{\partial u}{\partial t} = \nu \nabla^{2n} u \quad (1)$$

is spectrally,

$$\frac{\partial u}{\partial t} = \nu (ik2\pi)^{2n} u \quad (2)$$

which has the solution

$$u = e^{\nu(ik2\pi)^{2n}t}. \quad (3)$$

We want to convert this into an e-fold time, so we want

$$e^{-t/T} = e^{\nu(ik2\pi)^{2n}t}. \quad (4)$$

Using that $k = \frac{1}{2\Delta}$ where Δ is the sample interval and solving for ν in terms of the other variables,

$$\nu = \frac{(-1)^{n+1}}{T} \left(\frac{\Delta}{\pi} \right)^{2n} \quad (5)$$

Now add some forcing,

$$\frac{\partial \hat{u}}{\partial t} = \hat{F} + \nu(ik2\pi)^{2n} \hat{u} \quad (6)$$

solution

$$\hat{u} = u_0 e^{\nu(-1)^n(k2\pi)^{2n}t} - (-1)^n \frac{F}{\nu(k2\pi)^{2n}} \quad (7)$$

Then, in steady state,

$$\hat{u}\hat{u}^* = \frac{F^2}{\nu^2} (2\pi k)^{-4n} \quad (8)$$

- We can compute the wavenumber at which damping drops the amplitude more than 50 percent during the length of the simulation.

- We can compute, given U , the cfl criteria, and then ask that the Reynolds number be one at the grid scale.

How does this hyperviscous ν compare to the usual ν_0 ?

$$\nu(k2\pi)^{2n}u = \nu_0(k2\pi)^2u \quad (9)$$

$$\nu_0 = \nu(k2\pi)^{2n-2} \quad (10)$$

$$\nu_0 = \frac{1}{T} \left(\frac{\Delta}{\pi} \right)^2 \quad (11)$$

$$\frac{U\Delta}{\nu_0} = 1 \quad (12)$$

$$\frac{(-1)^{n+1}}{U\Delta} \left(\frac{\Delta}{\pi} \right)^2 = T \quad (13)$$

For one simulation, we have that $U = 0.0365$ m/s and $\Delta = 6750$ m with $n = 3$. This suggests a damping time scale of $T = 18000$ s. The actually simulation used twice that.

The linear wave mode propagation speed seems to be the maximum velocity in these simulations. Wait. There's no surface, so that can't matter.

1 Decorrelation time

$$R(\tau \geq 0) = \int_0^\infty u(t)u(t+\tau) dt \quad (14)$$

$$= \int_0^\infty e^{\nu(ik2\pi)^{2n}t} e^{\nu(ik2\pi)^{2n}(t+\tau)} dt \quad (15)$$

$$= \frac{e^{\nu(ik2\pi)^{2n}(2t+\tau)}}{2\nu(ik2\pi)^{2n}} \Big|_0^\infty \quad (16)$$

So what is $\nu(k2\pi)^{2n}$?

$$\nu(ik2\pi)^{2n} = \frac{(-1)^{n+1}}{T} \left(\frac{\Delta}{\pi} \right)^{2n} (ik2\pi)^{2n} \quad (17)$$

$$= - \frac{(k2\Delta)^{2n}}{T} \quad (18)$$

So then,

$$R(\tau) = - \frac{T}{2(k2\Delta)^{2n}} e^{-(k2\Delta)^{2n} \frac{\tau}{T}} \quad (19)$$

If we normalize by the total variance of the integrated velocity, then we have that,

$$R(\tau) = e^{-(k2\Delta)^{2n} \frac{\tau}{T}} \quad (20)$$

So how long does it take for each wavenumber to decay to $\epsilon = 0.5$?

$$\ln \epsilon = - (k2\Delta)^{2n} \frac{\tau}{T} \quad (21)$$

$$\tau = \frac{T \ln \epsilon}{-(k2\Delta)^{2n}} \quad (22)$$

I also want to know which wavenumbers take a certain amount of time to decay.

$$(k2\Delta)^{2n} = - \frac{T \ln \epsilon}{\tau} \quad (23)$$

$$k = \frac{1}{2\Delta} \left(- \frac{T \ln \epsilon}{\tau} \right)^{\frac{1}{2n}} \quad (24)$$

In terms of mode number,

$$\frac{j}{2N\Delta} = \frac{1}{2\Delta} \left(- \frac{T \ln \epsilon}{\tau} \right)^{\frac{1}{2n}} \quad (25)$$

$$j = N \left(- \frac{T \ln \epsilon}{\tau} \right)^{\frac{1}{2n}} \quad (26)$$

2 2D Decorrelation time

This should just be additive.

$$R(\tau \geq 0) = \int_0^\infty u(t)u(t+\tau) dt \quad (27)$$

$$= \int_0^\infty \left[e^{\nu_x(ik2\pi)^{2n}t} + e^{\nu_z(im2\pi)^{2n}t} \right] \left[e^{\nu_x(ik2\pi)^{2n}(t+\tau)} + e^{\nu_z(im2\pi)^{2n}(t+\tau)} \right] dt \quad (28)$$

$$= \frac{e^{\nu_x(ik2\pi)^{2n}(2t+\tau)}}{2\nu_x(ik2\pi)^{2n}} \Big|_0^\infty + \frac{e^{\nu_z(im2\pi)^{2n}(2t+\tau)}}{2\nu_z(im2\pi)^{2n}} \Big|_0^\infty \quad (29)$$

$$+ \left[e^{\nu_x(ik2\pi)^{2n}\tau} + e^{\nu_z(im2\pi)^{2n}\tau} \right] \frac{e^{(\nu_x(ik2\pi)^{2n} + \nu_z(im2\pi)^{2n})t}}{\nu_x(ik2\pi)^{2n} + \nu_z(im2\pi)^{2n}} \Big|_0^\infty \quad (30)$$

$$\frac{e^{\nu_x(ik2\pi)^{2n}\tau}}{2\nu_x(ik2\pi)^{2n}} + \frac{e^{\nu_z(im2\pi)^{2n}\tau}}{2\nu_z(im2\pi)^{2n}} + \frac{e^{\nu_x(ik2\pi)^{2n}\tau} + e^{\nu_z(im2\pi)^{2n}\tau}}{\nu_x(ik2\pi)^{2n} + \nu_z(im2\pi)^{2n}} \quad (31)$$