REVISE BTEC NATIONAL

Engineering

REVISION GUIDE

Series Consultant: Harry Smith

Authors: Andrew Buckenham, Kevin Medcalf, David Midgley and Neil Wooliscroft

A note from the publisher

While the publishers have made every attempt to ensure that advice on the qualification and its assessment is accurate, the official specification and associated assessment guidance materials are the only authoritative source of information and should always be referred to for definitive guidance.

This qualification is reviewed on a regular basis and may be updated in the future. Any such updates that affect the content of this Revision Guide will be outlined at **www.pearsonfe.co.uk/BTECchanges**. The eBook version of this Revision Guide will also be updated to reflect the latest guidance as soon as possible.

Pearson

Introduction

Which units should you revise?

This Revision Guide has been designed to support you in preparing for the externally assessed units of your course. Remember that you won't necessarily be studying all the units included here – it will depend on the qualification you are taking.

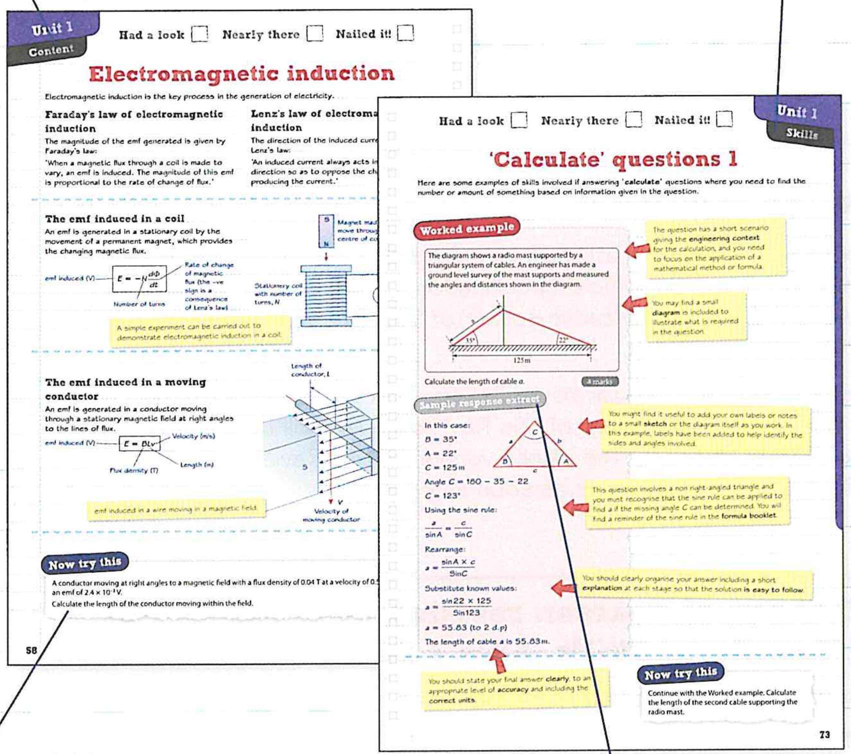
BTEC National Qualification	Externally assessed units
Certificate	1 Engineering Principles
For each of: Extended Certificate Foundation Diploma Diploma	1 Engineering Principles 3 Engineering Product Design and Manufacture
Extended Diploma	1 Engineering Principles 3 Engineering Product Design and Manufacture 6 Microcontroller Systems for Engineers

Your Revision Guide

Each unit in this Revision Guide contains two types of pages, shown below.

Content **pages** help you revise the essential content you need to know for each unit.

Skills **pages** help you prepare for your exam or assessed task. Skills pages have a coloured edge and are shaded in the table of contents.



Use the **Now try this** activities on every page to help you test your knowledge and practise the relevant skills.

Look out for the **sample response extracts** to exam questions or set tasks on the skills pages. Post-its will explain their strengths and weaknesses.

Contents

Workbook also available for externally assessed units
ISBN 9781292150277

Unit 1: Engineering Principles

Unit 3: Engineering Product Design and Manufacture

Unit 6: Microcontroller Systems for Engineers

A small bit of small print
Pearson publishes Sample Assessment Material and the Specification on its website. This is the official content and this book should be used in conjunction with it.
The questions in *Now try this* have been written to help you test your knowledge and skills. Remember: the real assessment may not look like this.

Laws of indices

The laws of indices make it possible to simplify and solve equations that contain indices.

Using the laws of indices

Numbers 1 to 3 below are the main laws of indices. Numbers 4 to 8 are useful applications and special cases of the main laws.

1. Multiplication $a^m \times a^n = a^{(m+n)}$
   e.g. $x^6 \times x^{-2} = x^4$
2. Division $a^m \div a^n = a^{(m-n)}$
   e.g. $x^5 \div x^3 = x^2$
3. Powers $(a^m)^n = a^{mn}$
   e.g. $(x^3)^5 = x^{15}$
4. Reciprocals $\frac{1}{a^m} = a^{-m}$
   e.g. $\frac{1}{x^5} = x^{-5}$
5. Index = 0 $a^0 = 1$
   e.g. $7^0 = 1$
6. Index = $\frac{1}{2}$ or 0.5 $a^{\frac{1}{2}} = a^{0.5} = \sqrt{a}$
   e.g. $16^{\frac{1}{2}} = 16^{0.5} = \sqrt{16} = 4$
7. Index = $\frac{1}{n}$ $a^{\frac{1}{n}} = \sqrt[n]{a}$
   e.g. $27^{-\frac{1}{3}} = \sqrt[3]{27} = 3$
8. Index = 1 $a^1 = a$
   e.g. $3.5^1 = 3.5$

Worked example

(a) Simplify the expression $a^{-2}b^{-1}a^{\frac{3}{2}}$.

Start at the beginning of the expression and look for terms that can be expressed as decimals to make simplification easier. e.g. $a^{\frac{3}{2}} = a^{-1.5}$.

$a^{-2}b^{-1}a^{\frac{3}{2}} = a^{-2}b^{-1}a^{1.5}$

Group together like terms.

$= a^{-2}a^{1.5}b^{-1}$

Add the indices for like terms.

$= a^{-0.5}b^{-1}$

Consider alternative ways to express the simplified expression.

$= \frac{1}{b\sqrt{a}}$

(b) Simplify the expression $\frac{a^{-1}b^{-1}b^{\frac{1}{2}}}{a^{-2}b}$.

Make all the indices decimals.

$= a^{-1}b^{-1}b^{0.5}a^{2}b^{-1}$

Group together like terms.

$= a^{-1}a^{2}b^{-1}b^{0.5}b^{-1}$

Add the indices for like terms.

$= a^{1}b^{-1.5}$

Consider alternative ways to express the simplfied expression.

$= \frac{a}{b^{1.5}} = \frac{a}{b\sqrt{b}}$

Information Booklet of Formulae and Constants

In your Unit 1 exam, you will be given an Information Booklet of Formulae and Constants and this includes the multiplication, division and powers laws of indices. Ideally, you should be confident in their use without reference to them.

The booklet is included in this Revision Guide on pages 81 to 85.

Now try this

1 Evaluate
(a) 19^0 (b) $16^{-\frac{1}{2}}$ (c) $64^{\frac{2}{3}}$ (d) $(\frac{1}{4})^{-2}$

2 Express $\sqrt{(x^a \times x^b)}$ as a power of x

3 Evaluate $\sqrt[3]{9} \times \sqrt[6]{9}$

Logarithms

Logarithms (or **logs**) are a way of writing facts about **powers**.

Logs

These two statements mean the same thing:

$$\log_a b = x \leftrightarrow a^x = b$$

You say, 'log to the base a of b equals x.'

$\log_a b = x \longleftrightarrow a^x = b$

a is the base of the logarithm.

Remembering the order

The key to being confident in log questions is remembering the *basic definition*. Start at the base, and work in a circle:

... to get b

$$\log_a b = x$$

Raise a to the power x ...

Laws of logarithms

Learn these four key laws for manipulating expressions involving logs. These laws work for all logarithms with *the same base*.

1 $\log_a x + \log_a y = \log_a(xy)$

$\log_4 8 + \log_4 2 = \log_4 16 = 2$

2 $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$

$\log_9 18 - \log_9 6 = \log_9 3 = \frac{1}{2}$

3 $\log_a\left(\frac{1}{x}\right) = -\log_a x$

$\log_8\left(\frac{1}{2}\right) = -\log_8 2 = -\frac{1}{3}$

4 $\log_a(x^n) = n\log_a x$

$\log_5 (25^3) = 3 \log_5 25 = 3 \times 2 = 6$

Worked example

Find:

(a) the positive value of x such that $\log_x 49 = 2$.

$x^2 = 49$

$x = 7$

(b) the value of y such that $\log_5 y = -2$.

$5^{-2} = y$

$y = \frac{1}{25}$

Changing the base

You can change the base of a logarithm using this formula:

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \log_9 27 = \frac{\log_3 27}{\log_3 9} = \frac{3}{2}$$

Common logs and natural logs

✓ Logarithms to base 10 are called common logs and the notation *log* x is sometimes used instead of $\log_{10} x$; e.g. $\log 100 = 2$ (note $100 = 10^2$).

✓ Logarithms to base e are called natural logs with the notation $\log_e x$ or $\ln x$.

Now try this

Write down the corresponding power fact. Work in a circle, starting at the base.

1 Find:

(a) the value of y such that $\log_3 y = -1$

(b) the value of p such that $\log_p 8 = 3$

(c) the value of $\log_4 8$.

2 Express as a single logarithm to base a.

(a) $2 \log_a 5$

(b) $\log_a 2 + \log_a 9$

(c) $3 \log_a 4 - \log_a 8$

Use law 4 to write $3 \log_a 4$ as $\log_a (4^3)$, then use law 1 to combine the two logarithms.

Exponential function

An exponential function (a^x) is one where the variable is the power, not the base. The Euler constant form of this expression (e^x) is found in many engineering disciplines such as aerodynamics, mechanics and electrical principles.

Exponential growth

If a population doubles or trebles every year, then it is said to be growing 'exponentially'.

We represent a population y that starts at 1000 and which doubles every year as $y = 1000 \times (2^x)$, where x represents the number of years.

Worked example

Predict the number of people (y) after 4 years for a population of 500 that is trebling every year.

$y = 500 \times (3^4) = 500 \times 81 = 40\,500$

Links You may need to use the laws of indices and logarithms when solving problems based on exponential growth or decay, see pages 1 and 2.

Using a calculator

Most calculators have the x^y button; for example, to find 3^4, press:

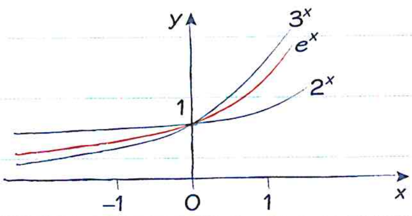
[3] [x^y] [4] [=]

and the calculator should show 81.

Exponential function e^x

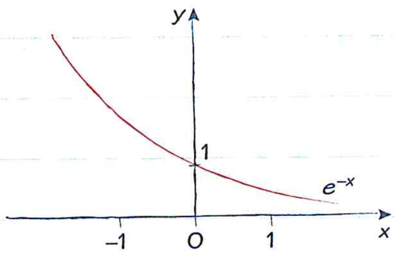
The Euler notation is a special form of the exponential function, written e^x. The exponential function e^x more than doubles, but does not quite treble over a period. The graph opposite shows $y = 2^x$, 3^x and e^x.

- A decaying population, such as radiated heat energy or radioactive particles, has the form $y = e^{-x}$.
- Recognise where the laws of indices can be used: $y = e^{-x}$ may also be written $y = \dfrac{1}{e^x}$



Graph of $y = 2^x$, $y = 3^x$ and $y = e^x$

For $x = 0$, 2^x, 3^x and e^x are all equal to 1. For e^x the gradient at any point on the curve $y = e^x$ is equal to e^x.



Graph of $y = e^{-x}$

Worked example

In a production process involving heat transfer, the temperature θ °C of a mould, at time t minutes, is given by $\theta = 250 + 150e^{-0.15t}$. Determine the temperature of the mould after 5 minutes.

$\theta = 250 + 150e^{-(0.15\times5)} = 250 + 150e^{-(0.75)}$

$= 250 + 70.8549$

$\theta = 320$°C (to 2 s.f.)

Using a calculator

Most calculators have the exponential button e^x button; it may be a secondary function of the 'ln' button. For example, to find $e^{2.5}$, press:

[SHIFT] [ln] [2.5] [=]

and the calculator should show 12.182....

Now try this

1 A manufacturer quadruples its production of a component from 2000 units per year every year for three years. Calculate the number of components produced at the end of the third year.

2 The voltage (V_c) across a capacitor in a RC circuit is given by $V_c = V_s(1 - e^{-\frac{t}{\tau}})$, where τ is the time constant and V_s is the supply voltage. Determine the value of V_c at $t = 5\tau$ when the supply voltage is 4.5 V.

Equations of lines

The equation of a straight line can be written in the form $y = mx + c$, where m is the **gradient** of the line, and c is the point where it crosses the y-axis.

Point and gradient

1. If you are given the gradient m of a straight line that passes through a point (x_1, y_1), then you can write its equation as:
   $y - y_1 = m(x - x_1)$ to obtain an expression of y in terms of x.

Worked example

A straight line passes through the point (−3, 2) and has a gradient −2. Find an equation for this line in the form $ax + by + c = 0$, where a, b and c are integers.

$y - y_1 = m(x - x_1)$

$y - 2 = -2(x - (-3))$

$y - 2 = -2x - 6$

$y + 2x + 4 = 0$

2. If you are given two points on a line, (x_1, y_1) and (x_2, y_2), you can calculate the gradient using:
   $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Worked example

The line L passes through the points (1, 1) and (2, 4). Find an equation for L in the form $y = mx + c$.

Gradient $(m) = \dfrac{4-1}{2-1} = 3$

$y = mx + c$

$y = 3x + c$

$1 = 3(1) + c$ (from point (1, 1))

$c = -2$ so $y = 3x - 2$

Check using point (2, 4):

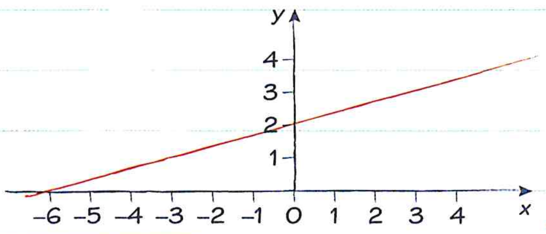
$4 = 3(2) - 2$

Once you've evaluated the value of c, you can substitute this in the general equation of the straight line $y = mx + c$

Intercepts

You can find where the line $y = mx + c$ intercepts both the x- and the y-axes.

The y- intercept is given by the value of c, and the x- intercept can be evaluated by setting the value of y to 0.



Graph of $y = \frac{1}{3}x + 2$

Worked example

Determine the intercepts for the x- and y-axes of the line $3y - x = 6$.

Rearranging the expression in the form $y = mx + c$ gives $y = \frac{1}{3}x + 2$, therefore the y-axis intercept is +2 (when $x = 0$).

Setting $y = 0$:

$\frac{1}{3}x + 2 = 0$

$x + 6 = 0$

$x = -6$ Therefore, the x- intercept is −6.

You can sketch a graph to check your answer. The gradient is positive because the m term $(\frac{1}{3})$ is positive and the line passes through the x-axis at −6 and the y-axis at +2.

Now try this

1. The line L passes through the point (6, −5) and has gradient $-\frac{1}{3}$. Find an equation for L in the form $ax + by + c = 0$, where a, b and c are integers.
2. The line L passes through (−4, 2) and (8, 11). Find an equation for L in the form $y = mx + c$, where m and c are constants.

Simultaneous linear equations

Linear equations have the form $y = mx + c$ (i.e. no x^2 or y^2). Simultaneous equations can be solved using either the substitution or the elimination method. Whichever method you use, remember to *number* the equations to keep track of your working.

Substitution

Solve the simultaneous equations:

$y - 2x = 2$ Call this equation (1).

$-2y + 5 = x$ Call this equation (2).

$y = 2x + 2$ Call this equation (3).

Rearrange the linear equation (1) to make y the subject.

$-2(2x + 2) + 5 = x$

$-4x - 4 + 5 = x$

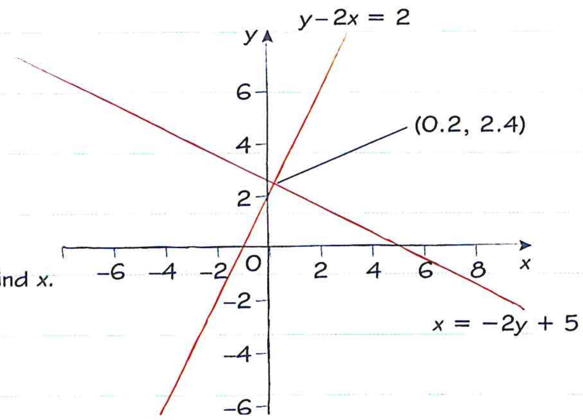
Substitute equation (3) into equation (2) and simplify to find x.

$5x = 1$

$x = \frac{1}{5} = 0.2$

You have found x. Substitute $x = 0.2$ into equation (1) to find y.

The solutions are $x = 0.2$, $y = 2.4$.



Graph showing simultaneous linear equations $y - 2x = 2$ and $-2y + 5 = x$.

Thinking graphically

✓ The solutions to a pair of linear simultaneous equations correspond to the point where the graphs of the equations intersect.

✓ The point of intersection has an x value and a y value.

Worked example

Solve the simultaneous equations:

$y - 3x = 8$ (1)

$2y + 11 = -9x$ (2)

You can substitute for x or y. It is easier to substitute for y because there will be no fractions.

Remember to number your equations.

From (1): $y = 3x + 8$ (3)

Substitute (3) into (2) and simplify to find x:

$2(3x + 8) + 11 = -9x$

$6x + 16 + 11 = -9x$

$15x = -27$, $x = -\frac{27}{15} = -1.8$

Substitute $x = -1.8$ into equation (1) to find y:

$y - 3(-1.8) = 8$, $y = 8 - 5.4 = 2.6$

Remember that the value of x and y represent the coordinates of the point where the simultaneous equations intersect.

Elimination

Manipulate one of the equations to make either the x or the y terms exactly the same in both equations.

Worked example

Solve the simultaneous equations

$6x + 6 = 5y$ (1)

$3y + 2x = 7$ (2)

Multiply equation (2) by -3 and rearrange to make the x terms the same.

$-6x + 21 = 9y$ (3)

Add equation (1) to equation (3) to eliminate the term $6x$ and $-6x$.

$27 = 14y$, $y = 1.93$; now substitute into (2) to obtain x:

$(3 \times 1.93) + 2x = 7$, $x = 0.61$

You can check your solution by substituting $x = 0.61$ into equation (1):

$(6 \times 0.61) + 6 = 5y$

$y = 1.93$

Now try this

Solve the following simultaneous equations:

1 $2x + 13 = -3.5y$
$-3x = -9y$

2 $14 = 3y + 5x$
$10x = 4y + 7$

Expanding and factorisation

Expanding brackets and factorising expressions are the basis for manipulation of engineering expressions and formulae and are therefore essential skills.

Expanding brackets

To expand the product of two factors you have to multiply EVERY TERM in the first factor by EVERY TERM in the second factor:

There are 2 terms in the first factor and 3 terms in the second factor, so there will be 2 × 3 = 6 terms in the expanded expression BEFORE you collect like terms.

2x ×

$$(2x + 3)(5x^2 - x + 4) = 10x^3 - 2x^2 + 8x + 15x^2 - 3x + 12$$
$$= 10x^3 + 13x^2 + 5x + 12$$

3 ×

Simplify your expression by collecting like terms: $-2x^2 + 15x^2 = 13x^2$

Factorising

1. For simple expressions you can extract the Highest Common Factor (HCF).
2. Some expressions may provide a common factor within the bracketed terms.

$(3x - 2)$ is common to both terms so can be extracted as a factor.

$$2x(3x - 2) - 7(3x - 2) = (3x - 2)(2x - 7)$$

Terms OUTSIDE the brackets form the second factor $(2x - 7)$.

3. In some cases it may be necessary to group the terms to identify a common factor.

Start by grouping the first two and last two terms. Then extract the HCF as in part 1 above for each group, which leaves a common factor: $(2a - 3)$. You can then extract the common factor as in 2 above.

Assuming no resistance, you can calculate the distance travelled by an object, using $s = ut + \frac{1}{2}at^2$.

Worked example

(a) Factorise the RHS of this equation.

$s = t(u + \frac{1}{2}at)$

Take out the common factor in each term. Don't forget to check your answer by multiplying back out. You should get the original expression.

(b) Factorise $3a(x + 4a) + 2(x + 4a)$.

$3a(x + 4a) + (x + 4a) = (x + 4a)(3a + 2)$

$(x + 4a)$ is common to both terms so can be taken as one of the factors. $(3a + 2)$ are the terms outside the brackets gathered together and which form the second factor.

(c) Factorise $6a^2 - 9a - 4ab + 6b$.

$(6a^2 - 9a) - (4ab - 6b)$

$3a(2a - 3) - 2b(2a - 3)$

$= (2a - 3)(3a - 2b)$

Special cases

There are some special cases that you should watch out for:

- Completing the square (See page 7 for how this works.)
  $x^2 + 2bx + c = (x + b)^2 - b^2 + c$
- The difference of two squares
  $(a + b)(a - b) = a^2 - b^2$

Now try this

1 Expand the brackets to show that $(3x - 4)(3x + 4) = 9x^2 - 16$.

2 Factorise $3x(y - 4) - 2(y - 4)$.

3 The thrust to speed efficiency of a jet engine is given by the expression $\dfrac{3V(V_j - V)}{V_j^2 - V^2}$.

Factorise and, hence, show that the complete expression is equal to $\dfrac{3V}{V_j + V}$.

Hint: to factorise $V_j^2 - V^2$, use the difference of two squares.

Quadratic equations 1

Quadratic equations occur throughout engineering in different forms. You must be able to identify them and know how to solve them using three methods. Factorisation and completing the square are shown below. Use of the formula is shown on page 8. You will find practical uses for these methods on page 20.

Factorising a quadratic

You can follow these steps to solve some quadratic equations:

1. **Rearrange** the equation into the form $ax^2 + bx + c = 0$.
2. **Factorise** the left-hand side.
3. Set each factor equal to zero and solve to find two values of t: $(t - 3)(t + 5) = 0$.

The valid solution is the value of t that is greater than 0. This is given by the factor $(t - 3)$, therefore the answer is $t = 3$ s.

You could indicate this in your answer by writing: where $t \geq 0$.

Worked example

The displacement of a car in metres (s) is given by $s = 2t + t^2$, where t is in seconds. Find how long it takes to travel 15 m.

$t^2 + 2t - 15 = 0$ Find two numbers with a sum of +2 and a product of −15.

$(t - 3)(t + 5) = 0$ The required numbers are '−3' and '+5'.

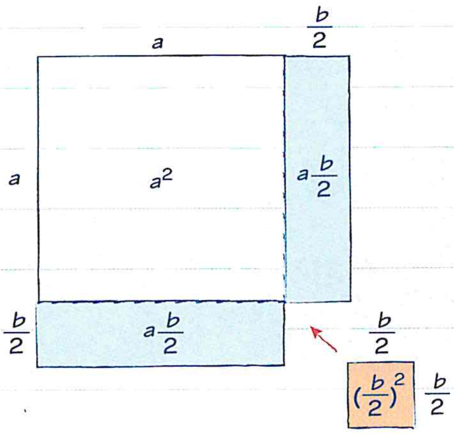
$t - 3 = 0$ or $t + 5 = 0$

$t = 3$ s or $t = -5$ s

Discount the negative root in which $t = -5$ s.

Completing the square

Useful for quadratics that do not factorise.



Complete the square with $(\frac{b}{2}) \times (\frac{b}{2})$; i.e. $(\frac{b}{2})^2$

The complete square is made of three terms:

1. $a^2 + ab + (\frac{b}{2})^2$
2. These are simplified as $(a + \frac{b}{2})^2$.
3. Remember to add $(\frac{b}{2})^2$ to *both* sides of the equation.

Worked example

The area of a rectangular building (length x) is given by $4 = x(5 - x)$, where the width in metres is $(5 - x)$. Find the roots and, hence, the length and width.

$x^2 - 5x = -4$

$x^2 - 5x + (\frac{-5}{2})^2 = -4 + (\frac{-5}{2})^2$ Complete the square by adding $(\frac{-5}{2})^2$ to both sides.

$(x - 2.5)^2 = (-2.5^2) - 4$

$x - 2.5 = \pm\sqrt{2.25}$

therefore $x = +1.5 + 2.5 = 4$ (length)

or $x = -1.5 + 2.5 = 1$ (width)

$x^2 - 5x + (\frac{-5}{2})^2$ may be written as $(a + \frac{b}{2})^2$ or $(x - 2.5)^2$.

The coefficient of the x term is the numerator of the 'complete the square' term.

Now try this

The distance x, in metres, along a beam where the bending moment = 0 is given by $5x^2 + 14x - 3$. Factorise this expression and, hence, find the position of x.

You need to put this expression = 0, find the roots and then discard negative values. Remember to include the units (metres).

Had a look ☐ Nearly there ☐ Nailed it! ☐

Quadratic equations 2

In some cases, you will be unable to solve quadratic equations by factorisation or completing the square. You will need to solve them using the formula instead. You will find practical uses for this method on page 72.

Solution by formula

You can solve any quadratic by use of the formula, but it *must* be in the form $ax^2 + bx + c = 0$, where:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

c is the 'c' term

b is the coefficient of x

a is the coefficient of x^2

Quadratics come in many forms, for example:

$120l = 10l^2 + 100$

$(y - 2)^2 = 18$

You need to recognise the different forms and use the appropriate method to find the solution.

Using the formula

The formula will be on the formulae sheet, but be confident in using the discriminant to check the nature of the roots.

- If $b^2 - 4ac > 0$ then there are two solutions.
- If $b^2 - 4ac = 0$ then the quadratic has one solution.
- If $b^2 - 4ac < 0$ then the quadratic doesn't have real roots.

You will not be asked to solve quadratics where $b^2 - 4ac < 0$.

Worked example

A duct manufacturer produces a rectangular ducting sheet of 28 m² in which the area is related to the width by the expression $28 = 1.6w^2 + 4.2w$. What is the width of the sheet?

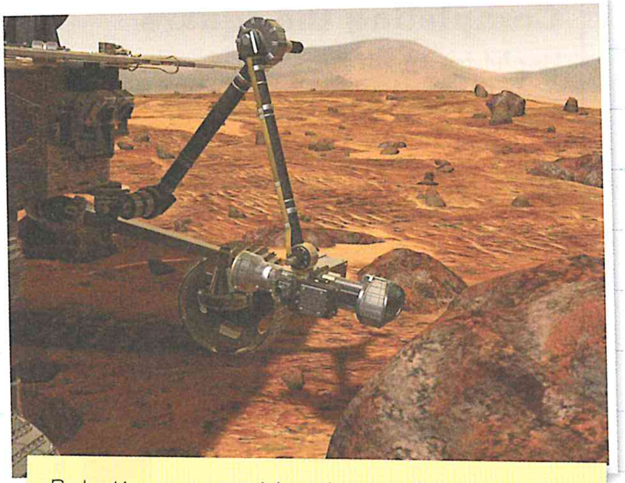
$28 = 1.6w^2 + 4.2w$ or $1.6w^2 + 4.2w - 28 = 0$

$a = 1.6$, $b = 4.2$, $c = -28$

$$w = \frac{-4.2 \pm \sqrt{4.2^2 - 4 \times (1.6 \times (-28))}}{2 \times 1.6}$$

$$w = \frac{-4.2 \pm 14.029}{3.2} = 3.07 \text{ or } -5.69$$

Reject negative answer, width = 3.07 m (to 2 d.p.)



Robotic arm on a Mars lander. Calculating the distance to turn a robotic arm in mid-motion is one use of the quadratic formula to solve $s = ut + \frac{1}{2}at^2$.

Now try this

The height, h, of a ball thrown vertically is given by

$h = -4.3t^2 + 54t + 13$

where t is time, measured in seconds. The time to reach the ground will be given when $h = 0$. Calculate the time taken for the ball to reach the ground, using the quadratic equation.

The equation will provide two solutions. In this example, one of them will be negative, which should be rejected.

Don't forget to specify the units.

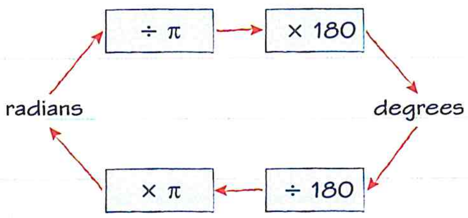
Radians, arcs and sectors

Radians are an alternative unit of angular measurement to degrees. They must be used in calculations involving **circular motion** and **rotational dynamics**.

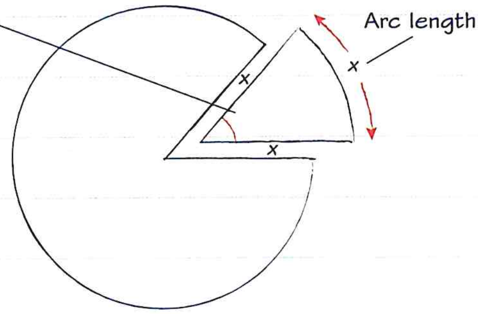
Links There is more about angular motion on page 27.

Radians

1 radian is the angle formed when the radius of a circle is 'wrapped' around the circumference. There are 2π radians in a full circle. Use these rules to convert between radians and degrees:



This sector has an arc length equal to its radius. The angle at the centre of the circle is 1 radian. You can also write '1 rad', or just '1'.

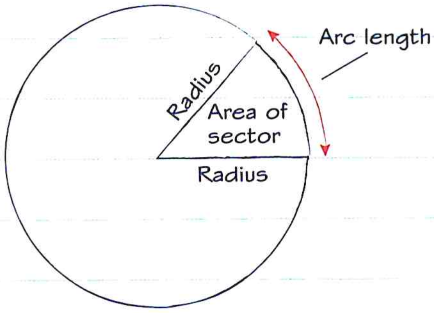


Arc length and sector area

You can calculate the area of a sector and the length of an arc easily if the angle is measured in radians.

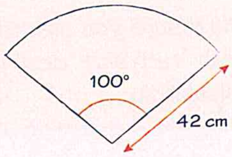
1. Length of an arc
   $s = r\theta$
2. Area of a sector
   $A = \frac{1}{2} r^2\theta$

Both these formulae are given in the formulae booklet. They work **only** if the angle θ is measured in **radians**.



Worked example

A microchip manufacturer uses silicon wafers in the shape of a sector of a circle, with the dimensions shown.



Calculate the area of one wafer.

$100° = \frac{100}{180} \times \pi = \frac{5}{9}\pi$ rad

Area $= \frac{1}{2} r^2\theta = \frac{1}{2} \times 42^2 \times \frac{5}{9}\pi = 1539.3804$

$\approx 1540\,\text{cm}^2$ (to 3 s.f.)

Convert the angle to radians then use the formula for the area of the sector. Remember to round your final answer to 3 significant figures and give the correct units.

Common angles

Save time converting between radians and degrees by learning these five common angles:

Degrees	45	60	90	180	360
Radians	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π

Now try this

1. Find the arc length and area of the sector of a circle, with radius 4 cm, which contains an angle of 30°.
2. A plasma cutter is used to cut sectors of a circle for ventilation trunking. The arc length of each sector is 450 mm and the radius is 1 m. Find the angle, in radians, of a sector and, hence, the number of complete sectors that can be obtained from a circle of sheet metal with radius 1 m.

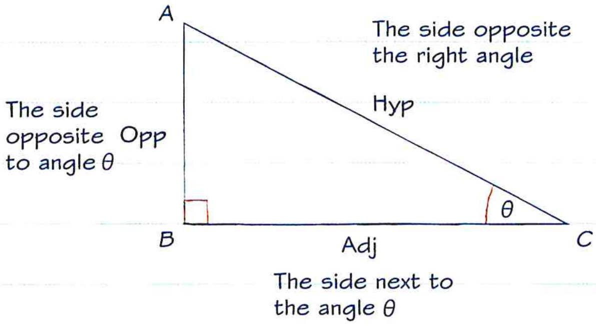
Trigonometric ratios and graphs

You need to be able to recall the trigonometric identities of SIN, COS and TAN and their respective graphs.

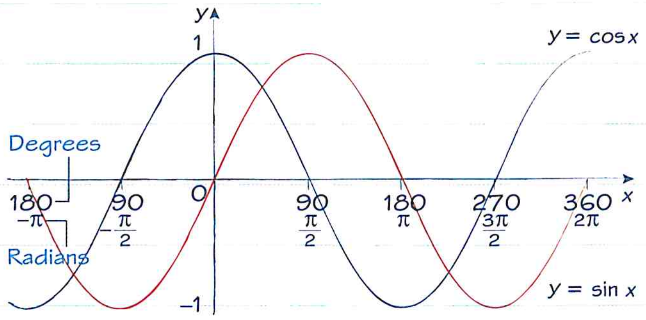
Basic trigonometric ratios

$\sin\theta = \frac{\text{Opp}}{\text{Hyp}}$ $\cos\theta = \frac{\text{Adj}}{\text{Hyp}}$ $\tan\theta = \frac{\text{Opp}}{\text{Adj}}$

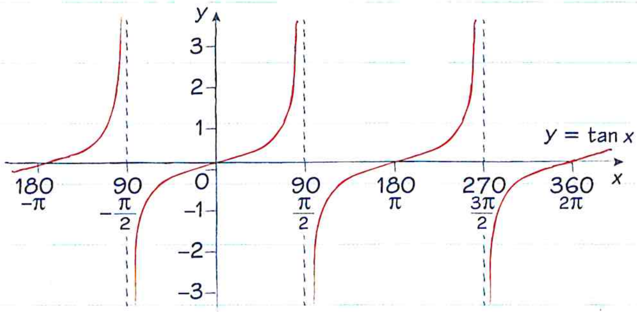
You can use the mnemonic 'SOH CAH TOA' to remember these ratios. (Sine: Opposite over Hypotenuse, Cosine: Adjacent over Hypotenuse, Tangent: Opposite over Adjacent.)



$y = \sin x$ and $y = \cos x$



$y = \tan x$



Trig values for θ

The value of θ (pronounced theta) may be represented in degrees, or as radians, in terms of π.

θ (°)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
45	$\frac{\pi}{4}$	0.707	0.707	1
90	$\frac{\pi}{2}$	1	0	$-\infty$
270	$\frac{3\pi}{2}$	−1	0	$-\infty$
360	2π	0	1	0

Make sure you are confident using both 'rad' and 'deg' modes on your calculator.

Now try this

1 Produce a table that states the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ at the following intervals: 0°, 30°, 45°, 60°, 90°, 135°, 180°, 270°, 360°. Include a column in the table for the radian equivalent of each of these angles.

2 Evaluate the length of BC in triangle ABC, in which angle B is a right-angle, angle A is $\frac{\pi}{4}$ rad and AB is 10 cm.

First make a rough sketch of the triangle.

Cosine rule

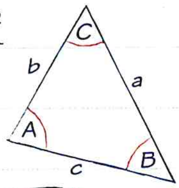
The cosine rule applies to *any triangle*. You usually use the cosine rule when you know two sides and the included angle (SAS) or when you are given three sides and you want to work out an angle (SSS). Although the cosine rule will be provided in the formulae sheet in your exam, you need to be confident in its use.

1 $a^2 = b^2 + c^2 - 2bc\cos A$

This version is useful for finding a missing side.

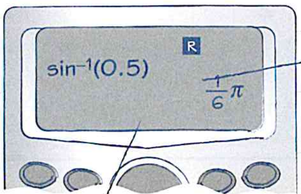
2 $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Use this version to find a missing angle.



Using a calculator

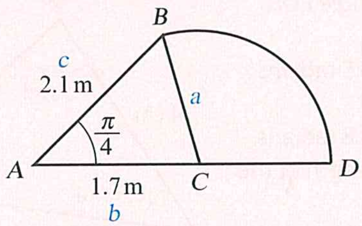
If you're using the sine rule or cosine rule you need to make sure that your CALCULATOR is in the correct mode (DEGREES or RADIANS).



This calculator is in radians mode.

On some calculators you need to press SHIFT and SETUP to change between degrees and radians mode.

Worked example



The diagram shows a triangle, *ABC*, and a sector, *BCD*, of a circle with centre *C*. Find the length of *BC*.

$a^2 = b^2 + c^2 - 2bc\cos A$

$BC^2 = 1.7^2 + 2.1^2 - 2 \times 1.7 \times 2.1 \times \cos\frac{\pi}{4}$

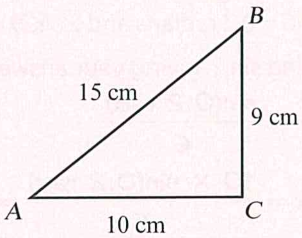
$= 2.2512\ldots$

$BC = 1.50\text{ m}$ (to 2 d.p.)

SAS → cosine rule. If the angle on the diagram is given in terms of π then it is in **radians**. Make sure your calculator is set to radians before working out $\cos\frac{\pi}{4}$.

Worked example

In the triangle *ABC*, *AB* = 15 cm, *BC* = 9 cm and *CA* = 10 cm. Find the size of angle *C*, giving your answer in radians to 2 decimal places.



$\cos C = \dfrac{BC^2 + AC^2 - AB^2}{2 \times BC \times AC}$

$= \dfrac{9^2 + 10^2 - 15^2}{2 \times 9 \times 10}$

$C = \cos^{-1}(-0.2444\ldots) = 1.82\text{ rad}$ (to 2 d.p.)

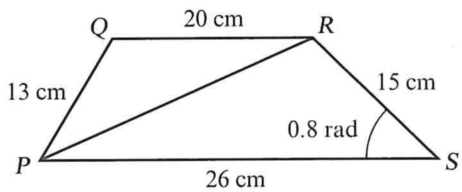
SSS → cosine rule. Be careful with the order. You add the squares of the sides **adjacent** to the angle, and subtract the square of the side **opposite**.

Now try this

The diagram shows two triangles, *PQR* and *PRS*. $\angle RSP = 0.8$ radians. Find:

(a) the length of *PR*

(b) the size of $\angle PQR$, giving your answer in radians to 3 significant figures.



Sine rule

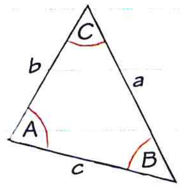
The sine rule applies to *any triangle*. The sine rule is useful when you know *two angles*, or when you know a side and the *opposite* angle. Although the sine rule will be provided in the formulae sheet in your exam, you need to be confident in its use.

1 $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

This version is useful for finding a missing side.

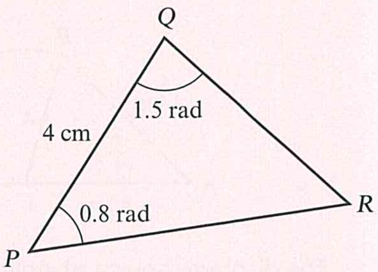
2 $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Use this version to find a missing angle.



Worked example

In the triangle PQR,
$PQ = 4$ cm,
$\angle PQR = 1.5$ radians
and
$\angle QPR = 0.8$ radians.
Find the length of the side PR.



$\angle QRP = \pi - 1.5 - 0.8 = 0.8415\ldots$

$$\frac{PR}{\sin(1.5 \text{ rad})} = \frac{4}{\sin(0.8415\ldots \text{ rad})}$$

$$PR = \frac{4 \times \sin(1.5 \text{ rad})}{\sin(0.8415\ldots \text{ rad})} = 5.35 \text{ cm (to 2 d.p.)}$$

The sine rule compares **opposite** sides and angles, so use the fact that the angles in a triangle add up to π radians (or 180°) to work out $\angle QRP$ first.

Using radians

Remember these key facts:

- Angles in a triangle add up to π radians.
- A right angle is $\dfrac{\pi}{2}$ radians.

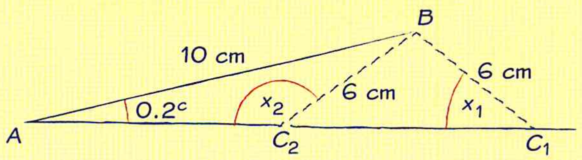
Links There is more about measuring angles in radians on page 9.

Two values

If you know $\sin x$, you might be asked to find *two possible values* for x. Use this rule for angles measured in *radians*:

$\sin x = \sin(\pi - x)$

You could draw a sketch to help you see what is going on.



Worked example

In the triangle ABC, $AB = 10$ cm, $BC = 6$ cm, $\angle BAC = 0.2$ radians and $\angle ACB = x$ radians.

(a) Find $\sin x$, giving your answer to 2 decimal places.

$$\frac{\sin x}{10} = \frac{\sin(0.2 \text{ rad})}{6}$$

$$\sin x = \frac{10 \times \sin(0.2 \text{ rad})}{6} = 0.33 \text{ (to 2 d.p.)}$$

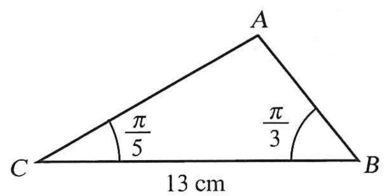
(b) Given that there are two possible values of x, find these values of x, correct to 2 decimal places.

$x_1 = \sin^{-1}(0.331) = 0.34$ radians (2 d.p.)

$x_2 = \pi - x_1 = \pi - 0.34 = 2.80$ radians (2 d.p.)

Now try this

In the triangle ABC, $BC = 13$ cm, $\angle ABC = \frac{\pi}{3}$ radians, and $\angle ACB = \frac{\pi}{5}$ radians. Find the length of AC.

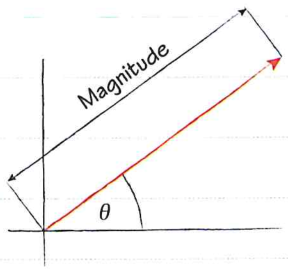


Start by finding the size of $\angle CAB$.

Vector addition

Vectors such as force, velocity and ac phasors have both magnitude and direction. You need to take both into account when finding the overall resultant of two or more vectors.

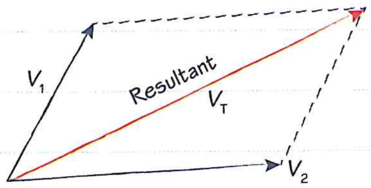
Visualising vectors



Here the magnitude of the vector is proportional to the length of the arrow. The direction is given by the angle (θ) and the sense by the arrowhead.

Vector diagrams

A vector diagram helps you visualise a system where two (or more) vector quantities are acting and find their sum or resultant vector.

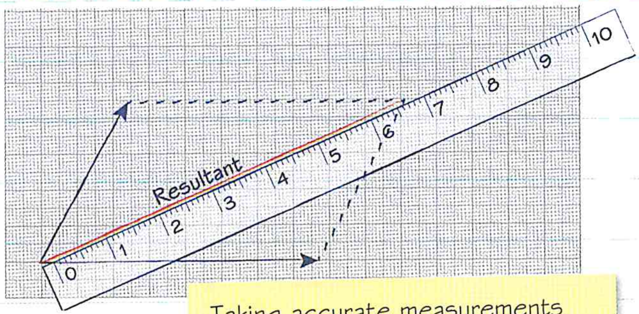


Vector diagram using the parallelogram law to show the resultant of two vectors.

Graphical vector addition

To find the resultant of a system of vectors:

- Draw them accurately and to scale on graph paper.
- Complete the parallelogram and draw in the diagonal that represents the resultant.
- Measure the resultant using a ruler and protractor.

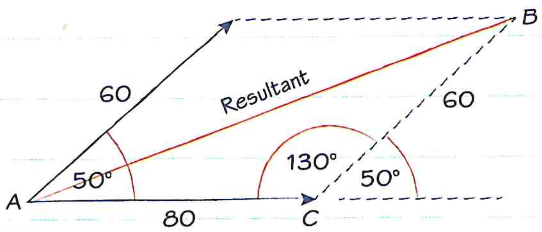


Taking accurate measurements to determine the magnitude and direction of a resultant vector.

Analytical vector addition

A more accurate solution can be found using trigonometry to find the resultant of two vectors.

Links You will find analytical methods of vector addition in specific applications are covered on page 16 Resolving forces and page 63 Addition of sinusoidal waveforms.



Here you can apply the cosine rule to calculate the resultant magnitude and direction.

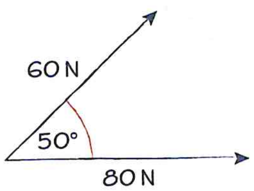
Links You will find the cosine rule on page 11 .

Now try this

The analysis of a system of forces produces the vector diagram shown. The diagram is not to scale.

Find, using a graphical approach, the magnitude and direction of the resultant vector.

Compare your graphical solution with that found using the analytical approach described above.

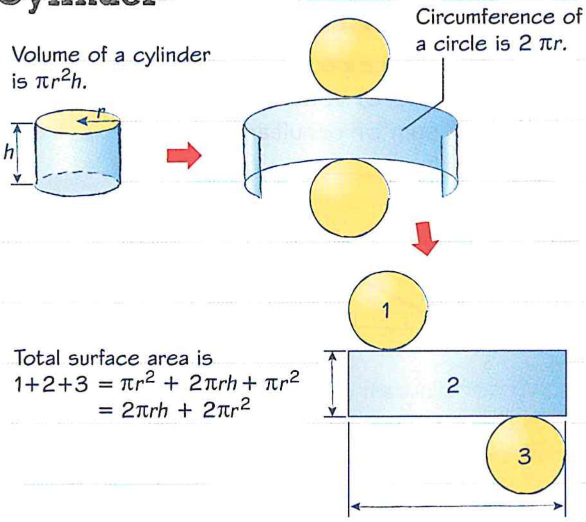


Had a look ☐ Nearly there ☐ Nailed it! ☐

Surface area and volume

You need to know how to calculate the surface areas and volumes of cylinders, spheres and cones.

Cylinder



Volume $= \pi r^2 h$, Total surface area $= 2\pi rh + 2\pi r^2$

Worked example

A weapon release unit on an aircraft is protected by a cover in the shape of a cylinder of radius 40 mm and height 60 mm, open at one end. Determine the total surface area of the component.

The total surface area of a complete cylinder $= 2\pi rh + 2\pi r^2$, but one end is open.

Total surface area of component

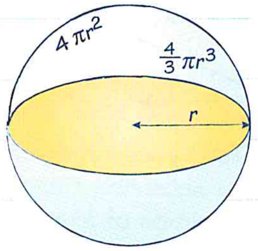
$= 2\pi rh + \pi r^2$

$= (2\pi \times 0.04 \times 0.06) + (\pi \times 0.04^2)$

$= 2.01 \times 10^{-2}\ m^2$ (to 3 s.f.)

Make sure that units are consistent. Here all measurements are converted to metres.

Sphere



Surface area $= 4\pi r^2$

Volume $= \frac{4}{3}\pi r^3$

Worked example

Calculate the minimum number of portable fuel pods, in the shape of 3 metre diameter polymer spheres, that are required to supply a town in a disaster-struck area, with 25 000 litres of fuel ($1\ m^3 = 1 \times 10^3$ litres).

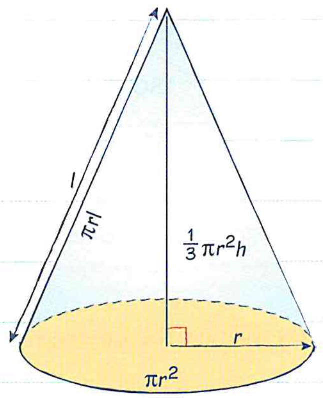
Volume of one fuel pod

$= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(\frac{3}{2})^3 = 9.42\ m^3 = 9420$ litres

Number of fuel pods $= 25000 \div 9420 = 2.65$

A minimum of 3 fuel pods are required.

Cone



Surface area $= \pi rl + \pi r^2$

Volume $= \frac{1}{3}\pi r^2 h$

Worked example

A food manufacturer wishes to market a new confectionary product, in the shape of a cone. Calculate the height (h) of the product if each cone contains 41.85 cm^3 of filling and has a diameter of 4.0 cm.

Volume $= \frac{1}{3}\pi r^2 h$

$41.85 \times 10^{-6} = \frac{1}{3}\pi \times 0.02^2 \times h$

$h = 0.10$ m or 10.0 cm (to 2 d.p.)

If the question doesn't specify which units to use for the answer then you can decide. Here all measurements are converted into metres.

Now try this

1 A cylindrical concrete piling has radius 0.7 m and height 12 m. Calculate the volume of concrete required.

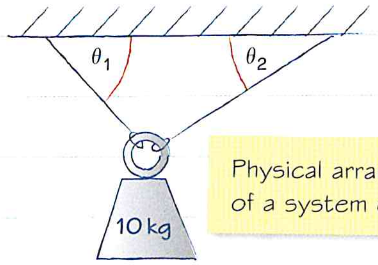
2 If you double the radius of a sphere, how does the volume of the new sphere compare with the volume of the original sphere?

First make a sketch with the known information.

Systems of forces

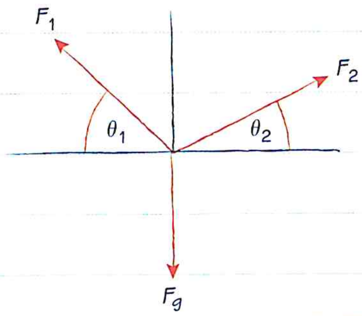
Mechanical systems can contain components that exert forces that push or pull other components. In a static system all the forces are balanced so, instead of a force causing a change in the motion of an object, the system remains at rest or in motion with constant velocity.

Space diagrams



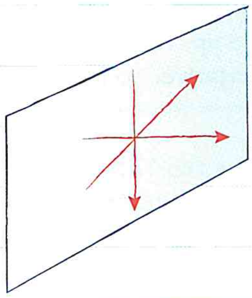
Physical arrangement of a system of forces.

Free body diagrams



You use a free body diagram to show only the forces acting in the system you are investigating.

Coplanar concurrent forces

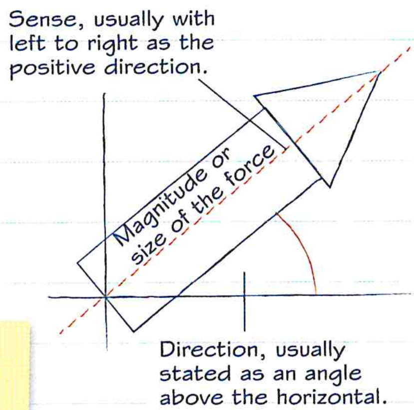


Coplanar concurrent forces act in a single plane and pass through a common point.

The resultant of coplanar concurrent forces

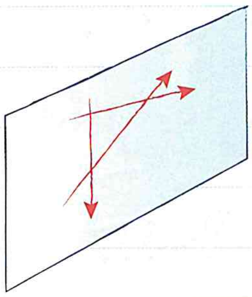
You can simplify coplanar concurrent forces to a single **resultant** force.

An **equilibrant** force can be used to balance the **resultant**. It has the same magnitude and direction but the opposite sense.



Fully defined resultant of a coplanar concurrent system of forces stating magnitude, direction and sense.

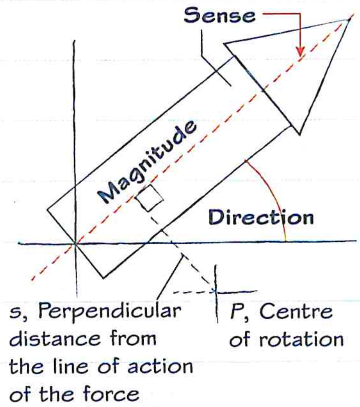
Coplanar non-concurrent forces



Coplanar non-concurrent forces act in a single plane but do not all pass through a common point.

The resultant of coplanar non-concurrent forces

You can also simplify coplanar **non**-concurrent forces to a single resultant. However, these commonly involve turning effects and so an extra piece of information is required to fully define them.

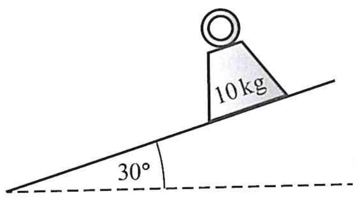


Fully defined resultant force of a non-concurrent system with magnitude, direction, sense and distance from a centre of rotation to the line of action.

Now try this

The space diagram shows a stationary weight on a slope or inclined plane. Friction is preventing the weight from sliding down the slope.

Draw a free body diagram for this system.



Note that, in your exam, you may find the terms 'free body diagram' and 'space diagram' are used interchangeably.

Resolving forces

To easily add forces together they must be acting in the same direction. First resolve them into **horizontal** and **vertical components**. Then add or subtract to find the resultant.

Formulae for resolving forces

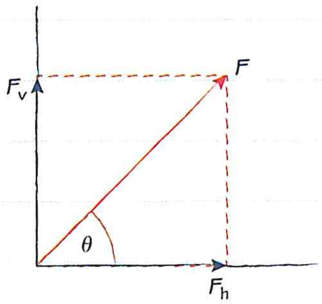
Imagine a right-angled triangle with the force as its hypotenuse. For a force of magnitude F acting at an angle θ to the horizontal:

$F_v = F \sin \theta$

$F_h = F \cos \theta$

$\frac{F_v}{F_h} = \tan \theta$

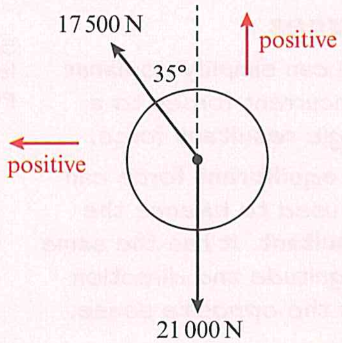
$F^2 = F_v^2 + F_h^2$



Worked example

The diagram shows coplanar forces acting on a demolition ball as it swings through the air.

Calculate the **magnitude** and **direction** of the resultant force.



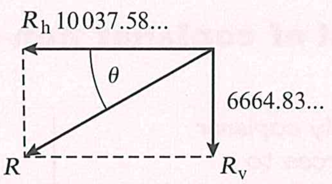
Choose a **positive direction** that makes your working easiest. Here the positive direction is **up**. The resultant vertical force is **negative**, so it acts downwards.

Resolving horizontally:

$17\,500 \sin 35° = 10\,037.58\ldots$ N

Resolving vertically:

$17\,500 \cos 35° - 21\,000 = -6664.83\ldots$ N



$R = \sqrt{10037.58\ldots^2 + 6664.83\ldots^2} = 12\,000$ N (3 s.f.)

$\tan\theta = \frac{6664.83\ldots}{10037.58\ldots}$

$\theta = 33.6°$ (3 s.f.)

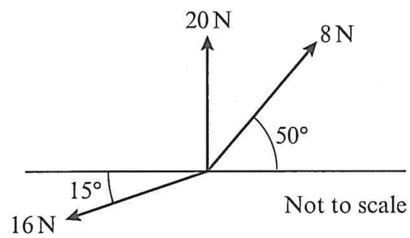
1. Resolve the 17 500 N force into vertical and horizontal components.
2. Find the **resultant** vertical and horizontal forces acting on the system.
3. **Sketch** a triangle with the overall resultant force as the hypotenuse.
4. Use Pythagoras' theorem to find the magnitude of the resultant.
5. Use trigonometry to find the direction of the resultant.

An engineer could resolve the tensions in the strings to work out the resultant force acting on this hot-air balloon basket.

Now try this

The diagram represents the tension forces acting at a single point in a structural framework.

Calculate the magnitude and direction of the resultant for this system of coplanar forces.



Moments and equilibrium

A moment is a measure of the **turning effect** of a force on a body. It is found from the **magnitude** of the force and the **perpendicular distance** from the line of action of the force to the centre of rotation.

Turning moment (*M*)

$$\boxed{M = Fs}$$

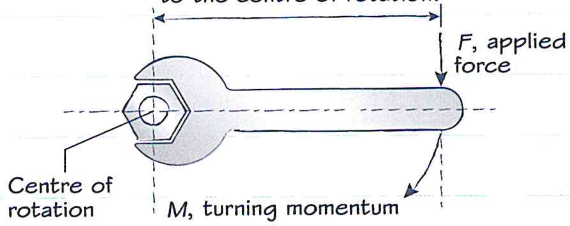
Turning moment (newton metres (N m))

Force, in newtons (N)

Perpendicular distance from the line of action of the force to the centre of rotation (*s*)

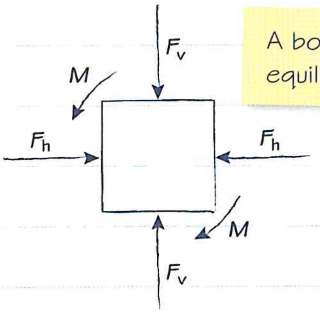
Turning moments can be used to describe the turning force applied to a nut or bolt head by a spanner.

The turning effect or moment depends on the applied force and the length of the handle.



Equilibrium

For a system to be in **static equilibrium** all forces and turning moments must balance each other.



A body in static equilibrium.

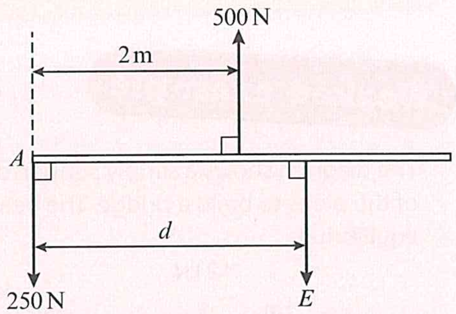
The following three conditions must all be satisfied:

1. The sum of the horizontal components is 0.
   $\sum F_h = A_h + B_h \ldots + n_h = 0$
2. The sum of the vertical components is 0.
   $\sum F_v = A_v + B_v \ldots + n_v = 0$
3. The sum of the moments is 0.
   $\sum M = M_A + M_B \ldots + M_n = 0$

Worked example

The diagram shows a simple system of forces acting on a beam that pivots at point *A* and is being held in static equilibrium by an equilibrant force *E*.

Calculate the magnitude, direction and sense of force *E* and distance *d*.



Consider the forces in equilibrium (where upwards is +ve):

$0 = 500 - 250 - E$

$E = 500 - 250$

$E = 250$ N acting vertically with –ve (downward) sense

Consider moments in equilibrium about *A* (where clockwise is +ve):

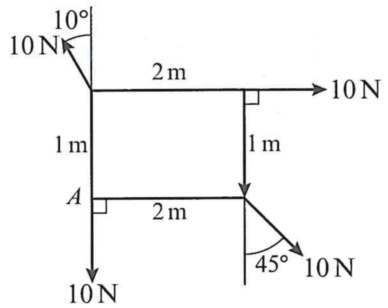
$0 = (250 \times 0) + (d \times 250) - (2 \times 500)$

$d = \dfrac{1000}{250}$

$d = 4$ m from point *A*

Now try this

Determine whether the system of forces acting on this square plate is in static equilibrium by finding the sum of the vertical and horizontal components of the forces present and taking moments about point *A*.

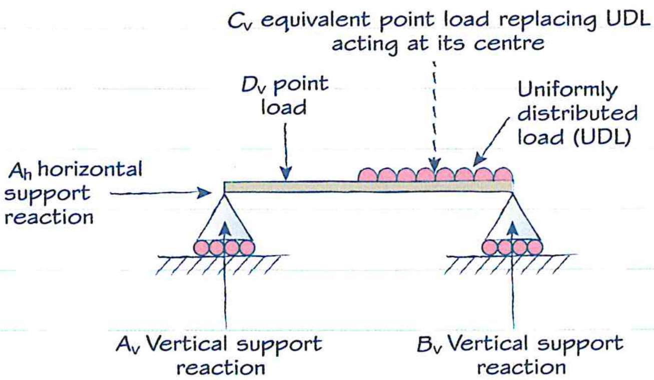


Links To revise resolving forces and resolving a force into horizontal and vertical components, see page 16.

Simply supported beams

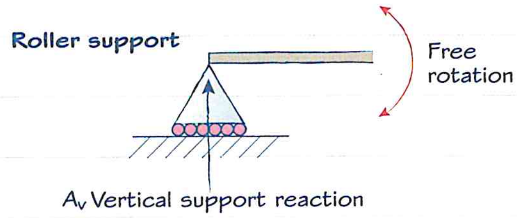
Simply supported beams are commonly used in engineering structures such as bridges, vehicle chassis and as supports over door and window openings in buildings. As an engineer, you must be able to analyse the forces required to support each end of a beam to make sure the supports are strong enough.

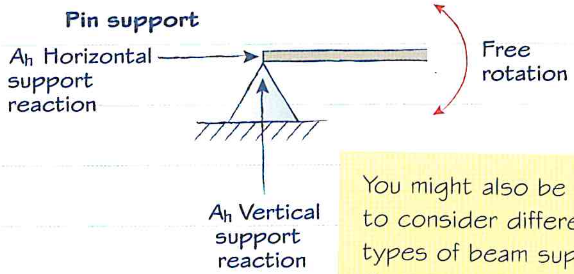
A simply supported beam



Several different types of forces can act on a simply supported beam. You may be asked to calculate the support reactions for a given system.

Types of beam supports

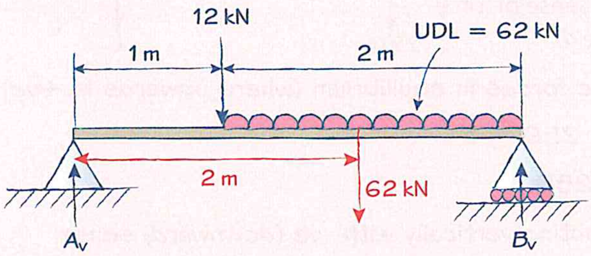




You might also be asked to consider different types of beam supports that are able to provide support reaction forces in the vertical or horizontal directions.

Worked example

The diagram shows a simply supported beam as part of the plans to build a bridge. The beam is in static equilibrium.



You can consider the uniformly distributed load (UDL) as a point load acting at the centre at the distribution. In this case, that would mean a point force acting vertically downwards 2 m from support A. Add this to the diagram.

Calculate the support reactions A_v and B_v.

Take moments about A to find B_v:

$(1 \times 12) + (2 \times 62) = (3 \times B_v)$

$B_v = 45.33$ kN (to 2 d.p.)

Take moments about B to find A_v:

$(3 \times A_v) = (1 \times 62) + (2 \times 12)$

$A_v = 28.67$ kN (to 2 d.p.)

1. Replace any uniformly distributed loads by equivalent point loads.
2. Take moments about support A. Anti-clockwise moments will equal clockwise moments, as the beam is in equilibrium.
3. Rearrange to find B_v.
4. Take moments about support B.
5. Rearrange to find A_v.
6. Check your answer.

Now try this

Check the solution in the Worked example by determining whether the beam still satisfies all the conditions of static equilibrium.

Refer back to page 17 Moments and equilibrium to find the conditions that must be met for static equilibrium.

Links You could also work through the additional beam problem given on page 69.