**II. Graph Theory: The Mathematical Framework** Enes

1. Discuss the significance of degree distribution in understanding network structure.

The degree distribution describes how the connections (degrees) of nodes are distributed in a network. It helps us to classify the network type such as random networks or scale-free networks.

2. How does clustering coefficient measure the local density of networks? Provide an example.

The clustering coefficient measures the local density of a network by quantifying how connected a node's neighbors are to each other. It is calculated as the ratio of the number of existing connections between a node's neighbors to the total possible connections between them.

For example, if you have two close friends, there’s a high chance that they know each other, which forms a "triangle" of connections. The more of these triangles exist, the higher the clustering coefficient for that person.

3. Describe the use of graph algorithms in identifying critical nodes or links in a network

Two common approaches are betweenness centrality and shortest path analysis.

* **Centrality Measures:** Algorithms like **betweenness centrality** identify which nodes or links play key roles in connecting different parts of the network. Nodes with high betweenness centrality are "bridges" that connect different groups.
* **Path-Finding Algorithms:** Algorithms like Dijkstra’s shortest path find the shortest routes in a network. This is useful for identifying crucial nodes in communication or transportation networks.

**For us to understand better**, in a social media network, if you remove a random person, it won’t affect the network much. But if you remove a celebrity or influencer (a node with high betweenness centrality), it could affect many people who were connected through them. Graph algorithms help identify these key players.

**V. The Barabási–Albert Model**  Enes

1. Describe the Barabási–Albert model for scale-free network generation and its core principles.

The Barabási–Albert (BA) model is a generative model used to create scale-free networks, which are characterized by a power-law degree distribution. In simpler terms, most nodes in the network have few connections, while a few "hubs" have a large number of connections.

Core principles :

* **Growth:** The network starts with a small number of connected nodes (typically

m0 nodes). New nodes are added one at a time.

* **Preferential Attachment**: Each new node connects to m existing nodes (where m <= m0) in the network. However, it does not connect randomly. Instead, it links to nodes that already have more connections with a probability proportional to the degree (number of links) of each node.

When a new node joins the network, it tries to connect with existing nodes that already have many connections. The more connections a node has, the more likely it is to attract even more connections. This is sometimes called the **"rich-get-richer"** effect, because popular nodes keep getting more popular over time.

2. Explain the "rich-get-richer" mechanism and its role in the growth of scale-free networks.

The **"rich-get-richer"** mechanism, also known as **preferential attachment**, is a critical concept in the BA model. It reflects how nodes with higher degrees (more connections) are more likely to receive new connections.

A small network of m0 nodes start the process. When a new node joins the network, it forms m connections to existing nodes. The likelihood of an existing node receiving a new connection is proportional to its current degree. Nodes that are already well connected (i.e., "richer" nodes) attract more new links, further increasing their connectivity.

As a result of that, some nodes become “hubs” with significantly more connections than others. This mechanism naturally produces a power-law degree distribution, with very few hubs and many nodes with low degrees.

3. How does the Barabási–Albert model address the limitations of random network models?

* **Power-Law Degree Distribution:**

In random networks, degrees follow a Poisson distribution, meaning most nodes have roughly the same number of links, and large hubs are extremely rare. In contrast, the BA model produces a **power-law degree distribution**, which matches the structure observed in many real-world networks. This allows for the presence of hubs, which are crucial for network resilience and the spread of information or diseases.

* **Growth and Preferential Attachment:**

In random networks, all nodes exist from the beginning, and edges are randomly assigned, which fails to capture the growth of real-world systems like social networks. The BA model incorporates network growth by adding nodes over time and introduces preferential attachment.

* **Hubs and Resilience:**

In random networks, if random nodes are removed from network, structure breaks down quickly. But scale-free networks (BA model), are robust against random failures because most nodes have low degrees. However, if the key “hub” nodes are removed, the network might break up. This is closer to real-world systems where attack on hubs can have a large impact.

1. Explain the Erdős–Rényi model and its key properties.

The **Erdős–Rényi (ER) model** is one of the simplest and most well-known models for generating random graphs. It describes a network that is built randomly according to specific rules. Here’s a breakdown in simple terms:

**Key Components**

1. **Nodes (vertices):** Think of these as points in the graph (e.g., people, cities, devices).
2. **Edges (connections):** Links between the nodes.

**Two Variants of the Model**

There are two main ways to generate an ER random graph:

1. **G(n,p) model:**
   * You start with n nodes.
   * For every possible pair of nodes, an edge is created with a fixed probability p.
   * p controls how dense or sparse the graph is.
   * Edges are independent of each other.
2. **G(n,m) model:**
   * You start with nnn nodes.
   * You randomly add mmm edges between nodes.

**Properties of the ER Model**

* **Degree distribution:** In G(n,p) the degree (number of connections) of each node follows a binomial distribution.
* **Connectivity:** As p increases, the graph transitions from being mostly disconnected to becoming fully connected (a "giant component" appears).
* **Clustering:** The model tends to have low clustering coefficients (nodes connected to a node are less likely to be connected to each other).

**Example**

Imagine a party with n=10 people (nodes), and there’s a 20% (p=0.2) chance that any two people know each other. The ER model would randomly connect pairs of people based on that probability, resulting in a random network of friendships.

The Erdős–Rényi model is widely used to study properties of random graphs, like thresholds for connectivity, but it may not capture features of real-world networks (e.g., hubs, clustering).

Degree Correlations

**Degree correlations** describe the relationship between the degrees of connected nodes in a network. In simple terms, they help us understand whether nodes with similar or different degrees tend to connect with each other.

### Types of Degree Correlations

1. **Assortative Mixing (Positive Correlation):**
   * High-degree nodes (hubs) are more likely to connect with other high-degree nodes.
   * Low-degree nodes tend to connect with other low-degree nodes.
   * Example: Social networks, where highly connected individuals (e.g., influencers) often interact with other highly connected individuals.
2. **Disassortative Mixing (Negative Correlation):**
   * High-degree nodes are more likely to connect with low-degree nodes.
   * Example: Biological networks, like protein-protein interaction networks, where a central protein (hub) interacts with many specialized, low-degree proteins.
3. **No Correlation (Neutral):**
   * The degrees of connected nodes are independent of each other.
   * Example: Random networks like those generated by the Erdős–Rényi model.

### Why Are Degree Correlations Important?

* **Network Resilience:** Disassortative networks are often more robust to random node failures, as hubs are spread across different parts of the network.
* **Epidemiology:** Assortative mixing can make disease spread faster, as highly connected individuals interact with each other.
* **Infrastructure Design:** Understanding degree correlations helps design efficient systems, such as traffic networks, where specific connections can minimize congestion.