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# CompAct Nets - A unifying tensor network architecture for probabilistic and logical reasoning

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## Abstract

We introduce Computation-Activation Networks (CompAct Nets), a novel architecture for tensor networks, which is adapted to represent propositional formulas and exponential distributions.

**Keywords:** Tensor Networks, Neuro-Symbolic AI

## 1 Introduction

Modern artificial intelligence is dominated by large-scale neural models that excel at pattern recognition but mostly remain black-box-solvers. Therefore, reliability and explainability are two main concerns when integrating these architecture into safety-critical processes. In contrast, classical symbolic approaches offer explicit logical structures and human-readable inference but can not handle uncertainty or scale to complex real-world data. Probabilistic models improved uncertainty handling but at the cost of explainability. Bridging these paradigms achieving both expressive architectures and transparent reasoning defines the central goal of *Neuro-Symbolic AI*, which seeks methods that combine the structural clarity of logic with the adaptability of neural computation within a single, mathematically coherent framework.

A central goal is to achieve *intrinsic explainability* rather than post-hoc interpretation. In conventional neural models, there are various ways to interpret a model after it has been trained, e.g. based on analyzing how changing input features influence the models prediction or based on fitting simpler surrogate models Barredo Arrieta et al. (2020); Lipton (2017). The proposed framework encodes symbolic relations that remain directly readable. Explainability is thus not added after training but built into the architecture itself.

The *tnreason framework* proposes tensor networks as a unifying mathematical foundation for reasoning and probabilistic inference. Tensor networks factor complex systems into interconnected logical and probabilistic components.

In this unifying mathematical framework, logical formulas corresponding to boolean tensors and probabilistic distributions corresponding to non-negative real tensors are combined.

Both can be manipulated through the same algebra of tensor contraction. This abstraction eliminates the traditional divide between symbolic and numerical representations: logical inference and probabilistic computations become different instances of the same underlying operation on structured tensors.

The *computation–activation architecture* organizes reasoning into two complementary tensor substructures. The computation network encodes the structural relations of a problem, such as logical dependencies or sufficient statistics, while the activation network assigns semantic or numerical values to these structures, representing truth assignments or probabilities. Their interaction defines a reasoning process as a tensor contraction between structure and activation. Logical inference emerges when activations are boolean, probabilistic inference when they are real-valued, and hybrid reasoning when both coexist within a shared tensor representation.

The logical tradition of artificial intelligence is motivated by the resemblance of human thought in logics McCarthy. Historic approaches to artificial intelligence have focused on models by vast knowledge bases and inference by logical reasoning. The main problem hindering the success of this approach is the inability of classical first-order logic to handle uncertainty of information, as present in realistic scenarios.

Towards extending the practical usage of logics, the field of Statistical Relational AI Nickel et al.; Getoor and Taskar studies statistical models of logical relations. This directly treats uncertainty and therefore unifies logics with statistical approaches. These aims have more recently reframed as neuro-symbolic AI Hochreiter; Sarker et al.; Colelough and Regli (2025), with close relations to statistical relational AI Marra et al.. Neuro-symbolic AI focuses on the unification of the neural and the symbolic paradigm Garcez et al., where early approaches are Towell and Shavlik; Avila Garcez and Zaverucha. While the symbolic paradigm is roughly understood as human understandable reasoning in formal logics, the neural paradigm is the computational benefit of decomposing a model into layers. These decompositions provide both expressive and efficiently inferable model architectures. While modern black-box AI focuses on large neural networks, whose size prevents human understanding of the inference process, neuro-symbolic AI aims at a re-implementation of the symbolic paradigm into such architectures.

Tensor networks have emerged as a highly efficient mathematical framework for handling data in high-dimensional spaces, effectively circumventing the "curse of dimensionality" that typically plagues grid-based methods Hackbusch. By decomposing high-order tensors into networks of low-rank components, these structures reduce the storage and computational complexity from exponential to polynomial with respect to the dimension Oseledets (2011); Hackbusch and Kühn (2009); Hitchcock (1927).

Historically rooted in quantum many-body physics White (1992), this framework found its first major success with Matrix Product States (MPS), originally developed to efficiently capture the quantum dynamics and ground states of one-dimensional spin chains Affleck et al. (1987). This format remains a standard tool in the field, with recent contributions refining it for tasks such as large-scale stochastic simulations and variational circuit operations Sander et al., 2025). To address the topological constraints of MPS, the landscape of architectures was subsequently expanded to include Projected Entangled Pair States (PEPS) for two-dimensional lattices and the Multi-scale Entanglement Renormalization Ansatz (MERA), which utilizes a hierarchical geometry to represent scale-invariant critical systems and has recently been adapted for simulating quantum systems Berezutskii et al. (2025); Orús (2019).

Beyond the quantum realm, these formats have been successfully adapted to applied mathematics, particularly for solving high-dimensional parametric PDEs, sampling problems, modeling complex continuous fields and learning dynamical laws Hagemann et al. (2025); Eigel et al. (2017); Goeßmann et al.. Furthermore, they exhibit properties helpful for handling these high-dimensional spaces, such as restricted isometry properties Goeßmann. Recent advancements have demonstrated the efficacy of these methods in capturing multiscale phenomena in fluid dynamics and turbulence, proving that the tensor network formalism offers a robust alternative to classical numerical schemes Gourianov et al. (2025).

Most significantly for the present work, we exploit the algebraic flexibility of tensor networks to bridge the gap between continuous numerical representation and discrete symbolic reasoning. By interpreting tensor contractions as logical operations within a basis calculus we can rigorously map propositional logic onto the linear-algebraic substrate of tensor networks. As shown in Goessmann (2025), this very versatile and flexible framework can be applied to unify logical and probabilistic modeling, enabling a single architecture to perform exact symbolic inference while retaining the efficient learnability of high-dimensional neural representations.

### 1.1 Related Works

The unification of symbolic and probabilistic approaches to interpretable model architectures has been a long-standing aim. Probabilistic grpahical models Pearl; Koller and Friedman are means to encode variable independences in graphs and are specific instances of exponential families Wainwright and Jordan. Markov Logic Networks Richardson and Domingos are specific instances of exponential families where also the dependencies are explicitly encoded in logical formulas. Further approaches treat uncertainties as generalized truth values **?**.

Tensor Networks have recently gained interest as a unifying language for AI, framed by Logical Tensor Networks Badreddine et al. and Tensor Logic Domingos. Different to the approaches therein we do not require non-linear transforms of tensors. Further, the MeLoCoToN approach Ali applies tensor network architectures similar to Computation-Activation Networks in combinatorical optimization problems.

### 1.2 Structure of the paper

The paper is organized as follows. Section 2 introduces the basic notation for categorical variables, tensors, and tensor networks, establishing the formal framework on which all subsequent reasoning structures are defined. Section 3 develops the probabilistic representation of reasoning through soft activation, showing how exponential-family distributions can be expressed as tensor networks based on independence assumptions and sufficient statistics. Section 4 turns to hard activation, formulating propositional logic within the same tensor framework and demonstrating how logical inference, entailment, and knowledge bases can be represented by boolean tensors and contractions. Section 5 unifies these two perspectives in the concept of Hybrid Logic Networks, which integrate hard logical constraints with soft probabilistic activations, thereby forming the core of the computation–activation architecture. The paper concludes with algorithmic considerations in section **??** and examples in section 6 that illustrate the expressive power and interpretability of this unified tensor-based reasoning approach.

## 2 Notation and Basic Concepts

We first, introduce the basic architecture and later show how the probabilistic and logical framework are covered. This is done in multiple steps. First, the considered variables and tensors are explained. Followed by the explanation of tensor calculations, mainly tensor products and contractions. Finally, the main architecture, the *Computation-Activation Network*, is defined.

### 2.1 Tensors

Tensors are multiway arrays and a generalization of vectors and matrices to higher orders. We will first provide a formal definition as real maps from index sets enumerating the coordinates of vectors, matrices and larger order tensors.

**Definition 1 (Tensor)** *For $k \in [d]$, let $m_k \in \mathbb{N}$ and let $X_k$ be categorical variables with values in $[m_k]$. A tensor $\tau[X_0, \ldots, X_{d-1}]$ of order $d$ and with leg dimensions $m_0, \ldots, m_{d-1}$ is defined through its coordinates*

$$\tau[X_0 = x_0, \ldots, X_{d-1} = x_{d-1}] \in \mathbb{R}$$

*for index tuples*

$$x_0, \ldots, x_{d-1} \in \bigtimes_{k \in [d]} [m_k] .$$

*Tensors $\tau[X_0, \ldots, X_{d-1}]$, also denoted by $\tau\left[X_{[d]}\right]$, are elements of the tensor space*

$$\bigotimes_{k \in [d]} \mathbb{R}^{m_k} ,$$

*which is a linear space, enriched with the operations of coordinate wise summation and scalar multiplication. We call a tensor $\tau\left[X_{[d]}\right]$ boolean, when $\mathrm{im}(\tau) \subset [2]$, i.e. all coordinates are either $0$ or $1$.*

This notation of tensors opposed to its notation through ordered indices as common in tensor calculus, facilitates writing down contractions along individual legs and other operations. Occasionally, when the categorical variables of a tensor are clear from the context, we will omit the notation of the variables.

**Example 1 (Trivial Tensor)** *The trivial tensor*

$$\mathbb{I}\left[X_{[d]}\right] \in \bigotimes_{k\in[d]} \mathbb{R}^{m_k}$$

*is defined by all coordinates being* 1*, that is for all* $x_0, \dots, x_{d-1} \in \times_{k\in[d]}[m_k]$

$$\mathbb{I}\left[X_{[d]} = x_{[d]}\right] = 1\,.$$

We are now ready to provide the link between tensors and states of systems with factored representations. To this end, we define the one-hot encoding of a state, which is a bijection between the states and the basis elements of a tensor space.

**Definition 2 (One-hot encodings to Atomic Representations)** *Given an atomic system described by the categorical variable* $X$*, we define for each* $x \in [m]$ *the basis vector* $\epsilon_x[X]$ *by the coordinates*

$$\epsilon_x[X = \tilde{x}] = \begin{cases} 1 & \text{if} \quad x = \tilde{x} \\ 0 & \text{else}\,. \end{cases} \tag{1}$$

*The* one-hot encoding of states $x \in [m]$ *of the atomic system described by the categorical variable* $X$ *is the map* $\epsilon : [m] \to \mathbb{R}^m$ *which maps* $x \in [m]$ *to the basis vectors* $\epsilon_x[X]$*.*

The basis vectors $\epsilon_x[X]$ are tensors of order 1 and leg dimension $m$ of the structure

$$\epsilon_x[X] = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T, \tag{2}$$

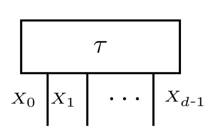
where the 1 is at the $x$th coordinate of the vector.

## 2.2 Contractions and Tensor Networks

Contractions are the central manipulation operation on sets of tensors. To introduce them, a graphical illustration of sets of tensors is explained, which we also call tensor networks.

### 2.2.1 Graphical Illustration

We will use the standard visualization of tensors, where they are represented by blocks with lines depicting the axes of the tensor blocks and each axis is assigned with a categorical variable $X_k$, or sometimes their index or dimension.




This depiction scheme has been established in the literature as wiring diagrams (see Landsberg and dates back at least to the work Penrose. Along this line, we represent vectors by blocks with one leg. The basis vectors being one-hot encodings of states are in this scheme represented by

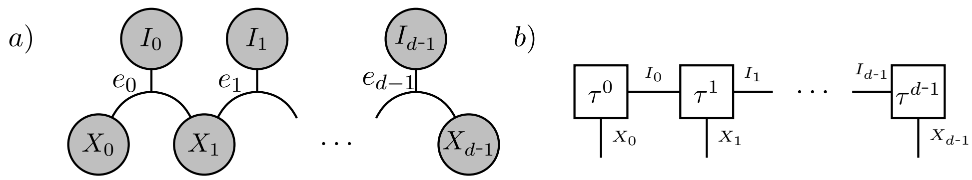
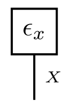



Figure 1: Hypergraph to a TT format. a) Node-centric design. b) Corresponding tensor-network on the edges of the hypergraph.




Assigning $x$ to the categorial variable $X$ will retrieve the $x$th coordinate (with value 1), whereas all other assignments will retrieve the coordinate values 0. Drawing on the interpretation of tensors by hyperedges we can continue with the definition of tensor networks.

**Definition 3 (Tensor Network)** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with nodes decorated by categorical variables $X_v$ with dimensions $m_v \in \mathbb{N}$ and hyperedges $e \in \mathcal{E}$ decorated by core tensors*

$$\tau^e [X_e] \in \bigotimes_{v \in e} \mathbb{R}^{m_v} ,$$

*where we denote by $X_e$ the set of categorical variables $X_v$ with $v \in e$. Then we call the set*

$$\tau^{\mathcal{G}} [X_{\mathcal{V}}] = \{\tau^e [X_e] \, : \, e \in \mathcal{E}\}$$

*the Tensor Network of the decorated hypergraph $\mathcal{G}$. The set of tensor networks on $\mathcal{G}$, such that all tensors have non-negative coordinates, is denoted by $\mathcal{T}^{\mathcal{G}}$.*

**Example 2 (The** TT **format)** *The Tensor-Train (*TT*) (see Oseledets (2011)) format corresponds in our notation with a hypergraph (see Figure 2)*

- *Nodes by $X_{[d]}$ and hidden variables $I_{[d-1]}$, each decorated by a dimension $m_{[d]}$ and $n_{[d-1]}$*
- *Edges by*

  $$\{e_0 = (X_0, I_0)\} \cup \{e_k = (I_{k-1}, X_k, I_{k+1}) \, : \, k \in \{1, \dots, d-2\}\} \cup \{e_{m-1} = (I_{d-2}, X_{d-1})\}$$

  *each decorated by a tensor of order 3 (respectively 2 for $k \in \{0, p-1\}$).*

**Example 3 (The** CP **format)** *The Candecomp-Parafac (*CP *(Hitchcock) tensor format corresponds in our notation with a hypergraph (see Figure 3)*

- *Nodes by $X_{[d]}$ and a single hidden variable $I$, decorated by dimensions $m_{[d]}$ and $n$.*
- *Edges by*

  $$\{e_k = (X_k, I) \, : \, k \in [d]\}$$

  *each decorated by a matrix $\tau^{e_k} [X_k, I]$.*

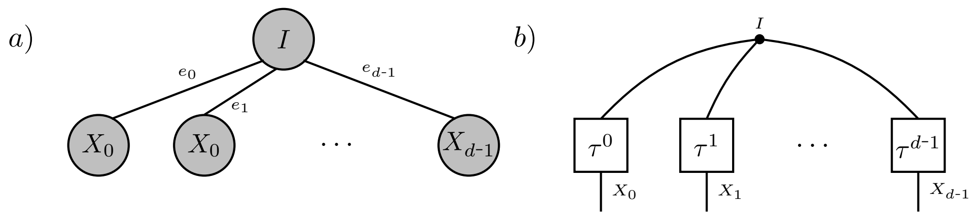



Figure 2: Hypergraph to a CP format. a) Node-centric design. b) Corresponding tensor-network on the edges of the hypergraph.

#### 2.2.2 Tensor Product

Let us now exploit the developed graphical representations to define contractions of tensor networks. The simplest contraction is the tensor product, which maps a pair of two tensors with distinct variables onto a third tensor and has an interpretation by coordinate wise products. Such a contraction corresponds with a tensor network of two tensors with disjoint variables.

**Definition 4 (Tensor Product)** *Let there be two tensors*

$$\tau\left[X_{[d]}\right] \in \bigotimes_{k\in[d]} \mathbb{R}^{m_k} \quad \text{and} \quad \tilde{\tau}\left[Y_{[p]}\right] \in \bigotimes_{l\in[p]} \mathbb{R}^{m_l}$$

*with different categorical variables assigned to its axes. Then their tensor product is the map*

$$\left\langle \tau\left[X_{[d]}\right], \tilde{\tau}\left[Y_{[p]}\right]\right\rangle_{\left[X_{[d]},Y_{[p]}\right]} \in \left(\bigotimes_{k\in[d]} \mathbb{R}^{m_k}\right) \otimes \left(\bigotimes_{l\in[p]} \mathbb{R}^{m_l}\right)$$

*defined coordinatewise for tuples of* $x_0,\ldots,x_{d-1} \in \bigtimes_{k\in[d]}[m_k]$ *and* $y_0,\ldots,y_{r-1} \in \bigtimes_{l\in[r]}[m_l]$ *as*

$$\begin{aligned}&\left\langle \tau\left[X_{[d]}\right], \tilde{\tau}\left[Y_{[p]}\right]\right\rangle_{\left[X_0=x_0,\ldots,X_{d-1}=x_{d-1},Y_0=y_0\ldots,Y_{p-1}=y_{p-1}\right]}\\ &\quad := \tau\left[X_0=x_0,\ldots,X_{d-1}=x_{d-1}\right]\cdot\tilde{\tau}\left[Y_0=y_0\ldots,Y_{p-1}=y_{p-1}\right].\end{aligned}$$

### 2.3 Generic Contractions

Contractions of Tensor Networks $\tau^{\mathcal{G}}$ are operations to retrieve single tensors by summing products of tensors in a network over common indices. We will define contractions formally by specifying just the indices not to be summed over.

When some of the variables are not appearing as leg variables, we define the contraction as being a tensor product with the trivial tensor $\mathbb{I}$ carrying the legs of the missing variables.

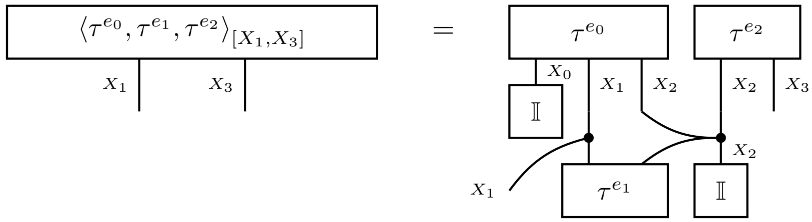



Figure 3: Example of a tensor network contraction of all but the variables $X_1, X_3$. Contraction of variables can always be depicted by closing the open legs with trivial tensors $\mathbb{I}$ performing index sums.

**Definition 5** *Let $\tau^{\mathcal{G}}$ be a tensor network on a decorated hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. For any subset $\tilde{\mathcal{V}} \subset \mathcal{V}$ we define the contraction to be the tensor (for an example see Figure 3)*

$$\left\langle \tau^{\mathcal{G}} \right\rangle_{[X_{\tilde{\mathcal{V}}}]} \in \bigotimes_{v \in \tilde{\mathcal{V}}} \mathbb{R}^{m_v} \tag{3}$$

*defined coordinatewise by the sum*

$$\left\langle \tau^{\mathcal{G}} \right\rangle_{[X_{\tilde{\mathcal{V}}} = x_{\tilde{\mathcal{V}}}]} = \sum_{x_{\mathcal{V}/\tilde{\mathcal{V}}} \in \times_{v \in \mathcal{V}/\tilde{\mathcal{V}}} [m_v]} \left( \prod_{e \in \mathcal{E}} \tau^e \left[ X_e = x_e \right] \right) . \tag{4}$$

*We call $X_{\tilde{\mathcal{V}}}$ the open variables of the contraction.*

To ease notation, we will often omit the set notation by brackets $\{\cdot\}$ and specify the tensors to be contracted with the delimiter "," (see e.g. Example **??**).

### 2.4 Directed Tensors and Normalizations

Directionality represents constraints on the structure of tensors, namely that the sum over outgoing trivializes the tensor.

**Definition 6** *A Tensor*

$$\tau \left[ X_{\mathcal{V}} \right] \in \bigotimes_{v \in \mathcal{V}} \mathbb{R}^{m_v}$$

*is said to be directed with incoming variables $\mathcal{V}^{\text{in}}$ and outgoing variables $\mathcal{V}^{\text{out}}$, where $\mathcal{V} = \mathcal{V}^{\text{in}} \dot{\cup} \mathcal{V}^{\text{out}}$, when*

$$\langle \tau \rangle_{[X_{\mathcal{V}^{\text{in}}}]} = \mathbb{I} \left[ X_{\mathcal{V}^{\text{in}}} \right]$$

*where $\mathbb{I} \left[ X_{\mathcal{V}^{\text{in}}} \right]$ denoted the trivial tensor in $\bigotimes_{v \in \mathcal{V}^{\text{in}}} \mathbb{R}^{m_v}$ which coordinates are all* 1.

By the normalization operation, tensors are turned into directed tensors.

**Definition 7** *A tensor $\tau[X_{\mathcal{V}}]$ is said to be normalizable on $\mathcal{V}^{\mathrm{in}} \subset \mathcal{V}$, if for any $x_{\mathcal{V}^{\mathrm{in}}} \in \bigtimes_{v \in \mathcal{V}^{\mathrm{in}}} [m_v]$ we have*

$$\left\langle \tau[X_{\mathcal{V}}], \epsilon_{x_{\mathcal{V}^{\mathrm{in}}}}[X_{\mathcal{V}^{\mathrm{in}}}] \right\rangle_{[\varnothing]} > 0 \,.$$

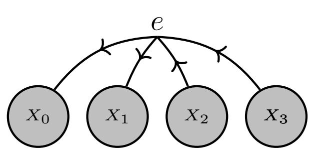
*The normalization of an on $\mathcal{V}^{\mathrm{in}} \subset \mathcal{V}$ normalizable tensor is the tensor*

$$\langle \tau[X_{\mathcal{V}}] \rangle_{[X_{\mathcal{V}^{\mathrm{out}}} | X_{\mathcal{V}^{\mathrm{in}}}]} = \sum_{x_{\mathcal{V}^{\mathrm{in}}} \in \bigtimes_{v \in \mathcal{V}^{\mathrm{in}}} [m_v]} \epsilon_{x_{\mathcal{V}^{\mathrm{in}}}}[X_{\mathcal{V}^{\mathrm{in}}}] \otimes \frac{\left\langle \tau[X_{\mathcal{V}}], \epsilon_{x_{\mathcal{V}^{\mathrm{in}}}}[X_{\mathcal{V}^{\mathrm{in}}}] \right\rangle_{[X_{\mathcal{V}^{\mathrm{out}}}]}}{\left\langle \tau[X_{\mathcal{V}}], \epsilon_{x_{\mathcal{V}^{\mathrm{in}}}}[X_{\mathcal{V}^{\mathrm{in}}}] \right\rangle_{[\varnothing]}}$$

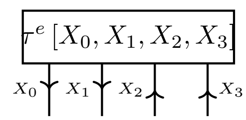
*where $\mathcal{V}^{\mathrm{out}} = \mathcal{V}/\mathcal{V}^{\mathrm{in}}$.*

In our graphical tensor notation, we depict directed tensors by directed hyperedges (a), which are decorated by directed tensors (b), for example:

*a)*




*b)*




### 2.5 Function encoding and Computation-Activation Networks

Let us now encode functions between the state sets of systems in factored representation.

**Definition 8 (Basis encoding of maps between state sets)** *Let there be two systems with factored representations by variables $X_{[d]}$ and $Y_{[p]}$, and a map*

$$q : \bigtimes_{k \in [d]} [m_k] \to \bigtimes_{l \in [r]} [m_l]$$

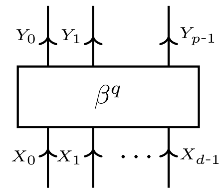
*between these state sets. Then the basis encoding of $q$ is a tensor*

$$\beta^q\left[Y_{[p]}, X_{[d]}\right] \in \left( \bigotimes_{l \in [p]} \mathbb{R}^{m_l} \right) \otimes \left( \bigotimes_{k \in [d]} \mathbb{R}^{m_k} \right)$$

*defined by*

$$\beta^q\left[Y_{[p]}, X_{[d]}\right] \quad = \sum_{x_0, \ldots, x_{d-1} \in \bigtimes_{k \in [d]} [m_k]} \epsilon_{q(x_{[d]})}\left[Y_{[p]}\right] \otimes \epsilon_{x_{[d]}}\left[X_{[d]}\right] \,.$$

Basis encodings are directed tensors (see (Goessmann, 2025, Theorem 14.10)) and are thus depicted as decorations of directed edges in hypergraphs: Explain the arrows. Haven't introduced directed graphs yet.




The entries of the basis encoding are defined by

$$\beta^q \left[Y_{[p]} = y_{[p]}, X_{[d]} = x_{[d]}\right] = \begin{cases} 1 & \text{if} \quad q(x_{[d]}) = y_{[p]} \\ 0 & \text{else} \end{cases}$$

Based on these concepts the main architecture can be defined.

**Definition 9 (Computation-Activation Network (CompAct Nets))** *Given a statistic $t : \bigtimes_{k\in[d]}[m_k] \to \mathbb{R}^p$, and a hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with nodes $[p] \subset \mathcal{V}$ containing the image coordinates of $t$, we define the by $t$ computable and by $\mathcal{G}$ activated family of distributions by*

$$\Lambda^{t,\mathcal{G}} = \left\{ \left\langle \beta^t \left[Y_{[p]}, X_{[d]}\right], \langle \xi \rangle_{\left[Y_{[p]}\right]} \right\rangle_{\left[X_{[d]}|\varnothing\right]} \,:\, \xi\left[Y_{\mathcal{V}}\right] \in \mathcal{T}^{\mathcal{G}} \right\} .$$

*We refer to any member $\mathbb{P}\left[X_{[d]}\right] \in \Lambda^{t,\mathcal{G}}$ as a Computation-Activation Network.*

To represent a Computation-Activation Network two tensor networks are needed: $\beta^{\mathcal{S}}$ to represent the basis encoding of the sufficient statistic (also called *computation network*) and $\xi$ to represent the tensor to contract with (also called *activation network*).

## 3 Decomposition of Probability Distributions

We here investigate tensor network decomposition mechanisms of probability distributions. After introducing probability distributions as tensors we derive tensor network decompositions based on conditional independencies (applying the Hammersley-Clifford theorem Clifford and Hammersley) to motivate graphical models. Further we present the Computation mechanism, in which the Fisher-Neyman Factorization Theorem is used to decompose distributions in the presence of sufficient statistics.

### 3.1 Basic concepts

Distributions $\mathbb{P}$ over a discrete state space can be represented by tensors, where each entry corresponds to the probability of a corresponding state. The joint probability distribution for a set of categorical variables as in definition **??** is defined here.

**Definition 10 (Joint Probability Distribution)** *Let there be for each $k \in [d]$ a categorical variable $X_k$ taking values in $[m_k]$. A joint probability distribution of these categorical variables is a function*

$$\mathbb{P}\left[X_{[d]}\right] \,:\, \bigtimes_{k\in[d]} [m_k] \to \mathbb{R}$$

*which is non-negative, that is for any* $x_{[d]} \in \times_{k\in[d]}[m_k]$ *it holds*

$$\mathbb{P}\left[X_{[d]} = x_{[d]}\right] \geq 0\,,$$

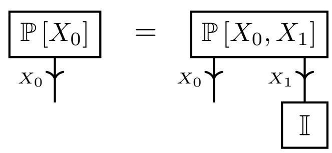
*and which is normalized, that is*

$$\left\langle \mathbb{P}\left[X_{[d]}\right] \right\rangle_{[\varnothing]} = 1\,.$$

**Example 4** *Coin toss Consider two coin tosses* $X_0, X_1 \in \{0,1\}$ *(1=heads). With* $p \in [0,1]$ *being the probability of heads. Then the probability for each toss and the joint probability have the form*

$$\mathbb{P}[X_0] = \begin{bmatrix} \mathbb{P}[X_0 = 0] \\ \mathbb{P}[X_1 = 1] \end{bmatrix} = \begin{bmatrix} 1-p \\ p \end{bmatrix}, \qquad \mathbb{P}[X_{[2]}] = \begin{bmatrix} (1-p)^2 & (1-p)p \\ p(1-p) & p^2 \end{bmatrix}$$

In most of the analysis, we assume that every state in the space is assigned a positive probability. We say, a distribution $\mathbb{P}\left[X_{[d]}\right]$ is *positive* if $\mathbb{P}[X_{[d]} = x_{[d]}] > 0$ for every configuration $x_{[d]}$. Note that in this tensor centered notation the calculation of marginal distributions reduces to contracting the open legs, not considered in the marginal with tensors $\mathbb{I}$ only containing ones, see (Goessmann, 2025, Section 2.1.3).




The number of coordinates in a tensor representation of probability distributions is the product

$$\prod_{k\in[d]} m_k\,,$$

and therefore scales exponentially in the number of coordinates. To find efficient representation schemes of probability distributions by tensor networks, we need to exploit additional properties of the distribution, such as Markov chain properties or independence. We further explore one property based on sufficient statistics.

### 3.2 The Independence mechanism: Graphical Model Factorization

Markov Networks on hypergraphs $\mathcal{G}$ are exactly those CompAct Nets, which are computable with respect to the identity and the graph $\mathcal{G}$. We here show how the sets of Markov Networks can be characterized by conditional independence assumptions.

Recall Def. 9. Given a statistic $t : \times_{k\in[d]}[m_k] \to \mathbb{R}^p$ and a hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ on the image coordinates $Y_{[p]}$, any by $t$ computable and by $\mathcal{G}$ activated CompAct Nets has the form

$$\mathbb{P}\left[X_{[d]}\right] = \left\langle \xi\left[Y_{[p]}\right], \beta^t\left[Y_{[p]}, X_{[d]}\right] \right\rangle_{\left[X_{[d]}|\varnothing\right]}$$

where $\xi\left[Y_{[p]}\right]$ is an arbitrary non-negative tensor. For graphical models we take the *identity statistic*

$$\delta(x_{[d]}) = x_{[d]},$$

so that the image coordinates coincide with the variables and there are no non-trivial computation cores. The associated basis encoding is just the identity tensor

$$\beta^{\delta}\left[Y_{[d]}, X_{[d]}\right] = \delta\left[X_{[d]}, Y_{[d]}\right] .$$

and therefore, for any activation tensor $\xi[Y[d]]$ we obtain

$$\mathbb{P}\left[X_{[d]}\right] = \left\langle \xi\left[Y_{[p]}\right], \beta^{\delta}\left[Y_{[d]}, X_{[d]}\right]\right\rangle_{\left[X_{[d]}|\varnothing\right]} \qquad = \left\langle \xi\left[X_{[d]}\right]\right\rangle_{\left[X_{[d]}|\varnothing\right]}$$

In other words, in the graphical–model case the activation tensor coincides with the joint distribution tensor. In this setting, structural properties of the distribution such as (conditional) independences can be read off as algebraic factorization patterns of the activation (and hence joint) tensor.

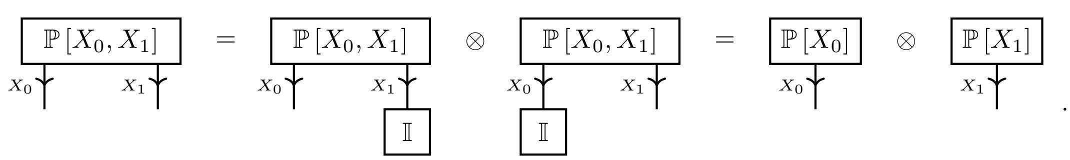
Independence leads to severe sparsifications of conditional probabilities and is therefore the key assumption to gain sparse decompositions of probability distributions. Before showing such decomposition schemes, we first provide a coordinatewise definition of independent variables.

**Definition 11 (Independence)** *We say that $X_0$ is independent of $X_1$ with respect to a distribution $\mathbb{P}[X_0, X_1]$, if for any values $x_0 \in [m_0]$ and $x_1 \in [m_1]$ the distribution satisfies*

$$\mathbb{P}[X_0, X_1] = \mathbb{P}[X_0] \otimes \mathbb{P}[X_1] ,$$

*In this case we denote $(X_0 \perp X_1)$.*

Thus, independence appears directly as a tensor–product decomposition of probability distribution. Using tensor network diagrams we depict this property by




Let us notice, that the assumption of independence reduces the degrees of freedom from $m_0 \cdot m_1 - 1$ to $(m_0 - 1) + (m_1 - 1)$. The decomposition into marginal distributions furthermore exploits this reduced freedom and provides an efficient storage. Having a joint distribution of multiple variables, which disjoint subsets are independent, we can iteratively apply the decomposition scheme. As a result, we can reduce the scaling of the degrees of freedom from exponential to linear by the assumption of independence.

Independence is, as we observed, a strong assumption, which is often too restrictive. Conditional independence instead is a less demanding assumption, which still implies efficient tensor network decompositions schemes. We introduce conditional independence as independence of variables with respect to conditional distributions.

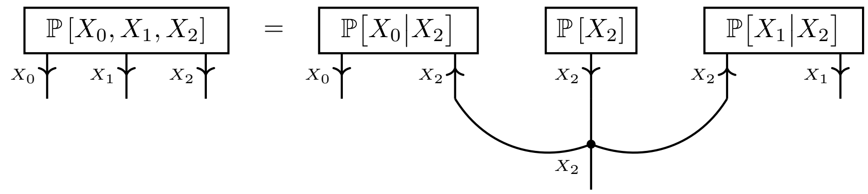



Figure 4: Diagrammatic visualization of the contraction equation in Cor. 13. Conditional independence of $X_0$ and $X_1$ given $X_2$ holds if the contraction on the right side is equal to the probability tensor on the left side.

**Definition 12 (Conditional Independence)** *Given a joint distribution of variables $X_0$, $X_1$ and $X_2$, such that $\mathbb{P}[X_2]$ is positive. We say that $X_0$ is independent of $X_1$ conditioned on $X_2$ if for any states $x_0 \in [m_0], x_1 \in [m_1]$ and $x_2 \in [m_2]$*

$$\mathbb{P}\big[X_0, X_1\big|X_2\big] = \big\langle \mathbb{P}\big[X_0\big|X_2\big], \mathbb{P}\big[X_1\big|X_2\big] \big\rangle_{[X_0, X_1, X_2]} .$$

*In this case we denote $(X_0 \perp X_1)|\, X_2$.*

There are different ways to decompose a probability distribution into hopefully smaller parts, which can be represented more efficiently.

Conditional independence stated in Def. 12 has a close connection with independence stated in Def. 11. To be more precise, $X_0$ is independent of $X_1$ conditioned on $X_2$, if and only if $X_0$ is independent of $X_1$ with respect to any slice $\mathbb{P}\big[X_0, X_1\big|X_2 = x_2\big]$ of the conditional distribution $\mathbb{P}\big[X_0, X_1\big|X_2\big]$. We further find a decomposition criterion for conditional independence. Since conditional independence can be regarded as a property of conditional probabilities, this decomposition criterion also involves conditional probabilities.

We can further exploit conditional independence to find tensor network decompositions of probabilities, as we show as the next corollary.

**Corollary 13** *If and only if $X_0$ is independent of $X_1$ conditioned on $X_2$ the probability distribution $\mathbb{P}$ satisfies (see Figure 4)*

$$\mathbb{P}[X_0, X_1, X_2] = \big\langle \mathbb{P}\big[X_0\big|X_2\big], \mathbb{P}\big[X_1\big|X_2\big], \mathbb{P}[X_2] \big\rangle_{[X_0, X_1, X_2]} .$$

This conditional–independence pattern is the basic local building block that is generalized in Markov networks, which we define in the following.

It is known that probabilistic graphical models are dual to tensor networks Robeva and Seigal; Glasser et al.. We define graphical models based on hypergraphs, to establish a direct connection with tensor network decorating the hypergraph. In a more canonical way, Markov Networks are instead defined by graphs, where instead of the edges the cliques are decorated by factor tensors (see for example Koller and Friedman).

**Definition 14 (Markov Network)** *Let $\tau^{\mathcal{G}}$ be a tensor network of non-negative tensors decorating a hypergraph $\mathcal{G}$. Then the Markov Network $\mathbb{P}^{\mathcal{G}}$ to $\tau^{\mathcal{G}}$ is the probability distribution of $X_v$ defined by the tensor*

$$\mathbb{P}^{\mathcal{G}}\left[X_{\mathcal{V}}\right] = \frac{\langle\{\tau^e : e \in \mathcal{E}\}\rangle_{[X_{\mathcal{V}}]}}{\langle\{\tau^e : e \in \mathcal{E}\}\rangle_{[\varnothing]}} = \left\langle \tau^{\mathcal{G}} \right\rangle_{[X_{\mathcal{V}}|\varnothing]} .$$

*We call the denominator*

$$\mathcal{Z}\left(\tau^{\mathcal{G}}\right) = \langle\{\tau^e : e \in \mathcal{E}\}\rangle_{[\varnothing]}$$

*the partition function of the tensor network $\tau^{\mathcal{G}}$.*

We can interpret the factors $\tau\left[X_{[d]}\right]$ as activation cores placed on the hyperedges $e$ of the graph. The global activation tensor (and hence the joint distribution) is obtained by contracting this activation network and normalizing by its partition function.

We call $\mathbb{P}\left[X_{[d]}\right]$ *positive* if $\mathbb{P}\left[X_{[d]} = x_{[d]}\right] > 0$ for all states $x_{[d]}$. The marginalization of a Markov Network to $\tau^{\mathcal{G}}$ on subsets of variables $X_{\tilde{\mathcal{V}}}$ is

$$\mathbb{P}^{\mathcal{G}}\left[X_{\tilde{\mathcal{V}}}\right] = \left\langle \tau^{\mathcal{G}} \right\rangle_{\left[X_{\tilde{\mathcal{V}}}|\varnothing\right]} .$$

Further, the distribution of $X_{\tilde{\mathcal{V}}}$ conditioned on $X_{\bar{\mathcal{V}}}$, where $\tilde{\mathcal{V}}, \bar{\mathcal{V}}$ are disjoint subsets of $\mathcal{V}$, is

$$\mathbb{P}^{\mathcal{G}}\left[X_{\tilde{\mathcal{V}}}|X_{\bar{\mathcal{V}}}\right] = \left\langle \tau^{\mathcal{G}} \right\rangle_{\left[X_{\tilde{\mathcal{V}}}|X_{\bar{\mathcal{V}}}\right]} .$$

While we have directly defined Markov Networks as decomposed probability distributions, we now want to derive assumptions on a distribution assuring that such decompositions exist. As we will see, the sets of conditional independencies encoded by a hypergraph are captured by its seperation properties, as we define next.

**Definition 15 (Separation of Hypergraph)** *A path in a hypergraph is a sequence of nodes $v_k$ for $k \in [d]$, such that for any $k \in [d-1]$ we find a hyperedge $e \in \mathcal{E}$ such that $(v_k, v_{k+1}) \subset e$. Given disjoint subsets $A$, $B$, $C$ of nodes in a hypergraph $\mathcal{G}$ we say that $C$ separates $A$ and $B$ with respect to $\mathcal{G}$, when any path starting at a node in $A$ and ending in a node in $B$ contains a node in $C$.*

To characterize Markov Networks in terms of conditional independencies we need to further define the property of clique-capturing. This property of clique-capturing established a correspondence of hyperedges with maximal cliques in the more canonical graph-based definition of Markov Networks Koller and Friedman.

**Definition 16 (Clique-Capturing Hypergraph)** *We call a hypergraph $\mathcal{G}$ clique-capturing, when each subset $\tilde{\mathcal{V}} \subset \mathcal{V}$ is contained in a hyperedge, if for any $a, b \in \tilde{\mathcal{V}}$ there is a hyperedge $e \in \mathcal{E}$ with $a, b \in \tilde{\mathcal{V}}$.*

Let us now show a characterization of Markov Networks in terms of conditional independencies.

**Theorem 17 (Hammersley-Clifford Factorization Theorem)** *Given a clique-capturing hypergraph $\mathcal{G}$, the set of positive Markov Networks on the hypergraph coincides with the set of positive probability distributions, such that for each disjoint subsets of variables $A$, $B$, $C$ we have $X_A$ is independent of $X_B$ conditioned on $X_C$, when $C$ separates $A$ and $B$ in the hypergraph.*

**Example 5 (I.i.d. Boolean Variables: Coin toss interpretation)** *Let there be $d$ boolean variables $X_{[d]}$, which are i.i.d. drawn from a positive distribution $\mathbb{P}[X]$. From the pairwise independencies of $X_k$ it follows with the Hammersley-Clifford Factorization Theorem that the distribution is representable by an elementary tensor network, that is*

$$\mathbb{P}\left[X_{[d]}\right] = \bigotimes_{k\in[d]} \mathbb{P}\left[X_k\right] .$$

Equivalently, Thm. 17 states that for any strictly positive joint distribution $\mathbb{P}[X_V]$ whose conditional independencies are exactly those encoded by a clique–capturing hypergraph $G = (V, E)$, there exist non–negative activation cores $\tau_e[X_e]$ such that

$$P[X_V] = \frac{1}{Z} \left\langle\right\rangle_{[]} \left\langle\{\tau_e : e \in E(G)\}\right\rangle[X_V],$$

for a suitable normalizing constant $Z > 0$. Thus, the conditional–independence structure of $\mathbb{P}$ determines a global tensor–network decomposition of its activation (and hence joint) tensor. We refer to this correspondence between independence structure and tensor–network factorization as the *independence mechanism*, in analogy to the computation mechanism provided by sufficient statistics in Section 3.1.

### 3.3 The Computation mechanism: Factorization in presense of Sufficient Statistics

**Definition 18** *Let $\mathbb{P}[X, Z]$ be a joint distribution of the $m$-dimensional variable $X$ and the $n$-dimensional variable $Z$ and let*

$$t\,:\,[m] \to [n]$$

*be a statistic. We are interested in the distribution $\mathbb{P}[X, Z, Y_t] = \left\langle \mathbb{P}[X, Z], \beta^t[Y_t, X]\right\rangle_{[X,Z,Y_t]}$. We say that $t$ is a sufficient statistic for $Z$ if and only if $X$ is independent of $Z$ conditioned on $Y_t$.*

Note that the independence is true if and only if

$$\mathbb{P}\left[X \middle| Z, Y_t\right] = \mathbb{P}\left[X \middle| Y_t\right] \otimes \mathbb{I}[Z] .$$

**Example 6 (Sufficient Statistics for the Probability)** *Let $Z$ be the value $\mathbb{P}\left[X_{[d]} = x_{[d]}\right]$, when drawing $X_{[d]}$ from $\mathbb{P}\left[X_{[d]}\right]$. Then $t$ is a sufficient statistic for $Z = \mathbb{P}\left[X_{[d]}\right]$, if for all $y$ in the image of $t$ we have*

$$\mathbb{P}\left[X_{[d]} = x_{[d]} \middle| t\left(x_{[d]}\right) = y\right] = \begin{cases} \frac{1}{\left|\{x_{[d]}\,:\,t\left(x_{[d]}\right)=y\}\right|} & \textit{if} \quad t(x_{[d]}) = y \\ 0 & \textit{else} \end{cases} .$$

*When knowing the value $tx_{[d]}$ of the sufficient statistic at a given index $x_{[d]}$, we then also know the probability $\mathbb{P}\left[X_{[d]} = x_{[d]}\right]$. The function $t$ is thus a sufficient statistic for $Z = \mathbb{P}\left[X_{[d]}\right]$, if and only if there is a tensor $\xi\left[Y_{[p]}\right]$ with*

$$\mathbb{P}\left[X_{[d]}\right] = \left\langle \beta^t\left[Y_{[p]}, X_{[d]}\right], \xi\left[Y_{[p]}\right]\right\rangle_{\left[X_{[d]}\right]} .$$

Example 6 hints at a connection between sufficient statistics and decompositions into CompAct Nets. More generally, such decompositions are provided by the Fisher-Neyman Factorization Theorem.

**Theorem 19 (Fisher-Neyman Factorization Theorem)** *Let $\mathbb{P}$ be a joint distribution of variables $Z, X$ with values $\mathrm{val}(Z), \mathrm{val}(X)$ and let $T(X)$ be a statistic. Then $t$ is a sufficient statistic for $Z$ if and only if there are tensors $\nu[X]$ and $\xi[Y_t, Z]$ such that*

$$\mathbb{P}[X, Z] = \left\langle \xi[Y_t, Z]\,\beta^t[Y_t, X], \nu[X]\right\rangle_{[X,Z]} .$$

The Fisher-Neyman Theorem is the fundamental motivation for the CompAct Nets Architecture:

- The tensors in a decomposition of $\xi\left[Y_{[p]}\right]$ are called *activation cores.*
- The tensors in a decomposition of $\beta^t\left[Y_{[p]}, X_{[d]}\right]$ are called *computation cores.*

**Example 7 (Order Statistic for Boolean Variables: Coin toss interpretation)** *Let there be $d$ boolean variables $X_{[d]}$ and a family $\{\mathbb{P}^\theta\left[X_{[d]}\right] : \theta \in \Theta\}$ of distributions. The order statistic assigns to each tuple $x_{[d]}$ the ordered tuple, which effectively counts the number of 1 coordinates in the tuple $x_{[d]}$, that is the statistic*

$$t^+ : \bigtimes_{k\in[d]} [m_k] \to [p] \quad , \quad t^+(x_{[d]}) = |\{k : x_k = 1\}| .$$

*When the order statistic is sufficient, the detailed order of the outcomes in uninformative about the member $\theta \in \Theta$ from which the random variables have been drawn. Let us now investigate those families for which $t^+$ is a sufficient statistic. By the Fisher-Neyman Factorization Theorem Thm. 19 $t^+$ is a sufficient statistic if and only if there are tensors $\nu\left[X_{[d]}\right]$ and $\xi[Y_+, \Theta]$ such that for each $\theta \in \Theta$*

$$\mathbb{P}^\theta\left[X_{[d]}\right] = \left\langle \xi^\theta[Y_+], \beta^{t^+}\left[Y_+, X_{[d]}\right], \nu\left[X_{[d]}\right]\right\rangle_{\left[X_{[d]}\right]} .$$

*The family of distributions, such that the variables $X_{[d]}$ are i.i.d. with respect to each (see Example 5) are the special case, where $\nu\left[X_{[d]}\right] = \mathbb{I}\left[X_{[d]}\right]$ and the family is labeled by $\theta \in [0,1]$ such that for $\theta \in (0,1)$ and $k \in [d+1]$*

$$\xi^\theta[Y_+ = k] = \theta^{d-k} \cdot \theta^k ,$$

*and for $\theta \in \{0,1\}$*

$$\xi^\theta[Y_+] = \begin{cases} \epsilon_0[Y_+] & \text{if} \quad \theta = 0 \\ \epsilon_d[Y_+] & \text{if} \quad \theta = 1 \end{cases} .$$

*The marginal distribution $\mathbb{P}^\theta[Y_+]$ is then the binominal distribution $B(d, \theta)$.*

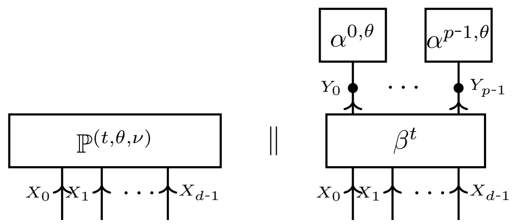



Figure 5: Missing base measure in the diagram! - Find convention. Tensor Network diagram of a member of an exponential family before normalization as an CompAct Net with elementary activation, that is an element in $\Lambda^{t,\mathrm{EL},\nu}$.

### 3.4 Exponential families in case of elementary activation tensors

A classical theorem by Pitman-Koopman-Darmois (see **?**) states, that whenever a family with constant support and a finite sufficient statistic for arbitrary large data sets is in an exponential family. We now restrict the activation cores to specific elementary tensors, which correspond with further assumptions on the dependence of $\mathbb{P}$ and $t$ made by exponential families. For a discussion of further universal properties of exponential families, such that the existence of priors and entropy maximizers, see **?**.

**Definition 20 (Exponential Family)** *from Thm.* **??** *Given any base measure $\nu$ and a sufficient statistic $t$ we enumerate for each coordinate $l \in [p]$ the image $\mathrm{im}(t_l)$ by a variable $Y_l$ taking values in $[|\mathrm{im}(t_l)|]$ (see for more details on this scheme Chapter* **??***), given an interpretation map*

$$I_l \,:\, [|\mathrm{im}(t_l)|] \to \mathrm{im}(t_l) \ .$$

*For any canonical parameter vector $\theta[L] \in \mathbb{R}^p$ we build the activation cores $\alpha^{l,\theta}[Y_l]$ for each coordinate $y_l \in [|\mathrm{im}(t_l)|]$ by*

$$\alpha^{l,\theta}[Y_l = y_l] = \exp\left[\theta[L=l] \cdot I_l(y_l)\right]$$

*and have*

$$\mathbb{P}^{(t,\theta,\nu)}\left[X_{[d]}\right] = \left\langle \{\nu\left[X_{[d]}\right]\} \cup \{\beta^{t_l}\left[Y_l, X_{[d]}\right] \,:\, l \in [p]\} \cup \{\alpha^{l,\theta}[Y_l] \,:\, l \in [p]\} \right\rangle_{\left[X_{[d]}|\varnothing\right]} .$$

*We then call the set*

$$\Gamma^{t,\nu} = \{\mathbb{P}^{(t,\theta,\nu)} \,:\, \theta[L] \in \mathbb{R}^p\}$$

*the exponential family with respect to the statistic $t$ and the base measure $\nu$.*

**Example 8 (Joint distributions of two booleans)** *In general, joint distribution of two Boolean variables $X_0, X_1$ are 2x2 matrices of non-negative coordinates summing to 1:*

$$\mathbb{P}\left[X_{[2]}\right] = \begin{bmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{bmatrix}$$

*In the following example, we will assume at different points that $X_0, X_1$ have a sufficient statistic, are independent and they have positive distributions. By the normalization constraint, $p_{1,1}$ is determined from $p_{0,0}, p_{0,1}$ and $p_{1,0}$, which leaves us with three free parameters.*

$$\mathbb{P}\left[X_{[2]}\right] = \begin{bmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & 1-(p_{0,0}+p_{0,1}+p_{1,0}) \end{bmatrix}$$

*Let us now restrict to those distributions, which have the sum $X_0 + X_1$ as a sufficient statistic. They need to satisfy $p_{0,1} = p_{1,0}$ (since in that cases the statistic is 1 and the definition of sufficiency is that the distribution conditioned on the statistic is uniform), leaving us with two free parameters.*

$$\mathbb{P}\left[X_{[2]}\right] = \begin{bmatrix} p_{0,0} & p_{0,1} \\ p_{0,1} & 1-p_{0,0}-2p_{0,1} \end{bmatrix}$$

*This symmetry also implies, that the distributions are identically distributed, i.e. for any $x \in \{0,1\}$ we have*

$$\mathbb{P}\left[X_0 = x\right] = \left\langle \mathbb{P}\left[X_0, X_1\right] \right\rangle_{[X_0=x]} = \left\langle \mathbb{P}\left[X_1, X_0\right] \right\rangle_{[X_1=x]} = \mathbb{P}\left[X_1 = x\right] .$$

*Restricting further to those, where $X_0$ and $X_1$ are independent and the distribution is everywhere supported, brings us to the rank one formulation of the distribution*

$$\mathbb{P}\left[X_0, X_1\right] = \begin{bmatrix} \mathbb{P}\left[X_0=0\right]\mathbb{P}\left[X_1=0\right] & \mathbb{P}\left[X_0=0\right]\mathbb{P}\left[X_1=1\right] \\ \mathbb{P}\left[X_0=1\right]\mathbb{P}\left[X_1=0\right] & \mathbb{P}\left[X_0=1\right]\mathbb{P}\left[X_1=1\right] \end{bmatrix} = \mathbb{P}\left[X_0\right] \otimes \mathbb{P}\left[X_1\right]$$

*In terms of an exponential family with the head count as a sufficient statistic, we parametrize the distribution by the canonical parameter $\theta \in \mathbb{R}$ as*

$$\mathbb{P}\left[X_0\right] = \frac{1}{1+\exp\left[\theta\right]} \begin{bmatrix} 1 \\ \exp\left[\theta\right] \end{bmatrix}$$

*Note, that with this parametrization the probabilities for head and tail automatically have the form $p, (1-p)$.*

$$\mathbb{P}\left[X_0, X_1\right] = \frac{1}{(1+\exp\left[\theta\right])^2} \begin{bmatrix} 1 \\ \exp\left[\theta\right] \end{bmatrix} \begin{bmatrix} 1 & \exp\left[\theta\right] \end{bmatrix}$$

*We can interpret this distribution as two independent coin tosses with outcome $X_0$ and $X_1$ and head probability*

$$\mathbb{P}\left[X_0 = 1\right] = \mathbb{P}\left[X_1 = 1\right] = \frac{\exp\left[\theta\right]}{1+\exp\left[\theta\right]}$$

*which is the sigmoid of $\theta$ and inverted by the logit*

$$\theta = \ln\left[\frac{\mathbb{P}\left[X_0=1\right]}{1-\mathbb{P}\left[X_0=1\right]}\right] .$$

*Consistent with the above parametrization, we have a uniform distribution of $X_0$ and $X_1$ in the fair coin toss case $\mathbb{P}\left[X_0 = 1\right] = 0.5$, where $\theta = 0$.*

*As a Computation-Activation Network we can represent any distribution* $\mathbb{P}[X_0, X_1]$ *with the head count* $+$ *as sufficient statistic by*

$$\mathbb{P}[X_0, X_1] = \left\langle \beta^+[Y_+, X_0, X_1], \xi[Y_+] \right\rangle_{[X_0, X_1|\varnothing]},$$

*such that*

$$\begin{aligned}\mathbb{P}[X_0 = x_0, X_1 = x_1] &= \frac{1}{Z}\left\langle \beta^+[Y_+, X_0, X_1], \xi[Y_+] \right\rangle_{[X_0 = x_0, X_1 = x_1]} \\ &= \frac{1}{Z}\sum_{y_+ \in [2]} \beta^+[Y_+ = y_+, X_0 = x_0, X_1 = x_1] \cdot \xi[Y_+ = y_+] \\ &= \frac{1}{Z}\xi[Y_+ = x_0 + x_1],\end{aligned}$$

*where the normalization constant* $Z$ *cancels out any multiplicative constant* $\lambda \in \mathbb{R}\backslash\{0\}$ *in* $\xi$ *and the equation above implies*

$$\xi[Y] = \lambda \cdot \begin{bmatrix} p_{0,0} \\ p_{0,1} \\ p_{1,1} \end{bmatrix}.$$

*We choose* $\lambda = 1/p_{0,0} = (1 + \exp[\theta])^2$ *in the following. Among these distribution, the exponential family with the head count statistic is then parametrized by activation tensors*

$$\xi[Y] = \begin{bmatrix} 1 \\ p_{0,1}/p_{0,0} \\ p_{1,1}/p_{0,0} \end{bmatrix} = \begin{bmatrix} 1 \\ \exp[\theta] \\ \exp[2\theta] \end{bmatrix},$$

*since* $p_{0,1} = \mathbb{P}[X_0 = 0] \cdot \mathbb{P}[X_0 = 1] = (1+\exp[\theta])^{-1} \cdot \exp[\theta](1+\exp[\theta])^{-1}$ *and* $p_{1,1} = (\exp[\theta](1+\exp[\theta])^{-1})^2$.

### 3.5 Reasoning based on contractions

The most central inference operation on probability distributions is the computation of marginals, which is just the contraction leaving only the marginalized variable open. Conditional probabilities are marginal proabilities of the distribution with added boolean tensors marking the evidence (e.g. one-hot encodings to states of some variables). With this formalism we can answer queries like "What is the probability of booking this account when having observed this feature on the invoice?".

## 4 Decompositions based on Propositional Syntax

A tensor-based representation of propositional logic is developed by encoding boolean variables into vectors, defining formulas as boolean tensors, and showing how logical connectives and normal forms can be expressed as tensor contractions.

### 4.1 Propositional Semantics by Boolean Tensors

**Definition 21** *A* propositional formula $f\left[X_{[d]}\right]$ *depending on* $d$ *atoms* $X_k$ *is a boolean-valued tensor*

$$f\left[X_{[d]}\right] : \bigtimes_{k\in[d]} [2] \to \{0,1\} \subset \mathbb{R}\,.$$

*We call a state* $x_{[d]} \in \bigtimes_{k\in[d]}[2]$ *a* model *of a propositional formula* $f$*, if*

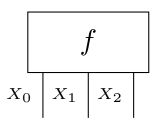
$$f\left[X_{[d]} = x_{[d]}\right] = 1\,,$$

*where we associate True* $\leftrightarrow 1$ *and False* $\leftrightarrow 0$*. If there is a model to a propositional formula, we say the formula is* satisfiable*.*

**Example 9** *Let there be* $d = 3$ *boolean variables* $X_{[3]}$ *and a propositional formula*

$$f\left[X_{[3]}\right] = (X_0 \vee X_1) \wedge \neg X_2\,.$$

*In a graphical depiction and in the coordinatewise representation this formula can be represented as*

$f\left[X_{[3]}\right] =$ 



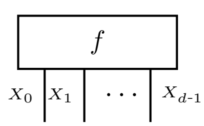
$$= \quad \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \,.$$

*In the state set* $\bigtimes_{k\in[d]}[2] = \{0,1\}\times\{0,1\}\times\{0,1\}$ *we have three models of the formula by the positions of the non-zero entries in the tensor, i.e.* $f\left[X_{[3]} = x_{[3]}\right] = 1$ *if and only if*

$$x_{[3]} \in \{(1,0,0),(0,1,0),(1,1,0)\}\,.$$

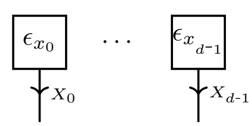
*The formula* $f$ *is therefore satisfiable.*

**CP decomposition** Since the tensor $f\left[X_{[d]}\right]$ is equal to one at index $x_{[d]}$ if and only if $x_{[d]}$ is a model of $f$, i.e. fulfills the formula, A propositional formula can be written as the sum over the one-hot encodings of its models.





$$= \sum_{\substack{x_0,\ldots,x_{d-1}\in \bigtimes_{k\in[d]}[2] \\ f(x_0,\ldots,x_{d-1}) = 1}}$$





This decomposition corresponds to the CP decomposition of a tensor.

**Example 10** *For the formula described in Example* **??***, we have*

$$\begin{aligned} f\left[X_{[3]}\right] &= (\epsilon_1[X_0]\otimes\epsilon_0[X_1]\otimes\epsilon_0[X_2]) + (\epsilon_0[X_0]\otimes\epsilon_1[X_1]\otimes\epsilon_0[X_2]) \\ &\quad + (\epsilon_1[X_0]\otimes\epsilon_1[X_1]\otimes\epsilon_0[X_2]), \end{aligned}$$

*where we denote the vectors* $\epsilon_1[Y] = [0,1]^T$ *and* $\epsilon_0[Y] = [1,0]^T$. *Then for the model* $x_{[3]} = (1,1,0)$ *it holds*

$$\begin{aligned} f\left[X_{[3]} = x_{[3]}\right] &= \left(\epsilon_1[X_0 = 1] \otimes \epsilon_0[X_1 = 1] \otimes \epsilon_0[X_2 = 0]\right) \\ &\quad + \left(\epsilon_0[X_0 = 1] \otimes \epsilon_1[X_1 = 1] \otimes \epsilon_0[X_2 = 0]\right) \\ &\quad + \left(\epsilon_1[X_0 = 1] \otimes \epsilon_1[X_1 = 1] \otimes \epsilon_0[X_2 = 0]\right) \\ &= 1 \cdot 0 \cdot 1 + 0 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 = 1. \end{aligned}$$

**Model counts by contraction** Each coordiante of the propositional formula is either a 1 or 0 encoding if the indexed state is a model of the formula or not. In this way, the contraction $\langle f \rangle_{[\varnothing]}$ counts the number of models of the propositional formula $f$. One can therefore decide the satisfiability of a formula by checking if $\langle f \rangle_{[\varnothing]} > 0$.

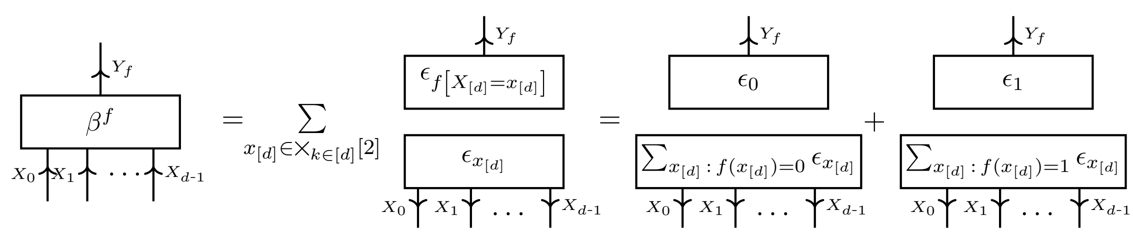
**Basis encoding** Representing booleans by elements in $\{0,1\}$ leads to the problem, that negation is an affine transformation and can not be represented by multilinear tensors (Goessmann, 2025, Section 4.1.1). Therefore, instead of using this *coordinate calculus* an approach based on *basis calculus* is employed, which is explained in this section. To be able to express different kinds of connectives and finally any propositional formula by multi-linear tensors, booleans are encoded by one-hot encodings as defined in Def. 2. Propositional formulas $f$ can be expressed in terms of a tensor describing the mapping and its negation by

$$\beta^f\left[Y_f = y_f, X_{[d]} = x_{[d]}\right] = \begin{cases} 1 & \text{if} \quad f\left[X_{[d]} = x_{[d]}\right] = y_f \\ 0 & \text{else} \end{cases} . \tag{5}$$

This basis encoding $\beta^f\left[Y_f, X_{[d]}\right] \in \{0,1\}^{2 \times 2^d}$ then has the form

$$\beta^f\left[Y_f, X_{[d]}\right] = \epsilon_1[Y_f] \otimes f\left[X_{[d]}\right] + \epsilon_0[Y_f] \otimes \neg f\left[X_{[d]}\right] . \tag{6}$$

In our graphical notation this property is visualized by




We further provide a more detailed example in coordinate sensitive notation in the following.

**Example 11 (Logical Negation and Conjunction)** *The basis encodings of the negation* $\neg : [2] \to [2]$ *is the matrix*

$$\beta^{\neg}[Y_\neg, X] = \begin{array}{cc} & \begin{array}{cc} \multicolumn{2}{c}{Y_\neg} \\ 0 & 1 \end{array} \\ X_0 \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

*The 2-ary conjunctions $\wedge : [2] \times [2] \rightarrow [2]$ is encoded by the order-3 tensor*

$$\beta^{\wedge}\left[Y_{\wedge}, X_0, X_1\right] = \begin{array}{c} \\ Y_{\wedge} \end{array} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{array}{cc} & \overset{X_1}{\begin{array}{cc} 0 & 1 \end{array}} \\ X_0 & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} + \begin{array}{c} \\ Y_{\wedge} \end{array} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{array}{cc} & \overset{X_1}{\begin{array}{cc} 0 & 1 \end{array}} \\ X_0 & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array} = \begin{array}{cc} & \overset{X_1}{\begin{array}{cc} 0 & 1 \end{array}} \\ X_0 & \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_{0} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{1}}_{Y_{\wedge}} \end{array}$$

*Further, the 2-ary disjunction $\vee : [2] \times [2] \rightarrow [2]$ is encoded by the order-3 tensor*

$$\beta^{\vee}\left[Y_{\vee}, X_0, X_1\right] = \begin{array}{cc} & \overset{X_1}{\begin{array}{cc} 0 & 1 \end{array}} \\ X_0 & \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{0} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}_{1}}_{Y_{\vee}} \end{array}$$

**Interpretation as CompAct Nets** The propositional formula and its negation can be represented by that tensor by

$$f\left[X_{[d]}\right] = \left\langle \epsilon_1\left[Y_f\right], \beta^f\left[Y_f, X_{[d]}\right] \right\rangle_{\left[X_{[d]}\right]} \quad \text{and} \quad \neg f\left[X_{[d]}\right] = \left\langle \epsilon_0\left[Y_f\right], \beta^f\left[Y_f, X_{[d]}\right] \right\rangle_{\left[X_{[d]}\right]} .$$

Both $f$ and $\neg f$ are thus Computation-Activation Networks to the statistic $\{f\}$ and the hard activation tensor $\epsilon_1\left[Y_f\right]$, respectively $\epsilon_0\left[Y_f\right]$. This representation of propositional formulas with respect to basis encoding thus leads to Computation-Activation Networks, which were also used to describe probability distributions in the last section. In this way the soft and hard logic can be combined in one framework.

### 4.2 Decomposition of Propositional Formulas

We now show, that the propositional formula allows for a decomposition into connective formulas, its basis encoding decomposes into the basis encodings of the connective formulas.

**Lemma 22** *Let $f\left[X_{[d]}\right]$ be a composition of a $p$-ary connective formula $\circ$ and propositional formulas $f_l\left[X_{[d]}\right]$, where $l \in [p]$, i.e. for $x_{[d]} \in \times_{k\in[d]}[2]$ we have*

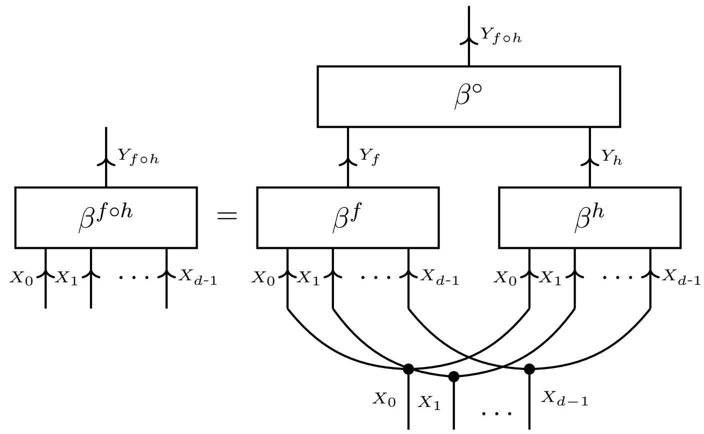
$$f\left[X_{[d]} = x_{[d]}\right] = \circ\left(f_0\left[X_{[d]} = x_{[d]}\right], \dots, f_{p-1}\left[X_{[d]} = x_{[d]}\right]\right) .$$

*Then we have*

$$\beta^f\left[Y_f, X_{[d]}\right] = \left\langle \left\{\beta^{\circ}\left[Y_f, Y_{[p]}\right]\right\} \cup \left\{\beta^{f_l}\left[Y_l, X_{[d]}\right] \,:\, l \in [p]\right\} \right\rangle_{\left[Y_f, X_{[d]}\right]} .$$

**Proof** This can be shown on each index $x_{[d]}$. ■

For the composition of two propositional formulas $f\left[X_{[d]}\right]$ and $h\left[X_{[d]}\right]$ the composition by some binary connective is pictured by:




Let us now define a more generic syntactical decomposition of propositional formulas.

**Definition 23** *A syntactical hypergraph is a directed acyclic hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ such that*

- *each hyperedge $e = (e^{\mathrm{in}}, e^{\mathrm{out}})$ has exactly one outgoing node, i.e. $\left|e^{\mathrm{out}}\right| = 1$*
- *each node $v \in \mathcal{V}$ carries a boolean variable $Y_v$ and appears at most once as the outgoing node of a hyperedge*
- *each hyperedge $(e^{\mathrm{in}}, \{v\})$ with $e^{\mathrm{in}} \neq \varnothing$ is decorated by a propositional formula*

$$\circ_v \left[Y_{e^{\mathrm{in}}}\right] : \bigtimes_{v \in e^{\mathrm{in}}} [2] \to [2]$$

- *the node not appearing as an outgoing node are labeled by $[d]$*

*We say that the syntactical hypergraph is single-rooted, if exactly one node $\tilde{v}$ does not appear as an incoming node of a hyperedge. In this case this unique node is called the root node. We assign atomic formulas to the nodes $[d]$ and recursively assign to each further node $v$ a node formula*

$$f_v \left[X_{[d]} = x_{[d]}\right] = \circ_v \left[\left[f_{\bar{v}} \left[X_{[d]} = x_{[d]}\right] \,:\, \bar{v} \in e^{\mathrm{in}}\right]\right] \quad \forall x_{[d]} \in \bigtimes_{k \in [d]} [m_k] \,,$$

*where $e^{\mathrm{in}}$ are the incoming nodes in the unique hyperedge with outgoing nodes $\{v\}$. We call the formula $f\left[X_{[d]}\right] \coloneqq f_{\tilde{v}}\left[X_{[d]}\right]$ to the root note $\tilde{v}$ the syntactical composition of $\mathcal{G}$ and $\mathcal{G}$ is a syntactical decomposition of $f$.*

**Theorem 24** *For any syntactical hypergraph $\mathcal{G}$ with composition $f$ we have*

$$f\left[X_{[d]}\right] = \Big\langle \left\{\beta^{\circ_v}\left[Y_v, Y_{e^{\mathrm{in}}}\right] \,:\, (e^{\mathrm{in}}, \{v\}) \in \mathcal{E}\right\} \cup \\ \left\{\delta\left[Y_k, X_k\right] \,:\, k \in [d]\right\} \cup \left\{\epsilon_1\left[Y_{\tilde{v}}\right]\right\} \Big\rangle_{\left[X_{[d]}\right]} .$$

**Proof** One can show this theorem by induction over the node formulas of the syntactical hypergraph, from the leafs to the root and iteratively applying Lem. 22. ■

Thus we have a tensor network representation of any propositional formula based on its syntactical decomposition, where the hypergraph of the syntactical decomposition equals the hypergraph of the representing tensor network.

### 4.3 Contractions to decide entailment

We have already seen that the contraction of a propositional formula counts its models. This allows to define entailment between two propositional formulas as follows.

**Definition 25 (Entailment of propositional formulas)** *Given two propositional formulas $\mathcal{KB}$ and $f$ we say that $\mathcal{KB}$ entails $f$, denoted by $\mathcal{KB} \models f$, if any model of $\mathcal{KB}$ is also a model of $f$, that is*

$$\langle \mathcal{KB}, \neg f \rangle_{[\varnothing]} = 0 \,.$$

*If $\mathcal{KB} \models \neg f$ holds, we say that $\mathcal{KB}$ contradicts $f$.*

Classically (see e.g. Russell and Norvig) entailment in propositional logics is defined as the the unsatisfiability of $\mathcal{KB} \wedge \neg f$. This is equivalent to Def. 25, since $\langle \mathcal{KB}, \neg f \rangle_{[\varnothing]} = 0$ is equivalent to $\langle \mathcal{KB} \wedge (\neg f) \rangle_{[\varnothing]} = 0$, which is the unsatisfiability of $\mathcal{KB} \wedge \neg f$.

Entailment is the central operation of "logical inference", i.e. deduce true statements from known statements. In the tensor network representation, these entailments can be decided by contracting the whole representing tensor with the statement, that needs to be checked.

**Example 12 ($n^2 \, x \, n^2$ Sudoku)** *We index the rows and the columns by tuples $(r0, r1)$ and $(co, c1)$, where $r0, r1, c0, c1 \in [n]$. The first index indicates the block and the second counts the row or column inside that block. For each $r0, r1, c0, c1 \in [n]$ and $i \in [n^2]$ we then define an atomic variable $X_{r0,r1,c0,c1,i} \in \{0, 1\}$ indicating whether in the row $(r0, r1)$ and column $(co, c1)$ the number $i$ is written. The Sudoku rules then amount to the formula*

$$\begin{aligned}\mathcal{KB}^n := &\left( \bigwedge_{r0,r1,c0,c1 \in [n]} \left( \bigoplus_{i \in [n^2]}^{(1)} X_{r0,r1,c0,c1,i} \right) \right) \wedge \left( \bigwedge_{r0,r1 \in [n], i \in [n^2]} \left( \bigoplus_{c0,c1 \in [n]}^{(1)} X_{r0,r1,c0,c1,i} \right) \right) \wedge \\ &\left( \bigwedge_{c0,c1 \in [n], i \in [n^2]} \left( \bigoplus_{c0,c1 \in [n]}^{(1)} X_{r0,r1,c0,c1,i} \right) \right) \wedge \left( \bigwedge_{r0,c0 \in [n], i \in [n^2]} \left( \bigoplus_{r1,c1 \in [n]}^{(1)} X_{r0,r1,c0,c1,i} \right) \right) ,\end{aligned}$$

*where $\bigoplus^{(1)}$ is the $n^2$-ary exclusive or connective (that is 1 if and only if exactly one of the arguments is 1). The four outer brackets in $\mathcal{KB}$ mark the constraints, that at each position exactly one number is assigned, further that in each row each number is assigned once, and similar for the columns and the squares of the board. When solving a specific Sudoku instance, one typically knows from an initial board assignment $E$ a collection of* *atomic variables, which hold, and needs to find further atomic variables, which are entailed. This means, we need to decide for each $(r_0, r_1, c_0, c_1, i) \notin E$ whether the Sudoku rules and the initial board imply that the atomic variable $X_{r0,r1,c0,c1,i}$ (i.e. assignment to the board) is true*

$$\mathcal{KB}^n \wedge \left( \bigwedge_{(r_0,r_1,c_0,c_1,i)\in E} X_{r0,r1,c0,c1,i} \right) \models X_{r0,r1,c0,c1,i}$$

*or false*

$$\mathcal{KB} \wedge \left( \bigwedge_{(r_0,r_1,c_0,c_1,i)\in E} X_{r0,r1,c0,c1,i} \right) \models \neg X_{r0,r1,c0,c1,i} \, . \tag{7}$$

*In other words, for each assignment to the board, that fulfills the Sudoku rules and the initial board, do we write the number $n$ in row $(r0, r1)$ and column $(c0, c1)$? If and only if the Sudoku has a unique solution given the initial board assignment $E$, exactly one of these entailment statements holds for each $(r_0, r_1, c_0, c_1, i) \notin E$. Deciding which is equivalent to solving of the Sudoku.*

*For example, let $n = 2$ and*

$$E = \{(0,0,0,0,0), (0,0,1,0,2), (0,0,1,1,1), (0,1,0,1,1), \\ (1,0,1,0,3), (1,1,0,0,3), (1,1,0,1,2)\} \, .$$

*We visualize this evidence by writing $i + 1$ in a grid cell $(r0, r1, c0, c1)$ to indicate that $(r0, r1, c0, c1, i) \in E$:*

| | | | |
|---|---|---|---|
| *1* | | *3* | *2* |
| | *2* | | |
| | | *4* | |
| *4* | *3* | | |

### 4.4 Efficient Representation of Knowledge Bases

We now investigate the representation of knowledge bases, which are conjunctions

$$\mathcal{KB}\left[X_{[d]}\right] = \bigwedge_{l\in[p]} f_l\left[X_{[d]}\right] \, .$$

To show efficient repesentations we will use the following identities.

**Lemma 26 (Computation Network Symmetries)** *We have for the $d$-ary $\wedge$-connective (where $d \in \mathbb{N}$) and the unary $\neg$-connective that*

$$\left\langle \epsilon_1\left[Y\right], \beta^{\wedge}\left[Y, X_{[d]}\right] \right\rangle_{\left[X_{[d]}\right]} = \bigotimes_{k\in[d]} \epsilon_1\left[X_k\right] \quad \textit{and} \quad \left\langle \epsilon_1\left[Y\right], \beta^{\neg}\left[Y, X\right] \right\rangle_{[X]} = \epsilon_0\left[X\right] \, .$$

**Proof** Follows directly from the definitions of the basis encodings and the connectives. ■

We use this to decompose knowledge bases into their individual formulas as follows.

**Theorem 27** *For any knowledge base $\mathcal{KB}\left[X_{[d]}\right] = \bigwedge_{l\in[p]} f_l\left[X_{[d]}\right]$ it holds that*

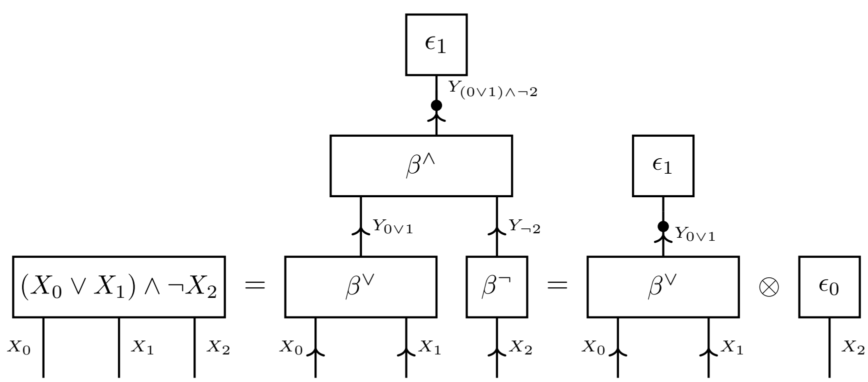
$$\mathcal{KB}\left[X_{[d]}\right] = \left\langle \left\{ f_l\left[X_{[d]}\right] \,:\, l\in[p] \right\} \right\rangle_{\left[X_{[d]}\right]}.$$

**Proof** With Lem. 26 we have

$$\begin{aligned}
\mathcal{KB}\left[X_{[d]}\right] &= \left\langle \left\{ \epsilon_1\left[Y_\wedge\right], \beta^\wedge\left[Y_\wedge, Y_{[p]}\right] \right\} \cup \left\{ \beta^{f_l}\left[Y_l, X_{[d]}\right] \,:\, l\in[p] \right\} \right\rangle_{\left[X_{[d]}\right]} \\
&= \left\langle \bigcup_{l\in[p]} \left\{ \epsilon_1\left[Y_l\right], \beta^{f_l}\left[Y_l, X_{[d]}\right] \,:\, l\in[p] \right\} \right\rangle_{\left[X_{[d]}\right]} \\
&= \left\langle \left\{ f_l\left[X_{[d]}\right] \,:\, l\in[p] \right\} \right\rangle_{\left[X_{[d]}\right]}.
\end{aligned}$$

■

**Example 13 (Computation Network Symmetries)** *For the propositional formula $f\left[X_{[3]}\right] = (X_0 \vee X_1) \wedge \neg X_2$ (see Example 9), we can write the formula in terms of a Computation-Activation Network with activation tensor $\epsilon_1$ and computation network decomposed by the basis encodings. First, it is written with one activation vector. Second, we see that it can also be interpreted with multiple features.*




### 4.5 Message-passing for Entailment

Since contracting the whole tensor is often infeasible and for instance for the Sudoku example would correspond to solving the whole problem, local contractions can be considered to decide in some cases. Here a local contraction describes the calculation of contractions along few closely connected legs in the tensor network. Now, if the local contraction of any legs leads to a zero-tensor in the network decomposition, the whole contraction amounts to zero, and the knowledge base entails $f$.

**Theorem 28 (Monotonocity of Propositional Logics)** *If* $\tilde{\mathcal{KB}} \subset \mathcal{KB}$ *and* $\tilde{\mathcal{KB}} \models f$ *then also* $\mathcal{KB} \models f$.

**Proof** Since $\tilde{\mathcal{KB}} \models f$ it holds that $\left\langle \tilde{\mathcal{KB}}, \neg f \right\rangle_{[\varnothing]} = 0$ and thus $\left\langle \tilde{\mathcal{KB}}, \neg f \right\rangle_{\left[X_{[d]}\right]} = 0\left[X_{[d]}\right]$. Denoting by $\mathcal{KB}/\tilde{\mathcal{KB}}$ the conjunctions of formulas in $\mathcal{KB}$ not in $\tilde{\mathcal{KB}}$, we have

$$\begin{aligned}
\left\langle \mathcal{KB}\left[X_{[d]}\right], \neg f\left[X_{[d]}\right] \right\rangle_{[\varnothing]} &= \left\langle \mathcal{KB}/\tilde{\mathcal{KB}}[X_{[d]}], \tilde{\mathcal{KB}}, \neg f\left[X_{[d]}\right] \right\rangle_{[\varnothing]} \\
&= \left\langle \mathcal{KB}/\tilde{\mathcal{KB}}[X_{[d]}], \left\langle \tilde{\mathcal{KB}}[X_{[d]}], \neg f[X_{[d]}] \right\rangle_{\left[X_{[d]}\right]} \right\rangle_{[\varnothing]} \\
&= \left\langle \mathcal{KB}/\tilde{\mathcal{KB}}[X_{[d]}], 0\left[X_{[d]}\right] \right\rangle_{[\varnothing]} \\
&= 0\,.
\end{aligned}$$

■

To decide entailment, we can therefore investigate entailment on smaller parts of the knowledge base. This is sound by the above theorem, but not complete, since it can happen that no smaller part of the knowledge base entails the formula, but the whole knowledge base does.

We can futhermore add entailed formulas to the knowledge base without the latter, as we show next.

**Theorem 29 (Invariance of adding Entailed Formulas)** *If* $\mathcal{KB} \models f$ *then*

$$\mathcal{KB}\left[X_{[d]}\right] = \left\langle \mathcal{KB}, f \right\rangle_{\left[X_{[d]}\right]}\,.$$

**Proof** We use that $f[X_{[d]}] + \neg f[X_{[d]}] = \mathbb{I}\left[X_{[d]}\right]$ and thus

$$\begin{aligned}
\mathcal{KB}\left[X_{[d]}\right] &= \left\langle \mathcal{KB}[X_{[d]}], (f[X_{[d]}] + \neg f[X_{[d]}]) \right\rangle_{\left[X_{[d]}\right]} \\
&= \left\langle \mathcal{KB}[X_{[d]}], f[X_{[d]}] \right\rangle_{\left[X_{[d]}\right]} + \left\langle \mathcal{KB}[X_{[d]}], \neg f[X_{[d]}] \right\rangle_{\left[X_{[d]}\right]} \\
&= \left\langle \mathcal{KB}[X_{[d]}], f[X_{[d]}] \right\rangle_{\left[X_{[d]}\right]}\,.
\end{aligned}$$

■

One can understand this theorem as "making the knowledge base more accessible": Adding deduced statements to a knowledge base does not change the knowledge base as a tensor, but one can interpret it in an easier way.

This motivates an message-passing approach to decide entailment by iteratively adding entailed formulas to the knowledge base and checking entailment on smaller parts of the knowledge base. Such inference methods are nown e.g. as Constraint Propagation.

**Example 14 (Constraint Propoagation for a $2^2\,x\,2^2$ Sudoku)** *We iteratively solve a Sudoku puzzle (see Example 12) by determining a possible value based on neighboring cells, rows and squares (using Thm. 28) and adding to our knowledge (using Thm. 29). For example, consider the following $r = 2$ Sudoku puzzle, where a first entailment step uses only the knowledge of the rules and the blue cells to determine the value 3 in the first square:*

$$\begin{array}{|c|c||c|c|}\hline 1 & & 3 & 2 \\ \hline & 2 & & \\ \hline\hline & & 4 & \\ \hline 4 & 3 & & \\ \hline\end{array} \;=\; \begin{array}{|c|c||c|c|}\hline 1 & & 3 & 2 \\ \hline 3 & 2 & & \\ \hline\hline & & 4 & \\ \hline 4 & 3 & & \\ \hline\end{array} \;=\; \dots \;=\; \begin{array}{|c|c||c|c|}\hline 1 & 4 & 3 & 2 \\ \hline 3 & 2 & 1 & 4 \\ \hline\hline 2 & 1 & 4 & 3 \\ \hline 4 & 3 & 2 & 1 \\ \hline\end{array}$$

*To illustrate the first reasoning step we make the following preliminary entailment steps applying Thm. 28:*

- *From $X_{0,1,0,1,1}$ (i.e. the 2 in the cell $(0,1,0,1)$) and the Sudoku rule that at the cell $(0,1,0,1)$ exactly one number is assigned, we get*

$$\left(\bigoplus_{i\in[n^2]}^{(1)} X_{0,1,0,1,i}\right) \land X_{0,1,0,1,1} \models \neg X_{0,1,0,1,2}\,,$$

  *That is, that the number 3 is not in the cell $(0,1,0,1)$.*

- *From $X_{0,0,1,0,2}$ (i.e. the 3 in the cell $(0,0,1,0)$) and the Sudoku rule that at the row $(0,0)$ exactly one number is assigned, we get*

$$\left(\bigoplus_{c0,c1\in[n]}^{(1)} X_{0,0,c0,c1,2}\right) \land X_{0,0,1,0,2} \models \neg X_{0,0,0,0,2} \land \neg X_{0,0,0,1,2}\,,$$

  *That is, that the number 3 is neither in the cell $(0,0,0,0)$ not in $(0,0,0,1)$.*

*We add these formulas to our knowledge base (justified by Thm. 29) and use the rule, that 3 appears exactly once in the first square*

$$\left(\bigoplus_{r1,c1\in[n]}^{(1)} X_{0,r1,0,c1,2}\right) \land (\neg X_{0,1,0,1,2}) \land (\neg X_{0,0,0,0,2} \land \neg X_{0,0,0,1,2}) \models X_{0,1,0,0,2}\,.$$

*That is, we conclude that the number 3 must be in the cell $(0,1,0,0)$, which information is also included in the updated knowledge base for further reasoning steps.*

## 5 Hybrid Logic Networks

Logical and probabilistic reasoning can be unified in a single tensor-network architecture called a Hybrid Logic Network (HLN).

In Markov logic networks, each state gets assigned a real-valued probability based on the basis encoding of a boolean statistic as a computation network and assigned probabilities of these features in terms of a positive elementary activation network.

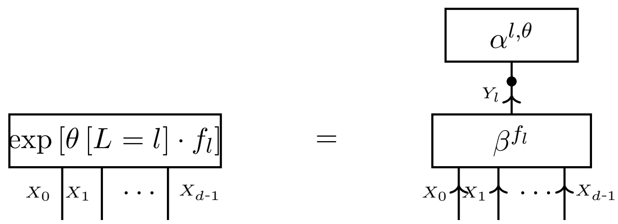



Figure 6: Factor of a Markov Logic Network to a formula $f_l$, represented as the contraction of a computation core $\beta^{f_l}$ and an activation core $\alpha^{l,\theta}$. While the computation core $\beta^{f_l}$ prepares based on basis calculus a categorical variable representing the value of the statistic formula $f_l$ dependent on assignments to the distributed variables, the activation core multiplies an exponential weight to coordinates satisfying the formula.

**Definition 30 (Markov Logic Network)** *A Markov Logic Networks is an element of and exponential family $\Gamma^{t,\mathbb{I}}$ with sufficient statistic defined coordinatewise by propositional formulas $f_\ell$*

$$t : \bigtimes_{k\in[d]}[2] \to \bigtimes_{l\in[p]}[2] \subset \mathbb{R}^p, \quad x_{[d]} \mapsto (f_\ell[X_{[d]} = x_{[d]}])_{\ell\in[d]}.$$

That means, they have the form

$$\mathbb{P}^{(\mathcal{F},\theta,\mathbb{I})}[X_{[d]}] = \left\langle \beta^t \left[Y_{[p]}, X_{[d]}\right], \alpha^\theta[Y_{[d]}] \right\rangle_{\left[X_{[d]}|\varnothing\right]}$$

with $\alpha^\theta[Y_{[d]}] = \bigotimes_{l\in[p]} \alpha^{\ell,\theta}[Y_\ell]$ with

$$\alpha^{l,\theta}\left[Y_l = y_l\right] = \exp\left[\theta\left[L = l\right] \cdot I_l(y_l)\right]$$

where $\theta[L] \in \mathbb{R}^p$, $I$ is an interpretation map. The architecture is visualized in Figure 6.

In Hard Logic Networks, on the other hand, states only get assigned true or false values 0/1. Basis encodings of propositional formulas can be interpreted as a CAN, where the basis encodings build the computation network and the boolean elementary activation network decides, which outcomes of the statistic are possible/ get assigned a true value.

**Definition 31 (Hard Logic Network)** *Given a boolean statistic $t : \bigtimes_{k\in[d]}[2] \to [2]^p$, a subset $A \subset [p]$ and a tuple $y_A \in \bigtimes_{l\in A}[2]$ the Hard Logic Network is the distribution*

$$\mathbb{P}^{t,(A,y_A)}\left[X_{[d]}\right] = \left\langle \beta^t \left[Y_{[p]}, X_{[d]}\right], \kappa^{(A,y_A)}\left[Y_{[p]}\right] \right\rangle_{\left[X_{[d]}|\varnothing\right]}$$

*where $\kappa^{(A,y_A)}\left[Y_{[p]}\right] = \bigotimes_{l\in[p]} \kappa^{l,(A,y_A)}\left[Y_l\right]$ and for $l \in [p]$*

$$\kappa^{l,(A,y_A)}\left[Y_l\right] = \begin{cases} \epsilon_{y_l}\left[Y_l\right] & \text{if} \quad l \in A \\ \mathbb{I}\left[Y_l\right] & \text{if} \quad l \notin A \end{cases},$$

*provided that the normalization exists.*

Note that this distribution is uniform on its support. Combining the Hard Logic Network and the Markov Logic Network leads to the representation of a probability density only over states, that fulfill an imposed hard logic, where the elementary activation tensors are allowed to take binary or real values.

**Definition 32 (Hybrid Logic Network)** *Given a boolean statistic $t$ we call any element of $\Lambda^{t,\mathrm{EL}}$ a Hybrid Logic Network (HLN). The extended canonical parameter set to $t$ is the set*

$$\mathcal{P}_p \coloneqq \{(A, y_A)) \,:\, A \subset [p],\, y_A \in \bigtimes_{l\in A} [2]\} \times \mathbb{R}^p \,.$$

*To each Hybrid Logic Network $\mathbb{P}^{t,(A,y_A,\theta)}\left[X_{[d]}\right]$ we find a tuple $(A, y_A, \theta)$ consistent of a subset $A \subset [p]$, a tuple $y_A \in \bigtimes_{l\in A}[2]$ and $\theta\left[L\right] \in \mathbb{R}^p$ such that*

$$\mathbb{P}^{t,(A,y_A,\theta)}\left[X_{[d]}\right] = \left\langle \beta^t\left[Y_{[p]}, X_{[d]}\right], \xi^{(A,y_A,\theta)}\left[Y_{[p]}\right]\right\rangle_{\left[X_{[d]}|\varnothing\right]}$$

*where the activation core is*

$$\xi^{(A,y_A,\theta)}\left[Y_{[p]}\right] = \left\langle \alpha^{\theta}\left[Y_{[p]}\right], \kappa^{(A,y_A)}\left[Y_{[p]}\right]\right\rangle_{\left[Y_{[p]}\right]} \,.$$

The architecture is demonstrated on an example in accounting logic.

**Example 15** *Hybrid Logic Network for Accounting Logic*

*Let us consider a system of three variables A1 Account 1 is booked, A2 Account 2 is booked, F a feature on an invoice. We respect two rules*

- *Exactly one account must be booked.*
- *If feature* F *is present on the invoice, the account* A1 *is typically booked.*

*We formalize this with the statistic*

$$t = \left(X_{A1} \oplus X_{A2}, X_F \Rightarrow X_{A1}\right).$$

*While the first formula is a hard feature, the second is soft since prone to exceptions. We parameterize the first output of the statistic with the hard parameters by setting the set of indices to be initialized with hard logic $A = \{0\}$ and the corresponding initialization $y_0 = 1$ meaning, that the first output of the statistic has to be true for the input to have positive probability. Then "hard logic activation tensor", should be indifferent to the second part of the statistic, and only impose rules on the first part, leading to*

$$\kappa^{(A,y_A)}[Y_0, Y_1] = \epsilon_{y_0}\left[Y_0\right] \otimes \mathbb{I}\left[Y_1\right] = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\1\end{bmatrix}.$$

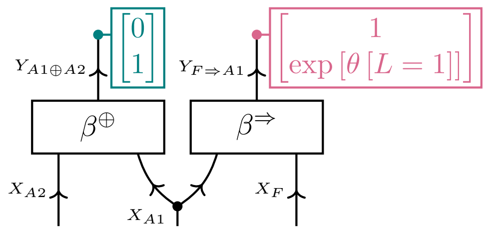
*Since the first feature is hard, the "soft logic activation tensor" should be invariant under the first coordinate of the canonical parameter and we set $\theta\left[L=0\right] = 0$. We choose the soft parameters as $\theta[L] = [0, \theta[L=1]]^{\intercal}$ to achieve*

$$\alpha^{\theta}[Y_0, Y_1] = \alpha^{0,0}\left[Y_0\right] \otimes \alpha^{1,\theta[L=1]}\left[Y_1\right] = \begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\ \exp\left[\theta\left[L=1\right]\right]\end{bmatrix}.$$

*The activation tensor of the hybrid network then has the from*

$$\xi^{(A,y_A,\theta)}[Y_0, Y_1] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \exp\left[\theta\left[L=1\right]\right] \end{bmatrix}.$$

*We get a tensor network representation of the Hybrid Logic Network representing the toy accounting example, before normalization to a distribution*




*The resulting Hybrid Logic Network is a tensor* $\mathbb{P}^{t,(A,y_A,\theta)}\left[X_{A1}, X_{A2}, X_F\right]$ *of order* 3. *With* $Y_{F\Rightarrow A_1} = 1$ *for* $F = 0$ *and any* $A_1$ *it has the coordinates*

$$\mathbb{P}^{t,(A,y_A,\theta)}\left[X_{A1}, X_{A2}, X_F = 0\right]$$
$$= \left\langle \beta^{\oplus}[Y_{A_1\oplus A_2}, X_{A_2}, X_{A_1}] \otimes \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}[Y_{F\Rightarrow A_1}, X_{A_1}], \begin{bmatrix} 0 \\ 1 \end{bmatrix}[Y_{A_1\oplus A_2}] \otimes \begin{bmatrix} 1 \\ \exp[\theta[L=1]] \end{bmatrix}[Y_{F\Rightarrow A_1}] \right\rangle$$

*where contraction along the* $Y$*-variables leads to*

$$\begin{aligned}\mathbb{P}^{t,(A,y_A,\theta)}\left[X_{A1}, X_{A2}, X_F = 0\right] &= \beta^{\oplus}[Y_{A_1\oplus A_2} = 1, X_{A_2}, X_{A_1}] \otimes \begin{bmatrix} \exp[\theta[L=1]] \\ \exp[\theta[L=1]] \end{bmatrix}[X_{A_1}] \\ &= \frac{1}{Z}\begin{bmatrix} 0 & \exp[\theta\left[L=1\right]] \\ \exp[\theta\left[L=1\right]] & 0 \end{bmatrix}\end{aligned}$$

*for the normalization constant* $Z = 1 + 3 \cdot \exp\left[\theta\left[L=1\right]\right]$ *and*

$$\mathbb{P}^{t,(A,y_A,\theta)}\left[X_{A1}, X_{A2}, X_F = 1\right] = \frac{1}{Z}\begin{bmatrix} 0 & 1 \\ \exp[\theta\left[L=1\right]] & 0 \end{bmatrix}.$$

Also entailment can be checked for Hybrid Logic Network. Assuming a positive probability of all models of the integrated Hard Logic Network, the entailment can be checked only considering the Hard Logic Network. Here a query is a formula to retrieve information from a given network.

**Theorem 33 ( (Goessmann, 2025, Theorem 8.12))** *Let* $\mathbb{P}^{\mathcal{F},(A,y_A,\theta)}[X_{[d]}]$ *be a Hybrid Logic Network. Given a query formula* $g$*, we have that* $\mathbb{P}^{\mathcal{F},(A,y_A,\theta)}[X_{[d]}] \models g$ *if and only if*

$$f^{\mathcal{F},(A,y_A)} \models g,$$

*where the tuple* $(A, y_A)$ *denote the hard logic part of the network and*

$$f^{t,(A,y_A)}[X_{[d]}] = \left(\bigwedge_{l\in A\,:\,y_l=1} f_l\left[X_{[d]}\right]\right) \wedge \left(\bigwedge_{l\in A\,:\,y_l=0} \neg f_l\left[X_{[d]}\right]\right).$$

**Example 16** *Entailment for Accounting Logic To check entailment of the query formula defined by*

$$g[X_{A_1} = 0, X_{A_2} = 1, X_F = 1] = 1$$

*and is set to zero otherwise, entailment can be checked by contracting*

$$f^{t,(A,y_A)}[X_{A_1}, X_{A_2}, X_F] = X_{A1} \oplus X_{A2} \otimes \mathbb{I}$$

*with $g$ arriving at*

$$\begin{aligned}
&\langle f^{t,(A,y_A)}[X_{A_1}, X_{A_2}, X_F], \neg g[X_{A_1}, X_{A_2}, X_F]\rangle \\
&= \sum_{x_{A_1}, x_{A_2}, x_F \in \{0,1\}} f^{t,(A,y_A)}[X_{A_1} = x_{A_1}, X_{A_2} = x_{A_2}, X_F = x_f](1 - g[X_{A_1} = x_{A_1}, X_{A_2} = x_{A_2}, X_F = x_f]) \\
&= \sum_{\substack{x_{A_1}, x_{A_2}, x_F \in \{0,1\} \\ (x_{A_1}, x_{A_2}, x_F) \neq (0,1,1)}} f^{t,(A,y_A)}[X_{A_1} = x_{A_1}, X_{A_2} = x_{A_2}, X_F = x_f] \\
&\geq f^{t,(A,y_A)}[X_{A_1} = 1, X_{A_2} = 0, X_F = 0] > 0.
\end{aligned}$$

*Therefore, $\mathbb{P}^{\mathcal{F},(A,y_A,\theta)}[X_{[d]}]$ does not entail $g$. In this case, this is due to the fact, that $g$ only assumes one model, despite the formula having multiple models. Doing the same calculations for*

$$\tilde{g}[X_{A_1} = 1, X_{A_2} = 1, X_F = 1] = 0$$

*and equal to 1 everywhere else leads to the constraction being equal to zero. Therefore, $\mathbb{P}^{\mathcal{F},(A,y_A,\theta)}[X_{[d]}]$ does entail $\tilde{g}$.*

## 6 Implementation

The architecture can be conveniently implemented with the python package tnreason. Multiple examples including graph-coloring, sat problems, Sudoku, and temporal clue are available[1]. To emphasize the intuitive implementation of CANs, a code snippet for the implementation of a CAN for the sat problem $f[X_a = a, X_b = b, X_c = c] = (a \vee b) \wedge \neg c$ described in example 11 is explained. This propositional formula is encoded by a dictionary of variables encoded by nested lists, that need to all be fulfilled.

```
expressionsDict = {"f0" :  ["and", ["or", "a", "b"], ["not", "c"]]}
```

After importing the package by

```
from tnreason import representation, application, engine
```

the CAN can then be build by defining all cores. Activation cores then have suffices `_aC` and computation cores are denoted with suffices `_cC` by

1. `https://github.com/EnexaProject/enexa-tensor-reasoning/tree/version1`

```
cores = application.create_cores_to_expressionsDict(expressionsDict)
computationCores = {key: value for key, value in cores.items()
                                                        if key.endswith("_cC")}
```

The generated `computationCores` is a dictionary with the following keys and tensors of given shapes as expected in example 11.

```
'f0_aC', [2]
'(and_(or_a_b)_(not_c))_cC', [2, 2, 2]
'(or_a_b)_cC', [2, 2, 2]
'(not_c)_cC', [2, 2]
```

Based on this dictionary, the Computation-Activation Network can be build by setting the activation network for the single output feature (the output of $f$) to a vector acting on the output of the basis encoding of $f$. Here a `SingleHybridFeature` is used, which allows for hard or sift activations. Then the value for the desired output of the feature is set to `True`.

```
caNet = representation.ComputationActivationNetwork(
    computationCoreDict=computationCores,
    featureDict={"f0" : representation.SingleHybridFeature(
                                    featureColor="(and_(or_a_b)_(not_c))_cV")},
    canParamDict={"f0" : True}
    )
```

As in example 11, the CAN then has the following form.

The other representation in example 11 can also be implemented. Both architectures can be found the linked notebook[2]. Note that more efficient representations of this network are possible and the one described here is mainly for pedagogic purposes. Furthermore the right implementation of the formula can be checked by contractions.

```
allCores = caNet.create_cores()
formula = engine.contract(coreDict = allCores,
                          openColors=["a_dV","b_dV", "c_dV"])
assert formula[{"a_dV": 1, "b_dV" : 1, "c_dV" : 0}] == 1
assert formula[{"a_dV": 1, "b_dV" : 1, "c_dV" : 1}] == 0
```

The notebook also shows how the network can be normalized to represent a uniform probability distribution over all models of the formula.

```
distribution = engine.normalize(coreDict = allCores,
                                outColors = ["a_dV","b_dV", "c_dV"],
                                inColors = [])
assert distribution[{"a_dV": 1, "b_dV" : 1, "c_dV" : 0}] == 1/3
assert distribution[{"a_dV": 1, "b_dV" : 1, "c_dV" : 1}] == 0
```

2. https://colab.research.google.com/drive/14knFuMJHI683DAmUgJ-G10MQFueoXR6q#scrollTo=vrOYBViZhX2S

## 7 Conclusion& Outlook

- Conclusion: tensor representation close mathematical structures -¿ strength to do analysis? (Janina)
- Contraction algorithms (Alex)
- Max: Combine with foundation models / LLMs for agentic models (Max)

This work has treated the representation of several models in tensor networks. Model inference such as the computation of marginal distributions and the decision of entailment are formulated by tensor network contractions. These contraction can become bottlenecks, which are known as

- tree-widths of graphical models Pearl
- intractability of generic logical reasoning Russell and Norvig

Approximation schemes can be derived based on variational inference Wainwright and Jordan, such as loopy message passing schemes and mean field methods. Further frequenctly applied schemes are particle-based inference schemes such as Gibbs sampling.

Message-passing schemes appear in particular as belief propagation in probability theory and syntactial inference algorithms in logics. We can understand them as approximation of (potentially intractible) contractions and will dedicate future work to study them in the tnreasonformalism.

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