Pricing Strategies for Online Dating Platforms

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Online dating is the most common way for new US couples to meet, with more than three-in-ten Americans having used dating apps, and with revenues from dating apps swelling to more than five billion annually. The majority of these dating apps earn revenue via subscription-based pricing, where subscriptions for a period of access to the app are sold at a fixed, reoccurring price. Subscription-based pricing is a ubiquitous way to monetize mobile apps, however in the context of online dating different subscription periods lengths can lead to very different outcomes for the platform, and for the market of users the platform serves.

In this work we study the revenue and welfare trade-offs associated with varying length subscription periods. In our model, we show that short period pricing (*freemium pricing*) always achieves at least 26.9% of the profit of the optimal policy. We then take a fine-grained approach and show that when the marginal cost is small, long period pricing (*contract pricing*) is most profitable. Further, under a natural slow matching condition, we show that in low marginal cost settings, long periods also lead to a higher percentage of the user-base getting matched, and allow the platform to incorporate user preference information in a way that aligns the interests of the platform and user. Overall, our results explain the prevalence of short period pricing in practice, but suggest that when the marginal cost is low, both the platform and the user-base may benefit from a switch to longer period pricing.

Key words: online dating, matchmaking, subscription, contract, pricing
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1. Introduction

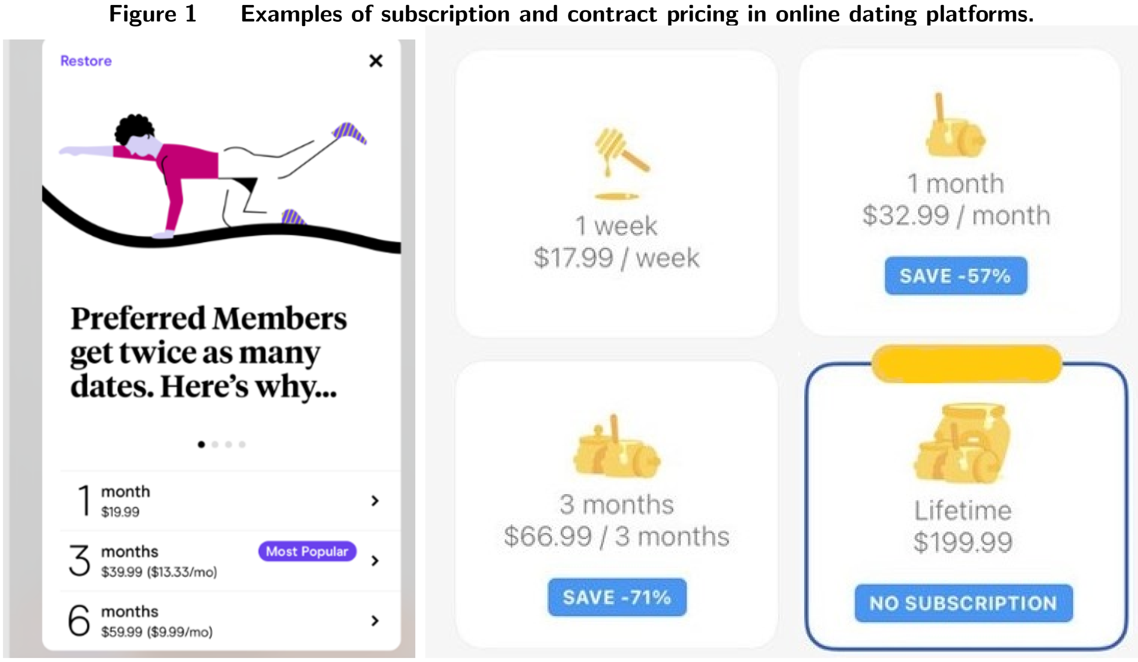
More and more people, especially young people, are meeting and falling in love online. Over the past two decades, online dating has displaced conventional mediums such as family, school, or the workplace, to become the most common way for new couples to meet (Shashkevich 2019). As of 2019, three-in-ten U.S. adults say they have used a dating site or app before, and that percentage rises to 48% for 18-to-29 year-old's (Pew Research Center 2020b). Moreover, dating apps have been enormously successful in connecting people otherwise left out of traditional dating culture. A full two-thirds of lesbian, gay, or bisexual Americans report using dating apps (Pew Research Center 2020a) and, with the COVID-19 pandemic severely restricting the venues in which people can meet, many leading dating apps are reporting record numbers of users and subscribers (Meisenzahl 2020).

The companies that run these apps have in turn grown with the increased demand. The dating services industry has swelled over the last five years, with an annualized growth rate of 12.9% (IBISWorld 2021), and revenues projected to rise 9.3% in 2021 to $5.3 billion as mobile services expand. In the United States, dating services are largely consolidated under one corporate entity, the Match Group, which as of 2020 is estimated to have cornered more than 60% of the dating app market with its suite of apps, including *Tinder*, *Hinge*, *OkCupid*, and *Match* (Meisenzahl 2020).

The majority of dating apps in the United States, including all apps owned by the Match Group, share a number of structural similarities. First, they are *swipe apps*, where a user on the dating platform is shown a curated queue of potentials matches, and interacts with the site by swiping left (to deny) or right (to accept) through the queue; only after two users mutually swipe right on each other can they then begin to communicate (Bocknek 2022). Second, they follow a subscription-based profit model where users pay a subscription fee per period (e.g. per week, per month, see Figs. 1 and EC.2 for examples) to use the full features of the platform.

While subscription-based profit models are an extremely common way to monetize apps, their use for dating platforms specifically is questionable. In the subscription model, the online dating platform's interest may be in conflict with it's users'. As (Wu et al. 2019) note, a profit-maximizing subscription-based dating app will strive to retain all of its subscribers, however upon finding a compatible partner, users will terminate their dating app subscription and leave the platform. Thus, the platform has an incentive not to provide

Figure 1 **Examples of subscription and contract pricing in online dating platforms.**



Note. Depicted are pricing options for two popular online dating platforms. On the left is an example of the types of subscription options at `https://hinge.co/`. On the right is an example of contract pricing (denoted Lifetime) at `https://bumble.com/`, among other options.

the best potential matches to their users so as to extend their time using the app. Of course, the severity of this incentive incompatibility depends on the length of the subscription period for the platform. For instance if the period length is a lifetime (i.e. a lifetime membership), such incentive issues vanish as the site can earn no further profit from the user after the intial payment.

In the context of online dating some matchmaking sites such as `selectivesearch.com` and `bumble.com` have already adopted lifetime memberships, what we refer to as a *contract based* pricing in this paper. In fact, in the pricing section of the selective service site (`https://www.selectivesearch.com/pricing`) they state explicitly "*While most dating apps and services are incentivized to keep members paying ongoing fees, Selective Search works with clients through a defined contract. Each contract is for a finite number of introductions over a defined period of time*". Ostensibly, one time payment, contract based pricing models can align the interests of the online dating platform and users. In Fig. 1 examples of both subscription priced (at various subscription period lengths) and contract priced dating apps are shown for reference.

In light of the above, in this work we study the monetization of a dating platform (swipe app) run by a profit maximizing monopolist. We focus specifically on the length of the subscription period, and attempt to understand the benefits and trade-offs of shorter or longer periods. We will give special attention to the following two policies which represent the extremes of subscription-based pricing: (i) *freemium pricing* (FP), where the online dating platform commits to a fixed price p, and users pay the price in a continuous fashion until they leave the platform, and (ii) *contract pricing* (CP), where the online dating platform commits to a fixed, one time price p. FP is very flexible, and requires no commitment between the platform and user making it easy to implement, understand, and deploy, even if it leads to possibly confusing incentives. CP is less flexible, especially in the online setting where it requires a large upfront payment, but may lead to simpler and more harmonious user platform interaction.

Given the ostensible advantages and disadvantages of both FP and CP for online dating platforms, the purpose of this work will be to understand the profit and welfare trade-offs associated with each. A summary of our key contributions and findings is as follows:

1. We give a novel, natural model to describe the operations of an online dating platform. In our model, we study the optimal profit a platform can obtain from a subscription policy parameterized by a period length, L, and a subscription price p. We examine two important extremes of this parameterization, freemium pricing ($L \to 0$) and contract pricing ($L \to \infty$). We show that freemium pricing is robust in the sense that it achieves a constant factor (26.8%) of the profit of the best-in-class policy for *all* distributions and market parameters, and no other policy can achieve such a guarantee (c.f. Theorem 1). We complement this result by noting that, if instead of robustness we focus on the parameter regime where per-user costs are vanishingly small, the profit optimal policy is in fact contract pricing (c.f. Theorem 2).
2. Given the unique advantages of freemium pricing and contract pricing, respectively, we next study the implications of the choice of between them for the users of the platform. Specifically, we look at which pricing strategy (when optimized to maximize profit) leads to a higher proportion of matches among the user-base. When model parameters satisfy a natural slow matching condition, we show a sharp relationship between the optimal freemium price and the optimal contract price (c.f. Lemma 6). Using this price characterization, and assuming marginal costs are low, we prove that not only is FP less profitable than CP, but it also matches a smaller percentage of the user-base (c.f. Theorem 3). Thus, in well-established online settings with low marginal

cost the use of contract pricing exhibits a "win-win" for both the platform and it's users.

3. Finally, we consider the scenario where the platform can incorporate heterogeneous potential match information to vary the match rate. We show that across periods the match rate should be increasing, and within periods the match rate should be decreasing (c.f. Theorem 4). Under FP, this translates to a profit maximizing online dating platform offering its users the worst possible potential matches to keep them on the platform longer, whereas under CP, the situation is reversed, formalizing the intuition of Wu et al. (2019). Moreover, not only does CP create an incentive for the platform to match the user as quickly as possible, the ability to learn user preferences also induces the platform to offer a lower contract price (c.f. Theorem 5).

1.1. Literature review

Our work is related to several streams of literature in economics, computer science, and operations. Here, we overview some of these streams and connect them to our work.

Platform design in operations management Our work contributes to a deep literature dealing with aspects of platform design using models from operations management. The most thematically relevant paper for our work is (Wu et al. 2018), who study competing matchmakers in a two-period, two-user model with Hotelling valuations. They model dating platforms as strategically investing in matching technologies, investigating the interaction between competition, and providing the best service for their users. (Wu et al. 2018) does not resolve the suitor-matchmaker incentive issues but does argue that via competition and perfect information about match quality, matchmaking platforms can be induced to act in the agents' best interests. (Ellison and Ellison 2009, Ellison and Wolitzky 2012, Dukes and Liu 2016, Halaburda et al. 2018, Basu et al. 2019) also study online platform incentives to provide less-than-perfect services. As the US dating market is largely non-competitive (Gilbert 2019), we instead focus on changing the pricing structure itself to understand profit, welfare, and incentive trade-offs, and in a significantly more general model.

Pricing strategy in operations management The closest to our paper, in terms of the framework and style of pricing strategy analysis, is (Ladas et al. 2021). The authors also consider two business models, pay-per-use selling, and product selling, which roughly correspond to our freemium pricing and contract pricing, respectively. They focus primarily on an equilibria analysis of the business model choice under duopoly. Their work explores the scenarios in which the pay-per-use model is more profitable than product selling. Other analyses of pay-per-use selling and product selling can be seen in (Varian 2000, Sundararajan 2004, Agrawal et al. 2012, Balasubramanian et al. 2015). Analysis of similar business models can be seen in (Niculescu and Wu 2014). In our work, we analyze the pricing strategies of profit maximizing monopolists running an online dating app.

Approximation analysis for online platforms On the technical side, in this work we approach pricing strategies for online dating through the well studied lens of approximation. Our work is in the spirit of (Kanoria and Saban 2021), who study the optimal design of matching platforms by introducing a stylized dynamic fluid model for the two-sided matching with strategic agents. They find, in unbalanced markets, the platform should force the short side to initiate contact with potential partners, therefore mitigating wasted searching effort. However, they focus on a setting without prices. (Johari et al. 2019, Immorlica et al. 2021) investigate the information disclosure problem for online platforms in two-sided matching markets. (Aouad and Saritaç 2020, Aouad and Saban 2021) shift their interests to optimization for online matching platforms in dynamic settings.

Matchmaking in other markets Outside of dating/marriage, online matchmaking is also a fundamental problem for labor markets (Bimpikis et al. 2020, Belavina et al. 2020) and in the video game industry. (Chen et al. 2021) study the problem of maximizing player engagement in video games through improved matchmaking. They focus on a stylized model with different skill levels of players, and where winning or losing influences the players' willingness to stay on the platform. (Chen et al. 2017, Huang et al. 2019, Deng et al. 2021) also investigate how to improve players' engagement through matching.

1.2. Paper outline

In Section 2, we introduce our notation and provide some preliminary results about the profit achievable from a pricing strategy parameterized by a period length L, and a subscription price p. In Section 3, we prove best-in-class profit guarantees for FP and CP. In Sections 4 and 5, we study the implications of committing to FP or CP for the platform user-base. Finally, in Section 6 we discuss the implications of our work for online dating platforms, their users, and potential regulators, as well as highlight some interesting avenues for future research.

2. Model and Preliminaries

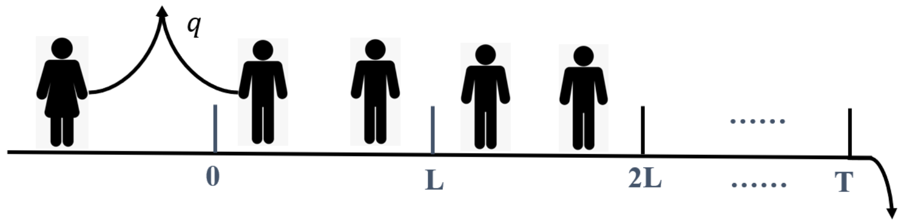
We consider a profit maximizing online dating platform serving a market of users seeking permanent partners, and refer to the formation of such a partnership as a *match*. Note that colloquially in online dating contexts, a "match" can refer to a candidate partner for the user, or someone who has shown preliminary interest but has not met the user. In this work, we will use *match* exclusively to mean the successful formation of a long-term partnership culminating in a departure from the platform. We model a random user's valuation (willingness-to-pay) for matching as described by a non-negative random variable V drawn from a known distribution F with density f. We use the notation $\overline{F}(x) := 1 - F(x)$ to denote the complementary cumulative distribution function.

We assume users are time-sensitive and their valuation for matching decays at a constant rate $\delta \in (0,1)$, so the user's valuation for matching after t time on the platform is $V_t = \delta^t V$. It is instructive to think of δ a decay in the valuation for matching *on the platform* specifically. Online dating is an arduous process (NPR 2019) with a high attrition rate, and four of five users experiencing emotional burnout or fatigue (Pearson 2022). The parameter δ captures this fatigue as a decay in the users willingness-to-pay.

Taking inspiration from swipe apps, we model the matching process as a stream of interactions between a user and potential match candidates displayed by the platform. To build intuition for our model, first consider an analogous discrete matching process i.e. one where the user interacts with candidate match 1, then candidate match 2, and so on (see Fig. 2 for a graphical depiction). If the probability of matching with each candidate is q, then a user's time until matching is geometrically distributed with expectation $1/q$. In this paper we consider the continuous analog of this process indexed by time t, with homogeneous match rate q, such that, over any period of time Δ on the platform, the probability of a user matching is an exponential random variable with rate q, *i.e.*, $\Pr(\text{Match} \in [t, t+\Delta)) = 1 - e^{-q\Delta}$. This formulation is the natural continuous time model for the operation of swipe apps like Tinder, Hinge, or Bumble, where users can continuously swipe through platform-prepared candidate matches.

On the platform's side, we assume the online dating platform commits to some fixed pricing strategy parameterized by a period length, L, and a fixed subscription price, p. We assume users are rational, and will pay the price if their expected utility from payment over the subscription period is non-negative. The platform prepares potential matches for

Figure 2 **Graphical depiction of the matching process.**



Note. Depicted is the matching process we consider in this paper. A user interacts with a stream of candidate matches. The user forms a successful match with someone in this stream at rate q. The candidate matches are grouped in bunches of size L, in practice there is a natural correspondence between the length of time L. and the number of candidates a user can see, stemming from caps on the number of users who can be seen in a period imposed by the platform.

the user at marginal operating cost c, paid continuously throughout the user's time on the platform. Finally, we assume the length of time the user can spend on the platform is upper bounded by T, which can be thought of as the size of the pool of potential matches (T may be nearly infinite in a city, or quite restrictive outside of cities), or as the maximum possible time the user can spend on the platform. When a user reaches time T on the platform, they leave, and we say in this case the platform has been *exhausted*.

We call a pricing strategy offering a period L of service at a fixed subscription price, p, a *period* L *pricing* (LP). Next we describe the profit of such a strategy.

Profit of Period L Pricing: Suppose the platform commits to a subscription price p with period length L, then each user pays the subscription price at the beginning of every subscription period until they either: 1) match, 2) their expected utility for another period on the platform decays below p, and they decide not to renew their subscription, or 3) the platform is exhausted. Let $\mathcal{R}(p, L)$ be the expected profit of the online dating platform using such a policy where the expectation is taken over the user's valuation, and the matching randomness, i.e.,

$$\mathcal{R}(p, L) := p\,\mathbb{E}\left[\#\text{ Periods on platform}\right] - c\,\mathbb{E}\left[\text{Time on platform}\right],$$

where c is the unit time operating cost. We emphasize the distinction between the number of periods and the total time, which can include the intra-period time to match. Let $\mathcal{R}(L) := \max_p \mathcal{R}_{FP}(p, L)$ denote the maximum achievable profit of the platform with payment period L.

In this paper our primary object of study is pricing strategies with varying predetermined payment period lengths L. We will give special consideration to two pricing strategies which we term *freemium pricing* (FP), where the user continuously pays a freemium price (while typically *freemium* implies the platform is free to use, users still pay with their attention, typically converted to profit via advertising or the sale of metadata, and in a continuous fashion) to participate on the platform, and *contract pricing* (CP), where the user and platform enter into a contract in which the user pays a fixed fee, and in return uses the platform until matched, or the platform is exhausted. These two strategies represent the extremes of the payment period length, with FP corresponding to $L \to 0$ and CP corresponding to $L \to \infty$ (equivalently $L = T$ when T is finite). Their profits are

$$\mathcal{R}_{FP}(p) := (p - c)\, \mathbb{E}\left[\text{Time on platform}\right],$$

$$\mathcal{R}_{CP}(p) := (p - c \times \mathbb{E}\left[\text{Time on platform}\right])\, \Pr(\text{User pays price } p).$$

We will use $\mathcal{R}_{FP} := \max_p \mathcal{R}_{FP}(p)$ and $\mathcal{R}_{CP} := \max_p \mathcal{R}_{CP}(p)$ to denote the maximum achievable profit under freemium pricing and contract pricing, respectively.

In the next subsection we explicitly characterize their profit in terms of model parameters, and provide some preliminary results.

2.1. Preliminaries results

In this subsection, we give full expressions for $\mathcal{R}(p, L)$, $\mathcal{R}_{FP}(p)$, and $\mathcal{R}_{SP}(p)$ in terms of the valuation distribution F and model primitives c, q, δ, and T. These expressions follow from integral representations for their profit based on value differentials and time differentials, respectively. We then discuss conditions for when the profit maximizing prices are unique which will aid our subsequent analysis in Sections 3 and 4.

To compute the profit for a platform with payment period L and subscription price p, consider a user with fixed valuation v, and note that such a user will pay to participate only if their time discounted expected valuation for a match in the period exceeds the subscription price. If the user pays the subscription price, they will stay on the platform until they either match and leave, which is distributed as an exponential random variable with rate q, or until the period ends and they again evaluate paying the period price. Let $X_L(q)$ be the random variable representing the time the user spends on the platform in a period of length L, assuming they paid the period price. Then $X_L(q) = \min\{\text{Exp}(q), L\}$, and a user with valuation $v\delta^{(i-1)L}$ at the beginning of the i^{th} period will pay if

$$\mathbb{E}\left[\overbrace{v\delta^{(i-1)L+X_L(q)}}^{\text{Valuation for match}} \quad \underbrace{\mathbf{1}_{X_L(q)<L}}_{\text{Chance of matching}}\right] \geq p$$

where $\mathbf{1}_{X_L(q)<L}$ is the indicator function which is 1 when the user matches in the period, and 0 when the user reaches the end of the period. The expectation can be evaluated as,

$$\mathbb{E}[v\delta^{(i-1)L+X_L(q)}\mathbf{1}_{X_L(q)<L}] = v\delta^{(i-1)L}\int_0^L \delta^t q e^{-qt} dt = v\delta^{(i-1)L}\left(\frac{q\left(1-\delta^L e^{-qL}\right)}{q-\log\left(\delta\right)}\right).$$

Therefore, a user with valuation $v\delta^{(i-1)L}$ at the beginning of the i^{th} period will pay the subscription price p only if

$$v\delta^{(i-1)L}\left(\frac{q\left(1-\delta^L e^{-qL}\right)}{q-\log\left(\delta\right)}\right) \geq p. \tag{1}$$

For simplicity of presentation, let

$$d(i) := \delta^{-(i-1)L}\frac{q-\log\left(\delta\right)}{q\left(1-\delta^L e^{-qL}\right)} \tag{2}$$

and define $\tau(v)$ as the maximum number of periods a user with initial valuation v could pay. When $\tau(v) \leq \left\lfloor \frac{T}{L} \right\rfloor$, $\tau(v)$ satisfies:

$$v \geq d(\tau(v))p, \quad v < d\left(\tau(v)+1\right)p.$$

Solving the condition for $\tau(v)$ we obtain,

$$\tau(v) = \left(\left\lfloor \left(\log\left(\frac{p}{v}\right) + \log\left(\frac{q-\log\left(\delta\right)}{q\left(1-\delta^L e^{-qL}\right)}\right)\right)/(L\log(\delta))\right\rfloor + 1\right)_+.$$

Let $i(p, L, v)$ be the random variable representing the number of times a user with valuation v pays the period price p. Then $i(p, L, v)$ will be geometrically distributed with parameter equal to the probability of successful match in a period $1 - e^{-qL}$, truncated at $\tau(v)$ i.e.,

$$i(p, L, v) = \min\left\{\text{Geo}(1-e^{-qL}), \tau(v)\right\}.$$

Thus, the expected number of periods the user with initial valuation v will pay is

$$\mathbb{E}\left[i(p,L,v)\right] = \sum_{j=1}^{\tau(v)} j(1-e^{-qL})e^{-(j-1)qL} + \tau(v)e^{-\tau(v)qL} = \frac{1-e^{-\tau(v)qL}}{1-e^{-qL}}.$$

If a user pays the period price, the expected operating costs of the platform will be c times the expected time the user spends on the platform for that period, *i.e.*,

$$c\mathbb{E}[X_L(q)] = c\left(\int_0^L qte^{-qt}dt + Le^{-qL}\right) = \frac{c(1-e^{-qL})}{q}. \tag{3}$$

Finally, the expected profit of the platform can be evaluated as,

$$\begin{aligned}
\mathcal{R}(p,L) &= \int_0^\infty \left(p - \frac{c(1-e^{-qL})}{q}\right)\mathbb{E}\left[i(p,L,v)\right]f(v)dv \\
&= \left(p - \frac{c(1-e^{-qL})}{q}\right)\sum_{i=1}^{\lfloor T/L\rfloor}\left(\frac{1-e^{-iqL}}{1-e^{-qL}}\right)\left(\overline{F}(d(i)p) - \overline{F}(d(i+1)p)\right) \\
&\quad + \left(p - \frac{c(1-e^{-qL})}{q}\right)\left(\frac{1-e^{-\lfloor T/L\rfloor qL}}{1-e^{-qL}}\right)\overline{F}\left(d\left(\lfloor T/L\rfloor + 1\right)p\right)
\end{aligned} \tag{4}$$

where $\overline{F}(d(i)p) - \overline{F}(d(i+1)p)$ is the proportion of users who can pay at most i periods, $p - \frac{c(1-e^{-qL})}{q}$ is the expected profit per period, and $\frac{1-e^{-iqL}}{1-e^{-qL}}$ the expected number of periods they will pay. Note when $i = \left\lfloor \frac{T}{L} \right\rfloor$, the users must leave as the platform is exhausted, which corresponds to the last term.

As mentioned earlier, FP and CP are two special cases of LP. In the following lemma, we derive formulations for the profit of FP and CP from the expression for the profit of LP in Eq. (4).

Lemma 1 **(Formulations of $\mathcal{R}_{FP}(p)$ and $\mathcal{R}_{CP}$(p)).** *For all positive valued distributions F, parameters $c,q,T>0$, and $\delta \in (0,1)$,*

$$\mathcal{R}_{FP}(p) = \left(\frac{p-c}{q}\right)\int_{\frac{p}{q}}^\infty \min\left\{1 - \left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1-e^{-qT}\right\}f(v)dv, \tag{5}$$

$$\mathcal{R}_{CP}(p) = \left(p - \frac{c(1-e^{-qT})}{q}\right)\overline{F}\left(\frac{p\,(q-\log(\delta))}{q\,(1-\delta^T e^{-qT})}\right). \tag{6}$$

In this work, we will find it convenient to alternatively represent the profit from LP, FP, and CP as integrals over the time on the platform. In the following lemma, we introduce equivalent integral formulations of Eq. (4), Eq. (5) and Eq. (6). We defer this proof and all subsequent proofs to Section B of the appendix.

Lemma 2 (**Integral Formulations of $\mathcal{R}_{FP}(p)$, $\mathcal{R}_{CP}(p)$, and $R(p,L)$**). *For all positive valued distributions F, parameters $c,q,T>0$, and $\delta\in(0,1)$,*

$$\mathcal{R}_{FP}(p) = \int_0^T (p-c)e^{-qt}\overline{F}(pq^{-1}\delta^{-t})dt, \tag{7}$$

$$\mathcal{R}_{CP}(p) = \int_0^T \left(\frac{p\delta^t\,(q-\log(\delta))}{1-\delta^T e^{-qT}} - c\right) e^{-qt}\overline{F}\left(\frac{p\,(q-\log(\delta))}{q\,(1-\delta^T e^{-qT})}\right) dt, \tag{8}$$

$$R(p,L) = \sum_{i=1}^{\lfloor T/L\rfloor} \left(p - \frac{c(1-e^{-qL})}{q}\right) e^{-(i-1)qL}\overline{F}\left(d(i)p\right). \tag{9}$$

We can interpret the integral formulations in Lemma 2 as describing the state of the platform at each period i and time t, respectively. For LP and FP, $e^{-(i-1)qL}\overline{F}\left(d(i)p\right)$ and $e^{-qt}\overline{F}(p/q\delta^{-t})$ are the fraction of the market at period i and time t, respectively, that is still not priced out, and still has not matched. For CP, $e^{-qt}\overline{F}\left(\frac{p(q-\log(\delta))}{q(1-\delta^T e^{-qT})}\right)$ is similarly the fraction of the market not priced out, times the fraction of those users who still have not matched at each time t, now paying a time dependant price.

While Eqs. (4) to (9) allow us to express the profit for different period lengths and subscription prices, in this work our focus is on the profit achieved using the optimal prices. Unfortunately, these profit functions can generally be quite badly behaved with multiple locally/globally optimal prices (see Example EC.1). To allow for tractable analysis, we will focus on a natural class of valuation distributions that simplifies the price optimization of FP and CP. Specifically, we look at monotone hazard rate valuation distributions (MHR).

Definition 1 (Monotone Hazard Rate (MHR) Distributions). A random variable $V\sim F$ with density f is MHR if $\frac{\overline{F}(x)}{f(x)}$ is non-increasing.

MHR distributions are commonly used to model valuations, where they strike the appropriate balance between structure (MHR distributions have sub-exponential tails) and generality. MHR includes many common distributions, including Normal, Uniform, Exponential, and more. In Lemma 3 we show that when valuations are MHR, the optimal freemium/contract prices are unique, and as the cost c increases the prices also increase.

Lemma 3 (**Uniqueness of Optimal Prices for FP and CP**). *For all positive valued, MHR distributions F, and parameters $c,q,T>0$, and $\delta\in(0,1)$, both the optimal freemium price and the optimal contract price are unique, and increasing in c.*

We note that although MHR distributions make the optimal freemium and contract prices unique, the same can not be said for general period L pricing which can exhibit multiple optimal prices even when the valuations are uniform (c.f. Example EC.1). Much of the technical difficulty in analyzing general period length L pricing stems from this intractability, we will return to this problem in Section 3 when we prove guarantees for these pricing policies.

2.2. Slow matching condition.

Finally, we introduce one condition on our market parameters that will be helpful when thinking about dating markets. Specifically, in this paper, it will be useful to focus on markets that do not quickly match users relative to the users' patience, and the size of the pool of potential matches. We describe such markets with the following condition.

Condition 1 ((C1) Slow Matching Condition). When the market parameters q, δ, and T are such that,

$$\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}} \geq \left(\frac{q}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right),$$

we say the market satisfies a *slow matching* condition.

To build intuition for (C1), consider the case when the pool of potential matches is large, *i.e.*, $T=\infty$. In this case, the condition reduces to $q \leq -\log(\delta)$. Recall, q is the rate at which users leave the platform due to matching, and also recall that user's valuations decay exponentially in δ. Condition (C1) compares these rates, and thus we understand (C1) as describing markets where more users leave due to lack of patience or market exhaustion than because they match.

According to Pew Research Center (2020b), while three-in-ten Americans have tried dating apps, less than one-in-ten have found a serious relationship using from them, suggesting that matching is less likely to cause users to leave the platform than other causes of user churn like loss of patience (valuation drops below price), or exhausting the platform. Thus, we believe (C1) to be a natural condition in our contexts of interest and likely to hold in practice. We will return to this condition in Section 4

3. Profit Analysis for Online Dating Platforms

In this section we study the optimal profits achieveable by an online platform in terms of the subscription period length L. To motivate the study of the choice of period length L, let us first emphasize that no fixed choice of L always yields more profit for all parameters and valuation distributions in our setting and moreover, for every choice of L, there is an instance for which that period length is uniquely profit optimal (see the right most panel of Fig. EC.1 for a construction). With this in mind, in theory if given parameters c, q, T, δ, and a valuation distribution F, a platform could attempt to directly optimize their choice of L from Eq. (9). We note however that such an optimization is computationally difficult with many locally optimal solutions (see the left panel of Fig. EC.1). Moreover, the conceptual parameters in our model may be difficult to estimate, and may vary over time as market dynamics and operational realities change.

Instead of focusing on this explicit optimization problem, we will instead direct our attention to proving robust guarantees for some fixed choice of L, and on examining natural parameter regimes where the profit maximizing choice of L can be identified. Both approaches will yield results which can directly inform operational decision making with respect to the setting of pricing policies. As one might imagine from the framing in the introduction, both FP ($L \to 0$) and CP ($L \to T$) will play a starring role as robust solutions or optimal solutions given some market conditions, respectively. In the next subsection we will show that FP is uniquely robust in the sense that for all market parameters, and any MHR valuation distributions, it achieves a constant factor of the optimal profit against the best-in-class pricing policy.

3.1. FP is approximately optimal

In this subsection, we compare the maximal profit of period L pricing with freemium pricing (FP) and demonstrate that FP achieves a constant factor of the optimal profit of the best-in-class pricing policies. Before stating our results, note that FP has several unique properties which make it a prime candidate as a robust pricing policy for an online dating platform. For one, by varying the costs c we can construct scenarios for which FP earns positive profit whereas *any* other period length $L > 0$ earns nothing when the costs are prohibitively high (see Fig. 3 for an example when valuations are exponentially distributed). This immediately precludes any other choice of L from guaranteeing a constant fraction of the optimal profit for all market parameters. Of course, this is just one way of varying the market parameters, in this subsection we will prove that no matter the market conditions, freemium pricing always earns at least 26.8% of the profit of any period length L subscription pricing, and for almost all parameters, the guarantee is much stronger.

To prove such a result we will require two lemmas. First, we will we show that the optimal profit of any period L pricing can be approximately decomposed into the profit of a freemium pricing policy, and the profit of a certain contract pricing policy where the maximal time on the platform T is reduced to L. Second, we will show that FP approximates CP. Carefully combining these two results will yield our promised robustness guarantee.

LEMMA 4 **(Profit decomposition)**. *For all positive valued distributions F, parameters $c, q, T > 0$, and $\delta \in (0,1)$, then:*

$$\mathcal{R}(L) \leq \mathcal{R}_{FP} + \mathcal{R}_{CP}^{L},$$

where we use $\mathcal{R}_{CP}^{L}$ to denote the optimal profit of contract pricing with T set to L.

Lemma 4 decomposes the profit from a period L pricing into the profit from a particular instance of FP and CP. The main idea of the lemma is to note that the profit earned by the end of each subscription period is exactly the same as an appropriately priced instance of freemium pricing, and profit earned in the final subscription period is upper bounded by the profit of a shortened bout of contract pricing. The upside of Lemma 4 is that, while the profit of the optimal best in class period L pricing policy is difficult to write down in closed form, and difficult to reason about, FP and LP are comparatively much easier to compute and reason about. By relating these quantities, we obtain an handle on the maximal achieveable profit by the best subscription policy (parameterized by L), and demonstrate that reducing our strategy space to just CP and FP maintains approximate optimality.

Armed with Lemma 4, our next task will be to show that FP approximates CP. The following lemma shows that freemium pricing always earns at least $1/e$ of the profit of contract pricing for all parameters, and often substantially more.

LEMMA 5 **($\mathcal{R}_{FP}$ approximates $\mathcal{R}_{CP}$)**. *For all positive valued distributions F, parameters $c, q, T > 0$, and $\delta \in (0,1)$, then:*

$$\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}} \geq \left(1 - \frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}.$$

Taking the minimum over $\frac{q}{-\log(\delta)}$ *yields a constant factor approximation,*

$$\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}} \geq \frac{1}{e}.$$

Further, this bound is tight.

Taking Lemmas 4 and 5 together, we show that freemium pricing can achieve at least $\frac{1}{1+e}$ of the optimal subscription profit of the optimal any period L pricing over any set of market parameters.

Theorem 1 (**$\mathcal{R}_{FP}$ approximates $\mathcal{R}(L)$**). *For all positive valued distributions F, parameters $c, q, T > 0$, $\delta \in (0,1)$, and for any period length L:*

$$\frac{\mathcal{R}_{FP}}{\mathcal{R}(L)} \geq \frac{1}{1+e},$$

and no other fixed choice of pricing strategy besides freemium pricing can guarantee any constant factor.

Theorem 1 demonstrates that in all parameter regimes, freemium pricing (FP) guarantees a substantial fraction of the profit garnered by period L pricing. Further, we note that our bound is parametric (via Lemma 5) and often guarantees more than $1/(1+e)$ of the optimal profit before considering anything about the valuation distribution. Thus in practice, we expect FP to earn an even greater fraction of the profit than the worst case guarantee in Theorem 1.

Practically, it is instructive to think of Theorem 1 as evidence that FP is a flexible and robust pricing strategy. This is in-line with its use cases in practice; often new dating apps are monetized in a freemium pricing fashion as they grow their user-base. Unlike contract pricing (CP), freemium pricing requires no extraordinary market powers or commitments from the users and thus may be the only available pricing mechanism for the an emerging online dating platform. These market realities, combined with our guarantee, provides a justification for the prevalence of freemium pricing in the US for emerging online dating platforms.

3.2. Fine-grained profit analysis

In Theorem 1, we proved the profit of FP always approximates the profit of optimally calibrated LP policy. While policy with good guarantees for all market are desirable, online dating markets parameters often are a certain way, and focusing and these specific cases may yield stronger guarantees. For instance, one interesting consequence of the Lemma 5 and Theorem 1 is that, when the match rate q is fixed and users are very patient *i.e.* $\delta \approx 1$, then the parameter $\frac{q}{-\log(\delta)}$ becomes very large, and the guarantee tends to 1/2. This implies that when users are patient, freemium pricing achieves essentially as much profit as subscription pricing and contract pricing regardless of cost or valuation distribution. In fact, when $\delta = 1$, $\mathcal{R}_{FP} = \mathcal{R}(L) = \mathcal{R}_{CP}$ (see Example EC.3 for full derivation) and thus Lemma 5 and Theorem 1 let us understand the performance of the strategies in markets with less than perfectly patient users.

In Theorem 2 we look at two highly business relevant instances where extremal versions of a market parameter, namely the marginal operating cost c, similarly imply strong profit relations.

THEOREM 2 **(Profit relationships as cost varies)**. *For any positive-valued, MHR distributions F, parameters $q, T > 0$, and $\delta \in (0,1)$, then:*

a) When $c = 0$, $\mathcal{R}_{CP} \geq \mathcal{R}(L)$ for any L.

b) When c is sufficiently large, $\mathcal{R}_{FP} \geq \mathcal{R}(L)$ for any L.

In Theorem 2 we find that the profit from contract pricing always dominates the profit of any other period L pricing when the marginal cost of operating the platform tends to zero. This is in stark contrast with our previous result which supports FP as the robust choice of pricing strategy. We note that in established online markets one can often expect the marginal cost of operating of the platform to be vanishingly small, and thus although FP is vastly more common for online dating, Theorem 2(a) suggests that CP may be more profitable for most online dating platforms. On the other hand, when the marginal cost of operation is high, as might be expected in the case of a emerging online dating platform trying to build its user base, we find the reverse is true. In this case the profit of the optimal freemium price dominates. Taking Theorems 1 and 2 together, we find the FP and CP have unique profit properties that make them particularly attractive among all period L pricing policies. FP is robust and optimal for emerging platforms with very high marginal costs, and CP is optimal in low cost scenarios which should describe most established online dating platforms. For the remainder of this paper we will give special focus to comparisons between FP and CP. Specifically, in the next section we study the impact of FP vs. CP on users when marginal costs are low.

4. Impact and incentives of profit maximizing online dating platforms

In the previous section we showed that FP and CP have unique properties with respect to profit which make them important to study in comparison with each other. In the remainder of this paper, we look beyond profit and study the structure, social welfare, and specific incentives induced by the choice of FP or CP. In Section 4.1, we prove a sharp relation between the optimal prices under FP and CP when a specific market condition (C1) holds, and use that relation to study the proportion of the market that ends up matched. In Section 5, we extend our study beyond our assumption that the population of candidate matches are all equally likely to match with the user. In a natural extension, we show that for LP, inside each period, the online dating platform has an incentive to show the user their least likely matches first, whereas between periods, the online dating platform the online dating platform has the opposite incentive, to display the most likely matches first.

4.1. Proportion of the market that matches

In this subsection, we will examine the consequences that profit maximizing sales practices have for users. Specifically, we study which of FP and CP leads to more of the user-base being matched. To help us characterize the percent of the user-base that matches under FP or CP, we will first prove a structural condition on the prices.

LEMMA 6 **(Price dominance for FP vs. CP)**. *For all positive valued, MHR distributions F, parameters $c, q, T > 0$, and $\delta \in (0,1)$ satisfying (C1), then:*

a) The optimal freemium price is greater than or equal to the optimal contract price divided by the expected amount of time a user who pays the contract price spends on the platform, $q^{-1}\left(1 - e^{-qT}\right)$, i.e.,

$$\arg\max_p \mathcal{R}_{FP}(p) \geq \left(\frac{q}{1 - e^{-qT}}\right) \arg\max_p \mathcal{R}_{CP}(p).$$

b) The ratio $\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}}$ is increasing, and there exists unique c^* such that $\mathcal{R}_{FP} = \mathcal{R}_{CP}$.

To build intuition for the inequality in Lemma 6(a), consider a user who pays some contract price p_{CP}. The expected time they will stay on the platform is $\mathbb{E}[X_{CP}(v,p)\mathbf{1}_{X_{CP}(v,p)\leq T}] = \frac{1-e^{-qT}}{q}$. Thus, $\left(\frac{q}{1-e^{-qT}}\right) \arg\max_p \mathcal{R}_{CP}(p)$ is the expected price per unit of time paid on the platform, and the claim is that under FP users pay a higher expected price per unit time than users do under CP. In Example EC.3 we demonstrate that this lemma is tight when (C1) holds.

Armed with this lemma, we will now investigate whether users are more likely to match under FP or CP. Let $\mathcal{M}_{FP}(p)$ and $\mathcal{M}_{CP}(p)$ be the proportion of the market that ultimately gets matched under freemium pricing or contract pricing, with price p, respectively. Note, this is the proportion of users who first pay the price, times the probability of then getting matched eventually. Like the profit, the match proportion can be expressed as a function of the model parameters. For FP, the probability of user with valuation v getting matched eventually is $1 - e^{q\tau(v)}$,

$$\mathcal{M}_{FP}(p) = \int_{\frac{p}{q}}^{\infty} \underbrace{\min\left\{1 - \left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1 - e^{-qT}\right\}}_{\text{Prob. of matching for valuation } v} f(v)dv, \tag{10}$$

$$\mathcal{M}_{CP}(p) = \underbrace{\left(1 - e^{-qT}\right)}_{\text{Prob. of matching}} \underbrace{\overline{F}\left(\frac{p\,(q - \log(\delta))}{q\,(1 - \delta^T e^{-qT})}\right)}_{\text{Prob. of paying price } p}, \tag{11}$$

Let $p_S^* = \arg\max_p \mathcal{R}_{FP}(p)$ and $p_C^* = \arg\max_p \mathcal{R}_{CP}(p)$ be the profit maximizing prices for FP and CP (recall these prices are unique by Lemma 3), and let $\mathcal{M}_{FP} = \mathcal{M}_{FP}(p_S^*)$, $\mathcal{M}_{CP} = \mathcal{M}_{CP}(p_C^*)$ denote the proportion of the user-base matched under the profit maximizing prices. Using the expression in Lemma 1, the optimal profit of the two strategies can be written in terms of the match rate then as,

$$\begin{aligned}
\mathcal{R}_{FP} &= \max_p \left(\frac{p-c}{q}\right) \int_{\frac{p}{q}}^{\infty} \min\left\{1 - \left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1 - e^{-qT}\right\} f(v)dv \\
&= \left(\frac{p_S^* - c}{q}\right) \mathcal{M}_{FP}, \\
\mathcal{R}_{CP} &= \max_p \left(p - \frac{c(1 - e^{-qT})}{q}\right) \overline{F}\left(\frac{p\,(q - \log(\delta))}{q\,(1 - \delta^T e^{-qT})}\right) \\
&= \left(\frac{p_C^*}{1 - e^{-qT}} - \frac{c}{q}\right) \mathcal{M}_{CP}.
\end{aligned}$$

Unfortunately as with profit, there is no universal match proportion relationship between the two strategies or in general period L pricing. When costs are sufficiently high, CP (and any other policy with $L > 0$) is not economically viable and thus matches none of the user-base, whereas FP stays in the market and still matches at least some users. However, as mentioned above, for online dating one can expect the marginal cost of operating a platform to be relatively small. To rule out these pathological instances and make a fair and relevant comparison between FP and CP, we will assume the cost is vanishingly small for our next result.

THEOREM 3 (**$\mathcal{M}_{CP}$ dominates $\mathcal{M}_{FP}$ when costs are low**). *For all positive valued, MHR distributions F, and parameters q, T, and δ satisfying (C1), if c is less than c^*, then:*

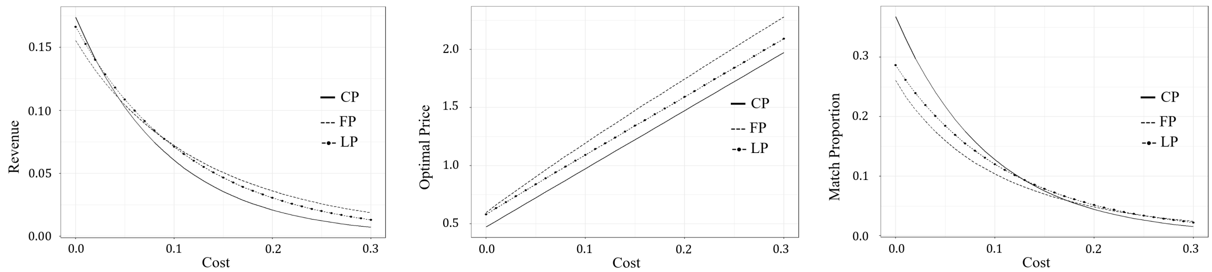
$$\mathcal{M}_{CP} \geq \mathcal{M}_{FP}.$$

Theorem 3 gives generic conditions for when CP matches a higher proportion of the market than FP. Specifically, it requires that the marginal costs be relatively small and (C1) hold, both of which are reasonable assumptions for the operational conditions of most large online dating platforms. Moreover, by Theorem 2(a) in these same conditions, the profit the platform earns from CP exceeds the profit of FP. Thus in online dating markets with low marginal costs, CP is a win-win: more users are matched, and more profit is made than under FP. In Fig. 3 we plot the relative profits, induced optimal prices, and match proportions as the cost varies for market parameters satisfying (C1), and when valuations drawn from an exponential distribution. We note that, in these numerics, c^* occurs quite far from 0, the prices are close in the sense of Lemma 6(a), and CP matches significantly more of the market, even when c is larger than c^* suggesting the result in Theorem 3 is relatively conservative. Similar results for Uniform valuations and Mixture of Log-normal distributions (which note are not MHR or even unimodal) are shown in Figs. EC.3 and EC.4, and similar "win-win" behaviour is exhibited.

5. Incentive considerations for online dating platforms

In all previous sections, we have assumed q, the rate at which a user matches on the platform, was fixed and constant. In this section, we will extend our framework to heterogeneous populations of potential matches, which we model as having varying match rates. Many dating platforms, including some of the most popular apps like Hinge, Tinder, Okcupid etc., implement features to segregate high probability and low probability potential matches (e.g. Hinge roses, Tinder top picks, etc.). It is thus clear that users have preferences over potential candidates that impact the match rate, and that platforms are

Figure 3 Relations between optimal price, profit, and match proportion when valuations are exponential.



Note. Here we plot the profit, optimal prices, and match proportions under FP, CP and period L pricing with $L = T/7$, when valuations are drawn from an exponential(1) distribution, and where $T = 50$, $\delta = 0.8$, $q = 0.2$, and c varies. In the left panel, we plot the profits of FP, CP and period L pricing with $L = T/7$, and the note relative profit ordering switches from $\mathcal{R}_{CP} > \mathcal{R}_{FP}$ when $c \le 0.04$, to $\mathcal{R}_{CP} < \mathcal{R}_{FP}$ for $c > 0.04$. In the middle panel, we plot the normalized optimal contract, freemium and period L prices, as in Lemma 6. In the right panel, we plot the proportion of the market that gets matched under FP, CP and period L pricing. Note that $\mathcal{M}_{CP}$ dominates $\mathcal{M}_{FP}$ for $c \le 0.17$.

able to learn (to some degree) these match rates and use them to determine the order in which potential matches are displayed.

Specifically, let there be k possible match rates $\{q_1, ..., q_k\}$, where $q_1 \le q_2 \le ... \le q_k$, and suppose the online dating platform can order the potential matches based on match rate. The population size of potential matches with rate q_i for the user is t_i, $i = 1, ..., k$, where $t_1 + ... + t_k = T$. We assume the user's belief about the match rate on the platform is the average of the potential candidates, $q = (\sum_{i=1}^{k} q_i t_i)/T$, which is independent of the order in which candidates are displayed. To denote the platforms decision for the order, we use $\overrightarrow{q}$ to denote $\{q_1, ..., q_k\}$, $\overleftarrow{q}$ to denote $\{q_k, ..., q_1\}$, and $\sigma(q)$ to denote any other order of match rates. Further, we use $\mathcal{R}_{FP}(p, \sigma(q))$ and $\mathcal{R}_{CP}(p, \sigma(q))$ to denote the profit of FP and CP, respectively, when potential matches are shown following match rate order $\sigma(q)$ with fixed price p, and we use $\mathcal{R}_{FP}(\sigma(q))$ and $\mathcal{R}_{CP}(\sigma(q))$ to denote the optimal profit of FP and CP under match rate order $\sigma(q)$. For an example of how profits change under different orders, see Example EC.4.

We emphasize that this model for match rate order is not game theoretic. Users cannot learn or respond to a platforms' choice of order, and valuations and beliefs about the match rate do not vary as users gain experience in our model. For this reason we certainly do not claim that the optimal ordering in this model necessarily translates to equilibrium behaviours for the platform. Instead, the purpose of our heterogeneous match rate model is the understand the *obvious* incentives of the online dating platforms with respect to match rate, around which much of the discussion about online dating platforms is centered, and which is being leveraged as a reason to innovate in the way dating apps are monetized (c.f. Fig. EC.2). The upshot of our analysis is that if (possibly under some conditions) the platform is incentivized to show the best possible matches up front, then the platform and users are in harmony and the space for strategic behaviour collapses.

In this vein, in Lemma 7 we characterize how a profit maximizing platform orders potential matches under FP and CP respectively.

LEMMA 7 **(Match rate ordering for CP and FP)**. *For all positive valued distributions F, parameters $c, T > 0$, and $\delta \in (0,1)$, and set of match rates $\{q_1, \ldots, q_k\}$, then for any ordering σ:*

a) $\mathcal{R}_{CP}(p, \overleftarrow{q}) \geq \mathcal{R}_{CP}(p, \sigma(q))$,

b) $\mathcal{R}_{FP}(p, \overrightarrow{q}) \geq \mathcal{R}_{FP}(p, \sigma(q))$.

Lemma 7 shows that a profit maximizing online dating platform has the incentive to show low match rate candidates first under FP, whereas it has the incentive to provide high match rate candidates first under CP. This is intuitive, for FP the longer users stay on the platform, the more profit an online dating platform can achieve. In contrast, for CP users pay the contract price p once and thus the strategy to maximize profit is to reduce the operating cost. Therefore online dating platforms using CP need to display high match rates candidates first so that users can get matched and leave the platform as soon as possible. In fact, the following theorem demonstrates that similar incentives hold true for general period L pricing.

THEOREM 4 **(Match rate ordering general period L pricing)**. *For all positive valued distributions F, parameters $c, T > 0$, $\delta \in (0,1)$, set of match rates $\{q_1, \ldots, q_k\}$, then $\sigma^* = \arg\max_\sigma \mathcal{R}_L(p, \sigma)$ is such that:*

a) Within each period, the match rate is decreasing.

b) Between periods, the average match rate is increasing if

$$c \leq p \frac{p\left(e^{-q_j L} - e^{-q_{j+1} L}\right) \overline{F}(d(i+1)p)}{\left(\frac{1-e^{-q_j L}}{q_j} - \frac{1-e^{-q_{j+1} L}}{q_{j+1}}\right) \overline{F}(d(i)p) + \left(\frac{(1-e^{-q_{j+1} L})e^{-q_j L}}{q_{j+1}} - \frac{(1-e^{-q_j L})e^{-q_{j+1} L}}{q_j}\right) \overline{F}(d(i+1)p)},$$

for $i = 1, ..., \lfloor T/L \rfloor - 1$.

Theorem 4 shows that, inside each period, the period L pricing behaves like CP, while between periods, the period L pricing behaves like FP as long the unit operating cost c is not too high, which is encoded in the complicated expression in part (b). Thus a profit maximizing online dating platform uses the same strategy to retain users between periods as FP, while at the same time, also reduces operation costs within each period like CP. Further, as L varies from 0 to T, Theorem 4 in a sense interpolates between CP and FP and implies that for longer periods, the platform is better align with the incentives of it's users, and those user will have an overall better experience.

Returning to the condition in part (b), we note that while complicated it is trivially true when c is vanishingly small, or when the match rate is constant, and so it strictly generalizes Lemma 7. The necessity of the condition follows from the fact that we are giving a bound for all feasible prices p. We conjecture that the result in (b) is unconditionally true for profit maximizing choices of p for LP, but given the non-uniqueness of the optimal price for LP we are not able to show this directly. To illustrate the complicating dynamics, in Example EC.5 we give an example where the operating cost is too high relative to the price, and thus the optimal average match rate is decreasing between period.

Finally, we note that there is some complicated interplay between the order of the match rates, and the profit maximizing subscription price. Certainly, if the match rate can vary, perhaps so too should a period L price to reflect the increase or decrease in the quality of service provided. In our final theorem, we show that for CP not only are the incentives of the platform and the users aligned, but also the pricing decision is simple and the improved efficiency is passed on to the user in the form of savings.

Theorem 5 **(CP aligns incentives at a lower price)**. *When the valuation distribution F is MHR, the optimal contract price is decreasing when the online dating platform can manipulate the order of potential match candidates, i.e.,*

$$\arg\max_p \mathcal{R}_{CP}(p, \overleftarrow{q}) \leq \arg\max_p \mathcal{R}_{CP}(p, \sigma(q))$$

Theorem 5 shows that strategic behavior under CP is not only beneficial for online dating platforms, but also improves users' surplus. The optimal contract price p will be smallest under the mutually beneficial, optimal match rate ordering, and this under CP the online dating platform can serve even more users by offering a lower price.

More generally, on dating apps, as the user interacts with the platform, the platform learns which potential matches the user would likely prefer. As mentioned in the introduction, ostensibly FP incentivizes the platform to hold likely matches back from the user, whereas CP incentivizes the platform to try and match the user as soon as possible. Taken together Lemma 7 and Theorem 5 formalizes this intuition and further shows that not only does CP incentivize the platform to use information about user preferences to help the user match, but also that information induces a profit maximizing platform to lower the contract price. This significantly simplifies the strategic considerations around revealing preference since the user and the platform are, in this regard, perfectly aligned.

6. Conclusions

In this paper we propose a novel and natural model to study the profit obtained by an online dating platform committing to subscription based pricing policies of varying lengths. In our model we gave a number of theoretical guarantees regarding the profit, welfare, and incentive issues induced by the pricing policy which translate directly to managerial insights for future dating platforms. In this final section we will elaborate on some of these managerial insights, and how they relate to current and future dating app operations. We will also touch on some drawbacks of our work, and future directions for research on these critical platforms.

For dating app users and designers, our work sheds light on the consequences of the pricing strategy a profit-maximizing dating apps commits to. As we emphasized in the introduction, a savvy user may recognize that for short subscription period lengths, there is an incentive for any online dating platform to act against their users best interests so as to earn additional subscription payments. Noting this compatibility mismatch, the strategic user may choose the cheaper option when it would be otherwise utility maximizing to purchase more expensive options. The strategic user may also misrepresent their true preferences, perhaps instead passing themselves off as a casual dater, further complicating the difficult task of finding a match for the user.

This difficult incentive mismatch is referred to as the“strategy puzzle” of online dating by Wu et al. (2019). One way of reading our work is as demonstrating that longer periods offer a way out of this puzzle. Specifically, contract pricing flips this script and aligns the platform and the user. When the costs are paid upfront, the platform and the users have the same goals and the incentive to behave strategically vanishes. We further note that the disincentive to strategize under long period pricing may have additional beneficial effects for the platform that are not accounted for by our model. When all users truthfully report preferences, this in turn may increase the accuracy of the platforms matching algorithms, and thus may generate more matches, which leads to better efficiency, better word of mouth advertising from happy couples, higher return rates, less burn-out among users and so on.

More generally, our work provide theoretical justification for longer periods as a true win-win in the online dating space. When marginal costs are low (as is often the case online) and matching is a slow and noisy process, we prove that contract pricing is simultaneously more profitable for the online dating platform, and more effective at matching a significant portion of the user-base. Of course, there certain barriers not captured in our model that may make longer period pricing strategies like CP more difficult to implement in an online environment. For instance, requiring a large up front sum from the customer may be difficult for budget constrained users, and requires substantial commitment and trust on the part of the user that after payment the site will be effective for them.

While these implementation issues are potentially burdensome, we believe they are worth the effort of surmounting. We note one emerging avenue for implementing CP is by working in collaboration with government agencies that can track marriage records, and thus enforce contracts where users pay *after* matching (or never charging them all). In fact, given the strong social value of efficient matchmaking, nationalized dating apps are being tested overseas in countries like Japan and Singapore (AKITA 2019, Afp 2010). For such platforms, CP could reasonably be implemented and may yield superior results. More generally, our work quantifies the limits of private online dating and provides motivation for dating app implementations which are not solely profit-driven.

Finally, we note that online dating in general is remarkably complicated and our model required a number of simplifications to ensure tractability. For instance, our work describes users solely looking for their life-long match, however on sites like Tinder or Grindr, many users may only be looking for short relationships. In this case, the incentives for the site are different, and even under FP the platform may still try to match users in the hopes of soliciting their repeat business. This heterogeneous population of user interests may provide some salve to the analysis presented in this paper, and would be interesting to study in future work. We also note that our paper studies platforms where a single subscription plan is offered. As seen in Figs. 1 and EC.2 however, many apps offer a menu of varying length subscriptions, where users can pay for a short, moderate, or long amount amount of time on the site. In future work, it would be interesting to study how these menus of subscription options effect the incentives and performance of dating apps.

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Appendix A: Omitted Examples

EXAMPLE EC.1 (PROPERTIES OF PERIOD L PRICING). In this example we demonstrate three properties about period L pricing. First, we show that the optimal price for period L pricing may not be unique with potentially many locally optimal prices, and thus we cannot rely on first order conditions alone in our analysis for general period L pricing. Second, we show that the profit of period L pricing may not be monotone over L, even for MHR valuation distributions. Third, we show that for any given period length L, there exists model parameters such that L is the optimal period length precluding the existence of a universally optimal choice of L.

To demonstrate the the non-uniqueness of the optimal prices for period L pricing, suppose a user's valuations are uniformly distributed on the range $[0,1]$, i.e., $V \sim U[0,1]$, and suppose the cost $c = 0.1$, the match rate $q = 0.8$, the valuation decays as $\delta = 0.25$, the total time users can stay on the platform $T = 1$, and the period length $L = T/2 = 0.5$. Then referring to Fig. EC.1 left panel, we can see there are two locally optimal subscription prices, one is near 0.12, the other near 0.15. To understand this phenomenon, note in this example that users can stay on the platform for at most two periods. Therefore, the online dating platform can set the price to be high, which will force users to stay on the platform for at most one period, or the online dating platform can set the price to be low, which will allow users stay for up to two periods. Each instance corresponds to a different locally optimal price.

To demonstrate the non-monotonicity of $\mathcal{R}(L)$ as a function of L, consider a market similar to the one described above where users' valuations are distributed as $U[0,1]$, $c = 0.05$, $q = 0.2$, and now valuations decay as $\delta = 0.8$, and $T = 50$. In such a market, the optimal choice of L can be numerically computed to be 4.55, and achieves profit 0.40. In the middle panel of Fig. EC.1, we see the non-monotonicity of profits in L, and further note that there also exists locally (but not globally) optimal choices of period L before the optimal period length L.

Finally, to demonstrate that every choice of L can be optimal given the right market parameters, again consider a market where user's valuations $V \sim U[0,1]$, $q = 0.2$, $\delta = 0.8$, $T = 50$. In the right panel of Fig. EC.1 we vary the cost c and compute the optimal choice of L via grid search. We note that L is decreasing in a smooth fashion as c increases.

□

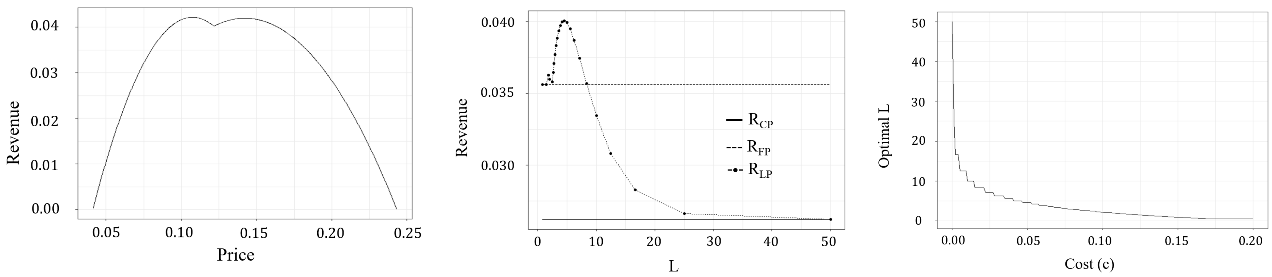
EXAMPLE EC.2 (TIGHTNESS OF LEMMA 5). In this example, we give an instance of our model such that Lemma 5 is tight. Specifically, suppose valuations are fixed and drawn from a point-mass distribution on v, let $c = 0$, $T = \infty$, and let q and δ be arbitrary. In this case the optimal freemium price is $p^* = vq\left(\frac{\log(\delta)-q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}$, and the profit of optimal freemium pricing is,

$$\mathcal{R}_{FP} = \left(\frac{vq}{q-\log(\delta)}\right)\left(\frac{\log(\delta)-q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}.$$

Similarly, the profit of optimal contract pricing is,

$$\mathcal{R}_{CP} = \left(\frac{vq}{q-\log(\delta)}\right).$$

Figure EC.1 Examples of non-unique optimal subscription price and LP dominates FP and CP.



Note. In the left panel, depicted are an online dating platform's profit vs. subscription prices, where user's valuations $V \sim U[0,1]$, unit operating cost is $c = 0.1$, match rate is $q = 0.8$, user's patience factor is $\delta = 0.25$, total market size is $T = 1$, and the period length is $L = T/2 = 0.5$. In the middle panel, depicted are an online dating platform's profit using LP, FP and CP, as period length L varies, where user's valuations $V \sim U[0,1]$, unit operating cost $c = 0.05$, match rate $q = 0.2$, user's patience factor $\delta = 0.8$, and total time $T = 50$. In the right panel, depicted is the optimal period length L when cost c changes, where user's valuations are $V \sim U[0,1]$, the match rate is $q = 0.2$, the user's patience factor is $\delta = 0.8$, and the market size is $T = 50$.

The ratio between $\mathcal{R}_{FP}$ and $\mathcal{R}_{CP}$ is then,

$$\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}} = \left(1 - \frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}},$$

therefore, the approximation ratio is tight. □

Example EC.3 (Tightness of Lemma 6). In this example, we give an instance of our model such that the price dominance in Lemma 6 is tight. Specifically, let $\delta = 1$ and let F, c, q, and T be arbitrary. In this case the profit of FP and CP can be written as,

$$\begin{aligned}
\mathcal{R}_{FP}(p) &= \max_p \int_0^T (p-c)\overline{F}(pq^{-1})e^{-qt}dt \\
&= \max_p (p-c)\overline{F}(pq^{-1}) \int_0^T e^{-qt}dt, \\
\mathcal{R}_{CP}(p) &= \max_p \int_0^T \left(\frac{pq}{1-e^{-qT}} - c\right)\overline{F}\left(\frac{p}{1-e^{-qT}}\right)e^{-qt}dt \\
&= \max_p \left(\frac{pq}{1-e^{-qT}} - c\right)\overline{F}\left(\frac{p}{1-e^{-qT}}\right)\int_0^T e^{-qt}dt.
\end{aligned}$$

Let $p_S^* = \arg\max_p \mathcal{R}_{FP}(p)$, $p_C^* = \arg\max_p \mathcal{R}_{CP}(p)$, then, solving the optimization for FP and CP, we obtain the optimal prices $(1-e^{-qT})\,p_S^* = qp_C^*$, and moreover $\mathcal{R}_{FP} = \mathcal{R}_{CP}$. □

Example EC.4 (Manipulation Gap). In this example, we consider FP and CP when the platform has access to two types of potential matches it can display. Suppose valuations are fixed and drawn from a point-mass distribution on v, let $c = 0$, $T = \infty$, $\delta = 1$, and suppose $q_1 = 0$, $q_2 = 1$, and $t_1 = t_2 = 1$. Then the match rate perceived by users is $q = \frac{q_1+q_2}{2} = 0.5$. The profit of contract pricing that chooses to show type 2 potential matches first is,

$$\mathcal{R}_{CP}(\{q_2, q_1\}) = v\left(1 - e^{-qT}\right) = 1 - \frac{1}{e},$$

The profit of contract pricing that chooses to show type 2 potential matches first is,

$$\mathcal{R}_{FP}(\{q_1,q_2\}) = vq\left(\frac{1-e^{-q_1}}{q_1} + e^{-q_1}\left(\frac{1-e^{-t_2 q_2}}{q_2}\right)\right) = 1 - \frac{1}{2e}.$$

By Example EC.3, when the match rate was homogeneous the profit of FP and CP was the same. Now, when the platform is allowed to choose the order, the difference in profit is,

$$\mathcal{R}_{FP}(\{q_1,q_2\}) - \mathcal{R}_{CP}(\{q_2,q_1\}) = \frac{1}{2e}.$$

□

Example EC.5 (Interactions between price and match rate for LP). In this example, we give an instance of period L pricing where under the profit maximizing ordering, the average match rate is not increasing. Specifically, let $T=2$, $L=1$, $q_1=0$, $q_2=2$, $t_1=t_2=1$, and thus the user's perceived match rate is $q=\frac{q_1+q_2}{2}=1$. Further, suppose the subscription price is p and the operating cost is $c=p$ so that the profit is 0 for a user that stays for an entire period. For the match rate order $\{q_1,q_2\}$, the profit for period L pricing is

$$\begin{aligned}\mathcal{R}(p,\{q_1,q_2\}) &= (p-c)\overline{F}(d(1)p) + \left(p - \frac{c(1-e^{-2})}{2}\right)\overline{F}(d(2)p)\\ &= \left(p - \frac{c(1-e^{-2})}{2}\right)\overline{F}(d(2)p),\end{aligned}$$

where the second equality follows from $c=p$. For the match rate order $\{q_2,q_1\}$, the profit for period L pricing is

$$\begin{aligned}\mathcal{R}(p,\{q_2,q_1\}) &= \left(p - \frac{c(1-e^{-2})}{2}\right)\overline{F}(d(1)p) + (p-c)e^{-2}\overline{F}(d(1)p)\\ &= \left(p - \frac{c(1-e^{-2})}{2}\right)\overline{F}(d(1)p)\\ &> \left(p - \frac{c(1-e^{-2})}{2}\right)\overline{F}(d(2)p) = \mathcal{R}(p,\{q_1,q_2\})\end{aligned}$$

where the inequality follows from $\overline{F}(d(1)p) > \overline{F}(d(2)p)$. Note that in this example the optimal match rate is increasing for the period L pricing. The reason is that the operating cost is too high, and thus offering users low match rate candidates doesn't bring any profit to the platform, and more some of those users will leave the platform after the first period due to a drop in valuation. Since the platform cannot get any benefit for offering low match rate candidates first, it is instead profit maximizing to offer high match rate candidate first.

We note this construction follows from the unrealistically low choice of p relative to c. In reality the operating is typically not high relative to the subscription price, since if otherwise the platform will simply raise the subscription price. □

Appendix B: Omitted Proofs

B.1. Omitted proofs from Section 2

Proof of Lemma 1 We will separately construct the desired expressions for FP and CP. First to derive FP we will reiterate, and simplifying where appropriate, the argument in Section 2.1 which derived $\mathcal{R}_{LP}$.

Specifically, when L goes to 0, Eq. (1) simplifies so the user with valuation v is now willing to pay the freemium price as long as $vq\delta^t \geq p$. Using this simplified payment condition, we can directly solve for the longest time a user with valuation v will stay which is $\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}$. Now let $X_{FP}(v,p)$ be the random variable representing the users time on the platform when the freemium price is p. Note $X_{FP}(v,p) = \min\left\{\text{Exp}(q), \frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}$, where $\text{Exp}(q)$ is the exponential random variable with rate q. The expectation over the exponential matching randomness is then,

$$\begin{aligned}
\mathbb{E}[X_{FP}(v,p)] &= \int_0^{\min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}} qte^{-qt}dt + \min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\} e^{-q\min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}} \\
&= \frac{1 - e^{-q\min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}}\left(1 + q\min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}\right)}{q} + \min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\} e^{-q\min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}} \\
&= \frac{1 - e^{-q\min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}}}{q} = \frac{\min\left\{1 - \left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1 - e^{-qT}\right\}}{q}
\end{aligned} \tag{EC.1}$$

Now taking expectation over the valuations, the average users expected time on the platform is then $E_{V\sim F}[X_{FP}(v,p)]$ and the expected profit for a given freemium price p is,

$$\begin{aligned}
\mathcal{R}_{FP}(p) &= (p-c)\mathbb{E}[\text{Time on platform}|\text{FP}] \\
&= \left(\frac{p-c}{q}\right)\int_{\frac{p}{q}}^{\infty} \min\left\{1 - \left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1 - e^{-qT}\right\} f(v)dv.
\end{aligned}$$

Now we will derive the expression for CP directly from Eq. (4) by noting that when $L = T$, LP will be the same as CP, i.e.,

$$\begin{aligned}
\mathcal{R}_{CP}(p) = \mathcal{R}(T,p) &= \left(p - \frac{c(1-e^{-qL})}{q}\right)\sum_{i=1}^{\lfloor T/T \rfloor}\left(\frac{1-e^{-iqL}}{1-e^{-qL}}\right)\left(\overline{F}(d(i)p) - \overline{F}(d(i+1)p)\right) \\
&+ \left(p - \frac{c(1-e^{-qL})}{q}\right)\left(\frac{1-e^{-\lfloor T/T \rfloor qL}}{1-e^{-qL}}\right)\overline{F}\left(d\left(\lfloor T/T \rfloor + 1\right)p\right) \\
&= \left(p - \frac{c(1-e^{-qT})}{q}\right)\left(\frac{1-e^{-qT}}{1-e^{-qT}}\right)\overline{F}(d(1)p) \\
&= \left(p - \frac{c(1-e^{-qT})}{q}\right)\overline{F}\left(\frac{p\left(q - \log(\delta)\right)}{q\left(1-\delta^T e^{-qT}\right)}\right),
\end{aligned}$$

where the equalities follow by simplifying. □

Proof of Lemma 2 In this proof, we separately derive alternative formulations for FP, CP, and LP.

First, we derive a new expression for FP. In Lemma 1, we derived an expression of FP as integration over user's valuations, here, we derive an alternative formulation of FP as an integration over time. Consider a platform offering subscription p to a user with fixed valuation v. Recall $X_{FP}(v,p)$ is the random variable representing the time the user spends on the platform. The expectation of $X_{FP}(v,p)$ is,

$$\mathbb{E}[X_{FP}(v,p)] = \int_0^{\min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}} e^{-qt}dt.$$

The expected profit (taken with respect to the users randomly drawn valuation, and the matching randomness) with freemium price p is then,

$$\begin{aligned}
\mathcal{R}_{FP}(p) &= \int_{\frac{p}{q}}^{\infty} (p-c) \left(\int_0^{\min\left\{\frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}, T\right\}} e^{-qt} dt \right) f(v) dv \\
&= \int_0^T \left(\int_{\frac{p}{q\delta^t}}^{\infty} (p-c) f(v) dv \right) e^{-qt} dt \\
&= \int_0^T (p-c) \overline{F}(pq^{-1}\delta^{-t}) e^{-qt} dt,
\end{aligned}$$

where the second equality follows from Fubini's theorem, where $t \le \frac{\log\left(\frac{p}{vq}\right)}{\log(\delta)}$ is rearranged as $v \ge \frac{p}{q\delta^t}$. Thus we obtain our desired expression for FP.

Next, consider a contract pricing with price p and again we derive an alternative formulation of CP as an integration over time. Specifically, let $\underline{v}$ be the minimum valuation for which a user will pay the contract price. Let $X_{CP}(v,p)$ is the random variable representing the time the user spends on the platform. Then the price is equal to the minimum valuation of a purchasing user times there expected time to match,

$$p = \underline{v}\mathbb{E}[\delta^{X_{CP}(\underline{v},p)} \mathbf{1}_{X(\underline{v},p)<T}] = \int_0^T \underline{v} q \delta^t e^{-qt} dt = \frac{\underline{v}\,(1-\delta^T e^{-qT})}{1-\frac{\log(\delta)}{q}}. \tag{EC.2}$$

The expected profit earned by contract price p can then be written as,

$$\begin{aligned}
\mathcal{R}_{CP}(p) &= \left(p - c\left(\int_0^T tqe^{-qt} dt + Te^{-qT} \right) \right) \overline{F}(\underline{v}) \\
&= \left(\int_0^T \underline{v}\delta^t q e^{-qt} dt - c\left(-te^{-qt}|_0^T + \int_0^T e^{-qt} dt + Te^{-qT} \right) \right) \overline{F}(\underline{v}) \\
&= \int_0^T (\underline{v}\delta^t q - c) \overline{F}(\underline{v}) e^{-qt} dt \\
&= \int_0^T \left(p\delta^t\,(q-\log(\delta))\,/(1-\delta^T e^{-qT}) - c\right) \overline{F}\left(p\left(1 - \frac{\log(\delta)}{q}\right) / \left(1-\delta^T e^{-qT}\right) \right) e^{-qt} dt,
\end{aligned}$$

where the second equality comes from integration by parts, and the final inequality follows from Eq. (EC.2).

Finally, for the period L pricing we simplify the formulation

$$\begin{aligned}
\mathcal{R}(p,L) &= \left(p - \frac{c(1-e^{-qL})}{q} \right) \sum_{i=1}^{\lfloor T/L \rfloor} \left(\frac{1-e^{-iqL}}{1-e^{-qL}} \right) \left(\overline{F}(d(i)p) - \overline{F}(d(i+1)p) \right) \\
&\quad + \left(p - \frac{c(1-e^{-qL})}{q} \right) \left(\frac{1-e^{-\lfloor T/L \rfloor qL}}{1-e^{-qL}} \right) \overline{F}\left(d\left(\lfloor T/L \rfloor + 1\right) p \right) \\
&= \sum_{i=1}^{\lfloor T/L \rfloor} \left(p - cq^{-1}(1-e^{-qL})^{-1} \right) \left(\frac{e^{-iqL} - e^{-(i+1)qL}}{1-e^{-qL}} \right) \overline{F}(d(i)), \\
&= \sum_{i=1}^{\lfloor T/L \rfloor} \left(p - cq^{-1}(1-e^{-qL})^{-1} \right) e^{-iqL} \overline{F}(d(i)),
\end{aligned}$$

the second equality comes from $\left(-(1-e^{-iqL}) + (1-e^{-(i+1)qL})\right) \overline{F}(d(i+1)p) = (e^{-iqL} - e^{-(i+1)qL}) \overline{F}(d(i+1)p)$, and the third equality follows by simplifying the equation. $\square$

Proof of Lemma 3 First, we consider the optimal freemium price. Using Lemma 2, we can upper bound the profit obtained a price p by,

$$\mathcal{R}_{FP}(p) = \int_0^T (p-c)\,\overline{F}\left(pq^{-1}\delta^{-t}\right)e^{-qt}dt \le \int_0^T (p-c)\,\overline{F}\left(p\right)e^{-qt}dt,$$

since $\overline{F}$ is decreasing. When p tends to infinity, the upper bound on the freemium profit goes to 0, which implies the optimal freemium price is finite. Now, we show optimal freemium price is unique. To do so, first consider equations of the form:

$$p - c = pg(p) \tag{EC.3}$$

where $g(p)$ is decreasing and $c \ge 0$. Eq. (EC.3) has at most one positive solution in p, denote it p^*. Since $g(p) = \frac{p-c}{p} \le 1$ for all $c, p \ge 0$, and $g(p) < g(p^*) \le 1$ for all $p > p^*$, when $g(p)$ is decreasing. Further, the first-order derivative of $pg(p)$ is

$$\frac{\partial pg(p)}{\partial p} = g(p) - pg'(p) \le g(p) < 1$$

for all $p > p^*$. If there is another solution $\bar{p} > p^*$ of Eq. (EC.3), by the mean value theorem, there exists $p \in (p^*, \bar{p})$, such that

$$g(p) - pg'(p) = 1,$$

which contradicts with the fact $g(p) - pg'(p) < 1$ for all $p > p^*$. Therefore, we conclude Eq. (EC.3) has at most one solution for $p > 0$, $c \ge 0$, when function $g(p)$ is decreasing in p. Rearrange Eq. (EC.3), we get

$$p\left(1 - g(p)\right) = c,$$

where $p\left(1 - g(p)\right)$ is increasing in p since $g(p)$ is decreasing in p. Therefore, the solution of Eq. (EC.3) is increasing in c.

Next, we show the first-order condition for the optimal freemium price is exactly of the form of Eq. (EC.3) implying uniqueness. Consider

$$\frac{\partial \mathcal{R}_{FP}(p)}{\partial p} = \int_0^T \overline{F}(pq^{-1}\delta^{-t})e^{-qt}dt - (p-c)\int_0^T q^{-1}\delta^{-t} f(pq^{-1}\delta^{-t})e^{-qt}dt = 0. \tag{EC.4}$$

There are two ways the above equation can be zero. The first is if p is such that $V \le p$ almost surely. In this case, both integrals are zero, and while p is a critical point, such a p cannot be the profit optimal freemium price as it earns no profit. Assume $\overline{F}(pq^{-1}) > 0$, then both integrals are positive and we can rearrange the expression Eq. (EC.4) to be

$$\begin{aligned}
p - c &= \frac{\int_0^T \overline{F}(pq^{-1}\delta^{-t})e^{-qt}dt}{\int_0^T q^{-1}\delta^{-t} f(pq^{-1}\delta^{-t})e^{-qt}dt} \\
&= \frac{\int_0^T \overline{F}(pq^{-1}\delta^{-t})e^{-qt}dt}{\int_0^T \frac{e^{-qt}}{p\log(\delta)}d\overline{F}(pq^{-1}\delta^{-t})} \\
&= \frac{p\log(\delta)\int_0^T \overline{F}(pq^{-1}\delta^{-t})e^{-qt}dt}{0 - \overline{F}(pq^{-1}) + \int_0^T q\overline{F}(pq^{-1}\delta^{-t})e^{-qt}dt} \\
&= \frac{p\log(\delta)}{q - \overline{F}(pq^{-1})\left(\int_0^T \overline{F}\left(pq^{-1}\delta^{-t}\right)e^{-qt}dt\right)^{-1}},
\end{aligned}$$

where the second follows from the identity $d\overline{F}(pq^{-1}\delta^{-t}) = f(pq^{-1}\delta^{-t})pq^{-1}\log(\delta)\delta^{-t}dt$, the third equation follows from integration by parts, and the fourth from simplifying. Now to apply Eq. (EC.3), we require the $\frac{\log(\delta)}{q-\overline{F}(pq^{-1})\left(\int_0^T \overline{F}(pq^{-1}\delta^{-t})e^{-qt}dt\right)^{-1}}$ to be decreasing in p. To show this, let $h(p) = \overline{F}(pq^{-1})\left(\int_0^T \overline{F}\left(pq^{-1}\delta^{-t}\right)e^{-qt}dt\right)^{-1}$, consider the derivative of the $h^{-1}(p)$,

$$\frac{\partial \int_0^T \overline{F}(pq^{-1}\delta^{-t})e^{-qt}\left(\overline{F}(pq^{-1})\right)^{-1}dt}{\partial p} = \int_0^T \frac{-\delta^{-t}f(pq^{-1}\delta^{-t})\overline{F}(pq^{-1}) + \overline{F}(pq^{-1}\delta^{-t})f(pq^{-1})}{q\left(\overline{F}(pq^{-1})\right)^2}e^{-qt}dt.$$

To show this derivative is negative, consider

$$\begin{aligned}-\delta^{-t}f(pq^{-1}\delta^{-t})\overline{F}(pq^{-1}) + \overline{F}(pq^{-1}\delta^{-t})f(pq^{-1}) &\le -f(pq^{-1}\delta^{-t})\overline{F}(pq^{-1}) + \overline{F}(pq^{-1}\delta^{-t})f(pq^{-1})\\ &= f\left(pq^{-1}\right)f\left(pq^{-1}\delta^{-t}\right)\left(-\frac{\overline{F}(pq^{-1})}{f(pq^{-1})} + \frac{\overline{F}(pq^{-1}\delta^{-t})}{f(pq^{-1}\delta^{-t})}\right)\\ &\le 0\end{aligned}$$

where the first inequality follows from $\delta^{-t} \ge 1$ and the second inequality from the fact that F is MHR. Then, $h^{-1}(p)$ is decreasing in p. Consequently $\frac{\log(\delta)}{q-\overline{F}(pq^{-1})\left(\int_0^T \overline{F}(pq^{-1}\delta^{-t})e^{-qt}dt\right)^{-1}}$ is also decreasing in p. Therefore, the first-order condition of the optimal freemium price is of the form in Eq. (EC.3). Combining with the existence of finite optimal freemium price, we conclude that the optimal freemium price is unique and increasing in c, and the profit of freemium pricing is unimodal.

The uniqueness of the optimal contract price comes from MHR directly. By Eq. (6), the first-order condition for the optimal contract price is

$$\overline{F}\left(\frac{p\left(q-\log(\delta)\right)}{q\left(1-\delta^T e^{-qT}\right)}\right) - \left(p - \frac{c(1-e^{-qT})}{q}\right)\left(\frac{q-\log(\delta)}{q\left(1-\delta^T e^{-qT}\right)}\right) f\left(\frac{p\left(q-\log(\delta)\right)}{q\left(1-\delta^T e^{-qT}\right)}\right) = 0. \tag{EC.5}$$

As above, assuming p is such that $\overline{F}\left(\frac{p(q-\log(\delta))}{q\left(1-\delta^T e^{-qT}\right)}\right) > 0$, we can rearrange Eq. (EC.5) to be

$$p - \frac{c(1-e^{-qT})}{q} = \left(\frac{q\left(1-\delta^T e^{-qT}\right)}{q-\log(\delta)}\right)\frac{\overline{F}\left(\frac{p(q-\log(\delta))}{q\left(1-\delta^T e^{-qT}\right)}\right)}{f\left(\frac{p(q-\log(\delta))}{q\left(1-\delta^T e^{-qT}\right)}\right)}.$$

Note $\overline{F}\left(\frac{p(q-\log(\delta))}{q\left(1-\delta^T e^{-qT}\right)}\right) f\left(\frac{p(q-\log(\delta))}{q\left(1-\delta^T e^{-qT}\right)}\right)^{-1}$ is non-increasing in p since F is MHR, while $p - \frac{c(1-e^{-qT})}{q}$ increasing in p. Therefore, we can conclude the optimal contract price is unique and increasing in c. $\square$

B.2. Omitted proofs from Section 3

Proof of Lemma 4 Let $p_L^* = \arg\max_p \mathcal{R}(p, L)$ be the optimal price for subscription policy with period length L, and consider the feasible freemium price $p = \frac{qp_L^*}{1-e^{-qL}}$. For any user with valuation v, let $\tau(v)$ be the largest number of periods the user will pay under period L pricing with price p_L^*. Note that $\frac{1-e^{-x}}{x}$ is decreasing in x, therefore,

$$\frac{1-e^{-qL}}{qL} \ge \frac{1-e^{-(q-\log(\delta))L}}{(q-\log(\delta))L}$$

for all $\delta \in (0,1)$. Thus after i periods, for any i less than $\tau(v)$, the user is willing to pay the period L price, and further would also pay the freemium price $p = \frac{qp_L^*}{1-e^{-qL}}$ at time iL since the purchasing condition for one implies the other i.e.,

$$v\delta^{iL}\left(\frac{q\left(1-\delta^L e^{-qL}\right)}{q-\log\left(\delta\right)}\right) \geq p_L^* \implies v\delta^{iL} q \geq \frac{qp_L^*}{1-e^{-qL}}.$$

Further, in the i^{th} period, FP with price $p = \frac{qp_L^*}{1-e^{-qL}}$ will yield the same expected profit as subscription pricing of period length L. To show this, note for a user a time t in $[(i-1)L, iL)$ (this corresponds to the i^{th} for period L pricing $i < \tau(v)$), a user will leave only if they matched with some candidate, and the expected profit the user will bring to the platform under FP over the time length $[(i-1)L, iL)$ is

$$\int_0^L \left(\frac{qp_L^*}{1-e^{-qL}}\right) e^{-qt} dt = p_L^*.$$

Thus freemium pricing with price $p = \frac{qp_L^*}{1-e^{-qL}}$ will yield the same expected profit for all the other period under period L pricing, with the exception of the last period the user would pay for. At the moment before the last period starts, the willingness to pay of a user with initial valuation v is $v\delta^{(\tau(v)-1)L}$. To bound the profit earned in this final period, consider a contract pricing with the same price p_L^* but with horizon $T = L$. The profit of this contract pricing with L is at least the profit for the last period of period L pricing, since a user is willing to pay the last period, they are also willing to pay the contract price p_L^*. Therefore, $\mathcal{R}(L) \leq \mathcal{R}_{FP} + \mathcal{R}_{CP}^L$. □

Proof of Lemma 5 Our proof will follow in three steps. First, we reduce the problem to the case where valuations are fixed and deterministic. Next, we bound the ratio for fixed valuations when (C1) holds by analyzing a feasible price, which is the optimal freemium price when $c = 0$. Finally, we bound the ratio when (C1) does not hold by analyzing a second feasible price, which is the average price of the contract pricing.

Step 1: Reduction to fixed valuations.

Define $\gamma(v) := \min\left\{1 - \left(\frac{p}{qv}\right)^{\frac{-q}{\log(\delta)}}, 1 - e^{-qT}\right\}$, and note $\gamma(v)$ is a non-decreasing function of v. Now, by Eq. (5) the profit of FP for some fixed price p is,

$$\begin{aligned}
\mathcal{R}_{FP}(p) &= \left(\frac{p-c}{q}\right)\int_{p/q}^{\infty} \min\left\{1 - \left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1 - e^{-qT}\right\} f(v)dv \\
&= \left(\frac{p-c}{q}\right)\int_{p/q}^{\infty} \gamma(v) f(v) dv \\
&= \left(\frac{p-c}{q}\right)\mathbb{E}[\gamma(V)\mathbf{1}_{V\geq p/q}].
\end{aligned}$$

By the generalized Markov's inequality, for any $x \geq 0$, we have,

$$\mathbb{E}[\gamma(V)\mathbf{1}_{V\geq p/q}] \geq \overline{F}(x)\gamma(x),$$

since $x \geq p/q$, $\gamma(v) \geq 0$, we can apply Markov's inequality, otherwise, $\gamma(v) \leq 0$. Applying the inequality to our expression for $\mathcal{R}_{FP}(p)$,

$$\mathcal{R}_{FP}(p) \geq \left(\frac{p-c}{q}\right)\min\left\{1 - \left(\frac{p}{xq}\right)^{\frac{-q}{\log(\delta)}}, 1 - e^{-qT}\right\}\overline{F}(x), \quad \text{for any } x \geq 0. \tag{EC.6}$$

Now, suppose the optimal price for $\mathcal{R}_{CP}(p)$ is p^*, and let $v^* = \frac{p^*(q-\log(\delta))}{q\left(1-\delta^T e^{-qT}\right)}$. Taking x as v^* in Eq. (EC.6) yields,

$$\mathcal{R}_{FP}(p) \geq \left(\frac{p-c}{q}\right) \min\left\{1-\left(\frac{p}{v^* q}\right)^{\frac{-q}{\log(\delta)}}, 1-e^{-qT}\right\} \overline{F}(v^*). \tag{EC.7}$$

Therefore,

$$\begin{aligned}
\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}} &\geq \frac{\mathcal{R}_{FP}(p)}{\mathcal{R}_{CP}(p^*)} \\
&\geq \frac{\max_p \left(\frac{p-c}{q}\right) \min\left\{1-\left(\frac{p}{v^* q}\right)^{\frac{-q}{\log(\delta)}}, 1-e^{-qT}\right\} \overline{F}(v^*)}{\left(\frac{v^*}{1-\frac{\log(\delta)}{q}}\left(1-\delta^T e^{-qT}\right) - \frac{c}{q}\left(1-e^{-qT}\right)\right) \overline{F}(v^*)} \\
&= \frac{\max_p \left(\frac{p-c}{q}\right) \min\left\{1-\left(\frac{p}{v^* q}\right)^{\frac{-q}{\log(\delta)}}, 1-e^{-qT}\right\}}{\left(\frac{v^*}{1-\frac{\log(\delta)}{q}}\left(1-\delta^T e^{-qT}\right) - \frac{c}{q}\left(1-e^{-qT}\right)\right)}
\end{aligned} \tag{EC.8}$$

where the first inequality follows from using the optimal price for CP and the second inequality follows by applying Eq. (EC.7) and plugging in the profit for CP from Eq. (6). Define F_v to be the point mass distribution for a random variable that is equal to some constant v with probability one. Note, then that the ratio in Eq. (EC.8) is exactly the same as the ratio for F_{v^*}, i.e.:

$$\frac{\max_p \left(\frac{p-c}{q}\right) \min\left\{1-\left(\frac{p}{v^* q}\right)^{\frac{-q}{\log(\delta)}}, 1-e^{-qT}\right\}}{\left(\frac{v^*}{1-\frac{\log(\delta)}{q}}\left(1-\delta^T e^{-qT}\right) - \frac{c}{q}\left(1-e^{-qT}\right)\right)} \geq \inf_v \frac{\mathcal{R}_{FP}^v}{\mathcal{R}_{CP}^v}. \tag{EC.9}$$

where we use $\mathcal{R}_{FP}^v$ and $\mathcal{R}_{CP}^v$ to donote the profit of FP and CP with point mass valuations v, respectively. For the remainder of our proof, we will lower bound Eq. (EC.9) by finding the worst case ratio over all point mass valuations v.

Step 2: Bounds when (C1) holds.

Fix some point mass valuation v. First assume, that $\delta^T > \left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}$, and consider the feasible freemium price be $\tilde{p} = vq\delta^T$. The corresponding freemium profit is

$$\begin{aligned}
\mathcal{R}_{FP}^v(\tilde{p}) &= \left(v\delta^T - \frac{c}{q}\right)\left(1-e^{-qT}\right) \\
&= \mathcal{R}_{CP}^v \left(\frac{\delta^T\left(1-e^{-qT}\right)}{\frac{q}{q-\log(\delta)}\left(1-\delta^T e^{-qT}\right)}\right) + \frac{c\left(1-e^{-qT}\right)}{q}\left[\frac{\delta^T\left(1-e^{-qT}\right)}{\frac{q}{q-\log(\delta)}\left(1-\delta^T e^{-qT}\right)} - 1\right].
\end{aligned}$$

Rearranging we have,

$$\begin{aligned}
\frac{\mathcal{R}_{FP}^v(\tilde{p})}{\mathcal{R}_{CP}^v(p^*)} &= \frac{\delta^T\left(1-e^{-qT}\right)}{\frac{q}{q-\log(\delta)}\left(1-\delta^T e^{-qT}\right)} + \frac{c\left(1-e^{-qT}\right)}{q}\left[\frac{\delta^T\left(1-e^{-qT}\right)}{\frac{q}{q-\log(\delta)}\left(1-\delta^T e^{-qT}\right)} - 1\right] \\
&\geq \left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}} \left(\frac{\left(1-e^{-qT}\right)}{\frac{q}{q-\log(\delta)}\left(1-\delta^T e^{-qT}\right)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c\left(1-e^{-qT}\right)}{q}\left[\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}\left(\frac{\left(1-e^{-qT}\right)}{\frac{q}{q-\log(\delta)}\left(1-\delta^{T}e^{-qT}\right)}\right)-1\right]\\
&\geq 1+\frac{c\left(1-e^{-qT}\right)}{q}\left[1-1\right]=1.
\end{aligned}
$$

where the first inequality follows from our assumption, and the second inequality follows from rearranging (C1).

Next consider the alternative assumption, $\delta^T \leq \left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}$. For this case, consider the feasible freemium price $\tilde{p}=vq\left(\frac{\log(\delta)-q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}$. By Eq. (5), the profit generated by $\tilde{p}$ is,

$$
\begin{aligned}
\mathcal{R}^v_{FP}(\tilde{p}) &= \frac{vq}{q-\log(\delta)}\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}-\frac{c}{q-\log(\delta)}\\
&=\frac{\mathcal{R}^v_{CP}}{1-\delta^T e^{-qT}}\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}+\frac{c}{q}\left[\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}\left(\frac{1-e^{-qT}}{1-\delta^T e^{-qT}}\right)-\frac{q}{q-\log(\delta)}\right].
\end{aligned}
$$

By (C1), we always have

$$
\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}\left(\frac{1-e^{-qT}}{1-\delta^T e^{-qT}}\right)-\frac{q}{q-\log(\delta)}\geq 0.
$$

Therefore,

$$
\mathcal{R}^v_{FP}\geq \mathcal{R}^v_{CP}\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}\left(1-\delta^T e^{-qT}\right)^{-1},
$$

as desired. Finally, letting $x=\frac{q}{-\log(\delta)}$, and minimizing the above expression for $0<x<1$, we have $\min_{x\in(0,1)}(1+x)^{\frac{-1}{x}}\geq \frac{1}{e}$ with the minimum occurring as x tends to 0. Thus in this case $\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}}\geq\frac{1}{e}$.

Step 3: Bounds when (C1) does not hold.

Again fix some point mass valuation v, and now consider the feasible freemium price, $\tilde{p}=vq^2(1-\delta^T e^{-qT})\left((q-\log(\delta))\left(1-e^{-qT}\right)\right)^{-1}$. By Eq. (5), the profit generated by $\tilde{p}$ is,

$$
\begin{aligned}
\mathcal{R}^v_{FP}(\tilde{p}) &= \left(\frac{vq\left(1-\delta^T e^{-qT}\right)}{\left(q-\log(\delta)\right)\left(1-e^{-qT}\right)}-\frac{c}{q}\right)\left[1-\left(\left(\frac{1}{1-\frac{\log(\delta)}{q}}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)\right)^{-\frac{q}{\log(\delta)}}\right]\\
&=\frac{\mathcal{R}^v_{CP}}{1-e^{-qT}}\left[1-\left(\left(\frac{1}{1-\frac{\log(\delta)}{q}}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)\right)^{-\frac{q}{\log(\delta)}}\right]
\end{aligned}
$$

Rearranging we have,

$$
\frac{\mathcal{R}^v_{FP}(\tilde{p})}{\mathcal{R}^v_{CP}}\geq \underbrace{\frac{1}{1-e^{-qT}}\left[1-\left(\left(\frac{1}{1-\frac{\log(\delta)}{q}}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)\right)^{-\frac{q}{\log(\delta)}}\right]}_{\phi(T)}.
$$

The derivative of $\phi(T)$ with respect to T is,

$$
\frac{\partial\phi(T)}{\partial T}=\left(\frac{qe^{-qT}}{\left(1-e^{-qT}\right)^2}\right)\left(\left(1-\frac{q}{\log(\delta)}\right)\left(\frac{1-\delta^T}{1-\delta^T e^{-qT}}\right)\left(\frac{1-\delta^T e^{-qT}}{\left(1-\frac{\log(\delta)}{q}\right)\left(1-e^{-qT}\right)}\right)^{-\frac{q}{\log(\delta)}}-1\right).
$$

When $\frac{q}{-\log(\delta)} \geq 1$, $\frac{\partial \phi(T)}{\partial T} \geq 0$, and thus,

$$\frac{\mathcal{R}^v_{FP}(\tilde{p})}{\mathcal{R}^v_{CP}} \geq \lim_{T\to 0} \phi(T) = \frac{1}{2}.$$

When $\frac{q}{-\log(\delta)} < 1$, $\frac{\partial \phi(T)}{\partial T} < 0$. Define T^* such that,

$$\left(1 - \frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}} = \left(\frac{q}{q - \log(\delta)}\right)\left(\frac{1-\delta^{T^*} e^{-qT^*}}{1-e^{-qT^*}}\right),$$

in this case, (C1) does not hold only for $T < T^*$. Thus to complete the proof we can consider

$$\frac{\mathcal{R}^v_{FP}(\tilde{p})}{\mathcal{R}^v_{CP}} \geq \lim_{T\to T^*} \phi(T) = \left(1 - \frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}} \left(1-\delta^{T^*} e^{-qT^*}\right)^{-1}.$$

Combining across all cases and taking the minimum yields the claimed bound.

Finally, for tightness, consider the case when $c = 0$, $T = \infty$, and valuations are a point mass v. In this case the optimal price for FP can be computed as $p^* = vq\left(\frac{\log(\delta)-q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}$ yielding optimal profit

$$\mathcal{R}^v_{FP}(0) = \frac{vq}{q - \log(\delta)}\left(1 - \frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}.$$

Then the ratio between FP and CP is

$$\frac{\mathcal{R}^v_{FP}(0)}{\mathcal{R}^v_{CP}(0)} = \left(1 - \frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}$$

matching the guarantee. Taking $\frac{q}{-\log(\delta)} \to 0$ gives the $1/e$ constant factor. □

Proof of Theorem 1 The proof will follow by applying Lemmas 4 and 5. First, in Lemma 4 we show that the optimal profit of period L pricing can be upper bounded by the profit of FP and CP where the time to exhaustion T equals L. Further, note the profit of FP, $\mathcal{R}_{FP}$ is monotonically increasing as the parameter T increases by looking at the integration formulation of FP in Eq. (7), i.e., $\mathcal{R}^{T_1}_{FP} \leq \mathcal{R}^{T_2}_{FP}$ for any $T_1 \leq T_2$, where we use $\mathcal{R}^{T}_{FP}$ to denote profit of FP with T. Therefore,

$$\begin{aligned}\mathcal{R}(L) &\leq \mathcal{R}_{FP} + \mathcal{R}^L_{CP}\\ &\leq \mathcal{R}_{FP} + e\mathcal{R}^L_{FP}\\ &\leq (1+e)\mathcal{R}_{FP},\end{aligned}$$

as desired. Finally, to show that no other fixed choice of pricing strategy besides freemium pricing can guarantee any constant factor of the optimal profit, we note that this will follow from the proof Theorem 2(b), which constructs instances where $\frac{\mathcal{R}_{FP}}{\mathcal{R}(L)}$ will go to infinity for any fixed L, as c goes to infinity.

Proof of Theorem 2 We will prove the two parts of the theorem separately.

Part a) First note that when c = 0, the optimal contract profit is

$$\mathcal{R}_{CP} = \max_p p\overline{F}\left(\frac{q-\log(\delta)}{q\left(1-\delta^T e^{-qT}\right)}p\right) = \frac{q\left(1-\delta^T e^{-qT}\right)}{q-\log(\delta)} p^* \overline{F}(p^*),$$

where p^* is the solution of $\max_p p\overline{F}(p)$. Now, using Lemma 2, we can upper bound $\mathcal{R}(L)$ as,

$$\begin{aligned}
\mathcal{R}(L) &= \max_p \sum_{i=1}^{\lfloor T/L \rfloor} p e^{-(i-1)qL} \overline{F}\left(d(i)p\right) \\
&\le \sum_{i=1}^{\lfloor T/L \rfloor} \max_p p e^{-(i-1)qL} \overline{F}\left(d(i)p\right) \\
&= \sum_{i=1}^{\lfloor T/L \rfloor} \frac{p^*}{d(i)} \overline{F}\left(p^*\right) e^{-(i-1)qL} \\
&= p^* \overline{F}\left(p^*\right) \sum_{i=1}^{\lfloor T/L \rfloor} \frac{e^{-(i-1)qL}}{d(i)} \\
&= p^* \overline{F}\left(p^*\right) \sum_{i=1}^{\lfloor T/L \rfloor} e^{-(i-1)qL} \left(\frac{\delta^{(i-1)L}\left(1-\delta^L e^{-qL}\right)}{q-\log(\delta)}\right) \\
&= p^* \overline{F}\left(p^*\right) \left(\frac{1-\delta^L e^{-qL}}{q-\log(\delta)}\right) \sum_{i=1}^{\lfloor T/L \rfloor} e^{-(i-1)qL} \delta^{(i-1)L} \\
&= p^* \overline{F}\left(p^*\right) \left(\frac{1-\delta^L e^{-qL}}{q-\log(\delta)}\right) \left(\frac{1-\delta^{L\lfloor T/L \rfloor} e^{-qL\lfloor T/L \rfloor}}{1-\delta^L e^{-qL}}\right) \\
&= \mathcal{R}_{CP},
\end{aligned}$$

the first inequality follows by exchanging the max and sum, the second equality follows by taking the maximum, the third and fourth equality comes from the definition of $d(i)$ in Eq. (2), the final equality follows if we assume $\lfloor T/L \rfloor$ is an integer.

□

Part b) We show that $\mathcal{R}(L) \le \mathcal{R}_{FP}$ when c is sufficiently large. Without loss of generality, assume $f(x) > 0$ for all $x > 0$, and let $\underline{f}(x) = \min\{f(y) : y \le x\}$. By definition, $\underline{f}(x)$ is non-increasing in x. For any MHR distribution F, $\frac{\overline{F}(x)}{f(x)}$ is non-increasing, and notice that for a small change Δ, $\underline{f}(x+\Delta) = f(x+\Delta)$ or $\underline{f}(x+\Delta) = \underline{f}(x)$, therefore, $\frac{\overline{F}(x)}{\underline{f}(x)}$ is also non-increasing. For subscription pricing with period length L, by Eq. (9), the optimal profit of LP is

$$\mathcal{R}(L) = \max_p \sum_{i=1}^{\lfloor T/L \rfloor} \left(p - \frac{c}{q}\left(1-e^{-qL}\right)\right) e^{-iqL} \overline{F}(d(i)p).$$

Let the optimal price of LP be $p^*(c)$, and note that when c goes to infinity, $p^*(c)$ also goes to infinity. Further, let $\tilde{p}(c) = \frac{qp^*(c)}{1-e^{-qL}}$, $v^* = d(1)p^*(c) = \left(\frac{q-\log(\delta)}{q\left(1-\delta^L e^{-qL}\right)}\right) p^*(c)$, and $\tilde{v} = \frac{1}{2}\left(\frac{p^*(c)}{1-e^{-qL}} + v^*\right)$. Now, the optimal profit of FP can be lower bounded by,

$$\begin{aligned}
\mathcal{R}_{FP} &= \max_p \left(\frac{p-c}{q}\right) \int_{\frac{p}{q}}^{\infty} \min\left\{1 - \left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1-e^{-qT}\right\} f(v)dv \\
&\ge \left(\frac{\tilde{p}(c)-c}{q}\right) \int_{\frac{\tilde{p}(c)}{q}}^{\infty} \min\left\{1 - \left(\frac{\tilde{p}(c)}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1-e^{-qT}\right\} f(v)dv
\end{aligned}$$

$$
\begin{aligned}
&= \left(1-e^{-qL}\right)\left(\frac{\tilde{p}(c)-c}{q}\right)\int_{\frac{\tilde{p}(c)}{q}}^{\infty} \min\left\{\left(1-\left(\frac{\tilde{p}(c)}{vq}\right)^{\frac{-q}{\log(\delta)}}\right)\left(1-e^{-qL}\right)^{-1}, \frac{1-e^{-qT}}{1-e^{-qL}}\right\} f(v)dv \\
&\geq \left(1-e^{-qL}\right)\left(\frac{\tilde{p}(c)-c}{q}\right)\int_{\frac{\tilde{p}(c)}{q}}^{\infty}\left(1-\left(\frac{\tilde{p}(c)}{vq}\right)^{\frac{-q}{\log(\delta)}}\right) f(v)dv \\
&= \left(p^*(c)-\frac{c}{q}\left(1-e^{-qL}\right)\right)\int_{\frac{p^*(c)}{1-e^{-qL}}}^{\infty}\left(1-\left(\frac{p^*(c)}{v\left(1-e^{-qL}\right)}\right)^{\frac{-q}{\log(\delta)}}\right) f(v)dv \\
&\geq \left(p^*(c)-\frac{c}{q}\left(1-e^{-qL}\right)\right)\int_{\tilde{v}}^{v^*}\left(1-\left(\frac{p^*(c)}{v\left(1-e^{-qL}\right)}\right)^{\frac{-q}{\log(\delta)}}\right) \underline{f}(v)dv \\
&\geq \left(p^*(c)-\frac{c}{q}\left(1-e^{-qL}\right)\right)\left(\left(1-\left(\frac{p^*(c)}{\tilde{v}\left(1-e^{-qL}\right)}\right)^{\frac{-q}{\log(\delta)}}\right)\underline{f}\left(v^*\right)\left(v^*-\tilde{v}\right)\right),
\end{aligned}
$$

where the first inequality follows from the fact that $\tilde{p}(c)=\frac{p^*(c)}{\left(1-e^{-qL}\right)}$ is only a feasible freemium price, the second equality follows from taking $(1-e^{-qL})$ out of the minimum, the second inequality follows from $\left(1-\left(\frac{\tilde{p}(c)}{vq}\right)^{\frac{-q}{\log(\delta)}}\right) \leq \left(1-\left(\frac{\tilde{p}(c)}{vq}\right)^{\frac{-q}{\log(\delta)}}\right)\left(1-e^{-qL}\right)^{-1}$, and, $\left(1-\left(\frac{\tilde{p}(c)}{vq}\right)^{\frac{-q}{\log(\delta)}}\right) \leq 1 \leq \frac{1-e^{-qT}}{1-e^{-qL}}$, the third inequality follows from $\underline{f}(v) \leq f(v)$ and $\left(1-\left(\frac{p^*(c)}{v\left(1-e^{-qL}\right)}\right)^{\frac{-q}{\log(\delta)}}\right) f(v) \geq 0$, the fourth inequality follows since $\underline{f}(x)$ is non-increasing and $1-\left(\frac{p^*(c)}{v\left(1-e^{-qL}\right)}\right)^{\frac{-q}{\log(\delta)}}$ is increasing in v. Therefore,

$$
\begin{aligned}
\frac{\mathcal{R}_{FP}}{\mathcal{R}(L)} &\geq \left(\left(1-\left(\frac{p^*(c)}{\tilde{v}\left(1-e^{-qL}\right)}\right)^{\frac{-q}{\log(\delta)}}\right)\left(v^*-\tilde{v}\right)\right)\frac{\underline{f}\left(v^*\right)}{\sum_{i=1}^{\lfloor T/L\rfloor} e^{-iqL}\overline{F}(d(i)p)} \\
&\geq \left(\left(1-\left(\frac{p^*(c)}{\tilde{v}\left(1-e^{-qL}\right)}\right)^{\frac{-q}{\log(\delta)}}\right)\left(v^*-\tilde{v}\right)\right)\frac{\underline{f}\left(v^*\right)}{\lfloor T/L\rfloor\overline{F}(v^*)} \\
&\geq \left(\left(1-\left(\frac{p^*(c)}{\tilde{v}\left(1-e^{-qL}\right)}\right)^{\frac{-q}{\log(\delta)}}\right)\left(v^*-\tilde{v}\right)\right)\frac{\underline{f}\left(0\right)}{\lfloor T/L\rfloor\overline{F}(0)},
\end{aligned}
$$

the second inequality follows from $e^{-iqL} \leq 1$ and $d(i)$ is increasing in i, the last inequality follows from $\frac{\overline{F}(x)}{\underline{f}(x)}$ is non-increasing. Note $\frac{p^*(c)}{\tilde{v}}$ is a constant with respect to q, δ, L, and $v^*-\tilde{v}$ tends infinity as $p^*(c)$ tends to infinity. Therefore, the ratio will be larger than 1 when $p^*(c)$ is sufficiently large, implying $\mathcal{R}(L) \leq \mathcal{R}_{FP}$ for all fixed L when c sufficiently large, as desired.

□

B.3. Omitted proofs of Section 4

Proof of Lemma 6 We'll prove each part separately.

Part a) First, let $\underline{v}$ be the lowest valuation for which a user will pay the contract price, we can rewrite the profit of contract pricing as

$$
\mathcal{R}_{CP} = \left(\frac{\underline{v}q\left(1-\delta^T e^{-qT}\right)}{q-\log(\delta)} - \frac{c}{q}\left(1-e^{-qT}\right)\right)\overline{F}(\underline{v}),
$$

The first-order condition for the optimal contract price is then,

$$
\frac{\partial \mathcal{R}_{CP}}{\partial \underline{v}} = \frac{q\left(1-\delta^T e^{-qT}\right)}{q-\log(\delta)}\overline{F}(\underline{v}) - \left(\frac{\underline{v}q}{q-\log(\delta)} - \frac{c}{q}\left(1-e^{-qT}\right)\right) f(\underline{v}) = 0. \tag{EC.10}
$$

By Lemma 3 the optimal contract price is unique in p and thus it is also unique in $\underline{v}$. Let $\underline{v}^*$ be the solution of Eq. (EC.10), and $\tilde{p}=\left(\frac{v^* q^2}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)$ be a feasible freemium price. In the proof for the uniqueness of optimal freemium price (Lemma 3), we proved that the freemium profit is unimodal over the freemium price p. If the first order derivative of the FP profit is larger than 0 at $\tilde{p}$, then by following the gradient, the optimal freemium price should be higher than $\tilde{p}$, and thus we have the price dominance.

Now, by Eq. (5) the profit of optimal freemium pricing is,

$$\mathcal{R}_{FP}=\max_p\left(\frac{p-c}{q}\right)\int_{\frac{p}{q}}^{\infty}\min\left\{1-\left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}},1-e^{-qT}\right\}f(v)dv.$$

For the case $1-\left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}\leq 1-e^{-qT}$, then $\min\left\{1-\left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}},1-e^{-qT}\right\}=1-\left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}$, and the first order derivative of Eq. (5) at $\tilde{p}$ is

$$\begin{aligned}
\frac{\partial\mathcal{R}_{FP}}{\partial p}|_{\tilde{p}}&=\frac{1}{q}\int_{\tilde{p}q^{-1}}^{\infty}\left[\left(1-\left(\frac{\tilde{p}}{vq}\right)^{\frac{-q}{\log(\delta)}}\right)+\frac{q}{\log(\delta)}\left(\frac{\tilde{p}-c}{\tilde{p}}\right)\left(\frac{\tilde{p}}{vq}\right)^{\frac{-q}{\log(\delta)}}\right]f(v)dv\\
&\geq\frac{1}{q}\int_{\tilde{p}q^{-1}}^{\infty}\left[1-\left(1-\frac{q}{\log(\delta)}\right)\left(\frac{\tilde{p}}{vq}\right)^{\frac{-q}{\log(\delta)}}\right]f(v)dv\\
&\geq\frac{1}{q}\left[1-\left(1-\frac{q}{\log(\delta)}\right)\left(\frac{\tilde{p}}{v^*q}\right)^{\frac{-q}{\log(\delta)}}\right]\overline{F}(v^*)\\
&=\frac{1}{q}\left[1-\left(1-\frac{q}{\log(\delta)}\right)\left(\left(\frac{q}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)\right)^{\frac{-q}{\log(\delta)}}\right]\overline{F}(v^*)\\
&\geq\frac{1}{q}\left[1-\left(1-\frac{q}{\log(\delta)}\right)\left(\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}\right)^{\frac{-q}{\log(\delta)}}\right]\overline{F}(v^*)\\
&\geq 0,
\end{aligned}$$

the first inequality is induced by $\frac{\tilde{p}-c}{\tilde{p}}\leq 1$ when $c\geq 0$, the second inequality follows from the fact that $\left[1-\left(1-\frac{q}{\log(\delta)}\right)\left(\frac{\tilde{p}}{vq}\right)^{\frac{-q}{\log(\delta)}}\right]$ is increasing in v and the generalized Markov's inequality, the second equality follows from plugging in the freemium price is $\tilde{p}$, the third inequality is derived from (C1), namely $\left(1-\frac{q}{\log(\delta)}\right)^{\frac{\log(\delta)}{q}}\geq\left(\frac{q}{q-\log(\delta)}\right)\left(\frac{1-\delta^t e^{-qt}}{1-e^{-qt}}\right)$, by simplifying we obtain the final inequality.

For the opposite case $1-\left(\frac{p}{vq}\right)^{\frac{-q}{\log(\delta)}}\geq 1-e^{-qT}$, and let $\tilde{v}=\frac{\tilde{p}}{q}=\left(\frac{v^* q}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)$, the first order derivative of Eq. (5) at $\tilde{p}$ is

$$\begin{aligned}
\frac{\partial\mathcal{R}_{FP}}{\partial p}|_{\tilde{p}}&=\left(\frac{1-e^{-qT}}{q}\right)\left(\overline{F}\left(\frac{\tilde{p}}{q}\right)-\left(\frac{\tilde{p}-c}{q}\right)f\left(\frac{\tilde{p}}{q}\right)\right)\\
&=\left(\frac{1-e^{-qT}}{q}\right)\left(\overline{F}\left(\tilde{v}\right)-\left(\left(\frac{v^*q}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)-\frac{c}{q}\right)f\left(\tilde{v}\right)\right)\\
&=f(\tilde{v})\left(\frac{1-e^{-qT}}{q}\right)\left(\frac{\overline{F}\left(\tilde{v}\right)}{f\left(\tilde{v}\right)}-\left(\frac{q}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)\left(\frac{\overline{F}(v^*)}{f(v^*)}\right)\right)\\
&\geq 0.
\end{aligned}$$

where the second equality follows from plugging $\tilde{p}$, the third follows from Eq. (EC.10), the inequality follows from the fact that $\frac{\overline{F}(x)}{f(x)}$ is non-increasing, $\tilde{v}\leq v^*$, and $\left(\frac{q}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)\leq 1$.

Thus in both cases, the first order derivative of the freemium profit at $\tilde{p} = \left(\frac{v^* q^2}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)$ is larger than 0. Thus the optimal freemium price should be higher than $\tilde{p} = \left(\frac{v^* q^2}{q-\log(\delta)}\right)\left(\frac{1-\delta^T e^{-qT}}{1-e^{-qT}}\right)$, which is in turn higher than the optimal contract price $\left(\frac{v^* q}{q-\log(\delta)}\right)(1-\delta^T e^{-qT})$. Hence, we show the price dominance

$$\arg\max_p \mathcal{R}_{FP}(p) \geq \left(\frac{q}{1-e^{-qT}}\right)\arg\max_p \mathcal{R}_{CP}(p,c,F).$$

when (C1) holds.

Part b) Note that by Eqs. (5) and (6), the derivative of $\mathcal{R}_{FP}$ and $\mathcal{R}_{CP}$ with respect to c is,

$$\begin{aligned}
\frac{\partial \mathcal{R}_{FP}}{\partial c} &= -\frac{1}{q}\int_{\frac{p_{FP}}{q}}^{\infty} \min\left\{1-\left(\frac{p_{FP}}{vq}\right)^{\frac{-q}{\log(\delta)}}, 1-e^{-qT}\right\} f(v)dv \\
&= -\frac{\mathcal{R}_{FP}}{p_{FP}-c}, \\
\frac{\partial \mathcal{R}_{CP}}{\partial c} &= -\left(\frac{1-e^{-qT}}{q}\right)\overline{F}\left(\frac{p_{CP}\,(q-\log(\delta))}{q\,(1-\delta^T e^{-qT})}\right) \\
&= -\frac{\mathcal{R}_{CP}}{\frac{qp_{CP}}{1-e^{-qT}}-c}.
\end{aligned}$$

where p_{FP} and p_{CP} are the optimal price for FP and CP respectively. Now, let's look at the derivative of the ratio $\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}}$ when the unit operating cost c changes,

$$\begin{aligned}
\frac{\partial \mathcal{R}_{FP}/\mathcal{R}_{CP}}{\partial c} &= \frac{\mathcal{R}_{CP}(\partial \mathcal{R}_{FP}/\partial c) - \mathcal{R}_{FP}(\partial \mathcal{R}_{CP}/\partial c)}{\mathcal{R}_{CP}^2} \\
&= \frac{-\mathcal{R}_{CP}\mathcal{R}_{FP}/\left(p_{FP}-c\right) + \mathcal{R}_{FP}\mathcal{R}_{CP}/\left(\frac{qp_{CP}}{1-e^{-qT}}-c\right)}{\mathcal{R}_{CP}^2} \\
&= \frac{\mathcal{R}_{CP}\mathcal{R}_{FP}/\left((p_{FP}-c)\left(\frac{qp_{CP}}{1-e^{-qT}}-c\right)\right)\left(p_{FP}-\frac{qp_{CP}}{1-e^{-qT}}\right)}{\mathcal{R}_{CP}^2} \\
&\geq 0,
\end{aligned}$$

where the second equality follows from the above calculation of $\partial \mathcal{R}_{FP}/\partial c$ and $\partial \mathcal{R}_{CP}/\partial c$, and the inequality follows from the price dominance in Part a). Therefore, the ratio $\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}}$ is increasing in c. By Theorem 2, when $c=0$, $\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}} \leq 1$, when c is sufficiently large, $\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}} \geq 1$. Thus there exists unique c^* such that $\frac{\mathcal{R}_{FP}}{\mathcal{R}_{CP}} = 1$. □

Proof of Theorem 3 By the definition of c^* in Lemma 6(b), for any $c \leq c^*$, $\mathcal{R}_{FP} \leq \mathcal{R}_{CP}$. Further, by Lemma 6(a) if p_S^* and p_C^* are the profit maximizing subscription and contract prices, respectively, then $\frac{p_S^*-c}{q} \geq \frac{p_C^*}{1-e^{-qT}} - \frac{c}{q}$. Therefore, by Eqs. (10) and (11)

$$\mathcal{M}_{CP} = \frac{\mathcal{R}_{CP}}{\frac{p_C^*}{\left(1-e^{-qT}\right)} - \frac{c}{q}} \geq \frac{q\mathcal{R}_{CP}}{p_S^*-c} \geq \frac{q\mathcal{R}_{FP}}{p_S^*-c} = \mathcal{M}_{FP},$$

as desired. □

B.4. Omitted Proofs from Section 5

Proof of Lemma 7. We will prove both parts separately.

Part a) First, we show the result for two match rates $\{q_1, q_2\}$ with population sizes $\{t_1, t_2\}$. Let $q = \frac{q_1 t_1 + q_2 t_2}{t_1 + t_2}$ be the average match rate and recall the profit of CP with contract price p is,

$$\mathcal{R}_{CP} = (p - c\mathbb{E}[\text{Time on platform}])\, \overline{F}\left(\left(\frac{q - \log(\delta)}{q - q\delta^T e^{-qT}}\right) p\right).$$

For any price p, the proportion of users which will pay the price depends only on the user supposed match rate q and is thus independent of the order in which candidate matches are shown. Thus, to compare the profit of the two possible orderings, $\{q_2, q_1\}$ and $\{q_1, q_2\}$, we only need to compare the expected matching cost to the platform from each user, or equivalently, the expected time on the platform. Recall $X_{CP}(q)$ was the expected time on the platform for a user who paid the contract price. When the platform chooses matching order $\{q_1, q_2\}$, the expected time on the platform for a user who paid the contract price is,

$$\begin{aligned} X_{CP}(\{q_1, q_2\}) &= \mathbb{E}[\text{Time on platform}] \\ &= \int_0^{t_1} t q_1 e^{-tq_1} dt + e^{-t_1 q_1} \int_0^{t_2} (t + t_1)\, q_2 e^{-tq_2} dt + T e^{-(t_1 q_1 + t_2 q_2)} \\ &= \int_0^{t_1} t q_1 e^{-tq_1} dt + e^{-t_1 q_1} \left(\int_0^{t_2} t q_2 e^{-tq_2} dt + t_1 \left(1 - e^{-q_2 t_2}\right)\right) + T e^{-(t_1 q_1 + t_2 q_2)}. \end{aligned}$$

Similarly, when the matching order is $\{q_2, q_1\}$, the expected time on the platform for a user who paid the contract price is

$$\begin{aligned} X_{CP}(\{q_2, q_1\}) &= \mathbb{E}[\text{Time on platform}] \\ &= \int_0^{t_2} t q_2 e^{-tq_2} dt + e^{-t_2 q_2} \int_0^{t_1} (t + t_2)\, q_1 e^{-tq_1} dt + T e^{-(t_1 q_1 + t_2 q_2)} \\ &= \int_0^{t_2} t q_2 e^{-tq_2} dt + e^{-t_2 q_2} \left(\int_0^{t_1} t q_1 e^{-tq_1} dt + t_2 \left(1 - e^{-t_2 q_2}\right)\right) + T e^{-(t_1 q_1 + t_1 q_1)}. \end{aligned}$$

Then,

$$\begin{aligned} & X_{CP}(\{q_1, q_2\}) - X_{CP}(\{q_2, q_1\}) \\ &= \left(1 - e^{-t_2 q_2}\right)\left(\int_0^{t_1} t q_1 e^{-tq_1} dt + t_1 e^{-t_1 q_1}\right) - \left(1 - e^{-t_1 q_1}\right)\left(\int_0^{t_2} t q_2 e^{-tq_2} dt + t_2 e^{-t_2 q_2}\right), \\ &= \frac{\left(1 - e^{-t_1 q_1}\right)\left(1 - e^{-t_2 q_2}\right)}{q_1} - \frac{\left(1 - e^{-t_1 q_1}\right)\left(1 - e^{-t_2 q_2}\right)}{q_2} \geq 0, \end{aligned}$$

where the final inequality follows from $q_2 \geq q_1$. Therefore, we conclude for any distribution F, $\mathcal{R}_{CP}(\{q_2, q_1\}) \geq \mathcal{R}_{CP}(\{q_1, q_2\})$.

Now, for k match rates $\{q_1, \ldots, q_k\}$, with associated populations $\{t_1, \ldots, t_k\}$ let $T_i = \sum_{j=1}^{i} t_j$, for $i = 1, \ldots k$, and $T_0 = 0$. The expected time on the platform for a user who paid the contract price is

$$X_{CP}(\{q_1, \ldots, q_k\}) = \sum_{i=1}^{k} \left(\int_0^{t_i} (t + T_{i-1})\, e^{-q_i (t + T_{i-1})} dt\right) + T e^{-qT}.$$

If we swap q_j and q_{j+1} in $\{q_1, \ldots, q_k\}$ where $1 \leq j \leq k-1$, the expected time on the platform for a user who paid the contract price becomes

$$\begin{aligned} X_{CP}(\{q_1, \ldots q_{j+1}, q_j, \ldots, q_k\}) = T e^{-qT} + \sum_{i=1}^{j-1} \left(\int_0^{t_i} (t + T_{i-1})\, e^{-q_i (t + T_{i-1})} dt\right) + \int_0^{t_{j+1}} (t + T_{j-1}) e^{-q_{j+}(t + T_{j-1})} dt \\ + \int_0^{t_j} (t + T_{j-1} + t_{j+1}) e^{-q_{j+1}\left(t + T_{j-1} + t_{j+1}\right)} dt + \sum_{i=j+2}^{k} \left(\int_0^{t_i} (t + T_{i-1})\, e^{-q_i (t + T_{i-1})} dt\right) \end{aligned}$$

Note all the other parts for the integrations do not change if we only swap two match rates next to each other. Therefore, we can generalize the proof for order $\{q_1,\dots,q_k\}$ by switching any two reverse orders that are next to each other. □

Part b) As in part a), we first show the result for two match rates $\{q_1,q_2\}$ with population sizes $\{t_1,t_2\}$. Again, let $q=\frac{q_1t_1+q_2t_2}{t_1+t_2}$ be the average match rate and recall profit of FP with freemium price p is,

$$\mathcal{R}_{FP}=\int_{\frac{p}{q}}^{\infty}(p-c)\times\mathbb{E}\left[\text{Time on platform}|\text{FP}\right]f(v)dv.$$

The profit comparison $\mathcal{R}_{FP}(c,F,\{t_1,...,t_k\}),\{q_1,...,q_K\})\le\mathcal{R}_{FP}(c,F,\sigma(\{t_1,...,t_k\})),\sigma(\{q_1,...,q_k\}))$ will follow if we can show that for each user, the expected time on the platform will be longer for $\{q_1,...,q_K\}$. Therefore, we only need to show that for price p and fixed user valuation v, the expected time the user will stay on platform satisfies

$$X_{FP}(v,p,\{q_1,q_2\})\ge X_{FP}(v,p,\{q_2,q_1\}).$$

If T is binding, the expected time on platform will be the same as contract model, therefore the conclusion is the same. Otherwise, let $\tau=\frac{\log(p/vq)}{\log(\delta)}$, when $\tau\le\min\{t_1,t_2\}$,

$$X_{FP}(v,p,\{q_1,q_2\})=\int_0^{\tau}e^{-tq_1}dt,$$
$$X_{FP}(v,p,\{q_2,q_1\})=\int_0^{\tau}e^{-tq_2}dt.$$

Note that $e^{-tq_1}\ge e^{-tq_2}$, therefore,

$$X_{FP}(v,p,\{q_1,q_2\})\ge X_{FP}(v,p,\{q_2,q_1\}).$$

When $t_1\le\tau\le t_2$

$$X_{FP}(v,p,\{q_1,q_2\})=\int_0^{t_1}e^{-tq_1}dt+e^{-qt_1}\int_0^{\tau-t_1}e^{-tq_2}dt,$$
$$X_{FP}(v,p,\{q_2,q_1\})=\int_0^{t_1}e^{-tq_2}dt+e^{-qt_2}\int_0^{\tau-t_1}e^{-tq_2}dt.$$

Similarly, we can show that

$$X_{FP}(v,p,\{q_1,q_2\})\ge X_{FP}(v,p,\{q_2,q_1\})$$

for $t_2\le\tau\le t_2$ or $\max\{t_1,t_2\}\le\tau\le t_1+t_2$. Therefore, we can conclude that for any distribution F, $\mathcal{R}_{FP}(\{q_2,q_1\})\le\mathcal{R}_{FP}(\{q_1,q_2\})$.

For matching order $\{q_1,\dots,q_k\}$, let $T_i=\sum_{j=1}^{i}t_j$, for $i=1,\dots k$, and $T_0=0$. If T is binding, the expected time on the platform for a user whose valuation is v under the freemium price p is

$$X_{FP}(v,p,\{q_1,\dots,q_k\})=\sum_{i=1}^{k}\left(\int_0^{t_i}(t+T_{i-1})\,e^{-q_i(t+T_{i-1})}dt\right)+Te^{-qT},$$

and we can apply the proof in part a). Otherwise, let $q_{k'}$ be the k'-th match rate such that $T_{k'}\le\tau\le T_{k'+1}$, the expected time on the platform for a user whose valuation is v under the freemium price p is

$$X_{FP}(v,p,\{q_1,\dots,q_k\})=\sum_{i=1}^{k'}\left(\int_0^{t_i}(t+T_{i-1})\,e^{q_i(t+T_{i-1})}dt\right)$$
$$+\int_0^{\tau-T_{k'}}(t+T_{k'})\,e^{q_i\left(t+T_{k'}\right)}dt+\tau e^{-\left(\sum_{i=1}^{k'}q_it_i+q_{k'+1}\left(\tau-T_{k'}\right)\right)}.$$

If we swap match rates q_j and q_{j+1} where $j<k'$, it will be the same as T is binding, if we swap match rates q_j and q_{j+1} where $j\geq k'+1$, it will cause no difference for user's the expected time on the platform, if we swap $q_{k'}$ and $q_{k'+1}$, the expected before k' will stay as $\sum_{i=1}^{k'-1}\left(\int_0^{t_i}(t+T_{i-1})\,e^{q_i(t+T_{i-1})}dt\right)$, we only need to consider the difference in $q_{k'}$ and $q_{k'+1}$, which is analyzed above. Therefore, we can generalize the proof for order $\{q_1,...,q_k\}$ by switching any two reverse orders that next to each other. $\square$

Proof of Theorem 4. Let $\{q_{i,j}\}_{j=1}^k$ be the match rate in period i of period L pricing, and correspondingly let $t_{i,j}$ be the population size of each $q_{i,j}$. Note that for a user, the probability of getting matched in period i is $\left(1-e^{-\sum_{i=j}^k q_{i,j}t_{i,j}}\right)$, which is independent of the match rate order in the period. Therefore, the match rate order inside period i doesn't affect the probability of future payments but may decrease the operating cost inside the period. By the same reasoning for CP in Lemma 7a), inside each period then the match rates of period L pricing should be decreasing.

Similarly, let $\bar{q}_i=\left(\sum_{j=1}^k q_{i,j}t_{i,j}\right)/\left(\sum_{j=1}^k t_{i,j}\right)$ be the average match rate of period i. Let c_i be the expected operating cost in period i, which is determined by the match rates inside period i. Then the profit of period L pricing can be rewritten as

$$\mathcal{R}(p,L)=\sum_{i=1}^{\lfloor T/L\rfloor}(p-c_i)e^{-\sum_{j=1}^{i-1}\bar{q}_i}\overline{F}(d(i)p).$$

Following the same swap argument of FP in Lemma 7b) consider period 1 and period 2, and assume $\bar{q}_1\leq\bar{q}_2$. The profit for period 1 and 2 is

$$\mathcal{R}(p,\{\bar{q}_1,\bar{q}_2\})=(p-c_1)\overline{F}(d(1)p)+(p-c_2)e^{-\bar{q}_1L}\overline{F}(d(2)p).$$

If we swap them, the profit for period 1 and 2 will be

$$\mathcal{R}(p,\{\bar{q}_2,\bar{q}_1\})=(p-c_2)\overline{F}(d(1)p)+(p-c_1)e^{-\bar{q}_2L}\overline{F}(d(2)p).$$

Since $q_1\leq\bar{q}_1\leq\bar{q}_2\leq q_k$, and $\frac{1-e^{-x}}{x}$ is decreasing, then,

$$\frac{c(1-e^{-q_kL})}{q_k}\leq c_2\leq c_1\leq\frac{c(1-e^{-q_1L})}{q_1},\ \text{and}\ e^{-q_kL}\leq e^{-\bar{q}_2L}\leq e^{-\bar{q}_2L}\leq e^{-q_1L}$$

i.e., high match rate will cause low operating cost inside each period. Therefore,

$$\begin{aligned}
\mathcal{R}(p,\{\bar{q}_1,\bar{q}_2\})-\mathcal{R}(p,\{\bar{q}_2,\bar{q}_1\})&=(c_2-c_1)\overline{F}(d(1)p)+\left((p-c_2)e^{-\bar{q}_1L}-(p-c_1)e^{-\bar{q}_2L}\right)\overline{F}(d(2)p)\\
&=(c_2-c_1)\,\overline{F}(d(1)p)+(c_1e^{-\bar{q}_2L}-c_2e^{-\bar{q}_1L})\overline{F}(d(2)p)+p(e^{-\bar{q}_1L}-e^{-\bar{q}_2L})\overline{F}(d(2)p)\\
&\geq p(e^{-\bar{q}_1L}-e^{-\bar{q}_2L})\overline{F}(d(2)p)-c\left(\frac{1-e^{-q_1L}}{q_1}-\frac{1-e^{-q_kL}}{q_k}\right)\overline{F}(d(1)p)\\
&\qquad -c\left(\frac{(1-e^{-q_2L})e^{-q_1L}}{q_2}-\frac{(1-e^{-q_1L})e^{-q_2L}}{q_1}\right)\overline{F}(d(2)p)\\
&\geq 0
\end{aligned}$$

where the first inequality follows from the fact that $c_1\leq\frac{c(1-e^{-q_1L})}{q_1}$ and $c_2\geq\frac{c(1-e^{-q_2L})}{q_2}$, the second inequality follows from the condition

$$c\leq\frac{p\,(e^{-q_jL}-e^{-q_{j+1}L})\,\overline{F}(d(i+1)p)}{\left(\frac{1-e^{-q_jL}}{q_j}-\frac{1-e^{-q_{j+1}L}}{q_{j+1}}\right)\overline{F}(d(i)p)+\left(\frac{(1-e^{-q_{j+1}L})e^{-q_jL}}{q_{j+1}}-\frac{(1-e^{-q_jL})e^{-q_{j+1}L}}{q_j}\right)\overline{F}(d(i+1)p)},$$

i.e., the unit operating cost is not too large. Thus, $\bar{q}_i$ should be increasing to retain users. Therefore, between periods, period L pricing should behave as freemium pricing and show users lower average matching rate periods first.

□

Proof of Theorem 5. As in Lemma 7(a), let $X_{CP}(\sigma(q))$ be the expected time a user who pays the contract price will spend on the platform under match rate order $\sigma(q)$, and let $X_{CP}(\{q_k,\dots,q_1\})$ be the expected time a user who pays the contract price will spend on the platform when the match rate order is $\{q_k,\dots,q_1\}$. First we will show $X_{CP}(\{q_k,\dots,q_1\}) \le X_{CP}(\sigma(q))$, i.e., compared with any other order $\sigma(q)$, users will leave the platform sooner when match rates are in descending order. To this end, the profit of contract pricing is

$$\mathcal{R}_{CP}(p,\{q_k,\dots,q_1\}) = (p - cX_{CP}(\{q_k,\dots,q_1\}))\,\overline{F}\left(\frac{q-\log(\delta)}{q\,(1-\delta^T e^{-qT})}p\right).$$

Let p^* be the optimal contract price when the match rate order is $\sigma(q)$, i.e. p^* such that,

$$\begin{aligned}\frac{\partial\mathcal{R}_{CP}(p,\sigma(q))}{\partial p}|_{p^*} &= \overline{F}\left(\frac{q-\log(\delta)}{q\,(1-\delta^T e^{-qT})}p^*\right) - \left(\frac{q-\log(\delta)}{q\,(1-\delta^T e^{-qT})}\right)(p^* - cX_{CP}(\sigma(q)))\,f\left(\frac{q-\log(\delta)}{q\,(1-\delta^T e^{-qT})}p^*\right)\\ &= 0.\end{aligned}$$

Let $v^* = \frac{q-\log(\delta)}{q\left(1-\delta^T e^{-qT}\right)}p^*$, then, the first order derivative of $\mathcal{R}_{CP}(p,\{q_k,\dots,q_1\})$ at p^* is

$$\begin{aligned}&\frac{\partial\mathcal{R}_{CP}(p,\{q_k,\dots,q_1\})}{\partial p}|_{p^*}\\ &= \overline{F}\,(v^*) - \left(\frac{q-\log(\delta)}{q\,(1-\delta^T e^{-qT})}\right)(p^* - cX_{CP}(\{q_k,\dots,q_1\}))\,f\,(v^*)\\ &= \frac{\partial\mathcal{R}_{CP}(p,\sigma(q))}{\partial p}|_{p^*} + c\left(\frac{q-\log(\delta)}{q\,(1-\delta^T e^{-qT})}\right)(X_{CP}(\{q_k,\dots,q_1\} - X_{CP}(\sigma(q)))\,f(v^*)\\ &= c\left(\frac{q-\log(\delta)}{q\,(1-\delta^T e^{-qT})}\right)(X_{CP}(\{q_k,\dots,q_1\} - X_{CP}(\sigma(q)))\,f(v^*) \le 0,\end{aligned}$$

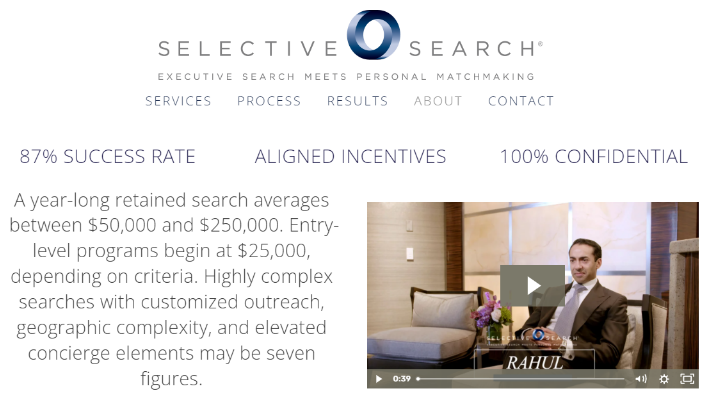
where the second equality follows from minus $X_{CP}(\sigma(q))$, then add $X_{CP}(\sigma(q))$ back, the third equality follows from $\frac{\partial\mathcal{R}_{CP}(p,c,F,\{T\},\sigma(q))}{\partial p}|_{p^*} = 0$, the inequality follows from $X_{CP}(\{q_k,\dots,q_1\}) \le X_{CP}(\sigma(q))$, which is shown in Lemma 7(a). Thus to maximize the profit, the optimal contract price for $\mathcal{R}_{CP}(\{q_k,\dots,q_1\})$ should be lower than p^*. □

Appendix C: Additional Figures

C.1. Additional examples

Figure EC.2 Examples of subscription and contract pricing in online dating platforms.

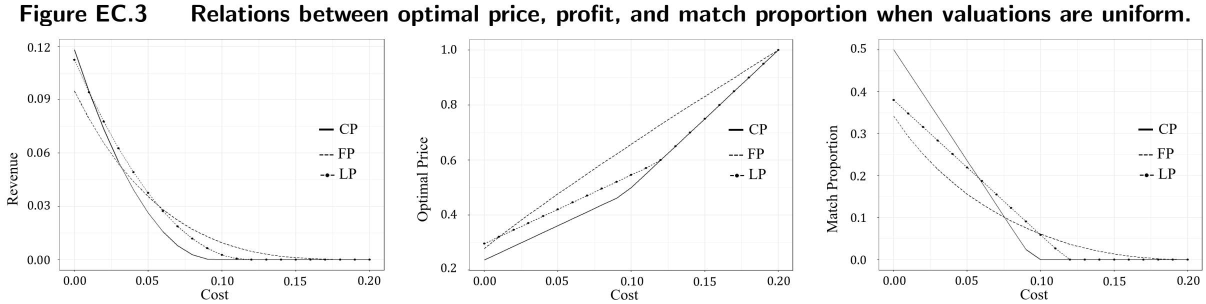
Membership	Term	Price Per Month
Basic	~	Free
Premium	1 month	$29.95
Premium	3 months	$19.98
Premium	6 months	$12.49
Premium	12 months	$12.49



Note. Depicted are price offerings for two online dating platforms. On the left is an example of freemium price at `https://www.zoosk.com/`. On the right is an example of contract pricing at `https://www.selectivesearch.com/pricing`.

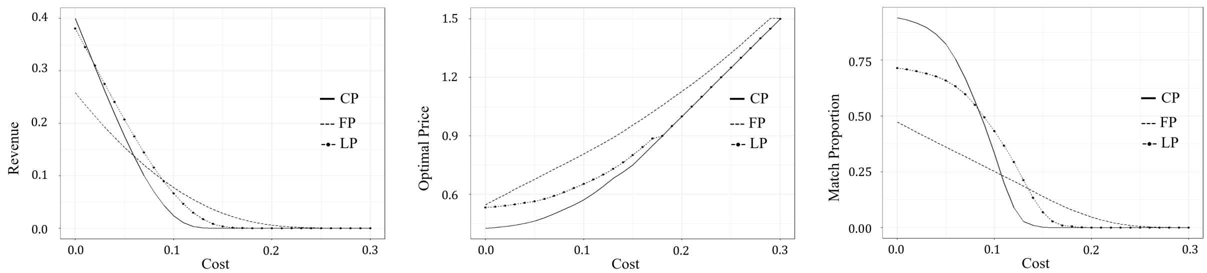
C.2. Additional numerics

Figure EC.3 Relations between optimal price, profit, and match proportion when valuations are uniform.



Note. Here we plot the profit, optimal prices, and match proportions under FP, CP and period L pricing with $L = T/7$, when valuations are drawn from an uniform(0, 1) distribution, and where $T = 50$, $\delta = 0.8$, $q = 0.2$, and c varies. In the left panel, we plot the profits of FP, CP and period L pricing, and the note relative profit ordering switches from $\mathcal{R}_{CP} > \mathcal{R}_{FP}$ when $c \leq 0.03$, to $\mathcal{R}_{CP} > \mathcal{R}_{FP}$ for $c > 0.03$. In the middle panel, we plot the normalized optimal contract, freemium and period L prices, as in Lemma 6. In the right panel, we plot the proportion of the market that gets matched under FP, CP and period L pricing. Note that $\mathcal{M}_{CP}$ dominates $\mathcal{M}_{FP}$ for $c \leq 0.07$.

Figure EC.4 Relations between optimal price, profit, and match proportion when valuations are a mixture of log-normal.



Note. Here we plot the profit, optimal prices, and match proportions under FP, CP and period L pricing with $L = T/7$, when valuations are drawn from mixed Log-normal distribution, where the two log-normal distributions Log-normal(0, 0.1) and Log-normal(0.2, 0.1), and the mixed probability is 0.6, and $T = 50$, $\delta = 0.8$, $q = 0.2$, and c varies. In the left panel, we plot the profits of FP and CP, and the note relative profit ordering switches from $\mathcal{R}_{CP} > \mathcal{R}_{FP}$ when $c \leq 0.06$, to $\mathcal{R}_{CP} > \mathcal{R}_{FP}$ for $c > 0.06$. In the middle panel, we plot the normalized optimal contract, freemium and period L prices, as in Lemma 6. In the right panel, we plot the proportion of the market that gets matched under FP and CP and note that $\mathcal{M}_{CP}$ dominates $\mathcal{M}_{FP}$ for $c \leq 0.105$.