Chapter 4.1, Problem 1E

Step-by-step solution

Step 1 of 12

Let V be the set of all ordered pairs of real numbers, addition and scalar multiplication operations on $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ are defined as follows.

$\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$ ...... (1)

$k\mathbf{u}=(0,ku_2)$ ...... (2)

Step 2 of 12

(a)

Let $\mathbf{u}=(-1,2)$, $\mathbf{v}=(3,4)$, and $k=3$

Need to compute $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$.

As $\mathbf{u}=(-1,2)$ and $\mathbf{v}=(3,4)$, we have that,

$\mathbf{u}+\mathbf{v}=(-1,2)+(3,4)$

$=(-1+3,2+4)$ [From (1), $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$]

$=(2,6)$

And

$k\mathbf{u}=3(-1,2)$

$=(0,3(2))$ [From (2), $k(u_1,u_2)=(0,ku_2)$]

$=(0,6)$

Therefore, $\boxed{\mathbf{u}+\mathbf{v}=(2,6)}$ and $\boxed{k\mathbf{u}=(0,6)}$.

Step 3 of 12

(b)

It is need to explain the reason for *V* is closed under addition and scalar multiplication.

On V, the addition and scalar multiplication are defined as components of real numbers and u_1,u_2,v_1,v_2 are real numbers, so u_1+v_1,u_2+v_2 and ku_2 are also real numbers.

Thus, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)\in\mathbb{R}$ and $k\mathbf{u}=(0,ku_2)\in\mathbb{R}$.

By general real numbers properties, it is clear that *V* is closed under addition and scalar multiplication.

Step 4 of 12

(c)

Addition operation on V is defined as,

$\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$

Need to find the axioms hold for V by using the addition operation.

Let $\mathbf{u}=(u_1,u_2),\mathbf{v}=(v_1,v_2)$

Axiom 1: Since $\mathbf{u}=(u_1,u_2),\mathbf{v}=(v_1,v_2)\in V$

So

$$\begin{aligned}\mathbf{u}+\mathbf{v}&=(u_1,u_2)+(v_1,v_2)\\&=(u_1+v_1,u_2+v_2)\in V\quad \text{Since } u_1,u_2,v_1,v_2\in\mathbb{R}\\&\qquad\qquad\qquad\qquad\qquad \text{So } u_1+v_1,u_2+v_2\in\mathbb{R}\end{aligned}$$

(Since the set of real numbers are closed)

Therefore, V is closed under addition.

Step 5 of 12

Axiom 2: Let $\mathbf{u}=(u_1,u_2),\mathbf{v}=(v_1,v_2)\in V$

Then

$$\begin{aligned}\mathbf{u}+\mathbf{v}&=(u_1,u_2)+(v_1,v_2)\\&=(u_1+v_1,u_2+v_2)\\&=(v_1+u_1,v_2+u_2)\end{aligned}$$

(Since the set of real numbers are commutative)

$=(v_1,v_2)+(u_1,u_2)$

$=\mathbf{v}+\mathbf{u}$

Therefore, the commutative property holds on V.

Step 6 of 12

Axiom 3: Let $\mathbf{u}=(u_1,u_2),\mathbf{v}=(v_1,v_2),\mathbf{w}=(w_1,w_2)\in V$

Then

$(\mathbf{u}+\mathbf{v})+\mathbf{w}=((u_1,u_2)+(v_1,v_2))+(w_1,w_2)$

$=(u_1+v_1,u_2+v_2)+(w_1,w_2)$

$=((u_1+v_1)+w_1,(u_2+v_2)+w_2)$

(Since the set of real numbers are Associative)

$=(u_1+(v_1+w_1),u_2+(v_2+w_2))$

$=(u_1,u_2)+(v_1+w_1,v_2+w_2)$

$=(u_1,u_2)+((v_1,v_2)+(w_1,w_2))$

$=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

Therefore, associative law is satisfied in V.

Step 7 of 12

Axiom 4: There is an object **0** in V, called zero vector for V, such that

$$\begin{aligned}\mathbf{0}+\mathbf{u}&=(0,0)+(u_1,u_2)\\&=(0+u_1,0+u_2)\\&=(u_1,u_2)\\&=\mathbf{u}\end{aligned}$$

Therefore, **0** is the additive identity of V.

Step 8 of 12

Axiom 5: For each $\mathbf{u}\in V$, there is an object $-\mathbf{u}\in V$, called negative of **u** such that

$\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$

And since u_1,u_2 are real numbers.

So $-u_1,-u_2\in\mathbb{R}$

Now

$$\begin{aligned}\mathbf{u}+(-\mathbf{u})&=(u_1,u_2)+(-u_1,-u_2)\\&=(u_1-u_1,u_2-u_2)\\&=(0,0)\\&=\mathbf{0}\end{aligned}$$

Thus, for every $\mathbf{u}\in V$, there is an additive inverse $-\mathbf{u}\in V$.

Step 9 of 12

(d)

Need to show that axioms 7,8, 9 are also satisfied in V.

Axiom 7: For every, $\mathbf{u},\mathbf{v}\in V$ and *k* is any scalar, $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

Let $\mathbf{u}=(u_1,u_2),\mathbf{v}=(v_1,v_2)\in V$ and *k* be any scalar

Then

$$\begin{aligned}k(\mathbf{u}+\mathbf{v})&=(0,k(u_2+v_2)) && \text{By using (2)}\\&=(0,ku_2+kv_2) && \text{Simplifying}\\&=(0,ku_2)+(0,kv_2)\\&=k\mathbf{u}+k\mathbf{v} && \text{By using (2)}\end{aligned}$$

Hence $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

Step 10 of 12

Axiom 8: For every, $\mathbf{u}\in V$ and *k*, *m* are any scalars, $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

Let $\mathbf{u}=(u_1,u_2)\in V$ and *k*, *m* are any scalars

$$\begin{aligned}(k+m)\mathbf{u}&=(0,(k+m)u_2) && \text{By using (2)}\\&=(0,ku_2+mu_2) && \text{Simplifying}\\&=(0,ku_2)+(0,mu_2)\\&=k\mathbf{u}+m\mathbf{u} && \text{By using (2)}\end{aligned}$$

Hence $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

Step 11 of 12

Axiom 9: For every $\mathbf{u}\in V$ and *k*, *m* are any scalars, $k(m\mathbf{u})=(km)\mathbf{u}$

Let $\mathbf{u}=(u_1,u_2)\in V$ and *k*, *m* are any scalars

By the definition in (2), $m\mathbf{u}=(0,mu_2)$

Now

$$\begin{aligned}k(m\mathbf{u})&=(0,k(mu_2)) && \text{By using (2)}\\&=(0,kmu_2) && \text{Simplification}\\&=(0,(km)u_2)\\&=(km)(\mathbf{u})\end{aligned}$$

Hence $k(m\mathbf{u})=(km)\mathbf{u}$

Thus, *V* satisfies the axioms 7, 8, and 9.

Step 12 of 12

(e)

Need to show that axiom 10 is failed on *V*.

Axiom 10: For every $\mathbf{u}\in V$, $1\mathbf{u}=\mathbf{u}$

Let $\mathbf{u}=(u_1,u_2)\in V$

By definition in (2),

$1\mathbf{u}=1(u_1,u_2)$

$=(0,1u_2)$ [From (2), $k(u_1,u_2)=(0,ku_2)$]

$=(0,u_2)$

$\neq(u_1,u_2)$

$=\mathbf{u}$

Hence the Axiom 10 does not hold on *V*.

Therefore, V is not a Vector Space under the given operations.

Chapter 4.1, Problem 2E

Step-by-step solution

Step 1 of 8

Let V be the set of all ordered pairs of real numbers and addition and scalar multiplication defined as shown below:

$\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$

$\mathbf{u}+\mathbf{v}=(u_1+v_1+1,u_2+v_2+1)$ and $k\mathbf{u}=(ku_1,ku_2)$

Step 2 of 8

(a)

Let $\mathbf{u}=(0,4)$ and $\mathbf{v}=(1,-3)$

Now find the vector $\mathbf{u}+\mathbf{v}$.

$$\begin{aligned}\mathbf{u}+\mathbf{v}&=(0,4)+(1,-3) && \text{given}\\&=(0+1+1,4-3+1) && \text{definition}\\&=(2,2) && \text{simplification}\end{aligned}$$

Therefore, $\mathbf{u}+\mathbf{v}=(2,2)$.

Now find the vector $k\mathbf{u}$, here $k=2$

$$\begin{aligned}2\mathbf{u}&=2(0,4)\\&=(0,8)\end{aligned}$$

Therefore, $2\mathbf{u}=(0,8)$

Step 3 of 8

(b)

Show that $(0,0)\neq\mathbf{0}$.

That is, the vector $(0,0)$ is not identity element in V.

Suppose $(0,0)$ is identity element in V, then it should be satisfies $(u_1,u_2)+(0,0)=(u_1,u_2)$.

Now consider $(u_1,u_2)+(0,0)$

$$\begin{aligned}(u_1,u_2)+(0,0)&=(u_1+0+1,u_2+0+1) \quad \text{(Definition)}\\&=(u_1+1,u_2+1)\\&\neq(u_1,u_2)\end{aligned}$$

Therefore, $(0,0)$ is not an identity element in V.

Hence, $(0,0)\neq\mathbf{0}$

Step 4 of 8

(c)

Show that $(-1,-1)=\mathbf{0}$.

That is, $(-1,-1)$ is the identity element in V.

Suppose $(-1,-1)$ is identity element in V, then it should be satisfies $(u_1,u_2)+(-1,-1)=(u_1,u_2)$.

Now consider $(u_1,u_2)+(-1,-1)$.

$$\begin{aligned}(u_1,u_2)+(-1,-1)&=(u_1-1+1,u_2-1+1) \quad \text{(Definition)}\\&=(u_1,u_2)\end{aligned}$$

Therefore, $(-1,-1)$ is the identity element in V.

Hence, $(-1,-1)=\mathbf{0}$

Comments (1)

Anonymous

(-1,-1) -> (1,1)

Step 5 of 8

(d)

Define $-\mathbf{u}$ as $-\mathbf{u}=(-u_1-2,-u_2-2)$ such that

$$\begin{aligned}\mathbf{u}+(-\mathbf{u})&=(u_1,u_2)+(-u_1-2,-u_2-2)\\&=(u_1-u_1-2+1,u_2-u_2-2+1) \quad \text{Deifinition}\\&=(-1,-1)\\&=\mathbf{0} \quad \text{From part }(c)\end{aligned}$$

Now consider $(-\mathbf{u})+\mathbf{u}$.

$$\begin{aligned}\mathbf{u}+(-\mathbf{u})&=(-u_1-2,-u_2-2)+(u_1,u_2)\\&=(-u_1-2+u_1+1,-u_2-2+u_2+1) \quad \text{Deifinition}\\&=(-1,-1)\\&=\mathbf{0} \quad \text{From part }(c)\end{aligned}$$

Therefore, $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$

Step 6 of 8

(e)

Let $k=1,m=2$ and $\mathbf{u}=(3,4)$

Consider $(k+m)\mathbf{u}$.

$$\begin{aligned}(k+m)\mathbf{u}&=(1+2)(3,4)\\&=3(3,4)\\&=(9,12)\end{aligned}$$

Step 7 of 8

Consider $k\mathbf{u}+m\mathbf{u}$.

$$\begin{aligned}k\mathbf{u}+m\mathbf{u}&=1(3,4)+2(3,4)\\&=(3,4)+(6,8)\\&=(3+6+1,4+8+1)\\&=(10,13)\end{aligned}$$

Therefore, $(k+m)\mathbf{u}\neq k\mathbf{u}+m\mathbf{u}$

Step 8 of 8

Let $k=2,\mathbf{u}=(3,4)$ and $\mathbf{v}=(-1,5)$.

Consider $k(\mathbf{u}+\mathbf{v})$

$$\begin{aligned}k(\mathbf{u}+\mathbf{v})&=2((3,4)+(-1,5))\\&=2(3-1+1,4+5+1)\\&=2(3,10)\\&=(6,20)\end{aligned}$$

Consider $k\mathbf{u}+k\mathbf{v}$

$$\begin{aligned}k\mathbf{u}+k\mathbf{v}&=2(3,4)+2(-1,5)\\&=(6,8)+(-2,10)\\&=(6-2+1,8+10+1)\\&=(5,19)\end{aligned}$$

Therefore, $k(\mathbf{u}+\mathbf{v})\neq k\mathbf{u}+k\mathbf{v}$.

Therefore, the set V does not satisfies the vector space axioms $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$ and $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.

Step-by-step solution

Step 1 of 11

Consider that V be the set of all pairs of real numbers of the form $(1,x)$ with the operations $(1,y)+(1,y')=(1,y+y')$ and $k(1,y)=(1,ky)$.

Then, $V=\{(1,x): x\in \ \}$.

The objective is to determine whether the set V equipped with the given operations is a vector space or not.

Step 2 of 11

Axiom 1:

Let $u=(1,x_1)$ and $v=(1,x_2)\in V$. Then,

$$\begin{aligned} u+v &= (1,x_1)+(1,x_2) \\ &= (1,x_1+x_2) && \text{By definition.} \\ &\in V && \text{As if } x_1\in \ ,x_2\in \ \text{, then } x_1+x_2\in \ . \end{aligned}$$

So, if u and $v\in V$, then $u+v\in V$.

Step 3 of 11

Axiom 2:

Let $u=(1,x_1)$ and $v=(1,x_2)\in V$. Then,

$$\begin{aligned} u+v &= (1,x_1)+(1,x_2) \\ &= (1,x_1+x_2) && \text{By definition.} \\ &= (1,x_2+x_1) && \text{The set } \ \text{ is commutative.} \\ &= (1,x_2)+(1,x_1) && \text{By definition.} \\ &= v+u \end{aligned}$$

Thus, $u+v=v+u$, i.e. the commutative property holds.

Step 4 of 11

Axiom 3:

Let $u=(1,x_1), v=(1,x_2)$, and $w=(1,x_3)\in V$. Then,

$$\begin{aligned} (u+v)+w &= ((1,x_1)+(1,x_2))+(1,x_3) \\ &= (1,x_1+x_2)+(1,x_3) && \text{By definition.} \\ &= (1,(x_1+x_2)+x_3) && \text{By definition.} \\ &= (1,x_1+(x_2+x_3)) && \text{The set } \ \text{ is associative.} \\ &= (1,x_1)+((1,x_2)+(1,x_3)) && \text{By definition.} \\ &= u+(v+w) \end{aligned}$$

So, $(u+v)+w=v+(u+w)$, i.e. the associative property holds.

Step 5 of 11

Axiom 4:

Let $u=(1,x)\in V$ and $0=(1,y)\in V$ such that

$$\begin{aligned} u+0 &= u \\ (1,x)+(1,y) &= (1,x) \\ (1,x+y) &= (1,x) \\ \Rightarrow x+y &= x \\ \Rightarrow \quad y &= 0 \end{aligned}$$

So, $0=(1,0)$ is the zero element in V.

Step 6 of 11

Axiom 5:

Let $u=(1,x)\in V$ and $v=(1,y)\in V$ such that

$$\begin{aligned} u+v &= 0 \\ (1,x)+(1,y) &= (1,0) \\ (1,x+y) &= (1,0) \\ \Rightarrow x+y &= 0 \\ \Rightarrow \quad y &= -x \end{aligned}$$

So, $v=(1,-x)$ is the inverse element of u in V.

Step 7 of 11

Axiom 6:

Let $u=(1,x)\in V$ and $k\in \ $. Then,

$$\begin{aligned} ku &= k(1,x) \\ &= (1,kx) && \text{By definition.} \\ &\in V \end{aligned}$$

So, if $u\in V$ and $k\in \ $, then $ku\in V$.

Step 8 of 11

Axiom 7:

Let $u=(1,x), v=(1,y)\in V$ and $k\in \ $. Then,

$$\begin{aligned} k(u+v) &= k((1,x)+(1,y)) \\ &= k(1,x+y) && \text{By definition} \\ &= (1,kx+ky) && \text{By definition} \\ &= (1,kx)+(1,ky) && \text{By definition} \\ &= k(1,x)+k(1,y) && \text{By definition} \\ &= ku+kv \end{aligned}$$

So, $k(u+v)=ku+kv$.

Step 9 of 11

Axiom 8:

Let $u=(1,x)\in V$ and $k,m\in \ $. Then,

$$\begin{aligned} (k+m)u &= (k+m)(1,x) \\ &= (1,(k+m)x) && \text{By definition} \\ &= (1,kx+mx) \\ &= (1,kx)+(1,mx) && \text{By definition} \\ &= k(1,x)+m(1,x) && \text{By definition} \\ &= ku+mu \end{aligned}$$

So, $(k+m)u=ku+mu$.

Step 10 of 11

Axiom 9:

Let $u=(1,x)\in V$ and $k,m\in \ $. Then,

$$\begin{aligned} k(mu) &= k(m(1,x)) \\ &= k(1,mx) && \text{By definition} \\ &= (1,kmx) \\ &= km(1,x) && \text{By definition} \\ &= (km)u \end{aligned}$$

So, $k(mu)=(km)u$.

Step 11 of 11

Axiom 10:

Let $u=(1,x)\in V$. Then,

$$\begin{aligned} 1u &= 1(1,x) \\ &= (1,1x) && \text{By definition} \\ &= (1,x) \\ &= u \end{aligned}$$

So, $1u=u$.

Thus, all the above axioms satisfy all the properties of a vector space.

Therefore, the set $(V,+,.)$ is a **vector space** with the given operations.