Hermite Polynomials

Steve Fisk
Bowdoin College
Brunswick, Me 04011

December 22, 1999

Abstract

We give an elementary proof that a transformation based on the Hermite polynomials preserves the property of having all real roots.

Consider the following example:

$$\begin{aligned} f &= (x-1)(x-3)(x+4)(x+6) \\ f &= 72 - 66x - 13x^2 + 6x^3 + x^4 \end{aligned}$$

$$\begin{aligned} H_0 &= 1 \\ H_1 &= 2x \\ H_2 &= 4x^2 - 2 \\ H_3 &= 8x^3 - 12x \\ H_4 &= 16x^4 - 48x^2 + 12 \end{aligned}$$

$$\begin{aligned} H(f) = {} & 72(\underline{1}) - 66(\underline{2x}) - 13(\underline{4x^2 - 2}) + \\ & 6(\underline{8x^3 - 12x}) + (\underline{16x^4 - 48x^2 + 12}) \end{aligned}$$

$$H(f) = (x+3.60)(x+1.96)(x-0.46)(x-2.10)$$

The goal of this paper is to show that the fact that $H(f)$ has all real roots is not an accident. The $H_i(x)$ are the *Hermite* polynomials. There are many ways of constructing the Hermite polynomials. They are the solutions to the differential equation

$$0 = y'' - 2xy' + 2ny$$

and are given explicitly by the Rodrigues' formula

$$H_n(x) = e^{x^2}(-1)^n \left(\frac{d}{dx}\right)^n e^{-x^2}.$$

They are also the polynomials orthogonal for the weight function e^{-x^2} on $(-\infty, \infty)$, but we will use the generating function definition.

$$e^{-y^2+2xy} = \sum_{i=0}^{\infty} H_i(x) \frac{y^i}{i!}. \tag{1}$$

If we differentiate (1) with respect to y then we find the recursive definition: $H_0 = 1, H_1 = 2x$ and

$$H_{n+1} = 2xH_n - 2nH_{n-1} \tag{2}$$

If we differentiate (1) with respect to x we find

$$H_n' = 2n\, H_{n-1} \tag{3}$$

An important consequence of (3) and (2) is that $H_{n+1} = 2xH_n - (H_n)' = (2x - \mathsf{D})H_n$ so we have an explicit formula:

$$H_n = (2x - \mathsf{D})^n(1) \tag{4}$$

This last formula is important in showing that the Hermite transformation preserves roots.

1 The Hermite Transformation

The *Hermite transformation* is the linear transformation T on polynomials that is defined by

$$T(x^n) = H_n$$

From (4) we find $T(x^n) = H_n = (2x - \mathsf{D})^n(1)$. If we use the linearity of T then

$$T(f) = f(2x - \mathsf{D})(1) \tag{5}$$

Theorem 1. *If f is a polynomial with all real roots then Tf also has all real roots.*

Proof. If we write $f(x) = a(x + a_1)\cdots(x + a_n)$ where all the a_i are real then by the representation (5) we have

$$Tf = a(2x - \mathsf{D} + a_1)\cdots(2x - \mathsf{D} + a_n)(1).$$

We can use induction on the degree of f. It thus suffices to show that if a polynomial g has all real roots then $h = (2x - \mathsf{D} + b)g = (2x + b)g - g'$ has all real roots for any choice of b.. This is a well known fact, whose argument we give in the next paragraph.

Write $g(x) = (x-d_1)\cdots(x-d_n)$ where $d_1 < \cdots < d_n$. Since $h(d_i) = -g'(d_i)$ it follows that the sign of $h(d_i)$ is $(-1)^{n+i+1}$. This shows that h has a root between each root of g, giving $n-1$ roots. Moreover, the sign of h on the largest root d_n is negative, and h has positive leading coefficient, so there is another root of h that is greater than d_n. Similarly there is a root of h that is smaller than d_1 and thus h has all real roots.

□

An elementary argument shows that the inverse of the Hermite transformation also has a representation as a composition:

$$T^{-1}(f) = f(x/2 + \mathsf{D})(1) \tag{6}$$

Unlike T, the inverse does not preserve the property of having all real roots. However, we do have

Theorem 2. *If f is a polynomial that has all negative roots, then $T^{-1}f$ has all real roots.*

Proof. Same as above. All that needs to be shown is that if g has all real roots then $(x/2+b)g + g'$ also has all real roots, where b is negative. This is left as an exercise. □