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Quadratically convergent method for simultaneously approaching the roots of polynomial solutions of a class of differential equations: application to orthogonal polynomials

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This paper investigates the application of the method introduced by L. Pasquini (1989) for simultaneously approaching the zeros of polynomial solutions to a class of second-order linear homogeneous ordinary differential equations with polynomial coefficients to a particular case in which these polynomial solutions have zeros symmetrically arranged with respect to the origin. The method is based on a family of nonlinear equations which is associated with a given class of differential equations. The roots of the nonlinear equations are related to the roots of the polynomial solutions of differential equations considered. Newton's method is applied to find the roots of these nonlinear equations. In (Pasquini, 1994) the nonsingularity of the roots of these nonlinear equations is studied. In this paper, following the lines in (Pasquini, 1994), the nonsingularity of the roots of these nonlinear equations is studied. More favourable results than the ones in (Pasquini, 1994) are proven in the particular case of polynomial solutions with symmetrical zeros. The method is applied to approximate the roots of Hermite–Sobolev type polynomials and Freud polynomials. A lower bound for the smallest positive root of Hermite–Sobolev type polynomials is given via the nonlinear equation. The quadratic convergence of the method is proven. A comparison with a classical method that uses the Jacobi matrices is carried out. We show that the algorithm derived by the proposed method is sometimes preferable to the classical QR type algorithms for computing the eigenvalues of the Jacobi matrices even if these matrices are real and symmetric.

Basic notation

The basic notations used in the following are:

C: the set of complex numbers;
C^n**:** the n-dimensional complex linear space;
R: the set of real numbers;
R^n**:** the n-dimensional real linear space;

N: the set of nonnegative integers;
N$^+$: the set of positive integers;
P: the linear space of the polynomials;
P^m: the linear space of the polynomials of degree less than or equal to m;
P$_R$: the linear space of the polynomials with real coefficients;
$[\,\cdot\,]$**:** integer part of.

Definition 0.1. Let $p \in \mathbf{P}_R$. We say that p is an *even* polynomial if we have $p(z) = p(-z)$, $z \in \mathbf{R}$.

Definition 0.2. Let $p \in \mathbf{P}_R$. We say that p is an *odd* polynomial if we have $p(z) = -p(-z)$, $z \in \mathbf{R}$.

Finally, we introduce the following notation:

P$_S$: the set of the polynomials with real coefficients which are even or odd.

1. Introduction

Let us consider the following differential equation:

$$\begin{aligned} &Lu(z) = 0, \quad z \in \mathbf{C}, \\ &L := \sum_{i=0}^{2} a_{2-i} \frac{\mathrm{d}^{2-i}}{\mathrm{d}z^{2-i}}, \quad a_j \in \mathbf{P}\ (j = 0, 1, 2), \quad a_2 \not\equiv 0. \end{aligned} \tag{1}$$

In [1–3] the author introduces a method for simultaneously finding the roots of polynomial solutions to differential equation (1). In this paper, we investigate the application of this method to the particular case in which the polynomial coefficients a_j, $j = 0, 1, 2$, satisfy conditions which assure the existence of polynomial solutions with real roots symmetrically arranged with respect to the origin. This method is based on a suitable nonlinear equation introduced in [1]. We study the nonsingularity of the roots of this nonlinear equation. We show more favourable results than the ones in [3] in the particular case considered in this article. We apply the method in the computation of the zeros of Hermite–Sobolev orthogonal polynomials (see [4–8]) and of the zeros of Freud polynomials (see [9,10]). In particular, we consider polynomials orthogonal with respect to the following Sobolev inner product (see [4–8]):

$$\tilde{\varphi}_\lambda(f, g) = \int_{-\infty}^{+\infty} f(z)g(z)\mathrm{e}^{-z^2}\,\mathrm{d}z + \lambda f'(0)g'(0). \tag{2}$$

In formula (2), the prime $'$ means the first derivative with respect to the variable z and λ is a nonnegative parameter. We deal also with the polynomials orthogonal with respect to the following semiclassical inner product (see [9]):

$$\tilde{\varphi}(f, g) = \int_{-\infty}^{+\infty} f(z)g(z)\mathrm{e}^{-z^4}\,\mathrm{d}z. \tag{3}$$

The roots these polynomials are symmetrically arranged with respect to the origin (see (2), (3) and section 3). We give a lower bound of the smallest positive root of Hermite–Sobolev type polynomials (see theorem 3.1) and prove that when λ goes to infinity, the root $z = 0$ of the polynomials with odd degrees becomes a triple root (see theorem 3.2). We prove that the method is quadratically convergent when applied to the computation of the roots of these two classes of polynomials (see theorem 3.1 and theorem 3.3). We compare the method with the classical method of computing the eigenvalues of the associated Jacobi matrices. We use both the QL algorithm and the QR algorithm (Lapack subroutine) for computing the eigenvalues of the Jacobi matrices. As we will show in section 5, the algorithm derived from the proposed method sometimes gives more accurate results than those obtained by using the QR type algorithms even if the Jacobi matrices are real and symmetric matrices. This fact is essentially due to the errors introduced in the computation of the entries of the Jacobi matrices and, in the case of Hermite–Sobolev type polynomials, to the behaviour of the smallest positive root of the polynomials.

Further developments of this work will be addressed to investigate how the method could be applied to a more general form of orthogonal polynomials (see [11]). Moreover, we will study how the nonlinear equation associated to the differential equation (1) could be used for some inverse problems of the type described in [12].

In section 2, we recall the method in [1] and we carry out an accurate study of the relationship between the null space of the operator L (see (1)) and the rank of the Jacobian matrix associated with the nonlinear equation which the method is based on. We refer to the method as the general reduced method (GRM) since it is a type of symmetrization procedure of the general method presented in [3]. We show that the method is applicable to a wide class of differential equations useful in the approximation theory and it guarantees a great accuracy on the numerical approximations.

In section 3, we describe two classes of differential equations on which we apply the method. These classes of differential equations are those satisfied by Hermite–Sobolev and Freud orthogonal polynomials respectively. We show some properties of the smallest positive root of Hermite–Sobolev polynomials and we prove that the method is quadratically convergent when applied to these classes of differential equations. In section 4, we describe the alternative methods based on the QR type algorithms and the Jacobi matrices associated with the recurrence relation satisfied by the orthogonal polynomials introduced in section 3. In section 5, we show numerical results obtained with the proposed method and the alternative ones. Finally, in section 6, we give proof of the stated theorems and include further results for the sake of completeness.

2. The general reduced method

In the following, we assume that the polynomials a_j, $j = 0, 1, 2$, given by (1), satisfy one of the following conditions:

$$\begin{aligned} &a_j \in \mathbf{P}_R, \quad j = 0, 1, 2, \\ &a_j(z) = a_j(-z), \quad j = 0, 2, \qquad a_1(z) = -a_1(-z), \quad z \in \mathbf{R}, \end{aligned} \tag{4}$$

or

$$\begin{aligned} &a_j \in \mathbf{P}_R, \quad j = 0, 1, 2, \\ &a_j(z) = -a_j(-z), \quad j = 0, 2, \qquad a_1(z) = a_1(-z), \quad z \in \mathbf{R}. \end{aligned} \tag{5}$$

For simplicity we assume that condition (4) holds. All the results hold when condition (4) is replaced with condition (5).

The general method presented in [1,3] consists of solving the nonlinear equation defined by

$$\underline{F}_n(\underline{z}) = \underline{0}, \quad \underline{z} = (z_1, z_2, \ldots, z_n) \in \mathcal{A} \subset \mathbf{C}^n, \ n \in \mathbf{N}^+, \tag{6}$$

$$\mathcal{A} := \{\underline{z} \in \mathbf{C}^n \colon i \neq j \to z_i \neq z_j \colon a_2(z_i) \neq 0, \ 1 \leqslant i, j \leqslant n\}, \tag{7}$$

$$\underline{F}_n : \mathcal{A} \subset \mathbf{C}^n \to \mathbf{C}^n, \quad \underline{F} = \big(F_{n,1}(\underline{z}), F_{n,2}(\underline{z}), \ldots, F_{n,n}(\underline{z})\big), \tag{8}$$

$$F_{n,i}(\underline{z}) := R(z_i) - \sum_{j=1,\ i\neq j}^{n} \frac{1}{z_i - z_j}, \quad i = 1, 2, \ldots, n, \tag{9}$$

$$R(s) := -\frac{a_1(s)}{2a_2(s)}, \quad s \in \mathbf{C}. \tag{10}$$

In [3] the relation between equation (6) and the polynomial solutions to differential equations in (1) is shown. Let $n \in \mathbf{N}^+$, $\nu = [n/2]$. The general reduced method consists of solving the following nonlinear equation:

$$\underline{G}_n(\underline{z}) = \underline{0}, \quad \underline{z} = (z_1, z_2, \ldots, z_\nu) \in \mathcal{D} \subset \mathbf{R}^\nu, \tag{11}$$

$$\mathcal{D} := \{\underline{z} \in \mathbf{R}^\nu \colon i \neq j \to z_i \neq z_j, z_j > 0 \colon a_2\big(z_j^{1/2}\big) \neq 0, \ 1 \leqslant i, j \leqslant \nu\}, \tag{12}$$

$$\underline{G}_n : \mathcal{D} \subset \mathbf{R}^\nu \to \mathbf{R}^\nu, \quad \underline{G}_n = \big(G_{n,1}(\underline{z}), G_{n,2}(\underline{z}), \ldots, G_{n,\nu}(\underline{z})\big), \tag{13}$$

$$G_{n,i}(\underline{z}) := -\frac{(2 + (-1)^{n+1})}{4} - z_i\left(\sum_{j=1,\ i\neq j}^{\nu} \frac{1}{z_i - z_j} - Q(z_i)\right), \quad i = 1, 2, \ldots, \nu, \tag{14}$$

$$Q\big(s^2\big) := \frac{R(s)}{2s}, \quad s \in \mathbf{R}, \tag{15}$$

where R is given by (10). The nonlinear equation (11) can be derived from (6) when we assume $a_i \in \mathbf{P}_R$, $i = 0, 1, 2$, and the existence of polynomial solutions p to (1) whose roots are symmetrically arranged with respect to the origin, i.e. $p \in \mathbf{P}_S^n$. The following theorem, that is theorem 2.1, shows the relationship between equation (6) and the polynomial solutions to the differential equation (1). Furthermore, theorem 2.1 gives conditions to reconstruct the polynomial a_0 from the knowledge of the polynomials a_1, a_2 and the roots of the polynomial solution to (1) (i.e. formula (20)). The first part of theorem 2.1 was already stated in [1] and under weaker assumptions in [3].

Theorem 2.1. Let be $n \in \mathbf{N}^+$ such that

$$Lp_n = 0, \tag{16}$$

$$p_n(z) := z^n + c_1 z^{n-1} + \cdots + c_n = (z - \zeta_{n,1})(z - \zeta_{n,2}) \cdots (z - \zeta_{n,n}), \tag{17}$$

and let the roots of p_n satisfy the following conditions:

$$\zeta_{n,i} \notin Z_2 := \{z \in \mathbf{C}\colon a_2(z) = 0\}, \quad i = 1, 2, \ldots, n, \tag{18}$$

$$\zeta_{n,i} \neq \zeta_{n,j}, \quad i \neq j, \ i, j = 1, 2, \ldots, n. \tag{19}$$

Then $\underline{\zeta}_n = (\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,n})$ is a root of equation (6).

Vice versa, let $n \in \mathbf{N}^+$ and $\underline{\zeta}_n = (\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,n}) \in \mathcal{A} \subset \mathbf{C}^n$ be a root of equation (6), then there exists a unique polynomial $a_0 \in \mathbf{P}$ satisfying (16) and the following relations:

$$\begin{aligned} a_0(\zeta_{n,i}) = {} & -a'(\zeta_{n,i}) + \frac{a_1(\zeta_{n,i})}{a_2(\zeta_{n,i})}\big(a_2'(\zeta_{n,i}) + a_1(\zeta_{n,i})\big) - \frac{3a_1^2(\zeta_{n,i})}{4a_2(\zeta_{n,i})} \\ & + 3a_2(\zeta_{n,i}) \sum_{j=1,\ j\neq i}^{n} \frac{1}{(\zeta_{n,i} - \zeta_{n,j})^2}, \ i = 1, 2, \ldots, n. \end{aligned} \tag{20}$$

From theorem 2.1, we can obtain the relationship between equation (11) and the polynomial solutions to differential equation (1) whose roots are symmetrically arranged with respect to the origin.

Corollary 2.2. Let $n \in \mathbf{N}^+$, $\nu = [n/2]$, such that

$$Lp_n = 0, \tag{21}$$

where $p_n \in \mathbf{P}_S^n$ is the following polynomial:

$$p_n(z) = \prod_{i=1}^{\nu}\big(z^2 - \zeta_{n,i}^2\big), \quad \text{if } n \text{ is even}, \tag{22}$$

$$p_n(z) = z\prod_{i=1}^{\nu}\big(z^2 - \zeta_{n,i}^2\big), \quad \text{if } n \text{ is odd}. \tag{23}$$

Moreover, let the assumptions (18) and (19) hold, then $\underline{\zeta}_n^* = (\zeta_{n,1}^2, \zeta_{n,2}^2, \ldots, \zeta_{n,\nu}^2)$ is a root of equation (11).

Vice versa, let $n \in \mathbf{N}^+$, $\nu = [n/2]$ and $\underline{\zeta}_n^* = (\zeta_{n,1}^2, \zeta_{n,2}^2, \ldots, \zeta_{n,\nu}^2) \in \mathcal{D} \subset \mathbf{R}^\nu$ be a root of equation (11). Then a unique $a_0 \in \mathbf{P}_R$ exists such that (20) and (21) hold.

Lemma 2.3. Let be $\mathrm{Ker}\,(L) \cap \mathbf{P}_R \neq \emptyset$, and let condition (4) hold. Then we have $\mathrm{Ker}\,(L) \cap \mathbf{P}_S \neq \emptyset$.

Remark 2.4. In the proof of lemma 2.3, we show that when polynomial solutions to (1) exist, whose roots are not symmetrically arranged with respect to the origin, their degrees are greater than the degree of symmetrical polynomial solutions to (1). This is not remarkable since the reduced method does not work with polynomial solutions to (1) not belonging in $\mathbf{P}_S$. That is, the method works also when nonsymmetrical solutions to (1) exist. Moreover, when two symmetrical polynomial solutions to (1) exist, the reduced method simultaneously approaches the zeros of the "lower" degree polynomial solution.

Therefore, the natural assumption under which the general reduced method works (see [3, formula (2.4)]) can be rewritten as follows:

$$\mathrm{Ker}\,(L) \cap \mathbf{P}_S^n = \mathrm{span}\{p_n\}. \tag{24}$$

Let the polynomial coefficients a_i, $i = 0, 1, 2$, satisfy condition (4) or otherwise condition (5). Then condition (24) means that the general reduced method is a method for simultaneously approaching the zeros of symmetrical lower degree solutions to a differential equation (1).

From now on we assume that conditions (4), (18), (19) and (24) hold. If condition (4) is replaced with condition (5) the results can be similarly proven. The following theorems give an analysis of the nonsingularity of the roots of (11). We assume the condition

$$a_j \not\equiv 0, \quad j = 0, 1, 2, \tag{25}$$

which is more restrictive than the one in (1), and simplifies the proof of the theorems. It is easy to see, that all differential equations like (1) which do not satisfy (25) are trivial or can be reduced into others satisfying (25). Let be

$$\begin{aligned} d_j &:= \deg(a_j), \quad j = 0, 1, 2, \\ a_j(z) &= \sum_{h=0}^{k+j} \alpha_{j,h} z^{k+j-h}, \quad j = 0, 1, 2, \ k := \max_{j=1,2}(d_j - j), \end{aligned} \tag{26}$$

and

$$\gamma_h \in \mathbf{P}^2, \quad \gamma_h(z) := z\big(\alpha_{2,h}(z-1) + \alpha_{1,h}\big) \quad (h = 0, 1, \ldots, k+2, \ \alpha_{1,k+2} = 0).$$

Remark 2.5. As underlined in [3], the condition

$$k = \max_{j=1,2}(d_j - j) = \max_{j=0,1}(d_j - j) = \max_{j=0,2}(d_j - j) = \max_{j=0,1,2}(d_j - j) \geqslant 0$$

is necessary to assure the existence of a polynomial solution of differential equation in (1).

We have the following results.

Theorem 2.6. Let condition (4) hold and let conditions (16), (18), (19) and (21) be satisfied for a given $n \in \mathbf{N}^+$. Let $\underline{\zeta}_n^* \in \mathcal{D} \subset \mathbf{R}^\nu$ be a root of nonlinear equation (11) and let $J_{\underline{G}_n}(\underline{\zeta}_n^*)$ be the Jacobian matrix of the function $\underline{G}_n$ at the point $\underline{\zeta}_n^*$.

In the case of $k = 0$ or $k = 1$, the inequality

$$\det J_{\underline{G}_n}(\underline{\zeta}_n^*) \neq 0 \tag{27}$$

is equivalent to condition (24). In the case $k > 1$, the inequality (27) is equivalent to the condition

$$Lw \in Q_{n,k}^*, \quad w \in \mathbf{P}_S^{n-2} \quad \Rightarrow \quad w \equiv 0, \quad \text{where} \tag{28}$$

$$Q_{n,k}^* := \{ p \in \mathbf{P}_S^{n+k-2} \colon\ p = qp_n,\ q \in \mathbf{P}_S^{k-2} \}, \quad k \geqslant 2. \tag{29}$$

Theorem 2.7. Let condition (4) hold and let conditions (18), (19) and (21) be satisfied for a given $n \in \mathbf{N}^+$. Let $\nu = [n/2]$, we have:

$$\nu - d^* - \left[\frac{k}{2}\right] \leqslant \operatorname{rank} J_{\underline{G}_n}(\underline{\zeta}_n^*) \leqslant \nu - d^*, \quad \text{where } d^* := \dim\big(\operatorname{Ker}(L) \cap \mathbf{P}_S^{n-2}\big).$$

Theorem 2.6 shows that in this specific case we are investigating, the condition for the nonsingularity of the roots of the nonlinear equation (11) comes down to condition (24) even for $k = 1$. This result can be stated in the general case only for $k = 0$ (see [3]). A simple example on which we can apply theorem 2.6 is given by the following:

$$z\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + \left(z^2 - \frac{5}{4}\right)\frac{\mathrm{d}u}{\mathrm{d}z} - 2zu = 0. \tag{30}$$

Note that in (30) we have $k = 1$ and condition (5) holds. A polynomial solution of (30) is $p_2(z) = z^2 - 1/4$.

Remark 2.8 (GRM algorithm). Due to theorems 2.6 and 2.7, we use Newton's method to solve the nonlinear equation (11). The choice of Newton's method is obvious in the case $k = 0$ and $k = 1$. In the case $k > 1$, the stronger condition (28) guarantees (27). However, when the roots are singular, theorems 2.6 and 2.7 give information about the nature of the singularity. This allows us to use some techniques based on Newton's method to retain quadratic convergence (see [13–15]).

Remark 2.9 (The stability of GRM algorithm). It is worth noting the interesting consequences of the inequality in (27). In fact, since $\underline{G}_n$ is an analytic function on $\mathcal{D}$ (see (11) and (12)), Newton's method will define a sequence $\{\underline{z}^{(m)}\}_{m=0,1,\dots}$ convergent to a root if the starting point $\underline{z}^0$ is appropriately chosen. Moreover, as in the case of the general method presented in [3], the sequence will be quadratically convergent to the root since condition (28) can be considered a generically satisfied condition on the parameters $\alpha_{j,h}$ in (26). In fact, we can express the equation $Lw = qp_n$, $w \in \mathbf{P}_S^{n-2}$, $q \in \mathbf{P}_S^{k-2}$, as a linear system of $[k/2]+[(n-2)/2]+2$ equations in the $[k/2]+[(n-2)/2]+2$ unknowns w, q, and we observe that (28) is equivalent to the nonsingularity of the coefficient matrix of this system.

As pointed out in [3], the quadratic convergence of the sequence $\{\underline{z}^{(m)}\}_{m=0,1,\ldots}$ leads stability properties to the computed sequence $\{\underline{z}^{(m)*}\}_{m=0,1,\ldots}$. That is, the error can be reduced to the roundoff error generated by a single iteration when m is large enough. The stability properties of $\{\underline{z}^{(m)*}\}_{m=0,1,\ldots}$ imply that the reduced general method gives appreciably accurate results.

We also use Newton's method for the simple expression of the Jacobian matrix $J_{\underline{G}_n}$, that is:

$$\left(J_{\underline{G}_n}(z)\right)_{i,i} = \sum_{j\neq i,\ j=1}^{\nu} \frac{z_j}{(z_i - z_j)^2} + A(z_i), \quad i = 1, \ldots, \nu, \tag{31}$$

$$\left(J_{\underline{G}_n}(z)\right)_{i,j} = -\frac{z_i}{(z_i - z_j)^2}, \quad i \neq j,\ i, j = 1, 2, \ldots, \nu, \tag{32}$$

and of its symmetrization:

$$\left(\widetilde{J}_{\underline{G}_n}(z)\right)_{i,j} = \frac{z_j}{z_i}\left(J_{\underline{G}_n}(z)\right)_{i,j}, \quad i, j = 1, 2, \ldots, \nu, \tag{33}$$

where $A(s)$, $s \in \mathbf{R}$, is given by

$$A(s) = Q(s) + s\frac{\mathrm{d}Q}{\mathrm{d}s}(s), \quad s \in \mathbf{R}. \tag{34}$$

As we will see later, the Jacobian matrix $J_{\underline{G}_n}(z)$ has further properties when the polynomial solutions to (1) belong to systems of orthogonal polynomials.

We conclude this section underlining that the introduced method is widely applicable. In fact, different systems of polynomials satisfy differential equations (1) and conditions (18), (19), (21) and (24) (see [16, theorem 6.8, p. 151]). For example, there are the classical systems of orthogonal polynomials such as Hermite polynomials, ultraspherical polynomials (see [1,2]), the class of polynomials orthogonal with respect to the discrete Sobolev inner product (see [4–6]) and the class of Freud polynomials (see [9, theorem 4.20.3, p. 130]). It is worth noting that, in most cases, the Jacobian matrices are Stieltjes matrices. That is, the method is quadratically convergent.

3. Two classes of orthogonal polynomials

We apply the method on two test cases: the polynomial systems introduced in [4, section 5] and in [9, section 4.21].

3.1. Test 1: Hermite–Sobolev type polynomials

Let λ be a positive constant, $\mathcal{C}^1[-\infty, +\infty]$ be the linear space of the real valued continuous differentiable functions defined in $[-\infty, +\infty]$, and f, $g \in \mathcal{C}^1[-\infty, +\infty]$.

We denote with $\{Q_n^\lambda\}_{n\in\mathbf{N}}$ the system of monic polynomials orthogonal with respect to the inner product (2), that is:

$$\tilde{\varphi}_\lambda(f,g) = \int_{-\infty}^{+\infty} f(z)g(z)\mathrm{e}^{-z^2}\,\mathrm{d}z + \lambda f'(0)g'(0).$$

Let $\{H_n\}_{n\in\mathbf{N}}$ be Hermite monic polynomial system, the following relation hold:

$$\left\{Q_n^0\right\}_{n\in\mathbf{N}} \equiv \{H_n\}_{n\in\mathbf{N}}.$$

The associated differential equation (1) is:

$$\begin{aligned}
&z^2\Delta(z)u'' + z\left[2\left(1-z^2\right)\Delta(z) - z\Delta'(z)\right]u' + \left\{2n(1+2\mu_n)z^4\right.\\
&\quad\left. + 2\mu_n\left[(n-2)(2n-2)\mu_n - 5n + 2\right]z^2 + 2\mu_n\left((2n-2)\mu_n - 3\right)\right\}u = 0, \quad (35)\\
&\Delta(z) = (1+2\mu_n)z^2 + \mu_n\left((2n-2)\mu_n - 3\right), \quad \mu_1 = 0,\\
&\mu_n = \frac{\alpha_n}{n-1}, \quad n > 1, \qquad (36)
\end{aligned}$$

$$\alpha_0 = \alpha_1 = 0, \quad \alpha_{2m} = 0, \quad \alpha_{2m+1} = \frac{\frac{2\lambda}{\sqrt{\pi}}m}{\frac{2^{2m}m!}{(2m+1)(2m)\ldots(m+1)} + \frac{4\lambda m}{3\sqrt{\pi}}}, \quad m \geqslant 1, \qquad (37)$$

and we have $k = 4$. As shown in [4], we have

$$z^2 Q_n^\lambda(z) = \left(z^2 - \frac{\alpha_n}{n-1}\right)H_n(z) + \frac{n\alpha_n}{n-1}zH_{n-1}(z), \quad n > 1,\ z \in \mathbf{R}. \qquad (38)$$

The relation (38) implies $Q_n^\lambda \in \mathbf{P}_S^n$, $\forall\lambda > 0$. That is, Q_n^λ is an even or odd polynomial according to whether n is even or odd. The zeros of the polynomials $\{Q_n^\lambda\}_{n\in\mathbf{N}}$ are real, distinct and symmetrically arranged with respect to the origin (see [6, Proposition 4.4, p. 750]) so that equation (11) can be used.

Let be

$$\delta_n = \left(3 - (2n-2)\mu_n\right)\mu_n, \qquad \tau_n = 1 + 2\mu_n, \quad n \in \mathbf{N}^+. \qquad (39)$$

It is easy to see that we have

$$\alpha_n < \frac{3}{2}, \qquad 0 < \delta_n \leqslant \frac{9}{2(n-1)}, \quad n > 1,$$

and when the parameter λ goes to $+\infty$, we have

$$\begin{aligned}
&\alpha_{2\nu+1} \to \frac{3}{2}, \qquad \mu_{2\nu+1} \to \frac{3}{4\nu}, \qquad \delta_{2\nu+1} \to 0, \qquad \tau_{2\nu+1} \to 1 + \frac{3}{2\nu},\\
&\quad \nu \geqslant 1. \qquad (40)
\end{aligned}$$

As a consequence of theorem 2.1, we can prove the following results interesting in the approximation theory.

Theorem 3.1. For any choice of $\lambda > 0$, we have that

$$J_{\underline{G}_n}(\underline{z}), \quad \underline{z} \in \mathcal{D}, \quad z_i \in \left(\delta_n/\tau_n + \sqrt{\delta_n/\tau_n}, +\infty\right), \quad i = 1, 2, \ldots, \nu,$$

is a strictly diagonally dominant matrix with positive diagonal entries and negative off-diagonal entries. Let $\underline{\zeta}_n^* = (\zeta_{n,1}^2, \zeta_{n,2}^2, \ldots, \zeta_{n,\nu}^2)$, $\zeta_{n,1} < \zeta_{n,2} < \cdots < \zeta_{n,\nu}$, $n = 2\nu + 1$, $\nu \geqslant 1$, be the squares of the positive roots of the n-degree polynomial solution to (35). Then we have

$$\zeta_{n,1}^2 > \xi_{\inf}^2 := \frac{\delta_n}{2\tau_n} + \frac{3}{4} - \frac{\sqrt{(3\tau_n + 2\delta_n)^2 - 40\delta_n\tau_n}}{4\tau_n}, \tag{41}$$

$$\zeta_{n,\nu}^2 > \xi_{\sup}^2 := \frac{\delta_n}{2\tau_n} + \frac{3}{4} + \frac{\sqrt{(3\tau_n + 2\delta_n)^2 - 40\delta_n\tau_n}}{4\tau_n}, \quad \text{and} \tag{42}$$

$$\zeta_{n,2}^2 > \zeta_{n,1}^2 + \frac{1}{3/(4\zeta_{n,1}^2) - Q(\zeta_{n,1}^2)}, \tag{43}$$

where Q is the function defined in (15). Finally, there exist two positive constants λ_0, λ_1 such that for any $\lambda \leqslant \lambda_0$ and $\lambda \geqslant \lambda_1$ we have that $\det J_{\underline{G}_n}(\underline{\zeta}_n^*) \neq 0$.

Theorem 3.2. Let $n = 2\nu + 1$, $\nu \in \mathbf{N}^+$, and let Q_n^λ be the polynomial satisfying equations (35) and (38). Then the root $z = 0$ of Q_n^λ becomes a triple root when $\lambda \to +\infty$.

We note that when n is even, i.e., $n = 2\nu$, we have that $\alpha_{2\nu} = 0$ and, consequently,

$$Q_{2\nu}^\lambda(z) = H_{2\nu}(z), \quad z \in \mathbf{R}, \lambda > 0. \tag{44}$$

On the other hand, when n is odd (i.e., $n = 2\nu + 1$) we have

$$\lim_{\nu \to +\infty} \alpha_{2\nu+1} = 0, \quad \lambda > 0. \tag{45}$$

Relations (44) and (45) suggest to us the use of the same starting point procedure as in Hermite polynomial case [2].

3.2. Test 2: Freud polynomials

Let $\mathcal{C}^0[-\infty, +\infty]$ be the linear space of the real valued continuous functions defined in $[-\infty, +\infty]$, and $f, g \in \mathcal{C}^0[-\infty, +\infty]$. We denote with $\{P_n^F\}_{n\in\mathbf{N}}$ the system of polynomials orthogonal with respect to the inner product $\tilde{\varphi}(\cdot, \cdot)$ (see (3)) given by

$$\tilde{\varphi}(f, g) = \int_{-\infty}^{+\infty} f(z)g(z)\mathrm{e}^{-z^4}\,\mathrm{d}z. \tag{46}$$

The associated differential equation (1) is

$$\varphi_n(z)u'' - \left(4z^3\varphi_n(z) + 2z\right)u' + 4s_n^2\left\{4\varphi_n^2(z)\varphi_{n-1}(z) + \varphi_n(z) - 4s_n^2z^2\varphi_n(z) - 4z^4\varphi_n(z) - 2z^2\right\}u = 0,$$

with

$$\varphi_n(z) = z^2 + s_n^2 + s_{n+1}^2 \quad \text{and}$$
$$n = 4s_n^2\big(s_{n+1}^2 + s_n^2 + s_{n-1}^2\big), \quad n = 1, 2, \ldots, \ s_0^2 = 0, \ s_1^2 = \Gamma(3/4)/\Gamma(1/4), \tag{47}$$

where $\Gamma(\cdot)$ is the Gamma function. We have $k = 4$. As shown in [9, formula (3.7) and theorem 4.20.1], the following recurrence relation is satisfied by $\{P_n^F\}_{n\in\mathbf{N}}$:

$$zP_n^F(z) = s_{n+1}P_{n+1}^F(z) + s_n P_{n-1}^F(z). \tag{48}$$

The roots of the orthogonal polynomials P_n^F are symmetrically arranged with respect to the origin as a consequence of (46) and (48).

The recurrence relation (47) has a unique nonnegative solution. Theoretically from s_0 and s_1 we can determine s_n via (47). This forward iteration is an exponentially unstable algorithm. A stable algorithm to compute the sequence $\{s_n\}_{n\in\mathbf{N}}$ is provided in [17, section 7]. This algorithm can be too time consuming. We note that the following expansion of the coefficient s_n^2 holds (see [17, pp. 373–374]):

$$s_n^2 = \tilde{s}_n^2 + \mathrm{O}\big(n^{-4-1/2}\big), \quad n \to +\infty, \quad \text{with} \tag{49}$$
$$\tilde{s}_n^2 = \sqrt{\frac{n}{12}}\left(1 + \frac{1}{24}\frac{1}{n^2} - \frac{7}{576}\frac{1}{n^4} + \mathrm{O}\left(\frac{1}{n^6}\right)\right), \quad n \to +\infty,$$

where $\mathrm{O}(\cdot)$ denotes the Landau symbol. The existence of the expansion (49) provides a fast computational algorithm that yields accurate values for s_n^2 for larger n (see [17, sections 7 and 9]). This fact plays in favour of the GRM algorithm. In fact, the GRM algorithm requires only the knowledge of $s_n^2+s_{n+1}^2$ to compute the roots of the orthogonal polynomial P_n^F. On the contrary, the classical method of Jacobi matrices needs the coefficients s_i, $i = 1, 2, \ldots, n+1$, as shown in formula (48). So that the GRM algorithm is preferable to the classical approach to compute the roots of P_n^F when n is large enough (see section 5, table 3). Moreover, the GRM algorithm is quadratically convergent as shown in the following theorem.

Theorem 3.3. Let $\underline{G}_n(\underline{z})$, $\underline{z} \in \mathcal{D}$, be the vector function defined in (11) associated with the system $\{P_n^F\}_{n\in\mathbf{N}}$, then the Jacobian matrix $J_{\underline{G}_n}(\underline{z})$ is a Stieltjes matrix for any $\underline{z} \in \mathcal{D}$.

Let $\nu = [n/2]$. We conclude this section recalling some relations on the greatest root $\zeta_{n,\nu}$ of P_n^F (see [9, theorem 4.18.7, p. 111, and formula (4.18.6), p. 106]), that we use to test the GRM algorithm. That is,

$$\lim_{n\to+\infty} n^{-1/4}\zeta_{n,\nu} = 2\left(\frac{1}{12}\right)^{1/4} \quad \text{and} \tag{50}$$
$$\zeta_{n,\nu} \geqslant s_{n-1}, \quad n = 1, 2, \ldots. \tag{51}$$

4. The alternative method

The alternative method is based on transforming the problem of approaching the zeros of a polynomial into that of approaching the eigenvalues of a suitable matrix associated to the polynomial itself.

4.1. Test 1

The matrix we associate with the system of test 1 (see section 3), comes from the recurrence relation satisfied by $\{Q_n^\lambda\}_{n\in\mathbf{N}}$, as shown in [4]. Let $\{\alpha_n\}_{n\in\mathbf{N}}$ be given by (37) and let $\{g_n\}_{n\in\mathbf{N}}$ and $\{l_n\}_{n\in\mathbf{N}}$ be the following sequences:

$$g_0 = g_1 = 0, \quad g_{2\nu} = 0, \quad \nu \geqslant 1,$$

$$g_{2\nu+1} = -\frac{2}{\sqrt{\pi}} \frac{(2\nu-1)(2\nu-2)\cdots(\nu+1)}{2^{2(\nu-1)}(\nu-1)!}, \quad \nu \geqslant 1,$$

$$l_0 = 1, \quad l_1 = 1 + \frac{2\lambda}{\sqrt{\pi}}, \quad l_{2\nu} = l_{2\nu-1}, \quad \nu \geqslant 1, \tag{52}$$

$$l_{2\nu+1} = l_{2\nu} + \frac{\frac{2\lambda}{\sqrt{\pi}}\nu}{\frac{2^{2\nu}\nu!}{(2\nu+1)(2\nu)\cdots(\nu+1)} + \frac{4\lambda\nu}{3\sqrt{\pi}}} = l_{2\nu-1} + \alpha_{2\nu+1}, \quad \nu \geqslant 1. \tag{53}$$

That is,

$$l_{2\nu+1} = l_1 + \sum_{j=1}^{\nu} \alpha_{2j+1}. \tag{54}$$

Then we have the following five-term recurrence relation for $\{Q_n^\lambda\}_{n\in\mathbf{N}}$:

$$z^2 Q_n^\lambda(z) = Q_{n+2}^\lambda(z) + \gamma_{n,n} Q_n^\lambda(z) + \gamma_{n,n-2} Q_{n-2}^\lambda(z), \quad z \in \mathbf{R},\ n \geqslant 1, \tag{55}$$

$$Q_0^\lambda(z) = 1, \qquad Q_{-1}^\lambda(z) = 0, \qquad Q_{-2}^\lambda(z) = 0, \qquad z \in \mathbf{R},$$

where $\{\gamma_{n,n}\}_{n\in\mathbf{N}}$ and $\{\gamma_{n,n-2}\}_{n\in\mathbf{N}}$ are given by

$$\gamma_{0,0} = \frac{1}{2}, \qquad \gamma_{1,1} = \frac{3}{2} - \alpha_3, \qquad \gamma_{2,2} = \frac{5}{2}, \tag{56}$$

$$\gamma_{n,n} = \frac{l_{n-1}}{l_n}\left(\alpha_n + n + \frac{1}{2}\right) - \frac{\lambda}{l_n} g_n \left(\frac{n(n-1+2\alpha_n)}{4}\right), \quad n \geqslant 3, \tag{57}$$

$$\gamma_{0,-2} = 0, \qquad \gamma_{1,-1} = 0, \qquad \gamma_{2,0} = \frac{1}{2}, \tag{58}$$

$$\gamma_{n,n-2} = \frac{n(n-1)}{4} \frac{l_n}{l_{n-1}} \frac{l_{n-3}}{l_{n-2}}, \quad n \geqslant 3. \tag{59}$$

It is easy to see that $\lim_{m\to+\infty} l_{2m+1} = +\infty$.

The recurrence relation (55) can be rewritten in a vector form distinguishing the cases $n = 2\nu$, $n = 2\nu + 1$. Let c be a constant, and $h \in \mathbf{N}$ be the integer defined by $h = n - 2[n/2]$. We have

$$z^2 \begin{pmatrix} Q_h^\lambda \\ Q_{h+2}^\lambda \\ Q_{h+4}^\lambda \\ \vdots \\ Q_{n-4}^\lambda \\ Q_{n-2}^\lambda \end{pmatrix} - c \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ Q_n^\lambda \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_{h,h} & 1 & 0 & 0 & \dots & 0 \\ \gamma_{h+2,h} & \gamma_{h+2,h+2} & 1 & 0 & \dots & 0 \\ 0 & \gamma_{h+4,h+2} & \gamma_{h+4,h+4} & 1 & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & \gamma_{n-4,n-6} & \gamma_{n-4,n-4} & 1 \\ 0 & \dots & \dots & \dots & \gamma_{n-2,n-4} & \gamma_{n-2,n-2} \end{pmatrix} \begin{pmatrix} Q_h^\lambda \\ Q_{h+2}^\lambda \\ Q_{h+4}^\lambda \\ \vdots \\ Q_{n-4}^\lambda \\ Q_{n-2}^\lambda \end{pmatrix}. \quad (60)$$

In section 5, we show the ill-conditioning of the eigenvalues of the matrix (60). This ill conditioning gives serious consequences when n is large (see figure 1). To compute the eigenvalues of the matrix in (60), the QR algorithm for Hessemberg matrices [18] is used.

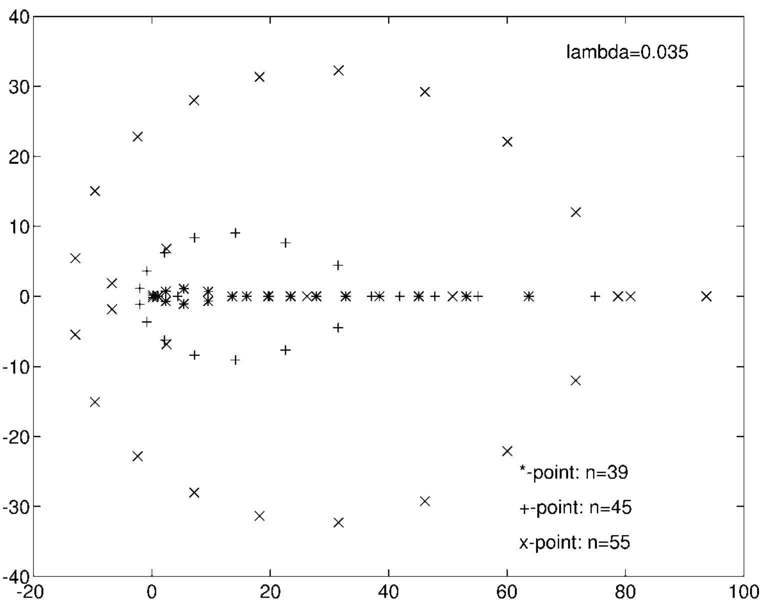

Figure 1.

From relation (60) we obtain that the Jacobi matrix associated to the system is the following real symmetric tridiagonal matrix:

$$\begin{pmatrix} \gamma_{h,h} & \sqrt{\gamma_{h+2,h}} & 0 & 0 & \dots & 0 \\ \sqrt{\gamma_{h+2,h}} & \gamma_{h+2,h+2} & \sqrt{\gamma_{h+4,h+2}} & 0 & \dots & 0 \\ 0 & \sqrt{\gamma_{h+4,h+2}} & \gamma_{h+4,h+4} & \sqrt{\gamma_{h+6,h+4}} & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & \sqrt{\gamma_{n-4,n-6}} & \gamma_{n-4,n-4} & \sqrt{\gamma_{n-2,n-4}} \\ 0 & \dots & \dots & \dots & \sqrt{\gamma_{n-2,n-4}} & \gamma_{n-2,n-2} \end{pmatrix}. \tag{61}$$

We use both the QL algorithm [19–21] and the QR Lapack subroutine to compute the eigenvalues of the matrix (61). In the following lemma, we show the ill conditioning of the coefficients $\{l_{2\nu+1}\}_{\nu\in\mathbf{N}}$ given by (52) and (53).

Lemma 4.1. Let $\{l_{2\nu+1}\}_{\nu\in\mathbf{N}}$ be the sequence defined by (52) and (53), and let $\{\tilde{l}_{2\nu+1}\}_{\nu\in\mathbf{N}}$ be the following sequence:

$$\tilde{l}_1 = l_1 = 1 + \frac{2\lambda}{\sqrt{\pi}}, \qquad \tilde{l}_{2\nu+1} = \left[\tilde{l}_{2\nu-1} + \alpha_{2\nu+1}\right](1+\tilde{\varepsilon}_\nu), \quad \nu = 1,2,\dots, \tag{62}$$

where $|\tilde{\varepsilon}_\nu| < \varepsilon$, $\nu = 1,2,\dots$, and ε is positive constant, $0 < \varepsilon \ll 1$. Then we have

$$\begin{gathered} \left|\tilde{l}_{2\nu+1} - l_{2\nu+1}\right| = \left|\sum_{j=1}^{\nu} l_{2j+1}\tilde{\varepsilon}_j + \mathrm{O}\left(\sum_{i,j}^{\nu} \tilde{\varepsilon}_i\tilde{\varepsilon}_j\right)\right|, \\ \tilde{\varepsilon}_j \to 0, \quad j = 1,2,\dots,\nu, \quad \text{and} \end{gathered} \tag{63}$$

$$\left|\frac{\tilde{l}_{2\nu+1} - l_{2\nu+1}}{l_{2\nu+1}}\right| \approx \varepsilon\mathrm{O}(\nu), \quad \nu \to +\infty, \quad \lambda \to +\infty. \tag{64}$$

Theorem 3.2, lemma 4.1 and formulae (57), (59) give reason for the failure of the alternative method in the computation of the root of Q_n^λ when n is large and λ goes to infinity (see table 1).

4.2. Test 2

The alternative method in the case of Freud polynomials consists of computing the eigenvalues of the following matrix:

$$\begin{pmatrix} 0 & s_1 & 0 & 0 & \dots & 0 \\ s_1 & 0 & s_2 & 0 & \dots & 0 \\ 0 & s_2 & 0 & s_3 & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & s_{n-2} & 0 & s_{n-1} \\ 0 & \dots & \dots & \dots & s_{n-1} & 0 \end{pmatrix}. \tag{65}$$

Table 1

λ	n	ε_n	$E_n(\underline{z}^*)$	$E_n(\underline{y}^*)$	$z_1^* - \xi_{\inf}$	$y_1^* - \xi_{\inf}$	z_1^*	y_1^*
0.01	199	6.993e–14	5.685e–13	5.752e–12	1.98e–03	1.98e–03	6.164e–02	6.164e–02
	299	2.125e–12	1.002e–12	3.478e–10	6.85e–04	6.85e–04	3.839e–02	3.839e–02
	399	7.583e–13	1.578e–12	1.861e–10	3.18e–04	3.18e–04	2.719e–02	2.719e–02
10	199	4.926e–12	6.960e–12	1.753e–08	7.00e–08	7.00e–08	2.108e–03	2.108e–03
	299	2.275e–11	1.251e–11	1.344e–07	2.30e–08	2.30e–08	1.269e–03	1.269e–03
	399	1.556e–11	1.940e–11	1.319e–07	1.05e–08	1.05e–08	8.851e–04	8.851e–04
10^4	199	7.667e–09	2.454e–10	8.624e–04	2.21e–12	2.72e–12	6.668e–05	6.668e–05
	299	2.146e–08	4.679e–10	4.012e–03	7.28e–13	1.59e–12	4.012e–05	4.012e–05
	399	4.298e–08	6.932e–10	1.152e–02	3.30e–13	1.53e–12	2.799e–05	2.799e–05
10^8	199	6.124e–05	2.886e–08	6.891e+02	2.50e–14	−4.08e–11	6.668e–07	6.667e–07
	299	1.680e–05	5.438e–08	3.143e+03	−2.46e–14	−6.74e–11	4.012e–07	4.011e–07
	399	3.456e–04	7.844e–10	9.279e+03	−5.89e–14	−9.68e–11	2.799e–07	2.798e–07

As observed in section 3, the computation of the roots of P_n^F involves the computation of the coefficients s_i, $i = 1, 2, \ldots, n-1$ (see (47)). Since this computation needs particular care when the index i is not large, the alternative method could give poor results.

5. The comparison

In this section we discuss the performance of the GRM algorithm and that of the QR type algorithms for symmetric matrices when applied to compute the eigenvalues of the Jacobi matrices defined by (61) or (65). We consider also the HQR method [18] applied to compute the eigenvalues of the matrices in (60). The algorithms are compared in the case of orthogonal polynomials described in section 3, which we refer to as MR polynomials (test 1) and FR polynomials (test 2). We consider only the MR polynomials with an odd degree since the MR polynomials with an even degree are Hermite polynomials. Numerical experiments involving Hermite polynomials can be found in [1,2].

All tests are carried out on a Pentium II using FORTRAN 77 standard language in double precision. To apply the general reduced method (GRM), we need $\underline{z}_0$ as a starting point for Newton's method. It is simple to provide a starting point. In particular, we use the starting point described in [2] for Hermite polynomials both for MR polynomials and for FR polynomials. With this choice of $\underline{z}_0$ the number of iterations required by Newton's method to satisfy the stopping rule is kept low. The number of iterations increases from 6 to 9 when $5 \leqslant n \leqslant 450$ in every attempted experiment.

First, we deal with the MR polynomials since the method in section 4 shows two interesting phenomena. The first is due to the ill-conditioning of the eigenvalues of the matrix in (60), and the other is due, more interestingly, to the errors in the entries of the matrix (61) when λ goes to infinity and n is large enough.

The first experiment involves the computation of the eigenvalues of the matrix (60) when $\lambda = 0.035$ and $n = 39, 45, 55$. We use the HQR routine [18]. Figure 1 shows a phenomenon of bifurcation of the computed eigenvalues. That is the computed eigenvalues are complex numbers and this is not possible since all of the zeros of Q_n^λ are real (see [6, proposition 4.4]). In the x-axes and y-axes of figure 1, we have the real part and the imaginary part of the eigenvalues, respectively. The results obtained by the HQR algorithm applied to the matrix (60) agree with those obtained by the GRM algorithm only for the small values of λ, $0 \leqslant \lambda \leqslant 0.02$, and for small the values of n, $1 \leqslant n \leqslant 35$.

The other experiments on the MR polynomials are carried out using the symmetric matrix (61). We used both the QL routine [16] and the QR Lapack routine in every experiment and the results obtained agree to at least ten decimal digits. We chose to use both algorithms to check if the observed phenomena depend on the computation of the coefficients of matrix (61) and/or on the QR type algorithm used. The results obtained using the GRM algorithm and the QR type algorithms sufficiently agree when $\lambda \leqslant 1000$ (see table 1).

Let $n = 2\nu + 1$, $\nu \in \mathbf{N}^+$, $\underline{z}^* = (z_1^*, z_2^*, \ldots, z_\nu^*, 0, -z_\nu^*, \ldots, -z_2^*, -z_1^*)^{\mathrm{T}} \in \mathbf{R}^n$, $0 < z_1^* < z_2^* < \cdots < z_\nu^*$, and $\underline{y}^* = (y_1^*, y_2^*, \ldots, y_\nu^*, 0, -y_\nu^*, \ldots, -y_2^*, -y_1^*)^{\mathrm{T}} \in \mathbf{R}^n$, $0 < y_1^* < y_2^* < \cdots < y_\nu^*$ be the approximations of the zeros of the MR polynomial Q_n^λ computed by the GRM and the alternative algorithm, respectively. Let ε_n be the following quantity:

$$\varepsilon_n = \max_{1 \leqslant i \leqslant \nu} \frac{|z_i^* - y_i^*|}{|z_i^*|}. \tag{66}$$

And, finally, for $\underline{z} = (z_1, z_2, \ldots, z_n)^{\mathrm{T}} \in \mathbf{R}^n$, let $E_n(\underline{z})$ be the function defined by

$$E_n(\underline{z}) = \max_{1 \leqslant i \leqslant n} \left| \sum_{j=1,\ j \neq i}^{n} \frac{1}{z_i - z_j} - \frac{a_1(z_i)}{2a_2(z_i)} \right|.$$

We note that $E_n(\underline{z})$ should be equal to zero when $\underline{z} = \underline{\zeta}_n$. That is, $\underline{z}$ is the vector whose components are the roots of the MR polynomials. This implies that $E_n(\underline{z}^*)$ and $E_n(\underline{y}^*)$ should be near zero. Let $\xi_{\inf}$ be given in formula (41). In table 1 we show the behaviour of ε_n^λ, $E_n(\underline{z}^*)$, $E_n(\underline{y}^*)$, $z_1^* - \xi_{\inf}$, $y_1^* - \xi_{\inf}$, z_1^* and y_1^* for $n = 199, 299, 399$ and $\lambda = 0.01, 10, 10^4, 10^8$. We can see that $E_n(\underline{y}^*)$ grows when λ goes to infinity and the quantity ε_n increases. This phenomenon is due to the fact that the smallest positive root goes to zero when λ goes to infinity (see theorem 3.2) and to the fact that the error in the computation of the entries of the Jacobi matrix increases when n becomes large and λ goes to infinity (see lemma 4.1 and formulae (56)–(59)).

Let λ' and η be two positive constants ($\lambda' \geqslant 10$). Table 1 shows that the smallest positive root of the MR polynomial with $\lambda = \lambda'\eta$ is equal to the smallest positive root of the MR polynomial with $\lambda = \lambda'$ multiplied by $\sqrt{\eta}$.

In tables 2 and 3 we show some results concerning test 2, the FR polynomials. In table 2 we show the values of ε_n (see formula (66)), with $n = 4, 5, 8, 9, 16, 19, 20$, obtained by computing $\underline{z}^*$ with the GRM algorithm and $\underline{y}^*$ with the alternative algorithm (QR Lapack subroutine). We consider the values of n less than 20 since the values of the coefficients s_i, $i = 1, 2, \ldots, 20$ (see (47)), are tabulated in [17, p. 378]. These coefficients have sixteen correct decimal digits. These values have been computed with

Table 2

n	ε_n
4	1.401e–16
5	1.694e–16
8	4.410e–16
9	2.596e–16
16	7.200e–16
17	2.669e–16
19	1.025e–15
20	7.136e–10

Table 3

n	d_n	$z_\nu^{*2} - s_{n-1}^2$	χ_n
55	−5.392e–02	5.604	6.442
100	−3.613e–02	7.911	8.675
199	−2.284e–02	11.542	12.227
300	−1.739e–02	14.366	15.008
399	−1.438e–02	16.693	17.310
450	−1.328e–02	17.773	18.378

the algorithm proposed in [17, section 7] using some computational refinements (see [17, section 9]).

The results in table 2 are obtained applying the GRM algorithm and using formula (49). Let us consider formulae (50) and (51). We denote with d_n and χ_n the following quantities:

$$d_n = z_\nu^* - 2\left(\frac{n}{12}\right)^{1/4} \quad \text{and}$$

$$\chi_n = \frac{15n+1}{\sqrt{12}(4\sqrt{n} + \sqrt{n-1})} = 2s_n - s_{n-1} + e(n), \qquad \lim_{n\to+\infty} e(n) = 0.$$

In table 3 we give the values of n (first column), d_n (second column), of $z_\nu^{*2} - s_{n-1}^2$ (third column) and of χ_n (last column). Note that by virtue of formula (50), we have

$$\lim_{n\to+\infty} d_n = \lim_{n\to+\infty} \chi_n.$$

6. Proofs

Proof of theorem 2.1. The proof is similar to the proof of theorem 2.1 in [3]. The proof of (20) follows differentiating (16) and using the relation $\underline{F}_n(\underline{\zeta}_n) = \underline{0}$. □

Proof of corollary 2.2. By virtue of (18) and (19), the point $\underline{\zeta}_n^* \in \mathcal{D}$ in (12). From (1) and (21) we have

$$\frac{p_n''}{2p_n'}(\zeta_{n,i}) - R(\zeta_{n,i}) = 0, \quad i = 1, \ldots, n, \tag{67}$$

where R is the rational function given by (10). Taking (21) into account, the relation (67) can be expressed as follows:

$$-\frac{(2+(-1)^{n+1})}{4\zeta_{n,i}^2} - \sum_{j=1,\ i\neq j}^{\nu} \frac{1}{\zeta_{n,i}^2 - \zeta_{n,j}^2} = Q(\zeta_{n,i}), \quad i = 1, 2, \ldots, \nu, \tag{68}$$

where Q is the rational function introduced in (15). This concludes the first part of the assertion.

Let (18), (19) and (21)–(23) hold. Since $\underline{\zeta}_n^* \in \mathcal{D}$, we have $\underline{\zeta}_n \in \mathcal{A}$ so that equality (67) follows from (68) and can be rewritten as

$$\big(a_2 p_n'' + a_1 p_n'\big)(\zeta_{n,i}) = 0, \quad i = 1, 2, \ldots, n.$$

The polynomial $a_2 p_n'' + a_1 p_n'$ vanishes at the zeros of p_n. Thus an $a_0 \in \mathbf{P}_R$ must exist such that $a_2 p_n'' + a_1 p_n' = -a_0 p_n$. This concludes the proof. □

Proof of lemma 2.3. Let $p_n \in \operatorname{Ker}(L) \cap \mathbf{P}_R$ be the lower degree solution to (1). If $p_n \in \mathbf{P}_S$, the proof is trivial. If $p_n \notin \mathbf{P}_S$, the proof is a straightforward consequence of (4) or (5) taking into account that the polynomial $v_n(z) = p_n(z) + p_n(-z)$ is a solution to (1) when (4) holds, and the polynomial $v_n(z) = p_n(z) - p_n(-z)$ is a solution to (1) when (5) holds. □

Now we give some notation and results needed for the proofs of theorems 2.6 and 2.7. We denote with $\omega_{n;i_1,i_2,\ldots,i_h}$, $i_j \in \mathbf{N}^+$, $i_1 \neq i_2 \neq \cdots \neq i_h$, $i \leqslant h \leqslant n$, the monic polynomial of degree $n-h$ defined by

$$\omega_{n;i_1,i_2,\ldots,i_h}(z) := \frac{\omega_n(z)}{\prod_{j=1}^h (z - z_{i_j})}, \quad z \in \mathbf{R}, \quad \text{where} \quad \omega_n(z) = \prod_{i=1}^n (z - z_i). \tag{69}$$

We denote with $\omega_n^{(h)}(z_i)$, the hth derivative of ω_n at the point z_i, and we regard it as a function of all coordinates $z_1, \ldots, z_n$ of $\underline{z}$. In this particular case we denote this function $\Omega_{h,i}$:

$$\Omega_{h,i} : \mathbf{R}^n \to \mathbf{R}, \quad \Omega_{h,i}(z) := \omega_n^{(h)}(z_i).$$

From (69) and (17), we denote with $p_{n;i_1,\ldots,i_h}$ the polynomial defined by

$$p_{n;i_1,\ldots,i_h}(z) := \frac{p_n(z)}{\prod_{j=1}^h (z - \zeta_{n,i_j})}. \tag{70}$$

The following two results can be found in [3]. We refer to them for the sake of clarity.

Lemma 6.1. We have

$$\omega_n^{(h)}(z_i) = h\omega_{n;i}^{(h-1)}(z_i), \qquad \frac{\partial \Omega_{h,i}}{\partial \underline{z}_j}(z) = h\omega_{n;i,j}^{(h-1)}(z_i), \quad i \neq j.$$

Proof. See [3, lemma 6.1]. □

Lemma 6.2. Let (16), (18) and (19) hold. Then we have

$$\big(J_{\underline{F}_n}(\underline{\zeta}_n)\big)_{i,j} = \mathcal{G}_i(\underline{\zeta}_n)(Lp_{n;j})(\zeta_{n,i}), \quad i, j = 1, 2, \ldots, n,$$

where $\mathcal{G}_i$, $i = 1, \ldots, n$, are the following functions:

$$\mathcal{G}_i : \mathcal{A} \subset \mathbf{C}^n \to \mathbf{C}, \quad G_i(z) := \frac{1}{2a_2(z_i)\Omega_{1,i}(z)} = \frac{1}{2a_2(z_i)\omega_n^{(1)}(z_i)}, \tag{71}$$

and $\mathcal{A}$ is the set in (7).

Proof. See [3, lemma 6.1]. □

Let $\nu = [n/2]$, $\zeta_{n,i}$, $i = 1, \ldots, \nu$, be given by (21)–(23), and $q_{n;j} \in \mathbf{P}_S^{n-2}$, $j = 1, \ldots, \nu$, be the polynomials defined by

$$q_{n;j} = \prod_{\substack{l=1 \\ l \neq j}}^{\nu} \left(z^2 - \zeta_{n,l}^2\right), \quad z \in \mathbf{R}, \quad j = 1, \ldots, \nu, \tag{72}$$

when n is even, and

$$q_{n;j} = z \prod_{\substack{l=1 \\ l \neq j}}^{\nu} \left(z^2 - \zeta_{n,l}^2\right), \quad z \in \mathbf{R}, \quad j = 1, \ldots, \nu, \tag{73}$$

when n is odd.

Lemma 6.3. Let conditions (18), (19) and (21)–(23) hold. Then we have

$$p_{n;j}(z) - p_{n;n-j+1}(z) = 2\zeta_{n,j} q_{n;j}(z), \quad z \in \mathbf{R}, \quad j = 1, \ldots, \nu, \tag{74}$$

and $q_{n;j}$, $j = 1, 2, \ldots, \nu$, are linearly independent polynomials.

Proof. Identity (74) follows from the definition (70) and the symmetry of the zeros. That is, $\zeta_{n,j} = -\zeta_{n,n-j+1}$, $j = 1, \ldots, \nu$. The polynomials $p_{n;j}$, $j = 1, \ldots, \nu$, are well-known to be linearly independent since we have $\zeta_{n,i} \neq \zeta_{n,j}$, $i \neq j$, so that $q_{n;j}$ are linearly independent polynomials as well. □

Lemma 6.4. Let conditions (18), (19) and (21)–(23) be satisfied and let $\underline{\zeta}_n^* = (\zeta_{n,1}^2, \ldots, \zeta_{n,\nu}^2) \in \mathcal{D}$ be a root of the function $\underline{G}_n$, given by (11). Let $\underline{\zeta}_n \in \mathcal{A}$ be a root of the function $\underline{F}_n$ (see (6)) given by

$$\underline{\zeta}_n = \begin{cases} (\zeta_{n,1}, \ldots, \zeta_{n,\nu}, -\zeta_{n,\nu}, \ldots, -\zeta_{n,1}), & \text{if } n \text{ is even}, \\ (\zeta_{n,1}, \ldots, \zeta_{n,\nu}, 0, -\zeta_{n,\nu}, \ldots, -\zeta_{n,1}), & \text{if } n \text{ is odd}. \end{cases}$$

Then we have

$$\begin{aligned} \left(J_{\underline{G}_n}\left(\underline{\zeta}_n^*\right)\right)_{i,j} &= \frac{\zeta_{n,i}\mathcal{G}_i(\underline{\zeta}_n)}{4\zeta_{n,j}}(Lp_{n;j} - Lp_{n;n-j+1})(\zeta_{n,i}) = \frac{1}{2}\zeta_{n,i}\mathcal{G}_i(\underline{\zeta}_n)Lq_{n;j}(\zeta_{n,i}), \\ &\quad i = 1, \ldots, \nu, \quad j = 1, \ldots, \nu, \end{aligned} \tag{75}$$

where $\mathcal{G}_i$ is given by (71) and $p_{n;j}$, $q_{n;j}$ by (70), (72) and (73), respectively.

Proof. First we consider $(J_{\underline{G}_n}(\underline{\zeta}_n^*))_{i,j}$, $i \neq j$, $i, j = 1, \dots \nu$. From the definition of the entries of $(J_{\underline{G}_n}(\underline{\zeta}_n^*))$ in (31), (32) and using $\zeta_{n,i} = -\zeta_{n,n-i+1}$, $i = 1, 2, \dots, n$, we get

$$\big(J_{\underline{G}_n}\big(\underline{\zeta}_n^*\big)\big)_{i,j} = \frac{-\zeta_{n,i}^2}{\big(\zeta_{n,i}^2 - \zeta_{n,j}^2\big)^2} = -\frac{1}{4}\frac{\zeta_{n,i}}{\zeta_{n,j}}\frac{1}{(\zeta_{n,i} - \zeta_{n,j})^2} + \frac{1}{4}\frac{\zeta_{n,i}}{\zeta_{n,j}}\frac{1}{(\zeta_{n,i} + \zeta_{n,j})^2}. \quad (76)$$

The Jacobian matrix of $\underline{F}_n$ is given by

$$\big(J_{\underline{F}_n}\big(\underline{\zeta}_n\big)\big)_{i,j} = -\frac{1}{(\zeta_{n,i} - \zeta_{n,j})^2}, \quad i \neq j,\ i, j = 1, 2, \dots, n, \quad (77)$$

$$\big(J_{\underline{F}_n}\big(\underline{\zeta}_n\big)\big)_{i,i} = R'(\zeta_{n,i}) - \sum_{j=1,\ j\neq i}^{n} \frac{1}{(\zeta_{n,i} - \zeta_{n,j})^2}, \quad i = 1, 2, \dots, n, \quad (78)$$

so that substituting (77) into (76) we obtain

$$\big(J_{\underline{G}_n}\big(\underline{\zeta}_n^*\big)\big)_{i,j} = \frac{1}{4}\frac{\zeta_{n,i}}{\zeta_{n,j}}\big(J_{\underline{F}_n}\big(\underline{\zeta}_n\big)\big)_{i,j} - \frac{1}{4}\frac{\zeta_{n,i}}{\zeta_{n,j}}\big(J_{\underline{F}_n}\big(\underline{\zeta}_n\big)\big)_{i,n-j+1}. \quad (79)$$

Identity (75) follows from (79), lemmata 6.2 and 6.3.

Now we consider the case $i = j$. From (32) we have

$$\big(J_{\underline{G}_n}\big(\underline{\zeta}_n^*\big)\big)_{i,i} = A\big(\zeta_{n,i}^2\big) + \sum_{j=1,\ j\neq i}^{\nu} \frac{\zeta_{n,j}^2}{\big(\zeta_{n,i}^2 - \zeta_{n,j}^2\big)^2}, \quad (80)$$

with A given by (34). From (34), we obtain

$$A\big(s^2\big) = \frac{1}{4}\frac{R(s)}{s} + \frac{1}{4}R'(s), \quad (81)$$

so that substituting (81) into (80) gives us

$$\big(J_{\underline{G}_n}\big(\underline{\zeta}_n^*\big)\big)_{i,i} = \frac{R(\zeta_{n,i})}{4\zeta_{n,i}} + \frac{1}{4}R'(\zeta_{n,i}) + \sum_{j=1,\ j\neq i}^{\nu} \frac{\zeta_{n,j}^2}{\big(\zeta_{n,i}^2 - \zeta_{n,j}^2\big)^2}. \quad (82)$$

Adding and subtracting $(1/4)\sum_{j\neq i}^{n} 1/(\zeta_{n,i} - \zeta_{n,j})^2$ to (82) and using (78), we obtain

$$\begin{aligned}\big(J_{\underline{G}_n}\big(\underline{\zeta}_n^*\big)\big)_{i,i} = {}& \frac{R(\zeta_{n,i})}{4\zeta_{n,i}} + \frac{1}{4}\big(J_{\underline{F}_n}\big(\underline{\zeta}_n\big)\big)_{i,i} - \frac{1}{4}\sum_{j=1,\ j\neq i}^{n} \frac{1}{(\zeta_{n,i} - \zeta_{n,j})^2} \\ & + \sum_{j=1,\ j\neq i}^{\nu} \frac{\zeta_{n,j}^2}{\big(\zeta_{n,i}^2 - \zeta_{n,j}^2\big)^2}.\end{aligned}$$

The thesis is a result of an easy computation when we apply lemmata 6.2 and 6.3 with the use of the equality $\zeta_{n,i} = -\zeta_{n,n-i+1}$, $i = 1, \dots, n$. $\square$

We underline two consequences of lemma 6.4.

Corollary 6.5. Let conditions (18), (19) and (21)–(23) be satisfied and L^* be the matrix defined by

$$L^* = [l_{i,j}], \quad l^*_{ij} = (Lq_{n;j})(\zeta_{n,i}), \quad i, j = 1, 2, \ldots, \nu, \tag{83}$$

where $\zeta_{n,i}$, $i = 1, \ldots, \nu$, are the positive roots of the polynomial solution $p_n \in \mathbf{P}^n_S$ to (1). Then we have

$$\operatorname{rank}\big(J_{\underline{G}_n}\big(\underline{\zeta}^*_n\big)\big) = \operatorname{rank}\big(L^*\big).$$

Finally, it is worth noting the following consequence of the definition (83) of the matrix L^*.

Lemma 6.6. Let L^* be the matrix in (83), let P belong to $L(\mathbf{P}^{n-2}_S)$, $w \in \mathbf{P}^{n-2}_S$ be a counter image of P and $\underline{c}^* =(c^*_1, \ldots, c^*_\nu) \in \mathbf{R}^\nu$ be the vector of the coefficients of w in the basis $\{q_{n;j}\}_{j=1,\ldots,\nu}$,

$$P := Lw \in L\big(\mathbf{P}^{n-2}_S\big), \quad w := \sum_{j=1}^{\nu} c^*_j q_{n;j} \in \mathbf{P}^{n-2}_S, \quad \underline{c}^* = \big(c^*_1, \ldots, c^*_\nu\big) \in \mathbf{R}^\nu.$$

Then $P(\zeta_{n,i}) = 0$, $P(-\zeta_{n,i}) = 0$, $i = 1, \ldots, \nu$, if and only if $L^*\underline{c}^* = \underline{0}$.

Proof. The proof is a direct consequence of the definition (83) of the matrix L^*. □

Proof of theorem 2.6. Let $Q^*_{n,k}$ be the set given by (29). We define $Q^*_{n,0} := \{0\}$, and $Q^*_{n,1} := \{0\}$, to unify the proof of the two assertions of the theorem. It is easy to see that for $k \geqslant 0$ we have

$$Lw = P \in Q^*_{n,k} \quad \Longleftrightarrow \quad P(\zeta_{n,i}) = 0, \quad i = 1, 2, \ldots, \nu.$$

Thus, by virtue of lemma 6.6, (28) implies $\det L^* \neq 0$ and vice versa. The proof follows from corollary 6.5. □

Proof of theorem 2.7. Let L^* be the matrix in (83). Then lemma 6.6 implies

$$\dim\big(\operatorname{Ker} L^*\big) = \dim\big(\big(\mathbf{P}^{n-2}_S\big) \cap Q^*_{n,k}\big) + d^*.$$

From corollary 6.5 and the standard argument of linear algebra we have

$$\begin{aligned}\nu &= \operatorname{rank}\big(L^*\big) + \dim\big(\ker L^*\big) = \operatorname{rank}\big(J_{\underline{G}_n}\big(\underline{\zeta}^*_n\big)\big) + \dim\big(L\big(\mathbf{P}^{n-2}_S\big) \cap Q^*_{n,k}\big) + d^* \\ &\leqslant \operatorname{rank}\big(J_{\underline{G}_n}\big(\underline{\zeta}^*_n\big)\big) + \left[\frac{k}{2}\right] + d^*.\end{aligned}$$

This proves the lower bound in the assertion. The upper bound is an obvious consequence of the following relation:

$$\nu = \operatorname{rank}\big(J_{\underline{G}_n}\big(\underline{\zeta}^*_n\big)\big) + \dim\big(L\big(\mathbf{P}^{n-2}_S\big) \cap Q^*_{n,k}\big) + d^* \geqslant \operatorname{rank}\big(J_{\underline{G}_n}\big(\underline{\zeta}^*_n\big)\big) + d^*.$$

This concludes the proof. □

Proof of theorem 3.1. Let $n = 2\nu + 1$, $\nu \geqslant 1$, and let $\underline{\zeta}_n^* = (\zeta_{n,1}^2, \dots, \zeta_{n,\nu}^2)^T \in \mathcal{D}$ be a root of the function $\underline{G}_n$ defined by (14). From equation $\underline{G}_n(\underline{\zeta}_n^*) = \underline{0}$, we obtain

$$\sum_{j=1,\ j\neq i}^{\nu} \frac{1}{\zeta_{n,i}^2 - \zeta_{n,j}^2} = -\frac{3}{4\zeta_{n,i}^2} + Q(\zeta_{n,i}^2), \quad i = 1, 2, \dots, \nu, \tag{84}$$

where $Q(s^2)$ is the function defined in (15). When we choose $i = 1$ in (84), we obtain

$$\frac{3}{4\zeta_{n,1}^2} > Q(\zeta_{n,1}^2). \tag{85}$$

From a simple computation, we have

$$a_1(s) = -2s(\tau_n s^4 - \delta_n s^2 + \delta_n), \qquad a_2(s) = s^2(\tau_n s^2 - \delta_n), \quad s > 0.$$

It is easy to see that $a_1(s) < 0$, $s > 0$, so that from relation (85) and the fact that $\zeta_{n,1}$ is the smallest positive root of an odd polynomial we have

$$\zeta_{n,1}^2 \in \left[\frac{\delta_n}{2\tau_n} + \frac{3}{4} - \frac{\sqrt{(3\tau_n + 2\delta_n)^2 - 40\delta_n\tau_n}}{4\tau_n}, \frac{\delta_n}{2\tau_n} + \frac{3}{4} + \frac{\sqrt{(3\tau_n + 2\delta_n)^2 - 40\delta_n\tau_n}}{4\tau_n}\right].$$

We see that formula (41) holds. Equation (84) implies the following relations:

$$\frac{3}{4\zeta_{n,\nu}^2} < Q(\zeta_{n,\nu}^2), \tag{86}$$

$$\frac{3}{4\zeta_{n,1}^2} > Q(\zeta_{n,1}^2) + \frac{1}{\zeta_{n,2}^2 - \zeta_{n,1}^2}. \tag{87}$$

Relations (42) and (43) follow from (86) and (87), respectively.

Now we prove that $J_{\underline{G}_n}(\underline{z})$, $\underline{z} = (z_1, z_2, \dots, z_\nu)^{\mathrm{T}} \in \mathcal{D}$, given in (31) and (32) is a diagonally dominant matrix when $z_i \in (\delta_n/\tau_n + \sqrt{\delta_n/\tau_n}, +\infty)$, $i = 1, 2, \dots, \nu$. Let $A(s)$, $s \in \mathcal{D}$, be given by (34) and let μ_n, α_n, δ_n, τ_n be given in (36), (37) and (39), respectively. From an easy computation, we have

$$A(s) = \frac{\tau_n^2 s^2 - 2\delta_n\tau_n s - \delta_n(\delta_n + \tau_n)}{2(\tau_n s - \delta_n)^2}.$$

The thesis of the first part of theorem 3.1 follows since $A(s)$ is a positive function when $s \in (\delta_n/\tau_n + \sqrt{\delta_n/\tau_n}, +\infty)$. The last assertion is proven through these three results: theorem 2.6, the fact that we have $A(s) = 1/2$, $s \in [0, +\infty)$ when $\lambda = 0$ or $\lambda \to +\infty$, and that the determinant of the following linear system in the unknowns w, q (see remark 2.9):

$$Lw(z) - q(z)Q_n^\lambda(z) = 0$$

is a rational function of λ (see formula (35) and [4, formula (5.5)]). □

Proof of theorem 3.2. This proof is a direct consequence of equations (1) and (38), relation (40) and the properties of Hermite polynomials. □

Proof of theorem 3.3. The thesis follows since we have $A(s)$, $s > 0$, as a positive function:

$$A(s) = \frac{4s^3 + 8(s_n^2 + s_{n+1}^2)s^2 + 4(s_n^2 + s_{n+1}^2)s + (s_n^2 + s_{n+1}^2)}{2(s + s_n^2 + s_{n+1}^2)^2}.$$

This concludes the proof. □

Proof of lemma 4.1. The proof of formula (63) follows from formulae (54) and (62) by recursion since we have

$$\tilde{l}_{2\nu+1} = l_{2\nu+1} + \tilde{\varepsilon}_1(l_1 + \alpha_3) + \tilde{\varepsilon}_2(l_1 + \alpha_3 + \alpha_5) + \cdots + \tilde{\varepsilon}_\nu\left(l_1 + \sum_{j=1}^{\nu}\alpha_{2j+1}\right) + \mathrm{O}\left(\sum_{i,j}^{\nu}\tilde{\varepsilon}_i\tilde{\varepsilon}_j\right).$$

Formula (64) is a direct consequence of the fact that

$$\lim_{\lambda\to+\infty}\frac{l_{2j+1}}{l_{2\nu+1}} = 1, \quad j = 1, 2, \ldots, \nu.$$ □

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