Combinatorics Tutorial

November 18, 2013

Notation

$$
\begin{aligned}
[n] &\triangleq \{1, 2, \ldots, n\} \\
(x)_k &\triangleq x \cdot (x-1) \cdot (x-2) \ldots (x-k+1) \\
|X| &\triangleq \text{ Number of elements in } X \text{ assuming that } X \text{is a finite set}
\end{aligned}
$$

1. Consider partitioning the set $[n]$ into k non empty subsets. Let the number $S(n,k)$ denote the number of ways in which this can be done.
   (a) What is $S(n,1)$?
   (b) What is $S(n,n)$?
   (c) What is $S(n,2)$?
   (d) What is $S(n,n-1)$?
2. Write down a recurrence for $S(n,k)$. (Hint. The element n could either be in a set of size 1 or more than 1).
3. What is the relation between $S(n,k)$ and number of onto functions from $[n]$ to $[k]$?
4. Prove the following identity
   $$x^n = \sum_{k=0}^{n} S(n,k)(x)_k$$
   (Hint: Imagine one side as the number of functions from a set A to a set B chosen appropriately).
5. Let A_i, $i \in \mathbb{N}$ be subsets of a finite set A?
   (a) What is $|A_1 \cup A_2|$?
   (b) What is $|A_1 \cup A_2 \cup A_3|$?
   (c) What is $|A_1 \cup A_2 \ldots \cup A_k|$?
6. Count the number of permutations of $[n]$ where i doesn't occur at the i^{th} position. (These are called dearragements).
7. Solve the following recurrences.

(a) $a_{n+2} = 3a_{n+1} - 2a_n, a_0 = 1, a_1 = 2$

(b) $f_n = g_{n-1} + f_{n-2}, g_n = f_{n-1} + g_{n-2}, f(0) = f(1) = 1, g(0) = g(1) = 2$

(c) $f(n) = n \times f(n-1), f(0) = 1$ (Try solving using generating functions)

(d) $f(n+1) = (n+1) \times (f(n) - (n-1)), f(0) = 1$