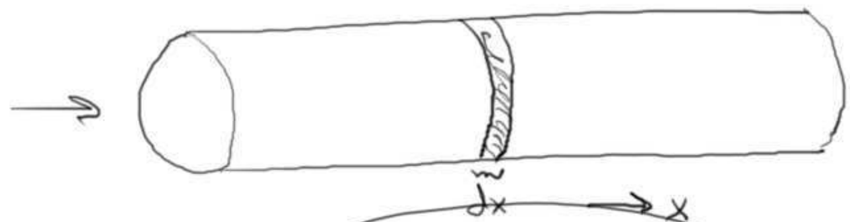
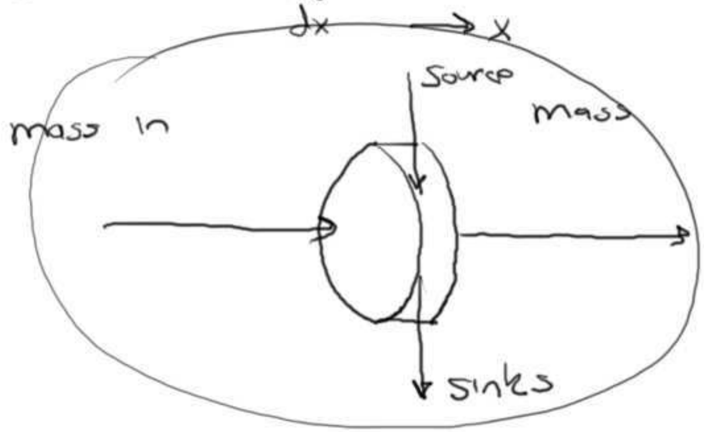
Conservation of Mass





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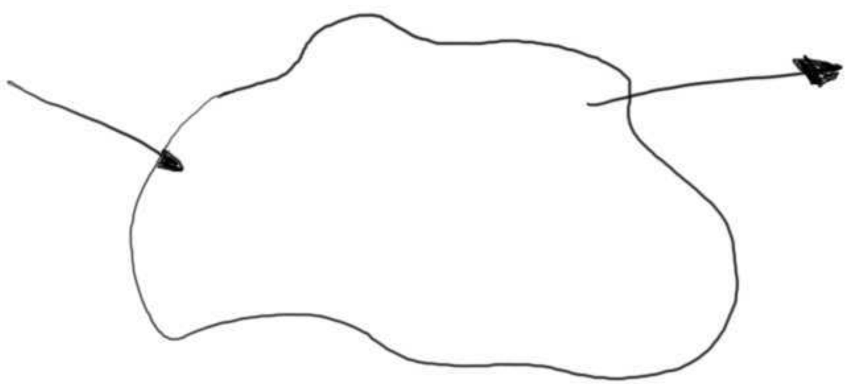
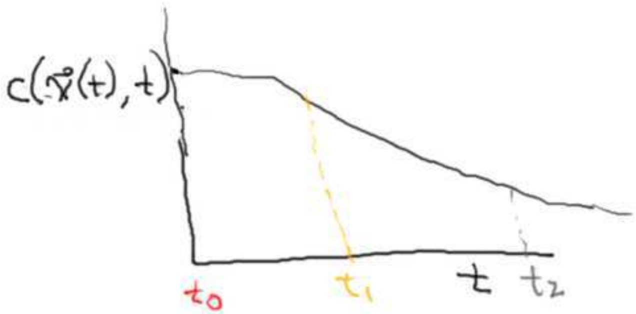
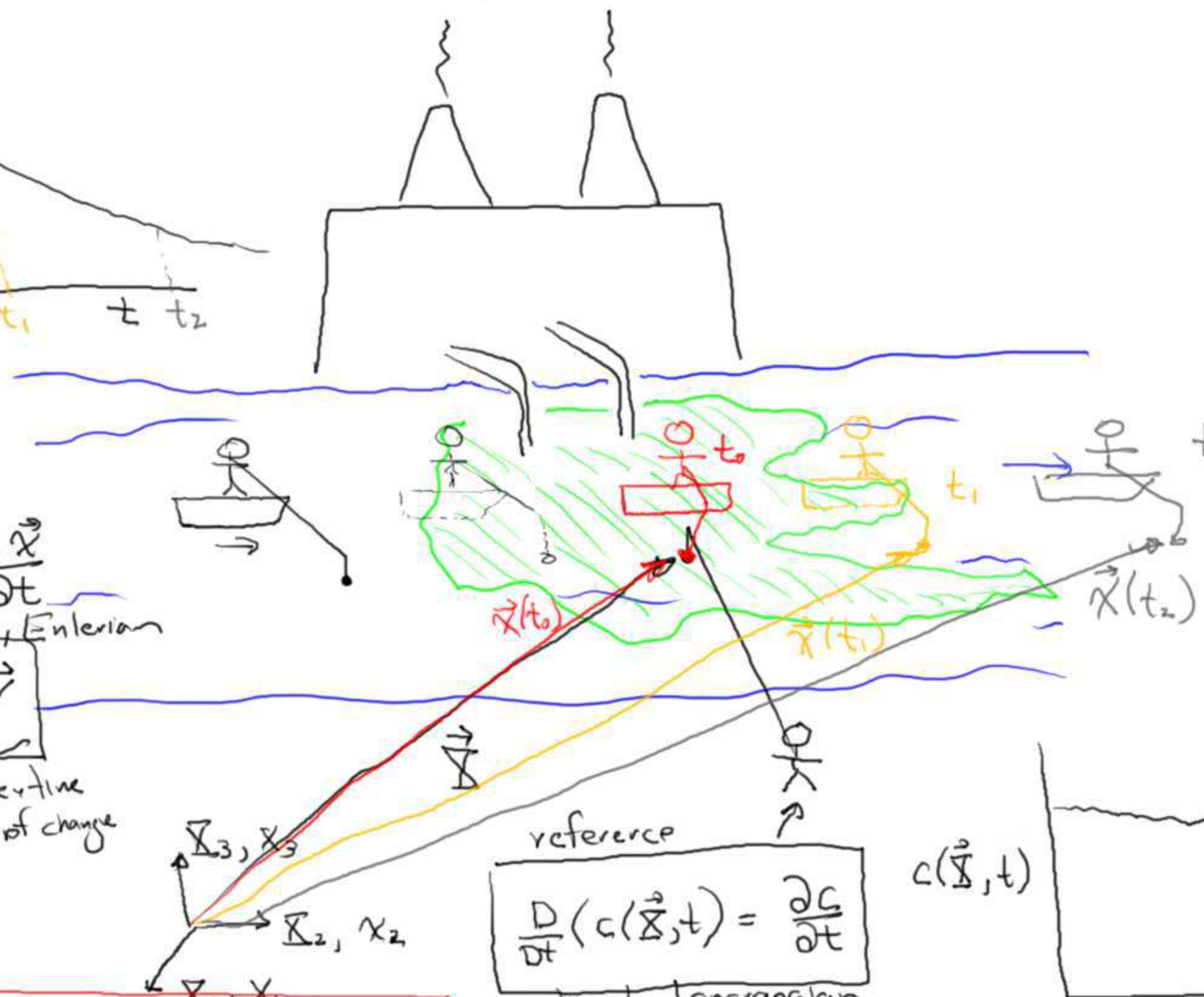
2D & 3D



Material Time Derivative (total, substantial, convective derivative)





$$\frac{D}{Dt}\left(c(\vec{x}(t),t)\right)$$

$$= \frac{\partial c}{\partial t} + \frac{\partial c}{\partial \vec{x}}\frac{\partial \vec{x}}{\partial t}$$

current, spatial, Eulerian

$$= \boxed{\frac{\partial c}{\partial t} + \nabla c \cdot \vec{v}}$$

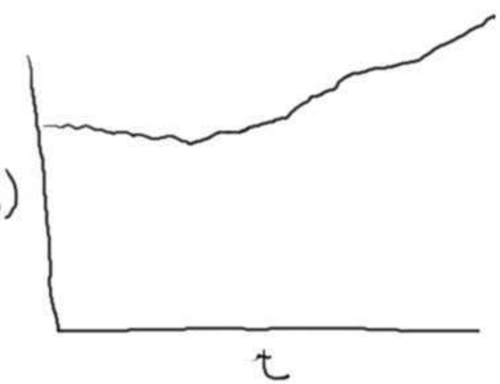
local rate of change

convective rate of change

reference

$$\boxed{\frac{D}{Dt}\left(c(\vec{X},t)\right) = \frac{\partial c}{\partial t}}$$

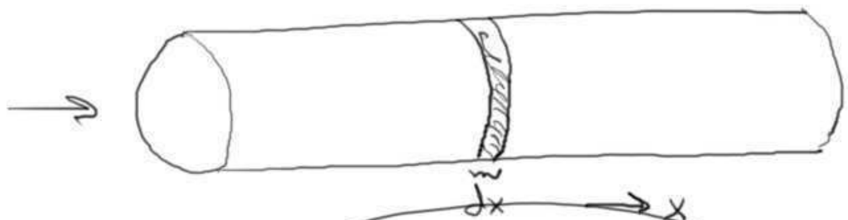
material, Langranglan

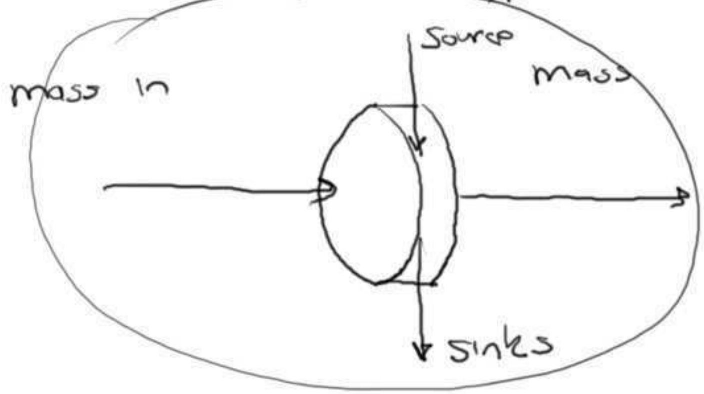


$$\boxed{\frac{D}{Dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \nabla(\cdot)\cdot\vec{v}}$$

← Material Time Derivative

Conservation of Mass





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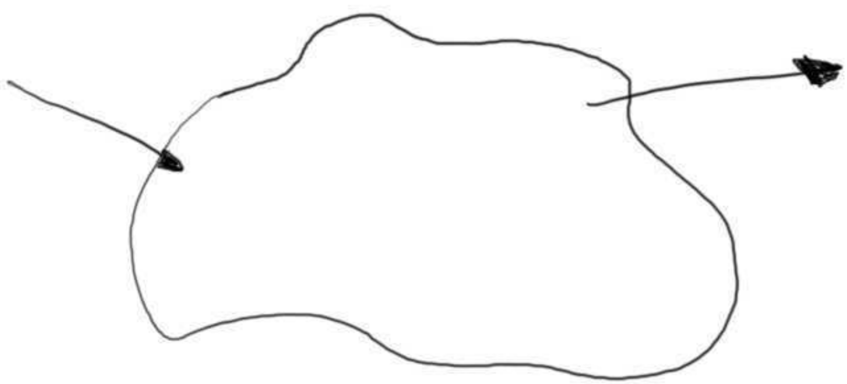
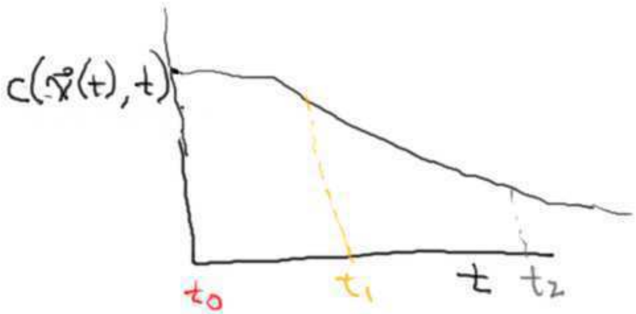
2D & 3D



Material Time Derivative (total, substantial, convective derivative)



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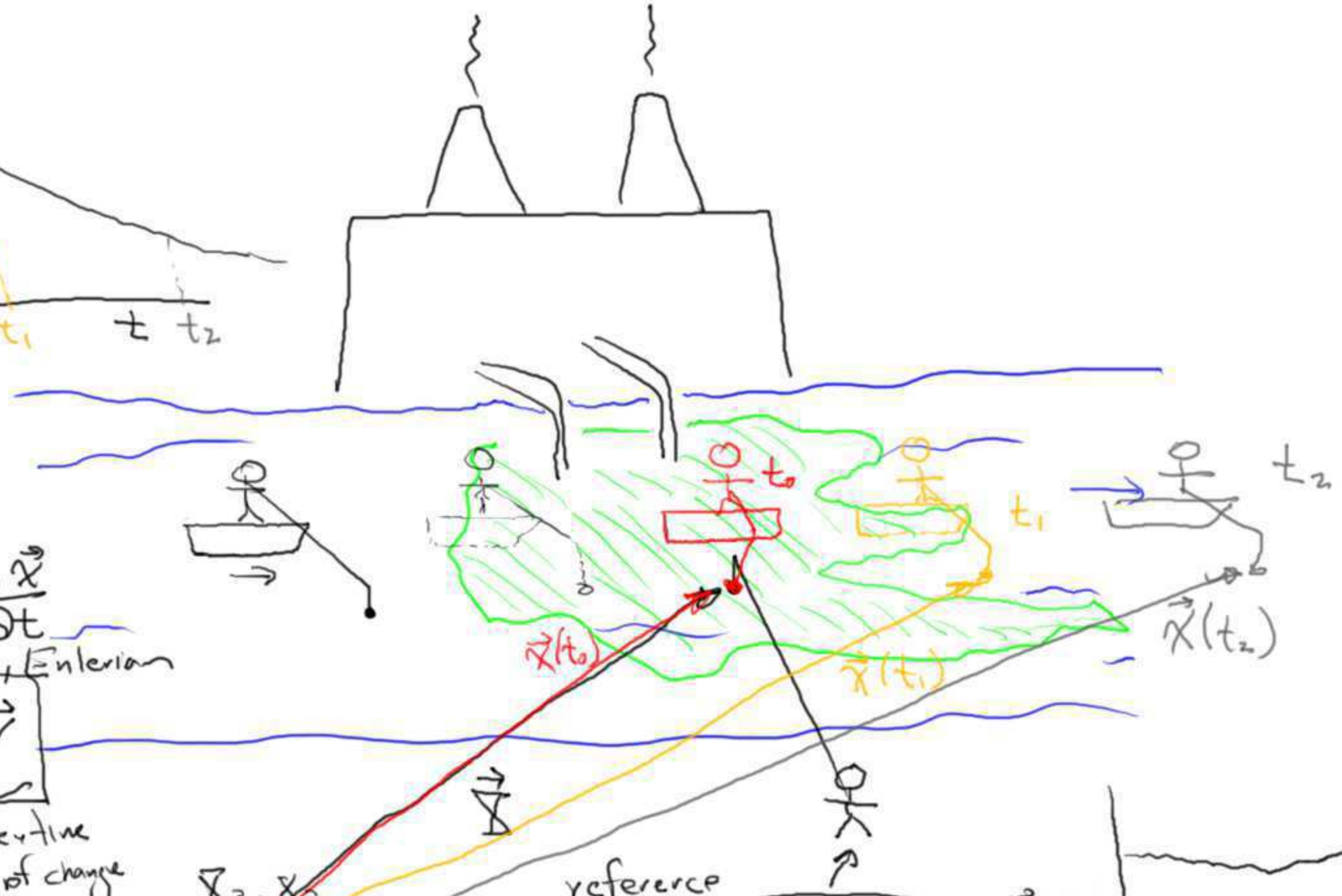
local rate of change

convective rate of change



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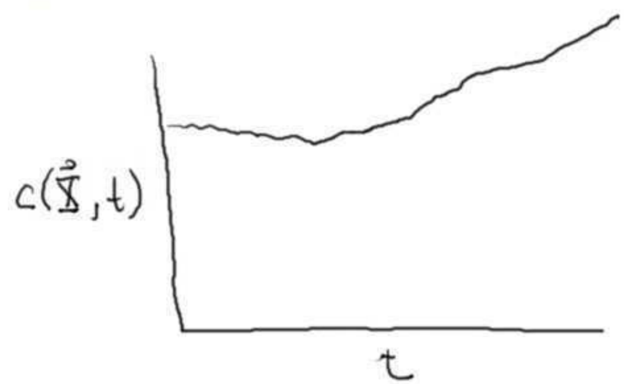
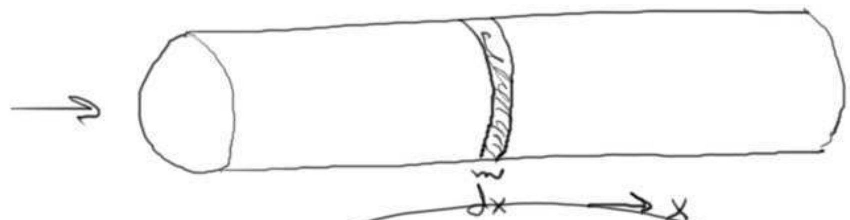
material, Langranglan

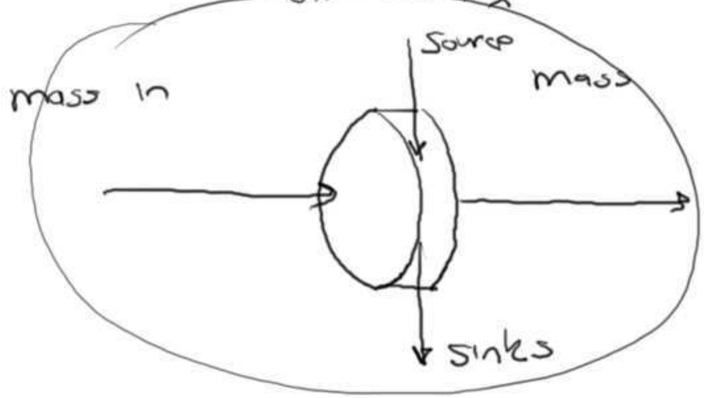


$$\boxed{\frac{D}{Dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \nabla(\cdot)\cdot\vec{v}}$$

← Material Time Derivative

Conservation of Mass





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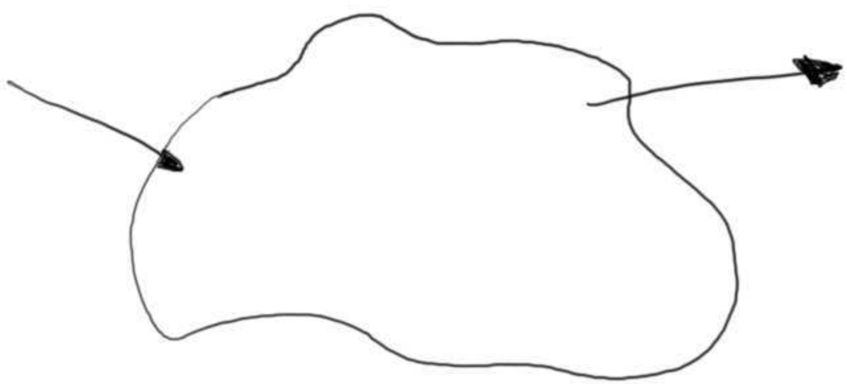
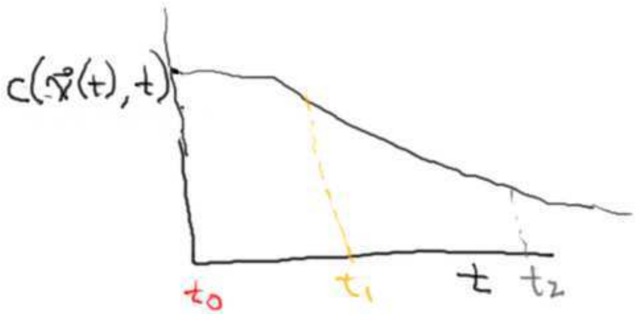
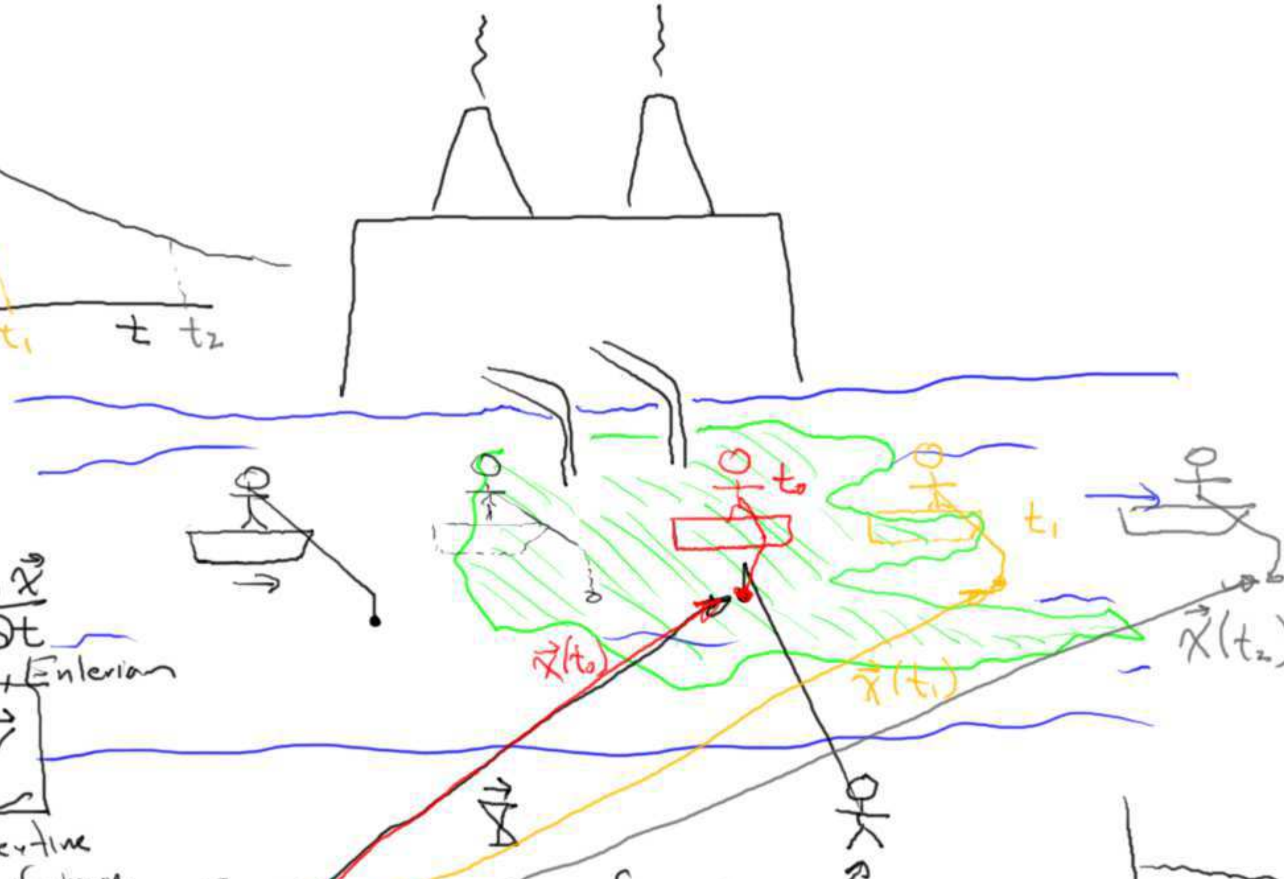
2D & 3D



Material Time Derivative (total, substantial, convective derivative)





$$\frac{D}{Dt}\left(c(\vec{x}(t),t)\right) = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial \vec{x}}\frac{\partial \vec{x}}{\partial t}$$

current, spatial, Eulerian

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local rate of change

convective rate of change

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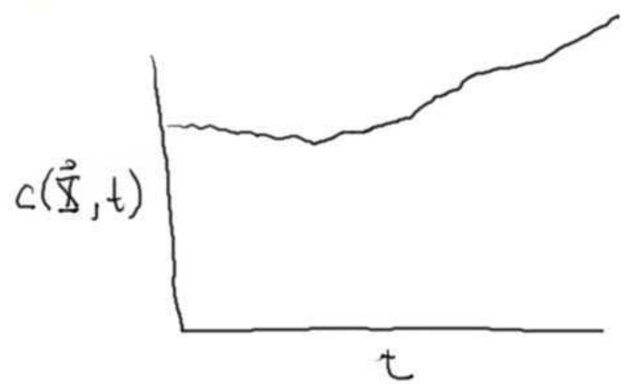
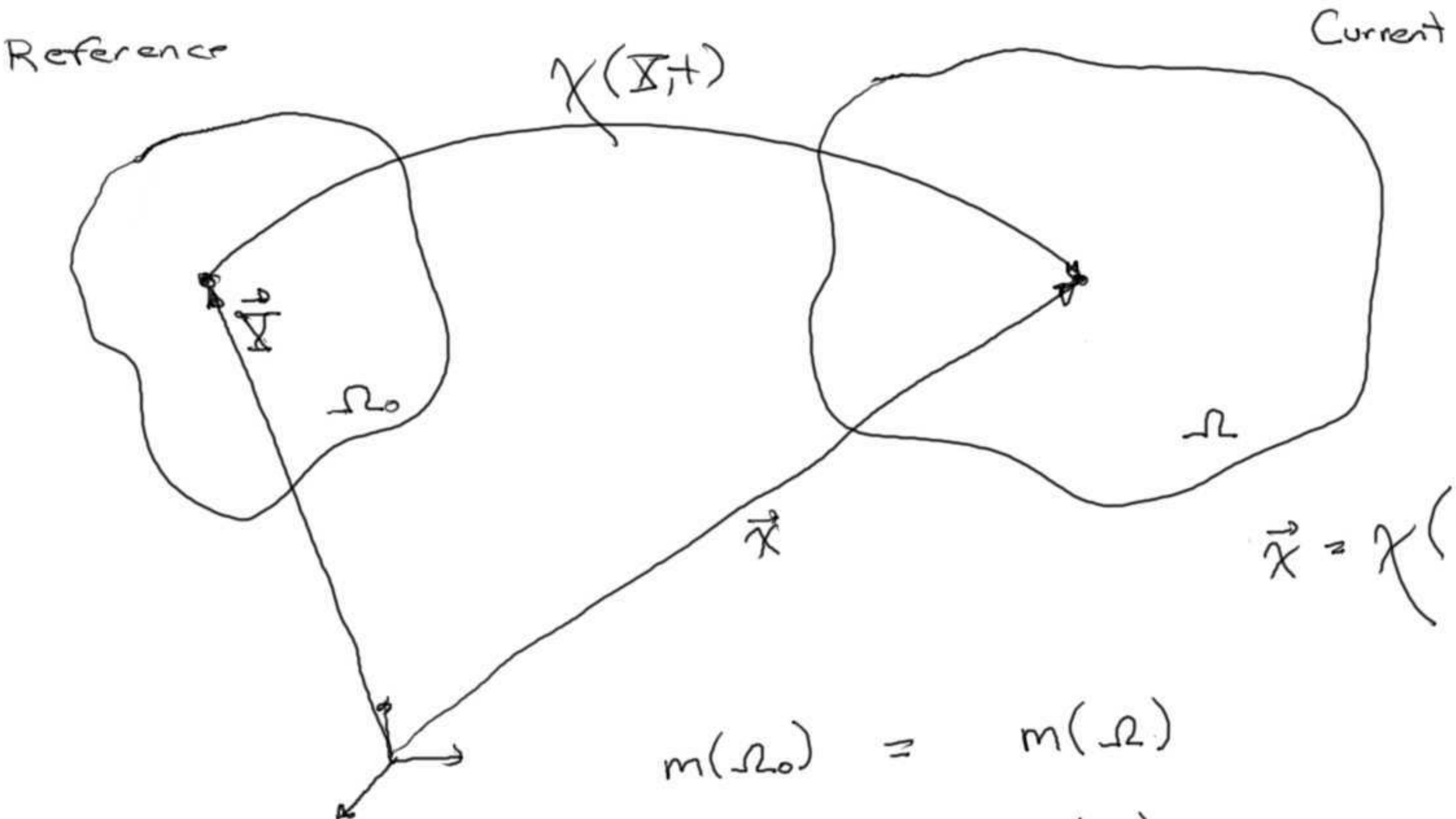
material, Langranglan



$$\boxed{\frac{D}{Dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \nabla(\cdot)\cdot\vec{v}}$$ ← Material Time Derivative



$$\vec{x} = \chi(\vec{X}, t)$$

$$m(\Omega_0) = m(\Omega)$$

$$dm(\vec{X}) = dm(\vec{x})$$

$$\int_{\Omega_0} \rho_0(\vec{X})\, \phi_0(\vec{X})\, dV_0 = \int_{\Omega} \rho(\vec{x}, t)\, \phi(\vec{x}, t)\, dV$$

$$\iiint_{\Omega_0} \rho_0(\vec{X})\, \phi_0(\vec{X})\, dX_1\, dX_2\, dX_3 = \iiint_{\Omega} \rho(\vec{x}, t)\, \phi(\vec{x}, t)\, dx_1\, dx_2\, dx_3$$

$$\longrightarrow \iiint_{\Omega_0} \rho_0(\vec{X})\, \phi_0(X)\, dX_1\, dX_2\, dX_3 = \iiint_{\Omega_0} \rho(\vec{x},t)\, \phi(\vec{x},t)\, J\, dX_1\, dX_2\, dX_3$$

$$J = \det(\bar{\bar{F}}) = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix}$$

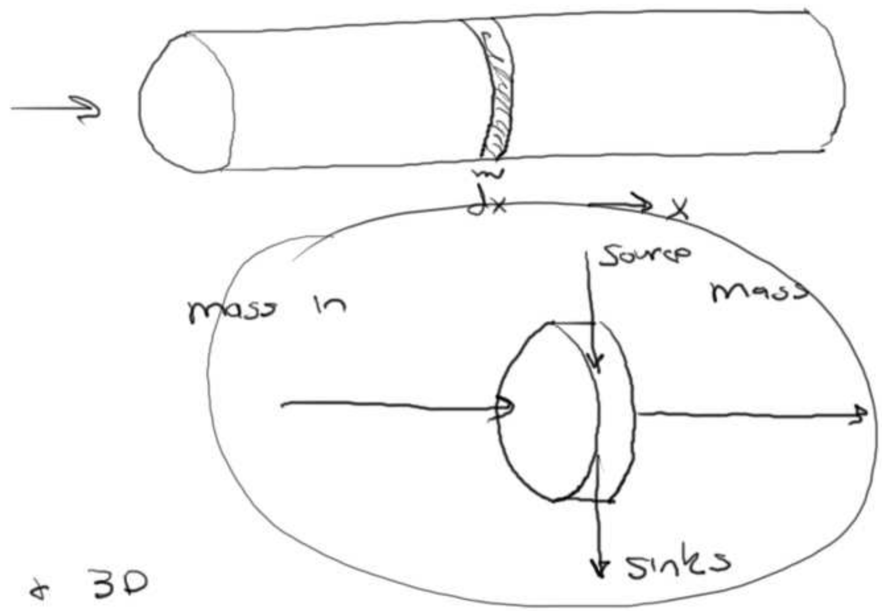
$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

$$\int_{\Omega_0} \rho_0(\vec{X})\, \phi_0(\vec{X})\, dV_0 = \int_{\Omega_0} \rho(\vec{x},t)\, \phi(\vec{x},t)\, J\, dV_0$$

$$\boxed{\rho_0(\vec{X})\, \phi_0(\vec{X}) = \rho(\vec{x},t)\, \phi(\vec{x},t)\, J}$$

Material for of conservation of mass

Conservation of Mass



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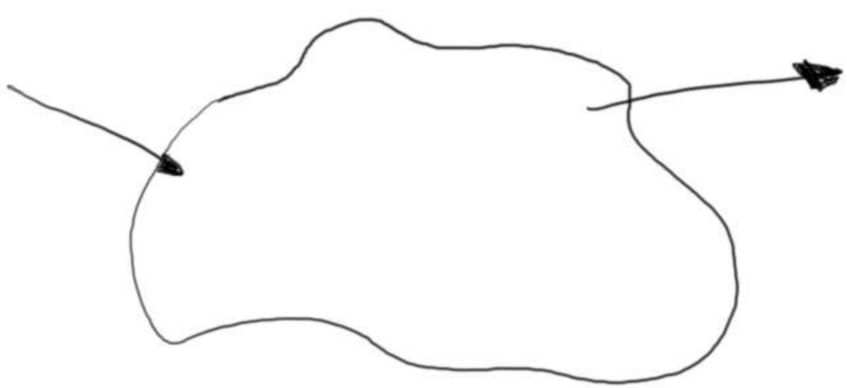
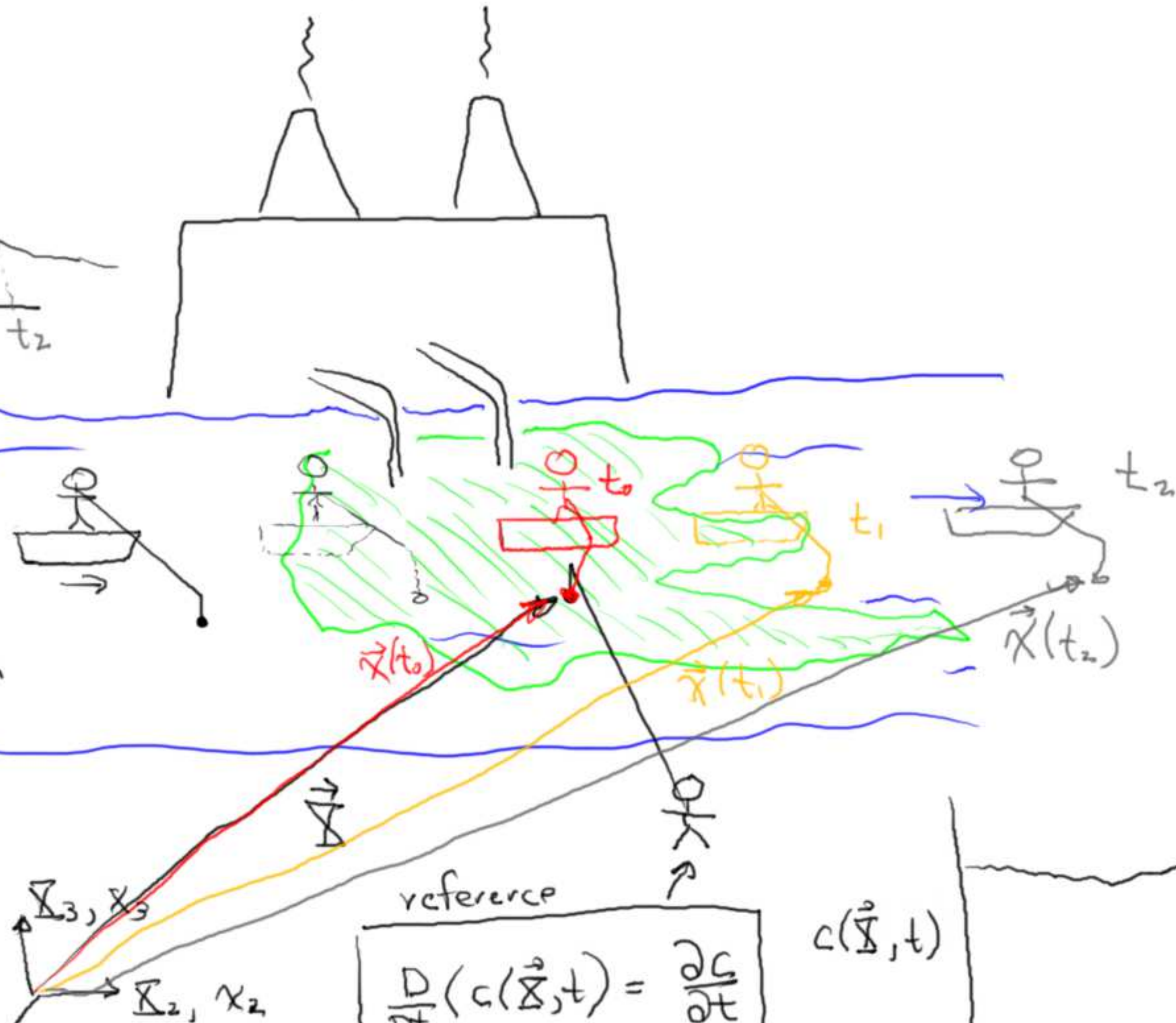
2D & 3D



Material Time Derivative (total, substantial, convective derivative)



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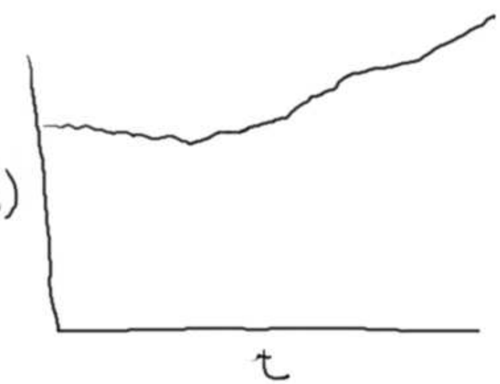
convective rate of change

reference

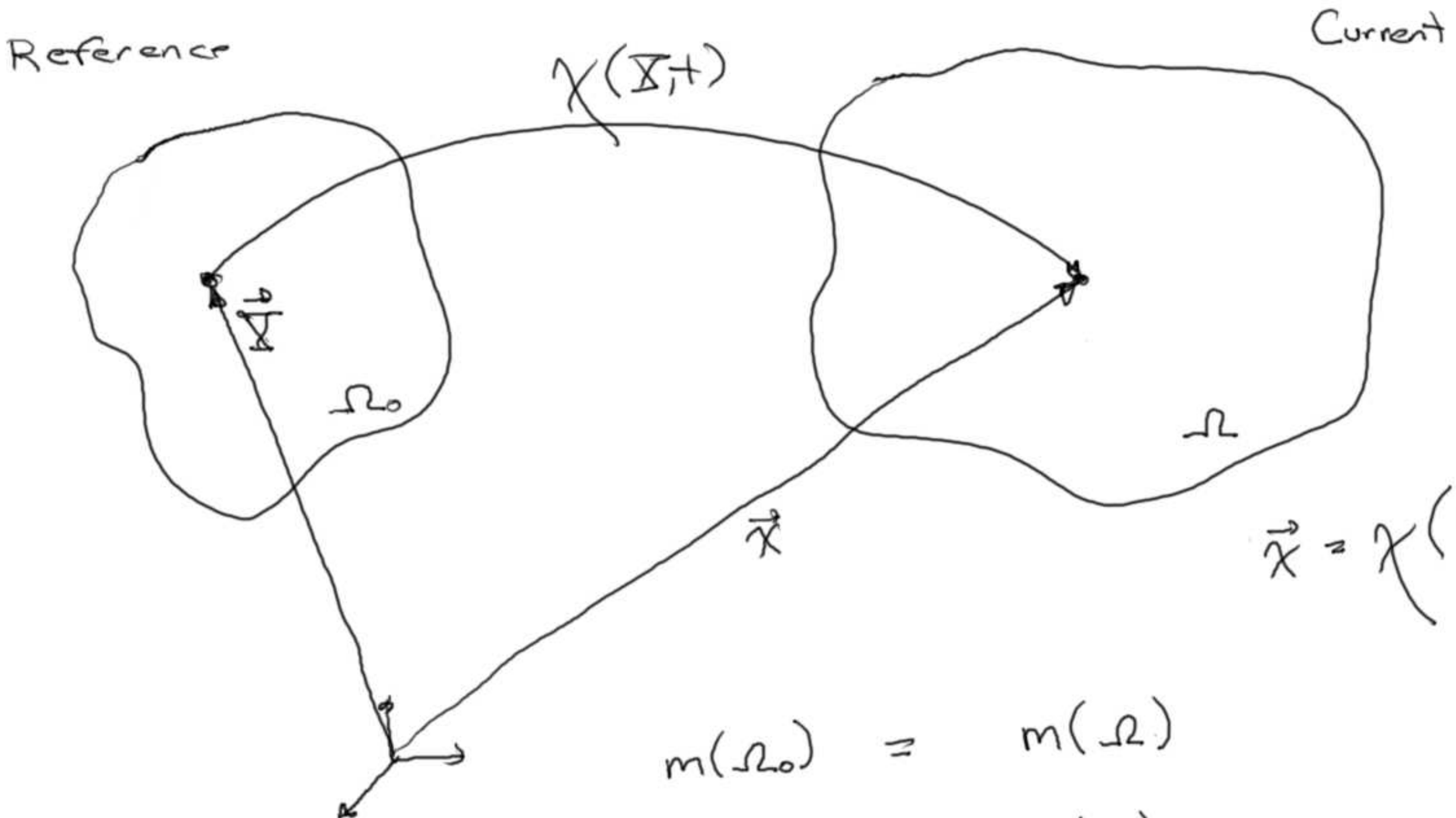
$$\boxed{\frac{D}{Dt}\left(c(\vec{X},t)\right) = \frac{\partial c}{\partial t}}$$

material, Langranglan



$$\boxed{\frac{D}{Dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \nabla(\cdot)\cdot\vec{v}}$$

← Material Time Derivative



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$$\iiint_{\Omega_0} \rho_0(\vec{X})\,\phi_0(\vec{X})\,dX_1\,dX_2\,dX_3 = \iiint_{\Omega} \rho(\vec{x},t)\,\phi(\vec{x},t)\,dx_1\,dx_2\,dx_3$$

$$\longrightarrow \iiint_{\Omega_0} \rho_0(\vec{X})\, \phi_0(X)\, dX_1\, dX_2\, dX_3 = \iiint_{\Omega_0} \rho(\vec{x},t)\, \phi(\vec{x},t)\, J\, dX_1\, dX_2\, dX_3$$

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$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

$$\int_{\Omega_0} \rho_0(\vec{X})\, \phi_0(\vec{X})\, dV_0 = \int_{\Omega_0} \rho(\vec{x},t)\, \phi(\vec{x},t)\, J\, dV_0$$

$$\boxed{\rho_0(\vec{X})\, \phi_0(\vec{X}) = \rho(\vec{x},t)\, \phi(\vec{x},t)\, J}$$

Material for of conservation of mass

$$\frac{D}{Dt}(\rho_0 \phi_0) = \frac{D}{Dt}(\rho \phi J)$$

$$0 = \frac{D}{Dt}(\rho \phi J) = \underbrace{\frac{D}{Dt}(\rho\phi)}\, J + \rho\phi \frac{D}{Dt}(J)$$

$$= J\left[\frac{\partial(\rho\phi)}{\partial t} + \nabla(\rho\phi)\cdot\vec{v}\right] + \rho\phi\frac{D}{Dt}(J)$$

$$= \cancel{J}\left[\frac{\partial(\rho\phi)}{\partial t} + \nabla(\rho\phi)\cdot\vec{v}\right] + \rho\phi\cancel{J}\,\nabla\cdot\vec{v}$$

$$= \frac{\partial(\rho\phi)}{\partial t} + \underbrace{\nabla(\rho\phi)\cdot\vec{v} + \rho\phi\,\nabla\cdot\vec{v}}_{\nabla\cdot(\rho\phi\vec{v})}$$

$$\boxed{0 = \frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot(\rho\phi\vec{v})}$$

$$\frac{D}{Dt}(J) = \frac{D}{Dt}(\det \bar{\bar{F}}) = \underbrace{\frac{\partial(\det \bar{\bar{F}})}{\partial F_{ij}}} \frac{\partial F_{ij}}{\partial t}$$

$$= \det \bar{\bar{F}} \, (F_{ji})^{-1} \frac{\partial F_{ij}}{\partial t}$$

$$= J \frac{\partial X_i}{\partial x_j} \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial X_j}\right)$$

$$= J \frac{\partial X_i}{\partial x_j} \frac{\partial}{\partial X_j}\left(\underbrace{\frac{\partial x_i}{\partial t}}_{v_i}\right)$$

$$= J \frac{\partial X_i}{\partial x_j} \frac{\partial v_i}{\partial X_j}$$

$$= J \frac{\partial v_i}{\partial x_j} \frac{\partial X_i}{\partial X_j}$$

$$= J \frac{\partial v_i}{\partial x_j} \delta_{ij}$$

$$= J \frac{\partial v_i}{\partial x_i}$$

$$\boxed{\frac{D}{Dt}(J) = J \, \nabla \cdot \vec{v}}$$

$$\nabla = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\}$$

$$\vec{v} = \{ v_1, V_2, V_3)$$

$$\nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

$$f = x$$

$$\frac{\partial f}{\partial x} = \frac{\partial x}{\partial x} = 1$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial v_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}$$

Einstien notation

$$0 = \frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\phi\rho v_1)}{\partial x_1} + \frac{\partial(\phi\rho v_2)}{\partial x_2} + \frac{\partial(\phi\rho v_3)}{\partial x_3}$$

$$x_1 \to x$$

$$x_2 \to y$$

$$x_3 \to z$$

$$\boxed{0 = \frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho\phi v_x)}{\partial x} + \frac{\partial(\phi\rho v_y)}{\partial y} + \frac{\partial(\rho\phi v_z)}{\partial z}}$$

Conservation of Mass