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# Chapter 3: Equations for Fluids - Chap. 2 of the book


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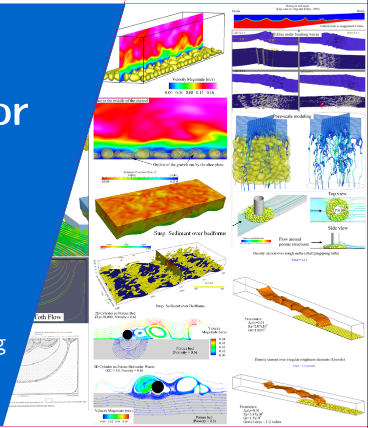


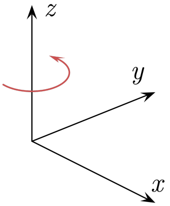
# Outline

# Tensor notations



Review of tensor notations (also see Chapter 1 of "OpenFOAM programmer's guide").

- OpenFOAM® uses a right handed cartesian coordinate system




- Tensors with different ranks: represent a property at a point
  - Scalars in lowercase: $a$ (rank 0)
  - Vectors in bold: $\mathbf{a} = (a_1, a_2, a_3)$, or $\vec{a}$ (rank 1)
  - Tensors in bold capitals: $\mathbf{T} = T_{ij}$, or $\vec{T}$ (rank 2)

$$\mathbf{T} = T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \quad (1)$$

## Tensor notations



Review of tensor notations (also see Chapter 1 of "OpenFOAM programmer's guide").

- *nabla* vector operator $\nabla$

$$\nabla \equiv \partial_i \equiv \frac{\partial}{\partial x_i} \equiv \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \tag{2}$$

$$\nabla$$

# Tensor notations



- Summation convention (Einstein notation):

$$a_i b_i = \sum_{i=1}^{3} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{3}$$

$$\frac{\partial u_i}{\partial x_i} = \nabla \cdot \mathrm{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \tag{4}$$

- Inner and outer product of vectors and tensors.
  - Scalar product: $a\mathbf{b} = ab_i$
  - Inner vector product, producing a scalar: $\mathbf{a} \cdot \mathbf{b} = a_i b_i$
  - Outer vector product, producing a second rank tensor: $\mathbf{a} \bigotimes \mathbf{b} = \mathbf{ab}^T$
  - Inner product of a vector and a tensor
    - product from the left: $\mathbf{a} \cdot \mathbf{T} = a_i \, T_{ij}$
    - product from the right: $\mathbf{T} \cdot \mathbf{a} = a_j \, T_{ij}$

# Tensor notations



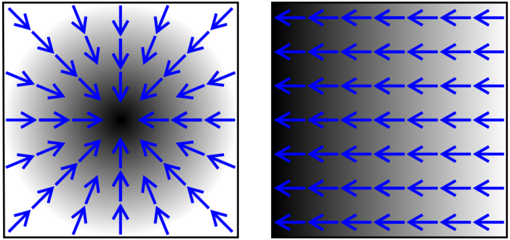
- Gradient of $s$

$$\nabla s = \left( \frac{\partial s}{\partial x_1}, \frac{\partial s}{\partial x_2}, \frac{\partial s}{\partial x_3} \right) \tag{5}$$

$\nabla$ can operate on any tensor field to produce a tensor field one rank higher.

$$s \text{ is a scalar} \quad -> \quad \nabla s \text{ is a vector} \tag{6}$$

$$\mathbf{s} \text{ is a vector} \quad -> \quad \nabla \mathbf{s} \text{ is a second order tensor} \tag{7}$$



Source: wikipedia

# Tensor notations



- Divergence of **a**

$$\nabla \cdot \mathbf{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \tag{8}$$

Divergence can operate on any tensor field (rank 1 and above) to produce a tensor field one rank lower.

$$\nabla \cdot \mathbf{T} = \partial_i T_{ij} = \begin{pmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{pmatrix} \tag{9}$$

- Curl of a vector field **a** (related to vorticity)

$$\begin{aligned} \nabla \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \left( \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}, \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}, \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) \end{aligned} \tag{10}$$

# Tensor notations



- Laplacian: related to diffusion; transform a tensor field into another tensor field of the same rank

$$\nabla^2 a = \nabla \cdot \nabla a = \frac{\partial^2 a}{\partial x_1^2} + \frac{\partial^2 a}{\partial x_2^2} + \frac{\partial^2 a}{\partial x_3^2} \tag{11}$$

  It is observed that the Laplacian operator is equivalent to "the divergence of the gradient field".

- Temporal derivative: change with time

$$\text{total or material time derivative: } \frac{D\phi}{Dt} = \lim_{\Delta t -> 0} \frac{\Delta \phi}{\Delta t} \tag{12}$$

$$\text{spatial time derivative: } \frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi \tag{13}$$

  where $\mathbf{u}$ is the advection (convection) velocity which carries the property $\phi$.

# Governing equations



- The governing equations represent mathematical statements of the following conservation laws of physics:
  - Conservation of mass.
  - Newton's second law: the change of momentum equals the sum of forces on a fluid particle.
  - First law of thermodynamics (conservation of energy): rate of change of energy equals the sum of rate of heat addition to and work done on fluid particle.
- The fluid is treated as a continuum: ignore the molecular structure

# Governing equations



Conservation of mass (continuity)

- Rate of increase of mass in a C.V. = net rate of mass flow into the C.V. (C.V. denotes control volume)

Unsteady, 3D:

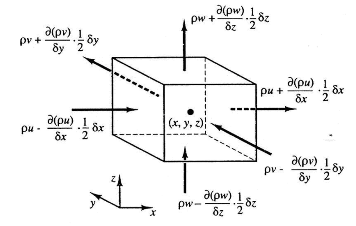
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{14}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{15}$$

If incompressible, then $\rho = \mathit{const}$, then

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{16}$$

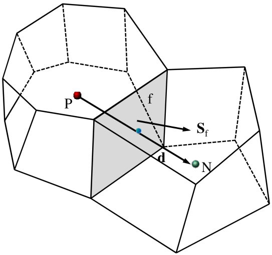


# Governing equations



More generally, we should use Reynolds Transport Theorem (RTT):

- Reynolds transport theorem is also known as the Leibniz-Reynolds' transport theorem.
- It is a 3D generalization of the Leibniz integral rule which is also known as differentiation under the integral sign.
- It can be used to assemble the standard transport equation for a generic property $\phi$.



# Governing equations



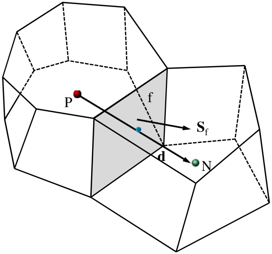
More generally, we should use Reynolds Transport Theorem (RTT):

- For an arbitrarily shaped control volume

  *The rate of change of a general property $\phi$ in the system is equal to the rate of change of $\phi$ in the control volume plus the rate of net outflow of $\phi$ through the surface of the control volume.*

$$\frac{d}{dt}\int_{V_m} \phi \, dV = \int_{V_m} \frac{\partial \phi}{\partial t} \, dV + \oint_{S_m} \phi(\mathbf{n} \cdot \mathbf{u}) dS$$

$$\frac{d}{dt}\int_{V} \phi \, dV = \int_{V} \left[ \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) \right] dV$$



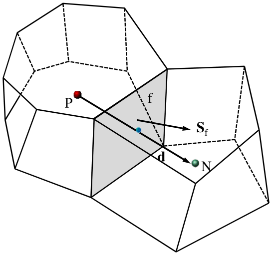
# Governing equations



- Transformation from the surface integral into the volume integral used above is called the Gauss' Theorem

$$\oint_{S_m} \phi(\mathbf{n} \cdot \mathbf{u})\, dS = \int_V [\nabla \cdot (\phi \mathbf{u})]\, dV$$

- **u** in the equation above represents the **convective velocity**: flux going in is negative ($\mathbf{u} \cdot \mathbf{n} < 0$).
- **u** is also a function of space and time.



# Governing equations



The generic equations in the previous two slides are integral forms. We can also write them in differential forms.

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) \tag{17}$$

where $\frac{d}{dt}$ is equivalent to $\frac{D}{Dt}$ (material differentiation).

| | | |
|---|---|---|
| continuity | $\phi = \rho$ | $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})$ |
| x-momentum | $\phi = \rho u$ | $\frac{d\rho u}{dt} = \frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \mathbf{u})$ |
| y-momentum | $\phi = \rho v$ | $\frac{d\rho v}{dt} = \frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho v \mathbf{u})$ |
| z-momentum | $\phi = \rho w$ | $\frac{d\rho w}{dt} = \frac{\partial \rho w}{\partial t} + \nabla \cdot (\rho w \mathbf{u})$ |
| concentration | $\phi = c$ | $\frac{dc}{dt} = \frac{\partial c}{\partial t} + \nabla \cdot (c \mathbf{u})$ |

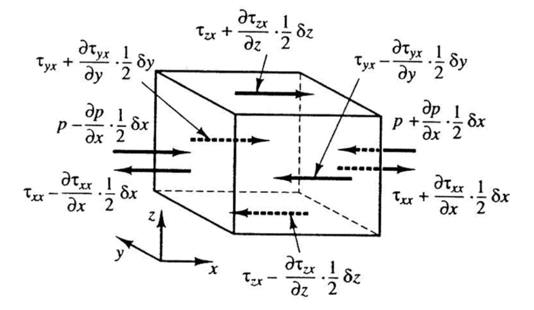
# Governing equations

Momentum equations

- Newton's second law:

  *The rate of change of momentum is equal to the sum of forces*

- Rate of changes of momentum in three directions: $\rho\frac{du}{dt}$, $\rho\frac{dv}{dt}$, and $\rho\frac{dw}{dt}$
- The forces could be:
  - surface forces: pressure force, viscous force, surface tension force, etc.
  - body forces: gravity force, centrifugal force, coriolis force, electromagnetic force.



# Governing equations

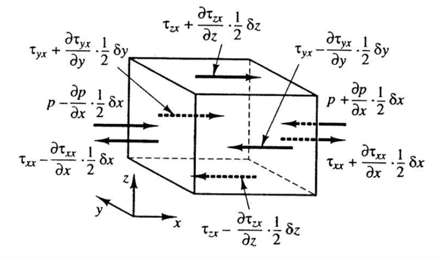
Using Newton's second law, momentum equations becomes:

$$\rho \frac{du}{dt} = \frac{\partial\left(-p+\tau_{xx}\right)}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx} \quad (18)$$

$$\rho \frac{dv}{dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial\left(-p+\tau_{yy}\right)}{\partial y} + \frac{\partial \tau_{zy}}{\partial y} + S_{My} \quad (19)$$

$$\rho \frac{dw}{dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial\left(-p+\tau_{zz}\right)}{\partial z} + S_{Mz} \quad (20)$$

where $S_{Mx}$, $S_{My}$, and $S_{Mz}$ are momentum sources terms, such as gravity. Note the negative sign in front of the pressure gradient due to the definition of positive tensile pressure.



# Governing equations



The equations can be written in tenor notations:

$$\rho \frac{d\mathbf{v}}{dt} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{S_M} \tag{21}$$

where $\boldsymbol{\sigma}$ is the sol-called Cauchy stress tensor. $\boldsymbol{\sigma}$ is a rank two symmetric tensor given by its covariant components:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \tag{22}$$

where the $\sigma$ are normal stresses and $\tau$ shear stresses.

# Governing equations



This tensor $\tau$ is split up into two terms:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} = - \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \sigma_{xx} + p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} + p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} + p \end{pmatrix} = -pI + \mathbb{T} \tag{23}$$

where $I$ is the 3 x 3 identity matrix and $\mathbb{T}$ is the deviatoric stress tensor. Note that the pressure $p$ is equal to minus the mean normal stress

# Governing equations



- Our goal here is to get the governing equations for Newtonian fluid motions, i.e., Navier-Stokes equation.
- To do that, we need to define the shear stresses, such as $\tau_{ij}$ by introducing a suitable rheological model.
- The simplest model is the linear model for Newtonian fluid: $\tau \propto \frac{\partial u}{\partial y}$, or more generally

$$\mathbb{T}_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \delta_{ij} \frac{\lambda}{\mu} \frac{\partial u_k}{\partial x_k} \right) = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \frac{\partial u_k}{\partial x_k} \qquad (24)$$

That is, the deviatoric of the deformation rate tensor is identified to the deviatoric of the stress tensor, up to a factor $\mu$.

Definition of viscosities:

- dynamic viscosity $\mu$: relates stresses to linear deformations
- second viscosity $\lambda$: relate stresses to the volumetric deformation. It only has small effect in practice. For incompressible flows, since $\frac{\partial u_k}{\partial x_x} = 0$, $\lambda$ does not matter.

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# Governing equations



Plug the Newtonian fluid relationship back to the momentum equations, we can get the Navier-Stokes equations for compressible flows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{25}$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \left( \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \right) + \rho \mathbf{g} \tag{26}$$

Similarly, we can get the Navier-Stokes equations for incompressible flows:

$$\nabla \cdot \mathbf{u} = 0 \tag{27}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{g} \tag{28}$$

where $\nu = \mu / \rho$ is the kinematic viscosity. Note the definition of the pressure $p$ here is a little different from the one for compressible flows, i.e., it has been divided by the density $\rho$.

# Governing equations

General Transport Equation

- General transport equation serves as our model equation for discretisation.
- Model equation may be more complex or with some source/sink terms. This leads to other forms, but the basics are still the same
- The common factor for all equations under consideration is the same set of operators: temporal derivative, gradient, divergence, Laplacian, curl, as well as various source and sink terms

$$\underbrace{\frac{\partial \phi}{\partial t}}_{\text{temporal derivative}} + \underbrace{\nabla \cdot (\phi \mathbf{u})}_{\text{convection term}} - \underbrace{\nabla \cdot (\gamma \nabla \phi)}_{\text{diffusion term}} = \underbrace{S_\phi}_{\text{source term}}$$

# Sources and Sinks

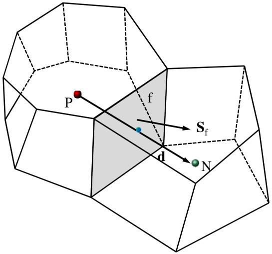


Volume and Surface sources (sinks)

- Apart from convection and diffusion (above), we can have local sources and sinks of $\phi$.
- Volume source: distributed through the volume
- Surface source: act on external surface $S$, *e.g.* heating. Typically modelled using gradient-based models

$$\frac{d}{dt}\int_V \phi \, dV = \int_V q_v \, dV - \oint_S (\mathbf{n} \cdot \vec{q_s}) dS$$

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) = q_v - \nabla \cdot \vec{q_s}$$

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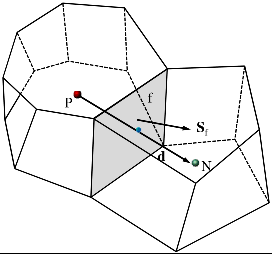
Modelling Diffusive Transport

- Gradient-based transport is a model for surface source/sink terms
- Consider a case where $\phi$ is a concentration of a scalar variable and a closed domain. Diffusion transport says that $\phi$ will be transported from regions of high concentration to regions of low concentration until the concentration is uniform everywhere.
- Taking into account that $\nabla\phi$ point up the concentration slope, and the transport will be in the opposite direction, we can define the following diffusion model

$$\vec{q}_s = -\gamma\,\nabla\phi$$

where $\gamma$ is the diffusivity



# Generic Transport Equation



Generic Transport

- Assembling the above yields the transport equation in the standard form

$$\underbrace{\frac{\partial \phi}{\partial t}}_{\text{temporal derivative}} + \underbrace{\nabla \cdot (\phi \mathbf{u})}_{\text{convection term}} - \underbrace{\nabla \cdot (\gamma \nabla \phi)}_{\text{diffusion term}} = \underbrace{S_\phi}_{\text{source term}}$$

- Temporal derivative represents inertia of the system
- Convection term represents the convective transport by the prescribed velocity field (coordinate transformation). The term has got **hyperbolic** nature: information comes from the vicinity, defined by the direction of the convection velocity
- Diffusion term represents gradient transport. This is an **elliptic** term: every point in the domain feels the influence of every other point instantaneously
- Sources and sinks account for non-transport effects: local volume production and destruction of $\phi$

# Classification for simple PDEs



- Elliptic: solutions are smooth. For example, steady heat equation and Laplace's equation. The motion of a fluid at subsonic speeds can be approximated with elliptic PDEs.
- Parabolic: For example, unsteady heat equation.
- Hyperbolic: retain or develop discontinuities in the solution. An example is the wave equation. The motion of a fluid at supersonic speeds can be approximated with hyperbolic PDEs, such as shallow water equations.

| Problem type | Equation type | Prototype equation | Conditions | Solution domain | Solution smoothness |
|---|---|---|---|---|---|
| Equilibrium problem | Elliptic | $\nabla \cdot \nabla \phi = 0$ | B.C. | Closed domain | Smooth |
| Marching problem with dissipation | Parabolic | $\frac{\partial \phi}{\partial t} = \nu \nabla \cdot \nabla \phi$ | I.C. and B.C. | Open domain | Smooth |
| Marching problem without dissipation | Hyperbolic | $\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla \cdot \nabla \phi$ | I.C. and B.C. | Open domain | May be discontinuous |

# Classification for simple PDEs



- More generally, a second-order PDE (in 2D):

$$a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + d\frac{\partial\phi}{\partial x} + e\frac{\partial\phi}{\partial y} + f\phi + g = 0 \tag{29}$$

- Classification depends on the roots of the characteristics equation, i.e. depending on the discriminant $b^2 - 4ac$:
  - $b^2 - 4ac < 0$: elliptic
  - $b^2 - 4ac = 0$: parabolic
  - $b^2 - 4ac > 0$: hyperbolic
- if $a$, $b$ and $c$ depends on $x$ and $y$, then the equations are termed quasilinear. The equation might exibith different types depending on the location (mixed type).

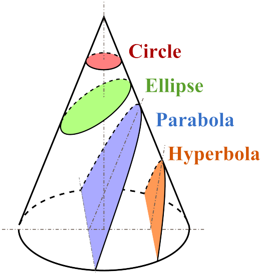
# Classification for simple PDEs



- The origin of the terms "elliptic", "parabolic", and "hyperbolic" is an analogy to the case of conic sections
- The general equation for a conic section from geometry

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{30}$$

- Classification depends on the roots of the characteristics equation, i.e. depending on the discriminant $b^2 - 4ac$:
  - $b^2 - 4ac < 0$: the conic is an ellipse
  - $b^2 - 4ac = 0$: the conic is a parabola
  - $b^2 - 4ac > 0$: the conic is a hyperbola



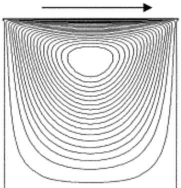

Source: wikipedia.org

# Classification for simple PDEs



Elliptic equations

- Many times, they are for equilibrium problems at steady state (no $\partial/\partial t$ term)
- Examples: steady potential flow, steady state temperature distribution.
- In fluid mechanics, a special case which demonstrates the elliptic equations is the potential flow equation. Define the velocity potential $\phi$ as $\mathbf{u} = \nabla\phi$. Since for incompressible flow, we have $\nabla \cdot \mathbf{u} = 0$, plugging in the definition, we have
$$\nabla^2\phi = 0 \tag{31}$$
which is also called the Laplace's equation.
- Only B.C. is needed. It is a boundary value problem.
- Important characteristics: disturbance travels in all direction and at infinite speed; as a result, solutions are smooth.



# Classification for simple PDEs

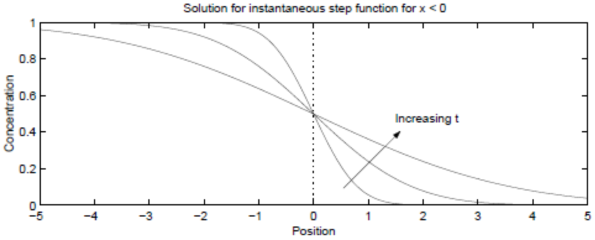


Parabolic equations

- Unsteady problems with dissipation (or diffusion).
- Examples: unsteady potential flow, unsteady temperature distribution.
- Model equation:

$$\frac{\partial \phi}{\partial t} = \nu \nabla^2 \phi \tag{32}$$

- Both B.C. and I.C. are needed. It is a initial-boundary-value problem.
- Important characteristics: disturbance only affects the solution at a later time; dissipation (diffusion) smooth out the solution.



# Classification for simple PDEs

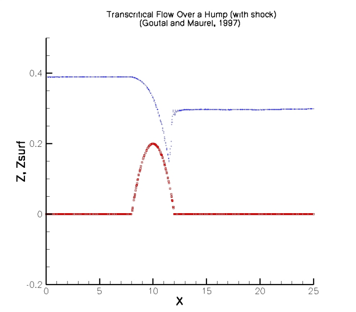


Hyperbolic equations

- Hyperbolic equations are for unsteady problem with negligible dissipation (diffusion)
- Examples: shallow water equation; wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi \tag{33}$$

- Both B.C. and I.C. are needed. It is an initial-boundary-value problem.
- Important characteristics: could have discontinuity in the solution (shocks and bores); disturbance travels at wave speed $c$.

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# Boundary Conditions



Role of boundary conditions

- The role of boundary conditions is to nail down the solutions. Without them, the possible solutions will be infinity.
- Position of boundaries and specified condition requires engineering judgement. Badly placed boundaries will compromise the solution or cause "numerical problems". Example: locating an outlet boundary across a recirculation zone.
- Choices need to be based on physical understanding of the system

Numerical boundary conditions

- Dirichlet condition: fixed boundary value of $\phi$
- Neumann: zero gradient or no flux condition: $\mathbf{n} \cdot \vec{q}_s = 0$
- Fixed gradient or fixed flux condition: $\mathbf{n} \cdot \vec{q}_s = q_b$. Generalisation of the Neumann condition, where

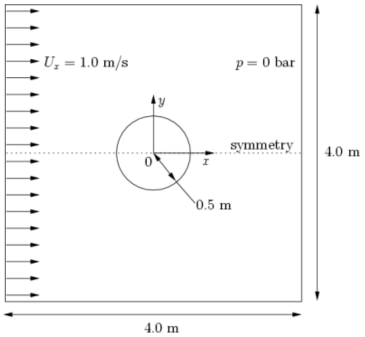
$$\vec{q}_s = -\gamma \nabla \phi$$

- Mixed condition: Linear combination of the value and gradient condition

# Boundary Conditions

Numerical boundary conditions

- Symmetry plane: $\mathbf{n} \cdot \nabla\phi = 0$

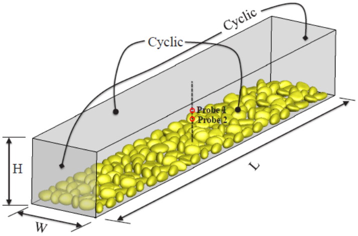


# Boundary Conditions



Numerical boundary conditions

- Cyclic (periodic) conditions $\phi_1 = \phi_2$. In cases of repeating geometry or fully developed conditions, the size of domain can be reduced by modelling only a representative segment of the geometry.
- In order to account for periodicity, a "self-coupled" condition can be set up on the boundary.
- In special cases, a jump condition can be specified for variables that do not exhibit cyclic behaviour. Example: pressure in fully developed channel flow
- Be careful with the size of the domain: e.g., in LES, the size will limit the size of eddies.



# Boundary Conditions



Physical boundary conditions:

- For a passive transport of a scalar variable, physical meaning of the boundary condition is trivial.
- In case of coupled equation sets or a clear physical meaning, it is useful to associate physically meaningful names to the sets of boundary conditions for individual equations. Examples:
  - stationary wall: no slip condition for velocity $\mathbf{u} = 0$
  - moving wall: *movingWallVelocity*
  - Turbulent inlet: *turbulentInlet*
  - Slip: zeroGradient if $\phi$ is a scalar. If it is a vector, normal component is fixedValue zero, tangential components are zeroGradient

# Boundary Condition



What happens when the B.C.s are not properly specified (Fletcher, 1991):

- Under-specification of B.C.s normally leads to failure to obtain a unique solution
- Over-specification of B.C.s gives rise to flow solutions with severe and unphysical 'boundary layers' close to the boundary where it is specified.

# Initial Condition



Specifying initial conditions

- **Initial conditions** specify the variation of each solution variable in space at the beginning.
- In some cases, this may be irrelevant: for example, steady-state
- but in other simulations it is essential.

# Summary



In this chapter, we **briefly** reviewed the following:

- Tensor notations and calculations
- Governing equations (PDEs)
  - Conservation laws: mass, momentum, concentration, etc.
  - Classification of PDEs
- Boundary conditions and initial conditions

# Questions?