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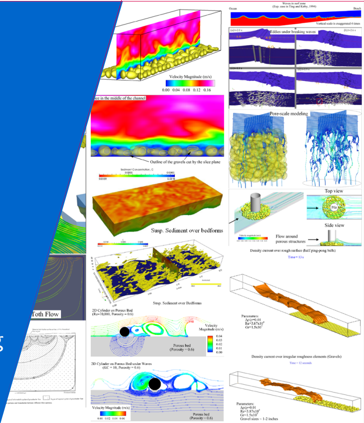
CE597D Special Topics, Fall, 2014

Chapter 4, Part 1: Numerical Solution of Diffusion Equations



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What will be covered in this chapter?

- FVM for diffusion equations
- FVM for advection-diffusion equations
- FVM for unsteady problems
- Linear system solvers
- Mesh generation: `blockMesh` and `snappyHexMesh`

Outline

What is Discretisation?

Numerical Discretisation Method

- Generic transport equation can very rarely be solved analytically: this is why we resort to numerical methods
- Discretisation is a process of representing the differential equation we wish to solve by a set of algebraic expressions of equivalent properties (typically a matrix)
- There are many diescretisation methods: FDM, FEM, FVM, spectral method, etc.
- In this course, we only introduce finite volume method (FVM).

What is Discretisation?

Numerical Discretisation Method

- This section demonstrate the simplest transport process: steady state diffusion.
- Diffusion: movement of a physical quantity from a region of high concentration to a region of low concentration.
  The full generic, unsteady, advection-diffusion equation

$$\underbrace{\frac{\partial \phi}{\partial t}}_{\text{temporal derivative}} + \underbrace{\nabla \cdot (\phi \mathbf{u})}_{\text{convection term}} - \underbrace{\nabla \cdot (\gamma \nabla \phi)}_{\text{diffusion term}} = \underbrace{S_\phi}_{\text{source term}}$$

  The simplified, steady state, diffusion equation

$$\underbrace{\nabla \cdot (\gamma \nabla \phi)}_{\text{diffusion term}} + \underbrace{S_\phi}_{\text{source term}} = 0$$

FVM Discretisation

FVM Discretisation

- The control volume (CV) integration of the steady state diffusion equation:

$$\int_{CV} \nabla \cdot (\gamma \nabla \phi)\, dV + \int_{CV} S_\phi \, dV = \int_A \mathbf{n} \cdot (\gamma \nabla \phi)\, dA + \int_{CV} S_\phi \, dV = 0$$

- This is the model diffusion equation we will be working with in this section.
- We will introduce the approximation method to get the discretized equations.
- We will use a 1D version of the equation as an example. The same discretization technique can be extended to 2D and 3D.

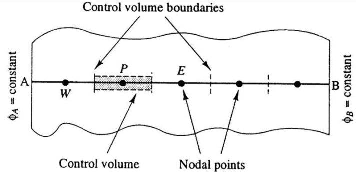
$$\frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) + S = 0$$

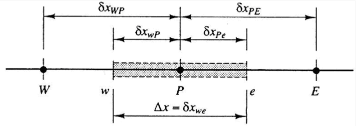
FVM Discretisation

Remember the typical steps in numerical solutions:

- Pre-processing including mesh generation
- Solution of the discretized equations
- Post-processing

So, the first step is to generate the mesh for this 1D problem:

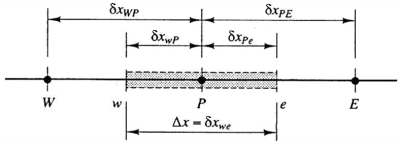




The notations: a general cell is noted as P and its neighbors, the cells to the west and east, are denoted as W and E.

For simplicity, we assume the boundary conditions on both ends are fixed values.

FVM Discretisation



Upon FVM discretization:

$$
\begin{aligned}
\int_{CV} \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) dV + \int_{CV} S\, dV &= \int_{A} \mathbf{n} \cdot (\Gamma \nabla \phi)\, dA + \int_{CV} S\, dV \\
&= \left(\Gamma A \frac{d\phi}{dx}\right)_e - \left(\Gamma A \frac{d\phi}{dx}\right)_w + \overline{S}\Delta V \\
&= 0
\end{aligned}
$$

where A is the cross-sectional area of the control volume, ΔV is the volume, and $\overline{S}$ is the average value of the source S over this control volume.
The physical meaning of above discretized equation is very clean and important: It says that the diffusive flux leaving the east face minus the diffusive flux entering through the west face is equal to the generation of ϕ, i.e., the conservation law.

FVM Discretisation - Diffusion term

We need to evaluate the flux at the west and east interfaces. In FVM, the variables are usually defined at cell centers. Assuming a linear distribution, the diffusion coefficients at the interfaces can be evaluated as

$$\Gamma_w = \frac{\Gamma_W + \Gamma_P}{2} \qquad \Gamma_e = \frac{\Gamma_P + \Gamma_E}{2} \tag{1}$$

We will also use the central difference to evaluate the derivatives

$$\left(\frac{d\phi}{dx}\right)_e = \frac{\phi_E - \phi_P}{\delta x_{PE}} \qquad \left(\frac{d\phi}{dx}\right)_w = \frac{\phi_P - \phi_W}{\delta x_{WP}} \tag{2}$$

So the diffusive fluxes at interfaces can be evaluated as

$$\left(\Gamma A \frac{d\phi}{dx}\right)_e = \Gamma_e A_e \frac{\phi_E - \phi_P}{\delta x_{PE}} \qquad \left(\Gamma A \frac{d\phi}{dx}\right)_w = \Gamma_w A_w \frac{\phi_P - \phi_W}{\delta x_{WP}} \tag{3}$$

FVM Discretisation - Diffusion term

In summary, we need an interpolation scheme to evaluate the diffusion coefficient at the interface

$$\Gamma_w = \frac{\Gamma_W + \Gamma_P}{2} \qquad \Gamma_e = \frac{\Gamma_P + \Gamma_E}{2}, \tag{4}$$

and a surface normal gradient scheme to evaluate the derivatives

$$\left(\frac{d\phi}{dx}\right)_e = \frac{\phi_E - \phi_P}{\delta x_{PE}} \qquad \left(\frac{d\phi}{dx}\right)_w = \frac{\phi_P - \phi_W}{\delta x_{WP}} \tag{5}$$

So in `fvSchemes` file, we have

```
Gauss <interpolationScheme> <snGradScheme>
```

For example:

```
laplacian(nu,U) Gauss linear corrected;
```

FVM Discretisation - Source term

The source term S may be a function of the dependent variable ϕ. In such case, a linear form can be used for the source term:

$$\overline{S}\Delta V = S_u + S_p\phi_P \tag{6}$$

Substituting both discretized terms into the equation, we have

$$\Gamma_e A_e \frac{\phi_E - \phi_P}{\delta x_{PE}} - \Gamma_w A_w \frac{\phi_P - \phi_W}{\delta x_{WP}} + (S_u + S_p\phi_P) = 0 \tag{7}$$

Re-arrange it to have

$$\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w - S_p\right)}_{a_P} \phi_P = \underbrace{\left(\frac{\Gamma_w}{\delta x_{WP}} A_w\right)}_{a_W} \phi_W + \underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}} A_e\right)}_{a_E} \phi_E + S_u \tag{8}$$

Or

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u \tag{9}$$

This is the discretized equation for a general control volume. For a control volume adjacent to the domain boundaries, the equation has be to modified to honor the specified B.C.

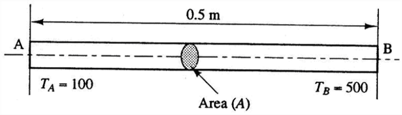
Examples

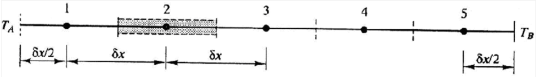
- Example 1: 1D steady-state heat conduction with no source

1D steady-state heat conduction with no source.

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{10}$$

with $k = 1000$ W/m/K and $A = 10 \times 10^{-3}$ m^2. The total length of the rod is 0.5 m and divided into 5 equal size pieces such that $\delta x = 0.1$ m





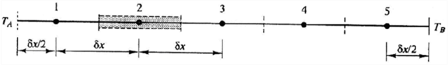
The mesh has five cells. Cells 2, 3, and 4 are internal. Cells 1 and 5 are adjacent to the boundaries.

Examples

- Example 1: 1D steady-state heat conduction with no source

General discretization equation:

$$\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e + \frac{\Gamma_w}{\delta x_{WP}}A_w - S_p\right)}_{a_P}\phi_P = \underbrace{\left(\frac{\Gamma_w}{\delta x_{WP}}A_w\right)}_{a_W}\phi_W + \underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)}_{a_E}\phi_E + S_u \quad (11)$$



The discretized equation for the internal cells 2, 3, and 4 can be written as

$$a_P T_P = a_W T_W + a_E T_E + S_u \quad (12)$$

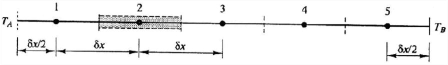
where $a_W = \frac{k}{\delta x}A$, $a_E = \frac{k}{\delta x}A$, and $a_P = a_W + a_E = 2\frac{k}{\delta x}A$

Examples

- Example 1: 1D steady-state heat conduction with no source

General discretization equation:

$$\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e + \frac{\Gamma_w}{\delta x_{WP}}A_w - S_p\right)}_{a_P}\phi_P = \underbrace{\left(\frac{\Gamma_w}{\delta x_{WP}}A_w\right)}_{a_W}\phi_W + \underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)}_{a_E}\phi_E + S_u \quad (13)$$



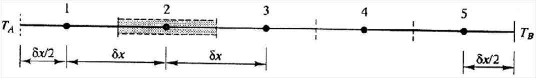
The discretized equation for the boundary cell 1 can be written as

$$kA\left(\frac{T_E - T_P}{\delta x}\right) - kA\left(\frac{T_P - T_A}{\delta x/2}\right) = 0 \quad (14)$$

where the flux through the boundary A has been approximated by assuming a linear distribution of temperature between boundary A and cell center 1. This equation can re-arranged to have

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right)T_P = 0 \cdot T_W + \left(\frac{k}{\delta x}A\right)T_E + \left(\frac{2k}{\delta x}A\right)T_A \quad (15)$$

- Example 1: 1D steady-state heat conduction with no source



This equation can be re-arranged to have

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right) T_P = 0 \cdot T_W + \left(\frac{k}{\delta x}A\right) T_E + \left(\frac{2k}{\delta x}A\right) T_A \tag{16}$$

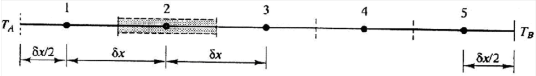
Observation: The fixed value B.C. at boundary A acts like a source term $(S_u + S_p T_P)$ with

$$S_u = \frac{2kA}{\delta x} T_A \qquad S_p = -\frac{2kA}{\delta x}, \tag{17}$$

and the link to the west interface has been suppressed by setting $a_W = 0$.

Examples

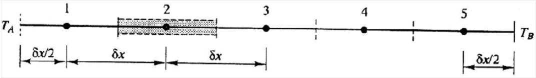
- Example 1: 1D steady-state heat conduction with no source



Do the same for the boundary cell 5, we have

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right) T_P = \left(\frac{k}{\delta x}A\right) \cdot T_W + 0 \cdot T_E + \left(\frac{2k}{\delta x}A\right) T_B \tag{18}$$

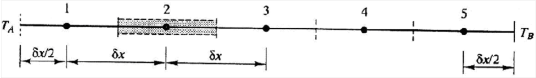
- Example 1: 1D steady-state heat conduction with no source



The above process yields one discretized equation for each cell. So there will be a total of 5 equations. Plugging the values of the parameters, we have the following set of algebraic equations

$$\begin{bmatrix} 300 & -100 & 0 & 0 & 0 \\ -100 & 200 & -100 & 0 & 0 \\ 0 & -100 & 200 & -100 & 0 \\ 0 & 0 & -100 & 200 & -100 \\ 0 & 0 & 0 & -100 & 300 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 200\,T_A \\ 0 \\ 0 \\ 0 \\ 200\,T_B \end{bmatrix} \tag{19}$$

- Example 1: 1D steady-state heat conduction with no source



Solving the set of algebraic equations (more on how to solve it later), we have the solutions

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 140 \\ 220 \\ 300 \\ 380 \\ 460 \end{bmatrix} \tag{20}$$

The exact solution is a linear distribution between the specified boundary temperatures: $T(x) = 800x + 100$. The numerical solution fits perfectly.

Examples

- Example 1: 1D steady-state heat conduction with no source

This example has been implemented in OpenFOAM and the solver name is *steadyDiffusionFoam*.

```
fvScalarMatrix TEqn
(
    fvm::laplacian(DT, T)
);

//save the matrix and source to a file
saveMatrix(TEqn);
```

Examples

Some notes about the `saveMatrix(TEqn)` part:

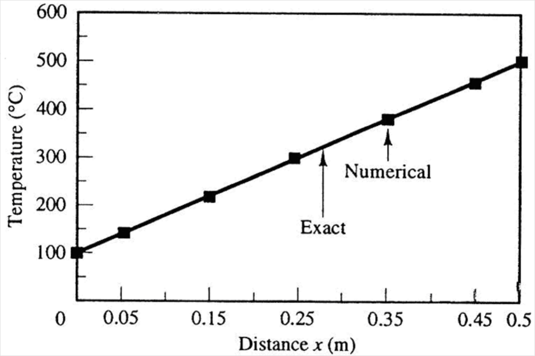
- Some auxiliary functions have been implemented in *extract_ matrix.H* to save matrix coefficients.
- To understand what is going on in these functions, one needs to know how matrix is stored in OpenFOAM® .
- Most important of all, for example, the class *fvScalarMatrix*.
- After discretization, the governing equation becomes a linear system $Ax = b$ stored in a *fvScalarMatrix* object.
- *fvScalarMatrix* usually does not consider the effect of boundary conditions.
- B.C.s are assembled to the matrix and source right before the solve step (in this example, *TEqn.solve()*).

Examples

- Example 1: 1D steady-state heat conduction with no source

This example case is called *1D_rod*. The setup is the same as the example. Note the following setup in *system/fvSchemes* file:

- *steadyState* for the *ddtSchemes* entry
- *Gauss linear corrected* for the *laplacian(DT,T)* entry: linear interpolation for `DT` and central difference for gradient.



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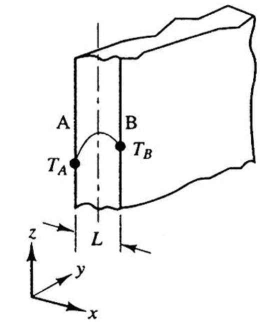
- Example 2: 1D steady-state heat conduction with source

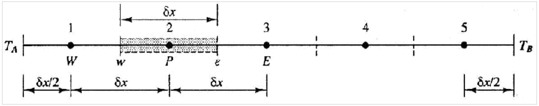
$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+q=0 \tag{21}$$

In this example, it is a large plate of thickness $L = 2$ cm, constant $k = 0.5$ W/m/K and uniform heat generation $q = 10^6\ W/m^3$. Face A and B have fixed temperatures of $100\,^\circ C$ and $200\,^\circ C$, respectively. It can be modeled as a 1D problem in x direction only.



- Example 2: 1D steady-state heat conduction with source

The total thickness of the plate L is divided into 5 equal size pieces such that $\delta x = 0.004$ m; a unit area is considered in the $y - z$ plane.

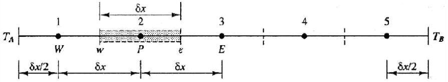


The mesh has five cells. Cells 2, 3, and 4 are internal. Cells 1 and 5 are adjacent to the boundaries.

- Example 2: 1D steady-state heat conduction with source

General discretization equation:

$$\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e + \frac{\Gamma_w}{\delta x_{WP}}A_w - S_p\right)}_{a_P}\phi_P = \underbrace{\left(\frac{\Gamma_w}{\delta x_{WP}}A_w\right)}_{a_W}\phi_W + \underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)}_{a_E}\phi_E + S_u \quad (22)$$



The discretized equation for the internal cells 2, 3, and 4 can be written as

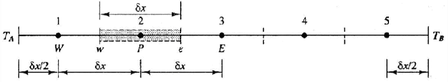
$$a_P T_P = a_W T_W + a_E T_E + S_u \quad (23)$$

where $a_W = \frac{k}{\delta x}A$, $a_E = \frac{k}{\delta x}A$, and $a_P = a_W + a_E - S_p = 2\frac{k}{\delta x}A$, $S_u = qA\delta x$, $S_p = 0$.

- Example 2: 1D steady-state heat conduction with source

General discretization equation:

$$\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e+\frac{\Gamma_w}{\delta x_{WP}}A_w-S_p\right)}_{a_P}\phi_P=\underbrace{\left(\frac{\Gamma_w}{\delta x_{WP}}A_w\right)}_{a_W}\phi_W+\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)}_{a_E}\phi_E+S_u \quad (24)$$



The discretized equation for the boundary cell 1 can be written as

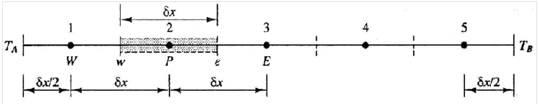
$$kA\left(\frac{T_E-T_P}{\delta x}\right)-kA\left(\frac{T_P-T_A}{\delta x/2}\right)+qA\delta x=0 \quad (25)$$

where the flux through the boundary A has been approximated by assuming a linear distribution of temperature between boundary A and cell center 1. This equation can be re-arranged to have

$$\left(\frac{k}{\delta x}A+\frac{2k}{\delta x}A\right)T_P=0\cdot T_W+\left(\frac{k}{\delta x}A\right)T_E+qA\delta x+\left(\frac{2k}{\delta x}A\right)T_A \quad (26)$$

Examples

- Example 2: 1D steady-state heat conduction with source



This equation can be re-arranged to have

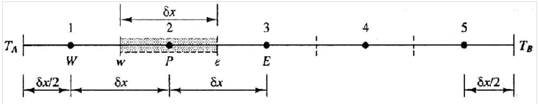
$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right) T_P = 0 \cdot T_W + \left(\frac{k}{\delta x}A\right) T_E + qA\delta x + \left(\frac{2k}{\delta x}A\right) T_A \qquad (27)$$

Observation: In addition to the source term q, the fixed value B.C. at boundary A acts like a source term $(S_u + S_p T_P)$ with

$$S_u = \frac{2kA}{\delta x} T_A \qquad S_p = -\frac{2kA}{\delta x}, \qquad (28)$$

and the link to the west interface has been suppressed by setting $a_W = 0$.

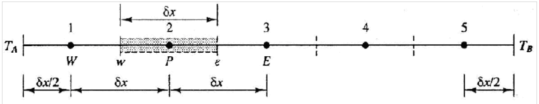
- Example 2: 1D steady-state heat conduction with source



Do the same for the boundary cell 5, we have

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right) T_P = \left(\frac{k}{\delta x}A\right) \cdot T_W + 0 \cdot T_E + qA\delta x + \left(\frac{2k}{\delta x}A\right) T_B \tag{29}$$

- Example 2: 1D steady-state heat conduction with source

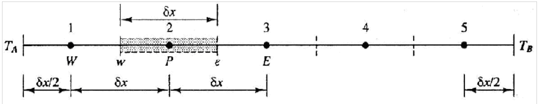


The above process yields one discretized equation for each cell. So there will be a total of 5 equations. Plugging the values of the parameters, we have the following set of algebraic equations

$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ -125 & 250 & -125 & 0 & 0 \\ 0 & -125 & 250 & -125 & 0 \\ 0 & 0 & -125 & 250 & -125 \\ 0 & 0 & 0 & -125 & 375 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29000 \\ 4000 \\ 4000 \\ 4000 \\ 29000 \end{bmatrix} \tag{30}$$

Examples

- Example 2: 1D steady-state heat conduction with source



Solving the set of algebraic equations, we have the solutions

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 150 \\ 218 \\ 254 \\ 258 \\ 230 \end{bmatrix} \tag{31}$$

The exact solution is a parabolic distribution between the specified boundary temperatures: $T(x) = \left[\frac{T_B - T_A}{L} + \frac{q}{2k}(L - x)\right] x + T_A$. The numerical solution fits well.

Examples

- Example 2: 1D steady-state heat conduction with source

This example has been implemented in OpenFOAM® and the solver name is *steadyDiffusionWithSourceFoam*.

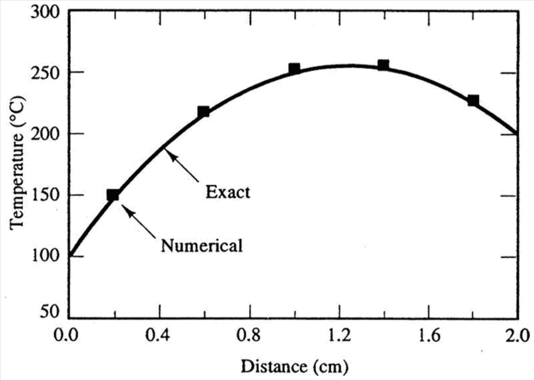
```
fvScalarMatrix TEqn
(
    fvm::laplacian(DT, T)
==
    -q
);

//save the matrix and source to a file
saveMatrix(TEqn);
```

The same auxiliary functions implemented in *extract_matrix.H* are used to save matrix coefficients.

- Example 2: 1D steady-state heat conduction with source

This example case is called *1D_plate_with_src*. The setup is the same as the example. Note the following setup in *system/fvSchemes* file:

- *steadyState* for the *ddtSchemes* entry
- *Gauss linear corrected* for the *laplacian(DT,T)* entry



Examples

- Example 3: 1D steady-state heat conduction with source and flux B.C.

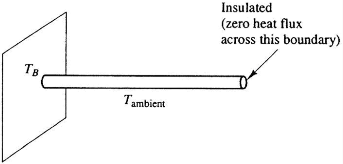
$$\frac{d}{dx}\left(\frac{dT}{dx}\right) - n^2\left(T - T_\infty\right) = 0 \tag{32}$$

where $n^2 = hp/(kA)$, h is the convective heat transfer coefficient, P the perimeter, k the thermal conductivity, and T_∞ the ambient temperature. In this example, a rod ($L = 1$ m) is placed in a space with constant ambient temperature $T_\infty = 20\,^\circ C$. $n^2 = 25/m^2$. One side (A) is insulated, which means zero heat flux, i.e.,

$$\bar{\mathbf{n}} \cdot (\nabla T) = 0 \tag{33}$$

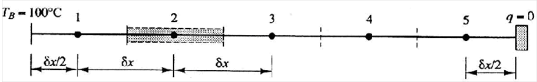
The other side (B) is kept at constant temperature $T_B = 100\,^\circ C$. The analytical solution is

$$\frac{T(x) - T_\infty}{T_B - T_\infty} = \frac{\cosh[n(L-x)]}{\cosh(nL)} \tag{34}$$



- Example 3: 1D steady-state heat conduction with source and flux B.C.

The total length of the rod L is divided into 5 equal size pieces such that $\delta x = 0.2$ m; a unit area is considered in the $y - z$ plane (area does not matter).



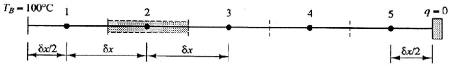
The mesh has five cells. Cells 2, 3, and 4 are internal. Cells 1 and 5 are adjacent to the boundaries.

Examples

- Example 3: 1D steady-state heat conduction with source and flux B.C.

General discretization equation:

$$\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e + \frac{\Gamma_w}{\delta x_{WP}}A_w - S_p\right)}_{a_P}\phi_P = \underbrace{\left(\frac{\Gamma_w}{\delta x_{WP}}A_w\right)}_{a_W}\phi_W + \underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)}_{a_E}\phi_E + S_u \quad (35)$$



The discretized equation for the internal cells 2, 3, and 4 can be written as

$$a_P T_P = a_W T_W + a_E T_E + S_u \quad (36)$$

where $a_W = \frac{1}{\delta x}$, $a_E = \frac{1}{\delta x}$, and $a_P = a_W + a_E - S_p = 2\frac{1}{\delta x} + n^2\delta x$, $S_u = n^2\delta x T_\infty$, $S_p = -n^2\delta x$.

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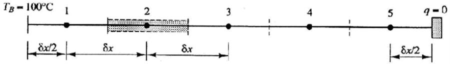
- Example 3: 1D steady-state heat conduction with source and flux B.C.

General discretization equation:

$$\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e+\frac{\Gamma_w}{\delta x_{WP}}A_w-S_p\right)}_{a_P}\phi_P=\underbrace{\left(\frac{\Gamma_w}{\delta x_{WP}}A_w\right)}_{a_W}\phi_W+\underbrace{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)}_{a_E}\phi_E+S_u \quad (37)$$



The discretized equation for the boundary cell 1 can be written as

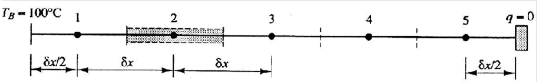
$$\left[\left(\frac{T_E-T_P}{\delta x}\right)-\left(\frac{T_P-T_B}{\delta x/2}\right)\right]-\left[n^2\left(T_P-T_\infty\right)\delta x\right]=0 \quad (38)$$

where the flux through the boundary B has been approximated by assuming a linear distribution of temperature between boundary B and cell center 1. This equation can re-arranged to have

$$\left(\frac{1}{\delta x}+\frac{2}{\delta x}+n^2\delta x\right)T_P=0\cdot T_W+\left(\frac{1}{\delta x}\right)T_E+n^2\delta x T_\infty+\frac{2}{\delta x}T_B \quad (39)$$

Examples

- Example 3: 1D steady-state heat conduction with source and flux B.C.



The re-arranged equation:

$$\left(\frac{1}{\delta x} + \frac{2}{\delta x} + n^2 \delta x\right) T_P = 0 \cdot T_W + \left(\frac{1}{\delta x}\right) T_E + n^2 \delta x T_\infty + \frac{2}{\delta x} T_B \qquad (40)$$

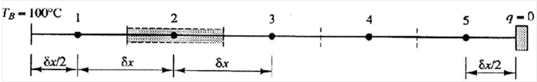
Observation: In addition to the source term resulting from the heat transfer with the ambient, the fixed value B.C. at boundary B acts like a source term $(S_u + S_p T_P)$ with

$$S_u = \frac{2}{\delta x} T_B \qquad S_p = -\frac{2}{\delta x}, \qquad (41)$$

and the link to the west interface has been suppressed by setting $a_W = 0$.

Examples

- Example 3: 1D steady-state heat conduction with source and flux B.C.



The boundary cell 5 is a little different since a flux B.C. is specified. Based on this, the flux through the east face of cell 5 is zero. So,

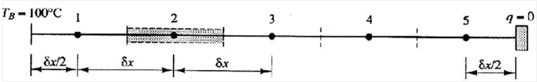
$$\left[0 - \left(\frac{T_P - T_W}{\delta x}\right)\right] - \left[n^2 \left(T_P - T_\infty\right) \delta x\right] = 0 \tag{42}$$

It can be re-arranged to:

$$\left(\frac{1}{\delta x} + n^2 \delta x\right) T_P = \frac{1}{\delta x} \cdot T_W + 0 \cdot T_E + n^2 \delta x T_\infty \tag{43}$$

Examples

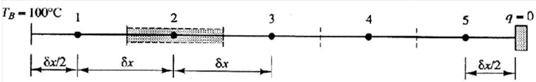
- Example 3: 1D steady-state heat conduction with source and flux B.C.



The above process yields one discretized equation for each cell. So there will be a total of 5 equations. Plugging the values of the parameters, we have the following set of algebraic equations

$$\begin{bmatrix} 20 & -5 & 0 & 0 & 0 \\ -5 & 15 & -5 & 0 & 0 \\ 0 & -5 & 15 & -5 & 0 \\ 0 & 0 & -5 & 15 & -5 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 1100 \\ 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} \tag{44}$$

- Example 3: 1D steady-state heat conduction with source and flux B.C.



Solving the set of algebraic equations, we have the solutions

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 64.22 \\ 36.91 \\ 26.50 \\ 22.60 \\ 21.30 \end{bmatrix} \tag{45}$$

Examples

- Example 3: 1D steady-state heat conduction with source and flux B.C.

This example has been implemented in OpenFOAM® and the solver name is *steadyDiffusionWithCoolingFoam*.

```
fvScalarMatrix TEqn
(
    fvm::laplacian(T)
  ==
    fvm::Sp(alpha,T)
  - alpha*Tinf
);

//save the matrix and source to a file
saveMatrix(TEqn);
```

Here $\alpha = n^2$.
The same auxiliary functions implemented in *extract_matrix.H* are used to save matrix coefficients.

Examples

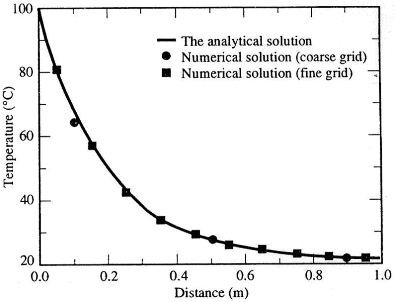
- Example 3: 1D steady-state heat conduction with source and flux B.C.

This example case is called *1D_rod_convective_cooling*. The setup is the same as the example. Note the following setup in *system/fvSchemes* file:

- *steadyState* for the *ddtSchemes* entry
- *Gauss linear corrected* for the *laplacian(T)* entry

The parameters are setup in *constant/transportProperties* file:

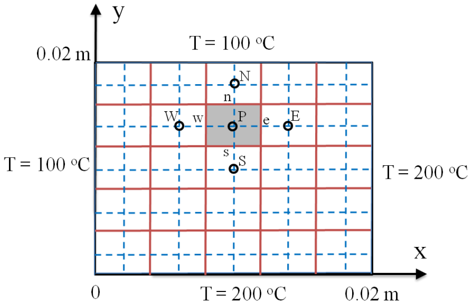
```
alpha              alpha [ 0 -2 0 0 0 0 0 ] 25;
Tinf               Tinf  [ 0  0 0 1 0 0 0 ] 20;
```



- Example 4: 2D steady-state heat conduction with source

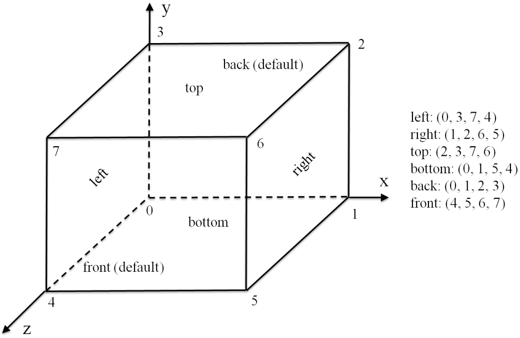
$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\frac{d}{dy}\left(k\frac{dT}{dy}\right)+q=0 \tag{46}$$

In this example, it is a large 2D plate $2\,cm \times 2\,cm$, constant $k = 0.5$ W/m/K and uniform heat generation $q = 10^6\ W/m^3$. West and north faces have fixed temperatures of $100\,^\circ C$ while east and south faces have fixed temperatures of $200\,^\circ C$. , respectively. It can be modeled as a 2D problem in $x - y$ direction only.



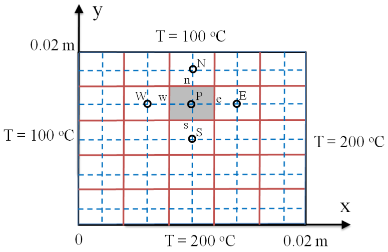
- Example 4: 2D steady-state heat conduction with source

OpenFOAM® setup for the 2D steady-state heat conduction with source case.



Examples

- Example 4: 2D steady-state heat conduction with source

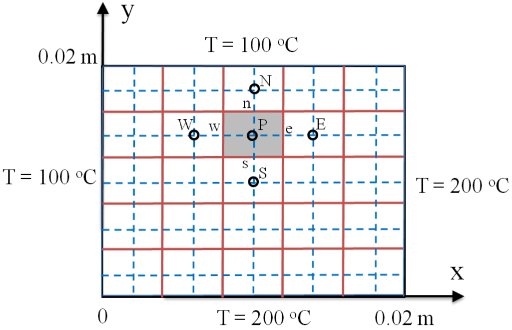


So there are four sides, noted n, e, s, and w. Correspondingly, there are four neighbors, namely N, E, S, and W (if P is not a boundary cell).

For each control volume centered at P, the integration should be carried out around the shaded area.

When dealing with each side, the calculation of fluxes and the interpolation schemes to get the value at the interface are treated as if they are 1D problems in that direction.

- Example 4: 2D steady-state heat conduction with source



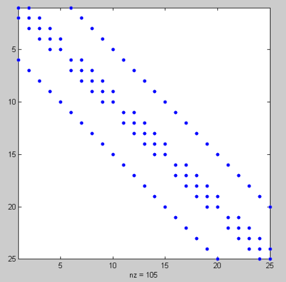
In this example, the x and y directions are divided into 5 equal-size cells,which results into a total of 25 control volumes in the domain.

Examples

- Example 4: 2D steady-state heat conduction with source

In OpenFOAM® , the solver does not need to be changed since it does not care about the dimensionality of the cases. So we still use the solver *steadyDiffusionWithSourceFoam*.

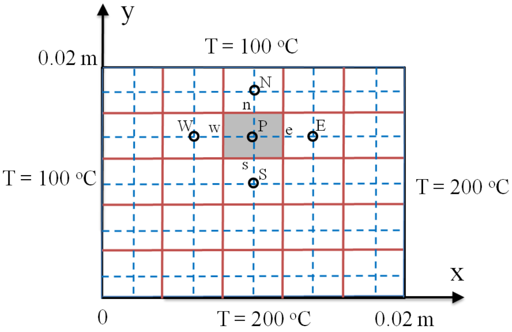
```
fvScalarMatrix TEqn
(
      fvm::laplacian(DT, T)
 ==
      -q
);

//save the matrix and source
//to a file
saveMatrix(TEqn);
```



The symmetrical structure of matrix A (plotted using the *spy* function in *Matlab*.)

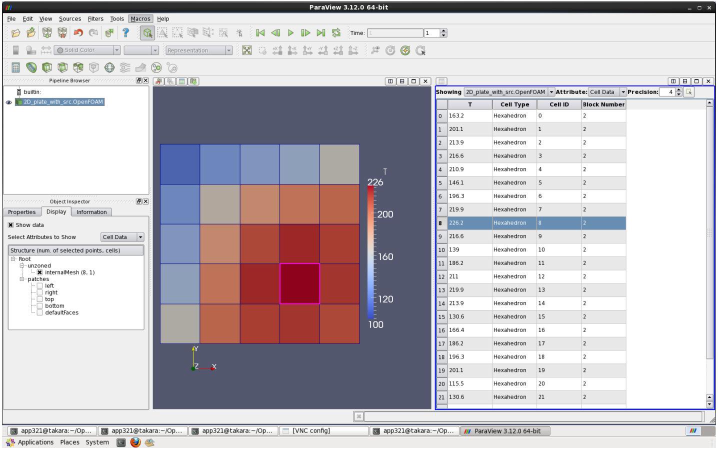
- Example 4: 2D steady-state heat conduction with source

This example case is called *2D_plate_with_src*.
To reflect the 2D nature of the case, you need to specify modify the *blockMeshDict* file and the field file "T" in the "0" directory. Two more boundaries, "top" and "bottom" have been added. Other setups are the same as Example 2.



- Example 4: 2D steady-state heat conduction with source

The simulation result is plotted in this figure. In ParaView, for each control volume, you can examine the *cellID* and field value as shown in the figure. The detailed steps have been documented in a companion tutorial.



	T	Cell Type	Cell ID	Block Number
0	163.2	Hexahedron	0	2
1	201.1	Hexahedron	1	2
2	213.9	Hexahedron	2	2
3	216.6	Hexahedron	3	2
4	210.9	Hexahedron	4	2
5	146.1	Hexahedron	5	2
6	196.3	Hexahedron	6	2
7	219.9	Hexahedron	7	2
8	226.2	Hexahedron	8	2
9	216.6	Hexahedron	9	2
10	139	Hexahedron	10	2
11	186.2	Hexahedron	11	2
12	211	Hexahedron	12	2
13	219.9	Hexahedron	13	2
14	213.9	Hexahedron	14	2
15	130.6	Hexahedron	15	2
16	166.4	Hexahedron	16	2
17	186.2	Hexahedron	17	2
18	196.3	Hexahedron	18	2
19	201.1	Hexahedron	19	2
20	115.5	Hexahedron	20	2
21	130.6	Hexahedron	21	2

Summary

From this section, the following statements can be made regarding the steady-state heat conduction problem $\nabla \cdot (\gamma \nabla \phi) + S_\phi = 0$ (regardless of the dimensionality):

- the general form of the discretized equation is

$$a_P \phi_P = \sum a_{nb} \phi_{nb} + S_u$$

  where $\sum$ indicates summation over all neighbouring nodes (nb).
- in all cases, the coefficient a_P satisfies the following relationship

$$a_P = \sum a_{nb} - S_p$$

- source term can be written generally as $S_\phi \Delta V = S_u + S_p \phi_P$.
- treatment of boundary condition: cutting the link and introducing boundary fluxes.

Questions?