Projection method for the Navier-Stokes Equations in OpenFOAM

1 Navier-Stokes Equations

The Navier-Stokes equations for a single-phase flow with a constant density and viscosity are the following:

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) - \nabla \cdot (\nu \nabla \mathbf{u}) = -\nabla p$$

The solution of this couple of equations is not straightforward because an explicit equation for the pressure is not available.

2 Projection method

The projection method is another class of numerical algorithm to solve the coupled Navier-Stokes equation. The original method was proposed independently by Chorin [1968] and Tmam [1969]. The general idea of the projection method is a fractional-step procedure where the flow velocity field is first estimated and then projected into the space of divergence-free vector fields with appropriate boundary conditions. The mathematical backbone of this method is the Helmholtz-Hodge Decomposition which states that any vector field can be decomposed into a divergence-free part and an irrotational part. Since the 1960s, the project method has been developed further and today it is still a popular method in computational fluid dynamics.

2.1 Helmholtz-Hodge decomposition

2.2 Non

Explicit projection method Chorin's projection method.

one first computes an intermediate velocity, $\mathbf{u}^*$, explicitly using the momentum equation by ignoring the pressure gradient term:

$$(1) \qquad \frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n + \nu \nabla^2 \mathbf{u}^n \tag{1}$$

where $\mathbf{u}^n$ is the velocity at $n - th$ time step. In the second half of the algorithm, the "projection" step, we correct the intermediate velocity to obtain the final solution of the time step $\mathbf{u}^{n+1}$:

$$(2) \qquad \mathbf{u}^{n+1} = \mathbf{u}^* - \frac{\Delta t}{\rho} \nabla p^{n+1}$$

One can rewrite this equation in the form of a time step as

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\frac{1}{\rho} \nabla p^{n+1}$$

to make clear that the algorithm is really just an operator splitting approach in which one considers the viscous forces (in the first half step) and the pressure forces (in the second half step) separately.

Computing the right-hand side of the second half step requires knowledge of the pressure, p, at the $(n+1)$ time level. This is obtained by taking the divergence and requiring that $\nabla \cdot \mathbf{u}^{n+1} = 0$, which is the divergence (continuity) condition, thereby deriving the following Poisson equation for p^{n+1},

$$\nabla^2 p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \mathbf{u}^*$$

It is instructive to note that the equation written as

$$\mathbf{u}^* = \mathbf{u}^{n+1} + \frac{\Delta t}{\rho} \nabla p^{n+1}$$

is the standard Hodge decomposition if boundary condition for p on the domain boundary, $\partial\Omega$ are $\nabla p^{n+1} \cdot \mathbf{n} = 0$. In practice, this condition is responsible for the errors this method shows close to the boundary of the domain since the real pressure (i.e., the pressure in the exact solution of the Navier-Stokes equations) does not satisfy such boundary conditions.

For the explicit method, the boundary condition for $\mathbf{u}^*$ in equation (1) is natural. If $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, is prescribed, then the space of divergence-free vector fields will be orthogonal to the space of irrotational vector fields, and from equation (2) one has

$$\frac{\partial p^{n+1}}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega$$

The explicit treatment of the boundary condition may be circumvented by using a staggered grid and requiring that $\nabla \cdot \mathbf{u}^{n+1}$ vanish at the pressure nodes that are adjacent to the boundaries.

A distinguishing feature of Chorin's projection method is that the velocity field is forced to satisfy a discrete continuity constraint at the end of each time step.

References Cited

References

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