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Chapter 5: Solution of N-S Equations-Part 2

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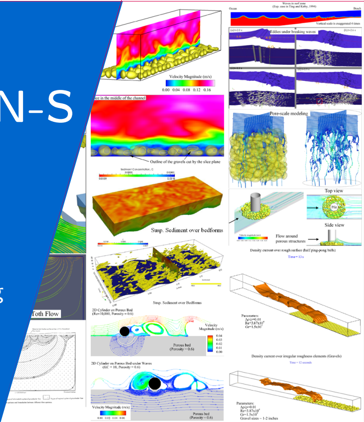

Outline

Solution of NS Equations
- Projection method

General overview

What will be covered in this chapter?

- General overview of the pressure–velocity coupling
- Solution algorithms
  - **Segregated** algorithms: u, v, w and p fields are solved separately. Coupling between these variables are through velocity and pressure corrections.
    - Iterative algorithms: SIMPLE, PISO, etc.
    - Projection method
  - **Coupled** algorithms: All fields are solved in one shot!
    - Not covered in this course.
    - Can be achieved through the new $block - matrix$

Projection method

Unsteady, 3D Navier-Stokes equations for incompressible flow on a closed domain V:

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p + \nu \nabla^2 \mathbf{u} \tag{2}$$

For simplicity, assume on the solid boundary conditions $\mathbf{u} = 0$ on ∂V.

Projection method

History and background:

- Another class of segregated algorithm to deal with the $\mathbf{u}$–p coupling
- Original method proposed in Chorin (1968) and Temam (1969)
- A lot of further development ever since
- I have seen/used projection method for high resolution simulations such as DNS for turbulence
  - Easy to be combined with high-order temporal discretization schemes such as RK4
- The projection method is a generic solution algorithm. It is not bound by any specific spatial discretization scheme.

Projection method

Important references:

- A. J. Chorin. Numerical solution of the Navier-Stokes equations. Mathematics of computation, 22(104):745-762, 1968.
- R. Temam. Sur l'approximation de la solution des quations de Navier-Stokes par la mthode des pas fractionnaires (ii). Archive for Rational Mechanics and Analysis, 33(5):377-385, 1969.
- J. Kim and P. Moin, Application of a fractional-step method to incompressible Navier-Stokes equations, J. Comput. Phys. 59, 308 (1985).
- J. van Kan. A second-order accurate pressure-correction scheme for viscous incompressible flow. SIAM Journal on Scientific and Statistical Computing, 7(3):870-891, 1986.
- J. B. Bell, P. Colella, and H. M. Glaz. A second-order projection method for the incompressible Navier-Stokes equations. Journal of Computational Physics, 85(2):257-283, 1989.
- D. L. Brown, R. Cortez, and M. L. Minion. Accurate projection methods for the incompressible Navier-Stokes equations. Journal of Computational Physics, 168(2):464-499, 2001.

Projection method

Basic idea of projection method:

- Basically a fractional-step method.
- First a flow velocity field is estimated from momentum equation
  - Non-incremental pressure correction scheme: if pressure is not included in the velocity estimation
  - Incremental pressure correction scheme: if pressure is included
- Then the velocity is projected into the space of divergence-free vectors with appropriate boundary condition
- The mathematical backbone of the projection method is the Helmholtz-Hodge decomposition

Projection method

Helmholtz-Hodge decomposition:

- A vector field $\mathbf{u}$ can be decomposed uniquely into a solenoidal (divergence-free) part and an irrotational part

$$\mathbf{u} = \mathbf{u}_S + \mathbf{u}_I \tag{3}$$

where

$$\nabla \cdot \mathbf{u}_S = 0, \qquad \nabla \times \mathbf{u}_I = 0 \tag{4}$$

- If we take divergence on both sides of Eqn. 3, we have

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \nabla \cdot \mathbf{u}_S + \nabla \cdot \mathbf{u}_I \qquad (5) \\ &= \nabla^2 \phi \qquad (6) \end{aligned}$$

because the irrotational part can be written as a gradient of a scalar function ϕ, i.e. $\mathbf{u}_I = \nabla \phi$.

- So we have:

$$\mathbf{u}_S = \mathbf{u} - \nabla \phi \tag{7}$$

Projection method

The original Chorin's projection method:

- First compute an intermediate velocity u^* using the momentum equation and ignoring the pressure gradient term

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n + \nu \nabla^2 \mathbf{u}^n \tag{8}$$

$$\mathbf{u}^* = 0 \text{ on } \partial V \tag{9}$$

  where $\mathbf{u}^n$ is the old velocity at the n-th time step.

- Then `project` (update) the intermediate velocity to get the new velocity $\mathbf{u}^{n+1}$

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \, \nabla p^{n+1} \tag{10}$$

  Compare with the Helmholtz-Hodge decomposition:

$$\mathbf{u}_S = \mathbf{u} - \nabla \phi \tag{11}$$

The original Chorin's projection method:

- As we mentioned, the projection method is in fact a two-step (fractional) method:

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n + \nu \nabla^2 \mathbf{u}^n \tag{12}$$

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = \nabla p^{n+1} \tag{13}$$

- But how do we get p^{n+1}? Answer: divergence-free condition.
- We want our new velocity field $\mathbf{u}^{n+1}$ to be divergence free. Take divergence on both sides of Eqn. 13:

$$\frac{\nabla \cdot \mathbf{u}^{n+1} - \nabla \cdot \mathbf{u}^*}{\Delta t} = \nabla \cdot (\nabla p^{n+1}) \tag{14}$$

then we get a familiar PPE:

$$\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \mathbf{u}^* \tag{15}$$

Projection method

The original Chorin's projection method:

- On a solid boundary, we take dot production of Eqn. 13 with surface normal $\mathbf{n}$, and can get the boundary condition for pressure p^{n+1}:

$$\mathbf{n} \cdot (\nabla p^{n+1}) = \frac{\mathbf{n} \cdot \mathbf{u}^{n+1} - \mathbf{n} \cdot \mathbf{u}^{*}}{\Delta t} = 0 \tag{16}$$

- Careful examination of the pressure B.C. reveals that it only guarantees no-penetration, not no-slip.
- As a result, there might be some small slip velocity on a solid boundary from projection method solution.

Projection method

To recap the original Chorin's projection method:

- First step: estimate $\mathbf{u}^*$

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n + \nu \nabla^2 \mathbf{u}^n \tag{17}$$

$$\text{B.C.: } \mathbf{u}^* = 0 \text{ on } \partial V \tag{18}$$

- Second step: solve PPE to project into divergence free vector space

$$\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \mathbf{u}^* \tag{19}$$

$$\text{B.C.: } \mathbf{n} \cdot (\nabla p^{n+1}) = 0 \tag{20}$$

- Update $\mathbf{u}^{n+1}$:

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \, \nabla p^{n+1} \tag{21}$$

Projection method

Implementation of the original Chorin's projection method in OpenFOAM® :

- Very easy!
- Just solve one unsteady, advection-diffusion equation and one diffusion equation.
- Demo: `ChorinProjectionFOAM`

Projection method

Drawback of the original projection method:

- Momentum is explicitly solved: limitation on time step size (`Courant` number)
- Ambiguity in specifying the B.C. for the intermediate velocity $\mathbf{u}^*$
  - In the prediction of $\mathbf{u}^*$, since pressure gradient is not included, $\mathbf{u}^*$ could be far away from $\mathbf{u}^{n+1}$
  - As a result, specifying the same B.C. for $\mathbf{u}^*$ as $\mathbf{u}^{n+1}$ is not accurate.
- Formal analysis reveals that the boundary condition for pressure lowers the overall solution accuracy to zero order!
- In projection method terminology, this will create an artificial numerical boundary layer and contaminate the whole solution field.

Projection method

Improvements to the original projection method:

- Increase the implicitness of momentum solution.
- For example, Kim and Moin (1986) used extrapolation scheme (second-order accurate Adams-Bashforth) for the advection term and a Crank-Nicholson scheme for the diffusion term:

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\frac{3\mathbf{u}^n \cdot \nabla \mathbf{u}^n - \mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^{n-1}}{2} + \frac{\nu}{2}\left(\nabla^2 \mathbf{u}^* + \nabla^2 \mathbf{u}^n\right) \tag{22}$$

- They also added some corrections to the $\mathbf{u}^*$ boundary condition
- A good paper to read is Brown et al., JCP, 2001.

Projection method

Development in OpenFOAM® :

- Official release of OpenFOAM® does not implement any projection method
- However, there has been some development by several researchers:
  - `projectionFoam`: https://github.com/asimonder/projectionFoam
    - Onder, A., Meyers, J. (2013). HPC realization of a controlled turbulent jet using OpenFOAM. Open Source CFD International Conference 2013. Hamburg, 24-25 October 2013
    - Onder, A., Meyers, J. (2014). Modification of vortex dynamics and transport properties of transitional axisymmetric jets using zero-net-mass-flux actuation, Physics of Fluids, 26, 075103 (2014)
  - v. Vuorinen at Aalto University School of Engineering, Finland
    - V. Vuorinen, J.-P. Keskinen, C. Duwig, B.J. Boersma, On the implementation of low-dissipative RungeŰKutta projection methods for time dependent flows using OpenFOAM, Computers & Fluids, Volume 93, 10 April 2014, Pages 153-163
    - V. Vuorinen, A. Chaudhari, J.-P. Keskinen, Large-eddy simulation in a complex hill terrain enabled by a compact fractional step OpenFOAM solver, Advances in Engineering Software, Volume 79, January 2015, Pages 70-80

Questions?