CONTROL SYSTEMS THEORY

Transfer Function of Physical Systems

CHAPTER 2

STB 35103

Objectives

- To learn about *transfer functions*.
- To develop mathematical models from schematics of physical system.

Overview

- Review on Laplace transform
- Learn about transfer function
  - Electric network
  - Translational mechanical system
  - Rotational mechanical system
- You will learn how to develop mathematical model.
- Present mathematical representation where the input, output and system are different and separate.
- Solving problems in group and individual

Introduction

- A differential equation
- An equation that involves the **derivatives** of a function as well as the function itself. If **partial derivatives** are involved, the equation is called a **partial differential equation**;

  If only ordinary **derivatives** are present, the equation is called an **ordinary differential equation**.

Introduction

- Differential equations. How to obtain?

Physical law of the process $\longrightarrow$ Differential equation

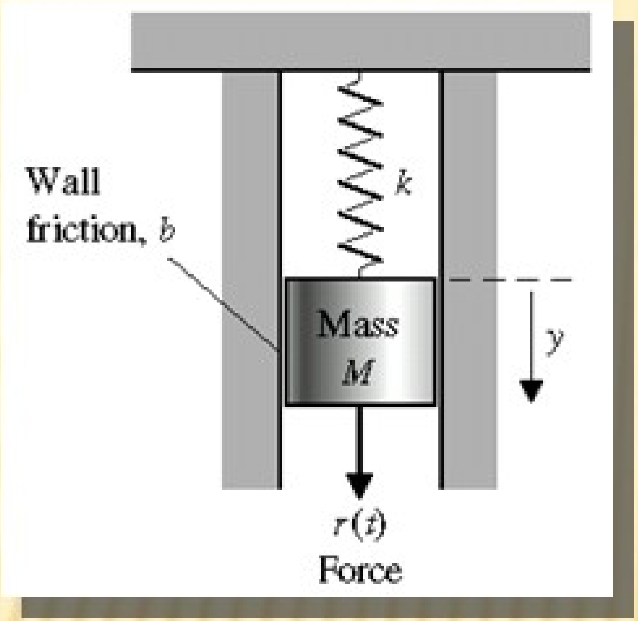
Examples :

Mechanical system (Newton’s laws)

Electrical system (Kirchhoff’s laws)

Introduction

- Example: Springer-mass-damper system



The time function of r(t) sometimes called forcing function

- Assumption: Wall friction is a viscous force.

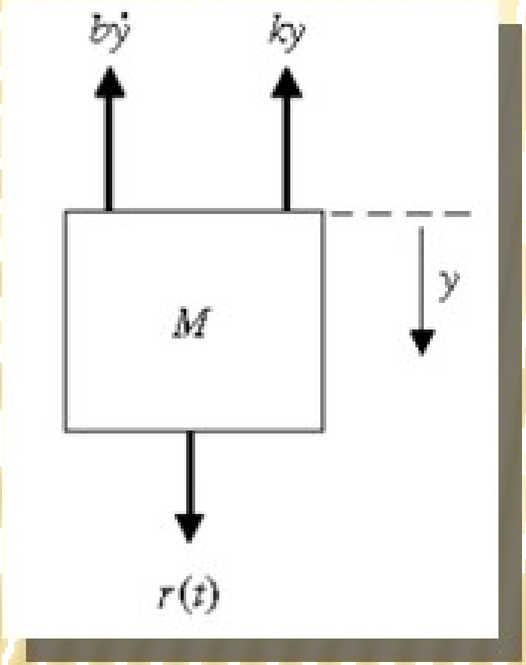
$$\boxed{f(t) = -bv(t)}$$

Linearly proportional to the velocity

Introduction

- Example: Springer-mass-damper system

- Newton's 2^{nd} Law:

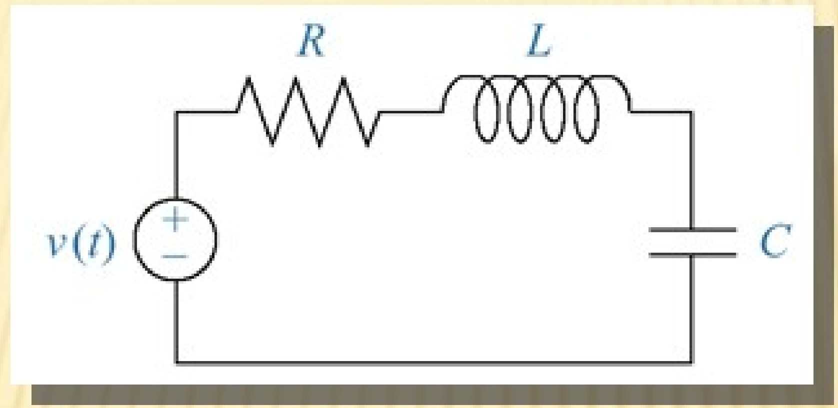


$$\boxed{-bv(t)-ky(t)+r(t)=Ma(t)}$$

$$\boxed{M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)}$$

Introduction

- RLC circuit
  ~KVL
  ~Ohm's law



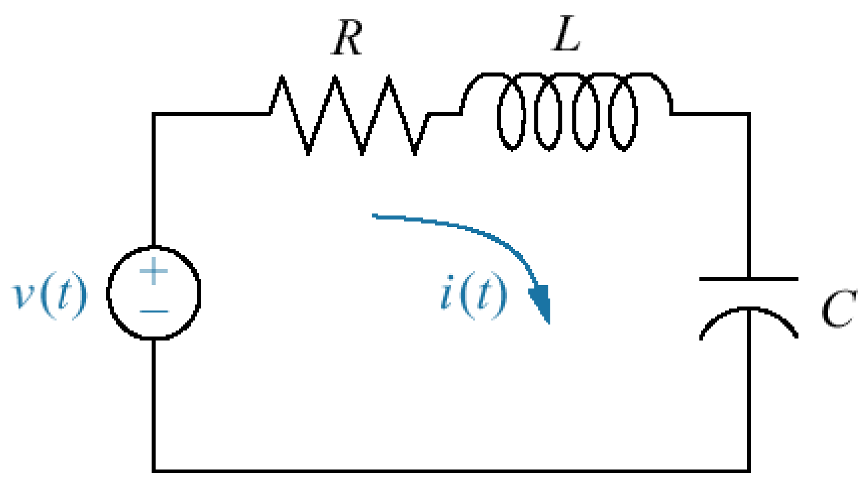
$$\boxed{-v(t)+V_R+V_L+V_c=0}$$

$$\boxed{L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau=v(t)}$$

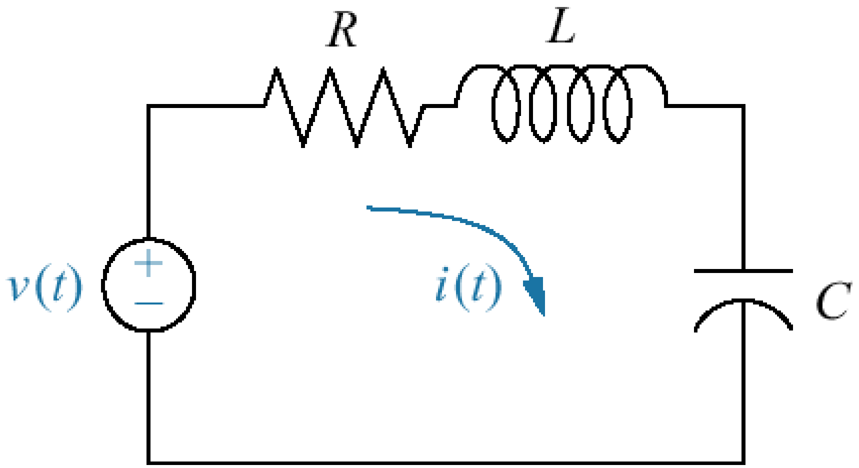
Introduction

- Control system can be represented using a mathematical model.
  - E.g. LED circuit

- Mathematical model is based on the schematic of physical systems.



Introduction



Differential equation

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_{0}^{t} i(\tau)d\tau = v(t)$$

Introduction

- Differential equation describes the relationship between the input and output of a system.

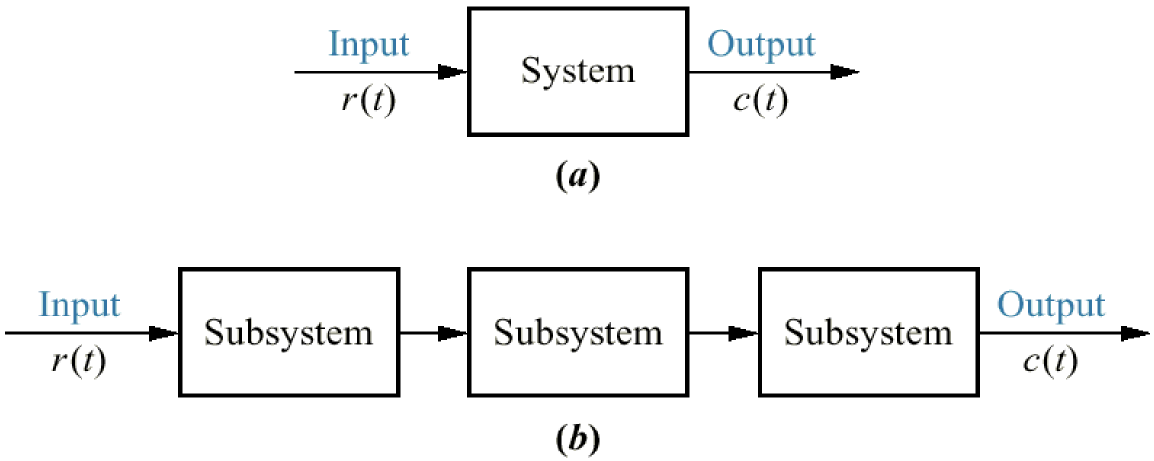
$$\frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

$c(t)$ output

$r(t)$ input

Introduction

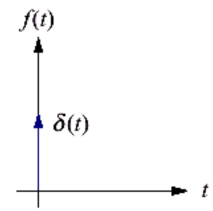
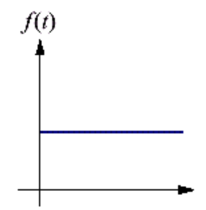
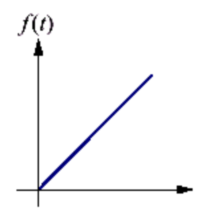
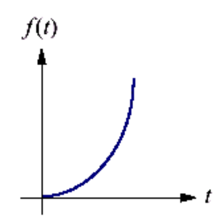
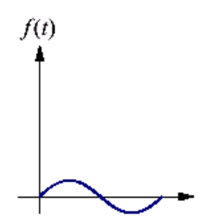
It is easier if we can see the input and the output clearly such as in the figure below.



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

Type of input test

Test waveforms used in control systems

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\,dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

Laplace transform review

Item no.	*f(t)*	*F(s)*
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Table 2.1
Laplace transform table

Laplace transform review

Table 2.2
Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

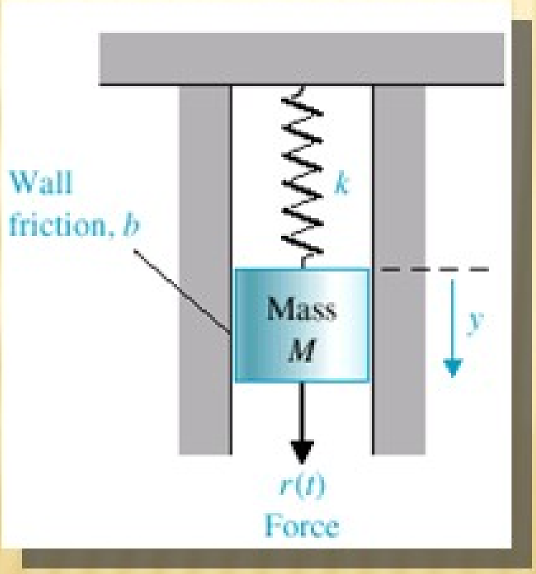
[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

Transfer function

The transfer function of a linear system is the ratio of the Laplace Transform of the output to the Laplace Transform of the input variable.

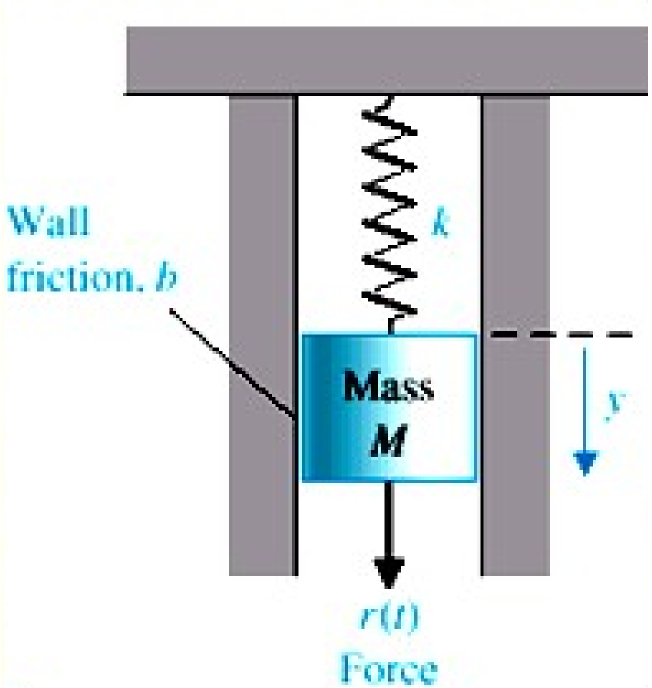
$$\boxed{G(s) = \frac{Output(s)}{Input(s)}}$$

Consider a spring-mass-damper dynamic equation with initial zero condition.

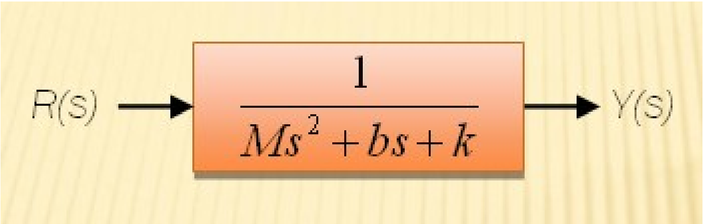


$$\boxed{Ms^2Y(s) + bsY(s) + kY(s) = R(s)}$$

Transfer function



$$Ms^2Y(s) + bsY(s) + kY(s) = R(s)$$

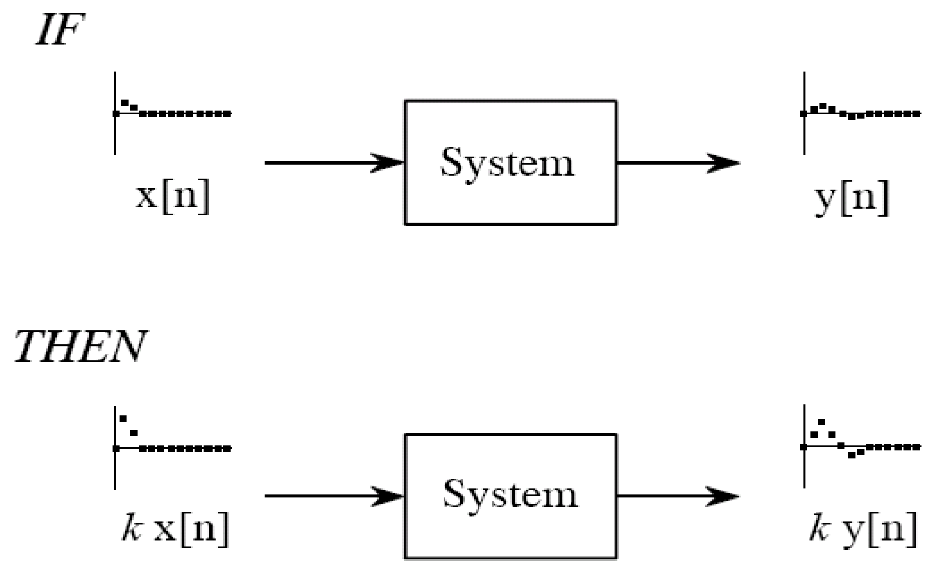


$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}$$

Transfer function

- Limited to linear system
  - What is linear system?
    - A system that has two mathematical properties: **Homogeneity** and **additivity**.

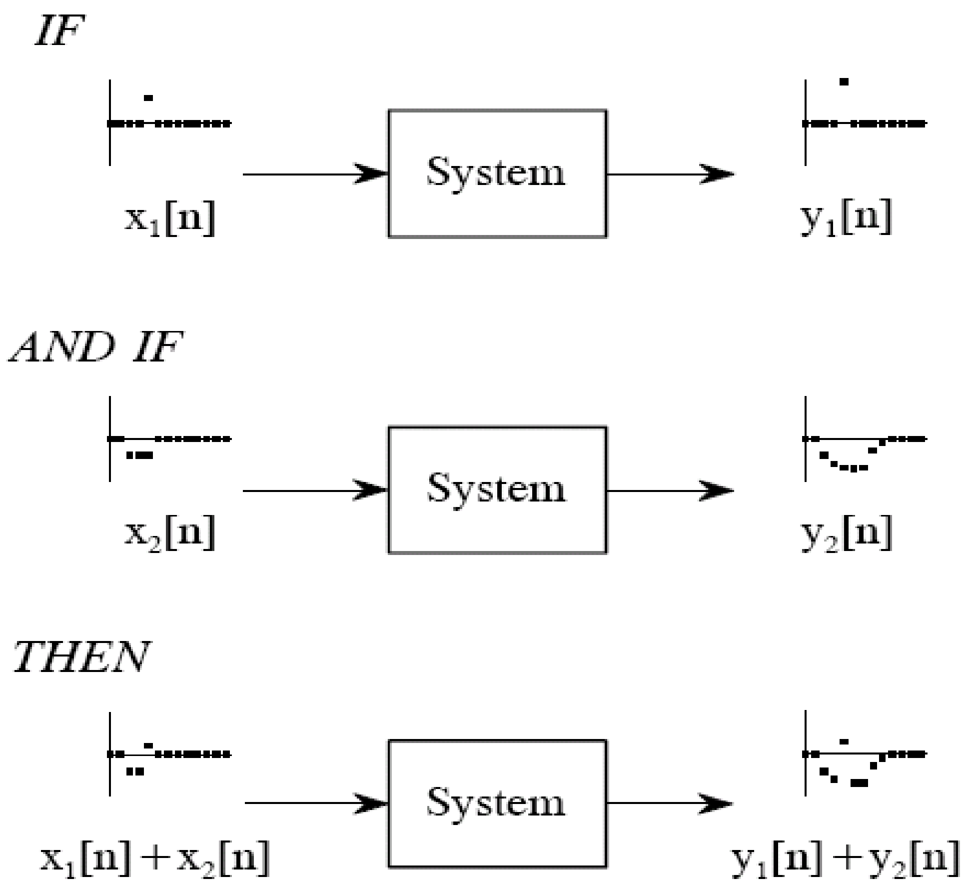
Homogeneity



Definition of homogeneity. A system is said to be *homogeneous* if an amplitude change in the input results in an identical amplitude change in the output. That is, if $x[n]$ results in $y[n]$, then $kx[n]$ results in $ky[n]$, for any signal, $x[n]$, and any constant, k.

Transfer function

Additivity



Definition of additivity. A system is said to be *additive* if added signals pass through it without interacting. Formally, if $x_1[n]$ results in $y_1[n]$, and if $x_2[n]$ results in $y_2[n]$, then $x_1[n]+x_2[n]$ results in $y_1[n]+y_2[n]$.

Transfer function

Example:

Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Solution:

$$sC(s) + 2C(s) = R(s)$$

The transfer function $G(s)$ is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Transfer function

Example:

Find the transfer function represented by

$$\frac{dc(t)}{dt} + 0.5c(t) = 0.3r(t)$$

Solution:

$$sC(s) + 0.5C(s) = 0.3R(s)$$

The transfer function *G*(*s*) is

$$G(s) = \frac{C(s)}{R(s)} = \frac{0.3}{s + 0.5}$$

Transfer function

Example:

Given the transfer function for a system is

$$G(s) = \frac{1}{(s+2)}$$

The input for the system, $r(t)$, is a unit step.

$r(t)=u(t)$, assuming zero initial conditions.

What is the output, $c(t)$, of the system?

Transfer function

Solution:

Refer to Table 2.1. Laplace transform for a unit step input is 1/*s.* We know that

$$\frac{Output}{Input} = Transfer\ Function$$

so

$$\frac{C(s)}{R(s)} = G(s)$$

$$C(s) = R(s)G(s)$$

$$= \frac{1}{s}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{s(s+2)}$$

Transfer function

Expanding the partial fraction, we get

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Taking the inverse Laplace transform (use Table 2.2) of each term,

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

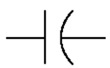
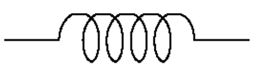
Transfer function

- You are going to apply transfer function in three types of mathematical modeling.
  - Electric network
  - Translational mechanical system
  - Rotational mechanical system

Electric Network Transfer Functions

- We are only going to apply transfer function to the mathematical modeling of electric circuits for passive networks (resistor, capacitor and inductor).
- We will look at a circuit and decide the input and the output.
- We will use Kirchhoff’s laws as our guiding principles.
  - Mesh analysis
  - Nodal analysis

Electric Network Transfer Functions

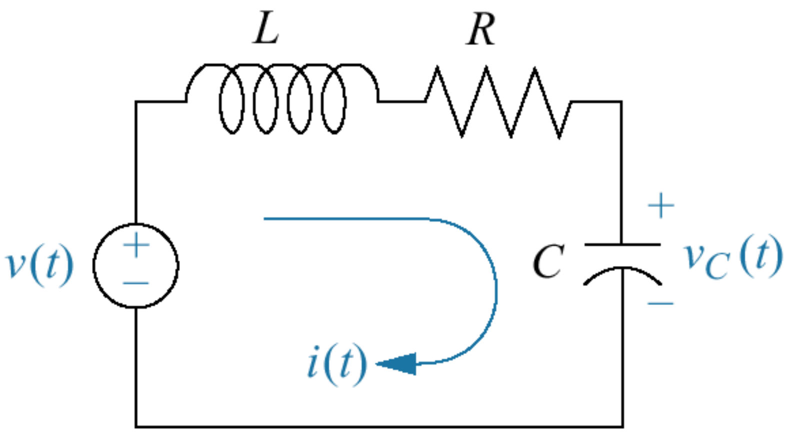
Component	Voltage-Current	Current-voltage	Voltage-charge	**Impedance Z(s)= V(s)/I(s)**	Admittance Y(s)= I(s)/V(s)
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Electric Network Transfer Functions

Simple circuit via mesh analysis

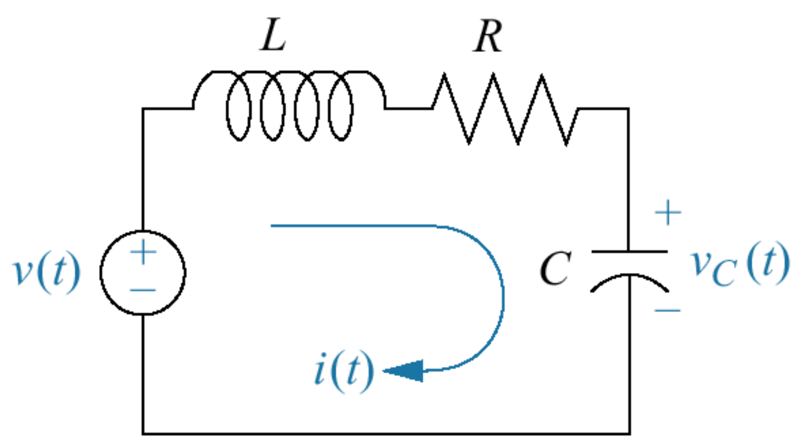
We can obtain transfer function using Kirchhoff's voltage law and summing voltages around loops or meshes. This method is called *loop* or *mesh analysis*.



Electric Network Transfer Functions

Example:

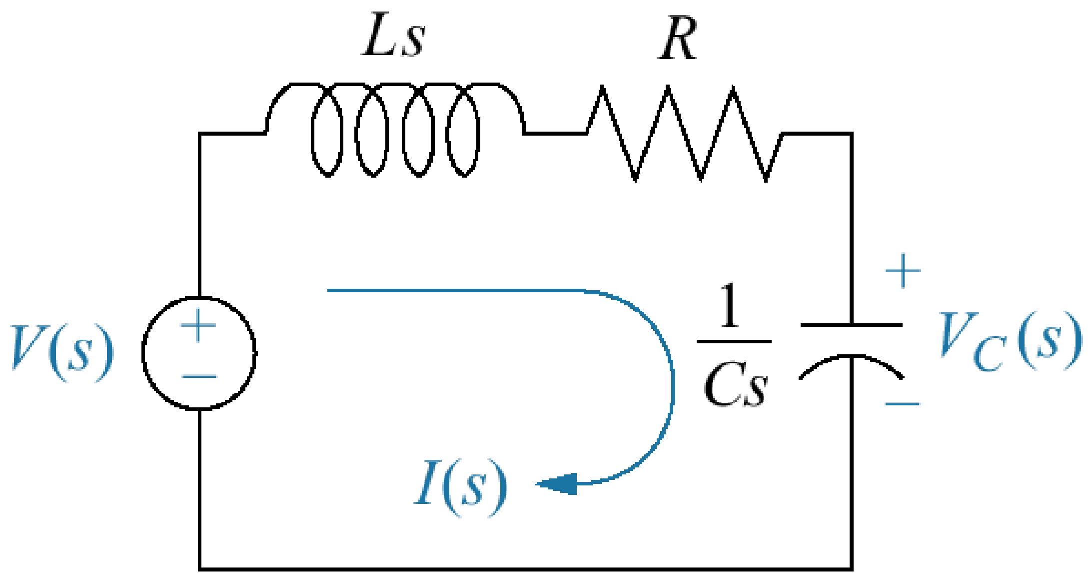
Find the transfer function relating the capacitor voltage, *Vc*(*s*), to the input voltage, *V*(*s*), in figure below



Electric Network Transfer Functions

Solution:

Redraw the circuit using Laplace transform. Replace the component values with their impedance values.



Electric Network Transfer Functions

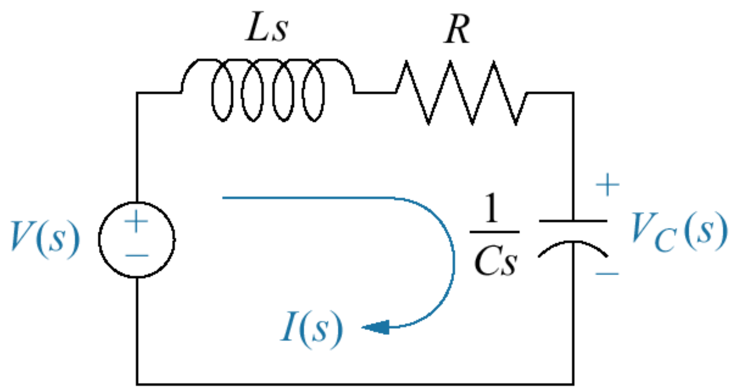
Determine the input and the output for the circuit. For this circuit,

Input is $V(s)$

Output is $V_C(s)$

Next, we write a mesh equation using the impedance as we would use resistor values in a purely resistive circuit.

Electric Network Transfer Functions



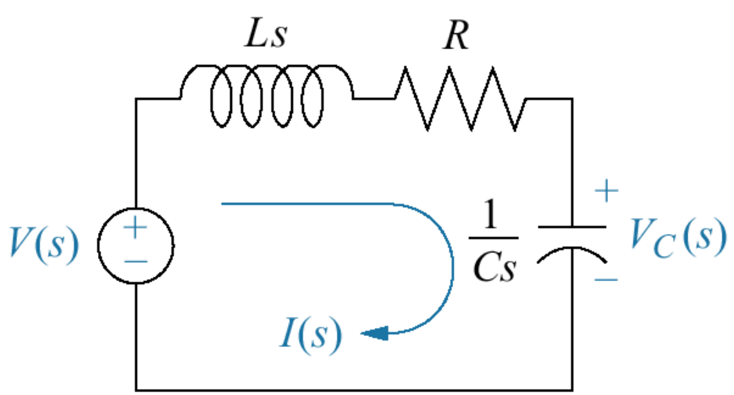
We obtain

$$\left(Ls + R + \frac{1}{Cs} \right) I(s) = V(s) \qquad (2.7)$$

Electric Network Transfer Functions

If we look at Eq.2.7, we can only find the input, $V(s)$. In order to calculate the transfer function we must have the output which is $V_c(s)$.

From the circuit,



$$V_C(s) = I(s)\frac{1}{Cs}$$

Electric Network Transfer Functions

$$V_C(s) = I(s)\frac{1}{Cs}$$

$$I(s) = V_C(s)Cs \qquad (2.8)$$

Substituting Eq.2.8 into Eq.2.7.

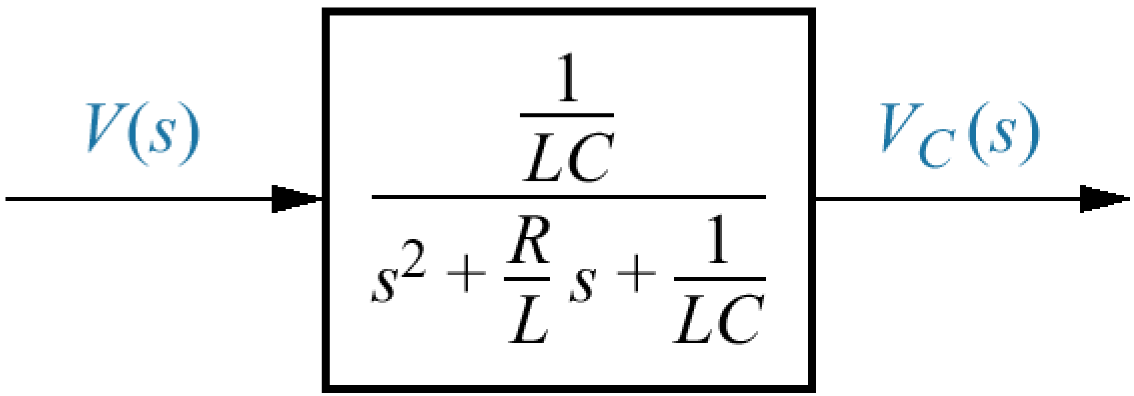
$$\left(Ls + R + \frac{1}{Cs}\right)V_C(s)Cs = V(s)$$

$$\frac{V_C(s)}{V(s)} = \frac{1}{Cs\left(Ls + R + \frac{1}{Cs}\right)} = \frac{1}{CLs^2 + RCs + 1}$$

$$= \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

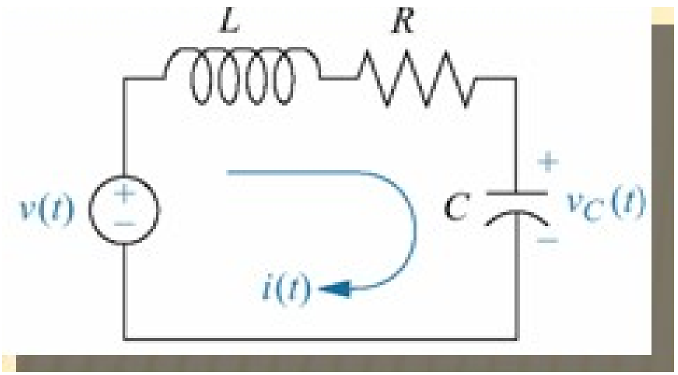
Electric Network Transfer Functions

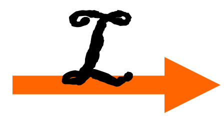
We can also present our answer in block diagram

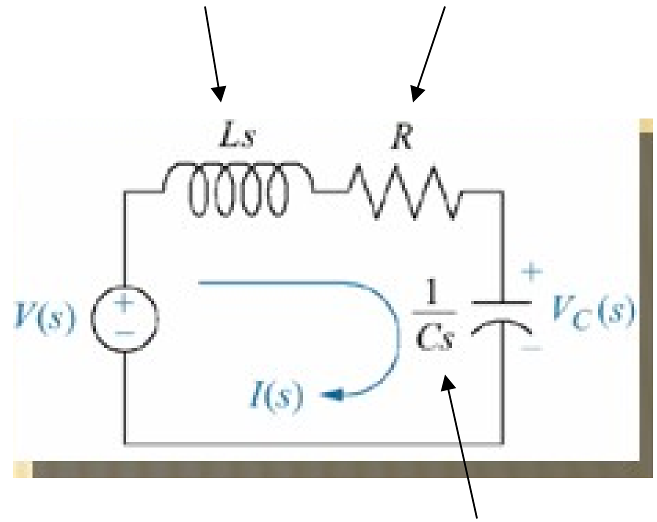


Electric Network Transfer Functions

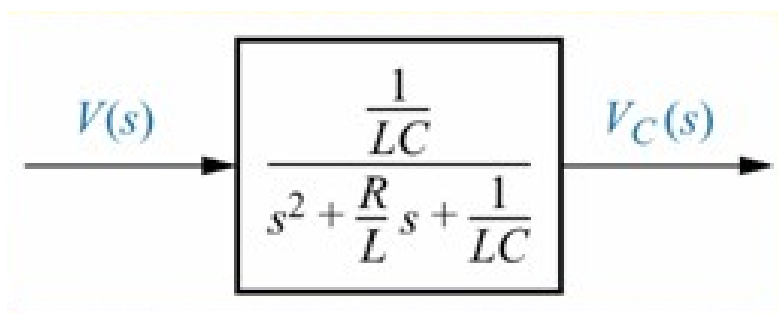
- Solution summary







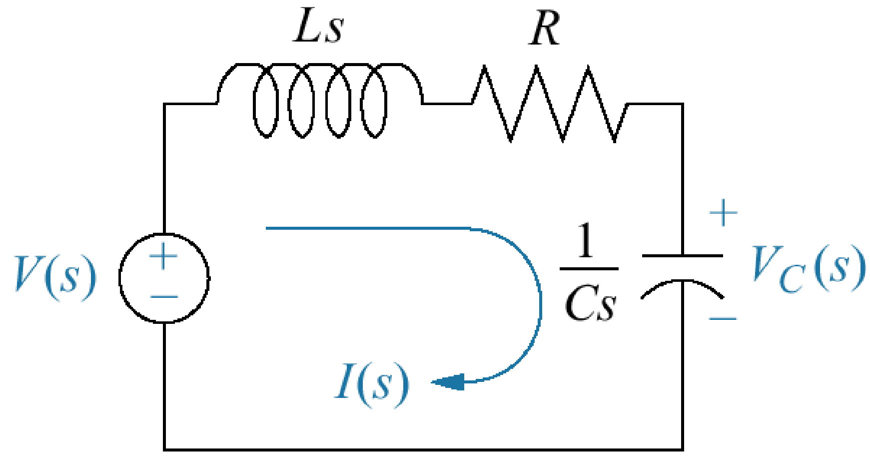
Using mesh analysis



Electric Network Transfer Functions

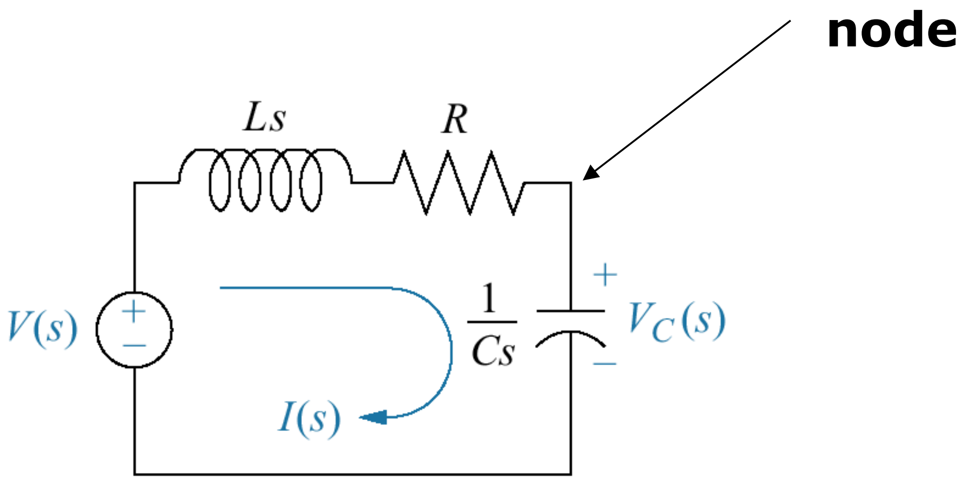
Simple circuit via nodal analysis

We obtain the transfer function using Kirchhoff's current law and summing current flowing from nodes. This method is called *nodal analysis*.



Electric Network Transfer Functions

Example:



We will try to solve the previous example but this time using nodal analysis.

Electric Network Transfer Functions

Solution:

We will look at the current flowing in and out of the node whose voltage is $V_c(s)$. We assume the current leaving the node is positive and current entering the node is negative.

$$\frac{V_C(s)}{1/Cs} = \frac{V(s) - V_C(s)}{R + Ls}$$

$$\frac{V_C(s)}{1/Cs} - \frac{V(s) - V_C(s)}{R + Ls} = 0$$

$$\vdots$$

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Electric Network Transfer Functions

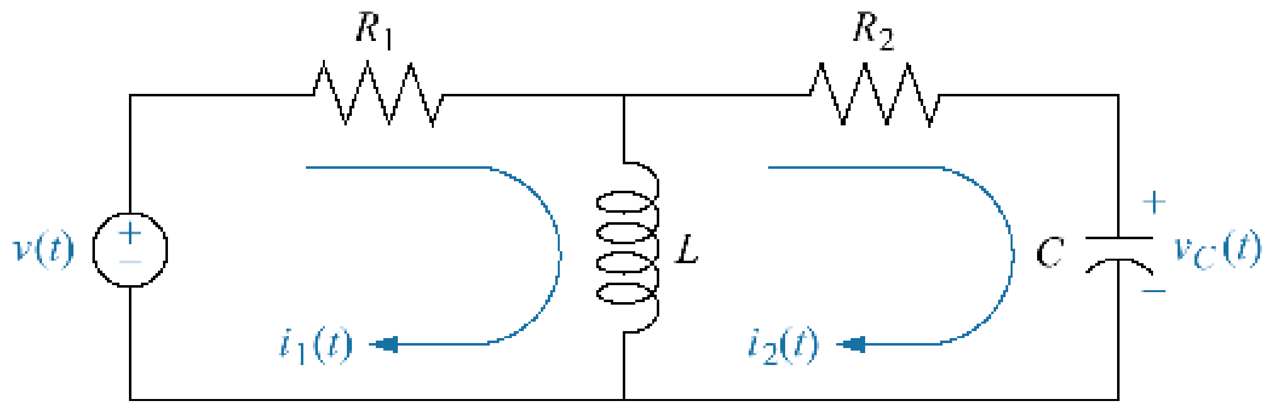
Complex circuit via mesh analysis

To solve a complex circuit we will perform the following steps.

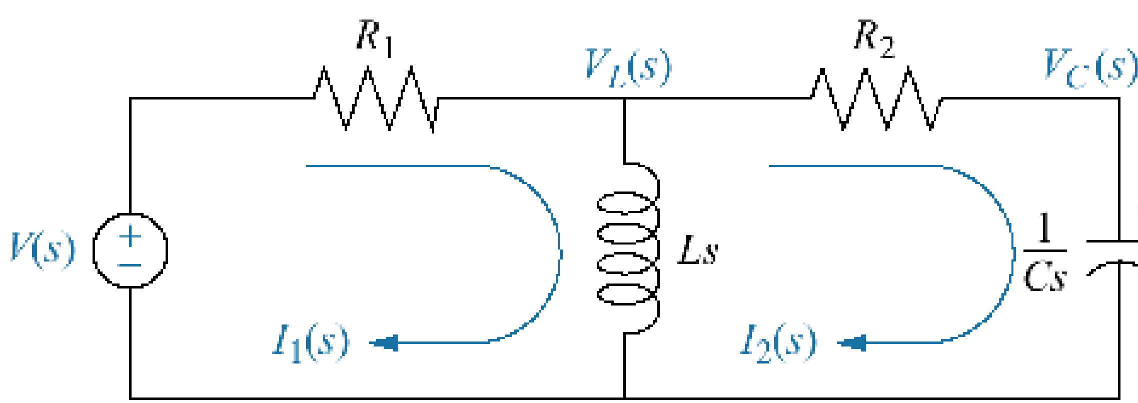
1. Replace passive elements values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.
4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Electric Network Transfer Functions

- Find the transfer function, $I_2(s)/V(s)$



(a)



Output $I_2(s)$

Input V(s)

(b)

Electric Network Transfer Functions

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 1} \end{bmatrix} I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{bmatrix}$$

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 2} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{bmatrix}$$

Electric Network Transfer Functions

$$(R_1 + Ls)\boxed{I_1(s)} - Ls\boxed{I_2(s)} = V(s) \qquad (1)$$

$$-Ls\boxed{I_1(s)} + \left(Ls + R_2 + \frac{1}{Cs}\right)\boxed{I_2(s)} = 0 \qquad (2)$$

We need to solve both equation (1) and (2) to get the value of $I_2(s)$ and V(s).

You can use substitution method or Cramer's rule

Electric Network Transfer Functions

It is easier if we use Cramer's rule

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

where

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls - R_2 + \dfrac{1}{Cs}\right) \end{vmatrix}$$

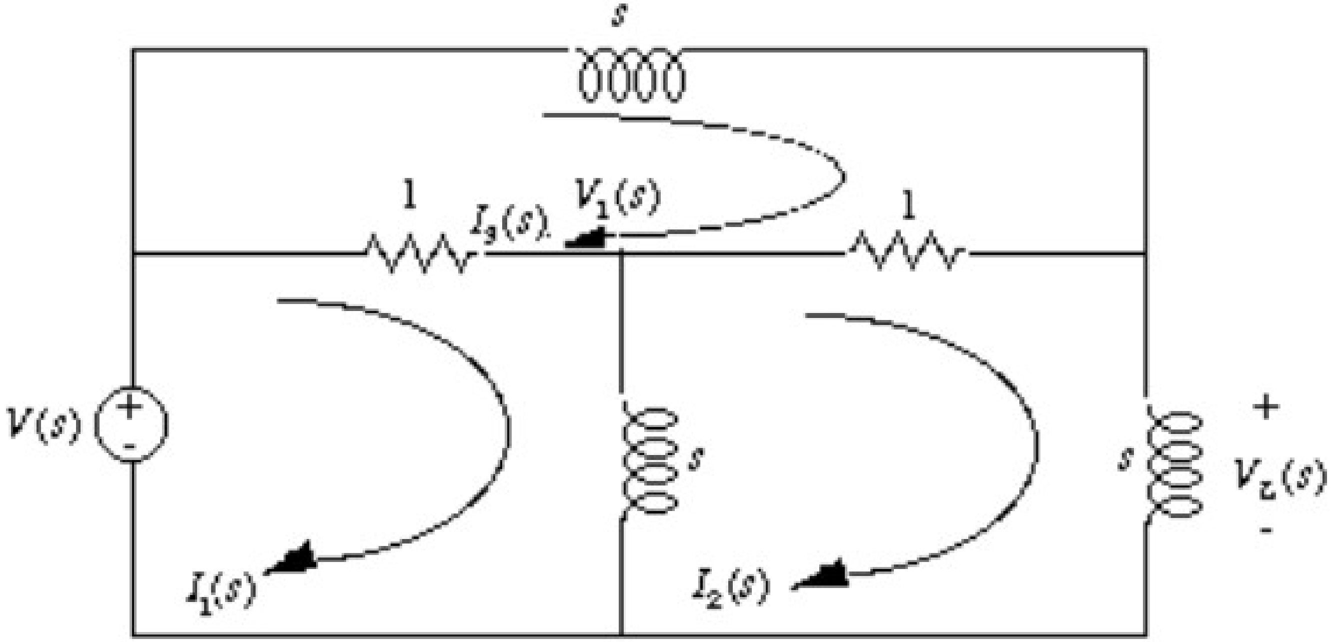
Electric Network Transfer Functions

Forming the transfer function, $G(s)$, yields

$$G(s)=\frac{I_2(s)}{V(s)}=\frac{Ls}{\Delta}=\frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$$

Example

- Solve using mesh analysis and Nodal analysis



Mesh analysis method

- Write the mesh equations :

$$(s+1)I_1(s) - sI_2(s) - I_3(s) = V(s)$$
$$-sI_1(s) + (2s+1)I_2(s) - I_3(s) = 0$$
$$-I_1(s) - I_2(s) + (s+2)I_3(s) = 0$$

- Solving the equations for $I_2(s)$:

$$I_2(s) = \frac{\begin{vmatrix} (s+1) & V(s) & -1 \\ -s & 0 & -1 \\ -1 & 0 & (s+2) \end{vmatrix}}{\begin{vmatrix} (s+1) & -s & -1 \\ -s & (2s+1) & -1 \\ -1 & -1 & (s+2) \end{vmatrix}} = \frac{(s^2+2s+1)V(s)}{s(s^2+5s+2)}$$

Mesh analysis method

But, $V_L(s) = sI_2(s)$

Hence,

$$V_L(s) = \frac{(s^2 + 2s + 1)V(s)}{(s^2 + 5s + 2)}$$

or

$$\frac{V_L(s)}{V(s)} = \frac{s^2 + 2s + 1}{s^2 + 5s + 2}$$

Nodal analysis method

- Write the nodal equations :

$$(\frac{1}{s}+2)V_1(s)-V_L(s)=V(s)$$

$$-V_1(s)+(\frac{2}{s}+1)V_L(s)=\frac{1}{s}V(s)$$

Nodal analysis method

Solving for $V_L(s)$,

$$V_L(s) = \frac{\begin{vmatrix} (\frac{1}{s}+2) & V(s) \\ -1 & \frac{1}{s}V(s) \end{vmatrix}}{\begin{vmatrix} (\frac{1}{s}+2) & -1 \\ -1 & (\frac{2}{s}+1) \end{vmatrix}} = \frac{(s^2+2s+1)V(s)}{(s^2+5s+2)}$$

or

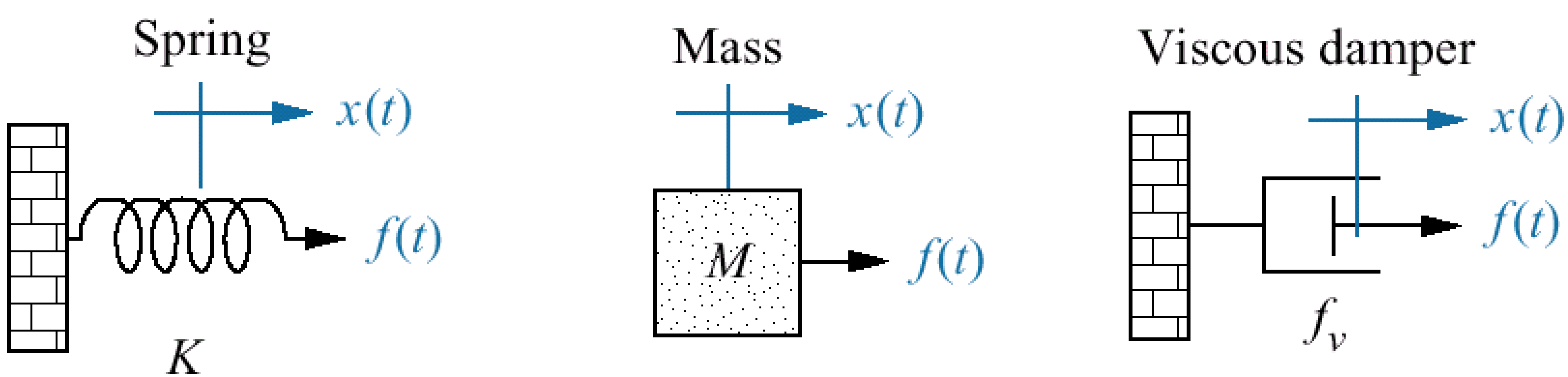
$$\frac{V_L(s)}{V(s)} = \frac{s^2+2s+1}{s^2+5s+2}$$

Mechanical Systems

- Types of Mechanical Systems
  - Translational Systems
  - Rotational Systems

Translational Mechanical System Transfer Function

- We are going to model translational mechanical system by a transfer function.
- In electrical we have three passive elements, resistor, capacitor and inductor. In mechanical we have spring, mass and viscous damper.



Translational Mechanical System Transfer Function

- In electrical we have resistance, capacitance and inductance but in mechanical we have, spring constant (K), viscous damper (f_v) and mass (M).

- We are going to find the transfer function for a mechanical system in term of force-displacement (i.e. forces are written in terms of displacement)

Translational Mechanical System

- Mass
  - f(t) represents the applied force, x(t) represents the displacement, and M represents the mass. Then, in accordance with Newton's second law,

$$f(t) = Ma(t) = \frac{Mdv(t)}{dt} = \frac{Md^2x(t)}{dt^2}$$

Where v(t) is velocity and a(t) is acceleration. It is assumed that the mass is rigid at the top connection point and that cannot move relative to the bottom connection point.

Translational Mechanical System

- Damper
  - Damper is the damping elements and damping is the friction existing in physical systems whenever mechanical system moves on sliding surface. The friction encountered is of many types, namely stiction, coulomb friction and viscous friction force
  - In friction elements, the top connection point can move relative to the bottom connection point. Hence two displacement variables are required to describe the motion of these elements, where B is the damping coefficient

$$f(t) = B\left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right)$$

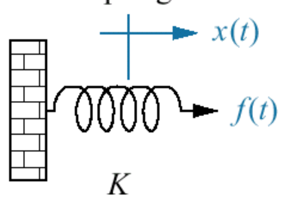
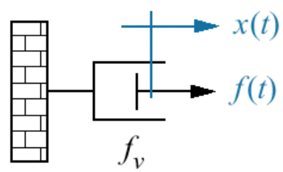
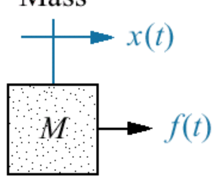
Translational Mechanical System

- Spring
  - The final translational mechanical element is a spring. The ideal spring gives the elastic deformation of a body. The defining equation from Hooke's law, is given by

$$f(t) = K\left(x_1(t) - x_2(t)\right)$$

Translational Mechanical System Transfer Function

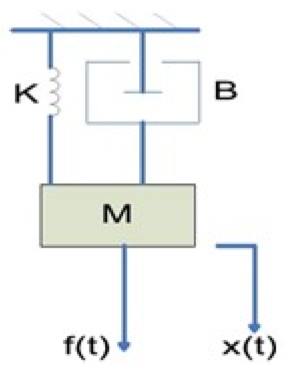
Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

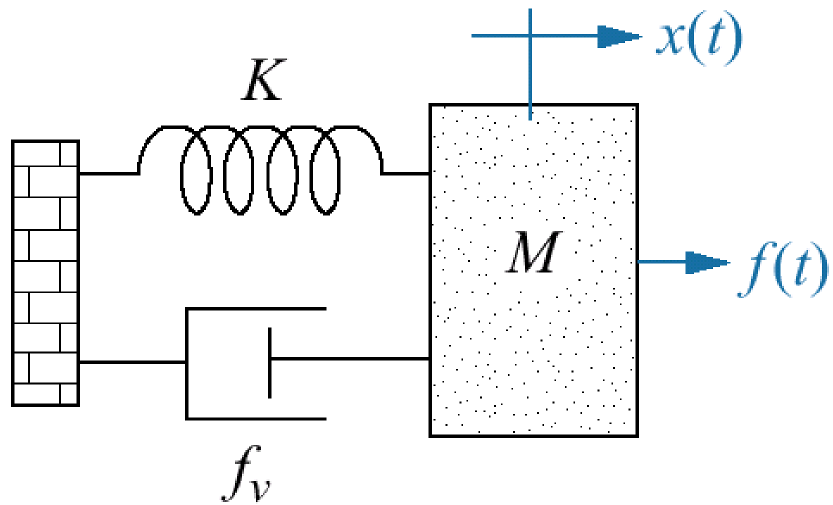
Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Translational Mechanical System Transfer Function

Simple system





(a)

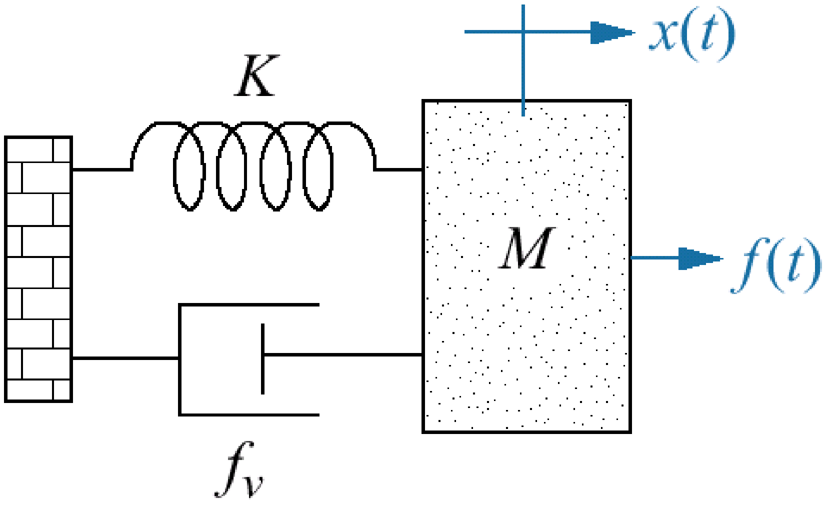
- Assumption
  - Movement to the left is assumed to be positive.
  - positive ⟵ +ve

Translational Mechanical System Transfer Function

- Draw a free body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it.
- Use Newton’s law to form a differential equation of motion by summing the forces and setting the sum equal to zero.
- Assume zero initial conditions, we change the differential equation into Laplace form.

Translational Mechanical System Transfer Function

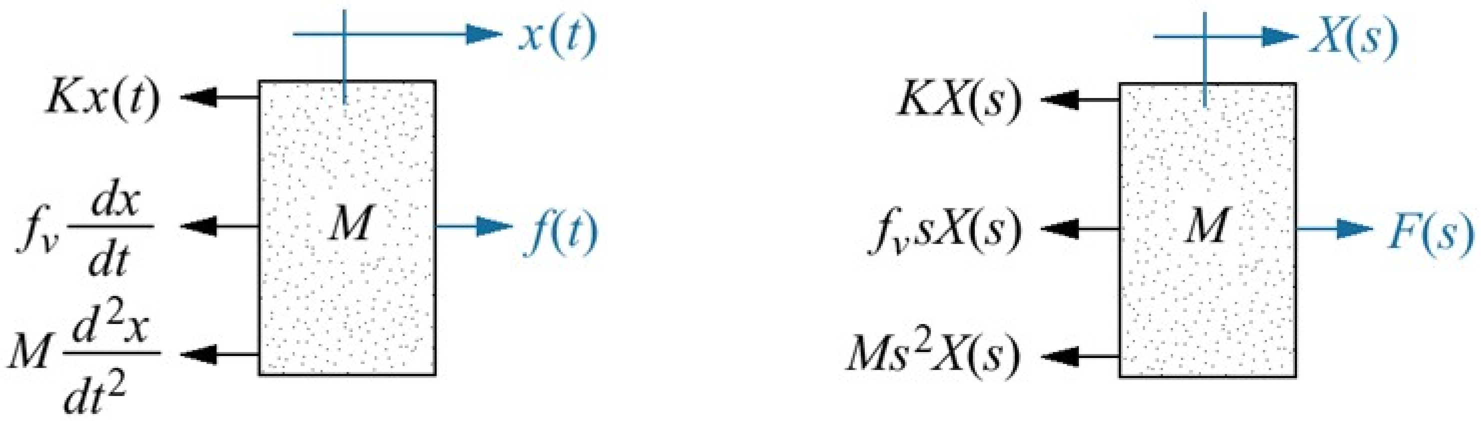
Example:



(a)

Find the transfer function, X(s)/F(s), for the system in Figure (a).

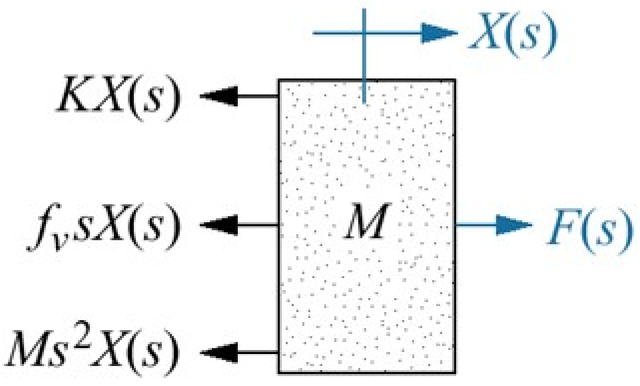
Translational Mechanical System Transfer Function



(a) **(b)**

- Draw the free body diagram
- Place on the mass all forces felt by the mass.
- Assume the mass is travelling toward the right.
- Use Laplace transform.

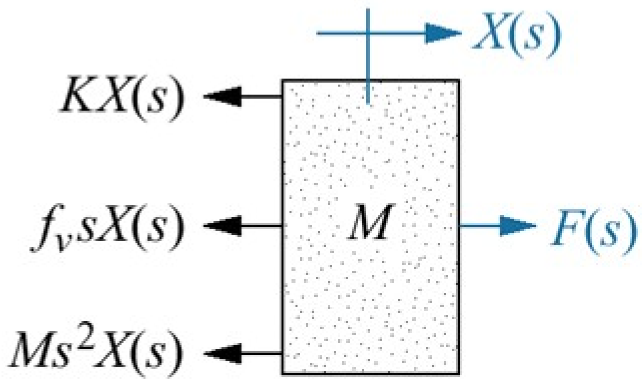
Translational Mechanical System Transfer Function



(b)

- How do we get the equation?
- Answer:
  - [sum of impedances]$X(s)$ = [sum of applied forces]
  - Movement to the left is positive

Translational Mechanical System Transfer Function

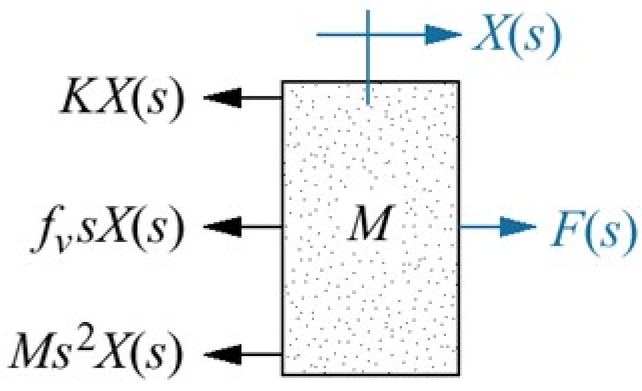


(b)

- The equation in Laplace form is

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

Translational Mechanical System Transfer Function



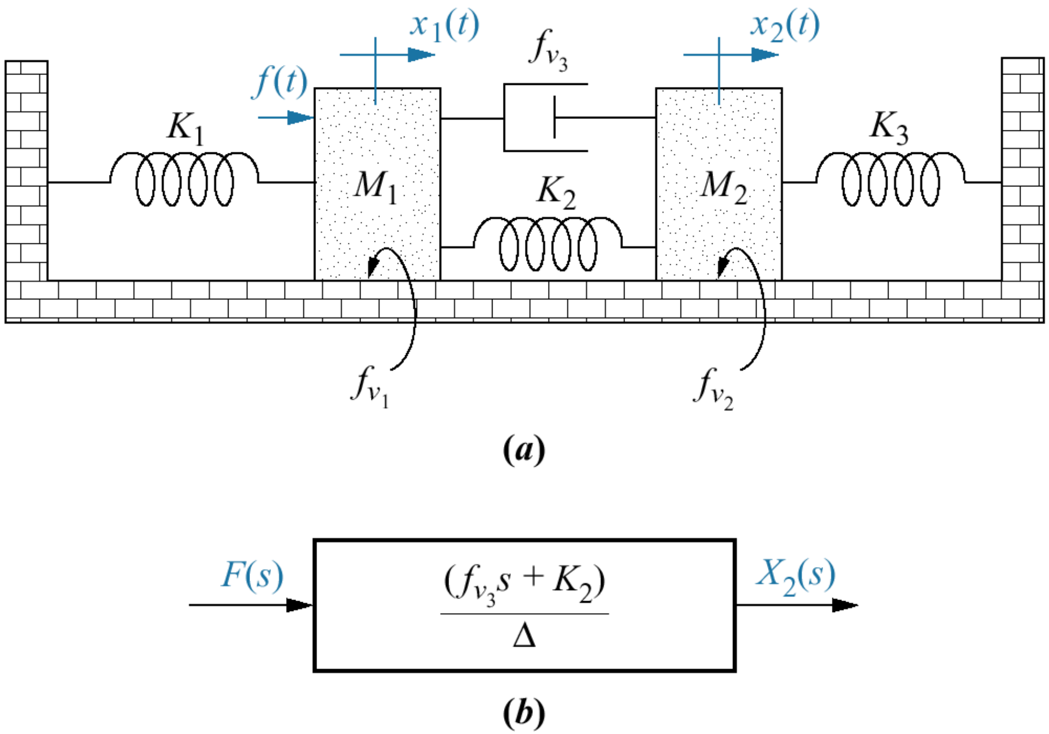
(b)

Solving for the transfer function

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

Translational Mechanical System Transfer Function

- **Complex system**
- Find transfer function, $X_2(s)/F(s)$

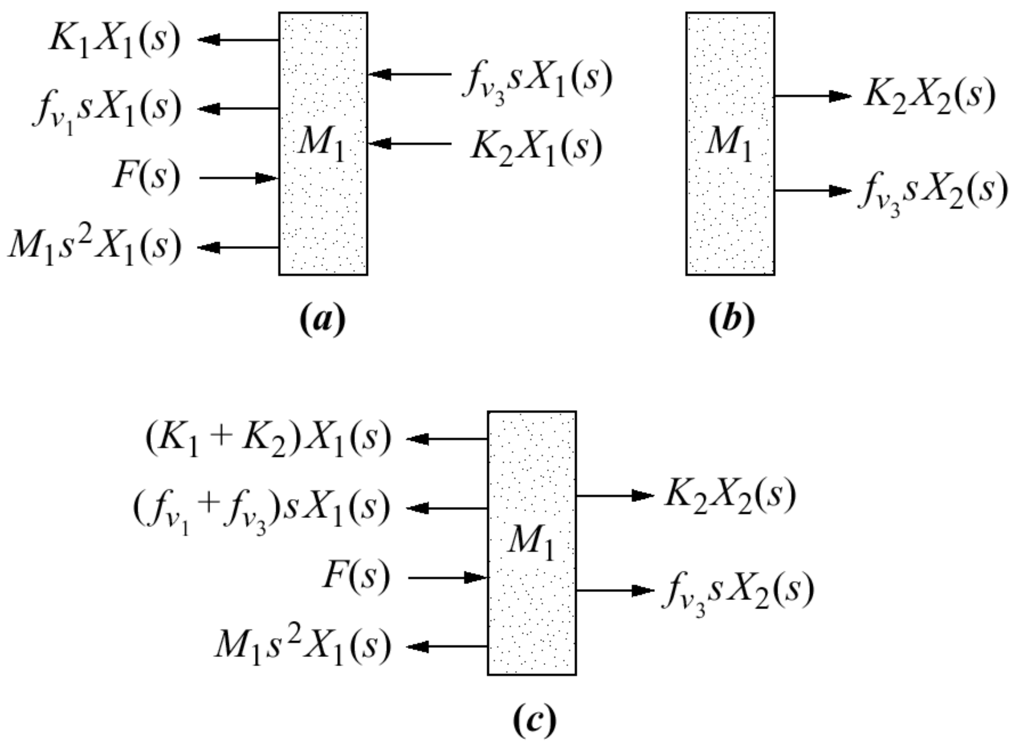


Translational Mechanical System Transfer Function

a. Forces on M_1 due **only** to motion of M_1

b. forces on M_1 due **only** to motion of M_2(sharing);

c. **all** forces on M_1

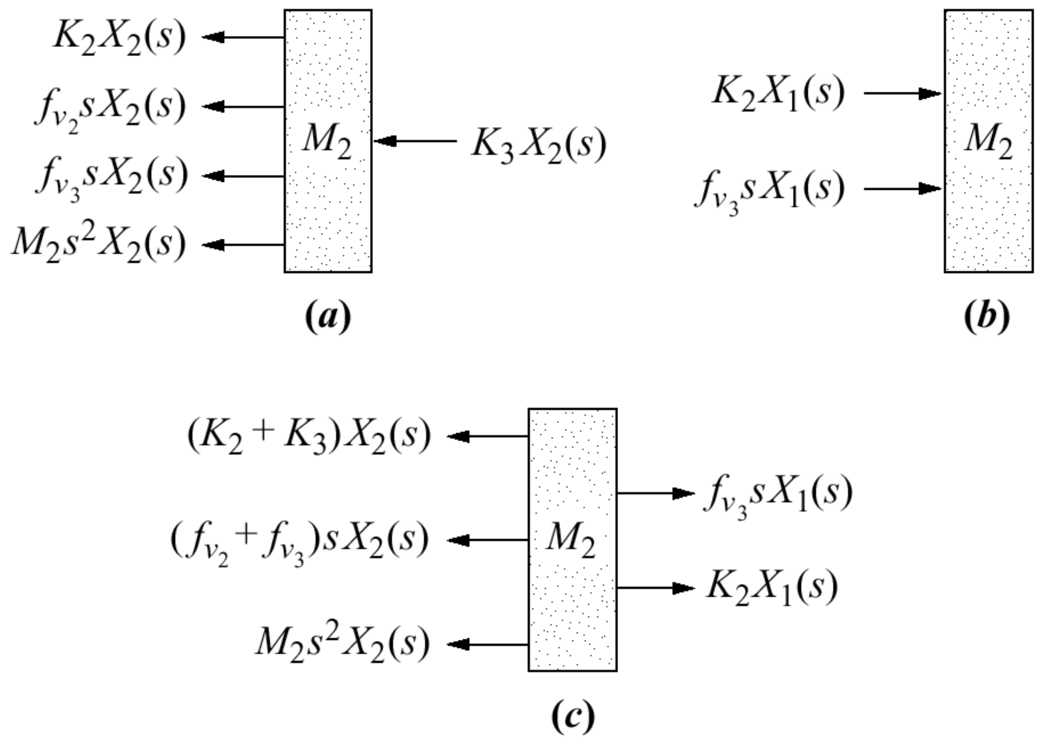


Translational Mechanical System Transfer Function

a. Forces on M_2 due **only** to motio of M_2;

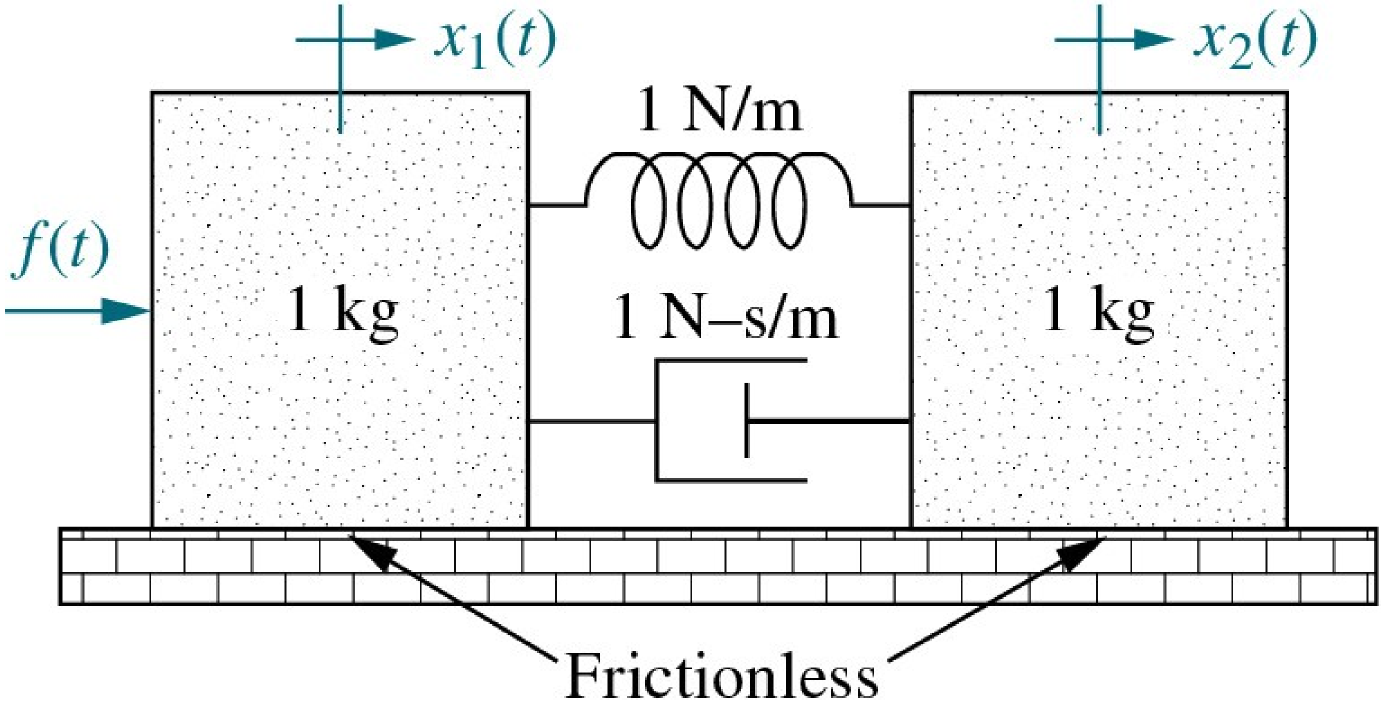
b. forces on M_2 du **only** to motion of M_1 (sharing);

c. **all** forces on M_2



Example

- Solve for $G(s) = X_2(s)/F(s)$



Writing the equations of motion,

$$(s^2+s+1)X_1(s)-(s+1)X_2(s)=F(s)$$

$$-(s+1)X_1(s)+(s^2+s+1)X_2(s)=0$$

Solving for $X_2(s)$,

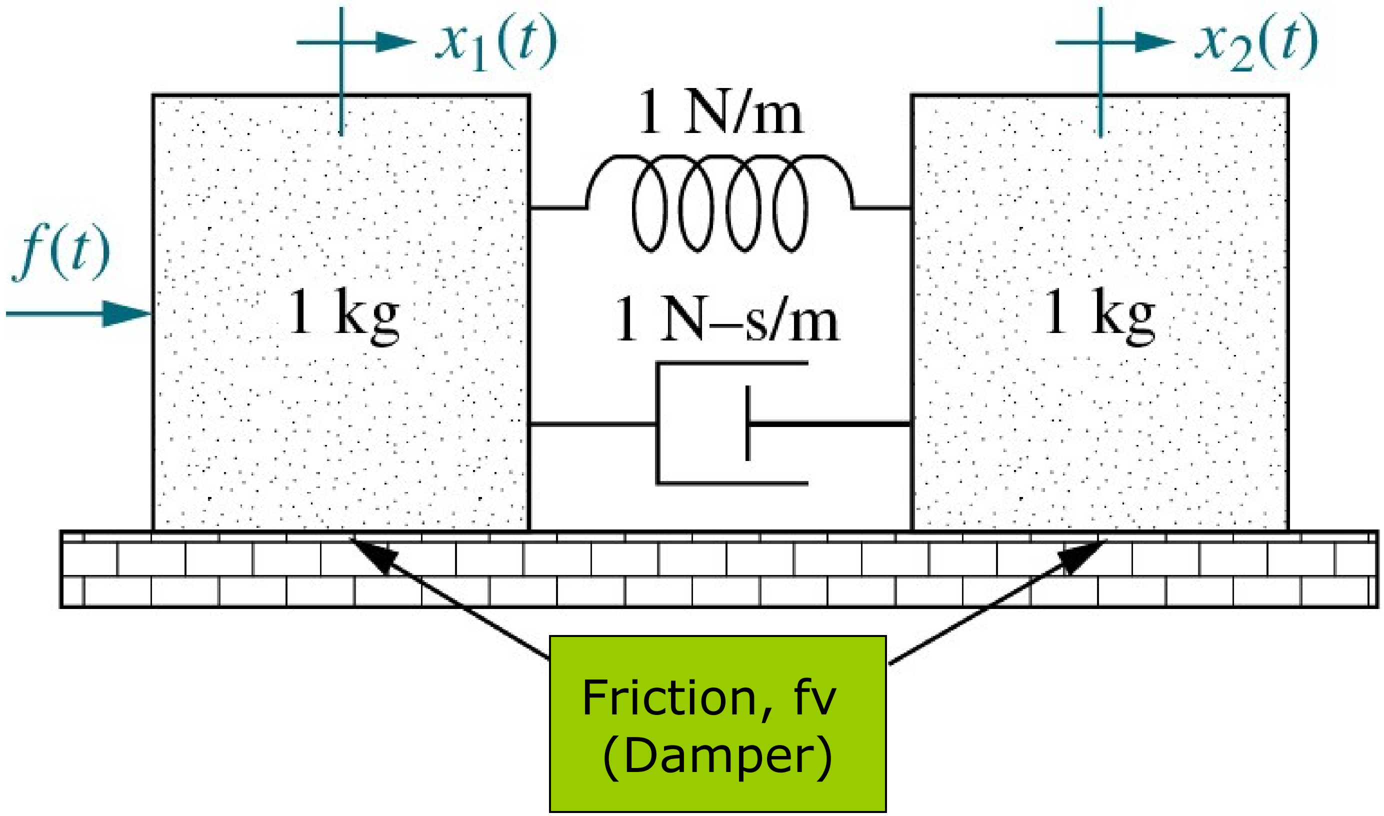
$$X_2(s)=\frac{\begin{vmatrix}(s^2+s+1) & F(s)\\ -(s+1) & 0\end{vmatrix}}{\begin{vmatrix}(s^2+s+1) & -(s+1)\\ -(s+1) & (s^2+s+1)\end{vmatrix}}=\frac{(s+1)F(s)}{s^2(s^2+2s+2)}$$

From which,

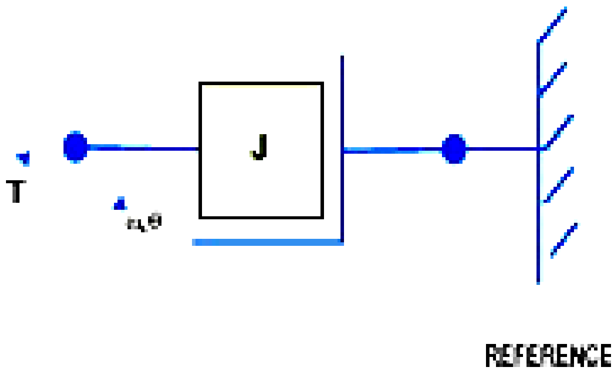
$$\frac{X_2(s)}{F(s)}=\frac{(s+1)}{s^2(s^2+2s+2)}.$$

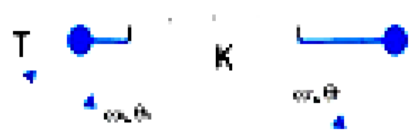
Exercise

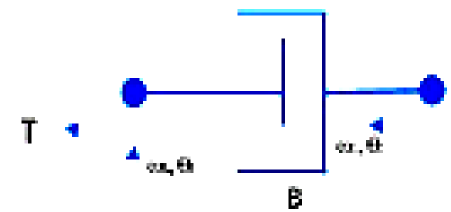
- Solve for $G(s) = X_2(s)/F(s)$



Rotational Mechanical System







$$T = \frac{J d^2\theta}{dt^2} = \frac{J d\omega}{dt^2}$$

$$T = K(\theta_1 - \theta_2)$$

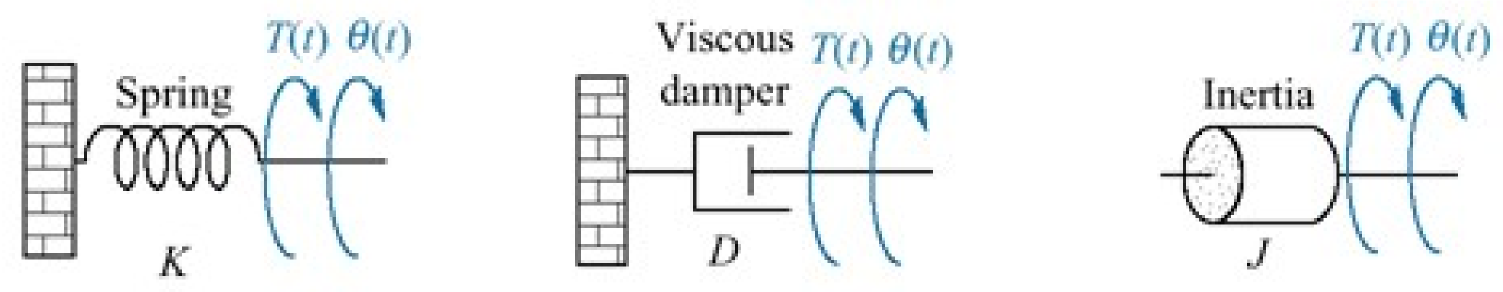
$$T = B\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right) = B(\omega_1 - \omega_2)$$

Rotational Mechanical System Transfer Functions

- We are going to solve for rotational mechanical system using the same way as the translational mechanical systems except
  - Torque replaces force
  - Angular displacement replaces translational displacement

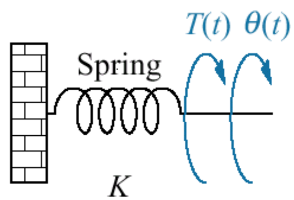
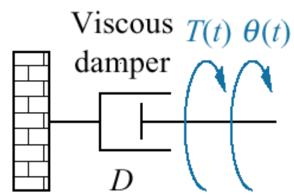
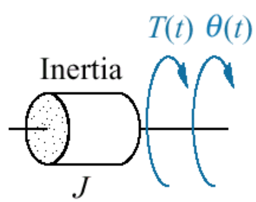
Rotational Mechanical System Transfer Functions

- In translational mechanical system we have three elements; **spring, damper and mass**. In rotational mechanical system we have; **spring, damper and inertia.**



Rotational Mechanical System Transfer Functions

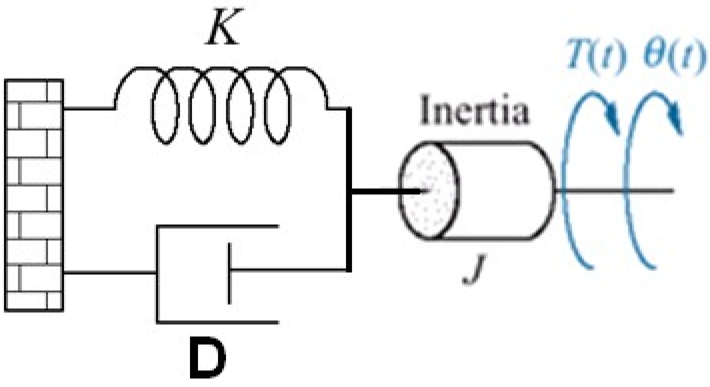
Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

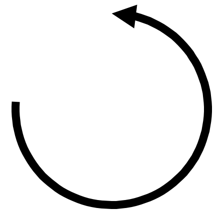
Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

Rotational Mechanical System Transfer Functions

Simple system

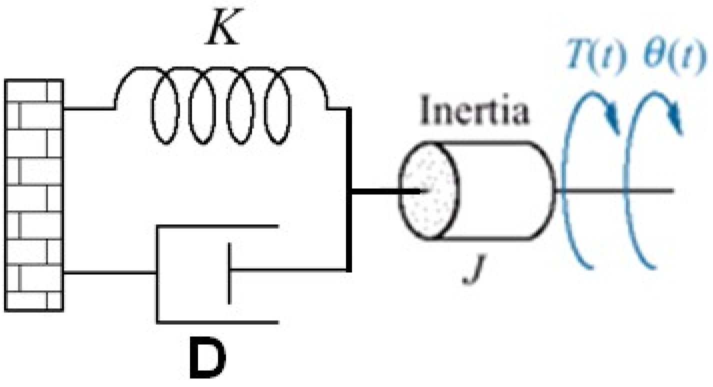


- Assumption
  - Anti-clockwise movement is assumed to be positive



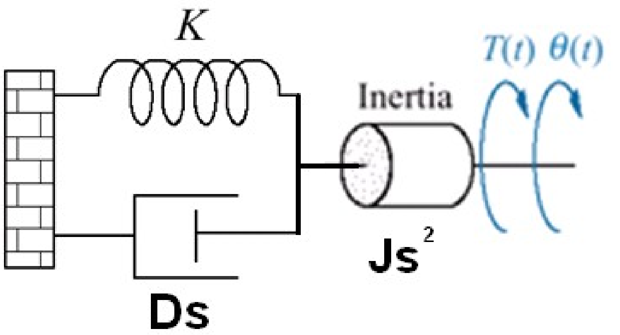
Rotational Mechanical System Transfer Functions

Example:

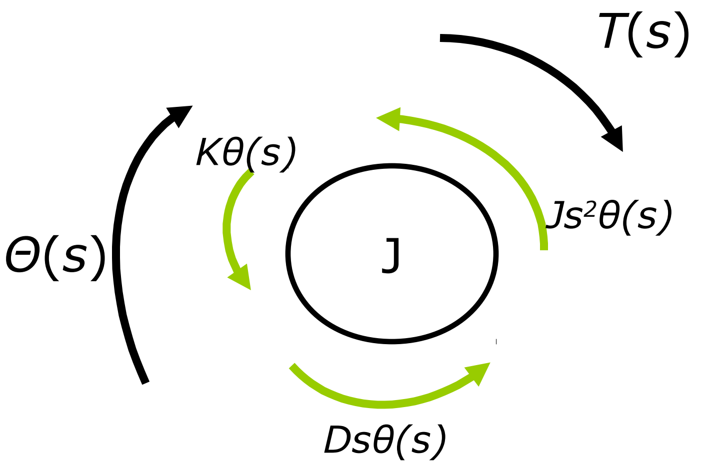


Find the transfer function, $\theta(s)/T(s)$

Rotational Mechanical System Transfer Functions



- Draw the free body diagram



Rotational Mechanical System Transfer Functions

- We will use
  - [sum of impedances]$\theta(s)$ = [sum of applied torque]
- Based on the free body diagram, the equation of motion is

$$\left(Js^2 + Ds + K\right)\theta(s) = T(s)$$

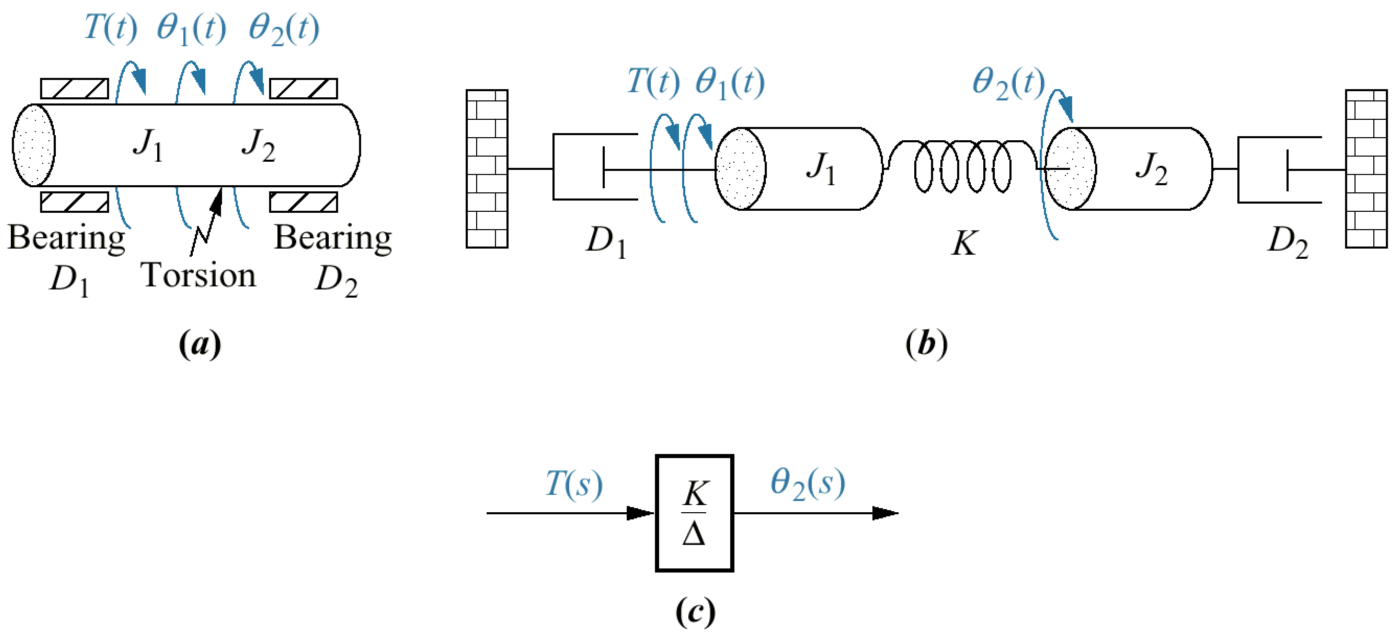
we know transfer function, $G(s)$, is $\dfrac{\theta(s)}{T(s)}$

$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Ds + K}$$

$$= \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

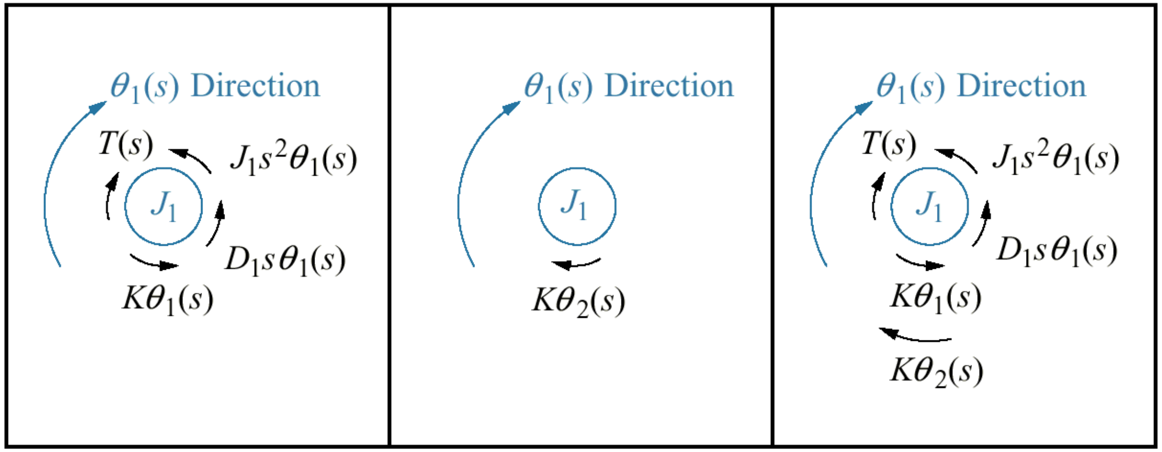
Rotational Mechanical System Transfer Functions

- **Complex system**
- Find transfer function $\theta_2(s)/T(s)$



Rotational Mechanical System Transfer Functions

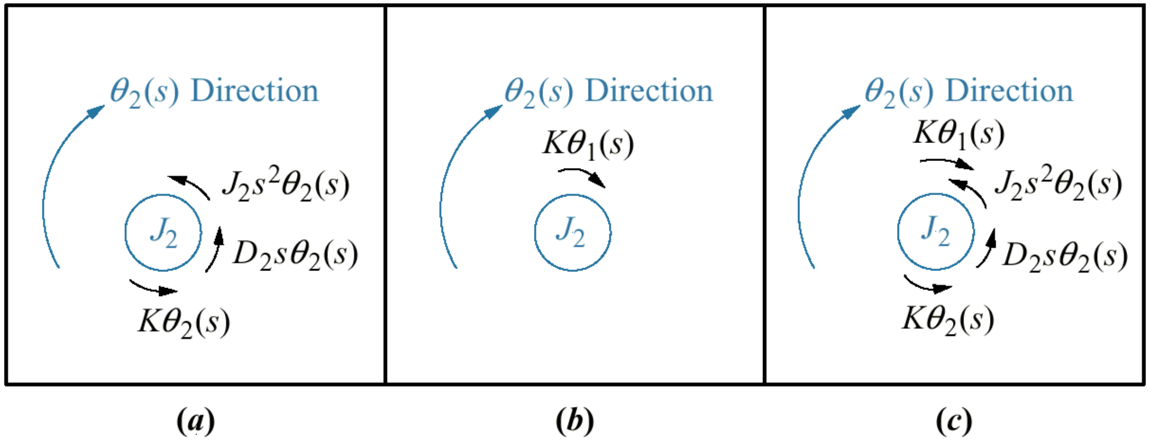
a. Torques on J_1 due only to the motion of J_1
b. torques on J_1 due only to the motion of J_2
c. final free-body diagram for J_1



(a) (b) (c)

Rotational Mechanical System Transfer Functions

a. Torques on J_2 due only to the motion of J_2;
b. torques on J_2 due only to the motion of J_1
c. final free-body diagram for J_2



Laplace transform review

- The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Where $s = \sigma + j\omega$

Inverse Laplace transform, to get f(t) given F(s), is

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$

Laplace transform review

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Table 2.1
Laplace transform table

Laplace transform review

Table 2.2
Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

Laplace transform review

Example

Find the inverse Laplace transform of

$F(s) = 1/s$

$F(s) = 1/s^2$

$F(s) = 1/(s+3)$

Laplace transform review

Example:

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s+3}$$

Answer

We use frequency shift theorem, item 4 in Table 2.2, and Laplace transform of $f(t)=u(t)$, item 3 in Table 2.1

If inverse transform of $F(s)=1/s^2$ is $tu(t)$, the inverse transform of $F(s+a)=1/(s+a)^2$ is $e^{-at}tu(t)$.

Hence, $f(t)=e^{-3t}tu(t)$

Laplace transform review

Example:

Find the inverse Laplace transform of

$$F(s) = 1/(s+3)^2$$

Answer

We use frequency shift theorem, item 4 in Table 2.2, and Laplace transform of $f(t)=tu(t)$, item 3 in Table 2.1

If inverse transform of $F(s)=1/s^2$ is $tu(t)$, the inverse transform of $F(s+a)=1/(s+a)^2$ is $e^{-at}tu(t)$.

Hence, $f(t)=e^{-3t}tu(t)$

Laplace transform review

Can you solve this question using Table 2.1 and Table 2.2?

$$F(s) = 1/(s+3)^3$$

Or this question

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

Laplace transform review

We can find the inverse Laplace transform for complicated function by changing the function to a sum of simpler term for which we know the Laplace transform of each term.

The result is called *partial-fraction expansion*

Partial fraction expansion

Let’s take $F(s)=N(s)/D(s)$

In order for us to do the partial-fraction expansion we need to make sure the order of N(s) is smaller than or equal to the order of D(s).

N(s) <= D(s)

Partial fraction expansion

What will we do if

N(s) >= D(s) ?

Answer.

We will divide N(s) by D(s) successively until the result has a remainder whose numerator is of order less that its denominator (polynomial division).

Partial fraction expansion

Polynomial long division

Example:

Divide $x^2+9x+14$ by $x+7$

Answer

$$
\begin{array}{r}
x + 2 \\
x+7 \,\overline{\big)\, x^2 + 9x + 14} \\
-x^2 - 7x \\
\hline
2x + 14 \\
-2x - 14 \\
\hline
0
\end{array}
$$

Partial fraction expansion

Divide $3x^3 - 5x^2 + 10x - 3$ by $3x + 1$

Answer:

$$
\begin{array}{r}
x^2 \quad - 2x \quad + 4 \\
3x+1 \overline{\big) 3x^3 - 5x^2 + 10x - 3} \\
-3x^3 - 1x^2 \\
\hline
-6x^2 + 10x - 3 \\
+6x^2 + 2x \\
\hline
12x - 3 \\
-12x - 4 \\
\hline
-7
\end{array}
$$

$$x^2 - 2x + 4 + \frac{-7}{3x+1}$$

Partial fraction expansion

Can be divided into 3 cases.

Case 1:

Roots of denominator A(s) are real and distinct.

Case 2:

Roots of denominator A(s) are real and repeated.

Case 3:

Roots of denominator A(s) are complex conjugate.

Partial fraction expansion: Case 1

Roots of the denominator of F(s) are real and distinct

- Example of an F(s) with real and distinct roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)}$$

Partial fraction expansion: Case 1

Solution

$$F(s)=\frac{2}{\left(s+1\right)\left(s+2\right)}=\frac{K_1}{(s+1)}+\frac{K_2}{(s+2)} \qquad (2.1)$$

To find K¹, we multiply Eq. (2.1) by (s+1), which isolates K¹. Thus

$$K_1=\left.\frac{2}{\left(s+2\right)}\right|_{s\to-1}=2$$

$$K_2=\left.\frac{2}{\left(s+1\right)}\right|_{s\to-2}=-2$$

Partial fraction expansion: Case 1

Change value K_1 and K_2 gives

$$F(s) = \frac{2}{\left(s+1\right)\left(s+2\right)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

$$F(s) = \frac{2}{\left(s+1\right)\left(s+2\right)} = \frac{2}{(s+1)} - \frac{2}{(s+2)}$$

Using Table 2.1, the inverse Laplace transform is ??

$$f(s) = (2e^{-t} - 2e^{-2t})u(t)$$

Partial fraction expansion: Case 1

Problem: Given the following differential equation, solve for y(t) if all initial condition are zero. Use Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

Partial fraction expansion: Case 2

Roots of the denominator of F(s) are real and repeated

Example of an F(s) with real and repeated roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

Roots of $(s+2)^2$ in the denominator are repeated.

Partial fraction expansion: Case 2

- We write the partial-fraction expansion as a sum of terms. Each factor of the denominator forms the denominator of each term.

- Each multiple root generates additional terms consisting of denominator factors of reduced multiplicity.

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)} \qquad (2.2)$$

Partial fraction expansion: Case 2

- We can solve for K_1 using method in case 1 in Eq.2.2.

$$K_1 = \left.\frac{2}{(s+2)^2}\right|_{s\to -1} = 2$$

- Next step is to isolate K2 by multiplying Eq. 2.2 by $(s+2)^2$

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3 \qquad (2.3)$$

Partial fraction expansion: Case 2

- Solve for K2 using the same method as Case 1 in Eq.2.3.

$$K_2 = \left.\frac{2}{(s+1)}\right|_{s\to -2} = -2$$

- To get the value of K_3 we need to differentiate Eq.2.3 with respect to *s*,

$$\frac{-2}{\left(s+1\right)^2} = \frac{\left(s+2\right)s}{\left(s+1\right)^2}K_1 + K_3 \qquad (2.4)$$

Partial fraction expansion: Case 2

- Using the method in case 1. K_3 can be found if we let *s* approach -2. (-2 comes from the denominator of K_3)

$$K_3 = \left.\frac{-2}{(s+1)^2}\right|_{s \to -2} = -2$$

- Changing K_1, K_2 and K_3 with their respective values

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

$$= \frac{2}{(s+1)} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)}$$

Partial fraction expansion: Case 3

Roots of the denominator of F(s) are complex or imaginary

Example of an F(s) with complex roots in the denominator is

$$F(s) = \frac{3}{s\left(s^2 + 2s + 5\right)}$$

We can expand the function in partial fractions as

$$F(s) = \frac{3}{s\left(s^2 + 2s + 5\right)} = \frac{3}{s\left(s + 1 + j2\right)\left(s + 1 - j2\right)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s + 1 + j2} + \frac{K_3}{s + 1 - j1}$$

Partial fraction expansion: Case 3

Solve for K1 using similar method in case 1.

$$K_1 = \left.\frac{3}{\left(s^2 + 2s + 5\right)}\right|_{s \to 0} = \frac{3}{5}$$

Solve for K2 using similar method in case 1.

$$K_2 = \left.\frac{3}{s\left(s + 1 - j2\right)}\right|_{s \to -1-j2} = -\frac{3}{20}\left(2 + j1\right)$$

Partial fraction expansion: Case 3

K3 is found to be the complex conjugate of K2

$$K_3 = \left.\frac{3}{s\left(s+1+j2\right)}\right|_{s\to -1+j2} = \frac{3}{20}\left(2+j1\right)$$

Changing K_1, K_2 and K_3 with their respective values

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2+j1}{s+1+j2} + \frac{2-j1}{s+1-j2}\right) \qquad (2.5)$$

Partial fraction expansion: Case 3

Inverse Laplace transform for Eq.2.5 is

$$f(t) = \frac{3}{5} - \frac{3}{20}\left[\left(2 + j1\right)e^{-\left(1+j2\right)t} + \left(2 - j1\right)e^{-\left(1-j2\right)t}\right]$$

$$= \frac{3}{5} - \frac{3}{20}e^{-t}\left[4\left(\frac{e^{j2t} + e^{-j2t}}{2}\right) + 2\left(\frac{e^{j2t} + e^{-j2t}}{2j}\right)\right] \qquad (2.6)$$

Using equation

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta \qquad \text{and} \qquad \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Into Eq.2.6

Partial fraction expansion: Case 3

The final result is

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) = 0.6 - 0.671e^{-t}\cos\left(2t - \phi\right)$$

where $\phi = \arctan 0.5 = 26.57^\circ$