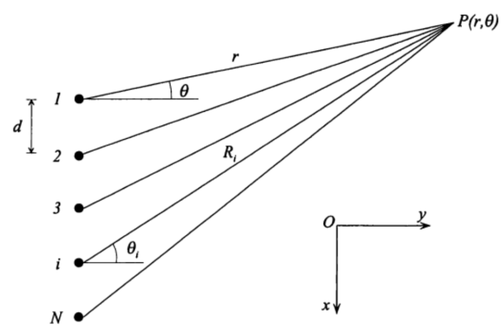
Contents

1. Radiation from an array of discrete point sources

Linear phased arrays can be approximated as an assembly of discrete point sources with time delays, in other words, the size of the elements is infinitely small. Under this assumption, discrete point sources can be thought as an array of simple point sources separated by a finite distances as shown in the figure below. The distance R_i from a point $P(r, \theta)$ in the medium to the i^{th} source can be easily expressed in the form of a simple trigonometric relationship in polar coordinates (r, θ).



$$R_i = \sqrt{r^2 + [(i-1)d]^2 - 2r(i-1)d\cos(\frac{\pi}{2} - \theta)}$$

Taking the clockwise angles with respect to the y-axis as positive, the angle shown in the figure is negative. If the distance r is sufficiently larger than the spacing d (i.e., r/d>>1), this relation can be approximated as

$$R_i \approx \mathrm{r} - (\mathrm{i}-1)\mathrm{d}\sin(\theta)$$

If the time delay between two adjacent point sources is given by $\Delta\tau$,the contribution of the i^{th} source to the acoustic field at position $P(r,\theta)$ and time t, can be obtained by substituting t by $[t-(i-1)\Delta\tau]$ and r by R_i. The pressure of the i^{th} source will be as follow,

$$p_i(r,\theta,t) = \frac{p_0 r_0}{R_i}\exp\{j[\omega(t-(i-1)\Delta\tau) - kR_i]\}$$

The pressure can be written in terms of r using the approximation given above for R_i, the pressure becomes,

$$p_i(r,\theta,t) = \frac{p_0 r_0}{R_i}\exp[\mathrm{j}(\omega\mathrm{t} - \mathrm{kr})]\exp\{-\mathrm{j}[\omega(i-1)\Delta\tau - k(i-1)dsin(\theta)]\}$$

Since the total sound pressure is the synthesis of all individual point sources in other words,

$$p(r,\theta,t) = \sum_{i=1}^{N} p_i(r,\theta,t)$$

It follows that the pressure field of the waves radiated from a linear array of simple sources can be represented as

$$p(r,\theta,t) = \sum_{i=1}^{N}\frac{P_0 r_0}{r}\exp[j(\omega t - kr)]\exp\{-j[\omega(i-1)\Delta\tau - k(i-1)dsin(\theta)]\}$$

$$= \frac{P_0 r_0}{r}\exp[\mathrm{j}(\omega t - kr)]\sum_{i=1}^{N}\exp[-j(\omega\Delta\tau - kdsin(\theta))(i-1)]$$

Expanding the summation term on the right hand side of the equation gives us,

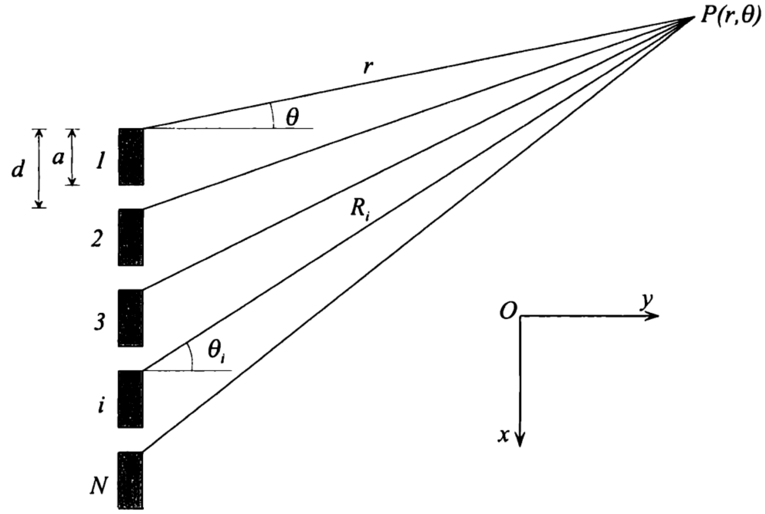
$$= \frac{\sin\left[\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)N\right]}{\sin\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)}\exp\left[-j\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)(N-1)\right]$$

By further simplification, the pressure distribution could be written as follow,

$$p(r,\theta,t) = \frac{p_0 r_0}{r}\frac{\sin\left[\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)N\right]}{\sin\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)}\exp\left[-j\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)(N-1)\right]\times\exp[j(\omega t - kr)]$$

2. Radiation from a phased array

A phased array is now considered as a linear array of single element sources as shown in the figure below, where (a) denotes the element size and others are same as the previous section. R_i and its approximation are the same as well to the previous section.



According to the previous section, the pressure distribution from the i^{th} element is,

$$p_i(r,\theta,t) = \frac{p_0 a}{R_i} \frac{\sin\left(\frac{kasin(\theta)}{2}\right)}{\frac{kasin(\theta)}{2}} \exp\left(-\frac{jkasin(\theta)}{2}\right) \exp\left[\mathrm{j}(\omega t_i - kR_i)\right]$$

The pressure for the phased array can now be written as,

$$p(r,\theta,t) = \sum_{i=1}^{N} p_i(r,\theta,t)$$

$$= \frac{p_0}{r} \frac{\sin\left(\frac{kasin(\theta)}{2}\right)}{\frac{ksin(\theta)}{2}} \frac{\sin\left[\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)N\right]}{\sin\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)} \exp\left[-j\left(\frac{kasin(\theta)}{2}\right)\right]$$
$$\times \exp\left[-j\left(\frac{\omega\Delta\tau - kdsin(\theta)}{2}\right)(N-1)\right] \exp\left[j(\omega t - kr)\right]$$

3. Definition of directivity pattern of sound pressure

The directivity pattern or directivity function is defined as the pressure $p(r,\theta,t)$ at any arbitrary angle θ normalized by the pressure $p(r,\theta_s,t)$ at steering angle θ_s. Those pressures can be found using the equations developed above.

$$H(\theta) = \left| \frac{p(r,\theta,t)}{p(r,\theta_s,t)} \right|$$

Note that $0 \le H(\theta) \le 1$ and the directivity at the steering angle θ_s corresponds to the condition $H(\theta_s) = 1$

1) Directivity for an array of discrete point sources

Using the above relations we find that the directivity for an array of discrete point sources is as follow,

$$H(\theta) = \left| \frac{\sin\left[\left(\frac{\omega\Delta\tau - kd\sin(\theta)}{2}\right)N\right]}{\mathrm{N}\sin\left(\frac{\omega\Delta\tau - kd\sin(\theta)}{2}\right)} \right|$$

Imposing the relationships

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$$

and

$$\sin(\theta_s) = \frac{c\Delta\tau}{d}$$

Now the directivity can be written as

$$H_2(\theta) = H(\theta) = \left| \frac{\sin\left[\frac{\pi d(\sin(\theta_s) - \sin(\theta))}{\lambda}N\right]}{\mathrm{N}\sin\left[\frac{\pi d(\sin(\theta_S) - \sin(\theta))}{\lambda}\right]} \right|$$

The subscript 2 above is only for comparison purposes with other directivities

2) Directivity for a phased array

$$H(\theta) = \left| \frac{\sin\left(\frac{ka\sin(\theta)}{2}\right)}{\frac{ka\sin(\theta)}{2}} \frac{\sin\left[\left(\frac{\omega\Delta\tau - kd\sin(\theta)}{2}\right)N\right]}{\mathrm{N}\sin\left(\frac{\omega\Delta\tau - kd\sin(\theta)}{2}\right)} \right|$$

Or it can be written in terms of λ and θ_s as

$$H(\theta) = \left| \frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{\frac{\pi a\sin(\theta)}{\lambda}} \frac{\sin\left[\frac{\pi d(\sin(\theta_s) - \sin(\theta))}{\lambda}N\right]}{\mathrm{N}\sin\left[\frac{\pi d(\sin(\theta_s) - \sin(\theta))}{\lambda}\right]} \right|$$

The code below services as an example to compute the directivities (or normalized pressure as a function of azimuthal angle) for the cases of discrete point sources, single element source and phased arrays respectively. The code specifies the steering angle and microphone array parameters.

We observe that the main lobe (direction of maximum pressure) appears exactly at the steering angle also, the shape of directivities of both line and phased arrays are quite similar due to the fact that we chose the element width (a) to be very small compared to the wavelength l.

This code confirms that the location of the main lobe exactly coincides with the steering angle. It also can be observed that the main lobe is surrounded by smaller lobes appearing at multiple locations along the azimuthal angle axis. These lobes are called side lobes. This means that the sound beam is propagated not only in the steering direction but in the other directions as well. This is called a leakage which may be significant if the microphone array is not well designed.

```
% Directivity of an array of discrete point sources f=2.3MHz
clc
clear
close all
theta = linspace(-90,90,1000);
N = 16;
d_l = 0.5;
h2oftheta = abs( ( sin( pi * d_l * ( sind(30) - sind(theta) ) * N ) ) ./ ( N * sin( pi
* d_l * ( sind(30) - sind(theta) ) ) ) ) ;
figure(1)
plot(theta,h2oftheta);
grid on;
axis([-90 90 0 1]);
xlabel('Angle (theta) [deg]');
ylabel('H_2(theta)');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Directivity for a single element source since it is even so the figure
%%is drawn with omitting the symmetry
clc
clear
close all
theta = linspace(-90,90,1000);
a_l = 0.25;
Q = ( (pi * a_l * sind(30) ) / ( sin(pi * a_l * sind(30) ) ) );
% Omit Q the normalization onstant for convienance in the coming analysis
h1oftheta =  abs( ( sin(pi * a_l * sind(theta) ) ) ./ ( pi * a_l * sind(theta) ) );
figure(2)
plot(theta,h1oftheta);
grid on;
axis([-90 90 0 1]);
xlabel('Angle (theta) [deg]');
ylabel('H_1(theta)');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% % Directivity for a phased array
clc
clear
close all
theta = linspace(-90,90,1000);
N = 10;
d_l = 0.5;
a_l = 0.25;
```

```
hoftheta = abs( ( (sin(pi * a_l * sind(theta))) .* (sin(pi * d_l * N * (sind(30) -
sind(theta) ) ))) ./ ((pi * a_l * sind(theta)) .* (N * sin(pi * d_l * (sind(30) -
sind(theta))))) );
figure(3)
plot(theta,hoftheta);
grid on;
axis([-90 90 0 1]);
xlabel('Angle (theta) [deg]');
ylabel('H(theta)');
```

The other code below shows a typical directivity plot provided to illustrate the concept of the optimization scheme. Besides the main and side lobes, it is particularly interesting to find that there is a lobe whose amplitude is the same as that of the main lobe appearing at the angle 60 degrees. Such a lobe is called a grating lobe. Grating lobes have a deleterious influence on the microphone array performance and hence should be avoided. If grating lobes exist, a strong secondary signal appears in the directions other than the steering angle, resulting in spurious and confusing signals. In this case, steerability is not well designed.

If the amplitudes of the side lobes are much smaller than that of the main lobe, the beam is well directed and the steering effect is pronounced. Therefore, it is desirable to suppress the side lobe amplitudes. Consequently, on of the main design objectives is to find a solution to satisfy the condition for the highest ration between the amplitudes of the main and side lobes

Furthermore, the main lobe width (the range of angles between the zero crossing points of the main lobe) is an important design objective. If the main lobe width is small or the shape of the main lobe is sharp, it means that the acoustic energy is primarily directed in the steering direction.

```
 %Plot illustrates the optimization process
clc
clear
close all
theta = linspace(-90,90,1000);
N = 16;
d_l = 2 / (1 + sqrt(3));
a_l = 1 / 100;
hoftheta = abs( ( (sin(pi * a_l * sind(theta))) .* (sin(pi * d_l * N * (sind(30) -
sind(theta) ) ))) ./ ((pi * a_l * sind(theta)) .* (N * sin(pi * d_l * (sind(30) -
sind(theta))))) );
plot(theta,hoftheta);
grid on;
axis([-90 90 0 1]);
xlabel('Angle (theta) [deg]');
ylabel('H(theta)');

% Plot the effect of steering angle on directivity
clc
clear
close all
theta = linspace(-90,90,1000);
thetas = [0 , 30 , 60 , 90];
N = 16;
d_l = 0.5;
a_l = 0.25;
for i=1:4
```

```
hoftheta = abs( ( (sin(pi * a_l * sind(theta))) .* (sin(pi * d_l * N *
(sind(thetas(i)) - sind(theta) ) ))) ./ ((pi * a_l * sind(theta)) .* (N * sin(pi * d_l
* (sind(thetas(i)) - sind(theta))))) );
subplot(2,2,i)
plot(theta,hoftheta);
grid on;
axis([-90 90 0 1]);
title(sprintf('Steering Angle: %g', thetas(i)))
xlabel('Angle (theta) [deg]');
ylabel('H(theta)');
end
```

4. Optimization of the microphone array parameters

The directivity patterns obtained previously provides the background of the optimization of the microphone array including number of elements, inter element spacing, element size and center frequency. The criteria are the minimization of the main lobe width, elimination of the grating lobes and suppression of side lobes.

1) Minimization of main lobe width

The main lobe width can be defined as the distance between the zero crossing points (or the angles) of the main lobe along the θ-axis. Hence, the condition for $H(\theta) = 0$ is that the numerator in directivity equation becomes zero or

$$\frac{\pi d(\sin(\theta_s) - \sin(\theta))N}{\lambda} = m\pi$$

Where (m) is an integer that is neither zero nor a multiple of N, so that the denominator in the above equation is non-zero with this condition. The corresponding locations may be written as

$$\theta = \sin^{-1}(\sin(\theta_s) - \frac{m\lambda}{Nd})$$

The angle corresponding to the value of m=-1 represents the first zero on the right hand side of the main lobe for positive steering angle and m=1 is for the let hand side, respectively. The values $m = \pm 2, \pm 3, \pm 4 \ldots .etc.$, represent zero crossing locations of the side lobes. The width of the main lobe is then obtained from the definition as follow,

$$\Delta\theta_m = \sin^{-1}(\sin(\theta_s) + \frac{\lambda}{Nd}) - \sin^{-1}(\sin(\theta_s) - \frac{\lambda}{Nd})$$

Where $0 \leq \Delta\theta_m \leq \pi$. We can create a dimensionless parameter q which will be called the main lobe sharpness factor

$$q = \frac{1}{\pi}[\sin^{-1}(\sin(\theta_s) + \frac{\lambda}{Nd}) - \sin^{-1}(\sin(\theta_s) - \frac{\lambda}{Nd})]$$

This parameter used as an indicator showing the quality of the main lobe n terms of sharpness at the intended steering angle and it is between 0 and 1. A small q-value means good directivity since the main lobe is narrow and sharply defined.

The below code services as an example of the lobe sharpness factor as a function of the steering angle. We can observe that as the steering angle increases, lobe sharpness also increases. This means that the directivity of acoustic beams steered at smaller angles are better than those steered at higher angles. Also we observe that the main lobe quality degrades with higher steering angles.

```
% Main lobe sharpness factor
clc
clear
close all
steeringanlge = 0:0.1:60;
N = 16;
l_d = 2;
q = 1/pi * ( asin(sind(steeringanlge) + l_d / N) - asin( sind(steeringanlge) - l_d / N
) ) ;
plot(steeringanlge,q);
grid on;
axis([0 70 0 0.3]);
xlabel('Steering Angle (theta_s) [deg]');
ylabel('Sharpness Factor (q)');
```

In addition, the q-value approaches zero for infinitely large number of elements ($N \rightarrow \infty$). This is an ideal condition but is impossible to achieve from a practical viewpoint mainly because the limitations of control electronics and increasing elements means increasing the dimensions of the design.

The code below however, shows the effect of number of elements on main lobe width. It proves that the sharpness decreases for a very low number of elements ($N < 8$) and it approaches asymptotically to zero. Therefore, it can be argued that a 16-element array is sufficient to ensure reasonably good directivity for this particular case. It is recommended to reasonably compromise between microphone array performance and cost.

```
% Main lobe sharpness factor as a function of number of element
clc
clear
close all
steeringanlge = 30;
N = 1:1:128;
l_d = 2;
q = 1/pi * ( asin(sind(steeringanlge) + l_d ./ N) - asin( sind(steeringanlge) - l_d ./
N ) ) ;
plot(N,q);
grid on;
axis([0 128 0 0.3]);
xlabel('Number of Elements (N)');
ylabel('Sharpness Factor (q)');
```

The code below, helps us observing that the lope sharpness factor improves with an increasing d/λ value. There are only two ways to achieve a high d/λ value. The first approach is to decrease the wavelength by using high frequency piezo-elements. However, there are some technical limitations associated with this approach. Firstly, there is a practical limit of frequency in fabrication piezo-ceramics. The second mode of difficulty is the problems associated with the microphone array fabrication. In order to achieve reasonably good microphone array performance, the inter-element spacing should be decreased accordingly as the frequency increases.

Using the same code we observe that the main lobe width approaches zero with large inter-element spacing. From these observations, we may argue that it is highly desirable to use larger d for improved beam directivity. However if (d) is selected too high, then grating lobes may be introduced.

```
% Main lobe sharpness factor as a function of inter-element spacing
clc
clear
close all
steeringanlge = 30;
N = 16;
l_d = 0.1:0.05:0.65;
q = 1/pi * ( asin(sind(steeringanlge) + (1 ./ l_d) / N) - asin( sind(steeringanlge) -
(1 ./ l_d) / N ) ) ;
plot(l_d,q);
grid on;
axis([0.1 0.65 0 0.7]);
xlabel('Inter Element Spacing (d/l)');
ylabel('Sharpness Factor (q)');
```

2) Eliminating grating lobes

The location of the peak of the main lobe can be determined by finding the condition such that the corresponding lobe amplitude reaches its maximum, $H(\theta) = 1$. The angles that satisfy this condition are given by the relationship

$$\frac{\pi d(\sin(\theta_s) - \sin(\theta))}{\lambda} = m'\pi$$

Where m' is an integer. Solving for the angles, we have

$$\theta = \sin^{-1}(\sin(\theta_s) - \frac{m'\lambda}{Nd})$$

There may be infinitely many solutions that satisfy the above equation depending on the selected value of θ_s, λ and d. In fact, there will be m' number of solutions that satisfy the condition

$$\left|\sin(\theta_s) - \frac{m'\lambda}{d}\right| < 1$$

Where the solution for $m' = 0$ represents the location of the main lobe and the remaining m' values corresponds to the locations of grating lobes. The condition to produce m'^{th} grating lobe is given by the relationship. This means also that grating lobes have the same amplitude as the main lobe.

$$d = \frac{m'\lambda}{1 + \sin(\theta_s)}$$

The first-order grating lobe appears when $m' = 1$ at the angle

$$\theta_{g1} = \sin^{-1}(\sin(\theta_s) - \frac{\lambda}{d})$$

If the condition

$$\sin(\theta_s) - \frac{\lambda}{d} < -1$$

Is met, the peak of the first grating lobe is expelled from the domain of real-valued angles between -90 and 90. The corresponding critical inter-element spacing is then written as

$$d_{cr} = \frac{\lambda}{1 + \sin(\theta_s)_{max}}$$

Where $\theta_{s_{max}}$ is the desired maximum operating steering angle of the microphone array without producing grating lobes. It will be referred to as the maximum steerable angle hereafter. It is obvious that the main lobe width becomes narrowest when the inter-element spacing reaches the critical value, however, it also obvious that the first grating lobe is introduced at the -90 angle under this condition.

Although the inter-element spacing is chosen as ($d < d_{cr}$), some portions of the grating lobe may still appear in the directivity pattern. In this case, the grating lobe is not completely eliminated and the pressure in the -90 direction may be still high. Consequently, the situation is not satisfactory yet.

Therefore, it is interesting to find the inter-element spacing that does not introduce any portion of grating lobe, which is referred to as maximum inter-element spacing. To do so, we need to stretch the directivity along the θ-axis, such that the first zero-crossing in the right hand side of the gratting lobe is pushed left all the wat to the position at the angle -90. This condition guarantees complete elimination of grating lobes and that there is no side lobe whose peak amplitude is higher than the peak side lobe.

Since there are exactly (N-2) number of side lobes between the main and grating lobes, this condition can be easily obtained by substituting $m = N - 1 \; and \; \theta = -\frac{\pi}{2}$ in

$$\frac{\pi d(\sin(\theta_s) - \sin(\theta))N}{\lambda} = m\pi$$

The maximum allowable inter-element spacing that completely eliminates grating lobes can be now determined from the relationship

$$d_{max} = \frac{\lambda}{1 + \sin(\theta_s)_{max}} \frac{N-1}{N}$$

In other words, grating lobes can be eliminated by choosing an inter-element spacing that is less than d_{max}.

The code below demonstrates the relation between grating lobes, inter-element spacing and the steering angles.

```
% Directivity plots for small inter-element spacings (d<=l/2)
clc
clear
close all
theta = linspace(-90,90,1000);
thetas = 30;
N = 16;
d_l = [0.02 , 0.1 , 0.3 , 0.5];
a_l = 0.25;
for i=1:4
hoftheta = abs( ( (sin(pi * a_l * sind(theta))) .* (sin(pi * d_l(i) * N *
(sind(thetas) - sind(theta) ) ))) ./ ((pi * a_l * sind(theta)) .* (N * sin(pi * d_l(i)
* (sind(thetas) - sind(theta))))) );
subplot(2,2,i)
plot(theta,hoftheta);
grid on;
axis([-90 90 0 1]);
title(sprintf('d/l: %g', d_l(i)))
xlabel('Angle (theta) [deg]');
ylabel('H(theta)');
end
for i=1:4
hoftheta = abs( ( (sin(pi * a_l * sind(theta))) .* (sin(pi * d_l(i) * N *
(sind(thetas) - sind(theta) ) ))) ./ ((pi * a_l * sind(theta)) .* (N * sin(pi * d_l(i)
* (sind(thetas) - sind(theta))))) );
subplot(2,2,i)
polarpattern(theta,hoftheta);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Directivity plots for large inter-element spacing (d>=2l/3
clc
clear
close all
theta = linspace(-90,90,1000);
thetas = 30;
N = 16;
d_l = [2/3 , 4/3 , 2 , 8/3];
a_l = 0.25;
for i=1:4
hoftheta = abs( ( (sin(pi * a_l * sind(theta))) .* (sin(pi * d_l(i) * N *
(sind(thetas) - sind(theta) ) ))) ./ ((pi * a_l * sind(theta)) .* (N * sin(pi * d_l(i)
* (sind(thetas) - sind(theta))))) );
figure(1)
subplot(2,2,i)
plot(theta,hoftheta);
grid on;
axis([-90 90 0 1]);
title(sprintf('d/l: %g', d_l(i)))
xlabel('Angle (theta) [deg]');
ylabel('H(theta)');
end
for i=1:4
hoftheta = abs( ( (sin(pi * a_l * sind(theta))) .* (sin(pi * d_l(i) * N *
(sind(thetas) - sind(theta) ) ))) ./ ((pi * a_l * sind(theta)) .* (N * sin(pi * d_l(i)
* (sind(thetas) - sind(theta))))) );
figure(2)
subplot(2,2,i)
polarpattern(theta,hoftheta);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
Directivity plots for the inter-element spacings of (a) critical value,
% (b) near the critical value, and (c)  the maximum value.
clc
clear
close all
theta = linspace(-90,90,1000);
thetasmax = 30;
N = 16;
dcr = 1/(1+sind(thetasmax));
dmax = (1/(1+sind(thetasmax))) * ((N -1)/N);
d_l = [dcr , 0.96*dcr , dmax];
a_l = 0.25;
for i=1:3
hoftheta = abs( ( (sin(pi * a_l * sind(theta))) .* (sin(pi * d_l(i) * N *
(sind(thetasmax) - sind(theta) ) ))) ./ ((pi * a_l * sind(theta)) .* (N * sin(pi *
d_l(i) * (sind(thetasmax) - sind(theta))))) );
figure(1)
subplot(2,2,i)
plot(theta,hoftheta);
grid on;
axis([-90 90 0 1]);
title(sprintf('d/l: %g', d_l(i)))
xlabel('Angle (theta) [deg]');
ylabel('H(theta)');
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Maximum inter-element spacing as a function of maximum steerable angle
%%for various number of elements

clc
clear
close all
thetasmax = 0:0.5:90;
N = [4 , 8 , 16 , 32 , 64 , 100000];
dcr = 1 ./ ( 1+ sind(thetasmax));
for i=1:6
dmax = (1 ./ ( 1 + sind(thetasmax))) * ((N(i) -1)/N(i));
plot(thetasmax,dmax)
hold on
title('d_m_a_x/l vs Maximum Steering Angle')
xlabel('Maximum Steering Angle');
ylabel('Maximum Inter-Element Spacing d_m_a_x/l');
legend('N=4','N=8','N=16','N=32','N=64','N=Infinity')
end
```

5. Suppression of side lobes

$$\theta = \sin^{-1}(\sin(\theta_s) - \frac{m\lambda}{Nd})$$

The values $m = \pm 2, \pm 3, \pm 4 \ldots . etc.$, represent zero crossing locations of the side lobes. Assuming that side lobes are reasonably symmetric around their midpoints, it is logical to assume that the peak amplitudes of side lobes appear in the middle of zero-crossings, so that approximate peak locations can be obtained from the relationship

$$\frac{\pi d(\sin(\theta_s) - \sin(\theta))N}{\lambda} = (2m'' + 1)\frac{\pi}{2}$$

Where $m'' = 1, \pm 2, \pm 3 \ldots etc.$, so that the solution for the peaks of side lobes can be written as

$$\theta = \sin^{-1}(\sin(\theta_s) - \frac{(2m'' + 1)\lambda}{2Nd})$$

It can be easily shown that the first side lobe in the left hand side of the main lobe ($m'' = 1$) has the maximum amplitude among the many side lobes in the directivity plot. This lobe is called the "peak side lobe", whose peak amplitude can be expressed as

$$H(\theta_{ps}) = \left|\frac{1}{Nsin(\frac{3\pi}{2N})}\right|$$

To quantify thee significance of the peak side lobe effect, it is convenient to introduce a parametric constant "ξ" defined by the ratio between the amplitudes of main and peak side lobes, such that

$$\xi = \frac{H(\theta_{ps})}{H(\theta_s)}$$

Since $H(\theta_s)$ is unity, the parameter can be written as

$$\xi = \frac{2}{3\pi}\left|\frac{\frac{3\pi}{2N}}{\sin(\frac{3\pi}{2N})}\right|$$

We can observe that the side lobe level is a function of (N) only, therefore, the only wat of manioulating the peak side lobe is to account for the number of elements

The code below represents an observation of peak side lobe amplitude as a function of number of elements.

```
% Peak side lobe amplitude as afunction of number of elements
clc
clear
close all
N = 0:1:128;
ratio = (2 / (3 * pi)) * abs( ((3 * pi) ./ (2 .* N)) ./ ( sin((3 * pi) ./ (2 .* N)) )
);
ratioindb = 20 * log10(ratio);
plot(N,ratioindb)
grid on
axis([0 128 -16 0])
title('Peak side lobe amplitude as afunction of number of elements')
xlabel('Number of Elements (N)')
ylabel('Peak side lobe amplitude H(theta_p_s) (dB)')
```

6. Assignment 12 Line Array (Delay and Sum) Design

Requirements and determination are as follow:

- Have a main lobe width of less than 30 degrees for a 1000 Hz signal when steered to 0 degrees
- Have a maximum sidelobe level of less than 13 dB
- Considerations for this design
    - Number of microphones
    - Microphone spacing
    - Avoiding aliasing
    - Keep the number of mics to minimum in order to reduce cost

To avoid aliasing, the following condition should be true,

$$\frac{d}{\lambda} \leq 0.5$$

Since the microphone array should work on 1000 Hz, then inter element spacings will be as follow,

$$\frac{d}{\lambda} = \frac{d}{(\frac{c}{f})} = \frac{d}{\frac{343}{1000}} = \frac{d}{0.343} \leq 0.5$$

$$d \leq 0.1715\ m$$

Another controlling parameter is the side lobe level. Side lobe level is connected directly with the number of elements, thus the following will help us determining the number of elements using side lobe level of 13 dB.

$$PSLL = 13 \ = \frac{1}{Nsin(\frac{3\pi}{2N})}$$

We run first iteration for N = 4 elements

$$\frac{1}{(4)sin(\frac{3\pi}{2(4)})} = 0.2705\ dB \ll 13$$

Now we verify if this element number will give us a main lobe width of less than 30 degrees for a 1000 Hz signal when steered to 0 degrees

The main lobe width is controlled by the following equation

$$\Delta\theta_m = \sin^{-1}(\sin(\theta_s) + \frac{\lambda}{Nd}) - \sin^{-1}(\sin(\theta_s) - \frac{\lambda}{Nd})$$

$$\Delta\theta_m = \sin^{-1}(\sin(0) + \frac{2}{4}) - \sin^{-1}\left(\sin(0) - \frac{2}{4}\right) = 60^o$$

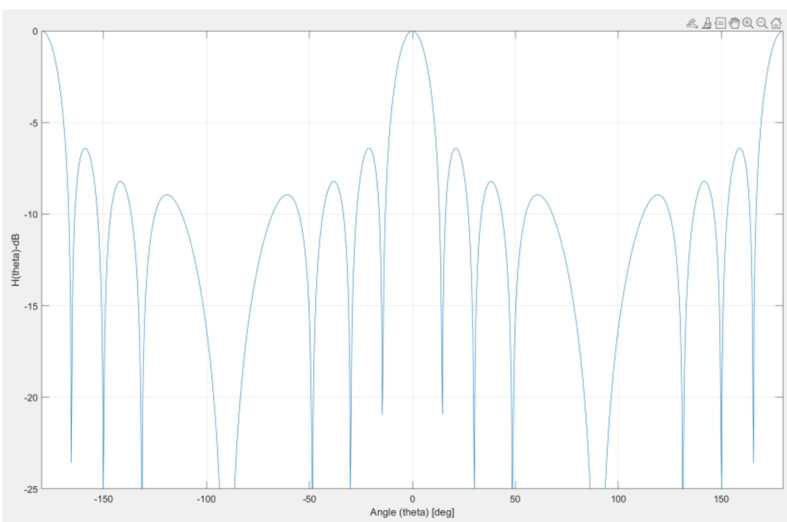
This angle is will yield a main lobe with width of 60 degrees, so we need to repeat the iteration process till we find the proper number of elements which generates the exact main lobe width.

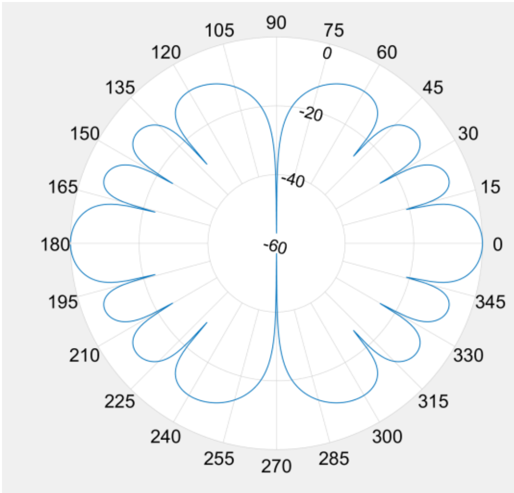
Doing these iterations gives us that the number of elements should be N = 8.

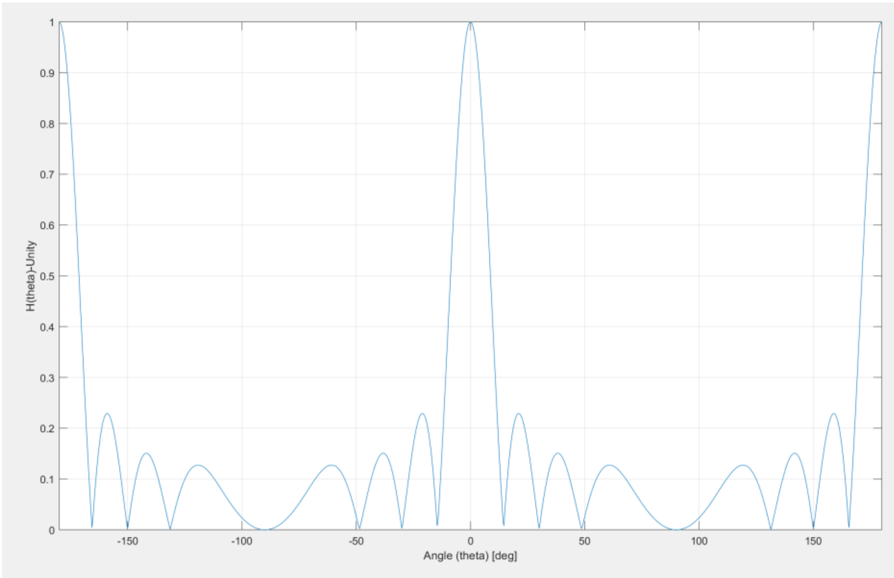
Now we can run the following code to plot required directivity patterns.

```
clc
clear
close all
theta = linspace(-180,180,1000);
N = 8;
d_l = 0.5;
%run two times separately and plug in 30 instead of 0 below in the second
%run
h2oftheta = abs( ( sin( pi * d_l * ( sind(0) - sind(theta) ) * N ) ) ./ ( N * sin( pi
* d_l * ( sind(0) - sind(theta) ) ) ) ) ;
figure(1)
plot(theta,h2oftheta)
grid on;
axis([-180 180 0 1]);
xlabel('Angle (theta) [deg]');
ylabel('H(theta)-Unity');
db = 10*log10(h2oftheta);
figure(2)
plot(theta,db)
grid on;
axis([-180 180 -25 0]);
xlabel('Angle (theta) [deg]');
ylabel('H(theta)-dB');
figure(3)
polarpattern(theta,db)
```

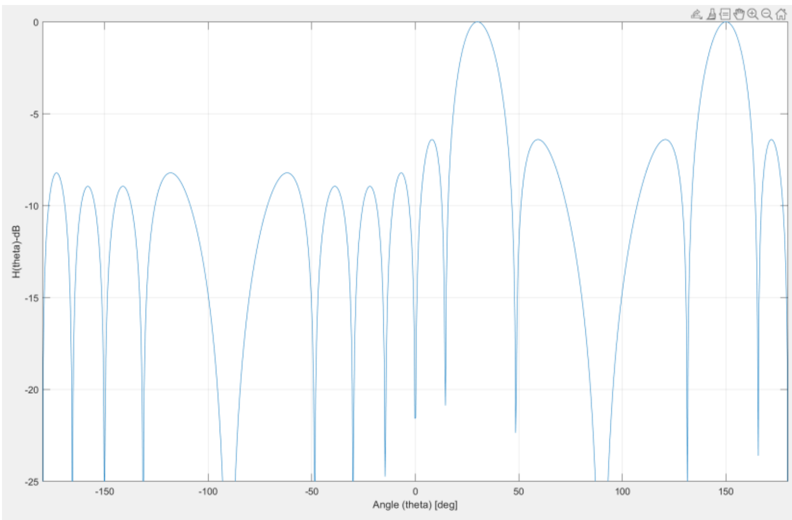
For steering angle 0 degree

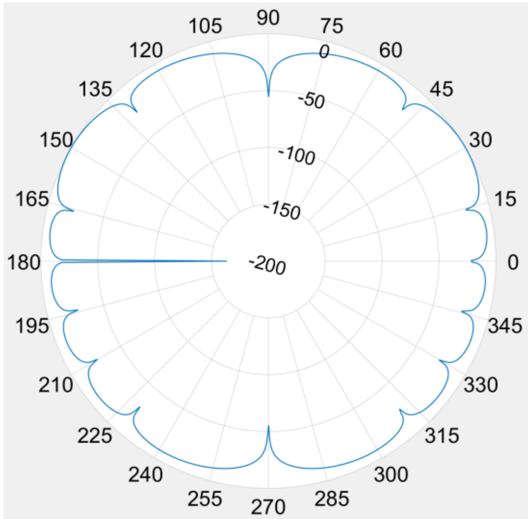


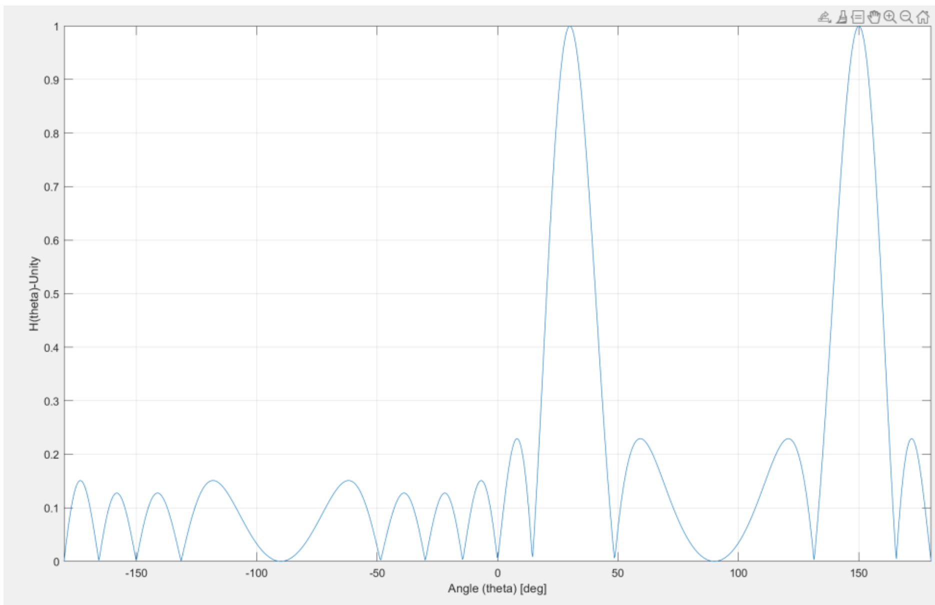




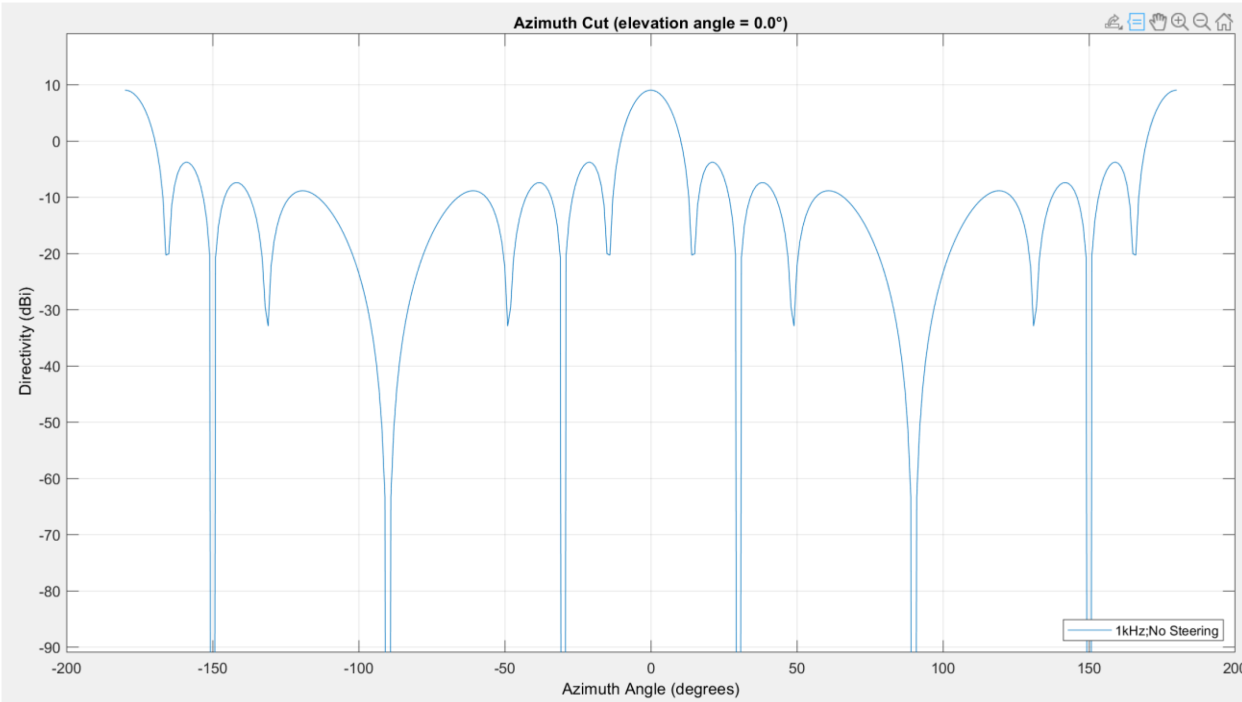
For steering angle of 30 degrees

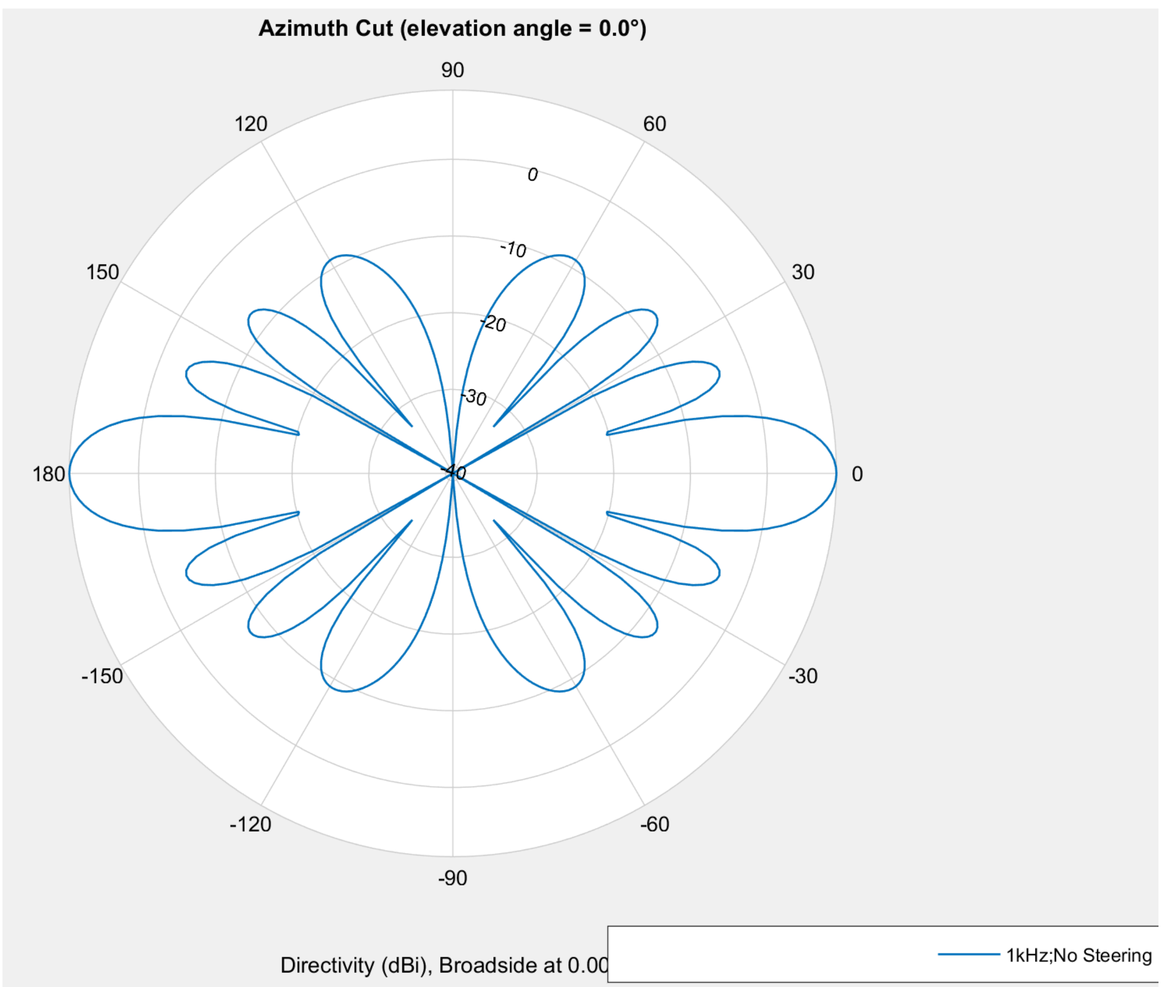






To verify our results, the sensor array app supported by MATLAB was run and following are the directivity patterns outputs.


Azimuth Cut (elevation angle = 0.0°)
Directivity (dBi)
Azimuth Angle (degrees)
1kHz;No Steering
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-200
-150
-100
-50
0
50
100
150
200


Azimuth Cut (elevation angle = 0.0°)
90
120
60
150
30
180
0
-150
-30
-120
-60
-90
0
-10
-20
-30
-40
Directivity (dBi), Broadside at 0.00
1kHz;No Steering