Probability and Statistics: Brief Review

Based on Prof. Ramana Grandhi's Seminar Lecture

Probability Theory

- Facilitates mathematical descriptions of uncertain events
- Began with attempts to analyze gambling during the 17th century
- Effective for both discrete and continuous events

➢ Probability is a ***measure of confidence*** that a specific event will occur

$0 \le P(E) \le 1$ E: any event

$P(\phi) = 0 \qquad P(\Omega) = 1$ ϕ: empty set
Ω: sample space (there is no element outside Ω)

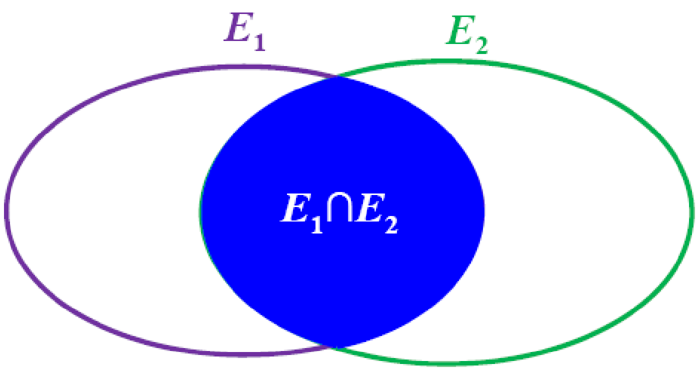
$P(A) \le P(B) \text{ if } A \subseteq B$ A *and* B: two sets

Joint Probability

Joint probability: Probability of two or more events ***happening together***

Ex: What is the probability that a die lands on 3 and a coin comes up heads?
What is the probability that it will rain and thunder tomorrow?

Consider two events E_1 and E_2



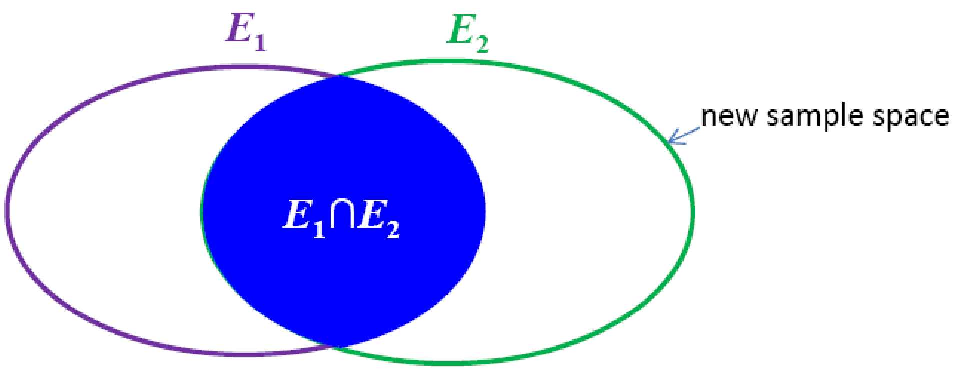
Venn Diagram

$P(E_1 \cap E_2)$: joint probability of E_1 and E_2

Conditional Probability

Conditional probability: probability of an event ***given the occurrence*** of another event

Ex: What is the probability that a die lands on 3 given that a coin comes up head?
What is the probability that it will rain tomorrow given that it will thunder?



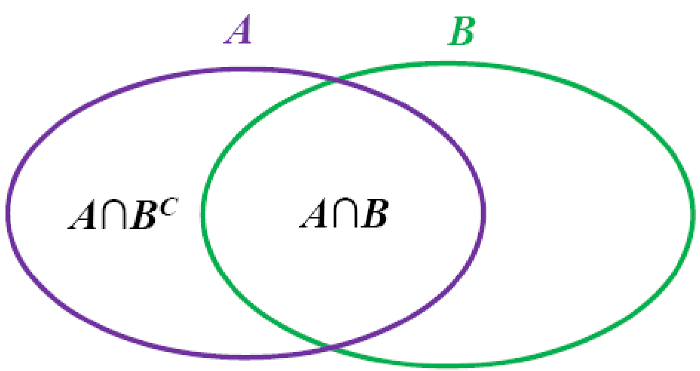
$P(E_1|E_2)$: conditional probability of E_1 given E_2

$$P\left(E_1 \middle| E_2\right) = \frac{P\left(E_1 \cap E_2\right)}{P\left(E_2\right)}$$

Marginal Probability

Marginal probability: probability of an event ***regardless of other events***
(unconditional probability)

Ex: What is the probability that a die lands on 3 (regardless of outcome of a coin toss)?
What is the probability that it will rain tomorrow (regardless of whether it will thunder)?



$$P(A) = P(A \cap B) + P\left(A \cap B^C\right)$$

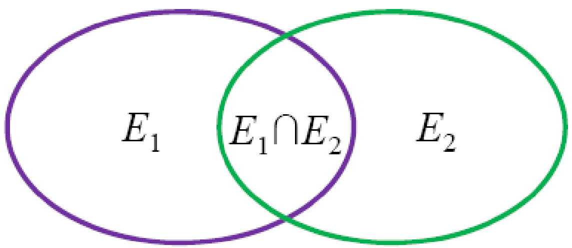
For multiple events $B_1, B_2, \ldots, B_N$

$$P(A) = \sum_{n=1}^{N} P(A \cap B_n)$$

Independent Event

Independent event: The occurrence of one event ***does not affect*** the result of another event

Ex: First Die lands on 3 and Second Die lands on 4 (*independent*)
It rains today and my chair breaks at work



$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

If two events E_1 and E_2 are ***independent***, then

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1)P(E_2)$$

$$P(E_1|E_2) = \frac{P(E_1)P(E_2)}{P(E_2)} = P(E_1)$$

$$P(E_2|E_1) = \frac{P(E_1)P(E_2)}{P(E_1)} = P(E_2)$$

Total Probability Theorem & Bayes' Rule

Total Probability Theorem

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \cdots + P(B \mid E_k)P(E_k) \end{aligned} \qquad (2\text{-}12)$$

Bayes' Rule

If $E_1, E_2, \ldots, E_k$ are $\boldsymbol{k}$ mutually exclusive and exhaustive events and B is any event,

$$P(E_1 \mid B) = \frac{P(B \mid E_1)P(E_1)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \cdots + P(B \mid E_k)P(E_k)} \qquad (2\text{-}16)$$

for $P(B) > 0$

Notice that the numerator always equals one of the terms in the sum in the denominator.

System Reliability

- Series system

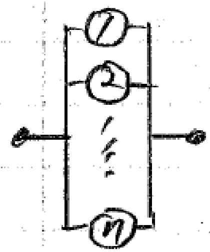
P_i : failure probability of the i^{th} component

$$P[\text{system failure}] = 1 - P[\text{system survival}]$$
$$= 1 - \prod_{i=1}^{n} (1 - P_i)$$

(when $P_i = P$ for all components)

$$= 1 - (1-P)^n$$

- Parallel system



P_i : failure probability

$$P[\text{system failure}]$$
$$= \prod_{i=1}^{n} P_i$$

(if all components are independent)

(when, $P_i = p$)

$$= p^n$$

- m-out-of-n system

when $P_i = p$, $P[\text{system failure}] = \binom{n}{m} p^m (1-p)^{n-m}$

if $m = 1$, then it becomes a series system.

if $m = n$, " " " a parallel system