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STT-6660 HW #5

In [211]:
```
import numpy as np
import scipy as sp
import sympy as sy
import warnings

%matplotlib inline
import matplotlib.pyplot as plt

sy.init_printing(use_latex='mathjax')
from IPython.display import display, Math, Latex
```

Problem 3.6

3.6. Refer to **Plastic hardness** Problem 1.22.

a. Obtain the residuals e_i and prepare a box plot of the residuals. What information is provided by your plot?

b. Plot the residuals e_i against the fitted values $\hat{Y}_i$ to ascertain whether any departures from regression model (2.1) are evident. State your findings.

c. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here? Use Table B.6 and $\alpha = .05$.

d. Compare the frequencies of the residuals against the expected frequencies under normality, using the 25th, 50th, and 75th percentiles of the relevant t distribution. Is the information provided by these comparisons consistent with the findings from the normal probability plot in part (c)?

e. Use the Brown-Forsythe test to determine whether or not the error variance varies with the level of X. Divide the data into the two groups, $X \leq 24$, $X > 24$, and use $\alpha = .05$. State the decision rule and conclusion. Does your conclusion support your preliminary findings in part (b)?

Part a: Get residuals and construct a box plot

In [212]:
```
#Define the data

Y = sy.Matrix([199, 205, 196, 200, 218, 220, 215, 223, \
               237, 234, 235, 230, 250, 248, 253, 246]).T
X = sy.Matrix([16, 16, 16, 16, 24, 24, 24, 24, 32, 32, \
               32, 32, 40, 40, 40, 40]).T
```

Get predicted values

In [213]:
```
Yhat = 2.0344*X + 168.6*np.ones((1,16))
```

Get residuals e_i

In [306]:
```
e = Y-Yhat
display(Math(r'e_i = '))
e.T.evalf(5)
```

$e_i =$

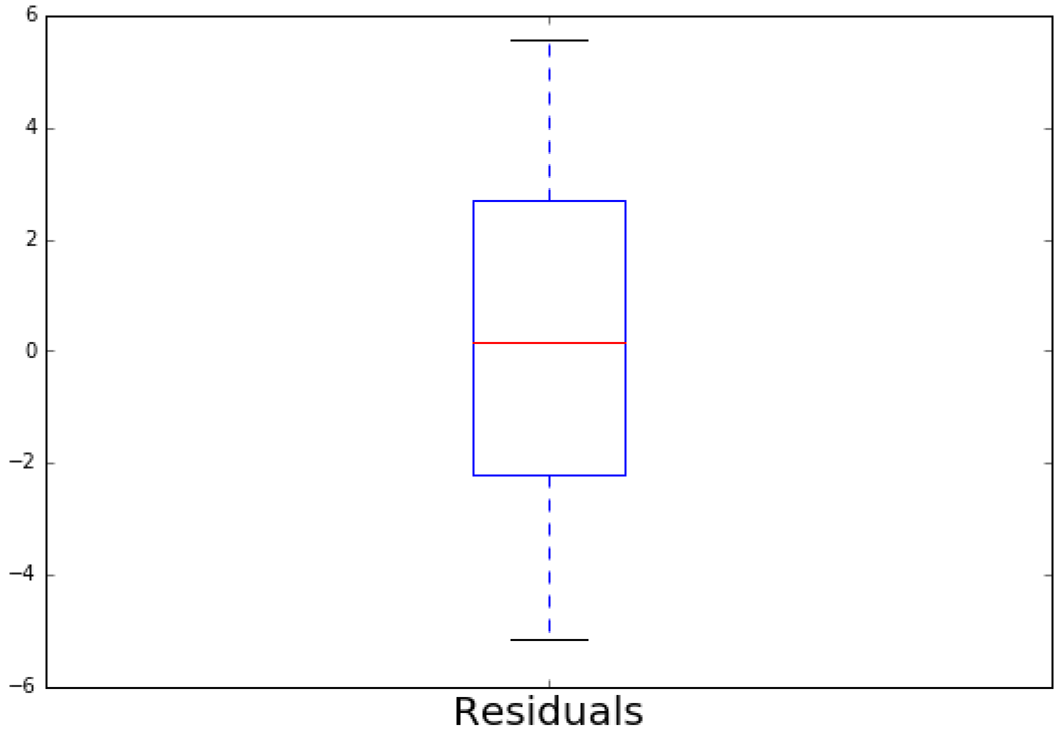
Out[306]:

$$\begin{bmatrix} -2.1504 \\ 3.8496 \\ -5.1504 \\ -1.1504 \\ 0.5744 \\ 2.5744 \\ -2.4256 \\ 5.5744 \\ 3.2992 \\ 0.2992 \\ 1.2992 \\ -3.7008 \\ 0.024 \\ -1.976 \\ 3.024 \\ -3.976 \end{bmatrix}$$

Here is the boxplot for the residuals

In [307]:
```
e2=np.array(e[:],dtype=float)
fig = plt.figure(1, figsize=(9, 6))
ax = fig.add_subplot(111)
ax.boxplot(e2)
ax.set_xticklabels(['Residuals'], fontsize=20)
```

Out[307]: [<matplotlib.text.Text at 0x23b68651eb8>]

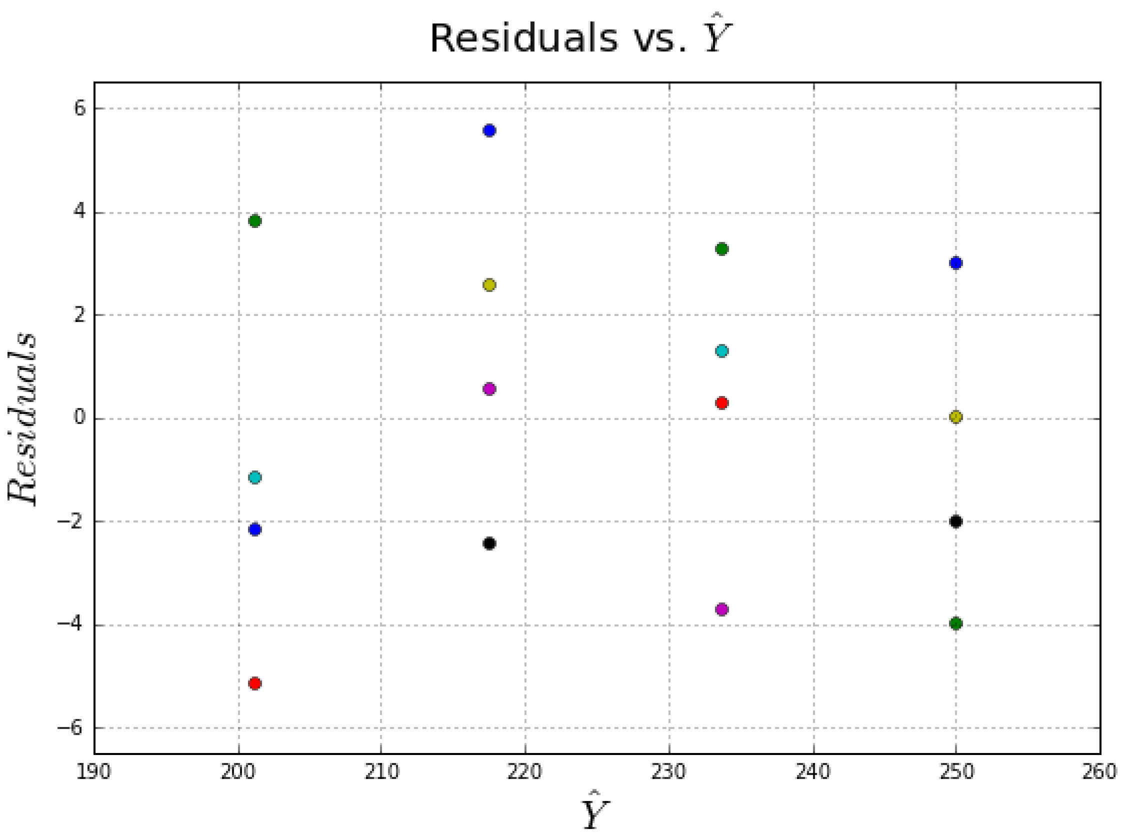


From the boxplot above, it appears that the residuals are normally distributed

Part b: plot residuals against fitted values

Below is plot of Residuals vs. $\hat{Y}$

In [308]:
```
fig=plt.figure(1, figsize=(9, 6))

ax = fig.add_subplot(111)
ax.plot(Yhat, e, "o")

#Plot annotations
axis_span=[190, 260, -6.5, 6.5]
plt.axis(axis_span)

fig.suptitle('Residuals vs. $\hat{Y}$', fontsize=20)
plt.xlabel('$\hat{Y}$', fontsize=20)
plt.ylabel('$Residuals$', fontsize=20)
plt.grid()
```



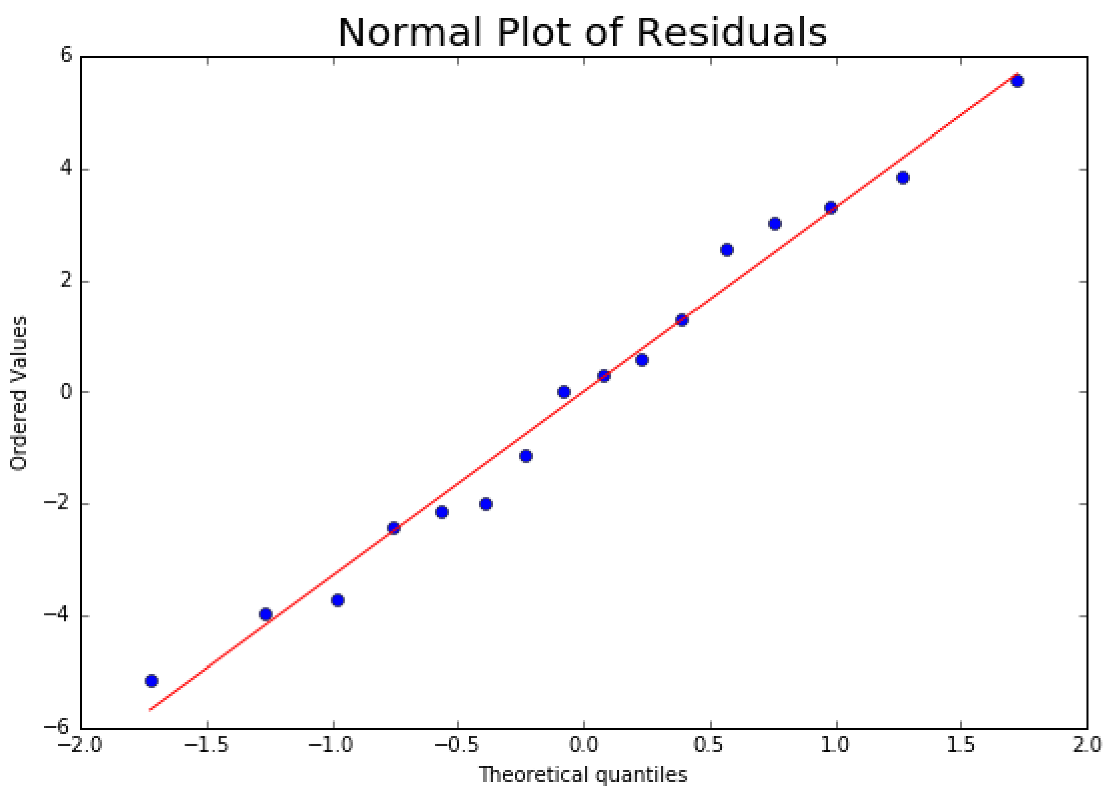
No unusual departures from the X-axis are noticable. The assumption of constant variance seems to apply for this data-set.

Part c: Plot normal plot for the residuals

Below is normal plot

In [217]:
```
from scipy import stats
fig=plt.figure(1, figsize=(9, 6))

ax = fig.add_subplot(111)
res = stats.probplot(e2, plot=ax)
ax.set_title("Normal Plot of Residuals", fontsize=20)
```

Out[217]: <matplotlib.text.Text at 0x23b6842a588>



Here is the correlation coefficient

In [287]:
```
ans = sp.stats.pearsonr(res[0][0], res[0][1])
display(Math(r'Correlation\ Coefficient = '))
display(ans[0])
```

$$Correlation\ Coefficient =$$

0.992196616622

Part e: Use Brown-Forsythe test to determine if error variance varies with X

In [309]:
```
display(Math(r"""The\ residuals\ are\ divided\ into\ two\ data\
sets\ set_1\ and\ set_2\ before\ the\ test\ is\ performed"""))
set1 = e2[0:8]
display(set1)
set2 = e2[8:17]
display(set2)
```

$The\ residuals\ are\ divided\ into\ two\ data\ sets\ set_1\ and\ set_2\ before\ the\ test\ is\ perfor$

```
array([-2.1504,  3.8496, -5.1504, -1.1504,  0.5744,  2.5744, -2.4256,
        5.5744])
```

```
array([ 3.2992,  0.2992,  1.2992, -3.7008,  0.024 , -1.976 ,  3.024 ,
       -3.976 ])
```

Get the medians for the two data sets

In [300]:
```
d1Med = np.median(set1)
d2Med = np.median(set2)

display(d1Mean)
display(d2Mean)
```

-1.1504

0.2992

In [314]:
```
d1 = abs(np.add(set1, -np.ones(8)*d1Med))
d2 = abs(np.add(set2, -np.ones(8)*d2Med))

display(d1)
display(d2)
```

```
array([ 1.    ,  5.    ,  4.    ,  0.    ,  1.7248,  3.7248,  1.2752,
        6.7248])
```

```
array([ 3.    ,  0.    ,  1.    ,  4.    ,  0.2752,  2.2752,  2.7248,
        4.2752])
```

Perform the variance test

In [317]:
```
stats.levene(d1, d2, center='median')
```

Out[317]: $(1.68009650173, \quad 0.215872449872)$

Formal test set-up:
$H_0: \sigma^2$ **is constant**
$H_a: \sigma^2$ **is not constant**

The above result gives the t statistic and the p-value.

t_{obs} **= 1.680**

$p-value$ **= 0.2158**

Since $p-value > 0.05$ **we cannot reject the null hypothesis. Therfore, the variance seems to be constant.**

Problem 3.11

3.11. **Drug concentration.** A pharmacologist employed linear regression model (2.1) to study the relation between the concentration of a drug in plasma (Y) and the log-dose of the drug (X). The residuals and log-dose levels follow.

i:	1	2	3	4	5	6	7	8	9
X_i:	−1	0	1	−1	0	1	−1	0	1
e_i:	.5	2.1	−3.4	.3	−1.7	4.2	−.6	2.6	−4.0

a. Plot the residuals e_i against X_i. What conclusions do you draw from the plot?

b. Assume that (3.10) is applicable and conduct the Breusch-Pagan test to determine whether or not the error variance varies with log-dose of the drug (X). Use $\alpha = .05$. State the alternatives, decision rule, and conclusion. Does your conclusion support your preliminary findings in part (a)?

In [220]:
```
# Define data

X2 = sy.Matrix([-1, 0, 1, -1, 0, 1, -1, 0, 1]).T
e = sy.Matrix([0.5, 2.1, -3.4, 0.3, -1.7, 4.2, -0.6, 2.6, -4.0]).T
E_sq = e.multiply_elementwise(e)
# Square the residuals
e_sq = np.array(np.power(e[:],2),dtype=float)
x2 = np.array(X2[:],dtype=float)

display(X2)
display(e)
display(E_sq)
```

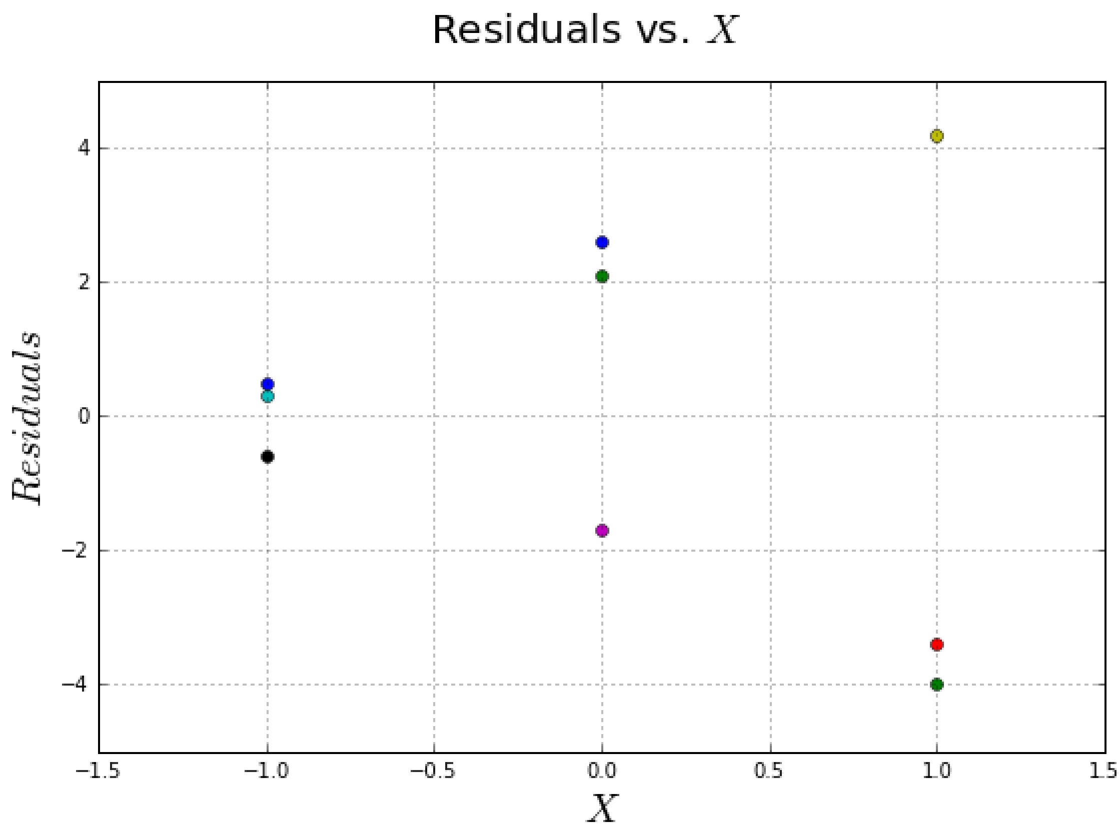
$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 2.1 & -3.4 & 0.3 & -1.7 & 4.2 & -0.6 & 2.6 & -4.0 \end{bmatrix}$$

$$\begin{bmatrix} 0.25 & 4.41 & 11.56 & 0.09 & 2.89 & 17.64 & 0.36 & 6.76 & 16.0 \end{bmatrix}$$

Part a: Plot residuals vs. X

In [221]:
```
fig=plt.figure(1, figsize=(9, 6))

ax = fig.add_subplot(111)
ax.plot(X2, e_311, "o")

#Plot annotations
axis_span=[-1.5, 1.5, -5, 5]
plt.axis(axis_span)

fig.suptitle('Residuals vs. $X$', fontsize=20)
plt.xlabel('$X$', fontsize=20)
plt.ylabel('$Residuals$', fontsize=20)
plt.grid()
```



Based on the residual plot above, the variance is not constant.

In [222]:
```
Xbar = np.mean(X2).evalf(5)
display(Math(r'\bar{X} = '))
display(Xbar)
```

$\bar{X} =$

0

In [223]:
```
e_bar = np.mean(E_sq).evalf(3)
display(Math(r'\bar{e^2} = '))
display(e_bar)
```

$\bar{e^2} =$

6.66

In [224]:
```
Sxy = np.sum(np.multiply(X2,E_sq)) - (1/9)*(np.sum(X2)*np.sum(E_sq))
display(Math(r'S_{xy} = '))
display(Sxy)
```

$S_{xy} =$

44.5

In [225]:
```
Sxx = np.sum(np.multiply(X2,X2)) - (1/9)*(np.sum(X2)*np.sum(X2))
display(Math(r'S_{xx} = '))
display(Sxx)
```

$S_{xx} =$

6

In [226]:
```
b1 = (Sxy/Sxx).evalf(5)
display(Math(r'b_{1} = '))
display(b1)
```

$b_1 =$

7.4167

In [227]:
```
b0 = (e_bar - b1*Xbar).evalf(4)
display(Math(r'b_{0} = '))
display(b0)
```

$b_0 =$

6.662

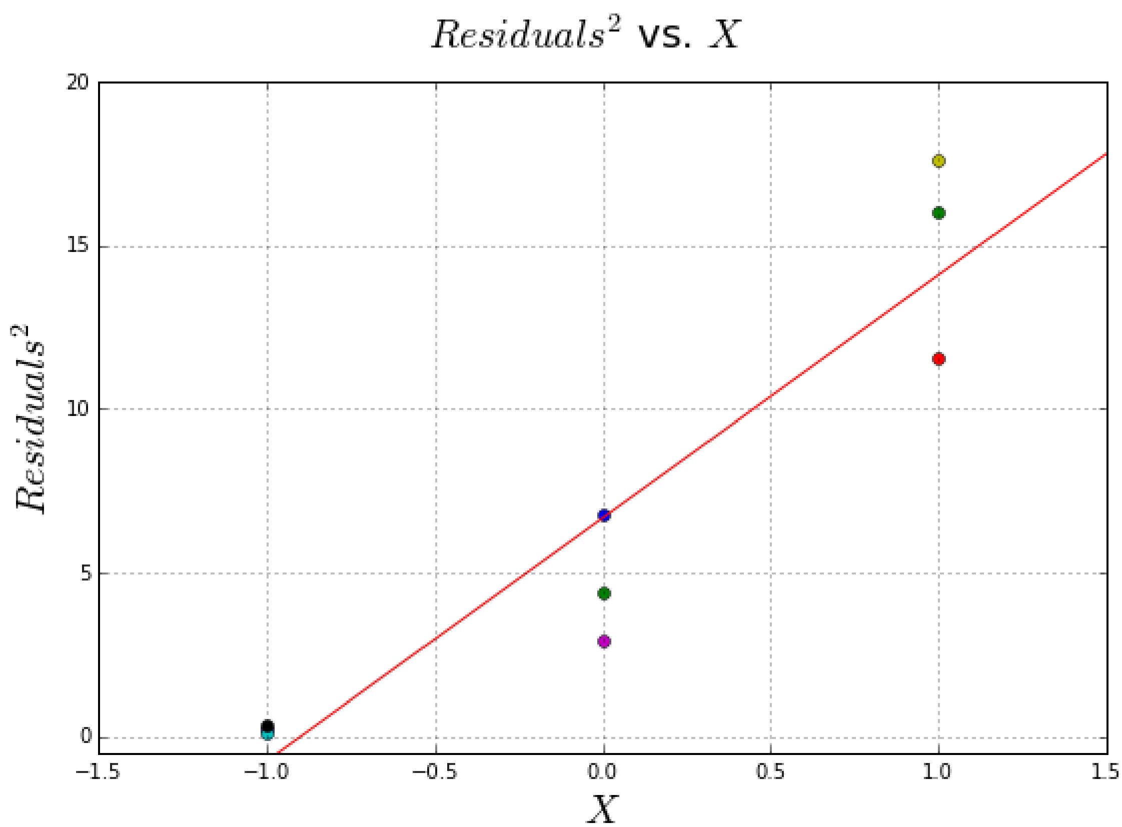
Regression Function is:

$$\hat{Y} = 7.4167X + 6.662$$

Next plot the data and the corresponding regression model.

From the figure below the regression model fits the data reasonably well.

In [228]:
```
Xd=np.linspace(-2.0,2.0,200)

Y1 = 7.4167*Xd + 6.662;

fig=plt.figure(1, figsize=(9, 6))

ax = fig.add_subplot(111)
ax.plot(X2, E_sq, "o")
ax.plot(Xd, Y1)

#Plot annotations
axis_span=[-1.5, 1.5, -0.5, 20]
plt.axis(axis_span)

fig.suptitle('$Residuals^2$ vs. $X$', fontsize=20)
plt.xlabel('$X$', fontsize=20)
plt.ylabel('$Residuals^2$', fontsize=20)
plt.grid()
```



In [229]:
```
SSTO = (np.sum(np.power((E_sq-np.ones((1,9))*e_bar),2))).evalf(4)
display(Math(r'SSTO = '))
display(SSTO)
```

$SSTO =$

375.0

In [230]:
```
e_est1 = np.multiply(np.ones((1,9))*7.4167, X2)
e_est = np.add(e_est1, np.ones((1,9))*6.662)
```

In [231]:
```
SSE1 = np.sum(np.power((E_sq-e_est),2))
display(Math(r'SSE1 = '))
display(SSE1)
```

$SSE1 =$

44.99268934

In [232]:
```
SSR = (SSTO-SSE1).evalf(6)
display(Math(r'SSR = SSTO - SSE'))
display(SSR)
```

$SSR = SSTO - SSE$

330.042

In [233]:
```
SSE = np.sum(E_sq)
display(Math(r'SSE = '))
display(SSE)
```

$SSE =$

59.96

Chi2Obs = ((SSR/2)/(SSE/9)**2).evalf(4) display(Math(r'\chi^{2}_{BP} = ')) display(Chi2Obs)

In [234]:
```
from scipy.stats import chi2
display(Math(r'\chi^{2}_{(0.95, 1)} = '))
chi2.ppf(0.95,1)
```

$\chi^{2}_{(0.95,1)} =$

Out[234]: 3.84145882069

Since $\chi^2_{BP} < \chi^2_{(0.95,1)}$ we fail to reject the null hypothesis. Therefore, based on this test variance is assumned to be equal. Looking at the plot of e_i^2 vs. X there is a clear slope so more samples are needed in order to improve the accuracy of the test

Problem 3.14

3.14. Refer to **Plastic hardness** Problem 1.22.

a. Perform the F test to determine whether or not there is lack of fit of a linear regression function; use $\alpha = .01$. State the alternatives, decision rule, and conclusion.

b. Is there any advantage of having an equal number of replications at each of the X levels? Is there any disadvantage?

c. Does the test in part (a) indicate what regression function is appropriate when it leads to the conclusion that the regression function is not linear? How would you proceed?

Part a: Perform F test for lack of fit

In [235]:

```
#Define the data

Y = sy.Matrix([199, 205, 196, 200, 218, 220, 215, 223, \
               237, 234, 235, 230, 250, 248, 253, 246]).T
X = sy.Matrix([16, 16, 16, 16, 24, 24, 24, 24, 32, 32, \
               32, 32, 40, 40, 40, 40]).T
```

In [236]:

```
Yhat = 2.0344*X + 168.6*np.ones((1,16))
```

In [237]:

```
mean1 = np.mean(Y[0:4])
mean2 = np.mean(Y[4:8])
mean3 = np.mean(Y[8:12])
mean4 = np.mean(Y[12:16])

display(mean1)
display(mean2)
display(mean3)
display(mean4)
```

200

219

234

$$\frac{997}{4}$$

SSPE calculated below

In [238]:
```
SSPE = np.sum(np.power((np.array(Y[0:4],dtype=float)-np.ones(4)*mean1),2)) + \
np.sum(np.power((np.array(Y[4:8],dtype=float)-np.ones(4)*mean2),2)) + \
np.sum(np.power((np.array(Y[8:12],dtype=float)-np.ones(4)*mean3),2)) + \
np.sum(np.power((np.array(Y[12:16],dtype=float)-np.ones(4)*mean4),2))

display(Math(r'SSPE = '))
SSPE.evalf(4)
```

$$SSPE =$$

Out[238]: 128.8

Now get SSLF from expression SSLF = SSE-SSPE

Start with obtaining S_{yy} in order to get $SSE = S_{yy} - b_1 S_{xy}$

b_1 and S_{xy} are obtained from HW #4:

- **b_1 = 2.0344**
- **S_{xy} = 2604.0**

In [239]:
```
Syy = np.sum(np.multiply(Y,Y)) - (1/16)*(np.sum(Y)*np.sum(Y))
display(Math(r'S_{yy} = '))
display(Syy)
```

$$S_{yy} =$$

$$5443.9375$$

In [240]:
```
display(Math(r'SSE = '))
SSE = Syy - 2.0344*2604.0
display(SSE.evalf(4))
```

$$SSE =$$

$$146.4$$

In [241]:
```
display(Math(r'SSLF = '))
SSLF = SSE-SSPE
display(SSLF.evalf(4))
```

$$SSLF =$$

$$17.61$$

Below is the ANOVA table

Source	df	SS	MS
regression	1	146.4	146.4
LOF	2	17.61	8.81
pure error	12	128.8	10.733
total	15	292.81	

In [242]:
```
display(Math(r'F_{obs} = '))
Fobs = (8.81/10.733)
display(Fobs)
```

$F_{obs} =$

0.8208329451225194

In [243]:
```
display(Math(r'Get\ F(0.99, 2, 12)'))
from scipy.stats import f
f.ppf(0.99, 2, 12)
```

$Get\ F(0.99, 2, 12)$

Out[243]: 6.92660814019

Since $F_{obs} = 0.8208 < F = 6.927$ and is therefore not in the rejection region we conclude that H_0 cannot be rejected.

Part c:

The F-test only tests if the chosen linear model is fitting the supplied data adequately. It does not provide, on its own, an idication of what model is appropriare should the linear model fail to fit the data.

One should plot the given data and the subject residuals in order to determine what fit looks appropriate and proceed accordingly.

Problem 3.15

3.15. **Solution concentration.** A chemist studied the concentration of a solution (Y) over time (X). Fifteen identical solutions were prepared. The 15 solutions were randomly divided into five sets of three, and the five sets were measured, respectively, after 1, 3, 5, 7, and 9 hours. The results follow.

i:	1	2	3	...	13	14	15
X_i:	9	9	9	...	1	1	1
Y_i:	.07	.09	.08	...	2.84	2.57	3.10

a. Fit a linear regression function.

b. Perform the F test to determine whether or not there is lack of fit of a linear regression function; use $\alpha = .025$. State the alternatives, decision rule, and conclusion.

c. Does the test in part (b) indicate what regression function is appropriate when it leads to the conclusion that lack of fit of a linear regression function exists? Explain.

Part a: Fit linear model

In [244]:
```
lines = np.loadtxt("CH03PR15.txt")
lines
```

Out[244]:
```
array([[ 0.07,  9.  ],
       [ 0.09,  9.  ],
       [ 0.08,  9.  ],
       [ 0.16,  7.  ],
       [ 0.17,  7.  ],
       [ 0.21,  7.  ],
       [ 0.49,  5.  ],
       [ 0.58,  5.  ],
       [ 0.53,  5.  ],
       [ 1.22,  3.  ],
       [ 1.15,  3.  ],
       [ 1.07,  3.  ],
       [ 2.84,  1.  ],
       [ 2.57,  1.  ],
       [ 3.1 ,  1.  ]])
```

In [245]:
```
Xbar = np.mean(lines[:,1])
display(Math(r'\bar{X} = '))
display(Xbar)
```

$\bar{X} =$

5.0

In [246]:
```
Ybar = np.mean(lines[:,0])
display(Math(r'\bar{Y} = '))
display(Ybar)
```

$\bar{Y} =$

0.955333333333

In [247]:
```
Sxy = np.sum(np.multiply(lines[:,1],lines[:,0])) \
    - (1/15)*(np.sum(lines[:,1])*np.sum(lines[:,0]))
display(Math(r'S_{xy} = '))
display(Sxy)
```

$S_{xy} =$

-38.88

In [248]:
```
Sxx = np.sum(np.multiply(lines[:,1],lines[:,1])) \
    - (1/15)*(np.sum(lines[:,1])*np.sum(lines[:,1]))
display(Math(r'S_{xx} = '))
display(Sxx)
```

$S_{xx} =$

120.0

In [249]:
```
b1 = (Sxy/Sxx)
display(Math(r'b_{1} = '))
display(b1)
```

$b_1 =$

-0.324

In [250]:
```
b0 = (Ybar - b1*Xbar)
display(Math(r'b_{0} = '))
display(b0)
```

$b_0 =$

2.57533333333

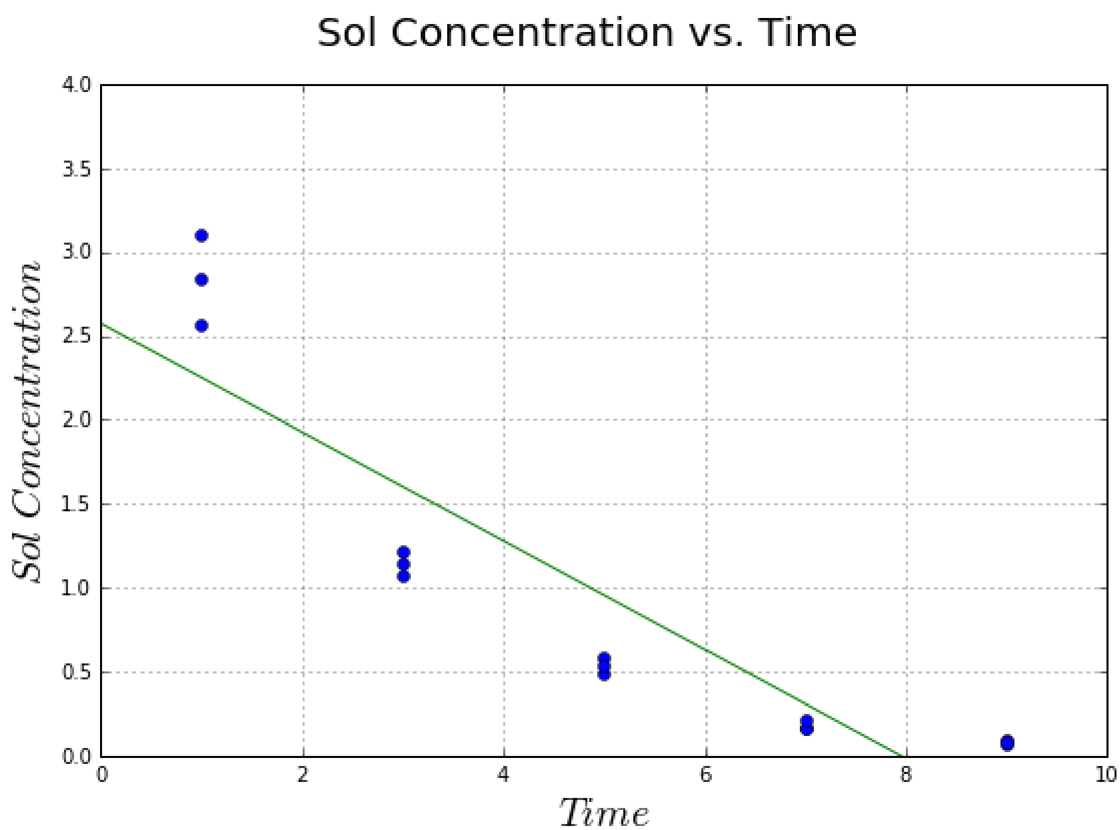
Regression Function is:

$$\hat{Y} = -0.324X + 2.5753$$

Next plot the data and the corresponding regression model.

From the figure below the regression model fits the data quite well.

In [251]:
```
X1=np.linspace(0.0,10.0,200)

Y1 = -0.324*X1 + 2.5753;

fig=plt.figure(1, figsize=(9, 6))

ax = fig.add_subplot(111)
ax.plot(lines[:,1], lines[:,0], "o")
ax.plot(X1, Y1)

#Plot annotations
axis_span=[0, 10, 0, 4]
plt.axis(axis_span)

fig.suptitle('Sol Concentration vs. Time', fontsize=20)
plt.xlabel('$Time$', fontsize=20)
plt.ylabel('$Sol\ Concentration$', fontsize=20)
plt.grid()
```



Part b: Perform F-test for lack of fit

In [252]:
```
mean1 = np.mean(lines[0:3,0])
mean2 = np.mean(lines[3:6,0])
mean3 = np.mean(lines[6:9,0])
mean4 = np.mean(lines[9:12,0])
mean5 = np.mean(lines[12:15,0])

display(mean1)
display(mean2)
display(mean3)
display(mean4)
display(mean5)
```

0.08

0.18

0.533333333333

1.14666666667

2.83666666667

In [253]:
```
SSPE = np.sum(np.power((lines[0:3,0]-np.ones(3)*mean1),2)) + \
np.sum(np.power((lines[3:6,0]-np.ones(3)*mean2),2)) + \
np.sum(np.power((lines[6:9,0]-np.ones(3)*mean3),2)) + \
np.sum(np.power((lines[9:12,0]-np.ones(3)*mean4),2)) + \
np.sum(np.power((lines[12:15,0]-np.ones(3)*mean5),2))

display(Math(r'SSPE = '))
SSPE
```

$SSPE =$

Out[253]: 0.1574

Now get SSLF from expression SSLF = SSE-SSPE

Start with obtaining S_{yy} in order to get $SSE = S_{yy} - b_1 S_{xy}$

In [254]:
```
Syy = np.sum(np.multiply(lines[:,0],lines[:,0])) \
    - (1/15)*(np.sum(lines[:,0])*np.sum(lines[:,0]))
display(Math(r'S_{yy} = '))
display(Syy)
```

$S_{yy} =$

15.5217733333

In [255]:
```
display(Math(r'SSE = '))
SSE = Syy - b1*Sxy
display(SSE)
```

$SSE =$

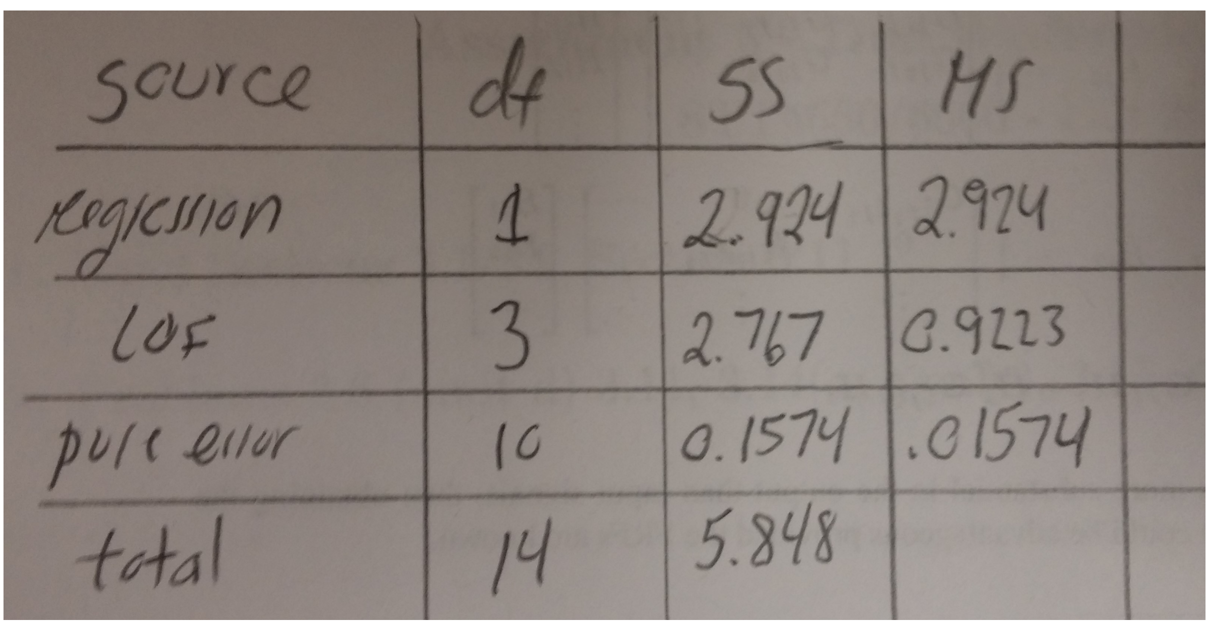
2.92465333333

In [256]:
```
display(Math(r'SSLF = '))
SSLF = SSE-SSPE
display(SSLF)
```

$SSLF =$

2.76725333333

Below is the ANOVA table



source	df	SS	MS
regression	1	2.924	2.924
lof	3	2.767	0.9223
pure error	10	0.1574	.01574
total	14	5.848	

In [257]:
```
display(Math(r'F_{obs} = '))
Fobs = (0.9223/0.01574)
display(Fobs)
```

$F_{obs} =$

58.595933926302415

In [258]:
```
display(Math(r'Get\ F(0.975, 3, 10)'))
from scipy.stats import f
f.ppf(0.975, 3, 10)
```

$Get\ F(0.975, 3, 10)$

Out[258]: 4.82562149341

Since $F_{obs} = 58.60 < F = 4.826$ and is therefore in the rejection region we conclude that H_0 cannot be accepted.

Part c:

The F-test only tests if the chosen linear model is fitting the supplied data adequately. It does not provide, on its own, an idication of what model is appropriare should the linear model fail to fit the data.

One should plot the given data and the subject residuals in order to determine what fit looks appropriate and proceed accordingly. For this case it looks like a second order fit is more appropriate.