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Topology Optimization of a Structure

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Abstract

Design optimization has become common place in both research and industry today. Topology optimization, without the requirement of an initial design, is quickly becoming a popular optimization method. A beam fixed at both ends with two required holes is subjected to a center load. Topology optimization is performed on the beam to determine the optimal material distribution of a design that is capable of supporting the load, but weighing no more than 60% of the original beam. A minimum displacement objective function is used with the modified SIMP interpolation method and the density of each element as the design variables. A sensitivity filter is applied to the results to ensure a solution void of grey material is obtained. This optimization procedure is performed with eleven different starting points to determine the possibility of a global minimum solution.

1. Introduction

1.1 Motivation

Design optimization has become commonplace in both research and industry today. Size, shape, and topology optimization are the three most common methods employed for structures. Size optimization and shape optimization require an initial design similar to the final solution. The size or shape is then refined into the optimal design. However, modern engineering problems incorporating multidisciplinary design are often unintuitive. This makes the formulation of an adequate initial design for shape or size optimization extremely difficult.

It is for this reason topology optimization is quickly increasing in popularity in the design industry. Without the need of an a priori assumption of the structure shape, the initial design can be simplified to a block of solid material. This is possible with topology optimization because the purpose of topology optimization is to determine the optimal layout of material.

1.2 Problem Description

A steel beam, fixed at both ends with two required circular holes in it, is loaded by force P at its center, as shown in Fig. 1 below. Common goals in many industries are reducing the weight of a structure as this often leads to reduced cost and/or increase performance. Therefore, the beam shown in Fig. 1 needs to be redesigned so the mass of the new design is no greater than 60% of the initial mass while still supporting the external load, P. This is accomplished using topology optimization.

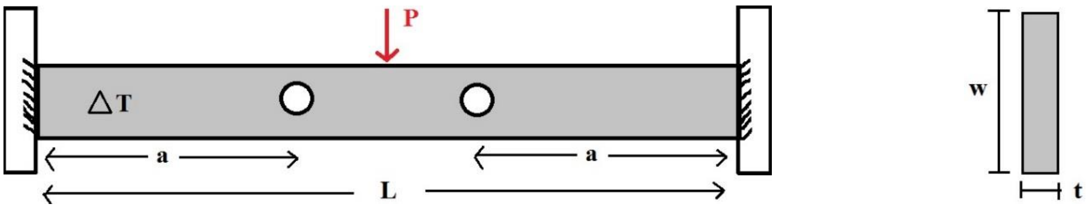

Figure 1: Fixed-Fixed Beam with Physical and Thermal Loading

P = 50,000 lbs		E = 29,000,000 psi	D_{hole} = 2 in
L = 60 in	a = 48 in	t = 0.75in	w = 6 in

Thermal structures have multidisciplinary design dependencies which makes the initial design of these structures unintuitive making them an ideal candidate for topology optimization. Although not considered for this project, the beam could also be subjected to a uniform temperature which is ΔT degrees above the ambient temperature. The restrained thermal expansion would result in additional thermal stresses developing in the beam. The amount of restrained expansion, and therefore stress magnitude, depends on the layout of the material. The loading then becomes design dependent and optimization also requires the sensitivity of the load with respect to the design variables. This, however, is beyond the scope of this paper.

1.3 Purpose

This project is intended to introduce the student to the basic methods of topology optimization. This project also introduces some concepts in numerical optimization. This is accomplished through writing a finite element analysis (FEA) code which also computes the gradient of the objective function. This project also stresses the importance of efficient sensitivity analysis as common numerical methods, such as finite-difference, results in code too slow to obtain results.

2. Methodology

As stated above in the problem description, the objective of this optimization is to find the optimal material layout that allows the mass of the beam to be reduced by at least 40% while remaining capable of supporting the loads. This can be accomplished through several methods. A very common topology objective function for structures is minimum compliance due to its simplicity [1]. Minimum compliance is alternatively known as maximum stiffness. However, the sensitivity analysis of minimum compliance is particularly easy. While this is often an advantage, for the demonstration of sensitivity analysis in this problem an alternative objective function is used.

2.1 Problem Formulation

2.1.1 Algorithm Choice (Design Variables)

A common topology optimization method is density-based optimization using the solid isotropic material with penalization (SIMP) to interpolate between solid and void material. This method is used in this problem because the material in this problem is given to be linear and isotropic. Using this method, the design variables, b, of the problem are the densities of each element. Due to the requirement of no material in the location of the holes, the densities at the elements in these locations need to be set to zero. This is accomplished by setting the initial value of the element density to zero and removing these densities from the design variables.

However, the finite element system is not actually dependent on the densities. In order to account for this, an interpolation function is used to relate Young's modulus, E, to the density. In this project, a modified SIMP approach is used. Instead of setting a very small limit on the densities, in order to avoid a singular stiffness matrix, a very small limit, E_{min}, is imposed on Young's modulus instead [3]. This modified SIMP approach is often easier to implement, among other advantages [3]. The interpolation used is:

$$E(b) = E_{min} + b^p(E_0 - E_{min}) \tag{1}$$

where p is a penalization factor, typically chosen to be three [2], and E0 is the actual Young's modulus of the material.

2.1.2 Objective Function

With the appropriate design variables chosen, the objective function needed to be formulated. An objective similar to minimum compliance is minimum displacement. This method ensures the strongest structure for the given mass constraint is generated and also provides sensitivity analysis complex enough for this project. Many optimizers require the objective function to evaluate to a single, scalar, value. The displacements from the FEA code are in the form of a column vector listing the x and y displacements of each node. In order to convert this to a single value, the sum of the squares of the displacement vector was calculated. This method takes both positive and negative displacements into consideration while remaining differentiable. This gives the final objective function the form of

$$Minimize\text{:}\ \sum_{i=1}^{P} U_i^2 = \boldsymbol{U}^T\boldsymbol{U} \qquad \text{where p} = \#\ \text{of DOFs} \tag{2}$$

2.1.3 Constraints

Next, the constraints on the objective function needed to be formulated. There are three constraints for this problem. The first constraint is the mass, or volume (because mass is proportional to volume), constraint. This is a linear inequality constraint because the volume must be less than or equal to the specified value of 60%. This constraint can be represented as

$$\frac{V(\boldsymbol{b})}{V_0} \leq 0.6 \tag{3}$$

Due to the density filter used for topology optimization, the volume constraint must also have the density filter applied to it. This is accomplished through the filter matrix, A. The actual volume constraint then becomes

$$g_1(\boldsymbol{b}) = \boldsymbol{Ab} - \boldsymbol{0.6} \leq 0 \tag{4}$$
$$A = filter * [\frac{1}{N_e}]$$

where N_e is the number of elements. The second constraint is a linear equality constraint for the finite element system.

$$h_1(\boldsymbol{b}) = \boldsymbol{KU} - \boldsymbol{F} = 0 \tag{5}$$

This constraint is not passed to the optimizer because this constraint is automatically satisfied by the FEA solver each time the optimizer evaluates the objective function.

The third constraint is the simple bounds of the design variables.

$$0 \le \boldsymbol{b} \le 1 \tag{6}$$

2.1.4 Optimization Problem

The objective function and the constraints are then combined into the optimization problem formulation shown below

Find:	**b**	where…
Minimize:	$\mathbf{U}(\mathbf{b})^T\mathbf{U}(\mathbf{b})$	**b** = ρ = Density of Element i
Subject to:	$g_1(\mathbf{b}) = \mathbf{Ab} \le 0.6$	**K** = Stiffness Matrix
	$h_2(\mathbf{b}) = \mathbf{KU}\text{-}\mathbf{F} = 0$	**U** = Displacement Matrix
	$0 \le \mathbf{b} \le 1$	**F** = Nodal Force Matrix

(7)

2.2 Solution Method

2.2.1 Finite Element Analysis

Topology optimization determines the optimal layout of material in a given domain that minimizes the objective function within some constraints. In order to do this, the domain must be discretized into a finite number of elements. Due to the symmetry of the geometry, boundary conditions, and loading, only half of this model is analyzed for computational efficiency. The mesh size for this problem is 120 x 24 elements. Four-node quadrilateral elements with an aspect ratio of one are used for the mesh to minimize error due to shear locking. Each of these elements has its own material properties such as the density. Accounting for the one hole that must remain in the structure, a total of approximately 2864 design variables are obtained. The finite element system is solved using MATLAB's direct solver, K\U, and the objective function value is calculated according to Eq. (2).

2.2.2 Sensitivity Analysis

The sensitivities of the objective and constraint functions then need to be calculated in order to check if the solution is optimal. Numerical differentiation methods, such as the finite-difference or complex-step method, are the easiest to implement, but are computationally very inefficient. Automatic differential programs, such as ADiMat, can be more efficient than the direct implementation of numerical techniques, but suffer from incompatibility with external libraries. The ideal solution is to obtain the analytical sensitivities, using a technique such as the adjoint method. The objective function of this problem is only a function of **U** which can be written as **U**(**K**(**E**(**b**)),**F**). The force, **F**, is not dependent on the design for this problem and so the sensitivities are zero. Applying the chain rule to the linear system, the sensitivity of **U** can then be calculated as:

$$\frac{d}{d\boldsymbol{b}}(\boldsymbol{KU} = \boldsymbol{F}) = \frac{d\boldsymbol{K}}{d\boldsymbol{b}}\boldsymbol{U} + \boldsymbol{K}\frac{d\boldsymbol{U}}{d\boldsymbol{b}} = \frac{d\boldsymbol{F}}{d\boldsymbol{U}} \xrightarrow{yields} \frac{d\boldsymbol{U}}{d\boldsymbol{b}} = \boldsymbol{K}^{-1} * \left(-\frac{d\boldsymbol{K}}{d\boldsymbol{b}}\boldsymbol{U}\right) \tag{8}$$

The element stiffness matrices are calculated as

$$\boldsymbol{K^e} = \boldsymbol{KE} * \boldsymbol{E} = \boldsymbol{KE} * \left(E_{min} + \boldsymbol{b}^p(E_0 - E_{min})\right) \tag{9}$$

where **KE** is the base stiffness matrix where E = 1. This step is performed for each element. Once all the element stiffness matrices have been obtained, they are assembled into the global stiffness matrix, **K**. Taking the derivative of this wrt b:

$$\frac{d\boldsymbol{K}}{d\boldsymbol{b}} = \left(p\boldsymbol{b}^{p-1}(E_0 - E_{min})\right) * \boldsymbol{KE} \tag{10}$$

This step is also performed iteratively for each design variable. The analytical sensitivities of the objective function wrt the design variables are then given as:

$$\frac{d\boldsymbol{U}}{d\boldsymbol{b}} = \boldsymbol{K}^{-1} * \left(-\left(\left(p\boldsymbol{b}^{p-1}(E_0 - E_{min})\right) * \boldsymbol{KE}\right)\boldsymbol{U}\right) \tag{11}$$

It is noted in these equations, the inverse of **K** is used for simplicity. Computing the inverse of **K** is expensive, and potentially unstable. In practice, alternative methods, such as Cholesky decomposition should be used to solve the system.

The densities, which have been specified as the design variables, are allowed to vary values between zero and one until the optimal of the current sub problem has been found. However, it is impossible to create a material that is partially solid and partially void at the same geometric location. This is commonly known as grey material. An example can be seen in Fig. 2a below.

Figure 2a: Grey Material

Figure 2b: Checkerboarding

Grey material is common in solutions that do not have a filter applied. In order to force the solution to a "black and white" solution of only completely solid and completely void material, a filter is used. A poor filter can also results in "checkerboarding" which occurs when solid and void material appear in an alternating pattern resembling that of a checkerboard. An example of this is shown in Fig. 2b above. Although no grey material exists, this is also a physical impossibility. Many different filters are available and many of them still have some grey material remaining. An example of this can be seen in Fig. 3 below.

Figure 3: Properly Filtered Results

It can be seen there is only a small amount of grey material and this could be replaced in reality by fillets. For this project, a filter was used on the objective function sensitivity which calculates the new sensitivity [2] as:

$$\frac{dU}{db}_{Filtered} = \frac{1}{\max(\gamma, b)\sum_{i\in N_e} H_{ei}} * \sum_{i\in N_e} H_{ei} * b_i * \frac{dU}{db} \tag{12}$$

H_{ei} is the weight factor for the filter which is calculated as:

$$H_{ei} = \max(\ 0, r_{min} - \Delta\ (e,i)\ \ ) \tag{13}$$

"where N_e is the set of elements i for which the center-to-center distance, Δ(e,i) to element e is smaller than the filter radius r_{min}"[2]. In the case of the sensitivity filter, the actual density and the density design variables remain the same and so the filter only needs to be applied to the sensitivities each time.

In some cases, the results of the optimization depends on the mesh size of the discretization. For this project, the results were made mesh independent by keeping the ratio of the filter radius, r_{min}, to the number of elements in the x direction constant between different mesh sizes. There are better methods of achieving mesh independence, but these are beyond the scope of this paper.

2.2.3 Optimization

The optimizer used for the solution of this problem is MATLAB's function, fmincon, with the "interior-point" algorithm. It is required to turn the gradient of the objective function on in order to pass the filtered sensitivities to fmincon. If this is not done, fmincon estimates the gradient using the forward finite difference method resulting in grey material due to the use of non-filtered sensitivities. For the same reason, fmincon's derivative check is also turned off.

Several initial design variables were tested on an example minimum compliance problem. Nine initial design variables sets, x0, were generated using MATLAB's "lhsdesign" function which resulted in an average number of 105 iterations with a standard deviation of 26. Nine initial design variables sets were generated using MATLAB's "rand" function which resulted in an average of 116 iterations with a standard deviation of 73. One initial design variables set was tested with all design variables equal to the volume fraction, 0.6 which resulted in 93 iterations consistently. This indicates when an educated initial design cannot be developed it is best to set all design variables equal to the volume fraction. For this project, all the initial design variables set equal to the volume fraction will be tested. Also, ten different Latin hypercube sampling designs (lhsdesign) will be tested in order to test for different solutions and determine the possibility of a global minimum.

The only constraints in this problem are linear, meaning a specific function file is not needed for them. Instead, they are passed to fmincon in the form of (**A**, volfrac) indicating the sum of the volumes, calculated as **Ab**, should be less than or equal to the volume fraction. The simple bounds on the design variables are passed to fmincon as lower and upper limits. The complete call to fmincon is written is

$x = fmincon($`@(x)objFun(x,data), x0, A, volfrac, [], [], xmin, xmax, [], options);` (14)

References

[1] Bendsoe, M. and Sigmund, O., *Topology Optimization Theory, Methods, and Applications*, Springer-Verlag, Berlin Heidelberg, 2003.

[2] Andreassen, E., Clausen, A., Schevenels, M., Lazarov, B. S., and Sigmund, O., "Efficient topology optimization in MATLAB using 88 lines of code," *Structural and Multidisciplinary Optimization*, Vol. 43, 2010, pp. 1-16.

[3] Sigmund, O., "Morphology-based black and white filters for topology optimization," *Structural and Multidisciplinary Optimization*, Vol. 33, 2007, pp. 401-424