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HW 10

Problem 1

Descent Direction. Determine whether the given direction at the point is a that of descent for the following functions (show your work):

(a) $f(\boldsymbol{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 4$ $\quad$ $\mathbf{d} = (2, 1)$ at $\mathbf{x} = (1, 1)$ $\quad$ Exercise 10.3

(b) $f(\boldsymbol{x}) = \frac{1}{2}x_1^2 + x_2^2 - 2x_1x_2 - 7x_1 - 7x_2$ $\quad$ $\mathbf{d} = (7, 6)$ at $\mathbf{x} = (1, 1)$ $\quad$ Exercise 10.9

For part **a** find the gradient of function first, which can be seen below:

```
Function for evaluation
```

$$x_1^2 - 2x_1 + x_2^2 - 2x_2 + 4$$

```
Calculated gradient
```

$$\begin{bmatrix} 2x_1 - 2, & 2x_2 - 2 \end{bmatrix}$$

```
After applying x=(1,1)
```

$$\begin{bmatrix} 0 & 0 \end{bmatrix}$$

```
For following d=(2, 1)
```

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

```
Descent condition is:
```

$$\begin{bmatrix} 0 \end{bmatrix}$$

```
Inequalty 10.9 is not satisfied, however, KKT condition is
```

For part **b** find the gradient of function first, which can be seen below:

```
Function for evaluation
```

$$\frac{x_1^2}{2} - 2x_1x_2 - 7x_1 + x_2^2 - 7x_2$$

```
Calculated gradient
```

$$[x_1 - 2x_2 - 7, \quad -2x_1 + 2x_2 - 7]$$

```
After applying x=(1,1)
```

$$\begin{bmatrix} -8 & -7 \end{bmatrix}$$

```
For following d=(7, 6)
```

$$\begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

```
Descent condition is:
```

$$\begin{bmatrix} -98 \end{bmatrix}$$

```
Inequalty 10.9 is satisfied, this is a descent direction
```

Problem 2

Search Direction Determination. Calculate the gradient of the following functions at the given points by the forward, backward, and central difference approaches with a 1% change in the point and compare them with the exact gradient (Exercise 10.64):

(a) $f(\boldsymbol{x}) = 12.096x_1^2 + 21.504x_2^2 - 1.7321x_1 - x_2$ at $\mathbf{x} = (5, 6)$

(b) $f(\boldsymbol{x}) = 50(x_2 + x_1^2)^2 + (2-x_1)^2$ at $\mathbf{x} = (1, 2)$

(c) $f(\boldsymbol{x}) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$ at $\mathbf{x} = (1, 2, 3)$

Below are listed equations to be used for finite difference.

Forward difference:

$$f_x(x,y) = \frac{f(x+h,y)-f(x,y)}{h}$$
$$f_y(x,y) = \frac{f(x,y+h)-f(x,y)}{h}$$

Backward difference:

$$f_x(x,y) = \frac{f(x,y)-f(x-h,y)}{h}$$
$$f_y(x,y) = \frac{f(x,y)-f(x,y-h)}{h}$$

Central difference:

$$f_x(x,y) = \frac{f(x+h,y)-f(x-h,y)}{2h}$$
$$f_y(x,y) = \frac{f(x,y+h)-f(x,y-h)}{2h}$$

Step h = 0.01

All calculations are seen below

```
Problem 2a
```

$$12.096x_1^2 - 1.7321x_1 + 21.504x_2^2 - x_2$$

```
Calculated gradient
```

$$[24.192x_1 - 1.7321, \quad 43.008x_2 - 1]$$

```
After applying x=(5,6)
```

$$\begin{bmatrix} 119.2279 & 257.048 \end{bmatrix}$$

```
Forward difference
```

$$[119.348860000014, \quad 257.263040000021]$$

```
Backward difference
```

$$[119.106939999983, \quad 256.832959999974]$$

```
Central difference
```

$$[119.227899999998, \quad 257.047999999998]$$

Problem 2b

$$(-x_1 + 2)^2 + 50\left(-x_1^2 + x_2\right)^2$$

Calculated gradient

$$\left[200x_1^3 - 200x_1x_2 + 2x_1 - 4, \quad -100x_1^2 + 100x_2\right]$$

After applying x=(1,2)

$$\begin{bmatrix} -202 & 100 \end{bmatrix}$$

Forward difference

$$[-200.96995, \quad 100.499999999998]$$

Backward difference

$$[-202.99005, \quad 99.5000000000005]$$

Central difference

$$[-201.98, \quad 99.9999999999993]$$

Problem 2c

$$x_1^2 + 2x_1x_2 + 2x_2^2 + 2x_2x_3 + 2x_3^2$$

Calculated gradient

$$[2x_1 + 2x_2, \quad 2x_1 + 4x_2 + 2x_3, \quad 2x_2 + 4x_3]$$

After applying x=(1,2,3)

$$\begin{bmatrix} 6 & 16 & 16 \end{bmatrix}$$

Forward difference

$$[6.00999999999985, \quad 16.0199999999996, \quad 16.0199999999996]$$

Backward difference

$$[5.9899999999999, \quad 15.9799999999997, \quad 15.9799999999997]$$

Central difference

$$[5.99999999999987, \quad 15.9999999999997, \quad 15.9999999999997]$$

Problem 3

Steepest-Descent Method. Verify the properties of the gradient vector for the following functions at the given point (Section 11.2):

(a) $f(\boldsymbol{x}) = 6x_1^2 - 6x_1x_2 + 2x_2^2 - 5x_1 + 4x_2 + 2$	at $\mathbf{x}^{(0)} = (-1, -2)$	Exercise 11.5
(b) $f(\boldsymbol{x}) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7$	at $\mathbf{x}^{(0)} = (5, 10)$	Exercise 11.6
(c) $f(\boldsymbol{x}) = 10(x_1^2 - x_2) + x_1^2 - 2x_1 + 5$	at $\mathbf{x}^{(0)} = (-1, 3)$	Exercise 11.7

Problem 3a

$$6x_1^2 - 6x_1x_2 - 5x_1 + 2x_2^2 + 4x_2 + 2$$

Calculated gradient

$$[12x_1 - 6x_2 - 5, \quad -6x_1 + 4x_2 + 4]$$

After applying x=(-1, -2) we get following c

$$\begin{bmatrix} -5 & 2 \end{bmatrix}$$

Magnitude of gradient

$$\sqrt{29}$$

Gradient unit vector

$$\begin{bmatrix} -\frac{5\sqrt{29}}{29} & \frac{2\sqrt{29}}{29} \end{bmatrix}$$

Define tangent vector T at point x=(-1, -2)

$$-5\frac{d}{ds}\,\mathrm{x}_1(s) + 2\frac{d}{ds}\,\mathrm{x}_2(s)$$

From above expression we get following tangent vector

$$[2, \quad 5]$$

Now check property 1, where dot product dot(c,T) = 0

$$\begin{bmatrix} -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \end{bmatrix}$$

Property 1 checks out = 0

Next define some random vector d as seen below:

$$\begin{bmatrix} -0.7071067 \\ 0.7071067 \end{bmatrix}$$

Next check Property 2, which states that gradient direction represents steepest direction

Calculate new x using c

$$\begin{bmatrix} -1.09284766908853 \\ -1.96286093236459 \end{bmatrix}$$

Calculate new x using d

$$\begin{bmatrix} -1.07071067 \\ -1.92928933 \end{bmatrix}$$

New function value using gradient direction c

$$1.61368889450655$$

New function value using gradient direction d

1.56497467392589

Function value using gradient c is greater than using direction d. Therefore, property 2 checks out

Next check property 3 by confirming that dot(c,c) is greater than dot(c,d)

First calculate dot(c,c):

1.0

Next calculate dot(c,d):

0.919144924486314

Property 3 checks out since dot(c,c)>dot(c,d)

Problem 3b

$3x_1^2 + 2x_1x_2 + 2x_2^2 + 7$

Calculated gradient

$[6x_1 + 2x_2, \quad 2x_1 + 4x_2]$

After applying x=(5, 10) we get following c

$\begin{bmatrix} 50 & 50 \end{bmatrix}$

Magnitude of gradient

$50\sqrt{2}$

Gradient unit vector

$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Define tangent vector T at point x=(5, 10)

$50\frac{d}{ds}\,\mathrm{x}_1(s) + 50\frac{d}{ds}\,\mathrm{x}_2(s)$

From above expression we get following tangent vector

$[-50, \quad 50]$

Now check property 1, where dot product dot(c,T) = 0

$\begin{bmatrix} 50 & 50 \end{bmatrix}$

$\begin{bmatrix} -50 & 50 \end{bmatrix}$

Property 1 checks out = 0

Next define some random vector d as seen below:

$\begin{bmatrix} 0.5 \\ 0.8660254 \end{bmatrix}$

Next check Property 2, which states that gradient direction represents steepest direction

Calculate new x using c

$\begin{bmatrix} 5.07071067811865 \\ 10.0707106781187 \end{bmatrix}$

Calculate new x using d

$\begin{bmatrix} 5.05 \\ 10.08660254 \end{bmatrix}$

New function value using gradient direction c

389.106067811866

New function value using gradient direction d

388.861287253869

Function value using gradient c is greater than using direction d. Therefore, property 2 checks out

Next check property 3 by confirming that dot(c,c) is greater than dot(c,d)

First calculate dot(c,c):

1.0

Next calculate dot(c,d):

0.965925823613066

Property 3 checks out since dot(c,c)>dot(c,d)

Problem 3c

$$11x_1^2 - 2x_1 - 10x_2 + 5$$

Calculated gradient

$$\left[22x_1 - 2, \quad -10\right]$$

After applying x=(-1, 3) we get following c

$$\begin{bmatrix} -24 & -10 \end{bmatrix}$$

Magnitude of gradient

$$26$$

Gradient unit vector

$$\begin{bmatrix} -\frac{12}{13} & -\frac{5}{13} \end{bmatrix}$$

Define tangent vector T at point x=(-1, 3)

$$-24\frac{d}{ds}\,\mathrm{x}_1(s) - 10\frac{d}{ds}\,\mathrm{x}_2(s)$$

From above expression we get following tangent vector

$$\left[-10, \quad 24\right]$$

Now check property 1, where dot product dot(c,T) = 0

$$\begin{bmatrix} -24 & -10 \end{bmatrix}$$

$$\begin{bmatrix} -10 & 24 \end{bmatrix}$$

Property 1 checks out = 0

Next define some random vector d as seen below:

$$\begin{bmatrix} -0.5 \\ -0.8660254 \end{bmatrix}$$

Next check Property 2, which states that gradient direction represents steepest direction

Calculate new x using c

$$\begin{bmatrix} -1.09230769230769 \\ 2.96153846153846 \end{bmatrix}$$

Calculate new x using d

$$\begin{bmatrix} -1.05 \\ 2.91339746 \end{bmatrix}$$

New function value using gradient direction c

$$-9.30627218934911$$

New function value using gradient direction d

-9.9064746

Function value using gradient c is greater than using direction d. Therefore, property 2 checks out

Next check property 3 by confirming that dot(c,c) is greater than dot(c,d)

First calculate dot(c,c):

1

Next calculate dot(c,d):

0.794625153846154

Property 3 checks out since dot(c,c)>dot(c,d)

Problem 4

Linearization of the Constrained Problem. Formulate the following design problem, transcribe into standard form, create a linear approximation at the given point, and plot the linearized sub-problem and the original problem on the same graph. Exercise 12.7, based on Exercise 2.3, given $(R, H) = (6, 15)$ cm.

Reference HW #3 solution: Design a beer mug, Figure E2.3, to hold as much beer as possible. The height and radius of the mug should be no more than 20 cm. The mug must be at least 5 cm in radius. The surface area of the sides must be no greater than 900 cm^2 (ignore the bottom area of the mug and mug handle). Formulate the optimum design problem.

Standard form for above problem:

find $x = [h, r]$

to max: $V = \pi r^2 h$

Subject to:

$g_1 : 2\pi rh + \pi r^2 - 900 \leq 0$

$5 \leq r \leq 20$
$0 \leq h \leq 20$

First scale costraint g1

$$\frac{\pi h}{450} r + \frac{\pi r^2}{900} - 1$$

Next check if g1 constraint is violated at x=(6, 15)

$$-0.246$$

per above results design point satisfies the constraint

next calculate gradient for function and g1 constraint

Calculated gradient

$$\left[2\pi hr, \quad \pi r^2\right]$$

After applying x=(6, 15) we get following gradient value

$$\left[565.486677646163 \quad 113.097335529233\right]$$

-Next calculate gradient for g1 constraint-

Calculated gradient for g1 constraint

$$\left[\frac{\pi h}{450} + \frac{\pi r}{450}, \quad \frac{\pi r}{450}\right]$$

Evaluate g1 constraint at x=(6, 15)

$$\left[0.146607657167524 \quad 0.0418879020478639\right]$$

Now form linearized sub-problem

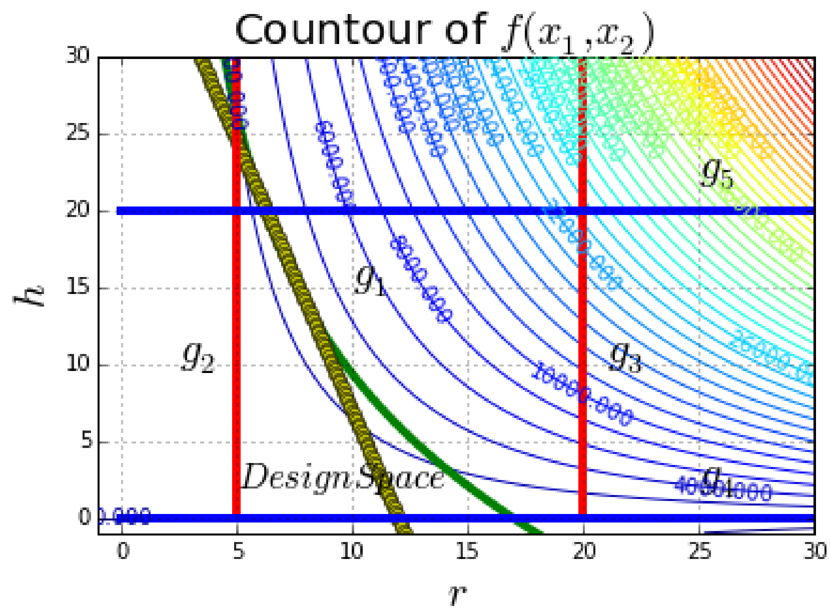
Maximize following function

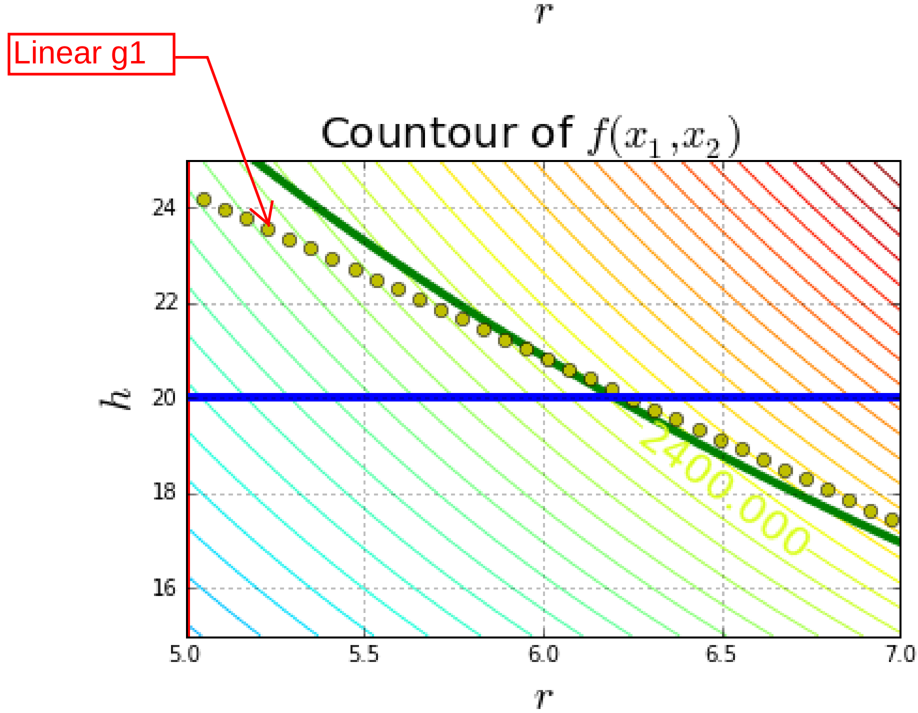
$$113.1h + 565.49r - 3392.98140625$$

Subject to:

$$0.042h + 0.147r - 1.757962890625$$

Next plot original problem along with linearized constraint $\bar{g}_1$





Linearized constraint $\bar{g}_1$ is very close to the original non-linear constraint g_1. This is most likely due to the location at which the linearization of g_1 was performed (i.e. $\bar{x} = (6, 15)$).

Optimized radius is $\approx$ 6.2
Optimized height is $\approx$ 20

Linearized cost function value is $\approx$ 2375.06
This linearized approximation is very close to original guess of 2400.00