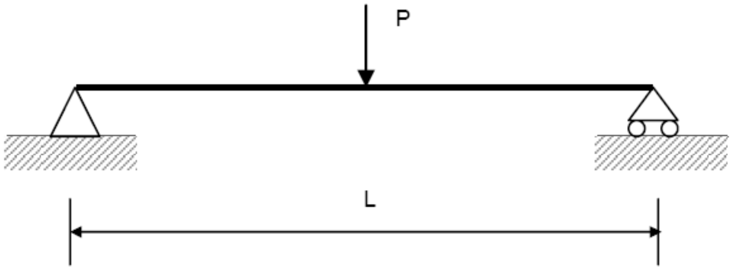
Fundamentals of Reliability Analysis

Reliability analysis evaluates the statistics of system behavior including the probability of structural failure by determining whether the limit state functions are exceeded.

Generally, the Limit State Function (LSF) indicates the margin of safety between the resistance and the load of structures



$g(X)=R(X)-S(X)<0$ (Failure)
$P_f=P[g(x)<0]$

$g(x)=0$ denotes the LS Boundary (LSB)

The *safety index* or *reliability index*, β, is defined as

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2 - 2\rho_{RS}\sigma_R\sigma_S}}$$

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$

Fundamentals of Reliability Analysis cont'd

Failure event: $g(X)=R(X)-S(X)<0$

With independent normal R & S,
LSF (g) is also normally distributed

$$f_g(g) = \frac{1}{\sigma_g\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{g-\mu_g}{\sigma_g}\right)^2\right]$$

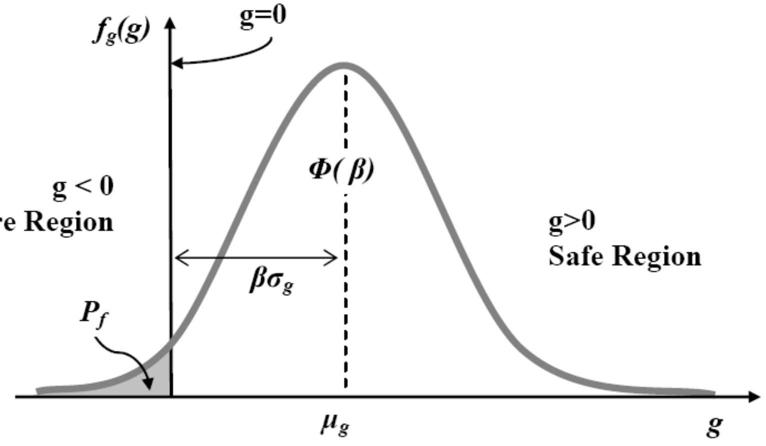


Figure 3.1. Probability Density for Limit-state $g(.)$

The *probability of failure* is

$$P_f = \int_{-\infty}^{0} f_g(g)\,dg$$

$$P_f = \int_{-\infty}^{0}\frac{1}{\sigma_g\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{0-\mu_g}{\sigma_g}\right)^2\right]dg$$

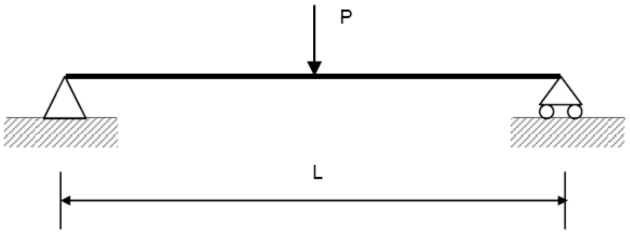
$$= 1-\Phi(\beta) = \Phi(-\beta)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

This example is taken from [17]. The figure below shows a simply-supported beam loaded at the midpoint by a concentrated force P. The length of the beam is L, and the bending moment capacity at any point along the beam is WT, where W is the plastic section modulus and T is the yield stress. All four random variables P, L, W, and T are assumed to be independent normal distributions. The mean values of P, L, W, and T are 10 kN, 8 m, 100×10^{-6} m^3, and 600×10^3 kN/m^2 respectively. The standard deviations of P, L, W, and T are 2 kN, 0.1 m, 2×10^{-5} m^3, and 10^5 kN/m^2, respectively. The limit-state function is given as

$$g(\{P,L,W,T\}) = WT - \frac{PL}{4}$$

Solve for the safety index, β, and the probability of failure, P_f, for this problem.



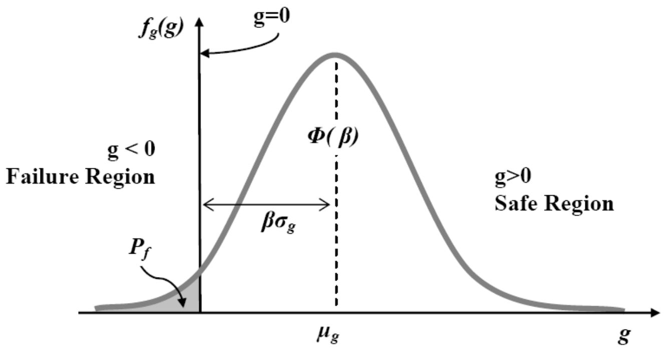


Figure 3.1. Probability Density for Limit-state $g(.)$

First and Second Order Reliability Method

Safety Index: $\beta = \dfrac{\mu_g}{\sigma_g}$ $g(X)=R(X)-S(X)$

To simplify the numerical treatment of the integration process, the Tylor series expansion is often used to linearize the Limit-State Function (LSF) g(X)=0.

First Order Second Moment (FOSM, also known as MVFOSM method)
Second Order Second Moment (SOSM)

The safety index approach to reliability analysis is actually a mathematical optimization problem for finding the point on the Limit State Boundary (LSB) that has the shortest distance from the origin in the standard normal space.

This point is called the Most Probable failure Point (MPP).

First and Second Order Reliability Method cont'd

Hasofer and Lind (HL) transformation btw the U and X spaces

$$P_f = \Phi(-\beta)$$

Taylor series expansion

If f is a function continuous and n times differentiable, the function can be expanded from a given point (a)

$$\sum_{n=1}^{\infty} \frac{f^{(n)}}{n!}(x-a)^n$$

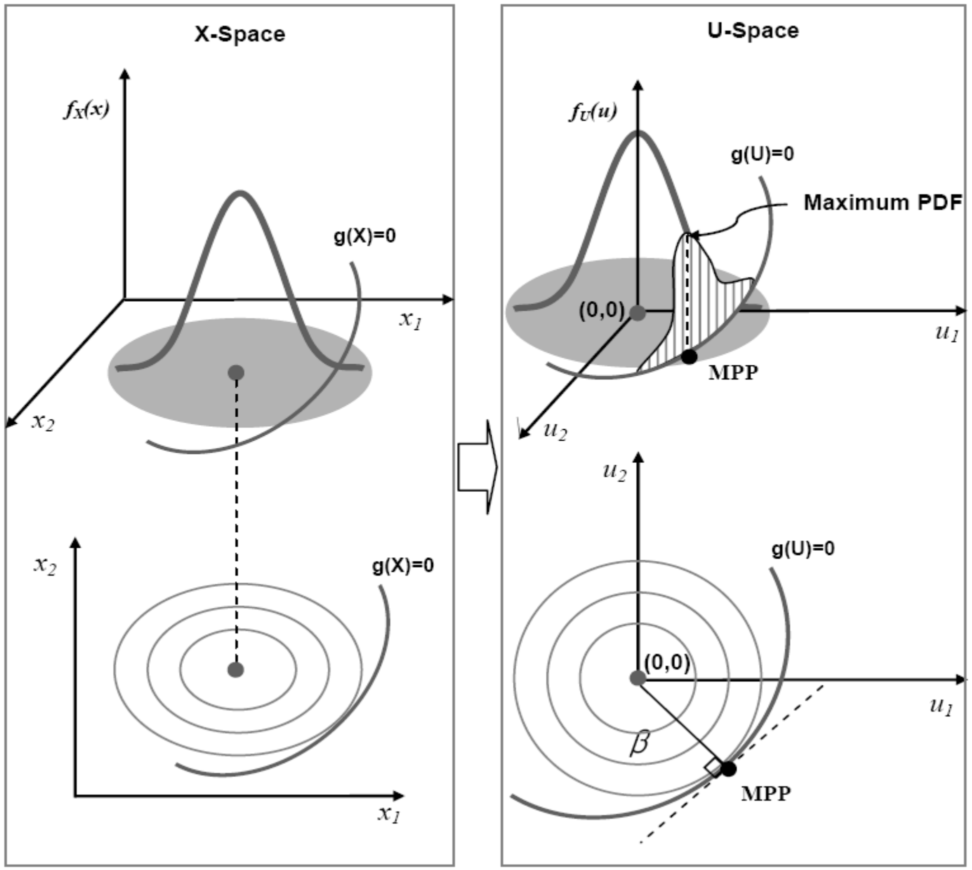


Figure 3.2. Transformation and MPP

Mean Value First Order Second Moment (MVFOSM)

- Linear approximation at the mean point

$$\tilde{g}(X) \approx g(\mu_X) + \nabla g(\mu_X)^T (X_i - \mu_{X_i})$$

$$\nabla g(\mu_X) = \left\{ \frac{\partial g(\mu_X)}{\partial x_1}, \frac{\partial g(\mu_X)}{\partial x_2}, \ldots, \frac{\partial g(\mu_X)}{\partial x_n} \right\}^T$$

MVFORM Reliability Index: $\beta = \dfrac{\mu_g}{\sigma_g}$

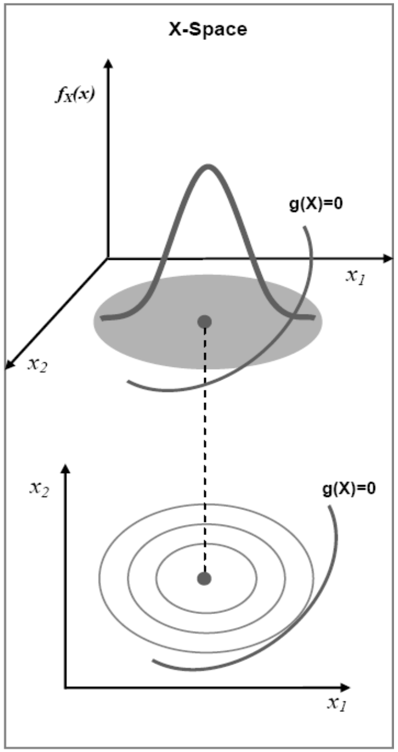


Figure 3.2. Tra

MVFOSM cont'd

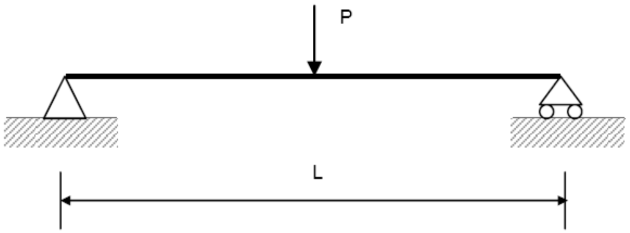
The performance function is

$$g(x_1, x_2) = x_1^3 + x_2^3 - 18$$

in which x_1 and x_2 are the random variables with normal distributions (mean $\mu_{x_1} = \mu_{x_2} = 10$, standard deviation $\sigma_{x_1} = \sigma_{x_2} = 5$). Find the safety-index β by using the mean-value FOSM method, and check the accuracy of the obtained result with the MCS.

$$\beta = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} = \frac{1982.0}{2121.32} = 0.9343$$

This example is taken from [17]. The figure below shows a simply-supported beam loaded at the midpoint by a concentrated force P. The length of the beam is L, and the bending moment capacity at any point along the beam is WT, where W is the plastic section modulus and T is the yield stress. All four random variables P, L, W, and T are assumed to be independent normal distributions. The mean values of P, L, W, and T are 10 kN, 8 m, 100×10^{-6} m^3, and 600×10^{3} kN/m^2 respectively. The standard deviations of P, L, W, and T are 2 kN, 0.1 m, 2×10^{-5} m^3, and 10^5 kN/m^2, respectively. The limit-state function is given as



$$g(\{P,L,W,T\}) = WT - \frac{PL}{4}$$

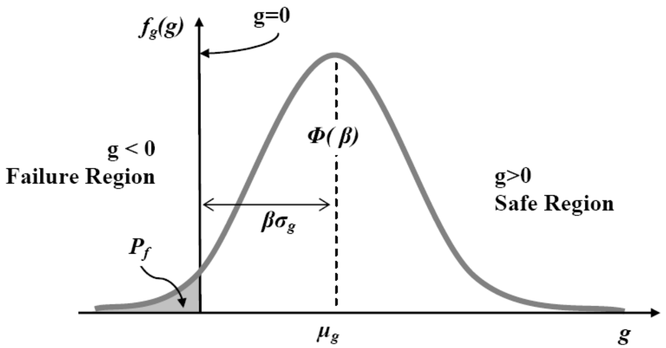


Figure 3.1. Probability Density for Limit-state $g(.)$

Find MVFORM Reliability Index

$$g_1(P,L,W,T) = WT - \frac{PL}{4}$$

$$g_2(P,L,W,T) = T - \frac{PL}{4W}$$

Ans.

β_1=2.48

β_2=3.48

A W16 × 31 steel section made of A36 steel is suggested to carry an applied deterministic bending moment of 1,140 kip-in. The nominal yield stress F_y of the steel is 36 ksi, and the nominal plastic modulus of the section Z is 54 in.^{3}. Consider that the distributions of these random variables are unknown; only the means, standard deviations, and COVs are known:

$$\mu_{F_y} = 38 \text{ ksi}, \ \sigma_{F_y} = 3.8 \text{ ksi}, \text{ and } \delta_{F_y} = 0.1$$

$$\mu_Z = 54 \text{ in.}^3, \ \sigma_Z = 2.7 \text{ in.}^3, \text{ and } \delta_Z = 0.05.$$

It is quite logical to assume that F_y and Z are statistically independent.

Hasofer and Lind (HL) Safety Index

- Linear approximation at the MPP

1. Find independent standard normal random variables

$$\hat{R} = \frac{R - \mu_R}{\sigma_R}, \quad \hat{S} = \frac{S - \mu_s}{\sigma_s}$$

2. Transform LSF into LSF in the std. normal (U) space

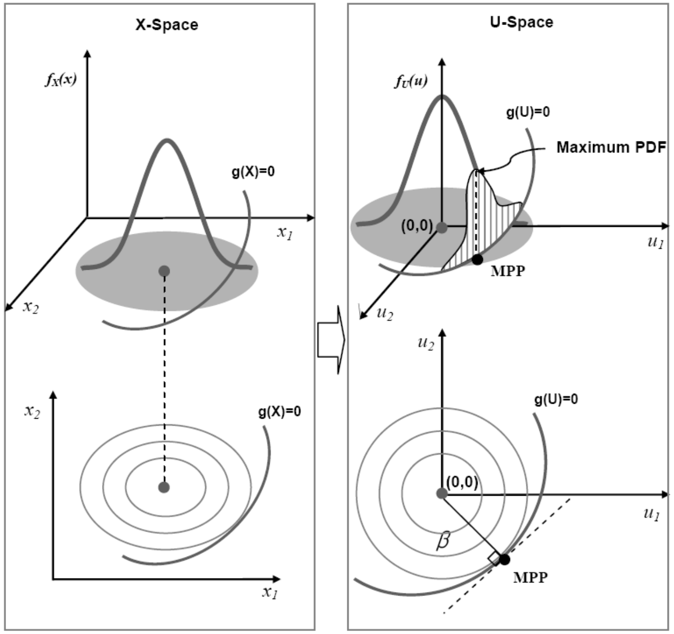


Figure 3.2. Transformation and MPP

$$g(R(\hat{R}), S(\hat{S})) = \hat{g}(\hat{R}, \hat{S}) = \hat{R}\sigma_R - \hat{S}\sigma_S + (\mu_R - \mu_S) = 0$$

3. Find the minimum distance, b, from the U origin

$$\text{Minimize:} \quad \beta(U) = (U^T U)^{\frac{1}{2}}$$

$$\text{Subject to:} \quad g(U) = 0$$

Hasofer and Lind (HL) Safety Index cont'd

1) Define the appropriate limit-state function of Equation 4.21

2) Set the mean value point as an initial design point, *i.e*, $x_{i,k} = \mu_{x_i}$ i=1,2,…,n, and compute the gradients $\nabla g(X_k)$ of the limit-state function at this point. Here, $x_{i,k}$ is the i^{th} element in the vector X_k of the k^{th} iteration

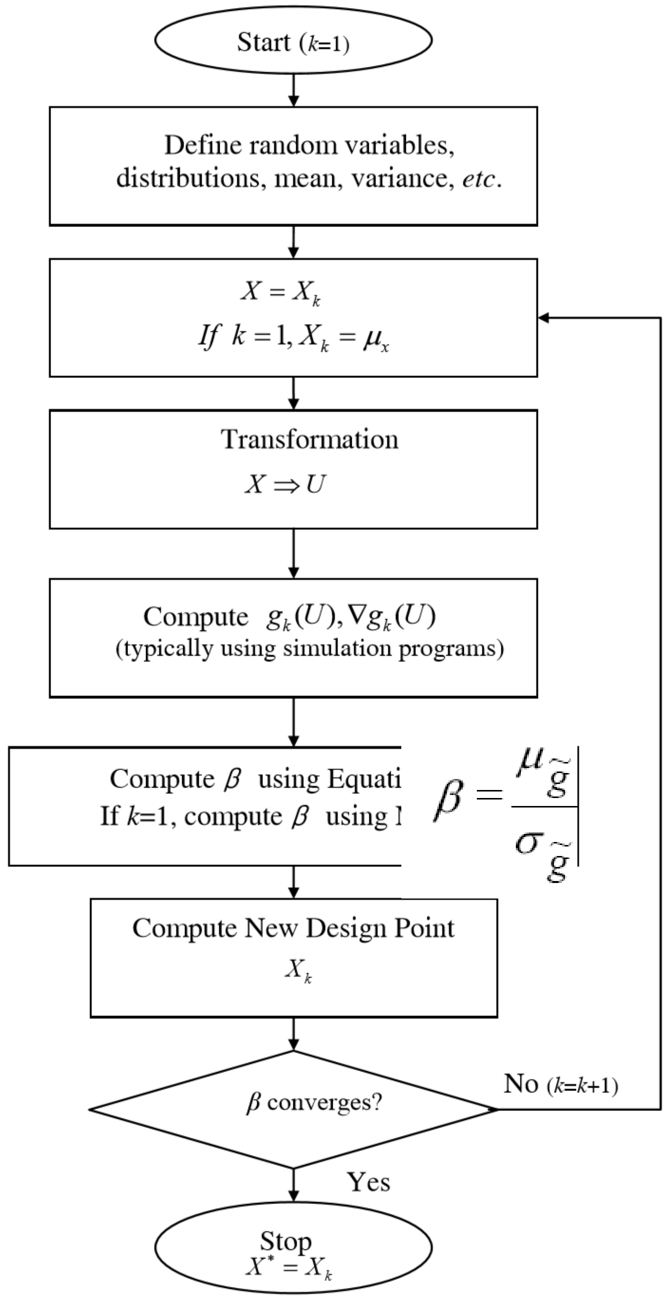
3) Compute the initial β using the mean-value method (Cornell safety-index), *i.e.*, $\beta = \mu_{\tilde{g}} / \sigma_{\tilde{g}}$ and its direction cosine

4) Compute a new design point X_k and U_k (Equations 4.27 and 4.28), function value, and gradients at this new design point

5) Compute the safety-index β using Equation 4.25 and the direction cosine or sensitivity factor from Equation 4.26

6) Repeate steps 4)~6) until the estimate of β converges

7) Compute the coordinates of the design point X_k or most probable failure point (MPP), X^*


Start (k=1)
Define random variables, distributions, mean, variance, *etc*.
$X = X_k$
$If\ \ k = 1, X_k = \mu_x$
Transformation
$X \Rightarrow U$
Compute $g_k(U), \nabla g_k(U)$
(typically using simulation programs)
Compute β using Equati
If k=1, compute β using I
$\beta = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}}$
Compute New Design Point
X_k
β converges?
No (k=k+1)
Yes
Stop
$X^* = X_k$

Hasofer and Lind (HL) Safety Index cont'd

The performance function is

$$g(x_1, x_2) = x_1^3 + x_2^3 - 18$$

in which x_1 and x_2 are the random variables with normal distributions (mean $\mu_{x_1} = \mu_{x_2} = 10$, standard deviation $\sigma_{x_1} = \sigma_{x_2} = 5$). Find the safety-index β by using the mean-value FOSM method, and check the accuracy of the obtained result with the MCS.

$$\beta = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} = \frac{1982.0}{2121.32} = 0.9343$$

Table 4.1. Iteration Results in the HL Method (Example 4.3a)

Iteration No.	1	2	3	4	5	6	7
$g(X_k)$	1982.0	582.63	168.08	45.529	10.01	1.1451	0.023
$\frac{\partial g}{\partial x_1}\|_{X_k}$	300	134.5374	61.598	30.0897	17.43	13.5252	12.9917
$\frac{\partial g}{\partial x_2}\|_{X_k}$	300	134.5374	61.598	30.0897	17.43	13.5252	12.9917
β	0.9343	1.5468	1.9327	2.1467	2.2279	2.2398	2.2401
α_1	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071	-0.771
α_2	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071
$x_{1,k}$	6.6967	4.5313	3.1670	2.4104	2.1233	2.0810	2.0801
$x_{2,k}$	6.6967	4.5313	3.1670	2.4104	2.1233	2.0810	2.0801
$u_{1,k}$	-0.6607	-1.0937	-1.3666	-1.5179	-1.5753	-1.5838	-1.5840
$u_{2,k}$	-0.6607	-1.0937	-1.3666	-1.5179	-1.5753	-1.5838	-1.5840
ε	-	0.6556	0.2495	0.1107	0.036	0.005	0.0001

Hasofer and Lind (HL) Safety Index cont'd

Table 4.2. Iteration Results in the HL Method (Example 4.3b)

Iteration No.	1	2	….	21	22	23
$g(X_k)$	1952.299	573.8398	….	678.9088	676.7346	677.655
$\frac{\partial g}{\partial x_1}\big\vert_{X_k}$	300	133.8982	….	218.0401	56.9049	217.6582
$\frac{\partial g}{\partial x_2}\big\vert_{X_k}$	294.03	132.5409	….	54.4352	216.2786	54.61056
β	0.9295	1.5387	….	1.1636	1.1650	1.1657
α_1	-0.7142	-0.7107	….	-0.9702	-0.2544	-0.9699
α_2	-0.7000	-0.7035	….	-0.2422	-0.9671	-0.2434
$x_{1,k}$	6.6808	4.5323	….	4.3553	8.5178	4.3468
$x_{2,k}$	6.6468	4.4877	….	8.4908	4.2666	8.4816
$u_{1,k}$	-0.6638	-1.0935	….	-1.1289	-0.2964	-1.1306
$u_{2,k}$	-0.6506	-1.0825	….	-0.2818	-1.1267	-0.2837
ε	-	0.6554	….	0.002	0.0012	0.0006

FORM: HL-RF method

1. Equivalent Normal Distribution Transformation

$$u_i = \Phi^{-1}[F_{x_i}(x_i)]$$

Taylor series expansion (First order)

$$u_i = \Phi^{-1}[F_{x_i}(x_i^*)] + \frac{\partial}{\partial x_i}([\Phi^{-1}F_{x_i}(x_i)])|_{x_i^*}\,(x_i - x_i^*)$$

$$u_i = \frac{x_i - [x_i^* - \Phi^{-1}[F_{x_i}(x_i^*)]\phi(\Phi^{-1}[F_{x_i}(x_i^*)])/f_{x_i}(x_i^*)]}{\phi(\Phi^{-1}[F_{x_i}(x_i^*)])/f_{x_i}(x_i^*)}$$

$$u_i = \frac{x_i - \mu_{x_i'}}{\sigma_{x_i'}}$$

where

$$\sigma_{x_i'} = \frac{\phi(\Phi^{-1}[F_{x_i}(x_i^*)])}{f_{x_i}(x_i^*)}$$

$$\mu_{x_i'} = x_i^* - \Phi^{-1}[F_{x_i}(x_i^*)]\sigma_{x_i'}$$

2. Rosenblatt Method

$$F_{x_i}(x_i^*) = F_{x_i'}(x_i^*)$$

$$\text{or } F_{x_i}(x_i^*) = \Phi\left(\frac{x_i^* - \mu_{x_i'}}{\sigma_{x_i'}}\right)$$

so

$$\mu_{x_i'} = x_i^* - \Phi^{-1}[F_{x_i}(x_i^*)]\sigma_{x_i'}$$

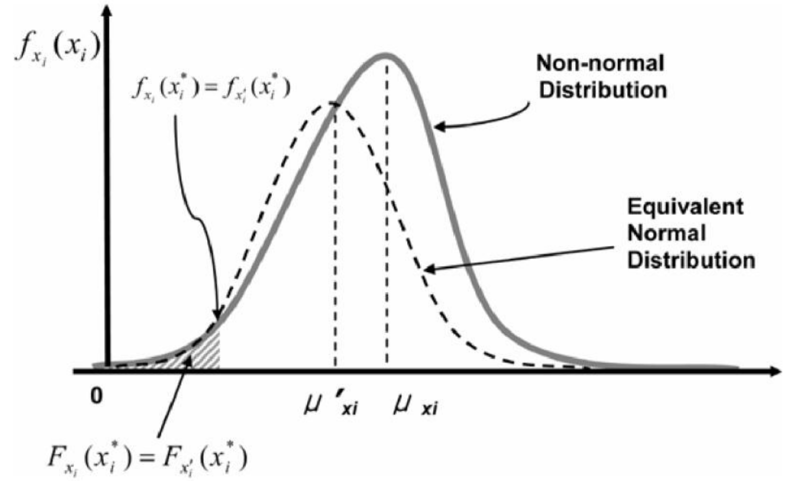


Figure 4.4. Normal Tail Approximation

The probability density function value of x and x_i' at x_i^* are equal:

$$f_{x_i}(x_i^*) = f_{x_i'}(x_i^*)$$

$$f_{x_i}(x_i^*) = \frac{1}{\sigma_{x_i'}}\phi\left(\frac{x_i^* - \mu_{x_i'}}{\sigma_{x_i'}}\right)$$

$$\sigma_{x_i'} = \frac{\phi(\Phi^{-1}[F_{x_i}(x_i^*)])}{f_{x_i}(x_i^*)}$$

$$\mu_{x_i'} = x_i^* - \Phi^{-1}[F_{x_i}(x_i^*)]\sigma_{x_i'}$$