Admir Makas

HW #7

Problem 4.51

Find points satisfying the neccessary conditions for the following problem, check they are optimum using graphical method.

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$

subject to $x_1 + x_2 = 4$

$L(x_1, x_2, v) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 + v(x_1 + x_2 - 4)$

Taking derivatives gives:

$\frac{dL}{dx_1} = 8x_1 - 5x_2 + v = 0$

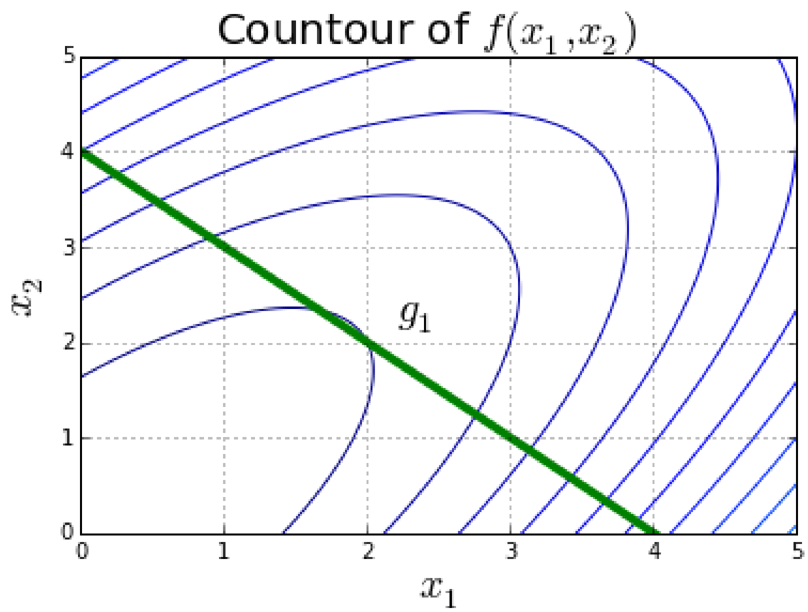
$\frac{dL}{dx_2} = 6x_2 - 5x_1 + v = 0$

$\frac{dL}{dv} = x_1 + x_2 - 4 = 0$

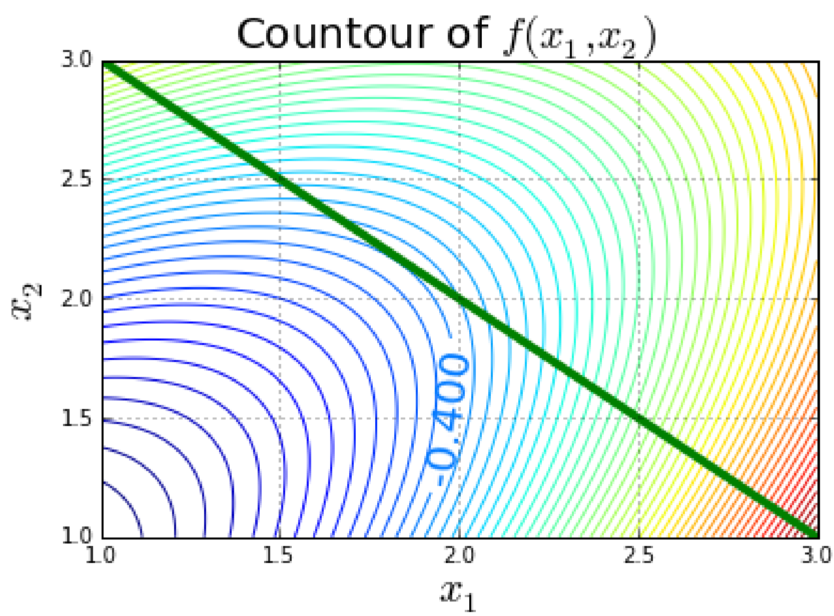
Next the above functions are solved to find points that satisfy the neccessary conditons.

```
x_1 = 1.833
x_2 = 2.167
v = -3.833
F(x_1, x_2) = -0.333
```

Next the function is plotted:



CLOSE UP ON THE OPTIMUM POINT



Using the graphical method the contour line that overlaps with the equality constrait has a value of -0.400. This is close to the calculated value of -0.333.

The equality contraint is active at the optimum point.

Problem 4.65

Find points satisfying the KKT conditions for the following problem, check they are optimum using graphical method.

$$\text{Minimize } f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$$
$$\text{subject to } x_1 + x_2 - 4 \leq 0$$

$L(x_1, x_2, u, s) = (x_1 - 1)^2 + (x_2 - 1)^2 + u(x_1 + x_2 - 4 + s_1^2)$

Taking derivatives gives:

$\frac{dL}{dx_1} = 2(x_1 - 1) + u = 0$

$\frac{dL}{dx_2} = 2(x_2 - 1) + u = 0$

$\frac{dL}{du} = x_1 + x_2 - 4 + s^2 = 0$

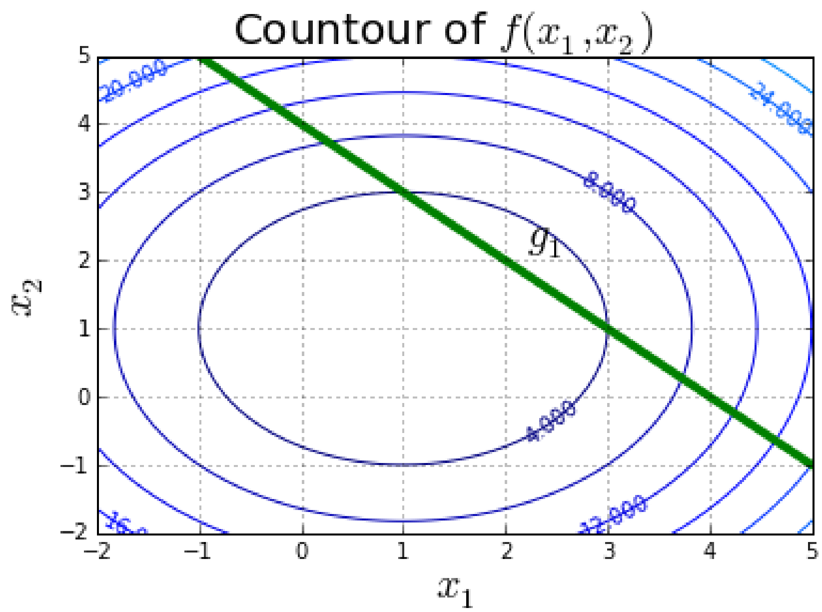
$\frac{dL}{ds} = 2su = 0$

Next the above functions are solved to find points that satisfy the neccessary conditons.

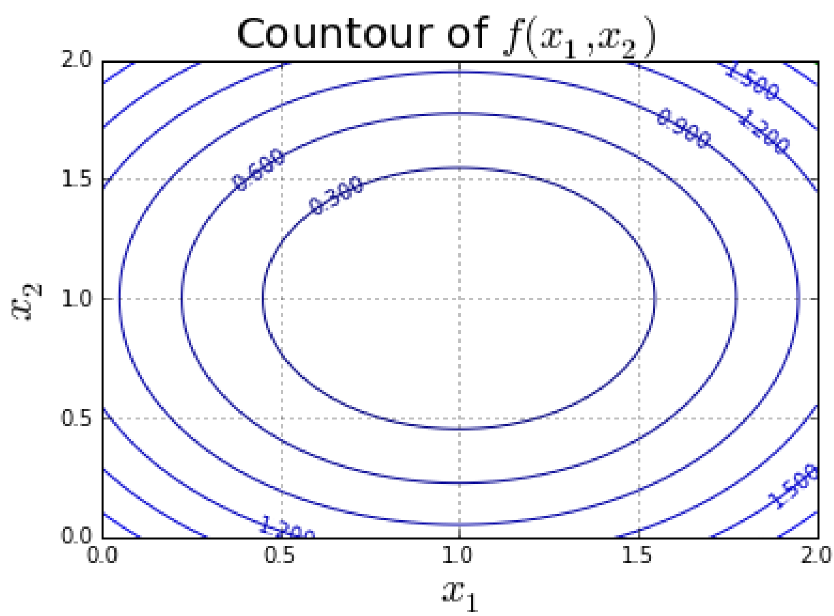
```
x_1 = 1.0
x_2 = 1.0
u = 0.0
s = 1.414
F(x_1, x_2) = 2.72294806678e-22
```

Based on KKT criterion $u = 0$, which statisfies requirement $u \geq 0$ and $s = 1.414$, which satisfies requirement $s \geq 0$.

Therefore x^* satisfies KKT conditions.



CLOSE UP ON THE OPTIMUM POINT



The graphical solutions seems to agree with the KKT analaysis. The optimum point lies at $x^* = (1, 1)$.

Problem 4.83

Solve following problem graphically. Verify the KKT conditions at the solution point and show gradients of the cost function and active constraints on the graph.

A 100×100 m lot is available to construct a multistory office building. At least 20,000 m^2 total floor space is needed. According to a zoning ordinance, the maximum height of the building can be only 21 m, and the area for parking outside the building must be at least 25 percent of the total floor area. It has been decided to fix the height of each story at 3.5 m. The cost of the building in millions of dollars is estimated at $0.6h + 0.001A$, where A is the cross-sectional area of the building per floor and h is the height of the building. Formulate the minimum cost design problem.

Standard form is:

- Find $\bar{x} = [h, A]$
- to minimize $f(\bar{x}) = 0.6h + 0.001A$

subject to:

- $g_1: 20000 - \frac{hA}{3.5} \leq 0$
- $g_2: \frac{0.25hA}{3.5} - 10000 + A \leq 0$

and simple constraints:

- $3.5 \leq h \leq 21$, meters
- $0 \leq A \leq 10,000\ m^2$

The Lagrangian is formed below:

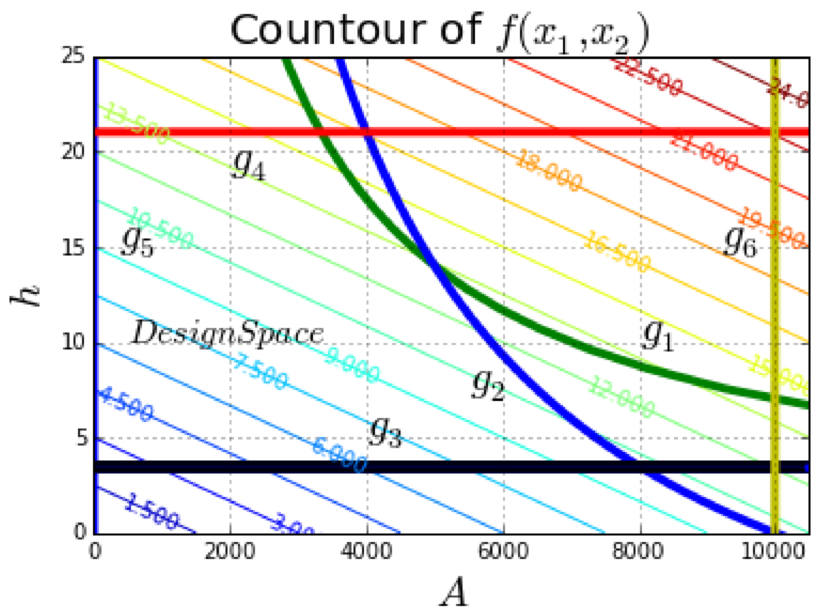
$$L = 0.6h + 0.001A + u_1(20000 - \frac{hA}{3.5} + s_1^2) + u_2(\frac{0.25hA}{3.5} - 10000 + A + s_2^2)$$
$$+ u_3(h - 3.5 + s_3^2) + u_4(h - 21 + s_4^2) + u_5(A - 10000 + s_5^2)$$

Taking the derivatives gives:

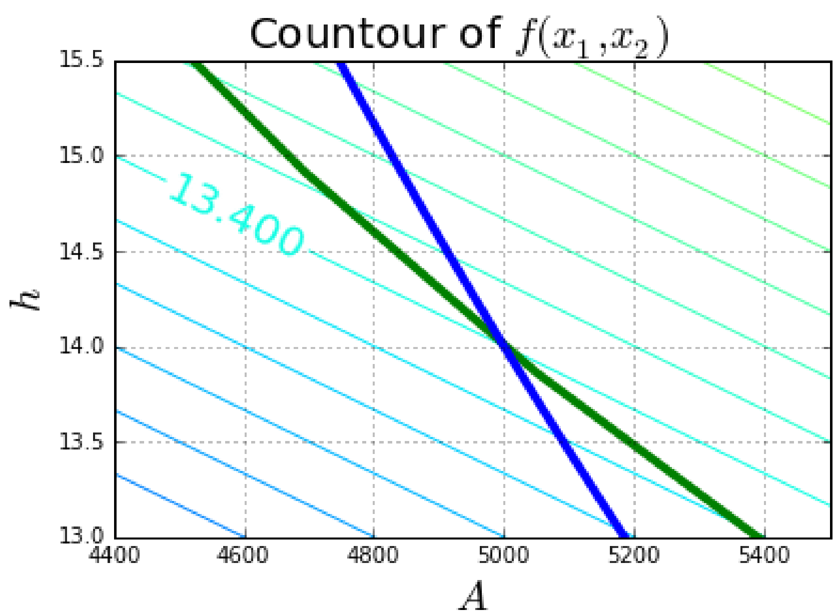
$$\frac{dL}{dh} = -0.2857Au_1 + 0.0714Au_2 + u_3 + u_4 + 0.6$$
$$\frac{dL}{dA} = -0.2857hu_1 + u_2(0.0714h + 1) + u_5 + 0.001$$
$$\frac{dL}{du_1} = -0.2857Ah + s_1^2 + 20000$$
$$\frac{dL}{du_2} = 0.0714Ah + A + s_2^2 - 10000$$
$$\frac{dL}{du_3} = h + s_3^2 - 3.5$$
$$\frac{dL}{du_4} = h + s_4^2 - 21$$
$$\frac{dL}{du_5} = A + s_5^2 - 10000$$
$$\frac{dL}{ds_1} = 2s_1u_1$$
$$\frac{dL}{ds_2} = 2s_2u_2$$
$$\frac{dL}{ds_3} = 2s_3u_3$$
$$\frac{dL}{ds_4} = 2s_4u_4$$
$$\frac{dL}{ds_5} = 2s_5u_5$$

The solution to the Lagrangian can be seen below:

```
h = 14.003
A = 4999.026
F(x_1, x_2) = 13.401
```



CLOSE UP ON THE OPTIMUM POINT



The graphical analysis confirms the KKT approach by confirming that optimum point is indeed 13.40.

Problem 4.142

Check for the convexivity of the following problem.

A 100×100 m lot is available to construct a multistory office building. At least $20{,}000\,m^2$ total floor space is needed. According to a zoning ordinance, the maximum height of the building can be only 21 m, and the area for parking outside the building must be at least 25 percent of the total floor area. It has been decided to fix the height of each story at 3.5 m. The cost of the building in millions of dollars is estimated at $0.6h + 0.001A$, where A is the cross-sectional area of the building per floor and h is the height of the building. Formulate the minimum cost design problem.

Standard form is:

- Find $\bar{x} = [h, A]$
- to minimize $f(\bar{x}) = 0.6h + 0.001A$

subject to:

- $g_1 : 20000 - \frac{hA}{3.5} \leq 0$
- $g_2 : \frac{0.25hA}{3.5} - 10000 + A \leq 0$

and simple constraints:

- $3.5 \leq h \leq 21$, meters
- $0 \leq A \leq 10,000\ m^2$

Inequalities g_1 and g_2 are non-linear. For these two inequalities we have following Hessian matrices and eigenvalues:

- For g_1:

  H_1 = $\begin{bmatrix} 0 & -0.2857 \\ -0.2857 & 0 \end{bmatrix}$

  Eigenvalues = [$\frac{-2}{7}$, $\frac{2}{7}$]

- For g_2:

  H_1 = $\begin{bmatrix} 0 & 0.0714 \\ 0.0714 & 0 \end{bmatrix}$

  Eigenvalues = [$\frac{-1}{14}$, $\frac{1}{14}$]

For both the inequality constraints the respective Hessians have negative eigenvalues. Therefore, the Hessians are not semi-positive definite or positive definite. As a result the problem is not convex.

Problem 4.150

Check for the convexity of the following problem.

The problem of minimum weight design of the symmetric three-bar truss of Figure 2.6 is formulated as follows:

Minimize $f(x_1, x_2) = 2x_1 + x_2$

subject to the constraints

$$g_1 = \frac{1}{\sqrt{2}}\left[\frac{P_u}{x_1} + \frac{P_v}{(x_1 + \sqrt{2}x_2)}\right] - 20{,}000 \le 0$$

$$g_2 = \frac{\sqrt{2}P_v}{(x_1 + \sqrt{2}x_2)} - 20{,}000 \le 0$$

$$g_3 = -x_1 \le 0$$

$$g_4 = -x_2 \le 0$$

where x_1 is the cross-sectional area of members 1 and 3 (symmetric structure) and x_2 is the cross-sectional area of member 2, $P_u = P\cos\theta$, $P_v = P\sin\theta$, with $P > 0$ and $0 \le \theta \le 90$. Check for convexity of the problem for $\theta = 60°$.

The Hessian for the cost function is.

H = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The eigenvalues for the above matrix is $\lambda_{1,2} = 0$. This implies a positive semi-definite matrix.

The Hessian for inequality costraint $g_1(\bar{x})$ takes the following form.

H = $\begin{bmatrix} \frac{1.22}{(x_1+1.41x_2)^3} + \frac{0.707}{x_1^3} & \frac{1.73}{(x_1+1.41x_2)^3} \\ \frac{1.73}{(x_1+1.41x_2)^3} & \frac{2.45}{(x_1+1.41x_2)^3} \end{bmatrix}$

First and second principal minors for the above matrix take following form:

$$D_1 = \frac{1.22}{(x_1+1.41x_2)^3} + \frac{0.707}{x_1^3}$$

$$D_2 = [(\frac{1.22}{(x_1+1.41x_2)^3} + \frac{0.707}{x_1^3}) * (\frac{2.45}{(x_1+1.41x_2)^3})] - \frac{1.73}{(x_1+1.41x_2)^3}$$

Since side constraints require x_1 and x_2 to be $\geq$ 0, Both principal minors will be $\geq$ 0. As a result contraint $g_1(\bar{x})$ is convex.

The Hessian for inequality costraint $g_2(\bar{x})$ takes the following form.

H = $\begin{bmatrix} \frac{2.45}{(x_1+1.41x_2)^3} & \frac{3.46}{(x_1+1.41x_2)^3} \\ \frac{3.46}{(x_1+1.41x_2)^3} & \frac{4.9}{(x_1+1.41x_2)^3} \end{bmatrix}$

First and second principal minors for the above matrix take following form:

$$D_1 = \frac{2.45}{(x_1+1.41x_2)^3}$$

$$D_2 = [(\frac{2.45}{(x_1+1.41x_2)^3}) * (\frac{4.9}{(x_1+1.41x_2)^3})] - \frac{3.46}{(x_1+1.41x_2)^3}$$

Since side constraints require x_1 and x_2 to be $\geq$ 0, Both principal minors will be $\geq$ 0. As a result contraint $g_2(\bar{x})$ is convex.

In closing since cost function and constraints are convex the optimization problem is convex.