Randomness in function response

Single random variables $Y = g(X)$ w/ $f_X(x)$ or $F_X(x)$

Find $F_Y(y) / f_Y(y)$

$$\begin{cases} F_Y(y) = P(Y \le y) = P(g(x) \le y) = \int_{g(x) \le y} f_X(x)\,dx \\ f_Y(y) = \frac{d}{dy}\left[F_Y(y) \right] \end{cases}$$

General Relationship $X = g^{-1}(y) = h(y)$

$$\begin{cases} F_Y(y) = \int_{-\infty}^{y} f_X(h(y))\,|h'(y)|\,dy \\ f_Y(y) = f_X(h(y))\,|h'(y)| \end{cases}$$

Two random variables $Y = g(X_1, X_2)$ w/ $f_{X_1,X_2}(x_1, x_2)$

$$F_Y(y) = \iint_{g(x_1, x_2) \le y} f_{X_1,X_2}(x_1, x_2)\,dx_1\,dx_2$$

If we can obtain $x_1 = g^{-1}(Y, x_2)$

$$F_Y(y) = \int_{x_2=-\infty}^{\infty} \int_{y=-\infty}^{y} f_{X_1 X_2}(g^{-1}, x_2) \left| \frac{dg^{-1}}{dy} \right| dy\,dx_2$$

$$f_Y(y) = \frac{dF_Y}{dy} = \int f_{X_1X_2}(g^{-1}, x_2)\left|\frac{dg^{-1}}{dy}\right| dx_2$$

Product of two random variables

— $Y = X_1 X_2$

$$F_Y(y) = \iint_{x_1x_2<y} f_{X_1X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}\left(\frac{y}{x_2}\right) f_{X_2}(x_2)\left|\frac{1}{x_2}\right| dx_2$$

— $Y = \frac{X_1}{X_2}$

$$f_Y(y) = \int_{-\infty}^{\infty} |x_2|\, f_{X_1X_2}(y \cdot x_2, x_2)\, dx_2$$

Generally, for random variables w/ covariance

$\boxed{Y = X_1X_2}$ $E[Y] = E[X_1X_2] = E[X_1]E[X_2] + Cov[X_1, X_2]$

$Var[Y] = Var[X_1X_2] = E[X_1^2X_2^2] - (E[X_1X_2])^2$

$\boxed{Y = \frac{X_2}{X_1}}$ $E[Y] = E[X_2]\, E[\frac{1}{X_1}]$

$$Var[Y] = Var[X_2]\,Var\left[\frac{1}{X_1}\right] + Var[X_2]\left(E\left[\frac{1}{X_1}\right]\right)^2 + Var\left[\frac{1}{X_1}\right]\left(E[X_2]\right)^2$$

w/ independent X_1 & X_2