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HW #6

Problem 4.6

Write Taylor expansion up to quadratic terms for the following expression.

4.6 e^x about the point $x^* = 0$

$f(x) = e^x \implies e^0 = 1$
$\frac{df}{dx} = e^x \implies e^0 = 1$
$\frac{d^2f}{dx^2} = e^x \implies e^0 = 1$

$\bar{f}\,(x) = 1 + 1(x-0) + 1(x-0)^2$
$\bar{f}\,(x) = 1 + x + \frac{x^2}{2}$

Problem 4.8

Write Taylor expansion up to quadratic terms for the following expression.

4.8 $f(x_1, x_2) = 10x_1^4 - 20x_1^2x_2 + 10x_2^2 + x_1^2 - 2x_1 + 5$ about the point (1, 1). Compare approximate and exact values of the function at the point (1.2, 0.8).

$$\nabla f(\bar{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 40x_1^3 - 40x_1x_2 + 2x_1 - 2 \\ -20x_1^2 + 20x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 120x_1^2 - 40x_2 + 2 & -40x_1 \\ -40x_1 & 20 \end{bmatrix} = \begin{bmatrix} 82 & -40 \\ -40 & 20 \end{bmatrix}$$

$$\bar{f}\,(x) = 4 + \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^T \begin{bmatrix} 82 & -40 \\ -40 & 20 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

$$\bar{f}\,(x) = 41x_1^2 - 40x_1x_2 + 10x_2^2 + 20x_2 + 15$$

At point (1.2, 0.8) the exact function and approximate function values are:

$$f(x) = 8.136$$
$$\bar{f}\,(x) = 7.64$$

Problem 4.11

Determine the nature of the quadratic form.

$$4.11 \quad F(\mathbf{x}) = x_1^2 + x_2^2 + 3x_1x_2$$

From the review of the last term in the above expression it can be seen that $F(\bar{x})$ can be both positive and negative depending on the values of x_1 and x_2.

Therefore $F(\bar{x})$ is indefinite.

Also eigenvalues for this system are repeated (i.e. $\lambda_{1,2} = 1$)

Problem 4.13

Determine the nature of the following quadratic forms.

$$4.13 \quad F(\mathbf{x}) = x_1^2 - x_2^2 + 4x_1x_2$$

From the review of the last term in the above expression it can be seen that $F(\bar{x})$ can be both positive and negative depending on the values of x_1 and x_2.

Therefore $F(\bar{x})$ is indefinite.

Also eigenvalues for this system are both negative and positive.
$\lambda_1 = 1$
$\lambda_2 = -1$

Problem 4.17

Determine the nature of the following quadratic forms.

$$4.17 \quad F(\mathbf{x}) = x_1^2 + 2x_1x_3 - 2x_2^2 + 4x_3^2 - 2x_2x_3$$

In matrix form:

$$F(\bar{x}) = [\, x_1 \; x_2 \; x_3 \,] \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The eigenvalues of the above matrix are $\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = -2$. Therefore $F(\bar{x})$ is indefinite.

Problem 4.23

Find fixed points for following expression and determine if the fixed points are min, max, or saddle.

$$4.23 \quad f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3$$

$$\nabla f(\bar{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix}$$

Roots for the above vector expression are $\bar{x}^* = [0, 0]$

The Hetian matrix for this system is constant and defined below.

$$H = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

For matrix H we have following eigenvalues: $\lambda_1 = -2$ $\lambda_2 = 6$

Based on the lambda values there is a inflection point at the fixed point.

Problem 4.32

Find fixed points for following expression and determine if the fixed points are min, max, or saddle.

4.32 The annual operating cost U for an electrical line system is given by the following expression

$$U = \frac{(21.9E+07)}{V^2 C} + (3.9E+06)C + (1.0E+03)V$$

where V = line voltage in kilovolts and C = line conductance in mhos. Find stationary points for the function, and determine V and C to minimize the operating cost.

$$\nabla f(\bar{x}) = \begin{bmatrix} \frac{\partial U}{\partial V} \\ \frac{\partial U}{\partial C} \end{bmatrix} = \begin{bmatrix} \frac{-438000000*V}{(C+V^2)^2} + 1000 \\ \frac{-219000000}{(C+V^2)^2} + 3900000 \end{bmatrix}$$

Roots for the above vector expression are:
$\bar{x}_1^* = [241.8, 0.0310]$
$\bar{x}_2^* = [-241.8, -0.0310]$

The Hetian matrix for this system is defined below.

$$H = \begin{bmatrix} \frac{1314000000}{CV^4} & \frac{438000000}{C^2V^3} \\ \frac{438000000}{C^2V^3} & \frac{438000000}{C^3V^2} \end{bmatrix}$$

For $\bar{x}_1^* = [241.8, 0.0310]$, H matrix eigenvalues are $\lambda_1 = 8.27$ and $\lambda_2 = 2.52e8$. Since both eigenvalues are positive there is a local minimum at the fixed point.

Conversly for $\bar{x}_2^* = [-241.8, -0.0310]$ the eigenvalues are $\lambda_1 = -8.27$ and $\lambda_2 = -2.52e8$. Therefore fixed point number two will be a local maximum.

Function value for the local min point is 483528.607.

Problem 4.42

Find fixed points for following expression and determine if the fixed points are min, max, or saddle.

$$4.42 \quad f(x_1, x_2, x_3, x_4) = (x_1 - 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4$$

$$\bigtriangledown f(\bar{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \\ \frac{\partial f}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 2x_1 - 20x_2 + 40(x_1 - x_4)^3 \\ -20x_1 + 200x_2 + 4(x_2 - 2x_3)^3 \\ 10x_3 - 10x_4 - 8(x_2 - 2x_3)^3 \\ -10x_3 + 10x_4 - 40(x_1 - x_4)^3 \end{bmatrix}$$

Roots for the above vector expression are:
$\bar{x}* = [0, 0, 0, 0]$
There were the only roots that satisfied the necessary condition of having $\bigtriangledown f = 0$

The Hetian matrix for this system is defined below.

$$H = \begin{bmatrix} 120(x_1 - x_4)^2 + 2 & -20 & 0 & -120(x_1 - x_4)^2 \\ -20 & 12(x_2 - 2x_3)^2 + 200 & -24(x_2 - 2x_3)^2 & 0 \\ 0 & -24(x_2 - 2x_3)^2 & 48(x_2 - 2x_3)^2 + 10 & -10 \\ -120(x_1 - x_4)^2 & 0 & -10 & 120(x_1 - x_4)^2 + 10 \end{bmatrix}$$

After applying the roots H becomes:

$$H = \begin{bmatrix} 2 & -20 & 0 & 0 \\ -20 & 200 & 0 & 0 \\ 0 & 0 & 10 & -10 \\ 0 & 0 & -10 & 10 \end{bmatrix}$$

The eigenvalues for the Hetian $\lambda_{1,2} = 0,\ \lambda_3 = 202,\ \lambda_4 = 20$

Since all eigenvalues are positive the root is a local minimum.

Min value of the function is 0.0.