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Take Home Exam #2

Problem 1

Below is the equation to be solved using methhod of multiple scales

$$\ddot{x} + \frac{k_1}{m}x = g - \epsilon(\frac{k_3}{m}x^3)$$

Since there is a constant forcing term the above expression will have to be shifted to the equilibrium position before method of multiple scales can be applied.

Equilibrium position position can be calculated by setting $\ddot{x} = 0$ and solving for equilibrium points.

For this instance the shifing term will be denoted as ϕ such that $x_1 = x + \phi$

Out[141]: $$\ddot{x} - g + \frac{k_1 x_1}{m} + \frac{k_3 x_1^3}{m}$$

Out[142]: $$\ddot{x} - g + \frac{k_1 \phi}{m} + \frac{k_1 x}{m} + \frac{k_3 \phi^3}{m} + \frac{3k_3}{m}\phi^2 x + \frac{3k_3}{m}\phi x^2 + \frac{k_3 x^3}{m}$$

In the above expression following terms sum to zero ($\frac{k_1\phi}{m} + \frac{k_3\phi^3}{m} - g$).
This leaves the following equation that can be used for analysis

Out[143]: $$\ddot{x} + \frac{3k_3}{m}\phi x^2 + \frac{k_3 x^3}{m} + x\left(\frac{k_1}{m} + \frac{3k_3}{m}\phi^2\right)$$

To make the algebra easier to deal with the above equation will be simplified to the following form:

Out[144]: $$\boxed{Ex^2 + Fx^3} + Gx + \ddot{x}$$

Non-linear parts multiplied by epsilon

Where $G = \frac{k_1}{m} + \frac{3k_3\phi^2}{m}$, $E = \frac{3k_3\phi}{m}$, and $F = \frac{k_3}{m}$

Next define definitions for the differential operators $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$

Out[145]: $D_1\epsilon + D_2\epsilon^2 + D_o$

Out[146]: $2D_1D_o\epsilon + D_o^2 + \epsilon^2\left(D_1^2 + 2D_2D_o\right)$

Next step is to define assumed solution for the system, which can be seen below:

Out[203]: $\boxed{\epsilon x_1 + x_o}$

Next define the complete expression using definitions above

Out[208]:

$$\boxed{\begin{gathered} G\left(\epsilon x_1 + x_o\right) + \epsilon\left(E(\epsilon x_1 + x_o)^2 + F(\epsilon x_1 + x_o)^3\right) \\ + \left(\epsilon x_1 + x_o\right)\left(2D_1D_o\epsilon + D_o^2 + \epsilon^2\left(D_1^2 + 2D_2D_o\right)\right) \end{gathered}}$$

Out[209]:

$$\boxed{\begin{gathered} D_1^2\epsilon^3x_1 + D_1^2\epsilon^2x_o + 2D_1D_o\epsilon^2x_1 + 2D_1D_o\epsilon x_o + 2D_2D_o\epsilon^3x_1 \\ + 2D_2D_o\epsilon^2x_o + D_o^2\epsilon x_1 + D_o^2x_o + E\epsilon^3x_1^2 + 2E\epsilon^2x_1x_o + E\epsilon x_o^2 + F\epsilon^4x_1^3 \\ + 3F\epsilon^3x_1^2x_o + 3F\epsilon^2x_1x_o^2 + F\epsilon x_o^3 + G\epsilon x_1 + Gx_o \end{gathered}}$$

Expanded version

From the expression above extract terms associated with ϵ^0, ϵ^1

Expression for x_o

Expression for xo

Out[206]: $\boxed{D_o^2x_o + Gx_o}$

Next define equation for x_1

Out[207]: $2D_1D_ox_o + D_o^2x_1 + Ex_o^2 + Fx_o^3 + Gx_1$

Out[152]: $D_o^2x_1 + Gx_1$

Out[153]: $2D_1D_o x_o + Ex_o^2 + Fx_o^3$

$$\boxed{\begin{array}{l} D_o^2 x_1 + Gx_1 \\ \texttt{'='} \\ -2D_1D_o x_o - Ex_o^2 - Fx_o^3 \end{array}}$$

Expression for x1

For the first ODE $D_o^2 x_o + Gx_o = 0$ assume following solution

Out[155]: $A(T_1, T_2)e^{i\sqrt{G}T_o} + e^{-i\sqrt{G}T_o}\overline{A(T_1, T_2)}$

Assumed solution satisfies the characteristic portion of the ODE

Out[156]: 0

Next substitute expression into x_1

Out[200]:

$$-2D_1D_oA(T_1,T_2)e^{i\sqrt{G}T_o} - 2D_1D_oe^{-i\sqrt{G}T_o}\overline{A(T_1,T_2)} - EA^2(T_1,T_2)e^{2i\sqrt{G}T_o} - 2EA(T_1,T_2)\overline{A(T_1,T_2)} - Ee^{-2i\sqrt{G}T_o}\overline{A(T_1,T_2)}^2 - FA^3(T_1,T_2)e^{3i\sqrt{G}T_o} - 3FA^2(T_1,T_2)e^{i\sqrt{G}T_o}\overline{A(T_1,T_2)} - 3FA(T_1,T_2)e^{-i\sqrt{G}T_o}\overline{A(T_1,T_2)}^2 - Fe^{-3i\sqrt{G}T_o}\overline{A(T_1,T_2)}^3$$

Has real and imaginary parts

Keep only real part

Out[199]:

$$\boxed{-2D_1D_oA(T_1,T_2)e^{i\sqrt{G}T_o} - EA^2(T_1,T_2)e^{2i\sqrt{G}T_o} - EA(T_1,T_2)\overline{A(T_1,T_2)} - FA^3(T_1,T_2)e^{3i\sqrt{G}T_o} - 3FA^2(T_1,T_2)e^{i\sqrt{G}T_o}\overline{A(T_1,T_2)}}$$

From above exprssion two terms will drive the system at resonance, therefore we must ensure that they sum to zero.

extract resonant term

Out[159]: $-2D_1D_oA(T_1,T_2)e^{i\sqrt{G}T_o} - 3FA^2(T_1,T_2)e^{i\sqrt{G}T_o}\overline{A(T_1,T_2)}$

Define non-resonant terms for later use:

Out[160]: $-EA^2(T_1,T_2)e^{2i\sqrt{G}T_o} - EA(T_1,T_2)\overline{A(T_1,T_2)} - FA^3(T_1,T_2)e^{3i\sqrt{G}T_o}$

Assume A is of the form $\frac{1}{2}\alpha e^{i\beta}$ and substitute into right hand side of equation for x_1

Out[161]: $\frac{1}{2}\alpha(T_1)e^{i\beta(T_1)}$

$\bar{A}$ definition

Out[162]: $\frac{1}{2}\alpha(T_1)e^{-i\beta(T_1)}$

Substitute assumed solution into the right hand side of equation for x_1

Out[163]: $-D_1D_o\alpha(T_1)e^{i\beta(T_1)}e^{i\sqrt{G}T_o} - \frac{3F}{8}\alpha^3(T_1)e^{i\beta(T_1)}e^{i\sqrt{G}T_o}$

Next get the derivatives per the expression above, there is only one.

To perform the differentiation, need to extract the values that need to be differentiated.

Term for D_oD_1 differentiation:

Out[164]: $-\alpha(T_1)e^{i\beta(T_1)}e^{i\sqrt{G}T_o}$

Next perform differentiation with respect to D_oD_1 term

Out[165]: $\sqrt{G}\left(\alpha(T_1)\frac{d}{dT_1}\beta(T_1) - i\frac{d}{dT_1}\alpha(T_1)\right)e^{i\left(\sqrt{G}T_o+\beta(T_1)\right)}$

Extract constant term: See next page

Out[166]: $\boxed{-\frac{3F}{8}\alpha^3(T_1)e^{i\beta(T_1)}e^{i\sqrt{G}T_o}}$

Constant term from the resonant expression

Combine all terms together to get differentiated expression

Out[197]: $$\left(-\frac{3F}{8}\alpha^3(T_1)+\sqrt{G}\alpha(T_1)\frac{d}{dT_1}\beta(T_1)-i\sqrt{G}\frac{d}{dT_1}\alpha(T_1)\right)e^{i\left(\sqrt{G}T_o+\beta(T_1)\right)}$$

Extract real and imaginary parts needed to solve for α and β.

Out[198]: $$-\frac{3F}{8}\alpha^3(T_1)+\sqrt{G}\alpha(T_1)\frac{d}{dT_1}\beta(T_1)$$

Real

Out[169]: $$-\sqrt{G}\frac{d}{dT_1}\alpha(T_1)$$

Imaginary

From the imaginary part α is a constant $\alpha = \alpha_c$

Next solve for β:

Out[170]: $$\left[\frac{3F}{8\sqrt{G}}\alpha^2(T_1)\right]$$

Out[171]: $$\frac{3F\alpha^2}{8\sqrt{G}}$$

Next solve for β by integrating

Out[172]: $$\frac{3FT_1\alpha^2}{8\sqrt{G}}$$

Solution for Beta

Next plug in the solution for A into the expression for x_1.

First start by defining the solution for A and $\bar{A}$

Out[173]: $$\frac{\alpha}{2}e^{\frac{3iFT_1}{8\sqrt{G}}\alpha^2}$$

Out[174]: $\frac{\alpha}{2} e^{-\frac{3iFT_1}{8\sqrt{G}}\alpha^2}$

Above solution for A checks out since the secular term in expression for x_1 sums to zero as seen below.

In short: $-2D_1D_oA(T_1,T_2)e^{i\sqrt{G}T_o} - 3FA^2(T_1,T_2)e^{i\sqrt{G}T_o}\overline{A(T_1,T_2)} = 0$

Out[175]: 0

Since solution for A above will make the secular terms sum to zero, expression for x_1 on the right hand side will become the following (i.e. non-resonant terms):

$$-EA^2(T_1,T_2)e^{2i\sqrt{G}T_o} - EA(T_1,T_2)\overline{A(T_1,T_2)} - FA^3(T_1,T_2)e^{3i\sqrt{G}T_o}$$

$$\boxed{\text{Out[176]: } -\frac{E\alpha^2}{4}e^{2i\sqrt{G}T_o}e^{\frac{3iFT_1}{4\sqrt{G}}\alpha^2} - \frac{E\alpha^2}{4} - \frac{F\alpha^3}{8}e^{3i\sqrt{G}T_o}e^{\frac{9iFT_1}{8\sqrt{G}}\alpha^2}}$$

After substituting for A and Abar

Extract 3 terms above for simplicity:

Out[177]: $-\frac{F\alpha^3}{8}e^{\frac{3i}{8\sqrt{G}}(3FT_1\alpha^2+8GT_o)}$

Out[178]: $-\frac{E\alpha^2}{4}e^{\frac{i}{4\sqrt{G}}(3FT_1\alpha^2+8GT_o)}$

Out[179]: $-\frac{E\alpha^2}{4}$

Since expression for x_1 is a linear ODE the particular solution can be solved in 3 parts.

- Part 1, particular solution for: $-\frac{F\alpha^3}{8}e^{\frac{3i}{8\sqrt{G}}(3FT_1\alpha^2+8GT_o)}$
- Part 2, particular solution for: $-\frac{E\alpha^2}{4}e^{\frac{i}{4\sqrt{G}}(3FT_1\alpha^2+8GT_o)}$
- Part 3, particular solution for: $-\frac{E\alpha^2}{4}$

For part 1 the solution becomes:

Out[180]: $\frac{F\alpha^3}{64G}e^{\frac{3i}{8\sqrt{G}}(3FT_1\alpha^2+8GT_o)}$

Check

Out[181]: $$-\frac{F\alpha^3}{8}e^{\frac{3i}{8\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)}$$

For part 2 solution becomes:

Out[182]: $$\frac{E\alpha^2}{12G}e^{\frac{i}{4\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)}$$

Check

Out[183]: $$-\frac{E\alpha^2}{4}e^{\frac{i}{4\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)}$$

For part 3 solution becomes:

Out[184]: $$-\frac{E\alpha^2}{4G}$$

Check

Out[185]: $$-\frac{E\alpha^2}{4}$$

Complete solution for x_1 becomes, keeping only real terms:

Out[186]: $$\frac{E\alpha^2}{12G}e^{\frac{i}{4\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)}-\frac{E\alpha^2}{4G}+\frac{F\alpha^3}{64G}e^{\frac{3i}{8\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)}$$

Solution for x_1 satisfies the condition on the right hand side as seen below

Out[187]: $$-\frac{E\alpha^2}{4}e^{2i\sqrt{G}T_o}e^{\frac{3iFT_1}{4\sqrt{G}}\alpha^2}-\frac{E\alpha^2}{4}-\frac{F\alpha^3}{8}e^{3i\sqrt{G}T_o}e^{\frac{9iFT_1}{8\sqrt{G}}\alpha^2}$$

Finally x_1 expressed in complete form takes the following expression:

Out[194]:

$$\frac{E\alpha^2}{6G}\cos\left(\frac{1}{4\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)\right)-\frac{E\alpha^2}{2G}+\frac{F\alpha^3}{32G}\cos\left(\frac{1}{8\sqrt{G}}\left(9FT_1\alpha^2+24GT_o\right)\right)$$

In similar form x_o is expressed as:

Out[195]:

$$\alpha\cos\left(\frac{1}{8\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)\right)$$

Finally $x(t,\epsilon)=x_o+\epsilon x_1$ can be expressed as:

Out[196]:

$$\epsilon\left(\frac{E\alpha^2}{6G}\cos\left(\frac{1}{4\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)\right)-\frac{E\alpha^2}{2G}+\frac{F\alpha^3}{32G}\cos\left(\frac{1}{8\sqrt{G}}\left(9FT_1\alpha^2+24GT_o\right)\right)\right)+\alpha\cos\left(\frac{1}{8\sqrt{G}}\left(3FT_1\alpha^2+8GT_o\right)\right)$$

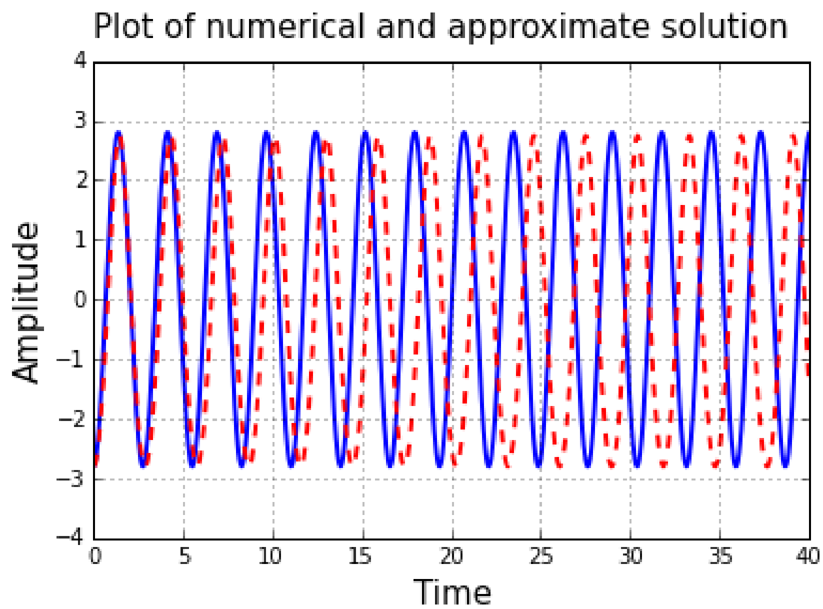
Where E, F, and G are the following:

- $E=\frac{3k_3\phi}{m}$
- $F=\frac{k_3}{m}$
- $G=\frac{k_1}{m}+\frac{3k_3\phi^2}{m}$

The above solution shall now be compared to the numerical solution. For simplicity only the first term will be ploted for the comparison. Numerical solution will be shifted by half amplitude to oscilate about 0.

Value for ϕ is the fixed point of the original expression with following values $m=1$, $k_1=2$, $k_3=0.1$, and $g=9.81$

With above values the fixed point value is $\phi=3.22$



Blue = 1st term of solution
Red dotted = numerical solution

The amplitudes match well, but the frequency has a slight shift. This will probably improve if second term is also included.

Problem 2)

$\dot{x} = x^3 + \delta x^2 - \mu x$

Find fixed points:

Out[56]: $$\left[0, \quad -\frac{\delta}{2} - \frac{1}{2}\sqrt{\delta^2 + 4\mu}, \quad -\frac{\delta}{2} + \frac{1}{2}\sqrt{\delta^2 + 4\mu}\right]$$

Next linearize the system to determine system behavior at fixed points:

Out[57]: $$2\delta x - \mu + 3x^2$$

Following the result from the linearization above the expression becomes $\dot{y} = (2\delta x - \mu + 3x^2)y$, so $\lambda = 2\delta x - \mu + 3x^2$.

At the trivial point $x = 0$, λ becomes:

Out[58]: $$-\mu$$

For $x = 0$ when $\mu \leq 0$ system is unstable and becomes stable when $\mu \geq 0$.

At $x = -\frac{\delta}{2} - \frac{1}{2}\sqrt{\delta^2 + 4\mu}$, λ becomes: See next page

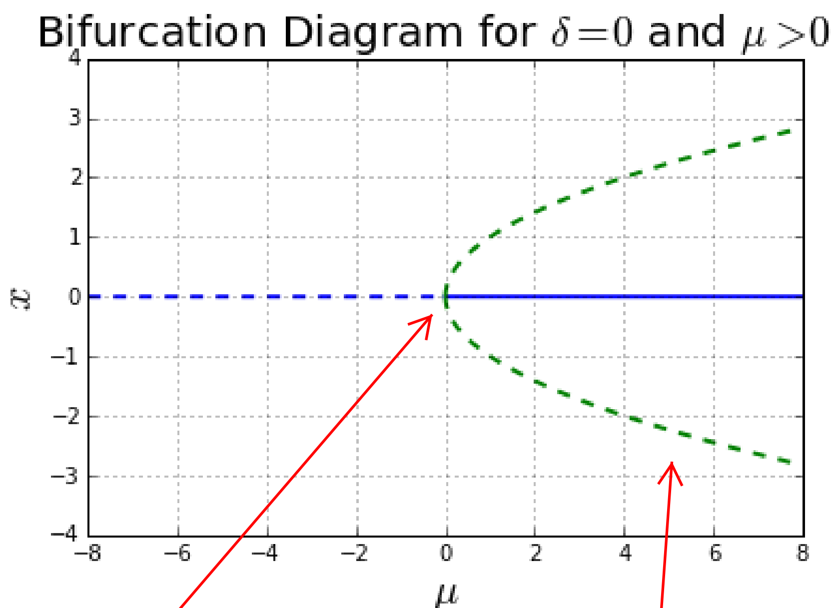
Out[59]: $$\frac{\delta^2}{2} + \frac{\delta}{2}\sqrt{\delta^2 + 4\mu} + 2\mu$$

At $x = -\frac{\delta}{2} + \frac{1}{2}\sqrt{\delta^2 + 4\mu}$, λ **becomes:**

Out[60]: $$\frac{\delta^2}{2} - \frac{\delta}{2}\sqrt{\delta^2 + 4\mu} + 2\mu$$

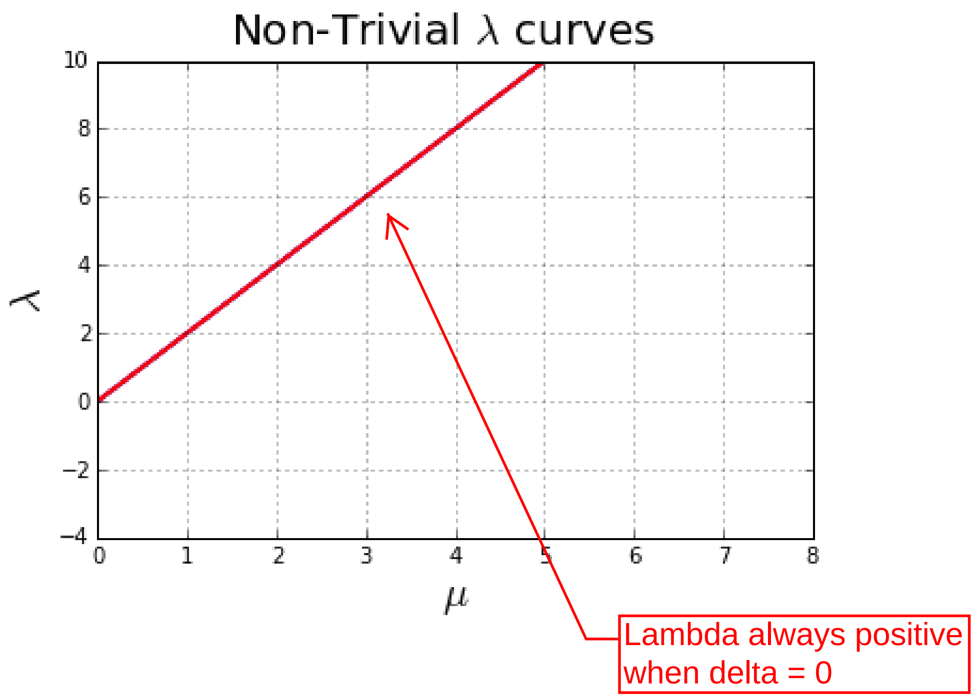
Plot bifurcation diagram for various values of δ:

For $\delta = 0$

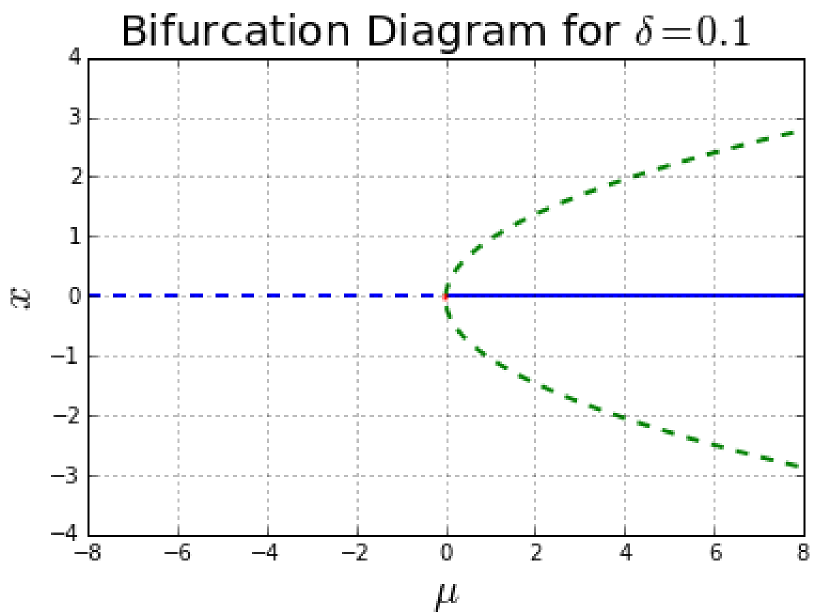


Turning point will be at 0,0. Since delta is zero nontrivial points will not exist until mu is zero or greater. This condition will not be transcritical

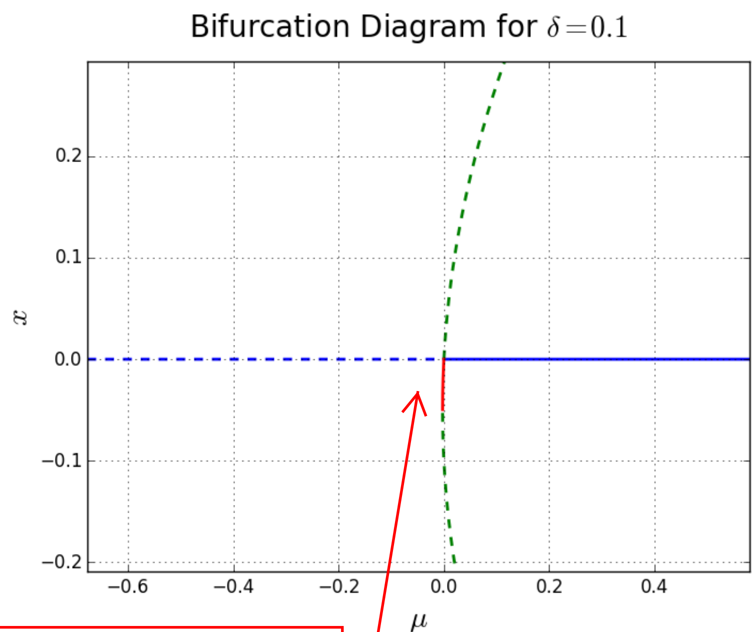
Both nodes are unstable since lambda is greater than zero, see next plot for lambda values.


Non-Trivial λ curves
λ
μ
Lambda always positive
when delta = 0

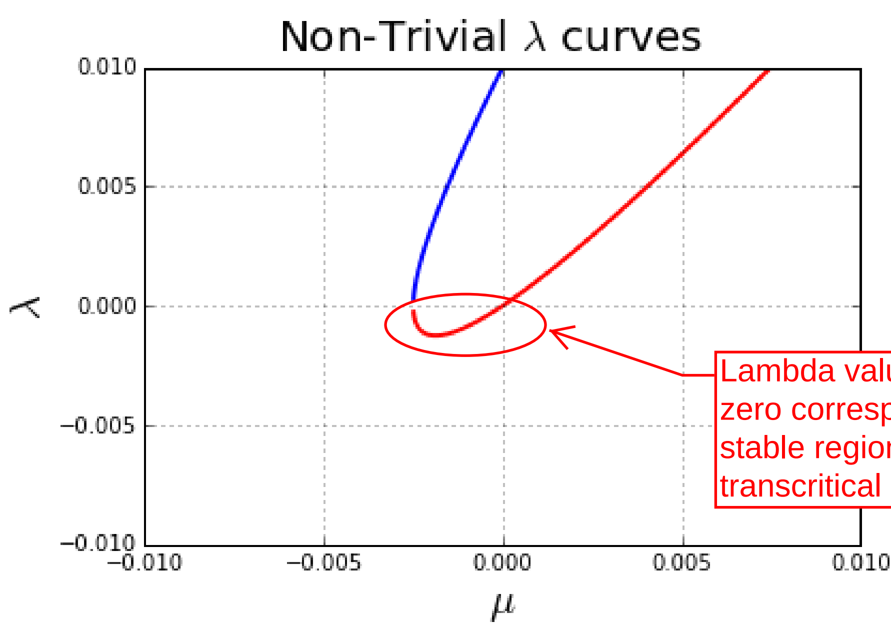
For $\delta = 0.1$


Bifurcation Diagram for $\delta = 0.1$
x
μ

Plot zooming on to $\mu = 0$ for $\delta = 0.1$

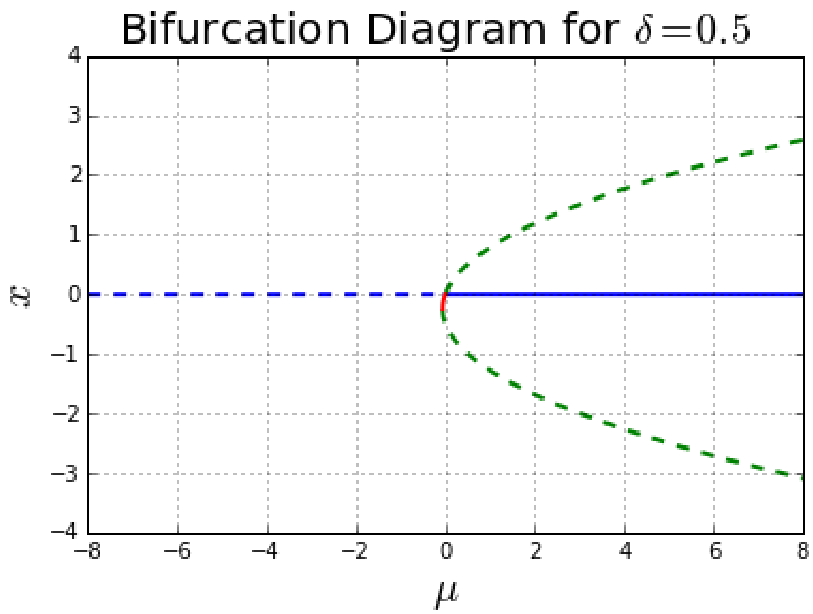


For delta = 0.1 there is a transcritical bifurcation where trivial and non-trivial solutions change from stable to unstable.
It can also be seen in the zoomed in figure that turning point occurs at approximately (-0.05, -0.0025)

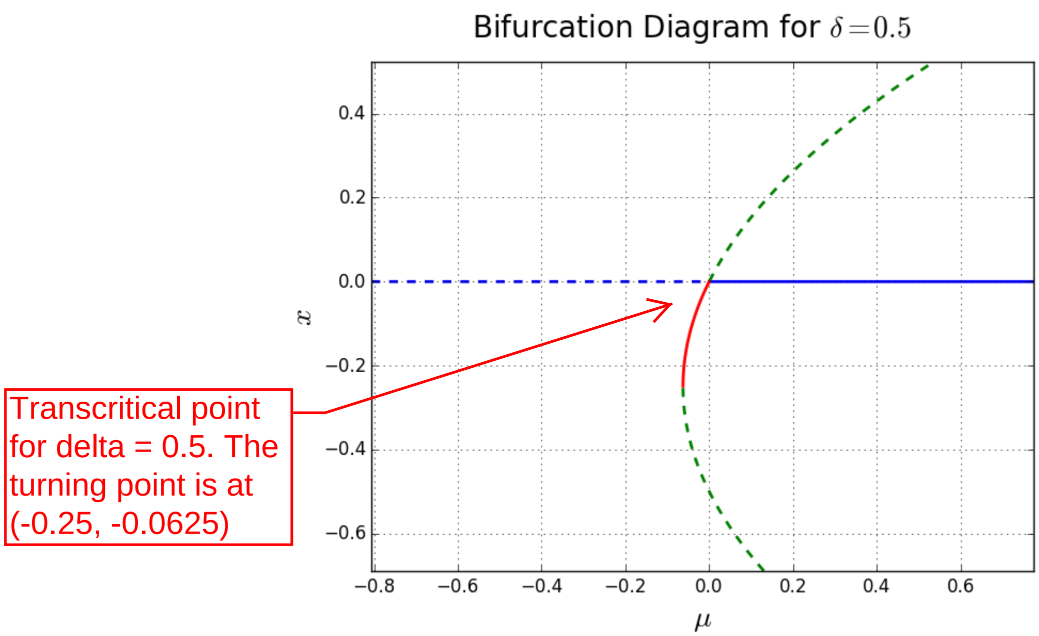


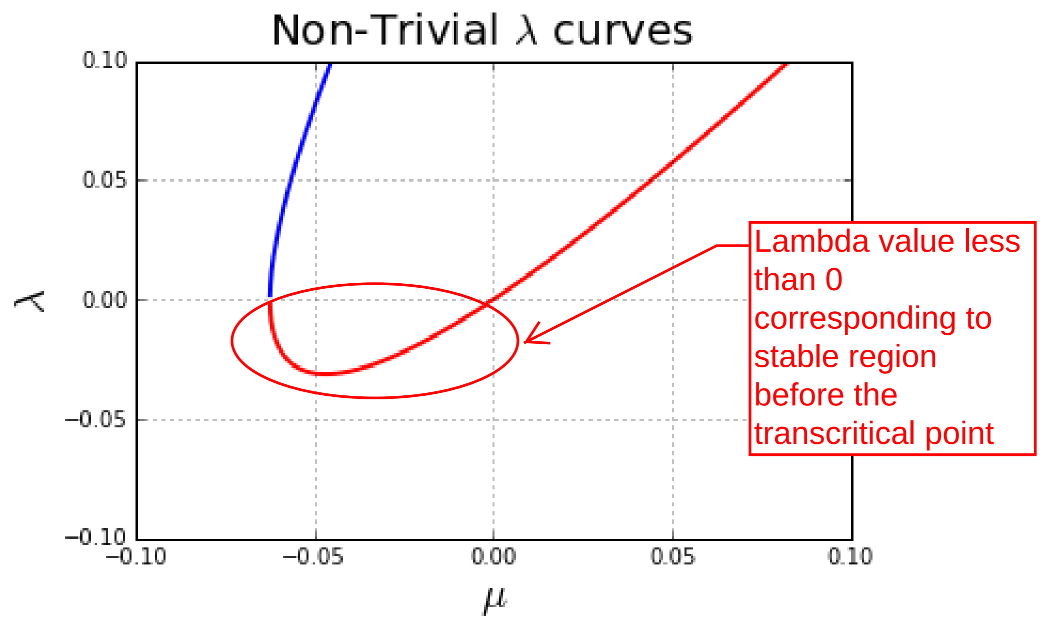
Lambda value less than zero corresponding to the stable region before the transcritical point.

For $\delta = 0.5$ See next page

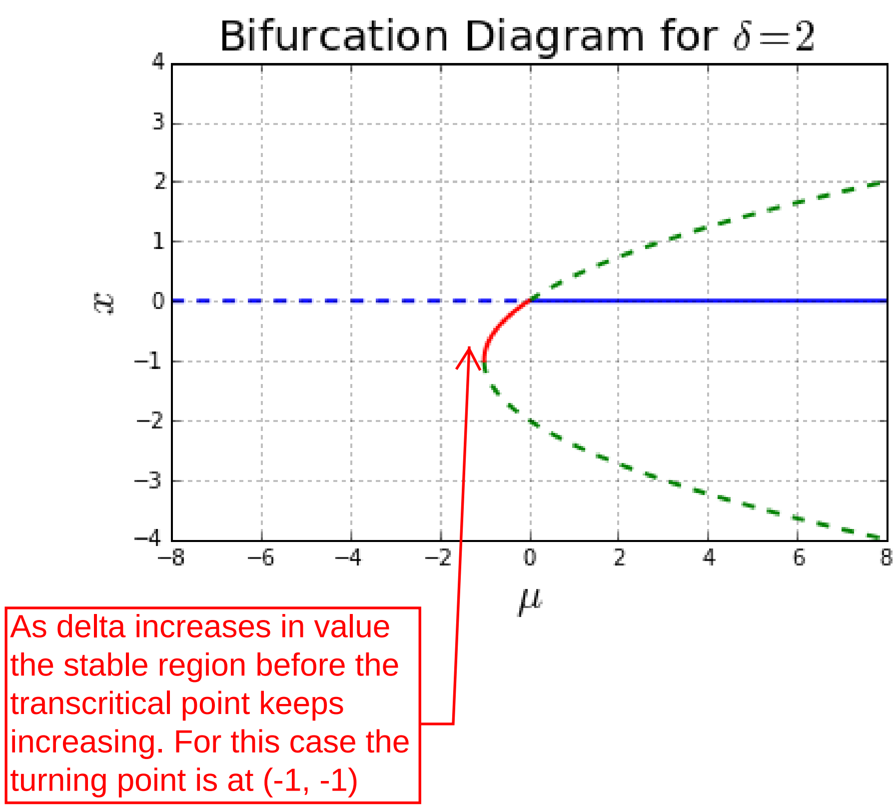


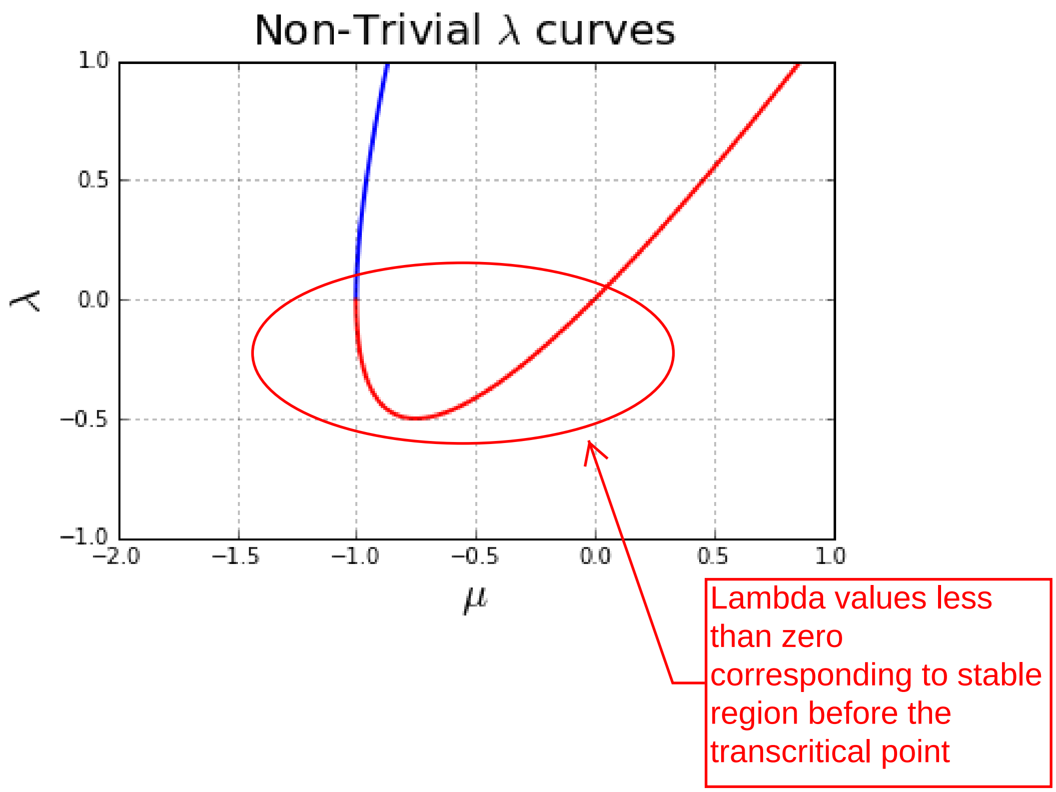
Plot zooming on to $\mu = 0$ for $\delta = 0.5$



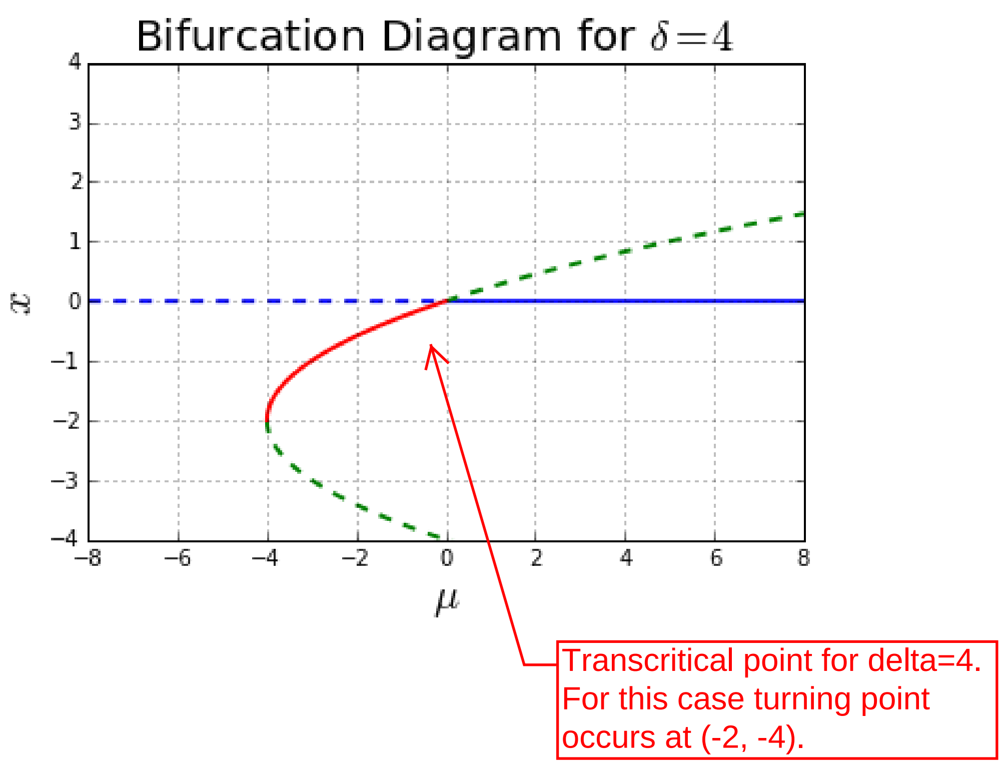

Non-Trivial λ curves
λ
μ
Lambda value less than 0 corresponding to stable region before the transcritical point

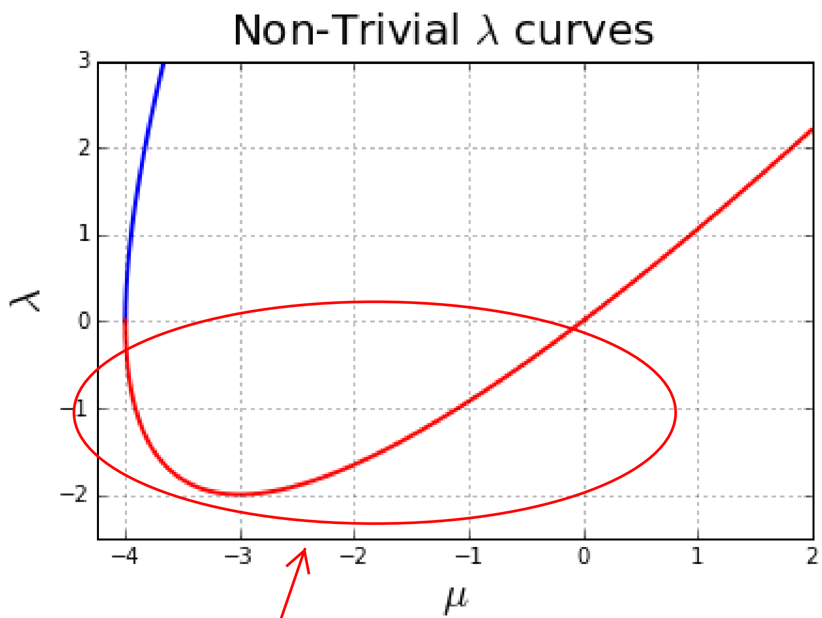
For $\delta = 2$


Bifurcation Diagram for $\delta=2$
x
μ
As delta increases in value the stable region before the transcritical point keeps increasing. For this case the turning point is at (-1, -1)


Non-Trivial λ curves
1.0
0.5
0.0
−0.5
−1.0
λ
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
μ
Lambda values less than zero corresponding to stable region before the transcritical point

For $\delta = 4$


Bifurcation Diagram for δ=4
4
3
2
1
0
−1
−2
−3
−4
x
−8
−6
−4
−2
0
2
4
6
8
μ
Transcritical point for delta=4. For this case turning point occurs at (-2, -4).



Lambda value less than zero corresponding to stable region before the transcritical point

TO SEE CODE FOR ALL THE OUTPUTS PLEASE REFER TO THE IPYTHON NOTEBOOK FILE SUPPLIED IN THE HW FOLDER