

atan2

The function **atan2**(*y*, *x*) or **arctan2**(*y*, *x*) (from "2-argument arctangent") is defined as the angle in the Euclidean plane, given in radians, between the positive *x* axis and the ray to the point (*x*, *y*) ≠ (0, 0).

The function **atan2**(*y*, *x*) first appeared in the programming language Fortran (in IBM's implementation FORTRAN-IV in 1961). It was originally intended to return a correct and unambiguous value for the angle *θ* in converting from cartesian coordinates (*x*, *y*) to polar coordinates (*r*, *θ*).

Equivalently, **atan2**(*y*, *x*) is the argument (also called *phase* or *angle*) of the complex number *x* + *i**y*.

atan2(*y*, *x*) returns a single value *θ* such that −*π* < *θ* ≤ *π* and, for some *r* > 0,

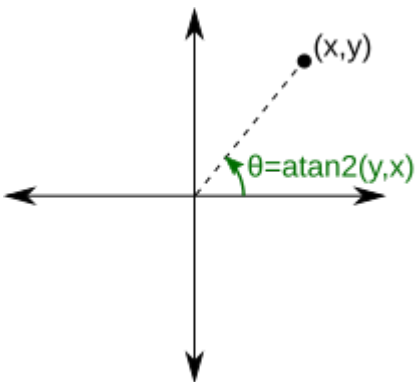
$$\begin{aligned}x &= r \cos \theta, \\ y &= r \sin \theta.\end{aligned}$$

While it is true that *r* = √(*x*² + *y*²), the following equivalence does not always hold:

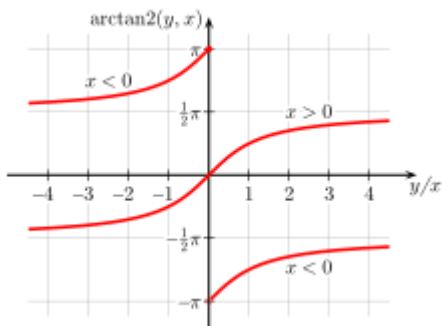
$$\theta = \arctan\left(\frac{y}{x}\right).$$

This only holds when *x* > 0. When *x* < 0, the angle apparent from the expression above is pointing in the opposite direction of the correct angle, and a value of *π* (or 180°) must be either added or subtracted from *θ* to put the cartesian point (*x*, *y*) into the correct quadrant of the Euclidean plane.^[1] This requires knowledge of the signs of *x* and *y* separately, which is information lost when *y* is divided by *x*.

Since any integer multiple of 2*π* can be added to the angle *θ* without changing either *x* or *y*, implying an ambiguous value for the returned value, the principal value of the angle, in the interval (−*π*, *π*] is returned. *θ* is signed, with counterclockwise angles being positive, and clockwise being negative. Specifically, **atan2**(*y*, *x*) is in the interval [0, *π*] when *y* ≥ 0, and in (−*π*, 0) when *y* < 0.



atan2(*y*, *x*) returns the angle *θ* between the ray to the point (*x*, *y*) and the positive *x* axis, confined to (−*π*, *π*].



Graph of **atan2**(*y*, *x*) over *y*/*x*

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History and motivation

The `atan2` function was first introduced in computer programming languages, but now it is also common in other fields of science and engineering. It dates back at least as far as the `FORTRAN` programming language^[2] and is currently found in many modern programming languages. Among these languages are: C's `math.h` standard library, the Java Math library, .NET's System.Math (usable from C#, VB.NET, etc.), the Python math module, the Ruby Math module, the Golang math package^[3] and elsewhere. In addition, many scripting languages, such as Perl, include the C-style `atan2(y, x)` function.

The single-argument arctangent function cannot distinguish between diametrically opposite directions. For example, the anticlockwise angle from the x axis to the vector $(1, 1)$, calculated in the usual way as $\arctan(1/1)$, is $\pi/4$ (radians), or 45° . However, the angle between the x axis and the vector $(-1, -1)$ appears, by the same method, to be $\arctan(-1/-1)$, again $\pi/4$, even though one might expect the answers $-3\pi/4$ (-135°) or $5\pi/4$ (225°). In addition, an attempt to find the angle between the x axis and the vectors $(0, y)$, $y \neq 0$ requires evaluation of $\arctan(y/0)$, which fails on division by zero.

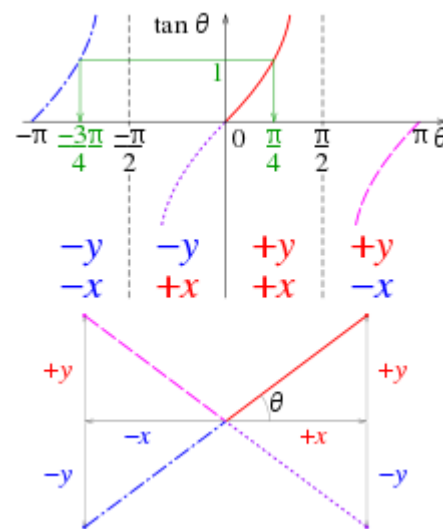
The `atan2` function calculates one unique arc tangent value from two variables y and x , where the signs of both arguments are used to determine the quadrant of the result, thereby selecting the desired branch of the arc tangent of y/x , e.g., $\text{atan2}(1, 1) = \pi/4$ and $\text{atan2}(-1, -1) = -3\pi/4$. Similarly, e.g., $\text{atan2}(1, 0) = \pi/2$.

When calculations are performed manually, the necessary quadrant corrections and exception handling can be done by inspection, but it is more useful to have a single function that always gives an unambiguous correct result. The `atan2` function is useful in many applications involving vectors in Euclidean space, such as finding the direction from one point to another. A principal use is in computer graphics rotations, for converting rotation matrix representations into Euler angles.

Definition and computation

The function `atan2` computes the principal value of the argument function applied to the complex number $x + iy$. That is, $\text{atan2}(y, x) = \text{Pr arg}(x + iy) = \text{Arg}(x + iy)$. The argument could be changed by an arbitrary multiple of 2π (corresponding to a complete turn around the origin) without making any difference to the angle, but to define `atan2` uniquely one uses the principal value in the range $(-\pi, \pi]$, that is, $-\pi < \text{atan2}(y, x) \leq \pi$.

In terms of the standard \arctan function, whose range is $(-\pi/2, \pi/2)$, it can be expressed as follows:



Graph of the tangent function from $-\pi$ to $+\pi$ with the corresponding signs of y/x . The green arrows point to the results of $\text{atan2}(-1, -1)$ and $\text{atan2}(1, 1)$.

$$\text{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

A compact expression with four overlapping half-planes is

$$\text{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & \text{if } y > 0, \\ -\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & \text{if } y < 0, \\ \arctan\left(\frac{y}{x}\right) \pm \pi & \text{if } x < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

The following expression derived from the tangent half-angle formula can also be used to define atan2:

$$\text{atan2}(y, x) = \begin{cases} 2 \arctan\left(\frac{y}{\sqrt{x^2+y^2}+x}\right) & \text{if } x > 0 \text{ or } y \neq 0, \\ \pi & \text{if } x < 0 \text{ and } y = 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

This expression may be more suited for symbolic use than the definition above. However it is unsuitable for general floating-point computational use, as the effect of rounding errors in $\sqrt{x^2+y^2}$ expand near the region $x < 0, y = 0$ (this may even lead to a division of y by zero).

A variant of the last formula that avoids these inflated rounding errors:

$$\text{atan2}(y, x) = \begin{cases} 2 \arctan\left(\frac{y}{\sqrt{x^2+y^2}+x}\right) & \text{if } x > 0, \\ 2 \arctan\left(\frac{\sqrt{x^2+y^2}-x}{y}\right) & \text{if } x \leq 0 \text{ and } y \neq 0, \\ \pi & \text{if } x < 0 \text{ and } y = 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Notes:

- This produces results in the range $(-\pi, \pi]$.^[1]
- As mentioned above, the principal value of the argument $\text{atan2}(y, x)$ can be related to $\arctan(y/x)$ by trigonometry. The derivation goes as follows:

If $(x, y) = (r \cos \theta, r \sin \theta)$, then $\tan(\theta/2) = y / (r + x)$. It follows that

$$\text{atan2}(y, x) = \theta = 2\theta/2 = 2 \arctan \frac{y}{\sqrt{x^2 + y^2} + x}.$$

Note that $\sqrt{x^2 + y^2} + x \neq 0$ in the domain in question.

Derivative

As the function atan2 is a function of two variables, it has two partial derivatives. At points where these derivatives exist, atan2 is, except for a constant, equal to $\arctan(y/x)$. Hence for $x > 0$ or $y \neq 0$,

$$\frac{\partial}{\partial x} \text{atan2}(y, x) = \frac{\partial}{\partial x} \arctan\left(\frac{y}{x}\right) = -\frac{y}{x^2 + y^2},$$

$$\frac{\partial}{\partial y} \text{atan2}(y, x) = \frac{\partial}{\partial y} \arctan\left(\frac{y}{x}\right) = \frac{x}{x^2 + y^2}.$$

Thus the gradient of atan2 is given by

$$\nabla \text{atan2}(y, x) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

Informally representing the function atan2 as the angle function $\theta(x, y) = \text{atan2}(y, x)$ (which is only defined up to a constant) yields the following formula for the total differential:

$$\begin{aligned} d\theta &= \frac{\partial}{\partial x} \text{atan2}(y, x) dx + \frac{\partial}{\partial y} \text{atan2}(y, x) dy \\ &= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy. \end{aligned}$$

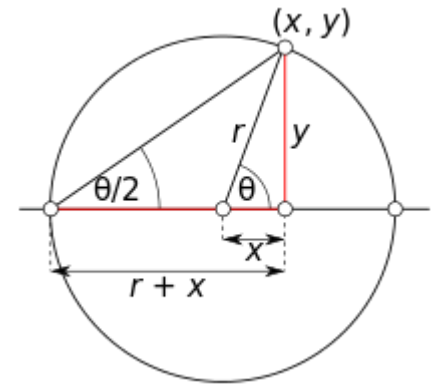
While the function atan2 is discontinuous along the negative x -axis, reflecting the fact that angle cannot be continuously defined, this derivative is continuously defined except at the origin, reflecting the fact that infinitesimal (and indeed local) *changes* in angle can be defined everywhere except the origin. Integrating this derivative along a path gives the total change in angle over the path, and integrating over a closed loop gives the winding number.

In the language of differential geometry, this derivative is a one-form, and it is closed (its derivative is zero) but not exact (it is not the derivative of a 0-form, i.e., a function), and in fact it generates the first de Rham cohomology of the punctured plane. This is the most basic example of such a form, and it is fundamental in differential geometry.

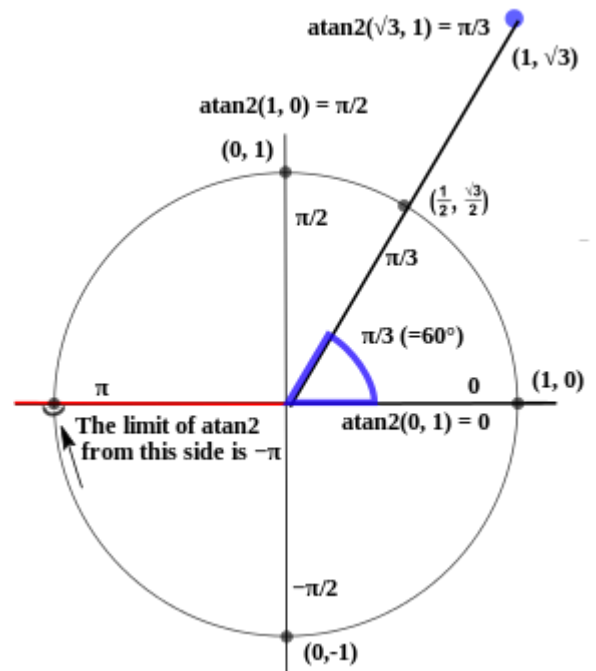
The partial derivatives of atan2 do not contain trigonometric functions, making it particularly useful in many applications (e.g. embedded systems) where trigonometric functions can be expensive to evaluate.

Illustrations

This figure shows values of atan2 along selected rays from the origin, labelled at the unit circle. The values, in radians, are shown inside the circle. The diagram uses the standard mathematical convention that angles increase counterclockwise from zero along the ray to the right. Note that the order of arguments is reversed; the function $\text{atan2}(y, x)$ computes the angle corresponding to the point (x, y) .



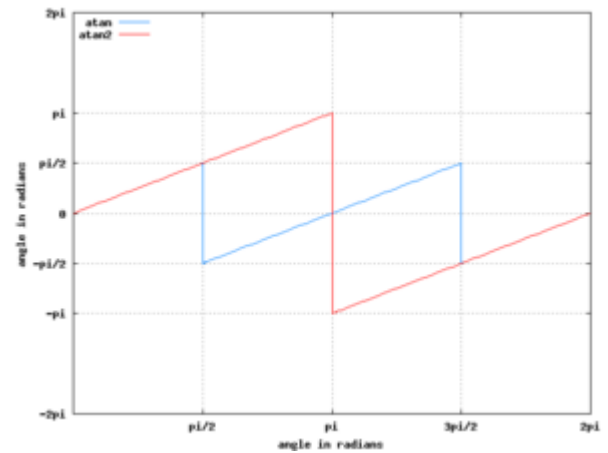
The derivation of the principal value of the argument refers to this figure



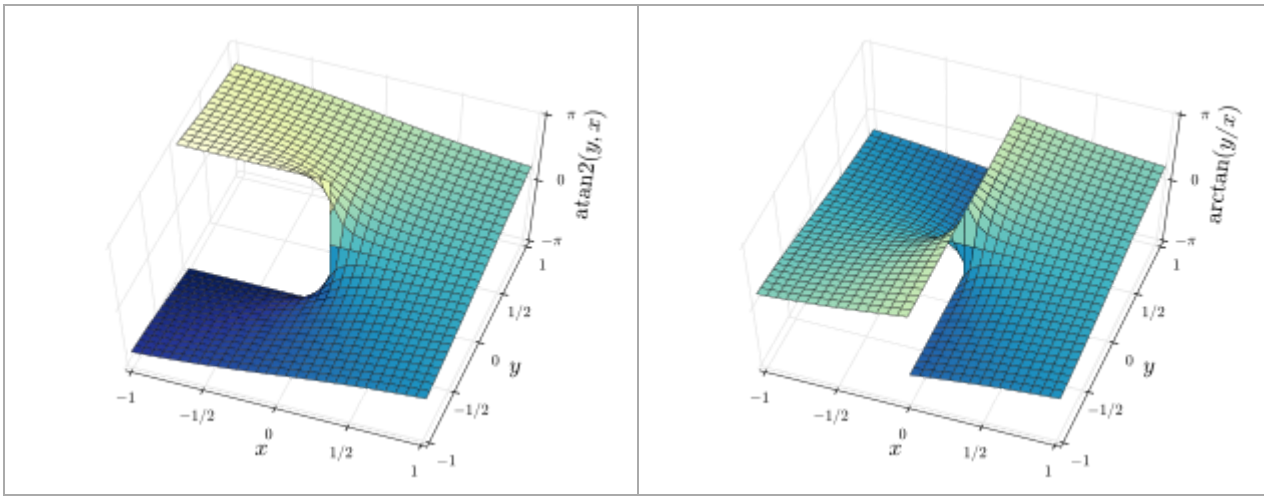
atan2 for selected rays

This figure shows the values of $\arctan(\tan(\theta))$ alongside with $\text{atan2}(\sin(\theta), \cos(\theta))$ for $0 \leq \theta \leq 2\pi$. Both functions are odd and periodic with periods π and 2π , respectively, and thus can easily be supplemented to any region of real values of θ . One can clearly see the branch cuts of the **atan2**-function at $\theta = \pi$, and of the **arctan**-function at $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.^[4]

The two figures below show 3D views of respectively $\text{atan2}(y, x)$ and $\arctan(\frac{y}{x})$ over a region of the plane. Note that for $\text{atan2}(y, x)$, rays in the X/Y -plane emanating from the origin have constant values, but for $\arctan(\frac{y}{x})$ lines in the X/Y -plane passing through the origin have constant values. For $x > 0$, the two diagrams give identical values.



Comparison of arctan and atan2 functions



Angle sum and difference identity

Sums of **atan2** may be collapsed into a single operation according to the following identity

$$\mathbf{atan2}(y_1, x_1) \pm \mathbf{atan2}(y_2, x_2) = \mathbf{atan2}(y_1 x_2 \pm y_2 x_1, x_1 x_2 \mp y_1 y_2)$$

...provided that $\mathbf{atan2}(y_1, x_1) \pm \mathbf{atan2}(y_2, x_2) \in (-\pi, \pi]$.

The proof involves considering two cases, one where $y_2 \neq 0$ or $x_2 > 0$ and one where $y_2 = 0$ and $x_2 < 0$.

We only consider the case where $y_2 \neq 0$ or $x_2 > 0$. To start, we make the following observations:

1. $-\mathbf{atan2}(y, x) = \mathbf{atan2}(-y, x)$ provided that $y \neq 0$ or $x > 0$.
2. $\mathbf{Arg}(x + iy) = \mathbf{atan2}(y, x)$, where **Arg** is the complex argument function.
3. $\theta = \mathbf{Arg} e^{i\theta}$ whenever $\theta \in (-\pi, \pi]$, a consequence of Euler's formula.
4. $\mathbf{Arg}(e^{i \mathbf{Arg} \zeta_1} e^{i \mathbf{Arg} \zeta_2}) = \mathbf{Arg}(\zeta_1 \zeta_2)$.

To see (4), we have the identity $e^{i \mathbf{Arg} \zeta} = \bar{\zeta} / |\zeta|$ where $\bar{\zeta} = \zeta / |\zeta|$, hence $\mathbf{Arg}(e^{i \mathbf{Arg} \zeta_1} e^{i \mathbf{Arg} \zeta_2}) = \mathbf{Arg}(\bar{\zeta}_1 \bar{\zeta}_2)$.

Furthermore, since $\mathbf{Arg} \zeta = \mathbf{Arg} a\zeta$ for any positive real value a , then if we let $\zeta = \zeta_1 \zeta_2$ and $a = \frac{1}{|\zeta_1| |\zeta_2|}$ then we have $\mathbf{Arg}(\bar{\zeta}_1 \bar{\zeta}_2) = \mathbf{Arg}(\zeta_1 \zeta_2)$.

From these observations have following equivalences:

$$\begin{aligned}
 \mathbf{atan2}(y_1, x_1) \pm \mathbf{atan2}(y_2, x_2) &= \mathbf{atan2}(y_1, x_1) + \mathbf{atan2}(\pm y_2, x_2) && \text{by (1)} \\
 &= \mathbf{Arg}(x_1 + iy_1) + \mathbf{Arg}(x_2 \pm iy_2) && \text{by (2)} \\
 &= \mathbf{Arg} e^{i(\mathbf{Arg}(x_1 + iy_1) + \mathbf{Arg}(x_2 \pm iy_2))} && \text{by (3)} \\
 &= \mathbf{Arg}(e^{i \mathbf{Arg}(x_1 + iy_1)} e^{i \mathbf{Arg}(x_2 \pm iy_2)}) \\
 &= \mathbf{Arg}((x_1 + iy_1)(x_2 \pm iy_2)) && \text{by (4)} \\
 &= \mathbf{Arg}(x_1 x_2 \mp y_1 y_2 + i(y_1 x_2 \pm y_2 x_1)) \\
 &= \mathbf{atan2}(y_1 x_2 \pm y_2 x_1, x_1 x_2 \mp y_1 y_2) && \text{by (2)}
 \end{aligned}$$

Corollary: if (y_1, x_1) and (y_2, x_2) are 2-dimensional vectors, the difference formula is frequently used in practice to compute the angle between those vectors with the help of **atan2**, since the resulting computation behaves benign in the range $(-\pi, \pi]$ and can thus be used without range checks in many practical situations.

Realizations of the function in common computer languages

The realization of the function differs from one computer language to another:

- The C function `atan2`, and most other computer implementations, are designed to reduce the effort of transforming cartesian to polar coordinates and so always define `atan2(0, 0)`. On implementations without signed zero, or when given positive zero arguments, it is normally defined as 0. It will always return a value in the range $[-\pi, \pi]$ rather than raising an error or returning a NaN (Not a Number).
- In Common Lisp, where optional arguments exist, the `atan` function allows one to optionally supply the *x* coordinate: `(atan y x)`.^[5]
- In Mathematica, the form `ArcTan[x, y]` is used where the one parameter form supplies the normal arctangent (note that the order of arguments is reversed relative to the convention used in the discussion above). Mathematica classifies `ArcTan[0, 0]` as an indeterminate expression.
- In Microsoft Excel,^[6] OpenOffice.org Calc, LibreOffice Calc,^[7] Google Spreadsheets,^[8] iWork Numbers,^[9] and ANSI SQL:2008 standard,^[10] the `atan2` function has the two arguments reversed.
- In the Intel Architecture assembler code, `atan2` is known as the `FPATAN` (floating-point partial arctangent) instruction.^[11] It can deal with infinities and results lie in the closed interval $[-\pi, \pi]$, e.g. $\text{atan2}(\infty, x) = +\pi/2$ for finite *x*. Particularly, `FPATAN` is defined when both arguments are zero:

```
atan2(+0, +0) = +0;
atan2(+0, -0) = +π;
atan2(-0, +0) = -0;
atan2(-0, -0) = -π.
```

This definition is related to the concept of signed zero.

- On most TI graphing calculators (excluding the TI-85 and TI-86), the equivalent function is called **R►Pθ** and has the arguments reversed.
- On TI-85 the `arg` function is called `angle(x, y)` and although it appears to take two arguments, it really only has one complex argument which is denoted by a pair of numbers: $x + yi = (x, y)$.
- In mathematical writings other than source code, such as in books and articles, the notations **Arctan**^[12] and **Tan**⁻¹^[13] have been utilized; these are uppercase variants of the regular `arctan` and `tan`⁻¹. This usage is consistent with the complex argument notation, such that $\text{Atan}(y, x) = \text{Arg}(x + yi)$.
- On HP calculators, treat the coordinates as a complex number and then take the `ARG`. Or `<< C -> R ARG >> 'ATAN2' STO`.
- On scientific calculators the function can often be calculated as the angle given when (x, y) is converted from rectangular coordinates to polar coordinates.
- Systems supporting symbolic mathematics normally return an undefined value for `atan2(0, 0)` or otherwise signal that an abnormal condition has arisen.
- For systems implementing signed zero, infinities, or Not a Number (for example, IEEE floating point), it is common to implement reasonable extensions which may extend the range of values produced to

include $-\pi$ and -0 . These also may return NaN or raise an exception when given a NaN argument.

- For systems implementing signed zero (for example, IEEE floating point), $\text{atan2}(-0, x)$, $x < 0$ poses the risk of returning the value $-\pi$, in the case where the implementation of $\text{atan2}(y, x)$ fails to properly handle -0 inputs.
- The free math library FDLIBM (Freely Distributable LIBM) available from [netlib](#) has source code showing how it implements atan2 including handling the various IEEE exceptional values.
- For systems without a hardware multiplier the function atan2 can be implemented in a numerically reliable manner by the [CORDIC](#) method. Thus implementations of $\text{atan}(y)$ will probably choose to compute $\text{atan2}(y, 1)$.

See also

- [hypot](#)

References

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External links

- ATAN2 Online calculator (https://www.medcalc.org/manual/atan2_function.php)
- Java 1.6 SE JavaDoc (<http://docs.oracle.com/javase/6/docs/api/java/lang/Math.html#atan2%28double,%20double%29>)
- `atan2` (http://everything2.com/index.pl?node_id=1008481) at Everything2
- PicBasic Pro solution (<http://www.picbasic.co.uk/forum/showthread.php?p=70269#post70269>) `atan2` for a PIC18F

Other implementations/code for atan2

- Bearing Between Two Points (<http://mathforum.org/library/drmath/view/55417.html>)
- Arctan and Polar Coordinates (<http://mathforum.org/library/drmath/view/54114.html>)
- What's 'Arccos'? (<http://mathforum.org/library/drmath/view/54172.html>)

Notes

- ¹ One can apply the periodicity of the result to map to any desired range. e.g. mapping to $[0, 2\pi)$ by adding 2π to negative results.

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