Equations for ray tracing

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For waves of a given type we have a dispersion function $D(\omega, \mathbf{k})$. The frequency ω and wavenumber $\mathbf{k}$ for a wave are not arbitrary, but are related by the dispersion function by $D(\omega, \mathbf{k}) = 0$. When solved explicitly for, say, $\omega = \omega(\mathbf{k})$ we call this the dispersion relation.

The dispersion function for EM waves of a particular polarization in an unmagnetized plasma is

$$D(\omega, \mathbf{k}) = -c^2 k^2 + \omega^2 - \omega_{\mathrm{p}}^2,$$

where ω_{p} is the electron plasma frequency, and c is the speed of light. The dispersion function for Langmuir (plasma) waves is similar:

$$D(\omega, \mathbf{k}) = -3 v_{\mathrm{Te}}^2 k^2 + \omega^2 - \omega_{\mathrm{p}}^2,$$

where v_{Te} is the electron thermal velocity. The dispersion function is used to construct the *ray Hamiltonian* D':

$$D' = \left(\frac{\partial D}{\partial \omega}\right)^{-1} D.$$

Using this as a Hamiltonian, the ray position $\mathbf{x}$ and its wavevector $\mathbf{k}$ evolve in time in the ray phase space $(\mathbf{x}, \mathbf{k})$ according to Hamilton's equation (just like a mechanical particle!)

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= -\nabla_{\mathbf{k}} D', \\ \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} &= \nabla_{\mathbf{x}} D'. \end{aligned}$$

Substituting for the EM wave Hamiltonian, we get the following **ray equations for EM waves**:

$$\boxed{\begin{aligned} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \frac{c^2 \mathbf{k}}{\omega}, \\ \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} &= -\frac{\omega}{2}\frac{n_{\mathrm{e}}}{n_c} \nabla_{\mathbf{x}} \log n_{\mathrm{e}}, \end{aligned}}$$

where n_{c} is the critical density for EM waves of frequency ω. In an inhomogeneous plasma that is stationary (or changes very slowly in time) the frequency ω is constant.

We can do the same thing to get the **ray equations for Langmuir waves** too:

$$\boxed{\begin{aligned} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \frac{3 v_{\mathrm{Te}}^2 \mathbf{k}}{\omega}, \\ \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} &= -\frac{\omega}{2}\frac{n_{\mathrm{e}}}{n_c} \nabla_{\mathbf{x}} \log n_{\mathrm{e}} - \frac{3}{2}\frac{k^2 v_{\mathrm{Te}}^2}{\omega} \nabla_{\mathbf{x}} \log T_{\mathrm{e}}, \end{aligned}}$$

where T_{e} is the spatially varying electron temperature.

Since these are a system of first order differential equations, the solutions are unique and ray trajectories cannot cross in the ray phase space. If we project these solutions to coordinate ($\mathbf{x}$) space, they can cross however.

All we need are the initial conditions for a ray (the initial point in the ray phase space) $(\mathbf{x}(0), \mathbf{k}(0))$ and we can find the ray at any other time. Of course we need to know the functional form for $n_{\mathrm{e}}(\mathbf{x})$, $T_{\mathrm{e}}(\mathbf{x})$ and their spatial derivatives. We get these from a radiation-hydrodynamics code. This means these quantities are only known at the spatial grid points. That's why in the Matlab program we need to use interpolation so that we can compute these quantities at arbitrary positions.

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