Physics 157 Written Homework 8

In this homework set, you'll practice analyzing systems in simple harmonic motion (SHM). Since we didn't have a homework set last week due to MT2, this HW set will give you some practice on last week's material as well as this week's material.

Here are the skills that we'd like you to develop.

- Given a graph of displacement vs time for a simple harmonic oscillator, determine the amplitude, period, angular frequency, and phase for the corresponding function (represented as a cosine).
- Given a graph of displacement, velocity, or acceleration vs time for a simple harmonic oscillator, sketch graphs for the other two quantities. Also, to say where each of these quantities is zero, maximum or minimum and give the maximum and minimum values.
- Given the displacement, velocity, or acceleration represented as a sinusoidal function of time, to calculate the functions corresponding to the other two quantities and sketch graphs of all three quantities.
- Given the graph or functional form of displacement, velocity, or acceleration for a simple harmonic oscillator, to determine the constant k in the force law given the mass of the object, or vice-versa.
- Given a physical system with a variety of relevant forces, to write an expression for the net force on some object in the system as a function of that object's position, and to use this to deduce the equilibrium position and the frequency for oscillations around equilibrium
- Given the description of the initial configuration for a physical system that will undergo simple harmonic motion, to predict the subsequent motion of the object, representing this as a sinusoidal function.
- Given the description of the initial configuration for a physical system that will undergo simple harmonic motion, to predict the amplitude or maximum velocity using conservation of energy.
- Given a graph representing the displacement vs time or the acceleration vs time for damped oscillation, to deduce the time constant for the system or (given the mass) the damping constant b, and to write an equation for the displacement vs time.
- Given a graph representing the displacement vs time or the acceleration vs time for damped oscillation, to determine the energy lost per cycle.
- To determine the damping required to produce critical damping given a graphical or mathematical representation of the oscillations in an undamped or underdamped system.
- To describe the mathematical description of sinusoidal travelling waves, and explain how the parameters in this mathematical description relate to wavelength, frequency, wave number, period and velocity.
- To relate wave speed with frequency and wavelength for a travelling sinusoidal wave.

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A summary of formulae you may need:

Restoring force leading to SHM: $F_{net} = -k\,\Delta x$

Finding k in a general system:

a) determine F_{net} as a function of position
b) find the equilibrium position via $F_{net}(x_{eq}) = 0$
c) -k is the slope of $F_{net}(x)$ at x_{eq}, $k = -dF_{net}/dx$ at $x = x_{eq}$

Newton's 2nd Law: $a = F_{net}/m$ (this allows us to predict the acceleration based on the net force)

Basic form of simple harmonic motion: $x(t) = A\cos(\omega t + \varphi)$ $\omega^2 = k/m$

Here A is the amplitude, ω is the angular frequency, related to the period by $\omega = 2\pi / T$, and φ is the phase.

The phase can be determined looking at what fraction of a period the graph is shifted to the left or right by (take the time that the function is maximum and divide by T), and then multiplying by ±2π, with + for shifts to the left and – for shifts to the right.

Velocity and acceleration: $v(t) = dx/dt$ (slope of displacement graph at time t)
$a(t) = dv/dt$ or $a = -\omega^2 x$

Energy is $\frac{1}{2} k x^2$ at times when x is maximum

Energy is proportional to amplitude squared

Displacement vs time for damped oscillation: $A e^{-t/\tau}\cos(\omega t + \varphi)$ (where amplitude decreases by an equal fraction for equal time intervals)

Energy is proportional to amplitude squared

Calculating τ for a system with damping forces $F = -b\,v$: $\tau = 2m/b$

Frequency for damped oscillation with $F_{net} = -k\,x - b\,v$: $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

Note: When damping is small (i.e. there are many visible oscillations), it's a good approximation just to use $\omega = \sqrt{\frac{k}{m}}$

Critical damping (no more oscillations) occurs for: $b = 2\sqrt{km}$

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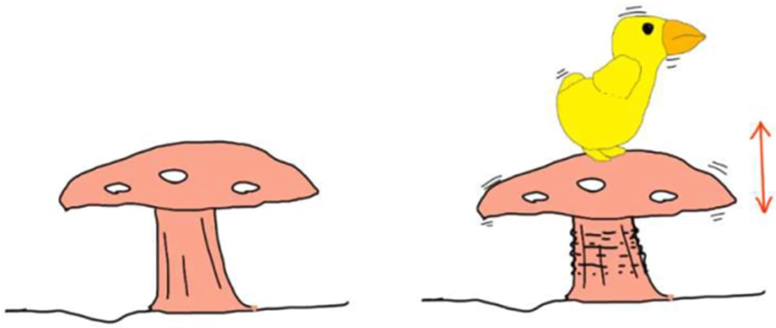
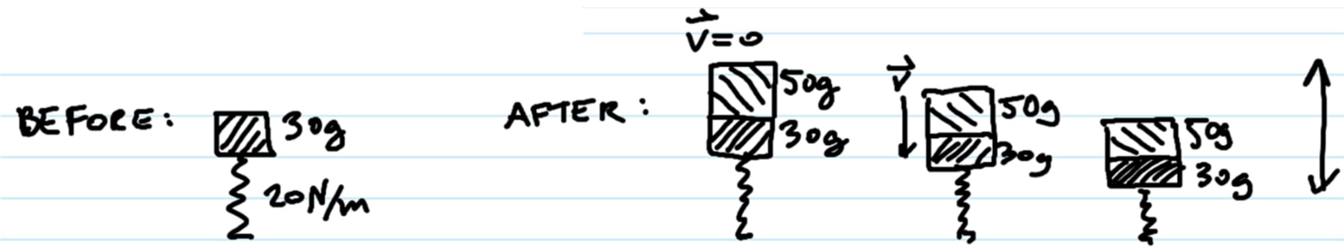
Problem 1: A very bouncy mushroom can be modeled as a mass of 30 g (the mushroom cap) on top of a spring (the mushroom stalk) with spring constant 20 N/m. A bird of mass 50 g lands on the mushroom gently so that its velocity is zero when it lands.



a) Relative to the initial height of the mushroom, what is the new equilibrium height of the mushroom about which oscillations will occur?



Before the bird lands, the force of gravity on the mushroom cap is equal in magnitude to the force of the stalk (the spring) on the cap. In the new equilibrium position, the additional force of gravity on the bird plus cap is balanced by the additional force form the spring. So, $k\,\Delta x = m_{bird}g$ and $\Delta x = \frac{m_{bird}g}{k} = \frac{0.05\ kg \cdot 9.8\ m/s^2}{20\ N/m} = 2.45\ cm$.

b) What is the frequency of the resulting oscillations?

The frequency of the oscillations is $\omega = \sqrt{\frac{k}{m_{tot}}} = \sqrt{\frac{20\ N/m}{0.08\ kg}} = 15.8\ s^{-1}$ (angular frequency), or $f = \frac{\omega}{2\pi} = 2.51\ s^{-1}$.

c) What is the maximum compression of the mushroom (i.e., how far down does the cap of the mushroom go relative to its location when the bird lands)?

The amplitude of the oscillations corresponds to the distance between the initial position and the equilibrium position, A = 2.45 cm. The maximum compression will be twice this amplitude, or 4.9 cm.

d) What is the equation that describes the oscillation of the mushroom as a function of time?

We can describe the height of the mushroom cap relative to its equilibrium position by $\Delta h = A\cos(\omega t + \varphi) = 2.45\ cm \cdot \cos(15.8\ s^{-1} \cdot t)$. Here $\varphi = 0$ since the displacement is maximum at the initial time.

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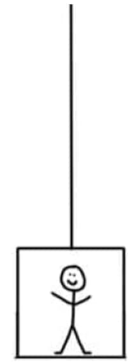


Problem 2: A cable with length 50m, Young's modulus 200GPa, and radius 0.5 cm is used for an elevator.

a) Determine the spring constant for this cable?

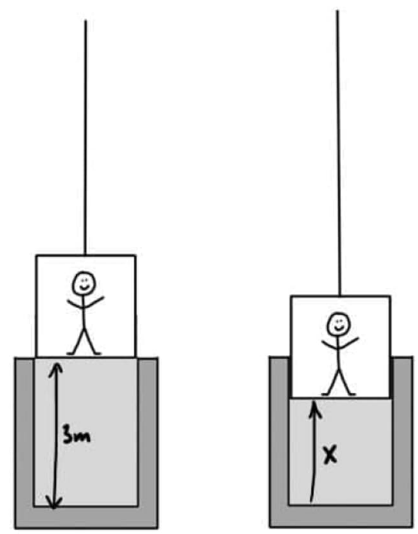
Hint: the spring constant is defined by F = - k ΔL. You want to find k in terms of the quantities given based on the definition of Young's modulus.

The Young's modulus relates the stress and strain via $\frac{F}{A} = Y\frac{\Delta l}{l}$. This gives $F = \frac{YA}{l}\Delta l$. We see that this is of the form $F = k\,\Delta x$ with $k = \frac{YA}{l} = \frac{200\cdot 10^9 \cdot \pi \cdot (0.005)^2}{50\ m} = 3.14x10^5\ N/m$

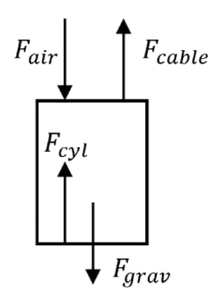
b) The total mass of the elevator and passenger is 500kg. What is equilibrium length of the cable and what is the oscillation frequency around this equilibrium point?

At the equilibrium length, $F_{net} = 0$, so $mg = k\,\Delta x$ where x is the amount the cable is stretched. Thus $x = \frac{mg}{k} = 0.0156\ m$. So, the equilibrium length of the cable is 50.0156 m. The oscillations frequency is $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.14x10^5\ N/m}{500\ kg}} = 25\ s^{-1}$ (angular frequency) or $f = \frac{\omega}{2\pi} = 3.98\ s^{-1}$.

Suppose that as a safety feature, the bottom of the elevator shaft acts as a cylinder of gas that is compressed adiabatically when the cable stretches longer than its normal length, as shown in the picture (first picture is with the cable unstretched, second picture is the equilibrium position). The area of the elevator bottom is 1 m^2 and the depth of the cylinder is 3 m. The outside air pressure and the uncompressed pressure of the gas in the cylinder are 100 kPa. For the gas, you can assume $\gamma = 1.4$



c) Defining x to be the height of the elevator above the bottom, find the net upward force on the elevator as a function of x. *(Hint: there are four forces you need to consider)*



The net force is $F_{net} = F_{cable} + F_{cyl} - F_{air} - F_{grav}$.
At height h, the stretching of the cable is $(3\,m - h)$, so $F_{cable} = k \cdot (3\,m - h) = 3.14\ x\ 10^5\,\frac{N}{m}(3\,m - h)$. We have $F_{grav} = m\,g = 4.91\ x\ 10^3 N$.

The force from the air on top is $F_{air} = P_o\,A = 10^5 Pa \cdot 1m^2 = 10^5 N$.

The force from the air in the cylinder is $F_{cyl} = P\,A$, but here the pressure depends on h. Since we have

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adiabatic compression, $PV^{\gamma} = constant$, so: $P_o(h_o \cdot A)^{\gamma} = P(h \cdot A)^{\gamma}$. This gives $P = P_o\left(\frac{h_o}{h}\right)^{\gamma}$. So, $F_{cyl} = P_oA\left(\frac{h_o}{h}\right)^{\gamma} = 10^5 N \cdot \left(\frac{3m}{h}\right)^{1.4}$.

Overall, $F_{net} = 3.14\ x\ 10^5 \frac{N}{m}(3\ m - h) + 10^5 N \cdot \left(\frac{3m}{h}\right)^{1.4} - 10^5 N - 4.91\ x\ 10^3 N$.

Hint: For the next two parts (d) and €, see the tips on the second page about finding k in a general system.

d) Sketch a graph of this net force as a function of x for positive values of x. Indicate the equilibrium position of the elevator on your graph.

e) What is the equilibrium value of x? *Hint: you won't be able to solve for the height by hand. You can get it by graphing, using a calculator, or Wolfram Alpha (www.wolframalpha.com: type in something like "solve x^2 + 1/x – 7 for x").*

Solving, $F_{net}(h) = 0$ using a computer, we find that $h_{eq} = 2.986\ m$.

f) What is the new oscillation frequency about this equilibrium point?

To find k for this system, we use $k = -\frac{dF_{net}}{dh}(h_{eq})$, the negative slop at the equilibrium position. Using our formula for F, we get $\frac{dF_{net}}{dh} = -3.14\ x\ 10^5 \frac{N}{m} - 4.66 \cdot 10^5 N/m \cdot 1.4 \cdot \left(\frac{1m}{h}\right)^{2.4}$. Plugging in $h = h_{eq} = 2.986\ m$, we get $\frac{dF_{net}}{dh}(h = h_{eq}) = -3.61\ x\ 10^5 \frac{N}{m}$. So, $k = 3.61\ x\ 10^5 \frac{N}{m}$, $\omega = \sqrt{\frac{k}{m}} = 26.9\ s^{-1}$ (angular frequency) and $f = \frac{\omega}{2\pi} = 4.28\ s^{-1}$

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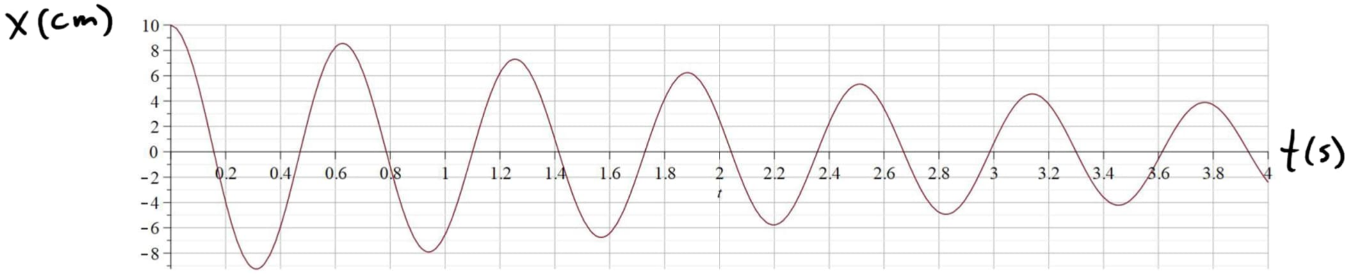
Problem 3: The graph shows the displacement vs time for an object of mass 2kg oscillating on a spring.



a) Assuming that we can represent the motion of this object by $x(t) = A\, e^{-t/\tau} \cos(\omega t + \varphi)$, what is the time constant τ?

We can use the peak at t = 3.7 s. At this time, the amplitude has decayed to 4 cm. So $10\text{ cm}\cdot e^{-3.7s/\tau} = 4$ cm, so $e^{-3.7s/\tau} = 0.4$, and $3.7s/\tau = -\ln(0.4)$ and $\tau = -3.7\text{ s}/\ln(0.4) \sim 4.1$ s.

b) What is the frequency ω?

We see that the period is T = 0.63 s, so $\omega = 2\pi/0.63s = 10\ s^{-1}$.

c) What is the initial amplitude A?

We have A = 10 cm.

d) What is the damping constant b?

We have $\tau = 2m/b$, so $b = 2m/\tau = 4\text{ kg}/4.1\text{s} \sim 1$ kg/s.

e) What is the spring constant k?

We can use $\omega = \sqrt{\frac{k}{m}}$ since the damping isn't large, so $k = m\,\omega^2 = 2\text{ kg}\cdot(10\ s^{-1})^2 = 200$ N/m

f) What is the fraction of energy lost in one complete oscillation?

We have $E_1/E_0 = (1/2\ k\ (x(t_1))^2) / (1/2\ k\ (x(t_2))^2) = (8.5\text{ cm}/ 10\text{ cm})^2 = 0.72$. So a fraction of 0.28 of the energy is lost in one oscillation.

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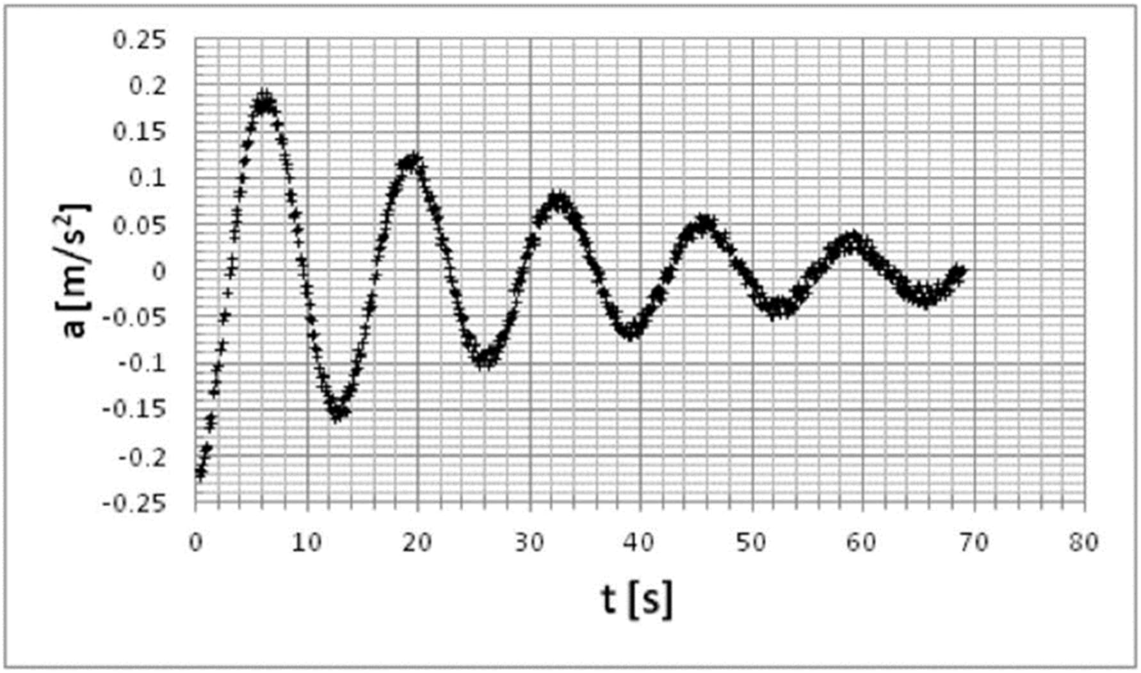
Problem 4: One day, a large drilling platform ($M = 9\times10^7$ kg) is floating in the ocean minding its own business when a tsunami passes by. The upward vertical acceleration vs time for the platform after the tsunami passes is measured by an accelerometer and plotted in the graph below.



a) What is the damping constant b for this oscillation?

The amplitude of acceleration falls off in the same way as the amplitude for displacement, since both are equalt to $e^{-t/\tau}$ times some sinusoidal function. We can get τ by using $a = 0.12$ m/s^2 at $t = 19.5$ s. Thus, 0.12 m/s^2 $= 0.225$ m/s^2 x $e^{-19.5s/\tau}$, so $\tau \sim 31$ s. Thus, $b = 2m/\tau \sim 5.8\text{x}10^6$ kg/s.

b) What was the initial vertical displacement of the platform (give a positive answer if it was upwards and a negative answer if it was downwards)?

We see $T \sim 13$ s, so $\omega = 2\pi/T \sim 0.486\text{s}^{-1}$. Initially, $v = 0$, so $ma = -kx$ and $x = -(m/k)a = -a/\omega^2 \sim 0.95$ m.