Teo. Doppio limite.

Sia $f_n: (a,b) \to \mathbb{R}$ una successione di funzioni limitate tali che

1) $f_n \to f$ uniformemente

2) $\lim_{x \to x_0} f_n(x) = \ell_n$. $\forall n$

Allora $\lim_{n \to +\infty} \ell_n = \ell$

Inoltre $\lim_{x \to x_0} f(x) = \ell$

$$\lim_{n \to +\infty} \lim_{x \to x_0} f_n(x) = \lim_{x \to x_0} \lim_{n \to +\infty} f_n(x)$$

$$|\ell_n - \ell_m| < \varepsilon \qquad \forall n, m >$$

$$|\ell_n - \ell_m| = |\ell_n - f_n(x) + f_n(x) - f_m(x) + f_m(x) - \ell_m|$$

$$\leq |\ell_n - f_n(x)| + |f_n(x) - f_m(x)| \; |f_m(x) - \ell_m|$$

$$\underset{\to 0}{} \qquad \underset{<\varepsilon}{} \qquad \underset{\to 0}{}$$

$$\ell_n \to \ell$$

$$|f(x) - \ell| = |f(x) - f_n(x) + f_n(x) - \ell_n + \ell_n - \ell|$$

$$\leq \underbrace{|f(x) - f_n(x)|}_{<\varepsilon} + \underbrace{|f_n(x) - \ell_n|}_{<\varepsilon} + \underbrace{|\ell_n - \ell|}_{<\varepsilon}$$

$$< 3\varepsilon$$

$C([a,b])$ $\qquad a < b$

$=$ funzioni continue $f: [a,b] \to \mathbb{R}$

$$\|f\| = \max_{x \in [a,b]} |f(x)|$$

$f_n \in C([a,b])$

$\forall \varepsilon > 0 \ \exists N \quad \|f_n - f_m\| < \varepsilon \quad \forall n,m > N$

$f_n \to f \in C([a,b]) \qquad \|f_n - f\| \to 0$

$\lim_{x \to x_0} f_n(x) = f_n(x_0) \qquad \forall x_0 \in [a,b]$

$$\lim_{x \to x_0} f(x) = \lim_{n \to +\infty} \lim_{x \to x_0} f_n(x)$$

$$= \lim_{n \to +\infty} f_n(x_0) = f(x_0)$$

Teo. Limite uniforme di funzioni continue è continuo.

$f_n \in C([a,b])$

Puntuale. $\forall x \in [a,b] \ \lim_{n \to +\infty} f_n(x) = f(x)$

$\forall x \ \forall \varepsilon > 0 \ \exists N \quad |f_n(x) - f_m(x)| < \varepsilon \quad \forall n,m \geq N$

Uniforme $\forall \varepsilon \; \exists N \quad \max_{x\in[a,b]} |f_n(x) - f_m(x)| < \varepsilon$

$\forall n, m \geq N$

$\forall \varepsilon \; \exists N \quad |f_n(x) - f_m(x)| < \varepsilon \quad \forall x \quad \forall n,m \geq N$

Se f_n è di Cauchy rispetto alla norma uniforme, allora $f_n \to f$ rispetto alla stessa norma.

$f_n \to f$ puntualmente.

$\forall \varepsilon > 0 \; \exists N \quad |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N$
$\forall x$

$f_n \to f$ uniformemente

f è continua $\quad f \in C([a,b])$

Teo. $C([a,b])$ con la norma uniforme

è uno spazio di Banach.

$$\|f\|_1 = \int_a^b |f(x)|\, dx$$

$C([a,b])$ $\| \ \|_1$

$f_n(x) = x^n$ $\|f_n\|_1 \to 0$

$[0,1]$

$$\int_0^1 x^n\, dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 \to 0$$

$f(x) = 0$ $\|f_n - f\| \to 0$

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$

$$\|f\|_1 = \int^b |f(x)|\, dx$$

$$\|f\|_p = \int_a^b |f(x)|^p \, dx \qquad p \geq 1$$

Teo. Sia $f_n \in C([a,b])$, sia $f_n \to f$ uniformemente.

Allora $\displaystyle\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$

$$\lim_{n\to+\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to+\infty} f_n(x)\,dx$$

$f_n \to f$ $\qquad \cancel{\frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to+\infty} \frac{d}{dx} f_n(x)}$

Teo. Siano $f_n : (a,b) \to \mathbb{R}$ derivabili

1) $f_n' \to g$ uniformemente

2) $\exists x_0 \in (a,b) \quad f_n(x_0) \to \ell$

Allora $f_n \to f$ uniformemente e

$$f'(x) = g(x) \quad \forall x \in (a,b)$$

$$\frac{d}{dx}\left(\lim_{n\to+\infty} f_n(x)\right) = \lim_{n\to+\infty}\left(\frac{d}{dx} f_n(x)\right)$$

$f_n' \to g$ unif.

$h_n = f_n + c_n \qquad h_n' = f_n'$

Serie di funzioni.

$f_n : (a,b) \to \mathbb{R}$

$$\sum_{n=0}^{\infty} f_n(x) = s(x)$$

$$s_k(x) = \sum^{k} f_n(x)$$

$n=0$

Se f_n sono continue e $\sum f_n$ converge uniformemente, allora $s(x)$ è continua.

Se $\sum f_n$ continue che converge uniformemente e f_n sono integrabili allora

$$\int_a^b \sum_{n=0}^{\infty} f_n(x)\, dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\, dx$$

Teo. Siano $f_n : (a,b) \to \mathbb{R}$ derivabili e tali che

$$\sum_{n=0}^{\infty} f_n'(x) = t(x) \text{ uniformemente}$$

e $\exists x_0$ $\sum_{n=0}^{\infty} f_n(x_0) = s$.

Allora $\sum_{n=0}^{\infty} f_n(x) = s(x)$ uniformemente

$s'(x) = t(x)$.

$$\sum_{n=0} f_n'(x) = \frac{d}{dx}\left(\sum_{n=0} f_n(x)\right)$$

$$\sum_{n=0}^{\infty} f_n(x) = s(x) \qquad \text{puntualmente}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x|<1$$

Criterio di Weierstrass

Sia $f_n : (a,b) \to \mathbb{R}$.

Sia $C_n = \sup_{x\in(a,b)} |f_n(x)|$.

Se $\sum_{n=0}^{\infty} C_n$ converge, allora $\sum_{n=0}^{\infty} f_n(x)$ converge uniformemente.

$f_n \to f \qquad \sum f_n = s$

Def. Se $\sum c_n$ converge diciamo che $\sum_{n=0}^{\infty} f_n(x)$ converge totalmente.

Dim. $\sum_{n=0}^{\infty} c_n$ converge.

$$S_K = \sum_{n=0}^{K} c_n \qquad S_K \to S$$

S_K è di Cauchy

$$\forall \varepsilon > 0 \ \exists N \quad |S_K - S_m| < \varepsilon \quad \forall K, m \geq N$$
$$K > m$$

$$S_K - S_m = c_K + c_{K-1} + c_{K-2} + \dots + c_{m+1}$$

$$c_K = \sup_{x \in (a,b)} |f_K(x)| \qquad |f_K(x)| \leq c_K \quad \forall x$$

$$\varepsilon > S_K - S_m \geq |f_K(x)| + |f_{K-1}(x)| + \dots + |f_{m+1}(x)|$$

$$\geq |f_K(x) + f_{K-1}(x) + \dots + f_{m+1}(x)| \quad \forall x$$

$$|f_K(x) + f_{K-1}(x) + \dots + f_{m+1}(x)| < \varepsilon \quad \forall x$$

$$|S_k(x) - S_m(x)| < \varepsilon \quad \forall x$$

$\Rightarrow S_k \to S$ uniformemente

Serie di potenze

Def. Data una successione $\{a_n\}$, chiamiamo SERIE DI POTENZE con coefficienti a_n, centrata in $x_0 \in \mathbb{R}$:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$z_0 \in \mathbb{C}$$

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad z \in \mathbb{C}$$

$$w = z - z_0 \qquad \sum_{n=0}^{\infty} a_n w^n$$

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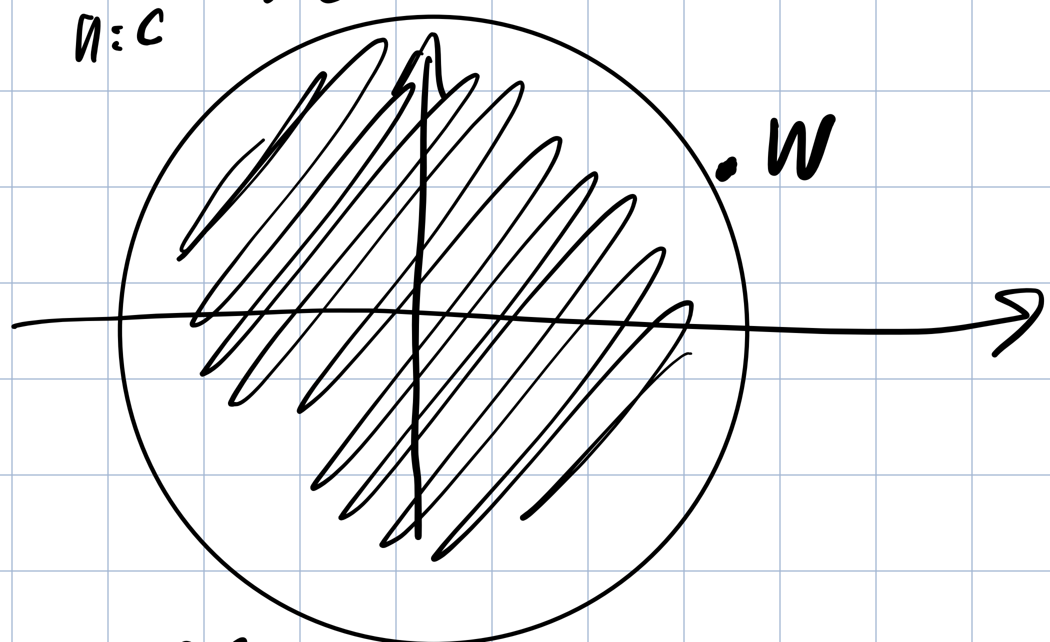
Lemma Sia $\sum_{n=0} a_n z^n$. Se $\exists w \neq 0$ tale che $\sum_{n=0}^{\infty} a_n w^n$ converge, allora la serie $\sum_{n=0}^{\infty} |a_n z^n|$ converge $\forall z \in \mathbb{C}$, $|z| < |w|$.

$$\sum_{n=0}^{\infty} a_n 0^n = a_0 \qquad 0^0 = 1$$



Dim. $\sum_{n=0}^{\infty} a_n w^n = W_0 \qquad a_n w^n \to 0$

$|a_n w^n| < 1$ definitivamente

$$\sum_{n=0}^{\infty} |a_n z^n| < +\infty$$

$n=0$

$$|a_n z^n| = \left|a_n w^n \frac{z^n}{w^n}\right| =$$

$$|a_n w^n| \left|\frac{z}{w}\right|^n < \left|\frac{z}{w}\right|^n \quad \text{def.}$$

$$|z|<|w| \qquad \left|\frac{z}{w}\right|<1 \qquad \sum_{n=0}^{\infty} \left|\frac{z}{w}\right|^n$$

Def. Data $\sum_{n=0}^{\infty} a_n z^n$ chiamo RAGGIO di convergenza della serie

$$R = \sup\left\{|z|, \sum_{n=0}^{\infty} a_n z^n \text{ converge}\right\}$$

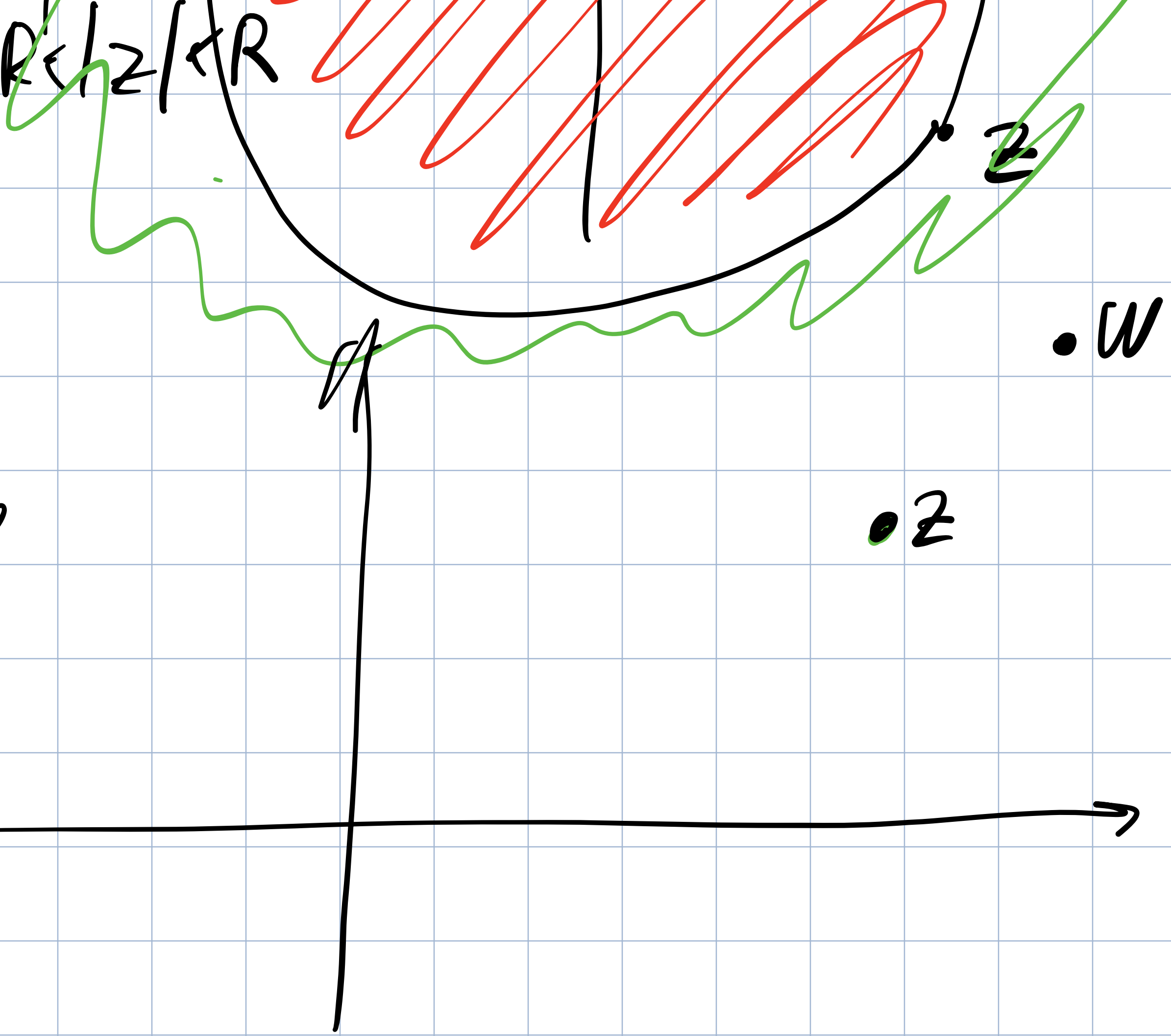
$R=0 \iff$ serie converge solo in $z=0$

$R=+\infty$

$R\in(0,+\infty)$

$R' < R \quad R' < |z| < R$

$|w| > R$



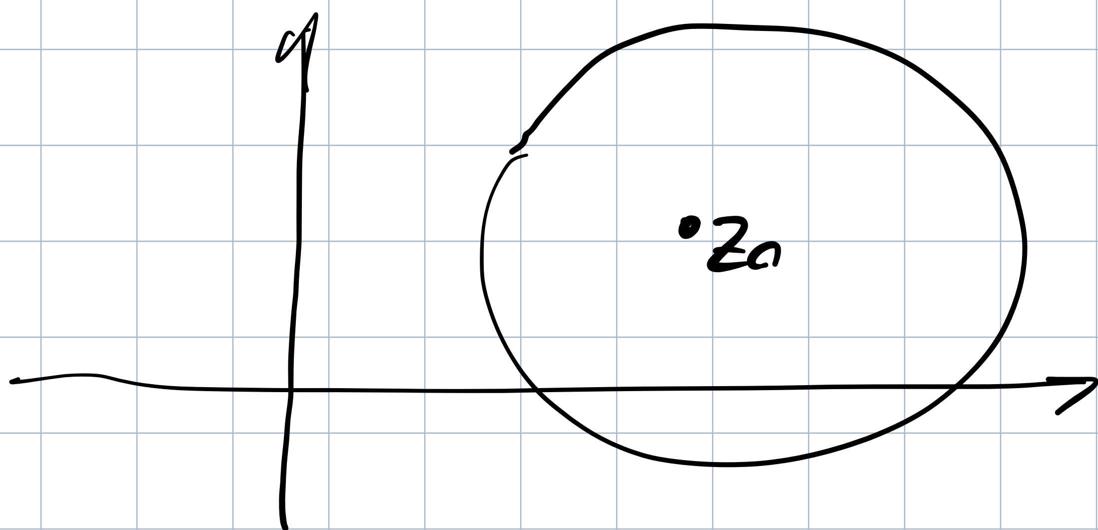
$R = +\infty$

Teo. Data $\sum_{n=0}^{\infty} a_n z^n$, $\exists R \in [0, +\infty]$ tale che 1) Se $R=0$ la serie converge solo nel centro $(z=0)$.

2) Se $R = +\infty$ la serie converge assolutamente (e quindi semplicemente) $\forall z \in \mathbb{C}$.

3) Se $R \in (0, +\infty)$, allora la serie converge assolutamente in ogni $|z| < R$,

non converge $|z| > R$.



$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

$|z| < 1$

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

$z = 1$ non conv.

$z = -1$ conv.

$R = 1$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

$|z| > 1$

$|z| < 1$

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n^2} < +\infty$$

Criterio della radice

Data $\sum_{n=0}^{\infty} a_n z^n$, sia

$$\ell = \limsup \sqrt[n]{|a_n|}.$$

Allora $R = \frac{1}{\ell}$ $\left(R = +\infty \text{ se } \ell = 0, \; R = 0 \text{ se } \ell = +\infty \right)$

$|z| > \frac{1}{\ell}$ la serie diverge

$|z| < \frac{1}{\ell}$ la serie converge

$\sum_{n=0}^{\infty} a_n z^n$ $\quad \limsup \sqrt[n]{|a_n z^n|} > 1$

$$\limsup \sqrt[n]{|a_n|} = \ell$$

$$\sqrt[n]{|a_n z^n|} > \sqrt[n]{|a_n| \frac{1}{\ell^n}} = \frac{\sqrt[n]{|a_n|}}{\ell}$$

$$\limsup \sqrt[n]{|a_n z^n|} > \ell \cdot \frac{1}{\ell} = 1$$

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Criterio del rapporto

$$\sum_{n=0}^{\infty} a_n z^n \qquad \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ell$$

$$R = \frac{1}{\ell} \qquad \begin{array}{l} R=0 \text{ se } \ell=+\infty \\ R=+\infty \text{ se } \ell=0 \end{array}$$

$$\ell \in (0,+\infty) \qquad |z| < \frac{1}{\ell}$$

$$\lim_{n\to+\infty} \left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| |z|$$

$$< \left(\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|\right) \frac{1}{\ell} = 1$$

Convergenza uniforme

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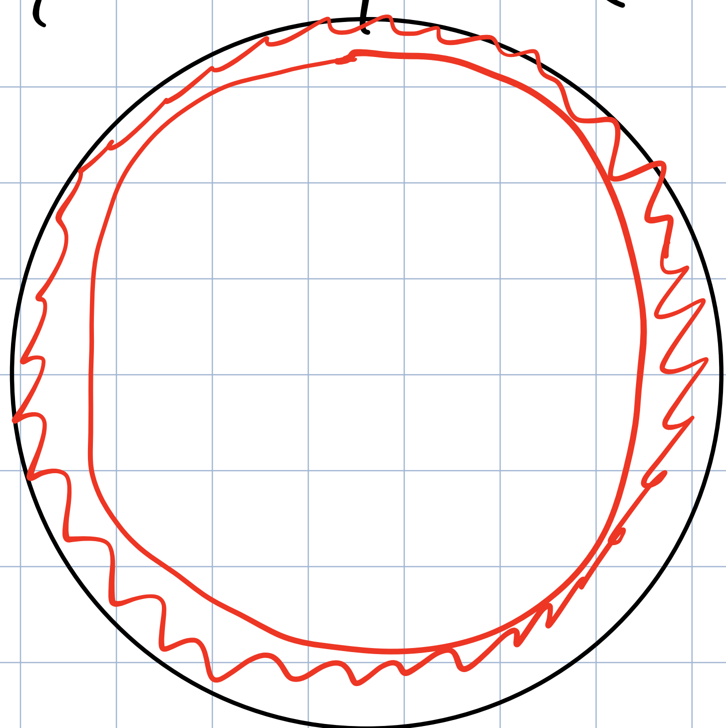
Teo. Data una serie $\sum_{n=0} a_n z^n$ di raggio $R>0$, sia $0<R'<R$. Allora la serie converge uniform. in $\{z \in \mathbb{C} \mid |z| \le R'\}$.

$|z|<R$



$\sum_{n=0}^{\infty} a_n z^n$ $\quad R \in (0, +\infty)$

$|z|<R$ la serie converge

$R'<R$ $\quad A = \{ z \in \mathbb{C}, \ |z| \le R' \}$

$$C_n = \max_{z \in A} |a_n z^n| \qquad \sum_{n=0} C_n < +\infty$$

$$R' < |w| < R \qquad \sum_{n=0}^{\infty} |a_n w^n| \text{ converge}$$

$|a_n w^n| < 1$ definitivamente

$$|a_n z^n| = |a_n w^n| \left|\frac{z^n}{w^n}\right| < \left|\frac{z}{w}\right|^n$$

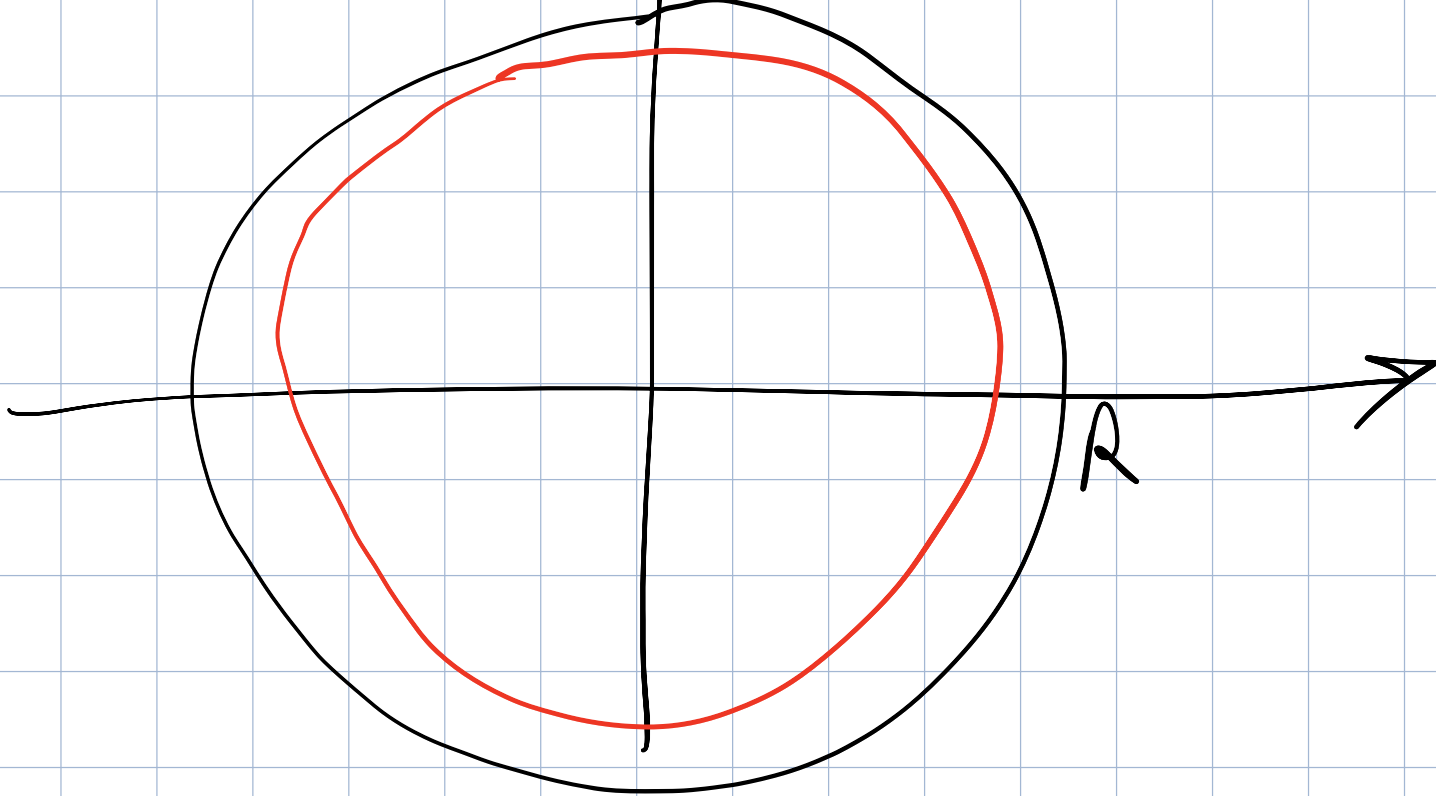
$$R'' = |w| > R'$$
$$|z| \leq R'$$

$$|a_n z^n| < \left|\frac{z}{w}\right|^n$$

$$C_n \max_{z \in A} |a_n z^n| < \max_{z \in A} \left|\frac{z}{w}\right|^n < \left(\frac{R'}{R''}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{R'}{R''}\right)^n \text{ converge}$$

1



$\forall x \in (a,b) \qquad f(x)$

$$\lim_{n\to+\infty} f_n(x) = f(x)$$

$\forall \varepsilon > 0 \quad \exists N \qquad \underline{|f_n(x) - f(x)| < \varepsilon \quad \forall x} \qquad \forall n \geq N$

$$\max_{x \in A} |f_n(x) - f(x)|$$