Parallel axes theorem

For a given inertia matrix w.r.t one reference frame, the inertia matrix for a different frame can be determined in two cases:

- Case – 1 (Parallel axis theorem)

A second reference frame, having a different origin from the first, but where the respective axes of the reference frames are parallel.

- Case – 2 (Principal directions and Principal moments of inertia)

A second reference frame, having the same origin as the first, but rotated w.r.t the first reference frame.

Case 1

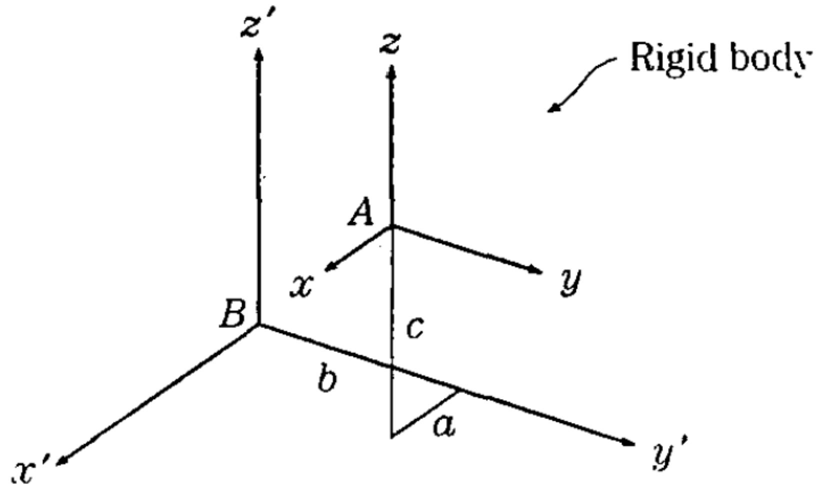
Useful for calculating the inertia matrix of the same body with respect to different reference frames having respective parallel axes and the inertia matrix of composite bodies.

Two reference frames are attached to the rigid body one with origin A, other with origin B.

Axes of the reference frames are parallel to each other.

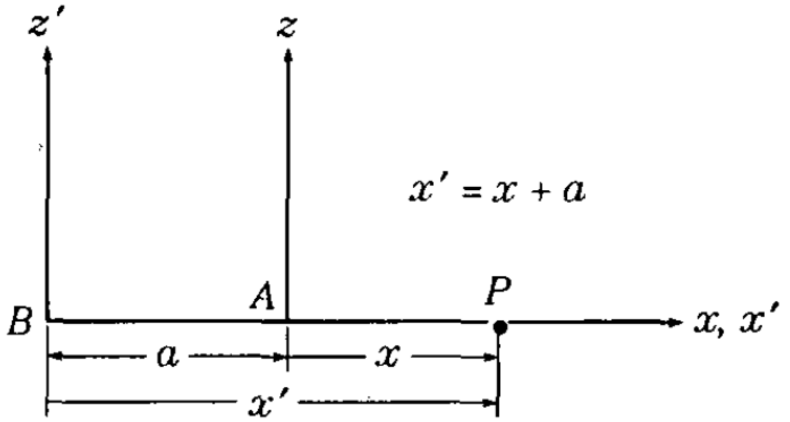
Points A and B are displaced w.r.t each other such that the transformations between their respective axes are

$$x' = x + a \qquad y' = y + b \qquad z' = z + c$$



Two sets of reference frames with parallel axes origins at A and B, attached to same rigid body.

Location of an arbitrary point P in $Axyz$ is simply x, while location of P in $Bx'y'z'$ is x', which is equal to $x + a$.



Given the inertia tensor $[I_A]$ for the $Axyz$ reference frame, find the inertia tensor $[I_B]$ for the $Bx'y'z'$ reference frame where the subscripts A and B denote the origins of the respective frames.

Moment of inertia, $I_{z'z'}$

$$
\begin{aligned}
I_{z'z'} &= \sum_{i=1}^{N} m_i ({x'_i}^2 + {y'_i}^2) \\
&= \sum_{i=1}^{N} m_i \left[(x_i + a)^2 + (y_i + b)^2 \right] \\
&= \sum_{i=1}^{N} m_i \left(x_i^2 + 2ax_i + a^2 + y_i^2 + 2by_i + b^2 \right) \\
&= \sum_{i=1}^{N} m_i \left(x_i^2 + y_i^2 \right) + \sum_{i=1}^{N} m_i \left(a^2 + b^2 \right) + \sum_{i=1}^{N} 2m_i (ax_i + by_i)
\end{aligned}
$$

Total mass of the body

$$M = \sum_{i=1}^{N} m_i$$

Coordinates of the center of mass of the body

$$x_c = \frac{\sum_{i=1}^{N} m_i x_i}{M} \qquad y_c = \frac{\sum_{i=1}^{N} m_i y_i}{M} \qquad z_c = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

$$I_{z'z'} = \underbrace{\boxed{\sum_{i=1}^{N} m_i \left(x_i^2 + y_i^2\right)}}_{I_{zz}} + \underbrace{\boxed{\sum_{i=1}^{N} m_i \left(a^2 + b^2\right)}}_{M\left(a^2 + b^2\right)} + \underbrace{\boxed{\sum_{i=1}^{N} 2 m_i \left(a x_i + b y_i\right)}}_{2M\left(a x_c + b y_c\right)}$$

$$I_{z'z'} = I_{zz} + M\left(a^2 + b^2\right) + 2M\left(a x_c + b y_c\right)$$

Product of inertia $I_{x'y'}$

$$
\begin{aligned}
I_{x'y'} &= -\sum_{i=1}^{N} m_i x_i' y_i' \\
&= -\sum_{i=1}^{N} m_i (x_i y_i + ab + b x_i + a y_i) \\
&= I_{xy} - Mab - M(b x_c + a y_c)
\end{aligned}
$$

Moments of inertia

$$I_{x'x'} = I_{xx} + M(b^2 + c^2) + 2M(by_c + cz_c)$$

$$I_{y'y'} = I_{yy} + M(a^2 + c^2) + 2M(ax_c + cz_c)$$

$$I_{z'z'} = I_{zz} + M(a^2 + b^2) + 2M(ax_c + by_c)$$

Products of inertia

$$I_{x'y'} = I_{y'x'} = I_{xy} - Mab - M(bx_c + ay_c)$$

$$I_{y'z'} = I_{z'y'} = I_{yz} - Mbc - M(cy_c + bz_c)$$

$$I_{z'x'} = I_{x'z'} = I_{xz} - Mac - M(az_c + cx_c)$$

In matrix form

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}_B = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}_A + M \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & a^2 + c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix}$$

$$+ M \begin{bmatrix} 2(by_c + cz_c) & -(bx_c + ay_c) & -(cx_c + az_c) \\ -(bx_c + ay_c) & 2(cz_c + ax_c) & -(cy_c + bz_c) \\ -(cx_c + az_c) & -(cy_c + bz_c) & 2(ax_c + by_c) \end{bmatrix}$$

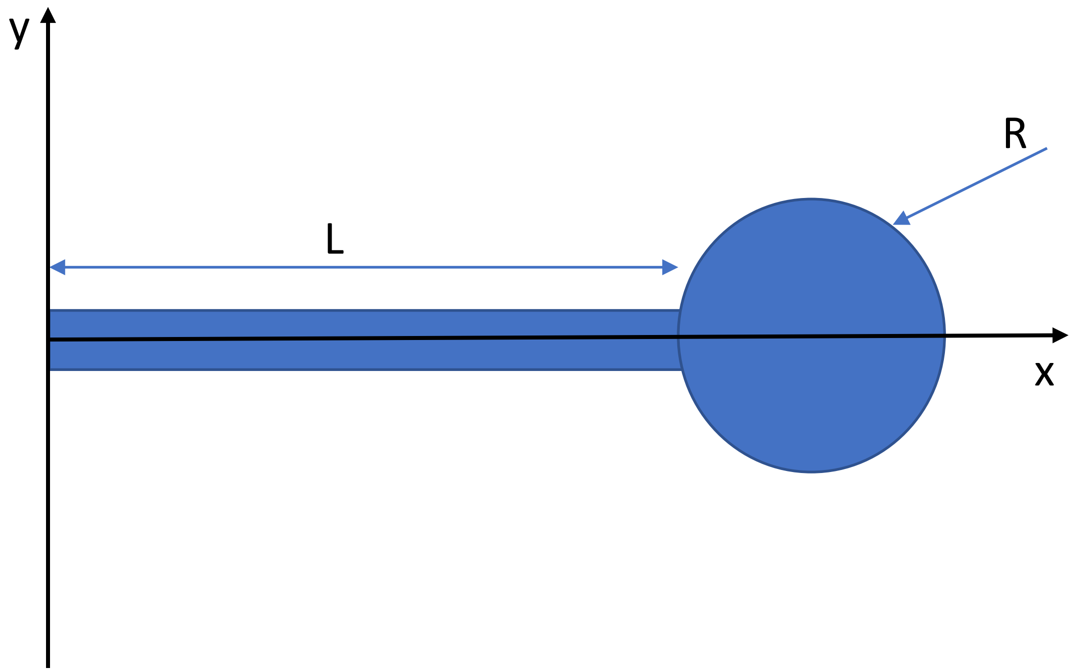
General form of the parallel axes theorem

If point A is at the center of mass of the rigid body

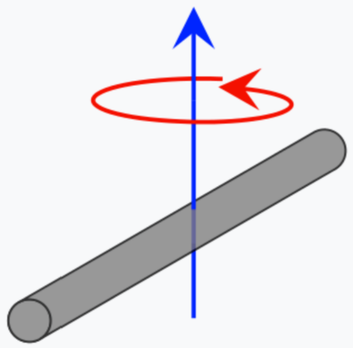
$$x_C = y_C = z_C = 0$$

$$[I]_B = [I]_C + M\begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & c^2 + a^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix}$$

Commonly expressed form of the parallel axes theorem.



A pendulum is made of a thin rod of length L, mass m and a sphere of radius R, mass M. Find the mass moment of inertia of the pendulum about y-axis.

Thin rod of length *L* and mass *m*, perpendicular to the axis of rotation, rotating about its center. This expression assumes that the rod is an infinitely thin (but rigid) wire. This is a special case of the thin rectangular plate with axis of rotation at the center of the plate, with *w* = *L* and *h* = 0.		$I_{\text{center}} = \frac{1}{12}mL^2$