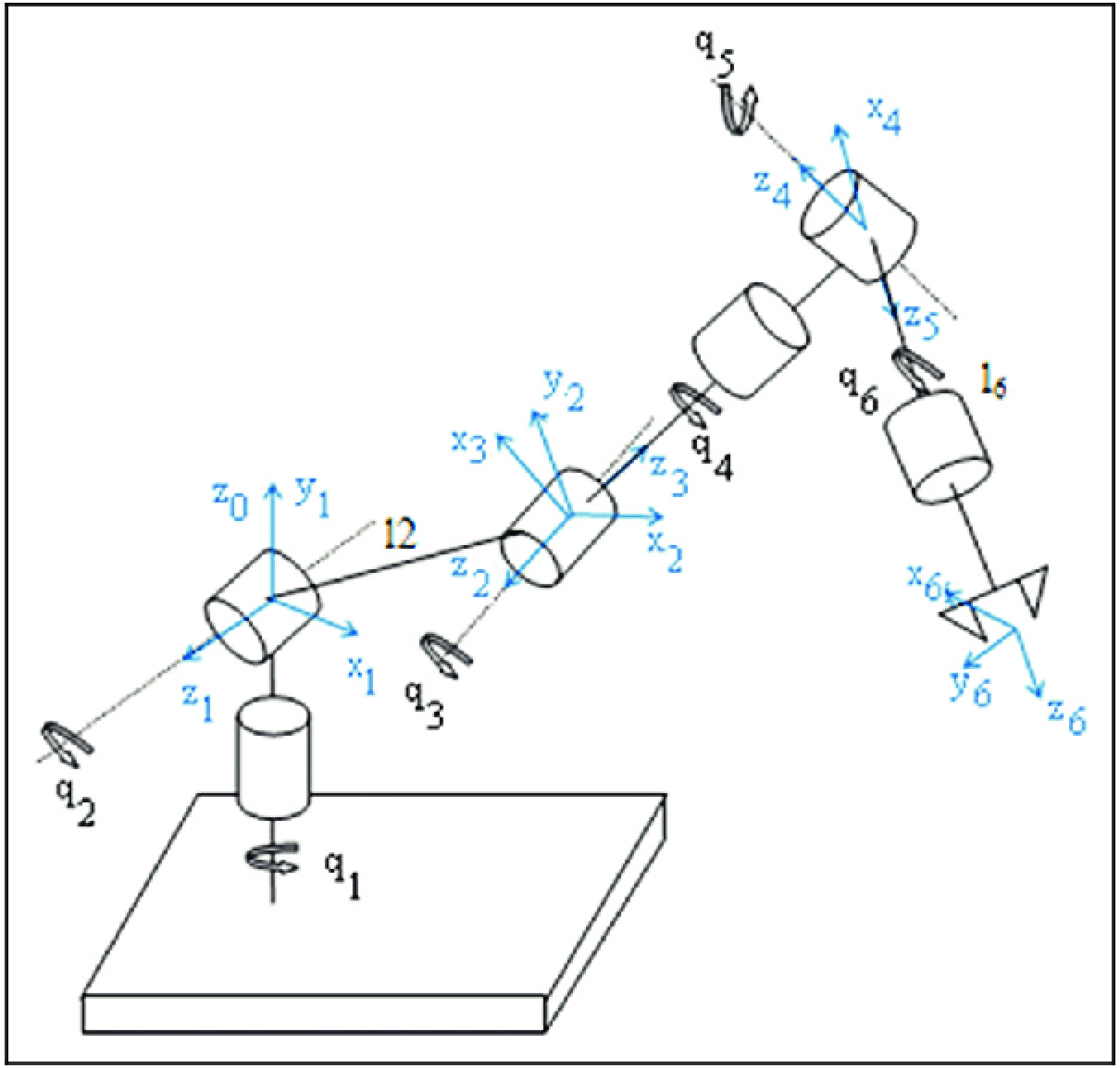
Coordinate frames, Mapping and Transformation matrix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y6
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z6
q2
q1

Coordinate frames

- Three orthogonal right handed axes X, Y, Z called *principal axes*.
- The origin of principal axes at O along with three unit vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ along these axes.
- The frame is labelled as $\{x\ y\ z\}$ or by a number as $\{1\}$ using a numbering scheme.
- Any point **P** can be defined w.r.t this coordinate frame by a vector $\overrightarrow{OP}$ (a directed line from origin to point P pointing towards P).

$$\boldsymbol{P} = \overrightarrow{OP} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z}$$

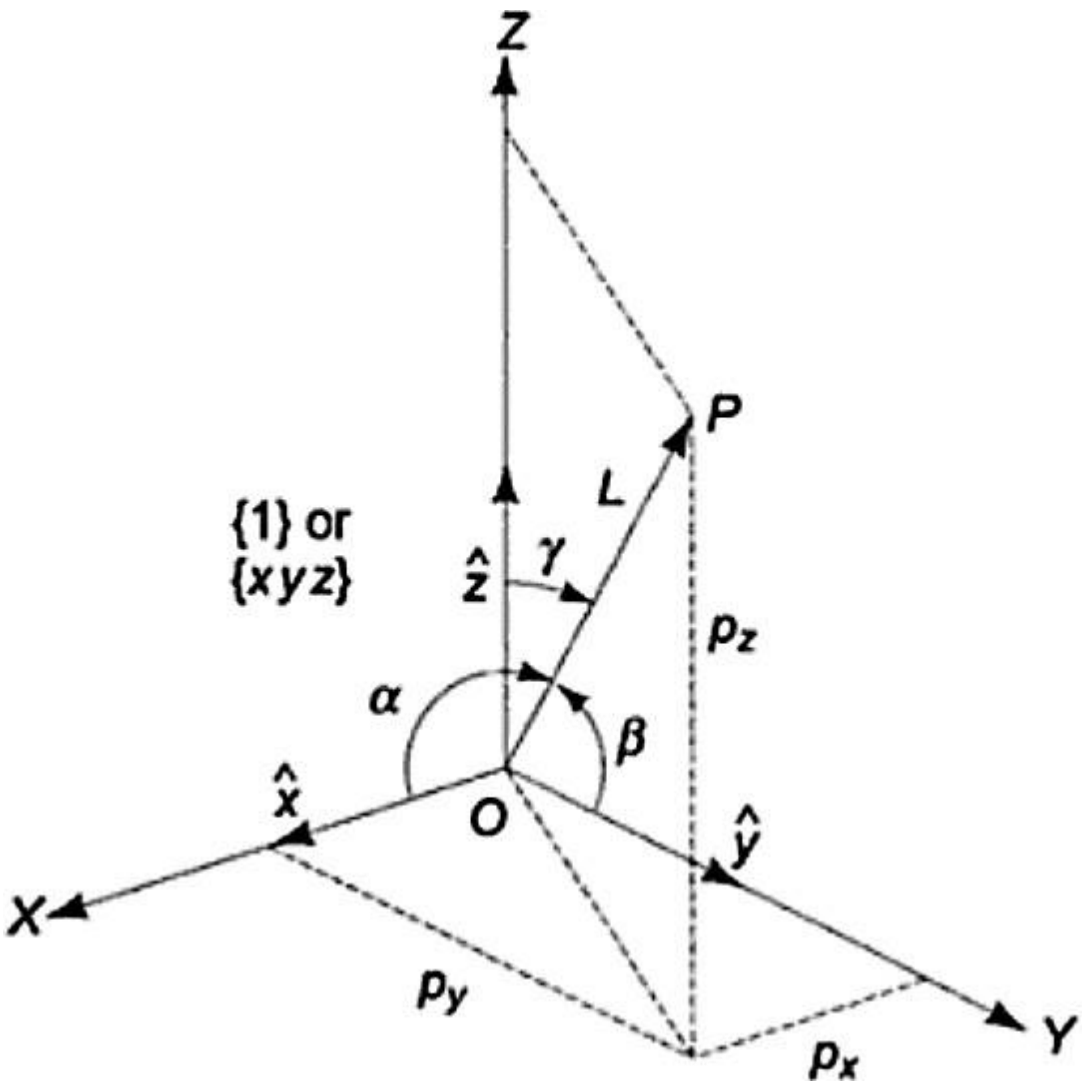

Figure 1: Position and orientation of a point P in a coordinate frame.

Frame – space notation

$$ {}^{1}\boldsymbol{P} = {}^{1}p_x\boldsymbol{x} + {}^{1}p_y\boldsymbol{y} + {}^{1}p_z\boldsymbol{z} $$

Vector – matrix notation

$$ {}^{1}P = \begin{bmatrix} {}^{1}p_x \\ {}^{1}p_y \\ {}^{1}p_z \end{bmatrix} $$

Leading superscript refers to the coordinate frame number.

Mapping

- Mapping refer to changing the description of a point (or vector) in space from one frame to another frame.

- The second frame has three possibilities in relation to the first frame:
  - Second frame is rotated w.r.t the first; the origin of both the frames is same.
    - changing orientation
  - Second frame is moved away from the first, the axes of both frames remain parallel, respectively.
    - translation of the origin of the second frame from the first
  - Second frame is rotated w.r.t the first and moved away from it.
    - second frame is translated and its orientation is also changed

Mapping between rotated frames.

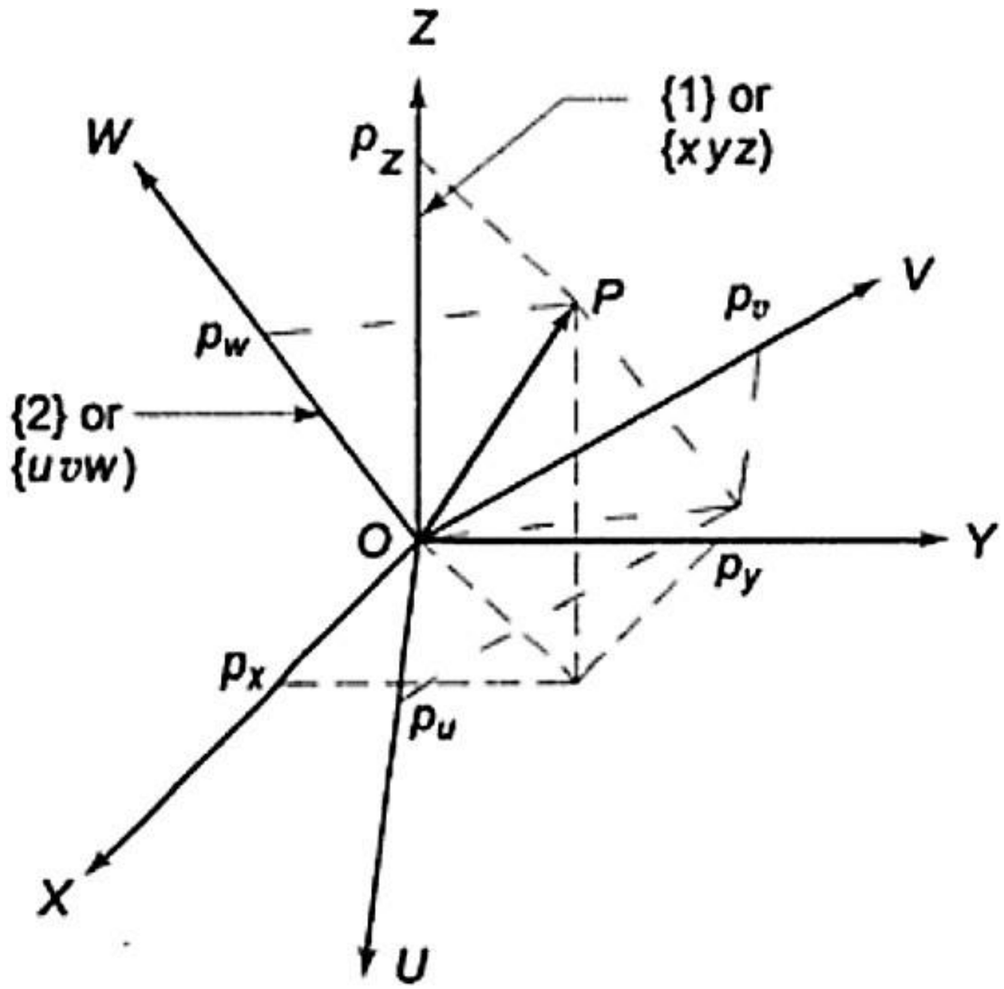

Figure 2: Representation of a point P in two frames {1} and {2} rotated w.r.t each other.

Consider two frames, frame {1} with axes X, Y, Z and frame {2} with axes U, V, W with a common origin.

A point P in space can be described by the two frames as vectors ${}^{1}\boldsymbol{P}$, ${}^{2}\boldsymbol{P}$.

$$ {}^{1}\boldsymbol{P} = {}^{1}p_x\boldsymbol{x} + {}^{1}p_y\boldsymbol{y} + {}^{1}p_z\boldsymbol{z} $$

$$ {}^{2}\boldsymbol{P} = {}^{2}p_u\boldsymbol{u} + {}^{2}p_v\boldsymbol{v} + {}^{2}p_w\boldsymbol{w} $$

Question

The description of point P in frame {2} is known and its description in frame {1} is to be found (or vice-versa).

Solution

Projecting the vector ${}^{2}\boldsymbol{P}$ on to the coordinates of frame {1}.

Projections of ${}^{2}\boldsymbol{P}$ on frame {1} are obtained by taking the dot product of ${}^{2}\boldsymbol{P}$ with the unit vectors of frame {1}.

$$
\begin{aligned}
{}^{1}p_x &= \hat{\boldsymbol{x}} \cdot {}^{2}\boldsymbol{P} = \hat{\boldsymbol{x}} \cdot {}^{2}p_u \hat{\boldsymbol{u}} + \hat{\boldsymbol{x}} \cdot {}^{2}p_v \hat{\boldsymbol{v}} + \hat{\boldsymbol{x}} \cdot {}^{2}p_w \hat{\boldsymbol{w}} \\
{}^{1}p_y &= \hat{\boldsymbol{y}} \cdot {}^{2}\boldsymbol{P} = \hat{\boldsymbol{y}} \cdot {}^{2}p_u \hat{\boldsymbol{u}} + \hat{\boldsymbol{y}} \cdot {}^{2}p_v \hat{\boldsymbol{v}} + \hat{\boldsymbol{y}} \cdot {}^{2}p_w \hat{\boldsymbol{w}} \\
{}^{1}p_z &= \hat{\boldsymbol{z}} \cdot {}^{2}\boldsymbol{P} = \hat{\boldsymbol{z}} \cdot {}^{2}p_u \hat{\boldsymbol{u}} + \hat{\boldsymbol{z}} \cdot {}^{2}p_v \hat{\boldsymbol{v}} + \hat{\boldsymbol{z}} \cdot {}^{2}p_w \hat{\boldsymbol{w}}
\end{aligned}
$$

In matrix form

$$\begin{bmatrix} {}^1P_x \\ {}^1P_y \\ {}^1P_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}\cdot\boldsymbol{u} & \boldsymbol{x}\cdot\boldsymbol{v} & \boldsymbol{x}\cdot\boldsymbol{w} \\ \boldsymbol{y}\cdot\boldsymbol{u} & \boldsymbol{y}\cdot\boldsymbol{v} & \boldsymbol{y}\cdot\boldsymbol{w} \\ \boldsymbol{z}\cdot\boldsymbol{u} & \boldsymbol{z}\cdot\boldsymbol{v} & \boldsymbol{z}\cdot\boldsymbol{w} \end{bmatrix} \begin{bmatrix} {}^2P_u \\ {}^2P_v \\ {}^2P_w \end{bmatrix}$$

In compressed vector-matrix notation

$${}^1\boldsymbol{P} = {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{P} \qquad \text{Eq (1)}$$

Where

$${}^1\boldsymbol{R}_2 = \begin{bmatrix} \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{w}} \end{bmatrix}$$

$\boldsymbol{R}$ is called *rotation matrix* or *rotational transformation matrix*.

It contains only the dot products of unit vectors of the two frames and is independent of the point P.

Rotation matrix ${}^{1}\boldsymbol{R}_{2}$ can be used for transformation of the coordinates of any point P in frame {2} (which is rotated w.r.t frame {1}) to frame {1}.

Rotation matrix ${}^{2}\boldsymbol{R}_1$

$$ {}^{2}R_1 = \begin{bmatrix} \boldsymbol{u}\cdot\boldsymbol{x} & \boldsymbol{u}\cdot\boldsymbol{y} & \boldsymbol{u}\cdot\boldsymbol{z} \\ \boldsymbol{v}\cdot\boldsymbol{x} & \boldsymbol{v}\cdot\boldsymbol{y} & \boldsymbol{v}\cdot\boldsymbol{z} \\ \boldsymbol{w}\cdot\boldsymbol{x} & \boldsymbol{w}\cdot\boldsymbol{y} & \boldsymbol{w}\cdot\boldsymbol{z} \end{bmatrix} $$

Point P in frame {1} is transformed to frame {2},

$$ {}^{2}\boldsymbol{P} = {}^{2}\boldsymbol{R}_1\, {}^{1}\boldsymbol{P} \qquad \text{Eq (2)} $$

As the vector dot product is commutative,

$$ {}^{2}\boldsymbol{R}_1 = \left[\, {}^{1}\boldsymbol{R}_2 \right]^T $$

$$ {}^{2}\boldsymbol{P} = \left[\, {}^{1}\boldsymbol{R}_2 \right]^T\, {}^{1}\boldsymbol{P} \qquad \text{Eq (3)} $$

Multiplying Eq(1) by $\left[\,{}^{1}\boldsymbol{R}_{2}\right]^{-1}$ in both sides

$$\left[\,{}^{1}\boldsymbol{R}_{2}\right]^{-1}\ {}^{1}\boldsymbol{P} = \left[\,{}^{1}\boldsymbol{R}_{2}\right]^{-1}\ {}^{1}\boldsymbol{R}_{2}\ {}^{2}\boldsymbol{P}$$

$$\left[\,{}^{1}\boldsymbol{R}_{2}\right]^{-1}\ {}^{1}\boldsymbol{P} = I\ {}^{2}\boldsymbol{P}$$

$${}^{2}\boldsymbol{P} = \left[\,{}^{1}\boldsymbol{R}_{2}\right]^{-1}\ {}^{1}\boldsymbol{P} \qquad \text{Eq (4)}$$

Eq(2), Eq(3) and Eq(4)

$$ {}^{2}\boldsymbol{P} = {}^{2}\boldsymbol{R}_{1}\ {}^{1}\boldsymbol{P} = \left[\ {}^{1}\boldsymbol{R}_{2}\right]^{T}\ {}^{1}\boldsymbol{P} = \left[\ {}^{1}\boldsymbol{R}_{2}\right]^{-1}\ {}^{1}\boldsymbol{P} $$

$$ {}^{2}\boldsymbol{R}_{1} = \left[\ {}^{1}\boldsymbol{R}_{2}\right]^{T} = \left[\ {}^{1}\boldsymbol{R}_{2}\right]^{-1} $$

In general, for any rotational transformation matrix $\boldsymbol{R}$

$$ \boldsymbol{R}^{-1} = \boldsymbol{R}^{T} $$

$$ \boldsymbol{R}\boldsymbol{R}^{T} = I $$

Mapping between translated frames.

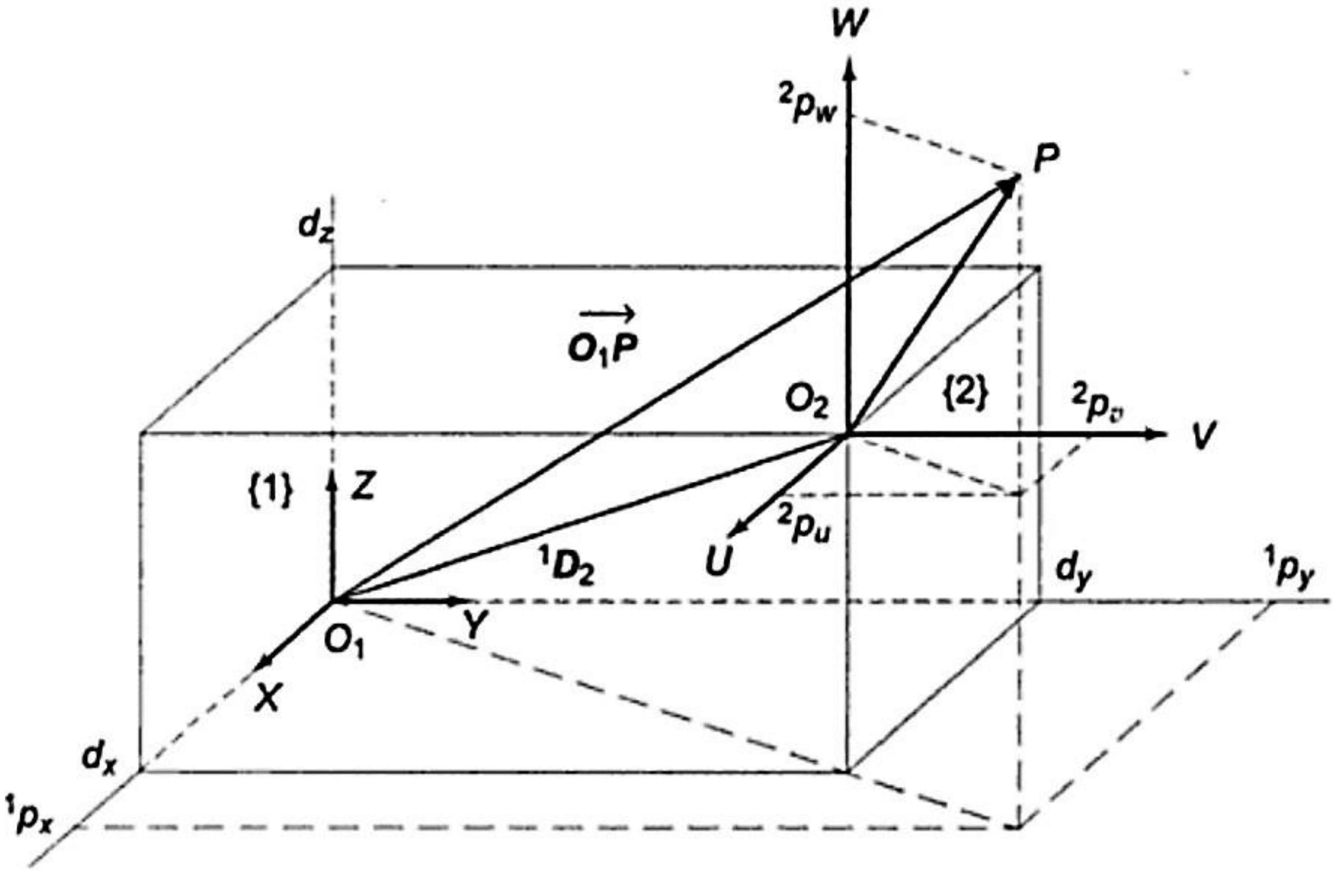

Figure 3: Translation of frames: frame {2} is translated w.r.t frame {1} by distance $^1\boldsymbol{D}_2$.

Consider two frames, frame {1} and frame {2}, with origins O_1 and O_2 such that the axes of frame {1} are parallel to axes of frame {2}.

A point P in space can be expressed as vectors $\overrightarrow{O_1P}$ and $\overrightarrow{O_2P}$ w.r.t the frames {1} and {2}, respectively.

The two vectors are related as,

$$\overrightarrow{O_1P} = \overrightarrow{O_2P} + \overrightarrow{O_1O_2}$$

$$^{1}\boldsymbol{P} = {}^{2}\boldsymbol{P} + {}^{1}\boldsymbol{D}_2 \qquad \text{Eq (5)}$$

The translation of origin of frame {2} w.r.t frame {1},

$$^{1}\boldsymbol{D}_2 = \overrightarrow{O_1 O_2}$$

The description of point P in frame {2} is $^{2}P = \begin{bmatrix} {}^{2}P_u & {}^{2}P_v & {}^{2}P_w \end{bmatrix}^T$ and $^{1}D_2 = \begin{bmatrix} d_x & d_y & d_z \end{bmatrix}^T$.

Substituting ^{2}P and $^{1}D_2$ in Eq (5) gives

$$^{1}\boldsymbol{P} = ({}^{2}P_u + d_x)\boldsymbol{x} + ({}^{2}P_v + d_y)\boldsymbol{y} + ({}^{2}P_w + d_z)\boldsymbol{z}$$

As,

$$ {}^{1}\boldsymbol{P} = {}^{1}P_x\boldsymbol{x} + {}^{1}P_y\boldsymbol{y} + {}^{1}P_z\boldsymbol{z} $$

This gives,

$$ {}^{1}P_x = {}^{2}P_u + d_x $$

$$ {}^{1}P_y = {}^{2}P_v + d_y $$

$$ {}^{1}P_z = {}^{2}P_w + d_z $$

The above relations can be verified using Figure 3.

In homogeneous coordinates, point P in space w.r.t frame {1} is denoted as

$$
{}^{1}\boldsymbol{P} = \begin{bmatrix} {}^{1}P_x \\ {}^{1}P_y \\ {}^{1}P_z \\ \sigma \end{bmatrix} = \begin{bmatrix} {}^{1}P_x & {}^{1}P_y & {}^{1}P_z & \sigma \end{bmatrix}^T
$$

σ is a non-zero positive scale factor. The physical coordinates are obtained by dividing each component in the homogeneous representation by the scale factor.

If the value of the scale factor σ is set to 1, the components of homogeneous and Cartesian representation are identical.

Scale factor can be used for magnifying (σ>1) or shrinking (0<σ<1) components of a vector in homogeneous coordinate representation.

Using the homogeneous coordinates, Eq(5) is written in the vector-matrix form as,

$$^{1}\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{2}p_u \\ ^{2}p_v \\ ^{2}p_w \\ 1 \end{bmatrix}$$

$$^{1}\boldsymbol{P} = {}^{1}\boldsymbol{T}_2\, {}^{2}\boldsymbol{P}$$

$^{1}T_2$ is a 4×4 homogeneous transformation matrix for translation of origin by $^{1}D_2 = \overrightarrow{O_1O_2} = [d_x \quad d_y \quad d_z \quad 1]^T$.

The 4×4 transformation matrix is called the basic homogeneous translation matrix.

Mapping between rotated and Translated frames

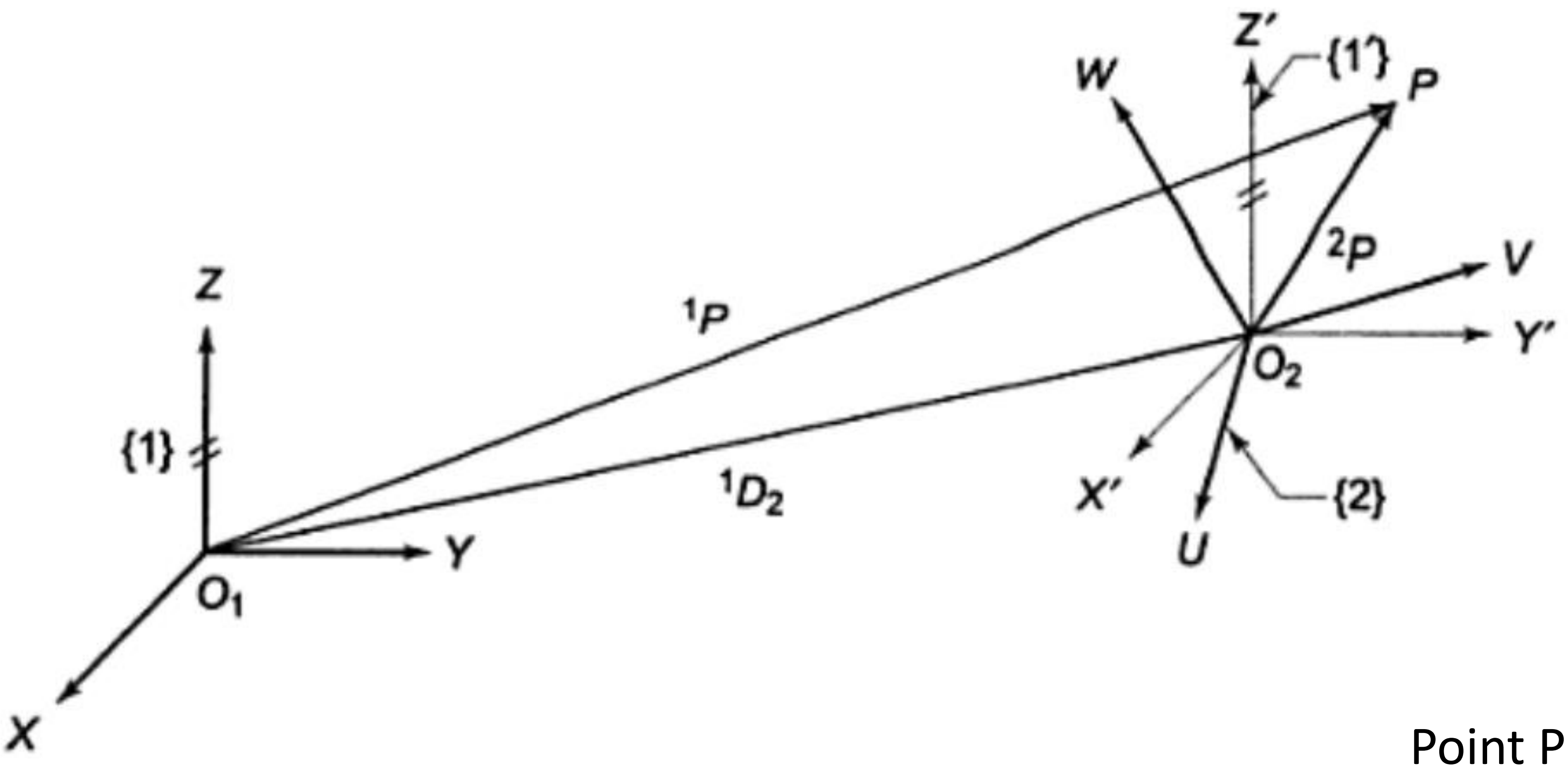

Figure 1: Mapping between two frames – translated and rotated w.r.t each other.

Frame {2} is rotated and translated w.r.t frame {1}.

The distance between the two origins is vector $\overrightarrow{O_1 O_2}$ or $^1D_2$.

Point P is described w.r.t frame {2} as ^2P, find ^1P (Point P described w.r.t frame {1}).

In terms of vectors

$$\overrightarrow{O_1P} = \overrightarrow{O_2P} + \overrightarrow{O_1O_2} \qquad \text{Eq(1)}$$

Vector $\overrightarrow{O_2P}$ in frame {2} is ${}^2P$, and it should be transformed to frame {1}.

Consider an intermediate frame {1'} with its origin coincident with O_2.

The frame {1'} is rotated w.r.t frame {2} such that its axes are parallel to axes of frame {1}.

Frame {1'} is related to frame {2} by pure rotation.

Point P is expressed in frame {1’] as

$$^{1'}P = {}^{1'}R_2 \, {}^{2}P$$

Frame {1’} is aligned with frame {1}, $^{1'}R_2 = {}^{1}R_2$

$$\overrightarrow{O_2P} = {}^{1'}P = {}^{1}R_2 \, {}^{2}P$$

Substituting in Eq(1),

$$^{1}P = {}^{1}R_2 \, {}^{2}P + {}^{1}D_2 \qquad \text{Eq(2)}$$

The vector $\overrightarrow{O_1O_2}$ or $^{1}D_2$ has components (d_x, d_y, d_z) in frame {1} as,

$$\overrightarrow{O_1O_2} = {}^{1}D_2 = \begin{bmatrix} d_x & d_y & d_z \end{bmatrix}^T$$

Using the homogeneous coordinates, Eq(2) can be written as a single 4×4 matrix.

$$ {}^1P = {}^1T_2 \, {}^2P $$

${}^1P$, ${}^2P$ are 4×1 vectors with a scale factor 1 and ${}^1T_2$ is 4×4 matrix referred as the *homogeneous transformation matrix*.

It describes both the position and orientation of frame {2} w.r.t frame {1}.

The components of ${}^1T_2$ matrix,

$$ {}^1T_2 = \begin{array}{ccc:c} \multicolumn{3}{r}{{}^1R_2} & {}^1D_2 \\ \hat{x}.\hat{u} & \hat{x}.\hat{v} & \hat{x}.\hat{w} & d_x \\ \hat{y}.\hat{u} & \hat{y}.\hat{v} & \hat{y}.\hat{w} & d_y \\ \hat{z}.\hat{u} & \hat{z}.\hat{v} & \hat{z}.\hat{w} & d_z \\ \hdashline 0 & 0 & 0 & 1 \end{array} $$

Scale factor σ

The four submatrices of a generalized homogeneous transform are,

$$T = \left[\begin{array}{c|c} \text{Rotation matrix} & \text{Translation vector} \\ (3 \times 3) & (3 \times 1) \\ \hline \text{Perspective} & \\ \text{transformation matrix} & \text{Scale factor} \\ (1 \times 3) & (1 \times 1) \end{array}\right]$$

Perspective transformation matrix is useful in vision systems and is set to zero vector wherever no perspective views are involved.

The scale factor σ has non-zero positive (σ>0) values and is called global scaling parameter.

σ>1 is useful for enlarging and 0< σ<1 is useful for reducing.

For describing the position and orientation of frame {2} w.r.t frame {1}, T takes the form

$$
{}^{1}\boldsymbol{T}_2 = \left[\begin{array}{ccc|c} & {}^{1}\boldsymbol{R}_2 & & {}^{1}\boldsymbol{D}_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

In reverse problem when ${}^{1}P$ is known and ${}^{2}P$ is to found, then

$$
{}^{2}P = {}^{2}T_1 \; {}^{1}P
$$

Where ${}^{2}T_1 = \left[{}^{1}T_2 \right]^{-1}$

$$
{}^{2}\boldsymbol{T}_1 = \left[{}^{1}\boldsymbol{T}_2 \right]^{-1} = \left[\begin{array}{ccc|c} & {}^{1}\boldsymbol{R}_2^{T} & & -{}^{1}\boldsymbol{R}_2^{T}\,{}^{1}\boldsymbol{D}_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$