Inertia tensor

Consider a rigid body consisting of a very large number of particles, and which is undergoing angular velocity $\boldsymbol{\omega}$.

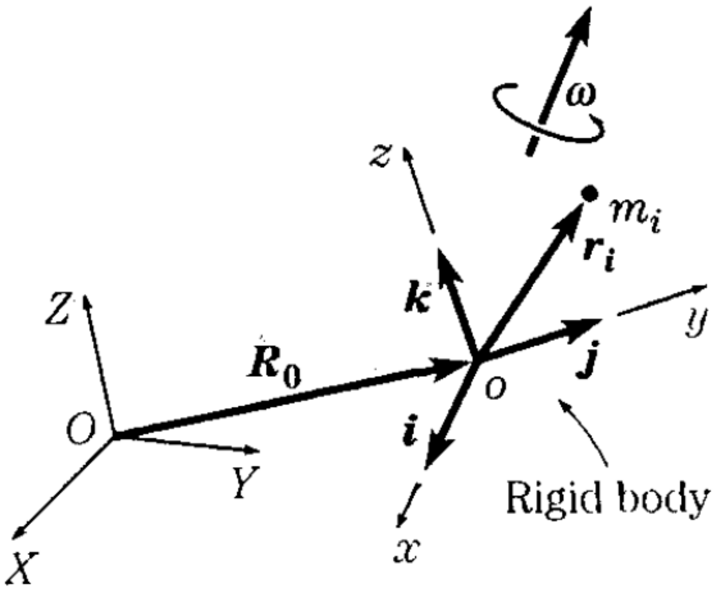
The body is attached to the $oxyz$ body-coordinate frame where the origin o of this frame is fixed.

The velocity of the i^{th} particle,

$$\boldsymbol{v}_i = \frac{d\boldsymbol{R}_o}{dt} + \boldsymbol{v}_{\mathrm{rel}} + \boldsymbol{\omega} \times \boldsymbol{r}_i$$

$$= \boldsymbol{\omega} \times \boldsymbol{r}_i$$

$\boldsymbol{r}_i$ is the position vector of m_i with respect to o.



Rigid body attached to $oxyz$ body-coordinate frame, undergoing angular velocity $\boldsymbol{\omega}$ about fixed point o.

The angular momentum of the rigid body about o,

$$\boldsymbol{H}_0 = \sum_{i=1}^{N} \boldsymbol{r}_i \times m_i \boldsymbol{v}_i$$

$$= \sum_{i=1}^{N} \boldsymbol{r}_i \times m_1 (\boldsymbol{\omega} \times \boldsymbol{r}_i)$$

$$= \sum_{i=1}^{N} m_i \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)$$

Vector triple product,

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$$

$$\boldsymbol{H}_0 = \sum_{i=1}^{N} m_i [(\boldsymbol{r}_i \cdot \boldsymbol{r}_i)\boldsymbol{\omega} - (\boldsymbol{r}_i \cdot \boldsymbol{\omega})\boldsymbol{r}_i]$$

In terms of the $oxyz$ unit vectors,

$$\boldsymbol{H}_o = H_x\boldsymbol{i} + H_y\boldsymbol{j} + H_z\boldsymbol{k}$$

$$\boldsymbol{r}_i = x_i\boldsymbol{i} + y_i\boldsymbol{j} + z_i\boldsymbol{k}$$

$$\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$$

$$H_o = \sum_{i=1}^{N} m_i[(x_i^2 + y_i^2 + z_i^2)(\omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k})$$

$$- (x_i\omega_x + y_i\omega_y + z_i\omega_z)(x_i\boldsymbol{i} + y_i\boldsymbol{j} + z_i\boldsymbol{k})]$$

$$
\begin{aligned}
H_o = \sum_{i=1}^{N} m_i [& \boxed{x_i^2 \omega_x \boldsymbol{i}} + x_i^2 \omega_y \boldsymbol{j} + x_i^2 \omega_z \boldsymbol{k} + \boxed{y_i^2 \omega_x \boldsymbol{i}} + y_i^2 \omega_y \boldsymbol{j} + y_i^2 \omega_z \boldsymbol{k} \\
& + \boxed{z_i^2 \omega_x \boldsymbol{i}} + z_i^2 \omega_y \boldsymbol{j} + z_i^2 \omega_z \boldsymbol{k} - (\boxed{x_i \omega_x x_i \boldsymbol{i}} + x_i \omega_x y_i \boldsymbol{j} \\
& + x_i \omega_x z_i \boldsymbol{k} + \boxed{y_i \omega_y x_i \boldsymbol{i}} + y_i \omega_y y_i \boldsymbol{j} + y_i \omega_y z_i \boldsymbol{k} + \boxed{z_i \omega_z x_i \boldsymbol{i}} \\
& + z_i \omega_z y_i \boldsymbol{j} + z_i \omega_z z_i \boldsymbol{k})]
\end{aligned}
$$

The sum of all the x components on the right hand side of the equation,

$$
\boldsymbol{H}_x = \sum_{i=1}^{N} m_i [\cancel{x_i^2 \omega_x \boldsymbol{i}} + y_i^2 \omega_x \boldsymbol{i} + z_i^2 \omega_x \boldsymbol{i} - \cancel{x_i^2 \omega_x \boldsymbol{i}} - x_i y_i \omega_y \boldsymbol{i} - x_i z_i \omega_z \boldsymbol{i}]
$$

By factoring out the angular velocity components,

$$\boldsymbol{H}_x = \left[\sum_{i=1}^{N} m_i(y_i^2 + z_i^2)\right]\omega_x + \left[-\sum_{i=1}^{N} m_i(x_i y_i)\right]\omega_y + \left[-\sum_{i=1}^{N} m_i(x_i z_i)\right]\omega_z$$

$$H_y = \left[-\sum_{i=1}^{N} m_i(y_i x_i)\right]\omega_x + \left[\sum_{i=1}^{N} m_i(x_i^2 + z_i^2)\right]\omega_y + \left[-\sum_{i=1}^{N} m_i(y_i z_i)\right]\omega_z$$

$$H_z = \left[-\sum_{i=1}^{N} m_i(z_i x_i)\right]\omega_x + \left[-\sum_{i=1}^{N} m_i(z_i y_i)\right]\omega_y + \left[\sum_{i=1}^{N} m_i(x_i^2 + y_i^2)\right]\omega_z$$

$$\boldsymbol{H}_x = \underbrace{\left[\sum_{i=1}^{N} m_i\left(y_i^2 + z_i^2\right)\right]}_{I_{xx}}\omega_x + \underbrace{\left[-\sum_{i=1}^{N} m_i(x_i y_i)\right]}_{I_{xy}}\omega_y + \underbrace{\left[-\sum_{i=1}^{N} m_i(x_i z_i)\right]}_{I_{xz}}\omega_z$$

$$H_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$H_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$H_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

Moments of inertia

$$I_{xx} = \sum_{i=1}^{N} m_i\left(y_i^2 + z_i^2\right) \qquad I_{yy} = \sum_{i=1}^{N} m_i\left(x_i^2 + z_i^2\right) \qquad I_{zz} = \sum_{i=1}^{N} m_i\left(x_i^2 + y_i^2\right)$$

Products of inertia

$$I_{xy} = -\sum_{i=1}^{N} m_i x_i y_i = I_{yx} \qquad I_{xz} = -\sum_{i=1}^{N} m_i x_i z_i = I_{zx} \qquad I_{yz} = -\sum m_i y_i z_i = I_{zy}$$

Moment of inertia

The resistance of a body to rotate about an axis

- Moment of inertia of a mass about x-axis $\boldsymbol{I_{xx}}$
- Moment of inertia of a mass about y-axis $\boldsymbol{I_{yy}}$
- Moment of inertia of a mass about z-axis $\boldsymbol{I_{zz}}$

List of moments of inertia – Wikipedia

Product of inertia

The symmetric measure for a body.

If any one of the three planes is a symmetric plane, then the product of inertia of the perpendicular planes are zero.

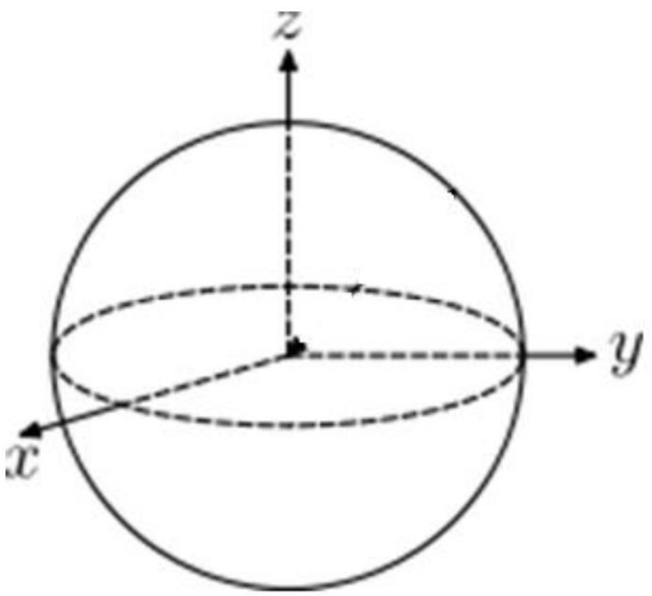
In matrix form,

$$\{H\}_o = [I]_o\{\omega\}$$

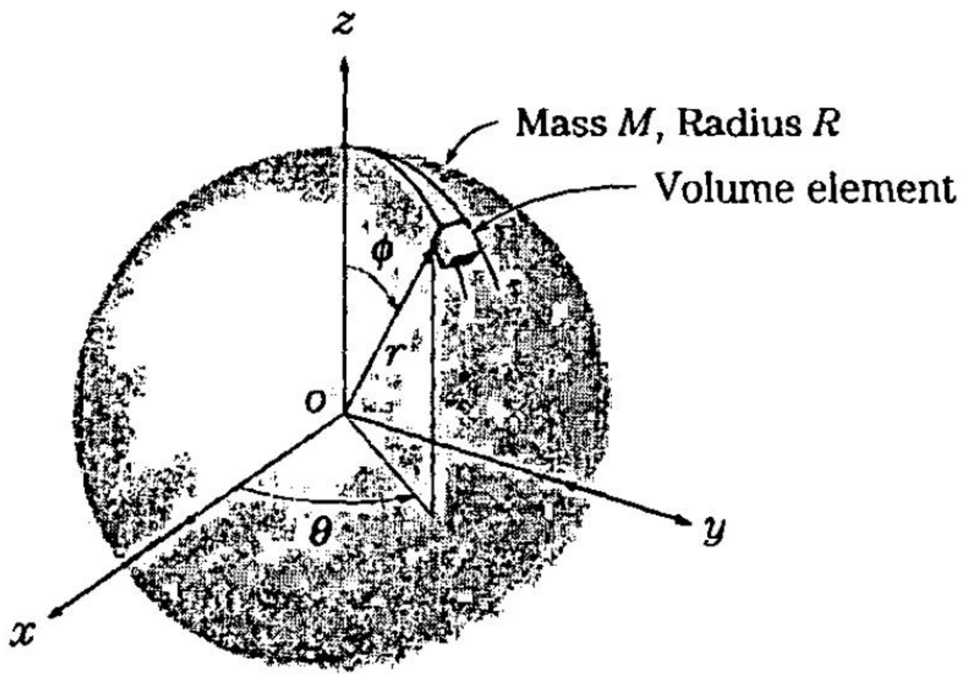
$$\{H\}_o = \begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix}_o \qquad \{\omega\} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

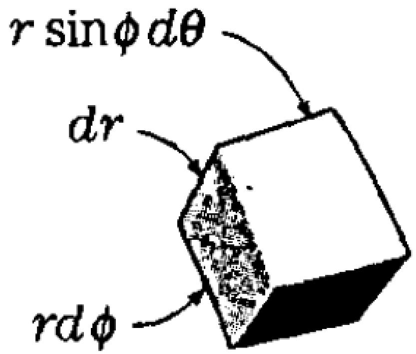
$$[I]_o = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}_o$$

Inertia tensor for a uniform sphere



Solid sphere with radius R and mass M





$$\rho = \frac{M}{\text{Volume}} = \frac{M}{\frac{4}{3}\pi R^3}$$

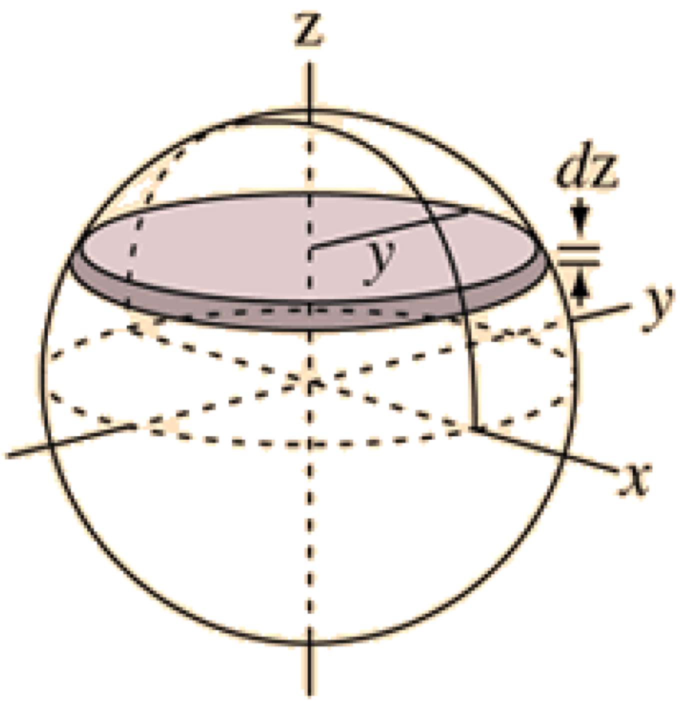
$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\phi$$

$$d\mathcal{V} = (r\,d\phi)(r\sin\phi\,d\theta)(dr) = r^2\sin\phi\,d\phi\,d\theta\,dr$$

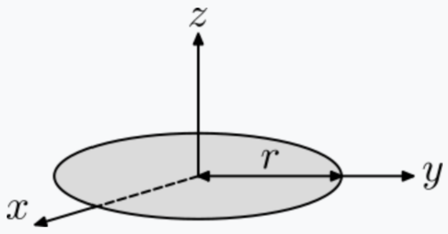
$$dm = \rho\,d\mathcal{V}$$

$$I_{yy} = \sum_{i=1}^{N} m_i(x_i^2 + z_i^2) = \int_M (x^2 + z^2)\,dm = \int_{\mathcal{V}} (x^2 + z^2)\rho\,d\mathcal{V}$$

Reference book, page 287, Example 6-3.



[Moment of Inertia, Sphere (gsu.edu)]

Thin, solid disk of radius *r* and mass *m*. This is a special case of the solid cylinder, with *h* = 0. That $I_x = I_y = \frac{I_z}{2}$ is a consequence of the perpendicular axis theorem.	 	$I_z = \frac{1}{2}mr^2$ $I_x = I_y = \frac{1}{4}mr^2$