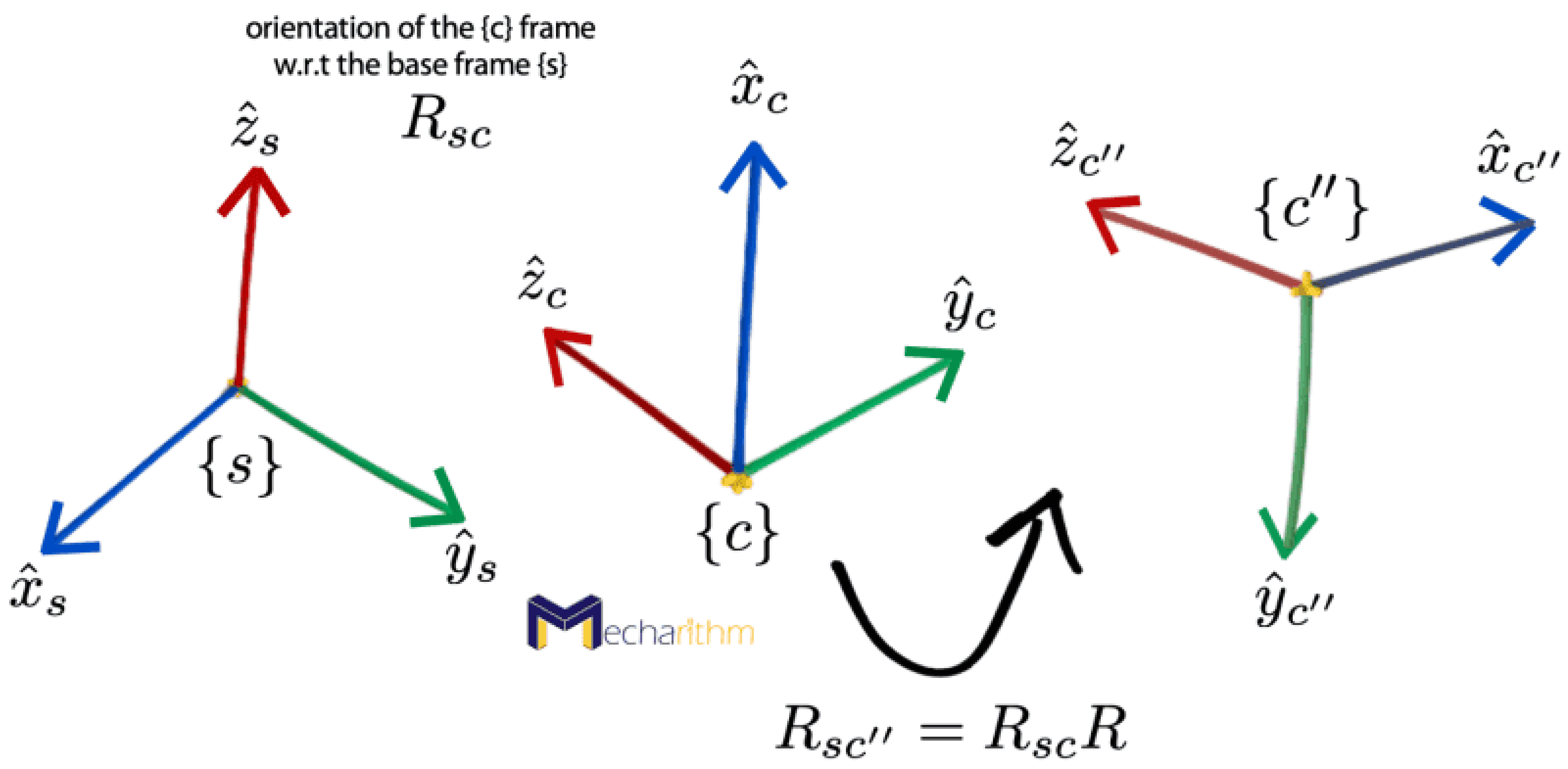
Kinematics and Dynamics

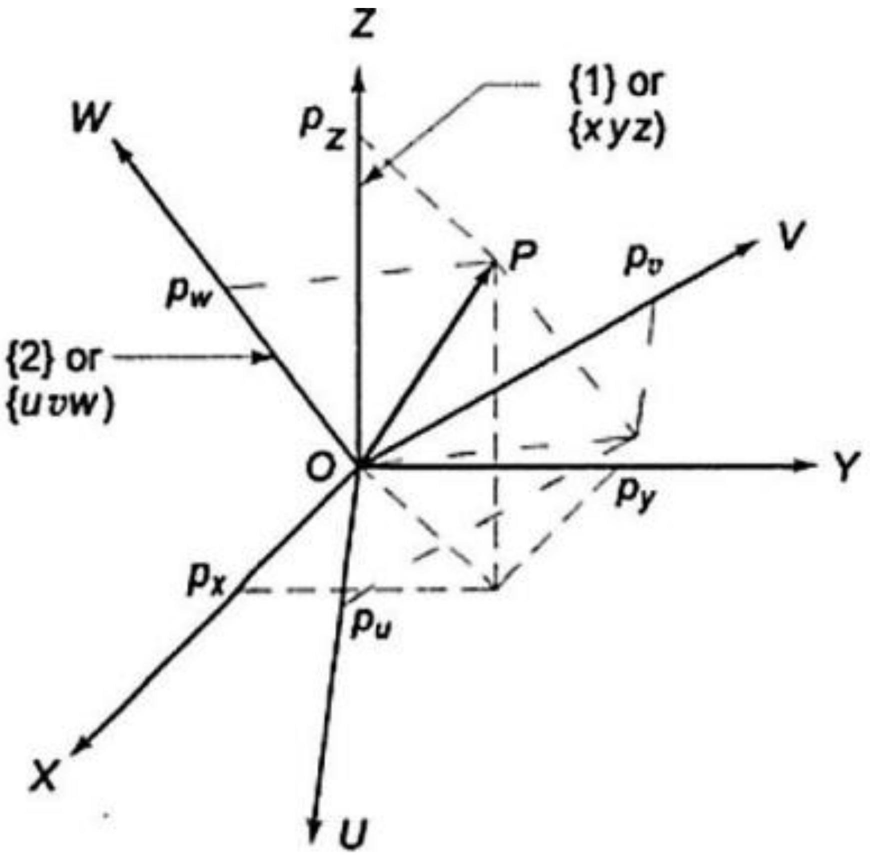
Day 02 | MP 3010



Coordinate frames, mapping and translation matrix

Section 1

Coordinate frames



$$^{1}\boldsymbol{P} = {}^{1}p_x\boldsymbol{x} + {}^{1}p_y\boldsymbol{y} + {}^{1}p_z\boldsymbol{z}$$

$$^{2}\boldsymbol{P} = {}^{2}p_u\boldsymbol{u} + {}^{2}p_v\boldsymbol{v} + {}^{2}p_w\boldsymbol{w}$$

$$^{1}p_x = \hat{\boldsymbol{x}}\cdot{}^{2}\boldsymbol{P} = \hat{\boldsymbol{x}}\cdot{}^{2}p_u\hat{\boldsymbol{u}} + \hat{\boldsymbol{x}}\cdot{}^{2}p_v\hat{\boldsymbol{v}} + \hat{\boldsymbol{x}}\cdot{}^{2}p_w\hat{\boldsymbol{w}}$$

$$^{1}p_y = \hat{\boldsymbol{y}}\cdot{}^{2}\boldsymbol{P} = \hat{\boldsymbol{y}}\cdot{}^{2}p_u\hat{\boldsymbol{u}} + \hat{\boldsymbol{y}}\cdot{}^{2}p_v\hat{\boldsymbol{v}} + \hat{\boldsymbol{y}}\cdot{}^{2}p_w\hat{\boldsymbol{w}}$$

$$^{1}p_z = \hat{\boldsymbol{z}}\cdot{}^{2}\boldsymbol{P} = \hat{\boldsymbol{z}}\cdot{}^{2}p_u\hat{\boldsymbol{u}} + \hat{\boldsymbol{z}}\cdot{}^{2}p_v\hat{\boldsymbol{v}} + \hat{\boldsymbol{z}}\cdot{}^{2}p_w\hat{\boldsymbol{w}}$$

Matrix form

In matrix form

$$\begin{bmatrix} {}^1P_x \\ {}^1P_y \\ {}^1P_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}\cdot\boldsymbol{u} & \boldsymbol{x}\cdot\boldsymbol{v} & \boldsymbol{x}\cdot\boldsymbol{w} \\ \boldsymbol{y}\cdot\boldsymbol{u} & \boldsymbol{y}\cdot\boldsymbol{v} & \boldsymbol{y}\cdot\boldsymbol{w} \\ \boldsymbol{z}\cdot\boldsymbol{u} & \boldsymbol{z}\cdot\boldsymbol{v} & \boldsymbol{z}\cdot\boldsymbol{w} \end{bmatrix} \begin{bmatrix} {}^2P_u \\ {}^2P_v \\ {}^2P_w \end{bmatrix}$$

In compressed vector-matrix notation

$${}^1\boldsymbol{P} = {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{P} \qquad \text{Eq (1)}$$

Where

$${}^1\boldsymbol{R}_2 = \begin{bmatrix} \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{w}} \end{bmatrix}$$

Rotation matrix ${}^{2}\boldsymbol{R}_1$

$$
{}^{2}R_1 = \begin{bmatrix} \boldsymbol{u}\cdot\boldsymbol{x} & \boldsymbol{u}\cdot\boldsymbol{y} & \boldsymbol{u}\cdot\boldsymbol{z} \\ \boldsymbol{v}\cdot\boldsymbol{x} & \boldsymbol{v}\cdot\boldsymbol{y} & \boldsymbol{v}\cdot\boldsymbol{z} \\ \boldsymbol{w}\cdot\boldsymbol{x} & \boldsymbol{w}\cdot\boldsymbol{y} & \boldsymbol{w}\cdot\boldsymbol{z} \end{bmatrix}
$$

Point P in frame {1} is transformed to frame {2},

$$
{}^{2}\boldsymbol{P} = {}^{2}\boldsymbol{R}_1 \, {}^{1}\boldsymbol{P} \qquad \text{Eq (2)}
$$

As the vector dot product is commutative,

$$
{}^{2}\boldsymbol{R}_1 = \left[\, {}^{1}\boldsymbol{R}_2 \right]^T
$$

$$
{}^{2}\boldsymbol{P} = \left[\, {}^{1}\boldsymbol{R}_2 \right]^T \, {}^{1}\boldsymbol{P} \qquad \text{Eq (3)}
$$

Multiplying Eq(1) by $[\,{}^{1}\boldsymbol{R}_2]^{-1}$ in both sides

$$[\,{}^{1}\boldsymbol{R}_2]^{-1}\,{}^{1}\boldsymbol{P} = [\,{}^{1}\boldsymbol{R}_2]^{-1}\,{}^{1}\boldsymbol{R}_2\,{}^{2}\boldsymbol{P}$$

$$[\,{}^{1}\boldsymbol{R}_2]^{-1}\,{}^{1}\boldsymbol{P} = I\,{}^{2}\boldsymbol{P}$$

$$ {}^{2}\boldsymbol{P} = [\,{}^{1}\boldsymbol{R}_2]^{-1}\,{}^{1}\boldsymbol{P} \qquad \text{Eq (4)}$$

Eq(2), Eq(3) and Eq(4)

$$ {}^{2}\boldsymbol{P} = {}^{2}\boldsymbol{R}_1\,{}^{1}\boldsymbol{P} = [\,{}^{1}\boldsymbol{R}_2]^{T}\,{}^{1}\boldsymbol{P} = [\,{}^{1}\boldsymbol{R}_2]^{-1}\,{}^{1}\boldsymbol{P}$$

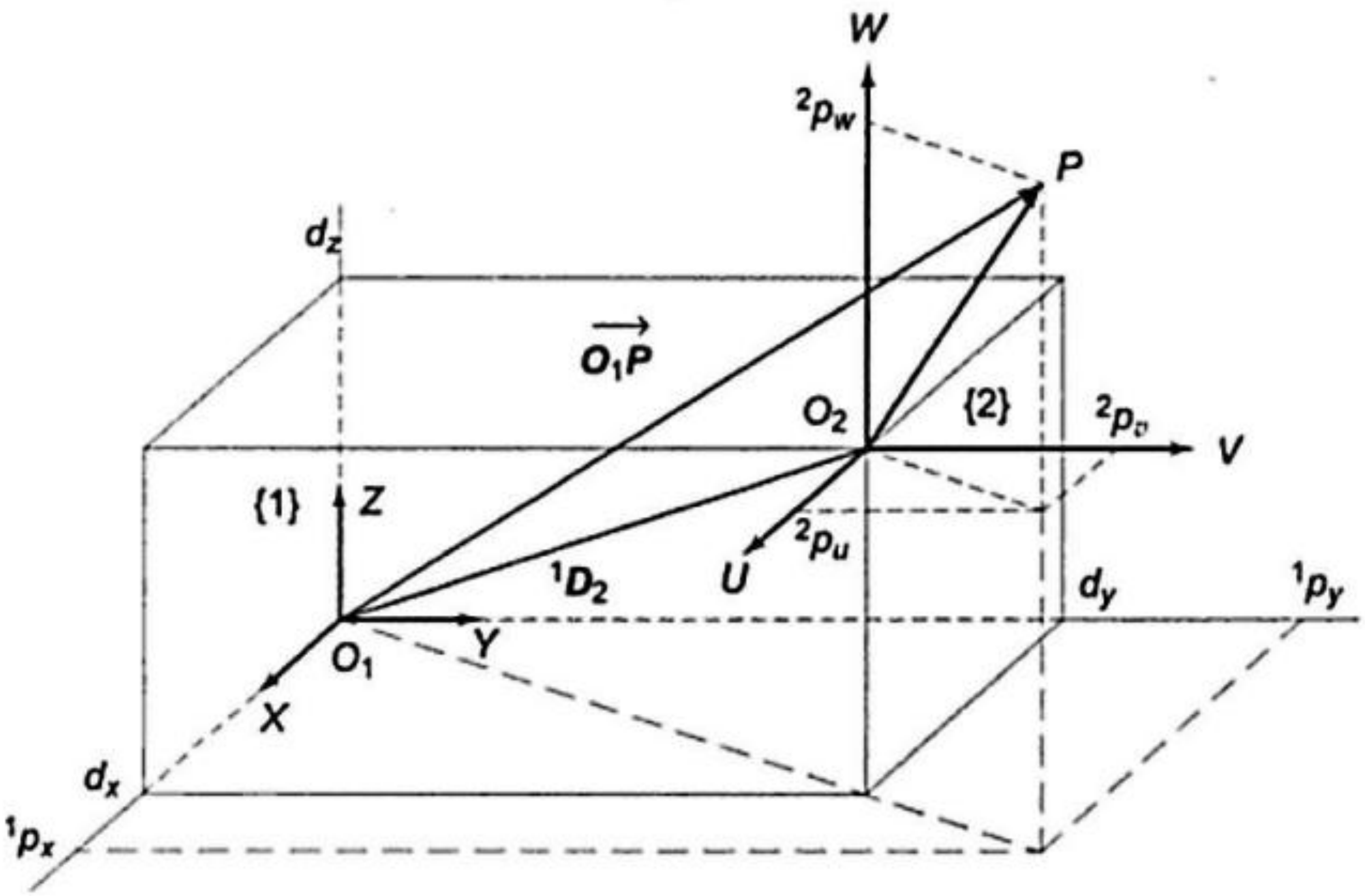
$$ {}^{2}\boldsymbol{R}_1 = [\,{}^{1}\boldsymbol{R}_2]^{T} = [\,{}^{1}\boldsymbol{R}_2]^{-1}$$

In general, for any rotational transformation matrix $\boldsymbol{R}$

$$\boldsymbol{R}^{-1} = \boldsymbol{R}^{T}$$

$$\boldsymbol{R}\boldsymbol{R}^{T} = I$$

Translated Frame



$$\overrightarrow{O_1P} = \overrightarrow{O_2P} + \overrightarrow{O_1O_2}$$

$$^1\boldsymbol{P} = {}^2\boldsymbol{P} + {}^1\boldsymbol{D}_2$$

The translation of origin of frame {2} w.r.t frame {1},

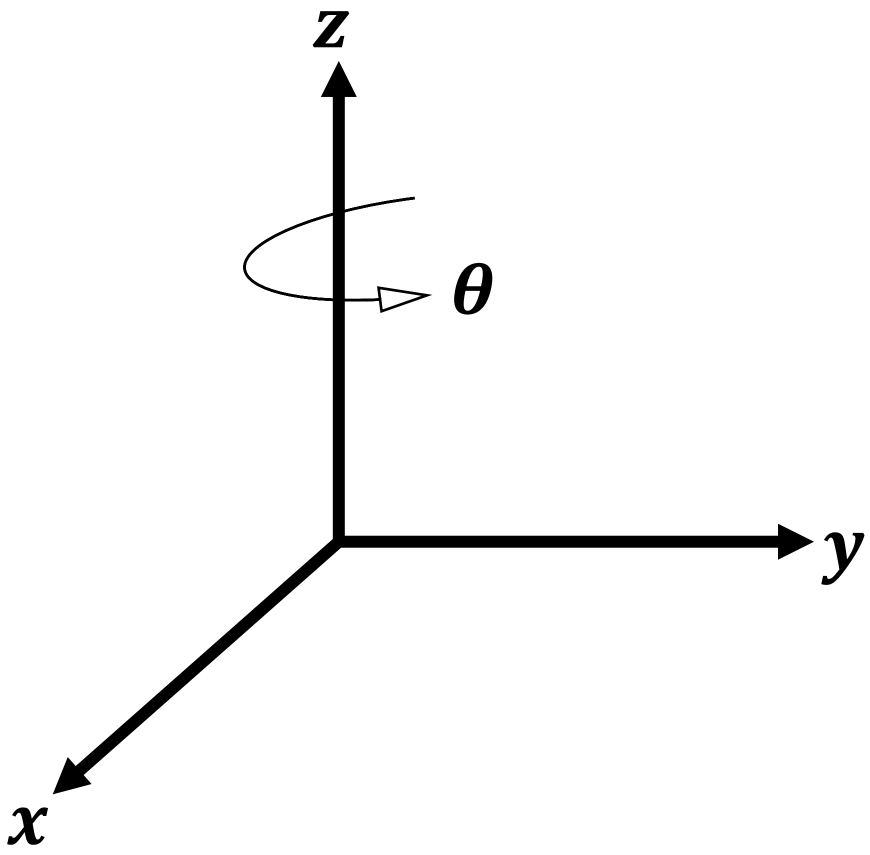
$$^{1}\boldsymbol{D}_2 = \overrightarrow{O_1 O_2}$$

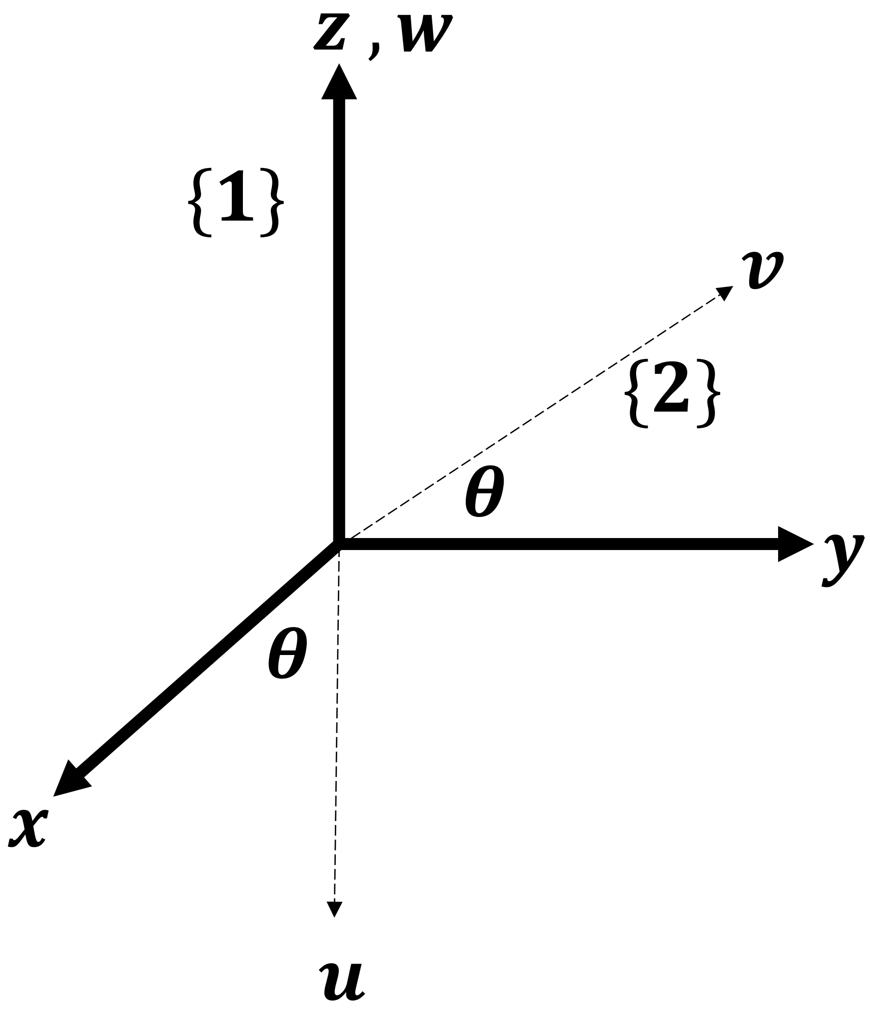
The description of point P in frame {2} is $^{2}P = [\,^{2}P_u \quad ^{2}P_v \quad ^{2}P_w]^T$ and $^{1}D_2 = [d_x \quad d_y \quad d_z]^T$.

Substituting ^{2}P and $^{1}D_2$ in Eq (5) gives

$$^{1}\boldsymbol{P} = (\,^{2}P_u + d_x)\boldsymbol{x} + (\,^{2}P_v + d_y)\boldsymbol{y} + (\,^{2}P_w + d_z)\boldsymbol{z}$$

Rotation Matrix





$$1_{R_2} = \begin{bmatrix} \hat{x}.\hat{u} & \hat{x}.\hat{v} & \hat{x}.\widehat{w} \\ \hat{y}.\hat{u} & \hat{y}.\hat{v} & \hat{y}.\widehat{w} \\ \hat{z}.\hat{u} & \hat{z}.\hat{v} & \hat{z}.\widehat{w} \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} cos\theta & \cos(90+\theta) & cos90 \\ \cos(90-\theta) & cos\theta & cos90 \\ cos90 & cos90 & cos0 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} c\theta & -\mathrm{s}\,\theta & 0 \\ \mathrm{s}\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} c\theta & 0 & \mathrm{s}\,\theta \\ 0 & 1 & 0 \\ -\mathrm{s}\,\theta & 0 & \mathrm{c}\,\theta \end{bmatrix}$$

As,

$$^{1}\boldsymbol{P} = {}^{1}P_x\boldsymbol{x} + {}^{1}P_y\boldsymbol{y} + {}^{1}P_z\boldsymbol{z}$$

This gives,

$$^{1}P_x = {}^{2}P_u + d_x$$

$$^{1}P_y = {}^{2}P_v + d_y$$

$$^{1}P_z = {}^{2}P_w + d_z$$

$$^{1}\boldsymbol{P} = \begin{bmatrix} {}^{1}P_x \\ {}^{1}P_y \\ {}^{1}P_z \\ \sigma \end{bmatrix} = [\,{}^{1}P_x \quad {}^{1}P_y \quad {}^{1}P_z \quad \sigma]^T$$

Scale Factor

$$^{1}D_2 = \overrightarrow{O_1O_2} = [d_x \quad d_y \quad d_z \quad 1]^T$$

$$ {}^1P = {}^1T_2 \; {}^2P $$

$$
{}^1T_2 = \begin{array}{c:c}
{}^1R_2 & {}^1D_2 \\
\begin{matrix} \hat{x}.\hat{u} & \hat{x}.\hat{v} & \hat{x}.\hat{w} \\ \hat{y}.\hat{u} & \hat{y}.\hat{v} & \hat{y}.\hat{w} \\ \hat{z}.\hat{u} & \hat{z}.\hat{v} & \hat{z}.\hat{w} \end{matrix} & \begin{matrix} d_x \\ d_y \\ d_z \end{matrix} \\
\hdashline
\begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \\
 & \text{Scale factor } \sigma
\end{array}
$$

$$
T = \left[\begin{array}{c:c}
\begin{matrix}\text{Rotation matrix} \\ (3 \times 3)\end{matrix} & \begin{matrix}\text{Translation vector} \\ (3 \times 1)\end{matrix} \\
\hdashline
\begin{matrix}\text{Perspective} \\ \text{transformation matrix} \\ (1 \times 3)\end{matrix} & \begin{matrix}\text{Scale factor} \\ (1 \times 1)\end{matrix}
\end{array}\right]
$$

Fundamental Rotation Matrix

The overall rotation matrix representing a rotation of angle θ_1 about x-axis followed by a rotation of angle θ_2 about y-axis is given by,

$$^{1}R_2 = R_y(\theta_2)R_x(\theta_1)$$

$$R = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta_2 & \sin\theta_2\sin\theta_1 & \sin\theta_2\cos\theta_1 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ -\sin\theta_2 & \cos\theta_2\sin\theta_1 & \cos\theta_2\cos\theta_1 \end{bmatrix}$$

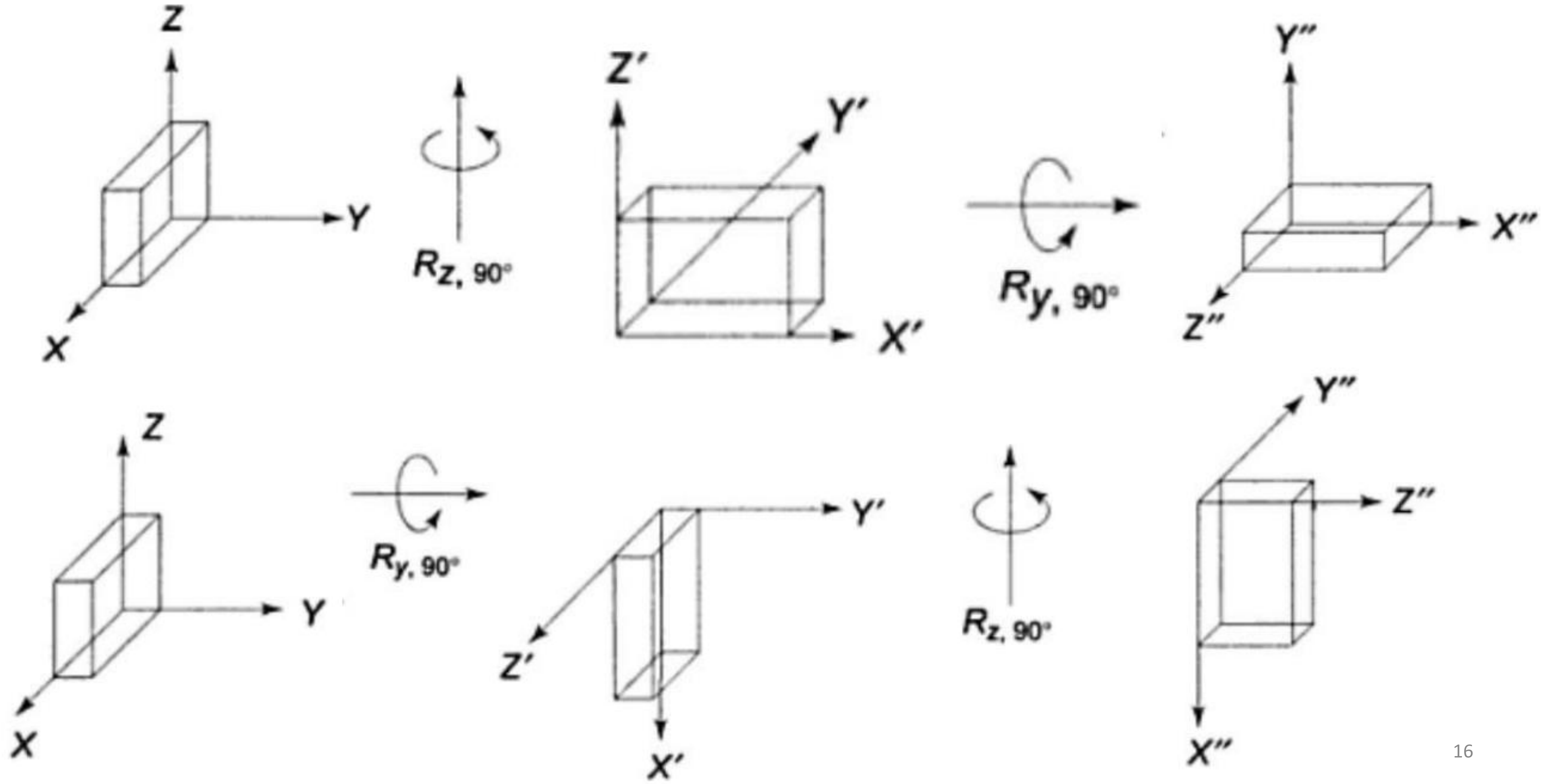
The sequence of multiplication of R matrices is very important.

- A different sequence may not give the same result and obviously will not correspond to same orientation of the rotated frame. This is because the matrix product is not commutative.
- Two rotations in general do not result in same orientation and the
- resultant rotation matrix depends on the order of rotations.

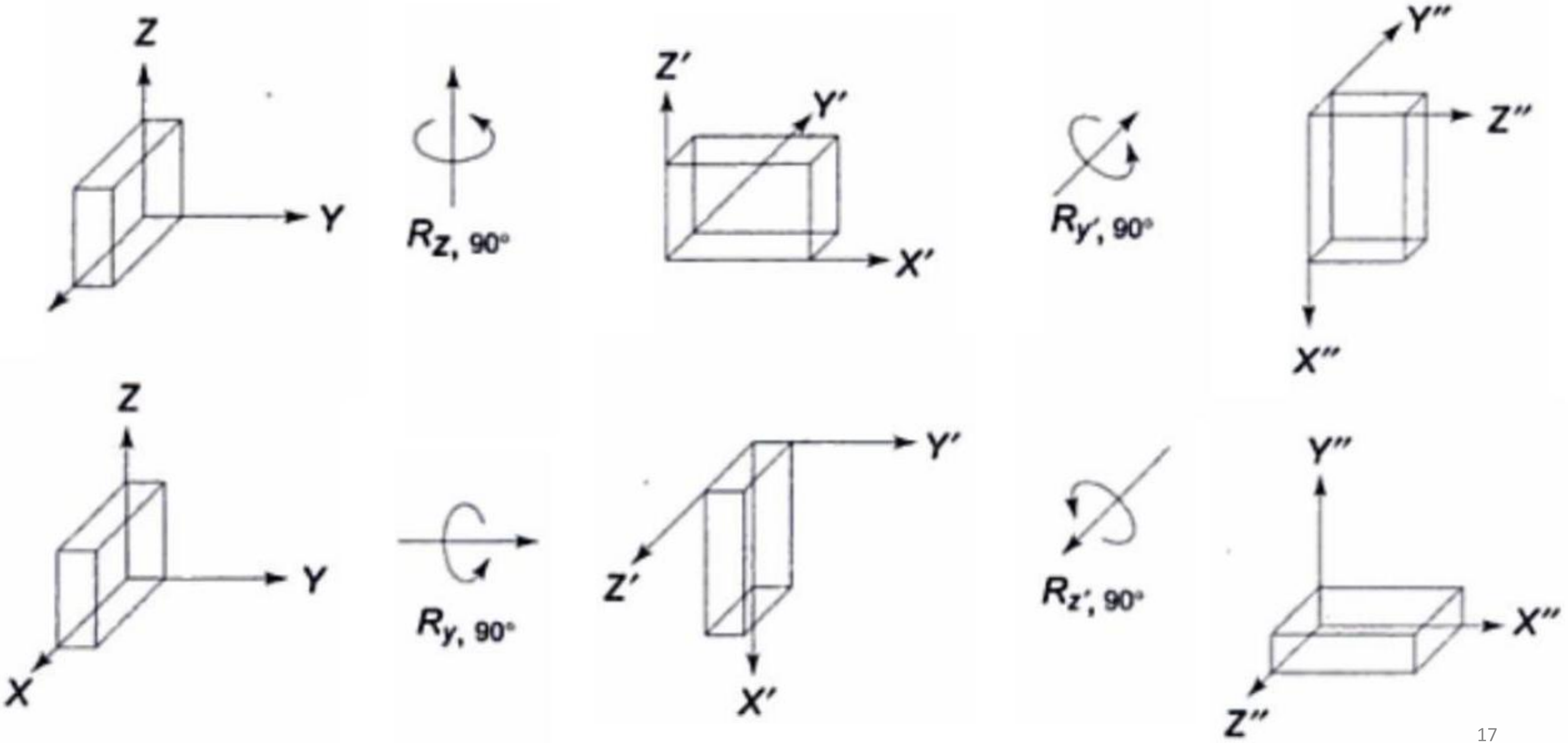
Another significant factor is how the rotations are performed. There are two alternatives.

1. To perform successive rotations about the principal axes of the fixed frame.
2. To perform successive rotations about the current principal axes of a moving frame.

Effect of order of rotations of a cuboid about principal axes of a fixed frame.



Effect of order of rotations of a cuboid about axes of the moving frame.



Fixed angle representation

Each rotation is specified about an axis of fixed reference frame.

1. Moving frame {2} is rotated by an angle θ_1 about X-axis to frame {2'}.

   This rotation is described by the rotation matrix $R_x(\theta_1)$.

2. The frame {2'} is rotated by an angle θ_2 about Y-axis to give frame {2"}.

   This rotation is described by the rotation matrix $R_y(\theta_2)$.

3. The frame {2"} is rotated by an angle θ_3 about Z-axis to give frame {2}.

   This rotation is described by the rotation matrix $R_z(\theta_3)$.

$$R_{xyz}(\theta_3\theta_2\theta_1) = {}^1R_2 = R_z(\theta_3)\mathrm{R_y}(\theta_2)R_x(\theta_1)$$

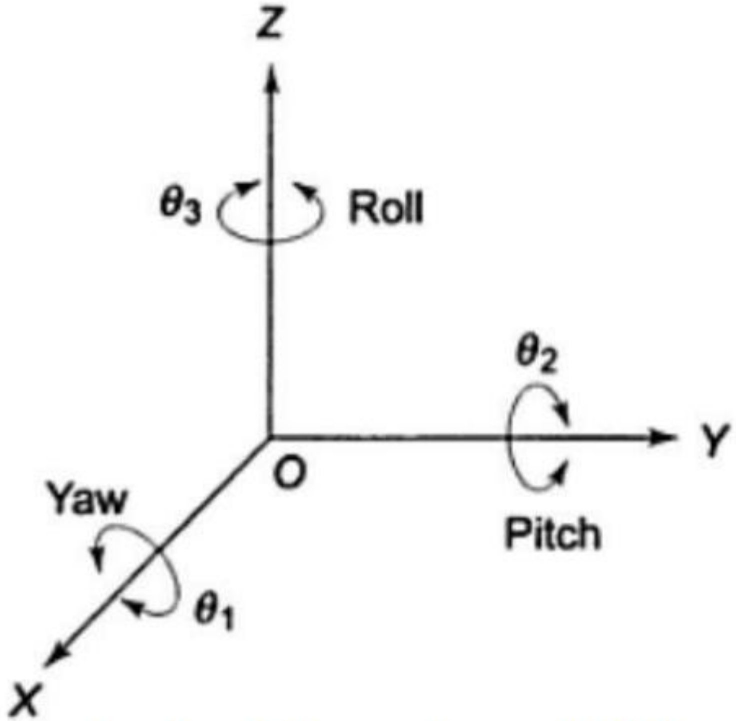
(rotation ordering right to left)

$$R_{xyz}(\theta_3\theta_2\theta_1)$$

$$= \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$R_{xyz}(\theta_3\,\theta_2\,\theta_1) = \begin{bmatrix} C_2C_3 & S_1S_2C_3 - C_1S_3 & C_1S_2C_3 + S_1S_3 \\ C_2S_3 & S_1S_2S_3 + C_1C_3 & C_1S_2S_3 - S_1C_3 \\ -S_2 & S_1C_2 & C_1C_2 \end{bmatrix}$$

Where $C_i = \cos\theta_i$ and $S_i = \sin\theta_i$



Representation of roll, pitch and yaw (RPY) rotations

Question

Base of the robot is frame {1} and the end-effector is frame {2}. The end-effector of a robot holds a tool with tool tip which is denoted by point ${}^{2}P$ having co-ordinates of $[5\ 8\ 13]^{T}$. The end-effector is rotated about the base frame z-axis by 90°, then about the base frame x-axis by 120°.

i. Find the rotational matrices R_{OX} and R_{OZ}.
ii. Obtain the equivalent rotation matrix ${}^{1}R_{2}$.
iii. Find the co-ordinates of point ${}^{1}P$ in frame {1}.

Question

Frame {1} and frame {2} have coincident origins and differ only in orientation. Frame {2} is initially coincident with frame {1}. Certain rotations are carried out about the axis of the fixed frame {1}: first rotation about x-axis by 45° then about y-axis by 30° and finally about x-axis by 60°.

Obtain the equivalent rotation matrix ${}^{1}R_{2}$.

Euler angle representation

The moving frame, instead of rotating about the principal axes of the fixed frame, can rotate about its own principal axes.

Consider the rotations of frame {2} w.r.t frame {1}, starting from the position when the two frames are coincident.

This convention for specifying orientation is called WVU-Euler angle representation.

1. Frame {2} is rotated by an angle θ_1 about its w-axis coincident with z-axis of frame {1}. The rotated frame is now {2'}.

This rotation is described by the rotation matrix $R_w(\theta_1)$.

2. Moving frame {2'} is rotated by an angle θ_2 about v'-axis, the rotated v-axis to frame {2"}.

This rotation is described by the rotation matrix $R_{v'}(\theta_2)$.

3. Frame {2"} is rotated by an angle θ_3 about its u"-axis, the rotated u-axis to give frame {2}.

This rotation is described by the rotation matrix $R_{u''}(\theta_3)$.

The equivalent rotation matrix is computed by post multiplication of the matrices of the elementary rotations.

$$R_{wvu}(\theta_1\theta_2\theta_3) = {}^{1}R_2$$

$$= R_w(\theta_1)R_{v'}(\theta_2)R_{u''}(\theta_3)$$

(rotation ordering left to right)

The rotations are performed about the current axes of the moving frame {uvw}.

$$R_{wvu}(\theta_1\theta_2\theta_3) = R_w(\theta_1)R_{v'}(\theta_2)R_{u''}(\theta_3)$$

$$\mathbf{R}_{wvu}(\theta_1\theta_2\theta_3) = \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & 0 & S_2 \\ 0 & 1 & 0 \\ -S_2 & 0 & C_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_3 & -S_3 \\ 0 & S_3 & C_3 \end{bmatrix}$$

$$\mathbf{R}_{wvu}(\theta_1\theta_2\theta_3) = \begin{bmatrix} C_2C_3 & S_1S_2C_3 - C_1S_3 & C_1S_2C_3 + S_1S_3 \\ C_2S_3 & S_1S_2S_3 + C_1C_3 & C_1S_2S_3 - S_1C_3 \\ -S_2 & S_1C_2 & C_1C_2 \end{bmatrix}$$

Question

Frame {B} is located as follows: initially coincident with frame {A}, then the origin of frame {B} is translated by ${}^{A}D_B = [5 \quad -4 \quad 3]^T$, then the translated frame is rotated about y_B axis by 30° and then the resulting frame is rotated about its own u_B axis by 45°.

i. Determine the rotational matrix, ${}^{A}R_B$.
ii. Find the description of point ${}^{A}P$ if ${}^{B}P$ is $[6 \quad 3 \quad 5 \quad 1]^T$.