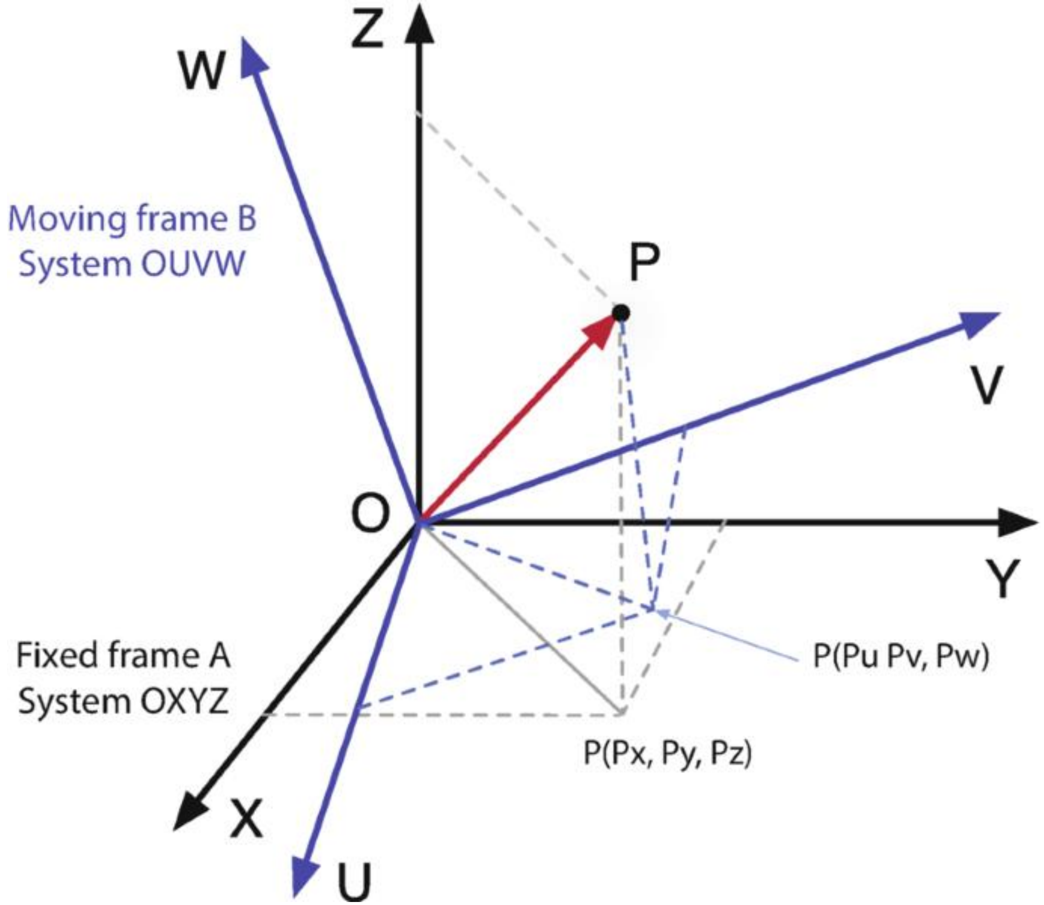
Kinematics and Dynamics

Day 01



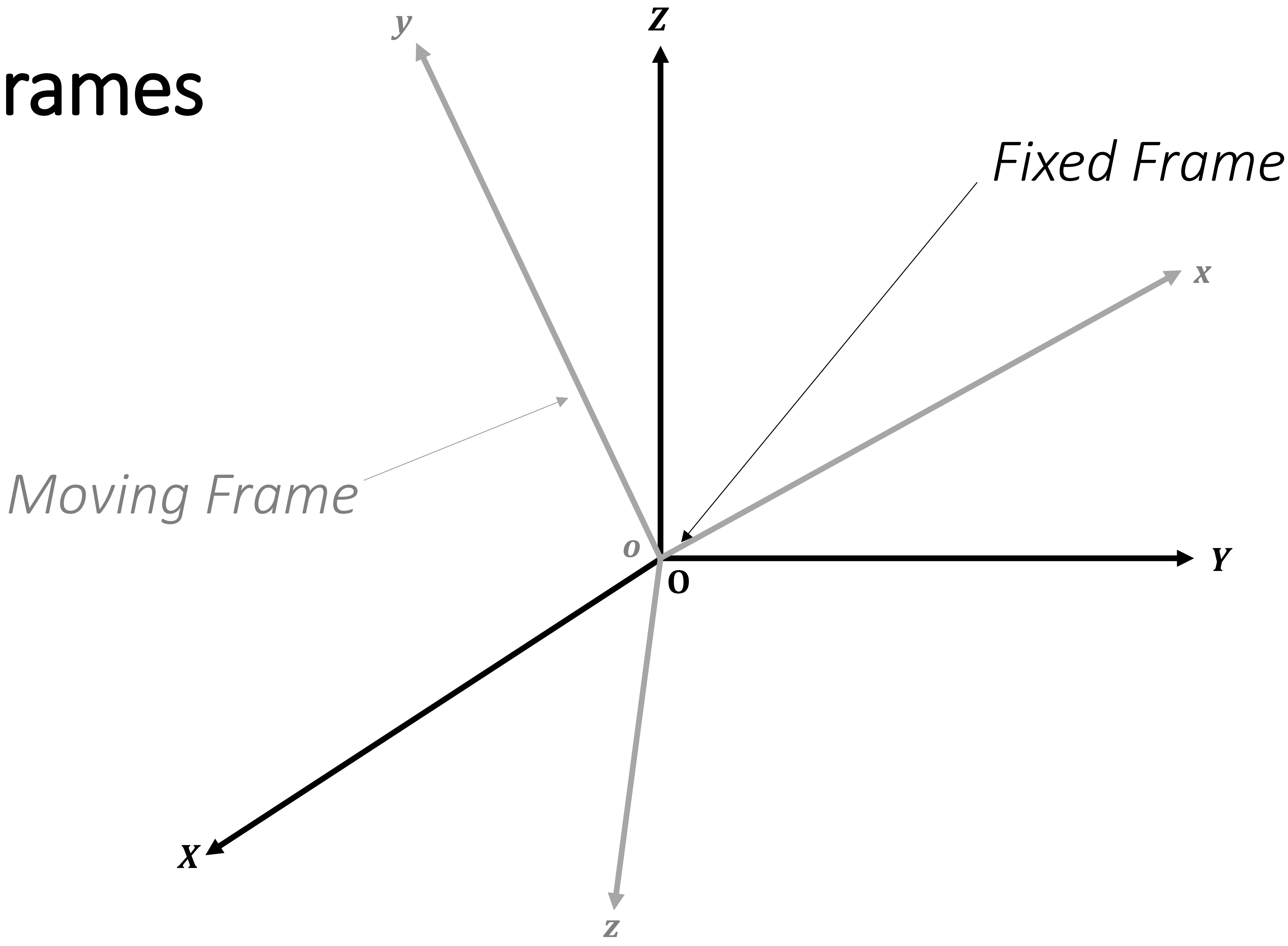
Structure of End Semester Examination

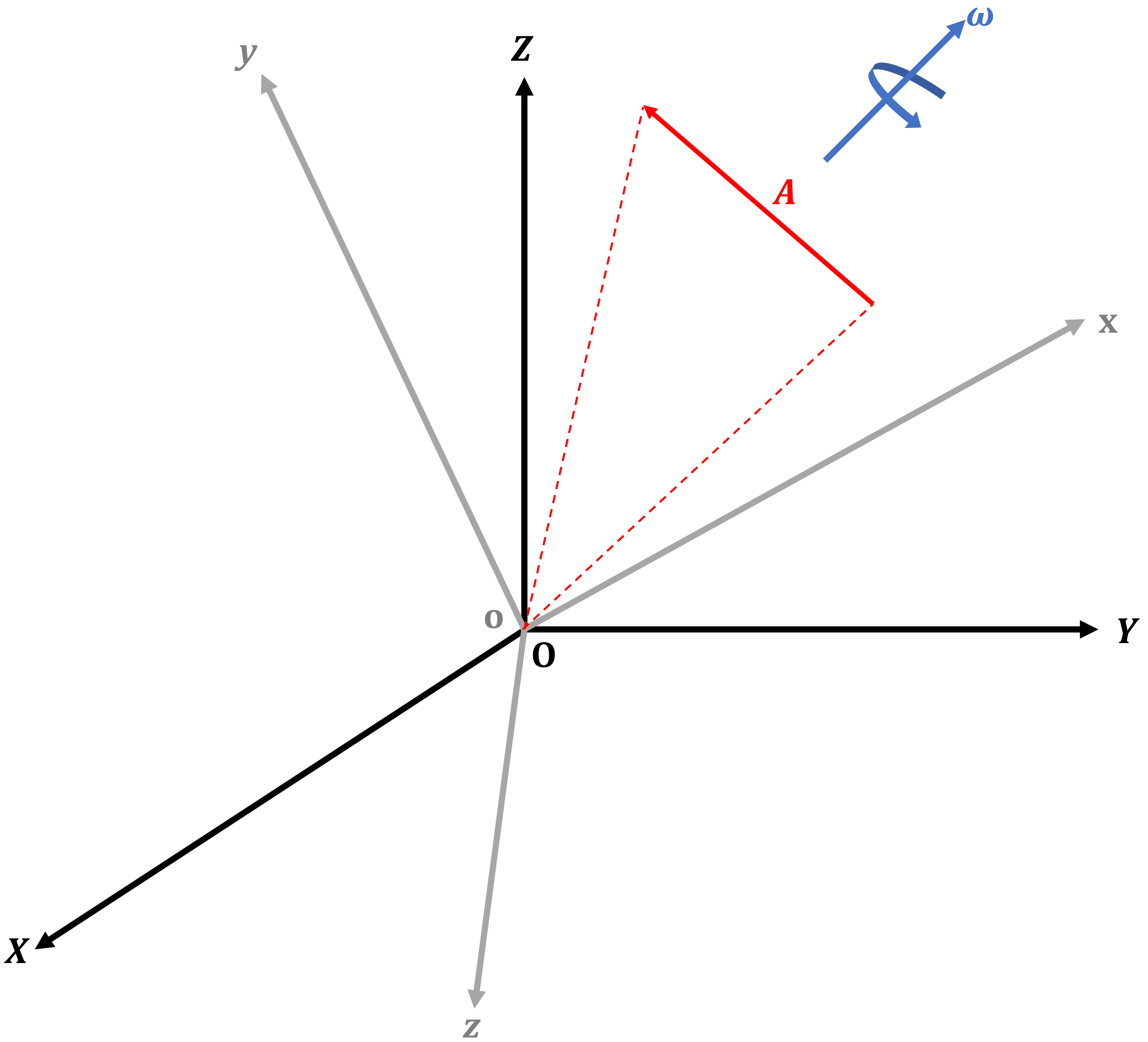
- Question 01 : MCQ
- Question 02 : Rotation Matrices, Euler Rotation
- Question 03 : VVD & AVD
- Question 04 : Gears
- Question 05 : Cam and Follower

Intermediate Frames

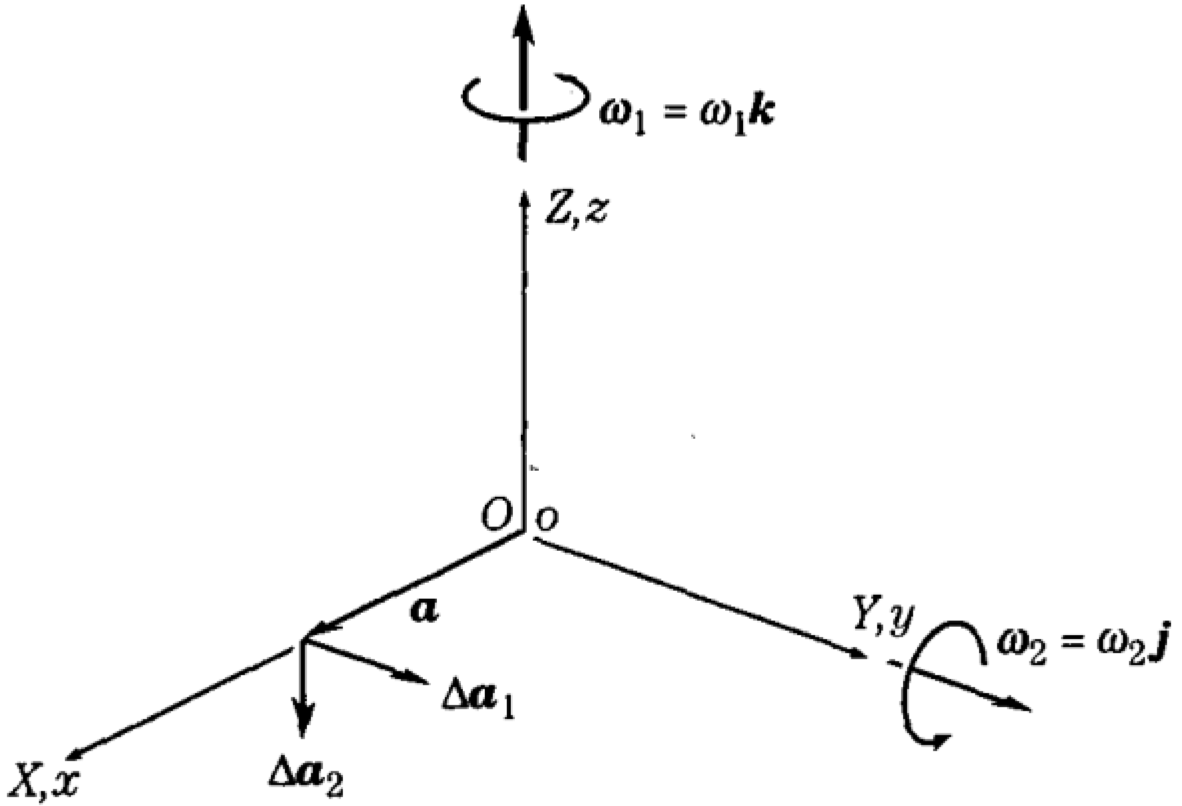
Section 1

Frames





$$\boxed{\frac{d\boldsymbol{A}}{dt} = \omega \times \boldsymbol{A}}$$



$$\omega = \omega_2 \boldsymbol{j} + \omega_1 \boldsymbol{k}$$

Arbitrary Vector (B)

$$\boldsymbol{B} = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$$

$$\frac{d\boldsymbol{B}}{dt} = \frac{dB_x}{dt}\boldsymbol{i} + \frac{dB_y}{dt}\boldsymbol{j} + \frac{dB_z}{dt}\boldsymbol{k} + B_x\frac{d\boldsymbol{i}}{dt} + B_y\frac{d\boldsymbol{j}}{dt} + B_z\frac{d\boldsymbol{k}}{dt}$$

$$= \underbrace{\dot{B}_x\boldsymbol{i} + \dot{B}_y\boldsymbol{j} + \dot{B}_z\boldsymbol{k}}_{\left(\frac{\partial \boldsymbol{B}}{\partial t}\right)_{\text{rel}}} + \underbrace{B_x\frac{d\boldsymbol{i}}{dt} + B_y\frac{d\boldsymbol{j}}{dt} + B_z\frac{d\boldsymbol{k}}{dt}}_{\boldsymbol{\omega}(\boldsymbol{B_x i} + \boldsymbol{B_y j} + \boldsymbol{B_z k})}$$

$$\boldsymbol{\omega} \times \boldsymbol{B}$$

$$\frac{d\boldsymbol{B}}{dt} = \left(\frac{\partial \boldsymbol{B}}{\partial t}\right)_{\text{rel}} + \boldsymbol{\omega} \times \boldsymbol{B}$$

General form:

$$\boxed{\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_{\text{rel}} + \boldsymbol{\omega}\times}$$

Intermediate Frames

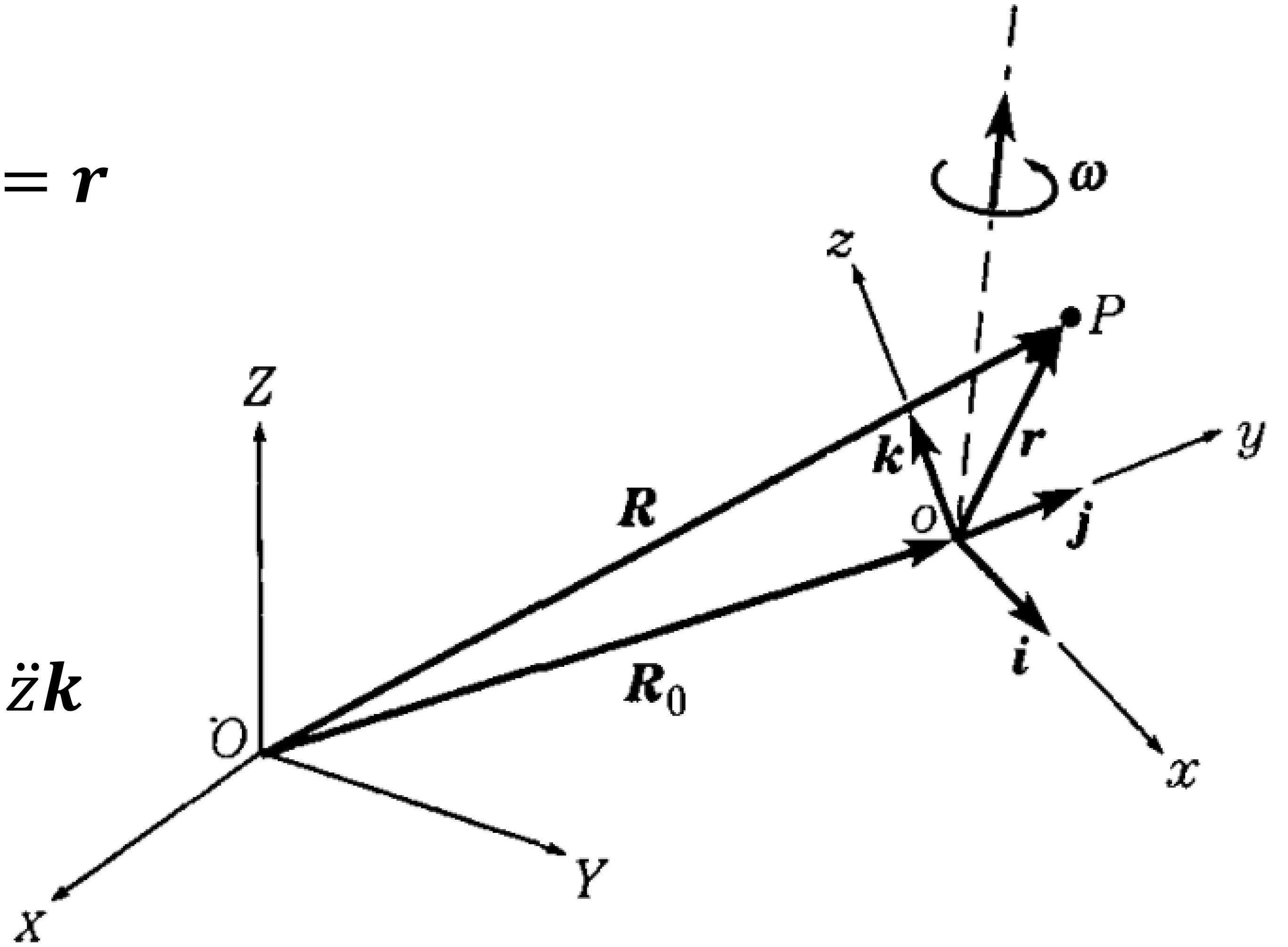
$Position\ vector\ of\ P = \boldsymbol{R}$

$Relative\ position\ vector = \boldsymbol{r}$

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\boldsymbol{v}_{\boldsymbol{rel}} = \frac{\partial \boldsymbol{r}}{\partial t} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$$

$$\boldsymbol{a}_{\boldsymbol{rel}} = \frac{\partial \boldsymbol{v}_{\boldsymbol{rel}}}{\partial t} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}$$



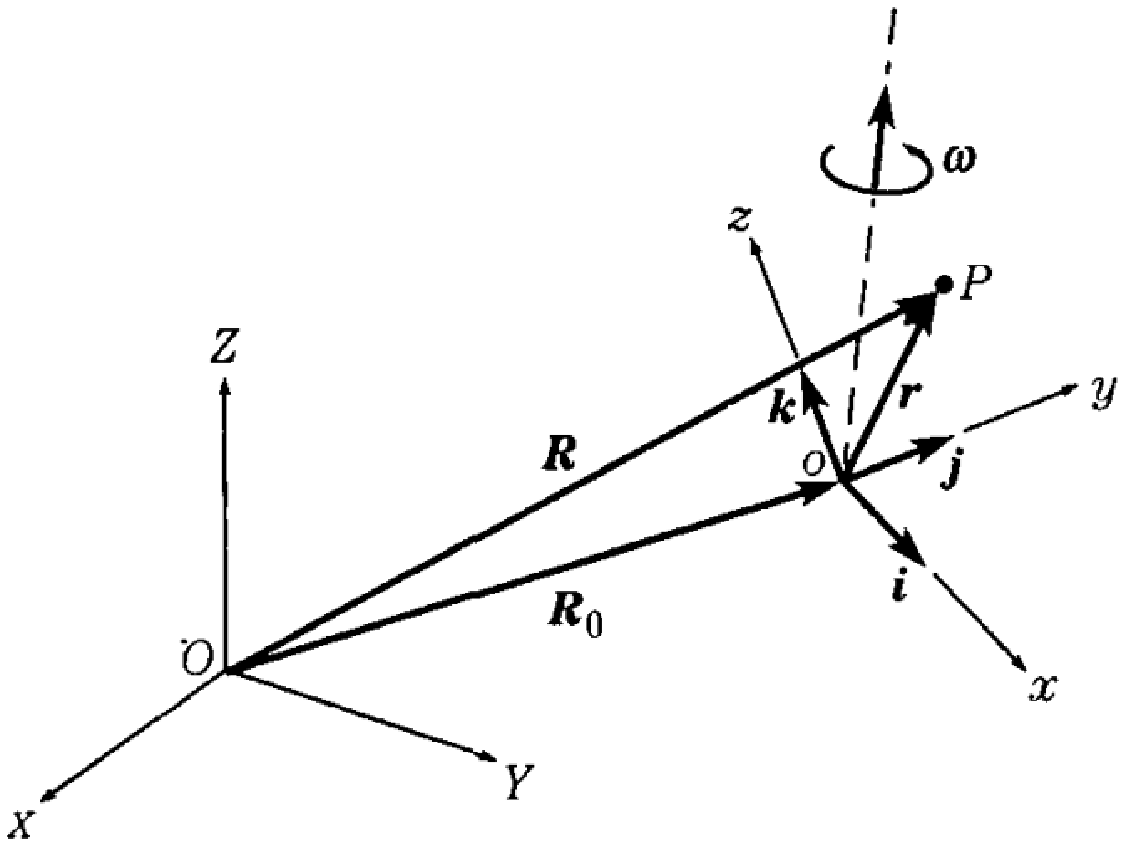
$$\boldsymbol{R} = \boldsymbol{r} + \ \boldsymbol{R_0}$$

$$\frac{d\boldsymbol{R}}{dt} = \frac{d}{dt}(\boldsymbol{r} + \ \boldsymbol{R_0})$$

$$\frac{d\boldsymbol{R}}{dt} = \frac{d\boldsymbol{r}}{dt} + \frac{d\boldsymbol{R_0}}{dt}$$

$$\frac{d\boldsymbol{r}}{dt} = \left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_{\text{rel}} + \omega \times \boldsymbol{r}$$

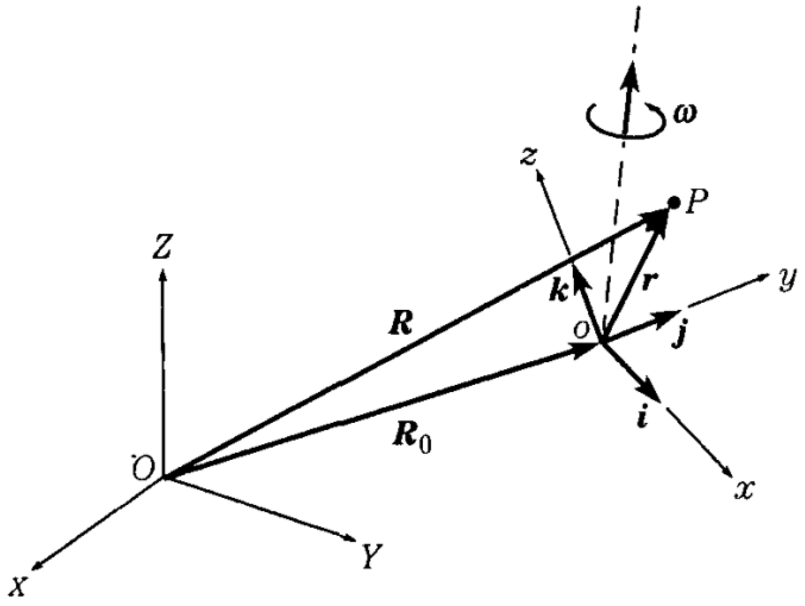
$$\boxed{v = \frac{d\boldsymbol{R_0}}{dt} + v_{rel} + \omega \times \boldsymbol{r}}$$



$$\boxed{v = \frac{d\boldsymbol{R_0}}{dt} + v_{rel} + \omega \times \boldsymbol{r}}$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{R_0}}{dt^2} + \frac{dv_{rel}}{dt} + \frac{d(\omega \times \boldsymbol{r})}{dt}$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{R_0}}{dt^2} + a_{rel} + \frac{d(\omega)}{dt} \times \boldsymbol{r} + \omega \times \frac{d(\boldsymbol{r})}{dt}$$



$$\frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{R_0}}{dt^2} + \boldsymbol{a_{rel}} + \dot{\omega} \times \boldsymbol{r} + \omega \times \frac{d(\boldsymbol{r})}{dt}$$

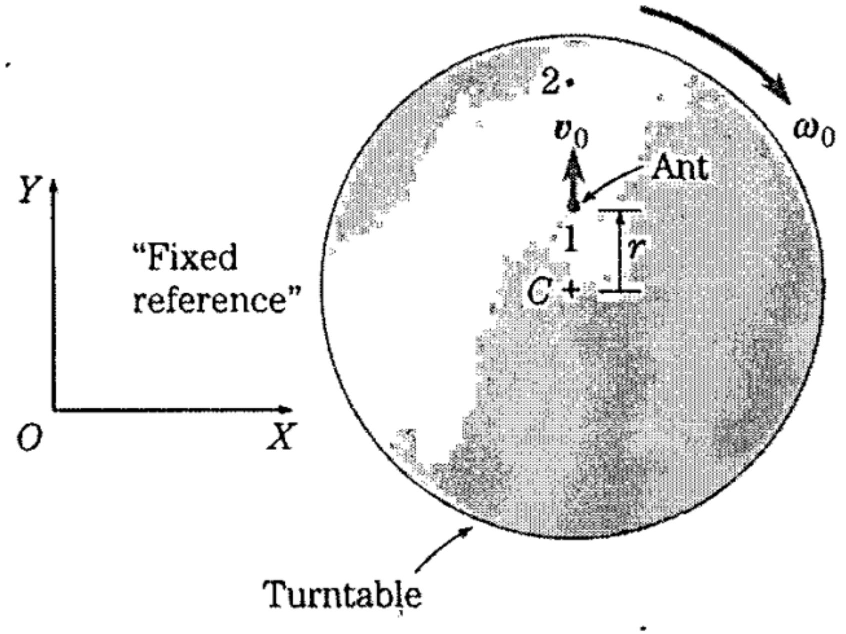
$$\frac{d\boldsymbol{r}}{dt} = \left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_{\mathrm{rel}} + \omega \times \boldsymbol{r}$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{R_0}}{dt^2} + \boldsymbol{a_{rel}} + \dot{\omega} \times \boldsymbol{r} + \omega \times \left[\left(\frac{\partial r}{\partial t}\right)_{\mathrm{rel}} + \omega \times \boldsymbol{r}\right]$$

$$\boxed{a = \frac{d^2\boldsymbol{R_0}}{dt^2} + \boldsymbol{a_{rel}} + \dot{\omega} \times \boldsymbol{r} + \omega \times v_{rel} + \omega \times (\omega \times \boldsymbol{r})}$$

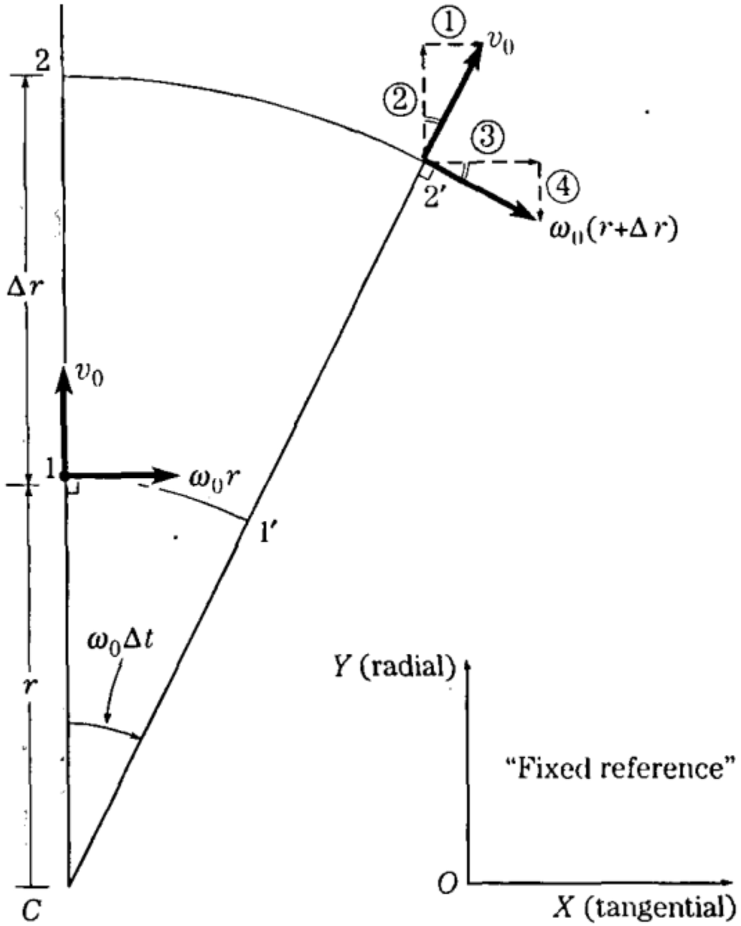
Question

A turntable is rotating at a *constant* angular velocity **ω_0** about its fixed center **C** (Figure 3-20). An ant (which we model as a point) is walking along a radius of the turntable at a constant velocity **v_0**, relative to the turntable. The ant is currently at point 1 and wants to get to the nearby point 2 (although it does not know why it wants to get to point 2). Although **ω_0** and **v_0** are constants, we seek the acceleration of the bug, relative to a fixed reference frame, which we denote as **OXYZ**.



Solution

$$\sin\omega_0\Delta t \approx \omega_0\Delta t \qquad \text{and} \qquad \cos\omega_0\Delta t \approx 1$$



1. Acceleration in radial direction ("fixed" Y direction): First, the change in radial velocity Δv_{radial} is simply the new radial velocity, which is at point 2′, minus the old radial velocity, which is at point 1.

$$\begin{aligned}\Delta v_{\text{radial}} &= [\text{New Radial Velocity}] - [\text{Old Radial Velocity}] \\ &= [\text{Term } ② + \text{Term } ④] - [v_0]\end{aligned} \qquad \text{(b)}$$

where, from Figure 3-21, term ② and term ④ refer to the radial velocity components at point 2′. Then

$$\Delta v_{\text{radial}} = [v_0\cos(\omega_0\Delta t) - \omega_0(r + v_0\Delta t)\sin(\omega_0\Delta t)] - [v_0] \qquad \text{(c)}$$

where $\Delta r = v_0\Delta t$ has been used in writing Eq. (c). Expanding Eq. (c) gives

$$\Delta v_{\text{radial}} = \left[v_0 - \omega_0^2 r\Delta t - \omega_0^2 v_0(\Delta t)^2\right] - [v_0] \qquad \text{(d)}$$

where Eqs. (a) have been used.

Dividing both sides of Eq. (d) by Δt and taking the limit as Δt goes to zero give

$$a_{\text{radial}} = \lim_{\Delta t\to 0}\frac{\Delta v_{\text{radial}}}{\Delta t} = -\omega_0^2 r$$

2. Acceleration in tangential direction ("fixed" X direction): Next, the change in the velocity in the tangential direction $\Delta v_{\text{tangential}}$ is simply the new tangential velocity, which is at point $2'$, minus the old tangential velocity, which is at point 1.

$$\begin{aligned}\Delta v_{\text{tangential}} &= [\text{New Tangential Velocity}] - [\text{Old Tangential Velocity}] \\ &= [\text{Term } ① + \text{Term } ③] - [\omega_0 r]\end{aligned} \tag{f}$$

where term ① and term ③ refer to the tangential velocity components at point $2'$ as sketched in Figure 3-21. Then

$$\Delta v_{\text{tangential}} = [v_0 \sin(\omega_0 \Delta t) + \omega_0 (r + v_0 \Delta t) \cos(\omega_0 \Delta t)] - [\omega_0 r] \tag{g}$$

where again $\Delta r = v_0 \Delta t$ has been used in writing Eq. (g). Expanding Eq. (g) gives

$$\Delta v_{\text{tangential}} = [\omega_0 v_0 \Delta t + \omega_0 r + \omega_0 v_0 \Delta t] - [\omega_0 r] \tag{h}$$

where Eqs. (a) have been used. Dividing both sides of Eq. (h) by Δt and taking the limit as Δt goes to zero give

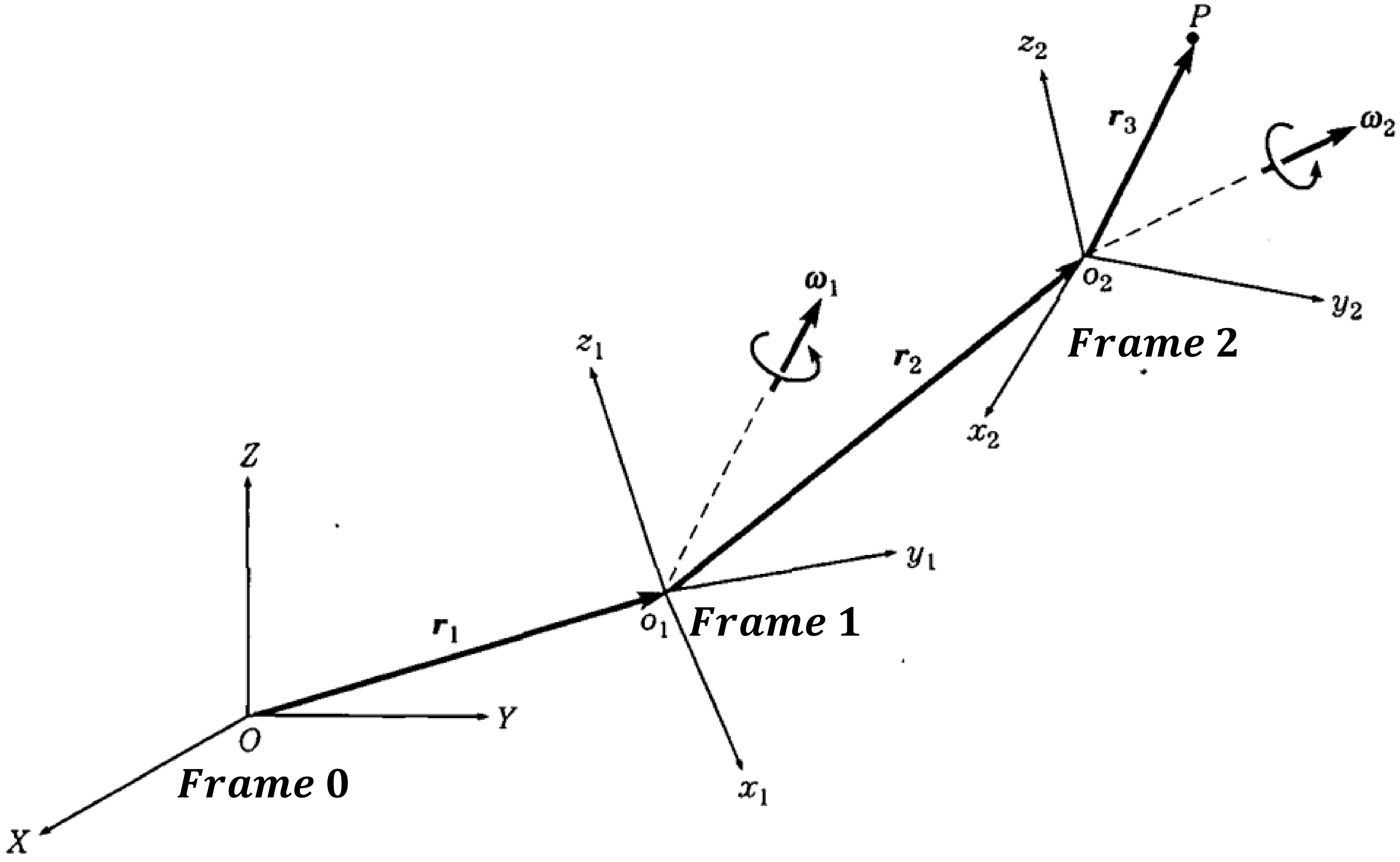
$$a_{\text{tangential}} = \lim_{\Delta t \to 0} \frac{\Delta v_{\text{tangential}}}{\Delta t} = \omega_0 v_0 + \omega_0 v_0 = 2\omega_0 v_0 \tag{i}$$

TABLE 3-1 Summary of Major Kinematic Results for Points

	Name	Variable or Equation
Motion of Point (defined in $OXYZ$) with respect to $OXYZ$. (Fig. 3-1)	Position Vector	$\boldsymbol{R}(t)$
	Velocity Vector	$\boldsymbol{v} = \dfrac{d\boldsymbol{R}}{dt}$
	Acceleration Vector	$\boldsymbol{a} = \dfrac{d^2\boldsymbol{R}}{dt^2}$
Motion of Point (defined in $oxyz$) with respect to $OXYZ$. (Fig. 3-19)	Position Vector	$\boldsymbol{R}(t) = \boldsymbol{R}_0(t) + \boldsymbol{r}(t)$
	Differential Operator	$\dfrac{d}{dt} = \left(\dfrac{\partial}{\partial t}\right)_{\mathrm{rel}} + \boldsymbol{\omega}\times$
	Velocity Vector	$\boldsymbol{v} = \dfrac{d\boldsymbol{R}_0}{dt} + \boldsymbol{v}_{\mathrm{rel}} + \boldsymbol{\omega} \times \boldsymbol{r}$
	Acceleration Vector	$\boldsymbol{a} = \dfrac{d^2\boldsymbol{R}_0}{dt^2} + \boldsymbol{a}_{\mathrm{rel}} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{rel}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$

Generalizations of kinematic expressions

Section 2

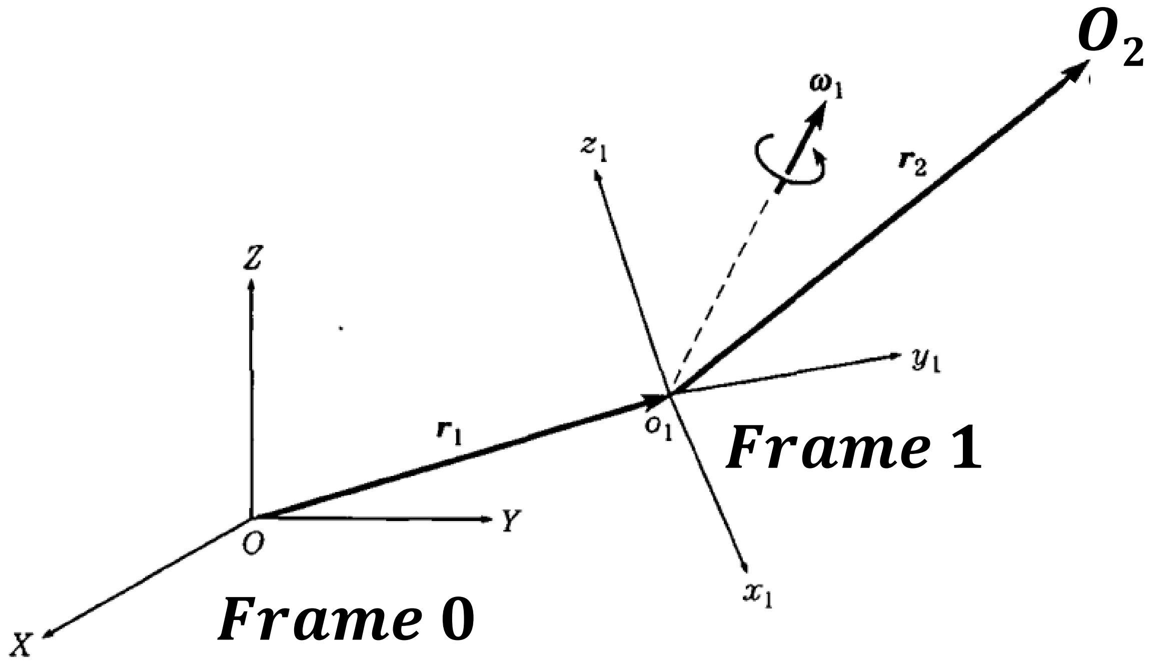


P
z_2
$\boldsymbol{r}_3$
ω_2
$\boldsymbol{\omega}_1$
o_2
y_2
z_1
$\boldsymbol{r}_2$
$Frame\ 2$
x_2
Z
y_1
$\boldsymbol{r}_1$
o_1
$Frame\ 1$
Y
O
$Frame\ 0$
x_1
X

Case I

The angular motions of $o_2 x_2 y_2 z_2$, ω_2 and $\dot{\omega}_2$ are defined w.r.t. frame OXYZ.

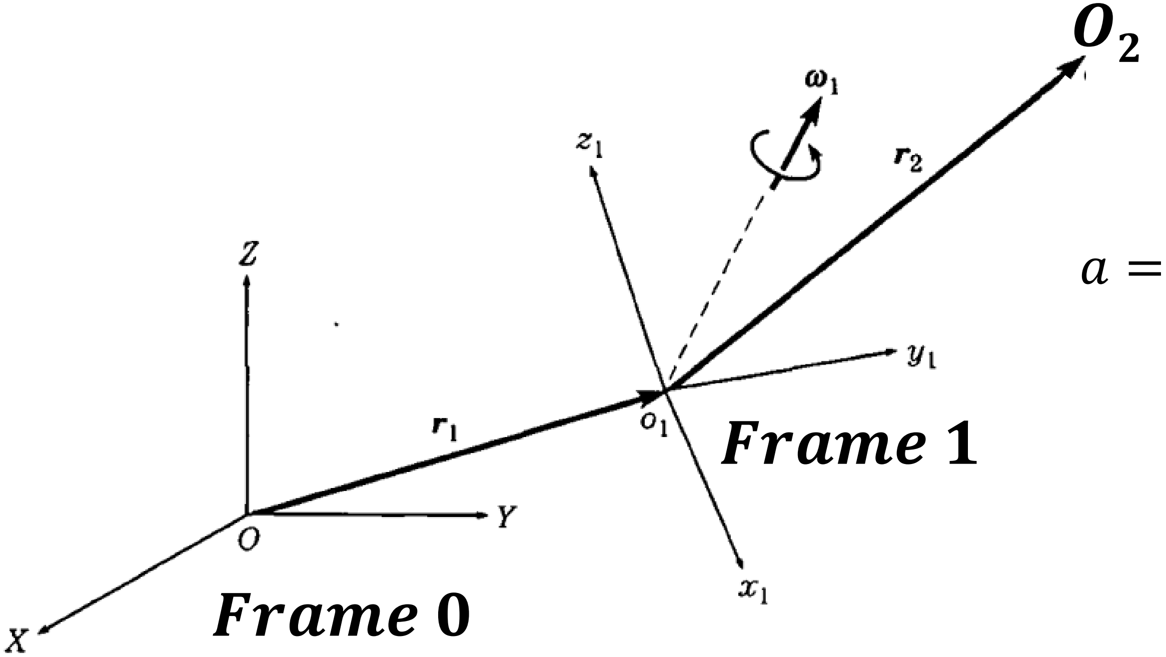
Step 01
Neglect the Frame 2, $o_2 x_2 y_2 z_2$ *and keep* o_2 *as a point.*



Case I

The angular motions of $o_2 x_2 y_2 z_2$, ω_2 and $\dot{\omega}_2$ are defined w.r.t. frame OXYZ.

Step 02
Write the velocity and acceleration equations for O_2 *point w.r.t. Frame 0.*



$$v = \frac{d\boldsymbol{R_0}}{dt} + v_{rel} + \omega \times \boldsymbol{r}$$

$$a = \frac{d^2\boldsymbol{R_0}}{dt^2} + \boldsymbol{a_{rel}} + \dot{\omega} \times \boldsymbol{r} + \omega \times v_{rel} + \omega \times (\omega \times \boldsymbol{r})$$

$$R_0 = r_1$$
$$r = r_2$$
$$\omega = \omega_1$$

$$\frac{d\boldsymbol{R_0}}{dt} = \dot{r}_1 \qquad \frac{d^2\boldsymbol{R_0}}{dt^2} = \ddot{r}_1$$

$$v_{rel} = \dot{r}_2 \qquad a_{rel} = \ddot{r}_2$$

$$\dot{\omega} = \dot{\omega}_{\boldsymbol{1}}$$

$$v = \frac{d\boldsymbol{R_0}}{dt} + v_{rel} + \omega \times \boldsymbol{r}$$

$$a = \frac{d^2\boldsymbol{R_0}}{dt^2} + \boldsymbol{a_{rel}} + \dot{\omega} \times \boldsymbol{r} + \omega \times v_{rel} + \omega^2 \times \boldsymbol{r}$$

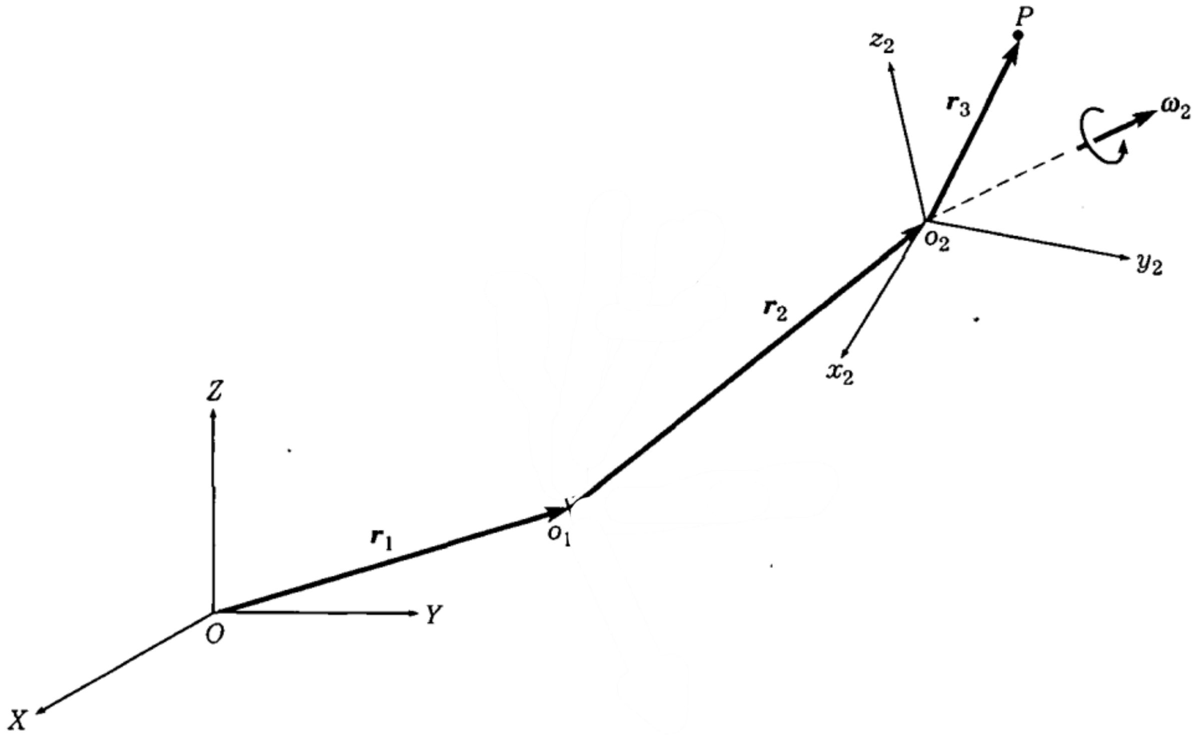
$$v_{O_2} = \dot{r}_1 + \dot{r}_2 + \omega_1 \times r_2$$

$$a_{O_2} = \ddot{r}_1 + \ddot{r}_2 + \dot{\omega}_{\boldsymbol{1}} \times r_2 + \omega_1 \times \dot{r}_2 + {\omega_1}^2 \times r_2$$

Case I

The angular motions of $o_2 x_2 y_2 z_2$, ω_2 and $\dot{\omega}_2$ are defined w.r.t. frame OXYZ.

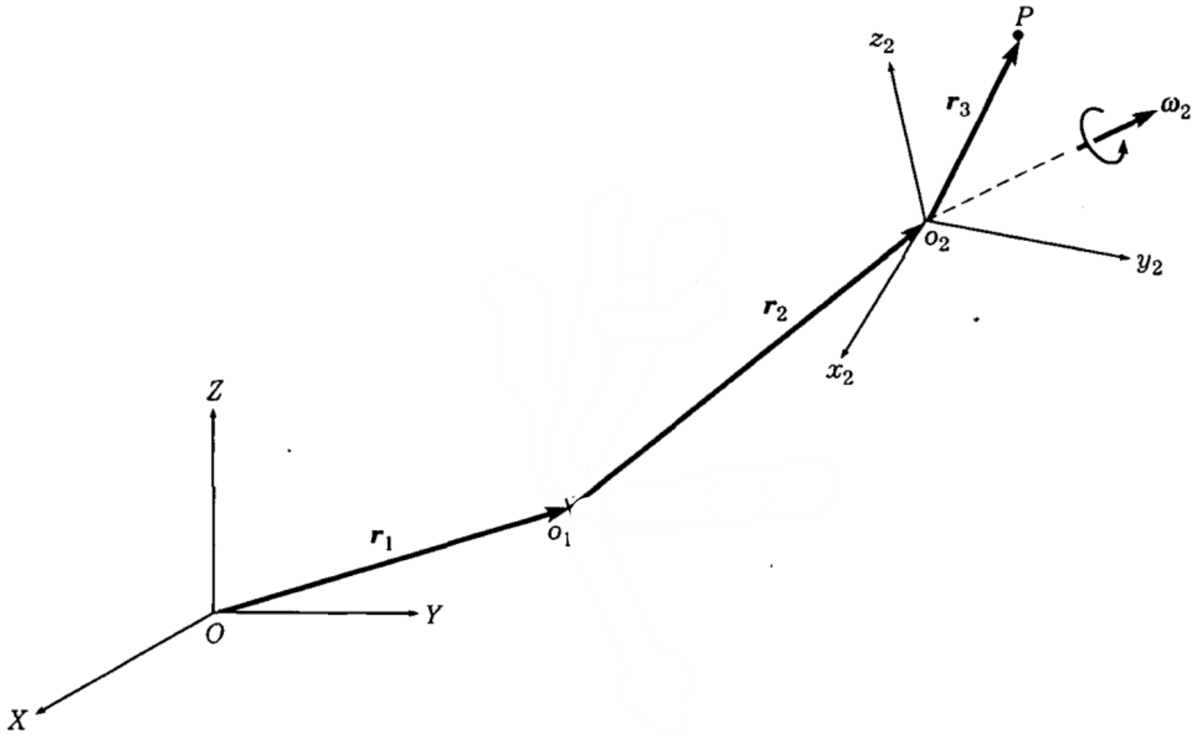
Step 03
Restore Frame 2 and neglect the Frame 1, $o_1 x_1 y_1 z_1$ *and keep* o_1 *as a point.*



Case I

The angular motions of $o_2 x_2 y_2 z_2$, ω_2 and $\dot{\omega}_2$ are defined w.r.t. frame OXYZ.

Step 04
As the step 02, apply velocity and acceleration equations for point P.



$$R_0 = r_1 + r_2$$
$$r = r_3$$
$$\omega = \omega_2$$

$$\frac{d\boldsymbol{R_0}}{dt} = v_{O_2} \qquad \frac{d^2\boldsymbol{R_0}}{dt^2} = a_{O_2}$$

$$v_{rel} = \dot{r_3} \qquad a_{rel} = \ddot{r_3}$$

$$\dot{\omega} = \dot{\omega}_{\boldsymbol{2}}$$

$$v = \frac{d\boldsymbol{R_0}}{dt} + v_{rel} + \omega \times \boldsymbol{r}$$

$$v_{O_2} = v_{O_2} + \dot{r_3} + \omega_1 \times r_3$$

$$v_{O_2} = \dot{r_1} + \dot{r_2} + \omega_1 \times r_2 + \dot{r_3} + \omega_1 \times r_3$$

$$a = \frac{d^2\boldsymbol{R_0}}{dt^2} + \boldsymbol{a_{rel}} + \dot{\omega} \times \boldsymbol{r} + \omega \times v_{rel} + \omega^2 \times \boldsymbol{r}$$

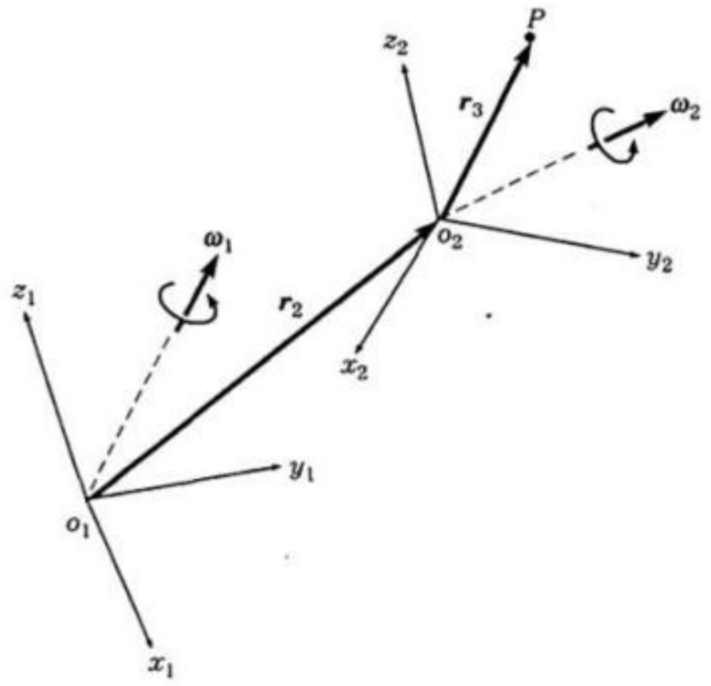
$$a_{O_2} = a_{O_2} + \ddot{r_3} + \dot{\omega}_{\boldsymbol{2}} \times r_3 + \omega_2 \times \dot{r_3} + {\omega_2}^2 \times r_3$$

$$a_{O_2} = \ddot{r_1} + \ddot{r_2} + \dot{\omega}_{\boldsymbol{1}} \times r_2 + \omega_1 \times \dot{r_2} + {\omega_1}^2 \times r_2 + \ddot{r_3} + \dot{\omega}_{\boldsymbol{2}} \times r_3 + \omega_2 \times \dot{r_3} + \omega_2 \times (\omega_2 \times r_3)$$

Case II

The angular motions of $o_2x_2y_2z_2$, ω_2 *and* $\dot{\omega}_2$, *are defined w.r.t frame* $o_1x_1y_1z_1$.

Step 01
Neglect the Frame 1, OXYZ and apply velocity and acceleration equations.



$$\boldsymbol{R}_0 = \boldsymbol{r}_2 \qquad \frac{d\boldsymbol{R}_0}{dt} = \dot{\boldsymbol{r}}_2 \qquad \frac{d^2\boldsymbol{R}_0}{dt^2} = \ddot{\boldsymbol{r}}_2$$

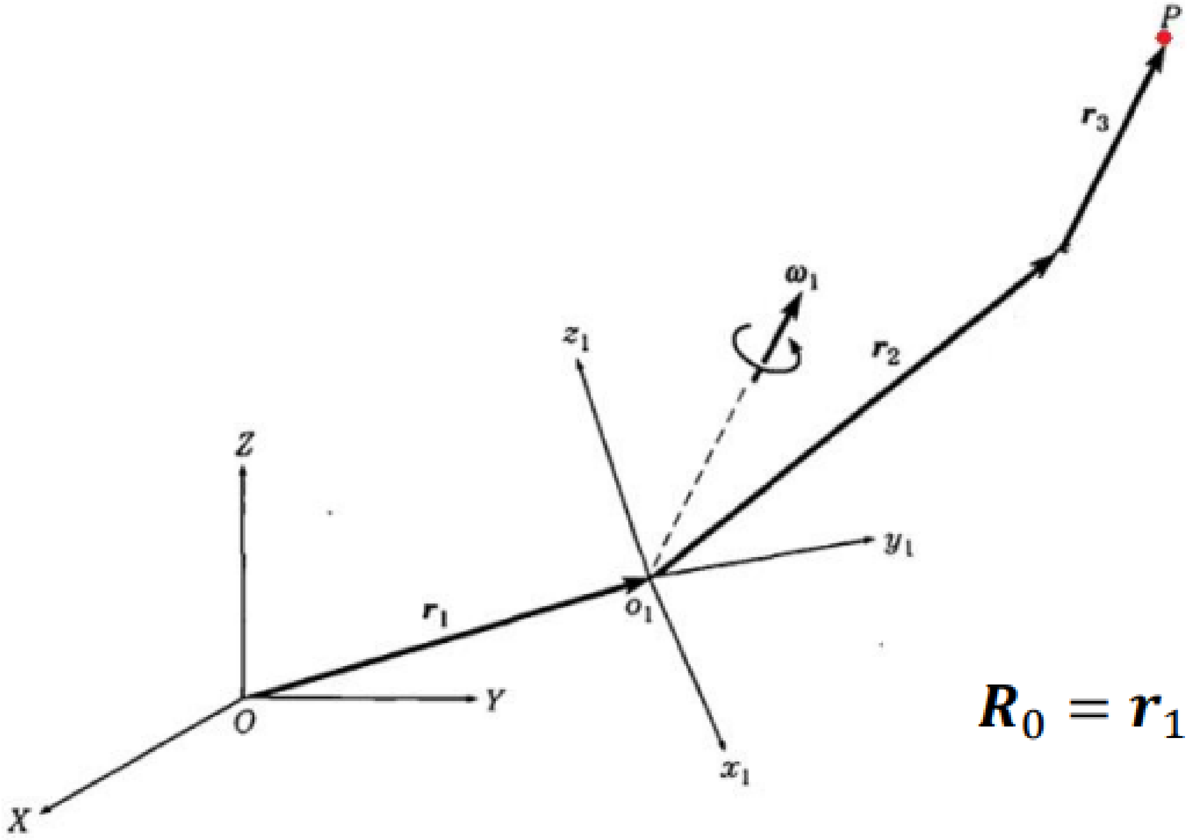
$$\boldsymbol{r} = \boldsymbol{r}_3 \qquad \boldsymbol{v}_{rel} = \dot{\boldsymbol{r}}_3 \qquad \boldsymbol{a}_{rel} = \ddot{\boldsymbol{r}}_3$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_2 \qquad \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_2$$

$$\boldsymbol{v}_{P(o_1x_1y_1z_1)} = \dot{\boldsymbol{r}}_2 + \dot{\boldsymbol{r}}_3 + \boldsymbol{\omega}_2 \times \boldsymbol{r}_3$$

$$\boldsymbol{a}_{P(o_1x_1y_1z_1)} = \ddot{\boldsymbol{r}}_2 + \ddot{\boldsymbol{r}}_3 + 2\boldsymbol{\omega}_2 \times \dot{\boldsymbol{r}}_3 + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_3 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_3)$$

Motion of point P (defined in $o_1x_1y_1z_1$) w.r.t OXYZ.



$$\boldsymbol{R}_0 = \boldsymbol{r}_1 \qquad \frac{d\boldsymbol{R}_0}{dt} = \dot{\boldsymbol{r}}_1 \qquad \frac{d^2\boldsymbol{R}_0}{dt^2} = \ddot{\boldsymbol{r}}_1$$

$$\boldsymbol{r} = \boldsymbol{r}_2 + \boldsymbol{r}_3 \qquad \boldsymbol{v}_{rel} = \boldsymbol{v}_{P(o_1x_1y_1z_1)} \qquad \boldsymbol{a}_{rel} = \boldsymbol{a}_{P(o_1x_1y_1z_1)}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 \qquad \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1$$

$$\boldsymbol{v}_{P(OXYZ)} = \dot{\boldsymbol{r}}_1 + \boldsymbol{v}_{P(o_1x_1y_1z_1)} + \boldsymbol{\omega}_1 \times (\boldsymbol{r}_2 + \boldsymbol{r}_3)$$

$$= \dot{\boldsymbol{r}}_1 + \dot{\boldsymbol{r}}_2 + \dot{\boldsymbol{r}}_3 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_2 + (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \boldsymbol{r}_3$$

$$\boldsymbol{a}_{P(OXYZ)}$$

$$= \ddot{\boldsymbol{r}}_1 + \boldsymbol{a}_{P(o_1x_1y_1z_1)} + 2\boldsymbol{\omega}_1 \times \boldsymbol{v}_{P(o_1x_1y_1z_1)} + \dot{\boldsymbol{\omega}}_1(\boldsymbol{r}_2 + \boldsymbol{r}_3) + \boldsymbol{\omega}_1$$

$$\times [\boldsymbol{\omega}_1 \times (\boldsymbol{r}_2 + \boldsymbol{r}_3)]$$

$$= \ddot{\boldsymbol{r}}_1 + \ddot{\boldsymbol{r}}_2 + \ddot{\boldsymbol{r}}_3 + 2\boldsymbol{\omega}_1 \times \dot{\boldsymbol{r}}_2 + 2(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \dot{\boldsymbol{r}}_3 + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_2$$

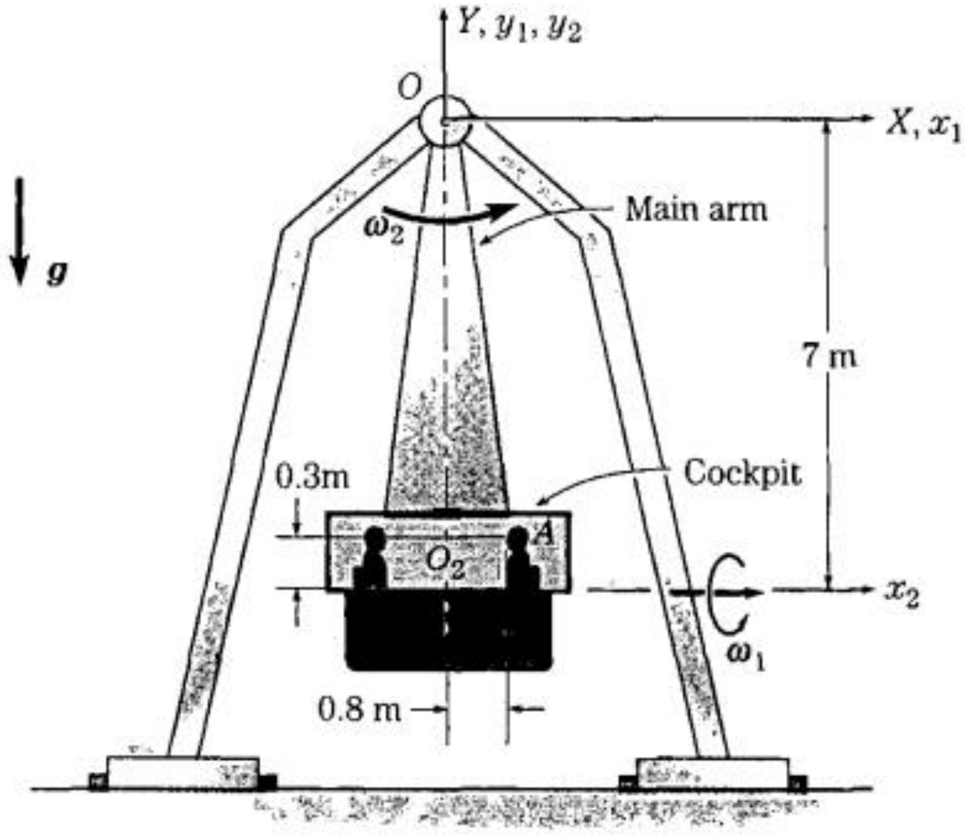
$$+ (\dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2) \times \boldsymbol{r}_3 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_3) + 2\boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_3) + \boldsymbol{\omega}_1$$

$$\times [\boldsymbol{\omega}_1 \times (\boldsymbol{r}_2 + \boldsymbol{r}_3)]$$

Questions

Problem 3-35: A ride at an amusement park is sketched in Figure P3-35. The cockpit containing two passengers rotates at an angular velocity $\omega_1 = 3$ rad/s, relative to the main arm, and the main arm rotates at an angular velocity $\omega_2 = 1$ rad/s relative to ground. Point A, which corresponds to an eye of a passenger, is located 0.8 m off the centerline of the main arm and 0.3 m above the cockpit's rotating axis, which is also 7 m from the main arm's rotating axis. Note that gravity acts. For the instant shown, find the velocity and acceleration of point A.

The reference frames defined in Figure P3-35 are as follows: $OXYZ$ is fixed to ground; $Ox_1y_1z_1$ is fixed to the main arm; and $ox_2y_2z_2$ is fixed to the cockpit.



Problem 3-30: A ride at an amusement park consists of four symmetrically located seats driven to rotate about the vertical A axis at a constant rate of 10 rev/min, with respect to the supporting arm AB which is 6 m long. The vertical B axis is driven by another motor at a constant rate of 6 rev/min, with respect to ground. All four seats are 2 m from the A axis, and the A and B axes are parallel, as sketched in Figure P3-30. The mass of each passenger is 75 kg. Note that gravity acts. Find the velocities and accelerations of all four passengers at the instant shown.

The reference frames defined in Figure P3-30 are as follows: $BXYZ$ is fixed to ground; $Bx_1y_1z_1$ is fixed to the arm AB; and $Ax_2y_2z_2$ is fixed to the frame that carries the seats.

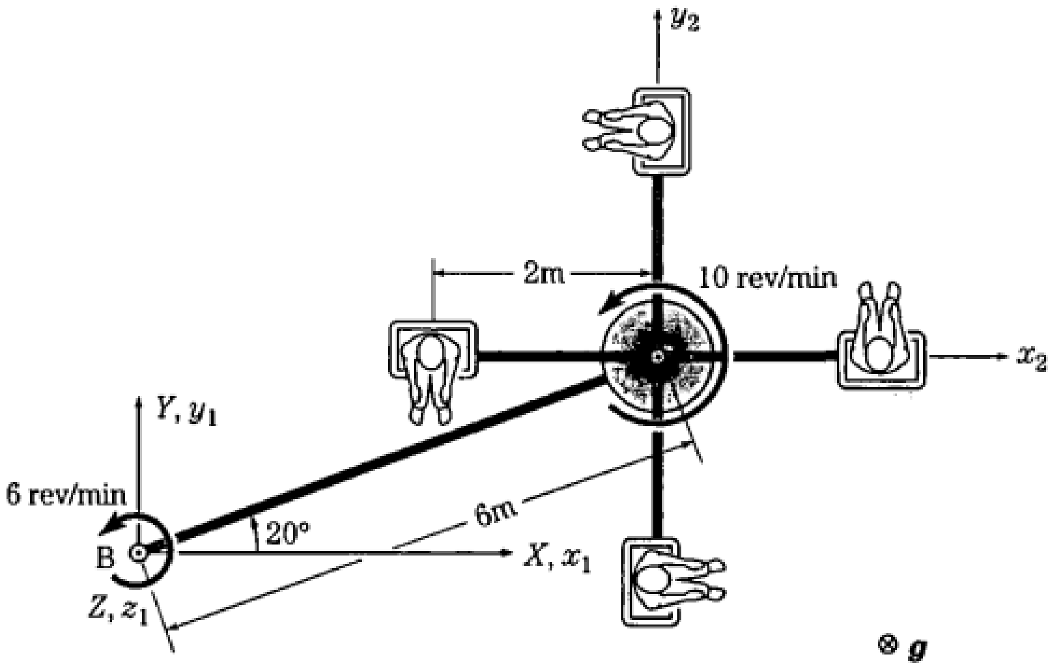


FIGURE P3-30

Momentum Analysis

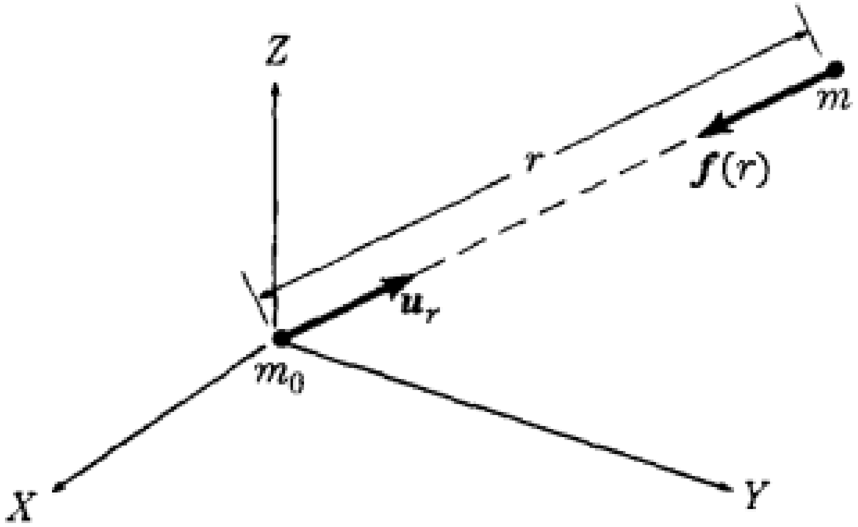
Section 3

Linear Momentum and Force

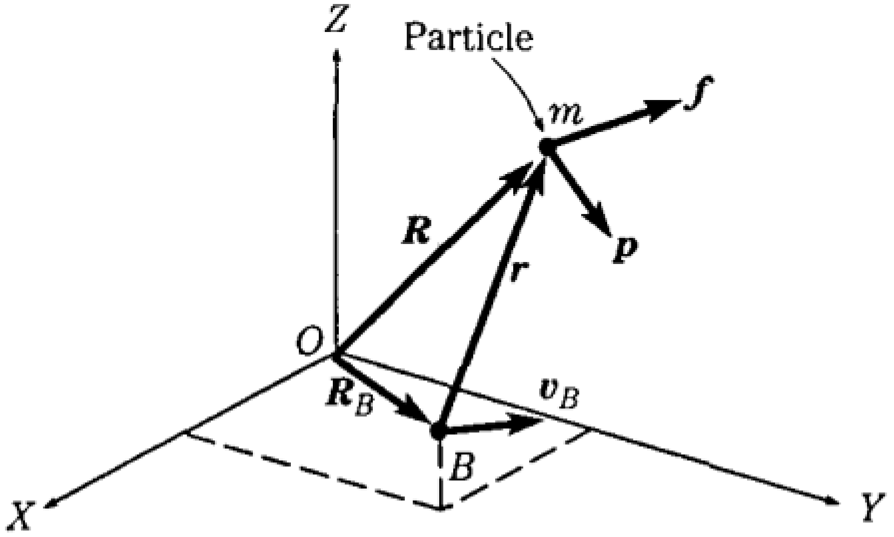
$$\boldsymbol{p} = m\boldsymbol{v}$$

$$\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt}$$

$$\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{f}_{\boldsymbol{i}} = \frac{d\boldsymbol{p}}{dt}$$



Torque and angular momentum for a particle



$$Velocity\ of\ point\ B = \frac{d\boldsymbol{R_B}}{dt}$$

$$\boldsymbol{\tau_B} = \boldsymbol{r} \times \boldsymbol{f}$$

$$\boldsymbol{h_B} = \boldsymbol{r} \times \boldsymbol{p}$$

$$\boldsymbol{r} \times \boldsymbol{f} = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt}$$

$$\frac{d(\boldsymbol{r} \times \boldsymbol{p})}{dt} = \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt}$$

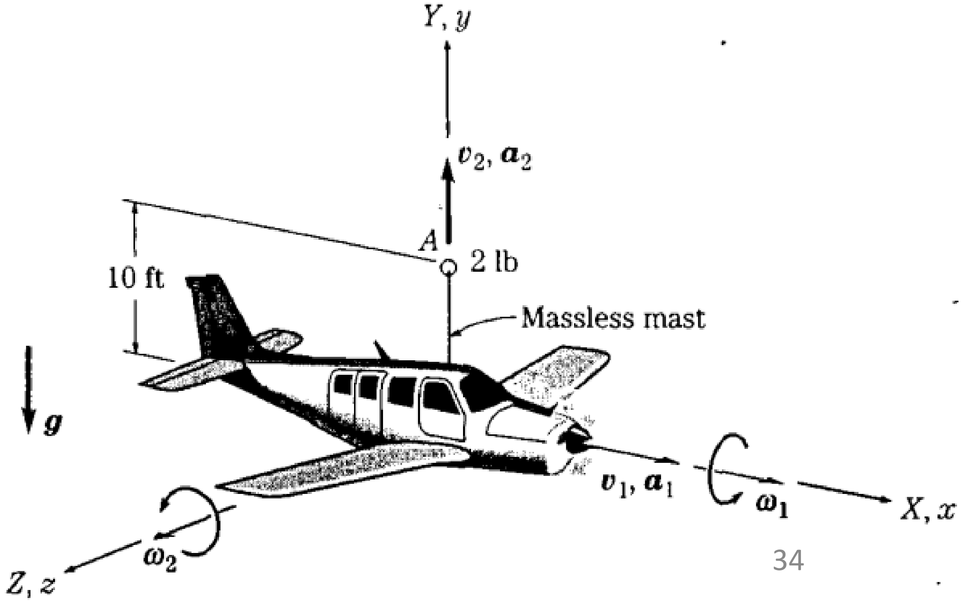
$$\boldsymbol{r}\ \frac{d\boldsymbol{p}}{dt} = \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} - \frac{d(\boldsymbol{r} \times \boldsymbol{p})}{dt}$$

$$\begin{aligned} \boldsymbol{r} \times \boldsymbol{f} &= \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) - \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} \\ &= \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) - \left(\frac{d\boldsymbol{R}}{dt} - \frac{d\boldsymbol{R}_B}{dt}\right) \times \boldsymbol{p} \\ &= \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) - (\boldsymbol{v} - \boldsymbol{v}_B) \times \boldsymbol{p} \end{aligned}$$

$$\boxed{\boldsymbol{\tau}_B = \frac{d\boldsymbol{h}_B}{dt} + \boldsymbol{v}_B \times \boldsymbol{p}}$$

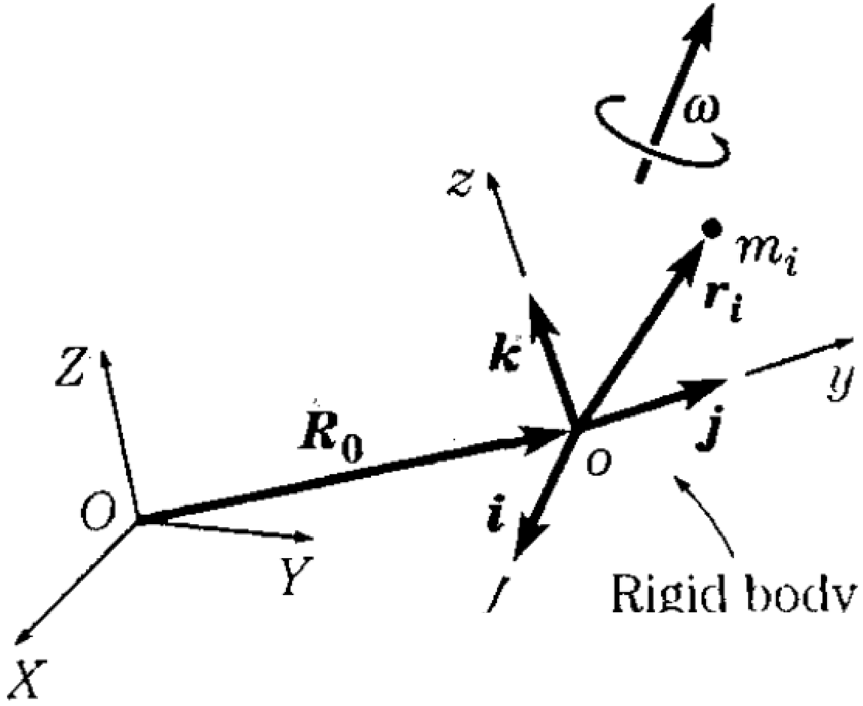
Question

An airplane moving with velocity v_1 and acceleration a_1, both with respect to ground (i.e., "fixed" space), is banking at $\boldsymbol{\omega}_1$ and climbing at $\boldsymbol{\omega}_2$, both with respect to "fixed" space, as sketched in Figure 3-24. An antenna A (weight 2 lb) is being deployed from the airplane. The airplane is in a horizontal orientation (with respect to gravity) and the antenna A is at a vertical distance of 10 ft from the centerline of the airplane, moving with velocity v_2 and acceleration a_2, both defined with respect to the airplane. Find the velocity and acceleration of the antenna A relative to "fixed" space, at the instant shown. Assume that v_1 is 200 ft/sec, a_1 is 100 ft/sec^2, v_2 is 6 ft/sec, a_2 is 0.1 ft/sec^2, $\boldsymbol{\omega}_1$ is 3 rad/min, and $\boldsymbol{\omega}_2$ is 2 rad/min. $OXYZ$ is a reference frame that is "fixed" in space, and $oxyz$ is an intermediate frame that is attached to the airplane.



Inertia Tensor

Section 4



$$\boldsymbol{H}_o = \sum_{i=1}^{N} \boldsymbol{r}_i \times m_i \boldsymbol{v}_i$$

$$= \sum_{i=1}^{N} \boldsymbol{r}_i \times m_i (\boldsymbol{\omega} \times \boldsymbol{r}_i)$$

$$= \sum_{i=1}^{N} m_i \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$$

$$\boldsymbol{H}_o = \sum_{i=1}^{N} m_i [(\boldsymbol{r}_i \cdot \boldsymbol{r}_i)\boldsymbol{\omega} - (\boldsymbol{r}_i \cdot \boldsymbol{\omega})\boldsymbol{r}_i]$$

$$\boldsymbol{H}_o = H_x \boldsymbol{i} + H_y \boldsymbol{j} + H_z \boldsymbol{k}$$

$$\boldsymbol{r}_i = x_i \boldsymbol{i} + y_i \boldsymbol{j} + z_i \boldsymbol{k}$$

$$\boldsymbol{\omega} = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k}$$

$$\boldsymbol{H}_o = \sum_{i=1}^{N} m_i \left[(x_i^2 + y_i^2 + z_i^2)(\omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k}) - (x_i \omega_x + y_i \omega_y + z_i \omega_z)(x_i \boldsymbol{i} + y_i \boldsymbol{j} + z_i \boldsymbol{k}) \right]$$

$$H_x = \sum_{i=1}^{N} m_i \left[x_i^2 \omega_x + y_i^2 \omega_x + z_i^2 \omega_x - x_i^2 \omega_x - x_i y_i \omega_y - x_i z_i \omega_z\right] \quad (6\text{-}28)$$

By factoring out the angular velocity components in Eq. (6-28), H_x may be written as

$$H_x = \left[\sum_{i=1}^{N} m_i (y_i^2 + z_i^2)\right] \omega_x + \left[-\sum_{i=1}^{N} m_i (x_i y_i)\right] \omega_y + \left[-\sum_{i=1}^{N} m_i (x_i z_i)\right] \omega_z \quad (6\text{-}29)$$

Equation (6-27) contains analogous terms for H_y and H_z, which we may identify as

$$H_y = \left[-\sum_{i=1}^{N} m_i (y_i x_i)\right] \omega_x + \left[\sum_{i=1}^{N} m_i (x_i^2 + z_i^2)\right] \omega_y + \left[-\sum_{i=1}^{N} m_i (y_i z_i)\right] \omega_z \quad (6\text{-}30)$$

and

$$H_z = \left[-\sum_{i=1}^{N} m_i (z_i x_i)\right] \omega_x + \left[-\sum_{i=1}^{N} m_i (z_i y_i)\right] \omega_y + \left[\sum_{i=1}^{N} m_i (x_i^2 + y_i^2)\right] \omega_z \quad (6\text{-}31)$$

Equations (6-29), (6-30) and (6-31) may be collected in the form

$$
\begin{aligned}
H_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\
H_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\
H_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z
\end{aligned} \tag{6-32}
$$

where

$$
I_{xx} = \sum_{i=1}^{N} m_i\left(y_i^2 + z_i^2\right) \qquad I_{yy} = \sum_{i=1}^{N} m_i\left(x_i^2 + z_i^2\right) \qquad I_{zz} = \sum_{i=1}^{N} m_i\left(x_i^2 + y_i^2\right) \tag{6-33}
$$

and

$$
I_{xy} = -\sum_{i=1}^{N} m_i x_i y_i = I_{yx} \qquad I_{xz} = -\sum_{i=1}^{N} m_i x_i z_i = I_{zx} \qquad I_{yz} = -\sum m_i y_i z_i = I_{zy} \tag{6-34}
$$

The terms in Eqs. (6-33) are called the *moments of inertia* and the terms in Eqs. (6-34) are called the *products of inertia.**

Matrix Form

Equations (6-32) may be expressed in matrix notation as

$$\{H\}_o = [I]_o\{\omega\}$$

where

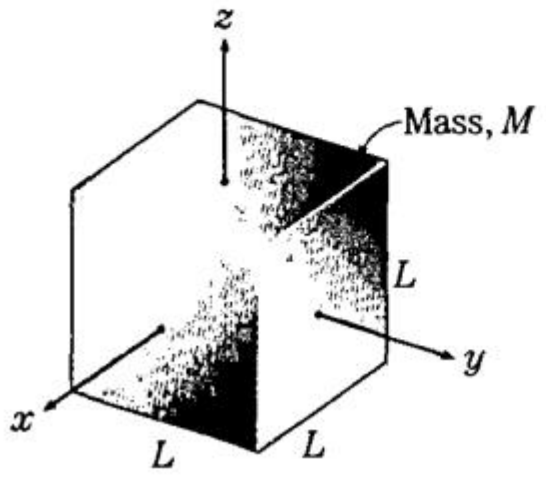
$$\{H\}_o = \begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix}_o \qquad \text{and} \qquad \{\omega\} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

are *column matrices* (also called *column vectors* because they are vectors) and

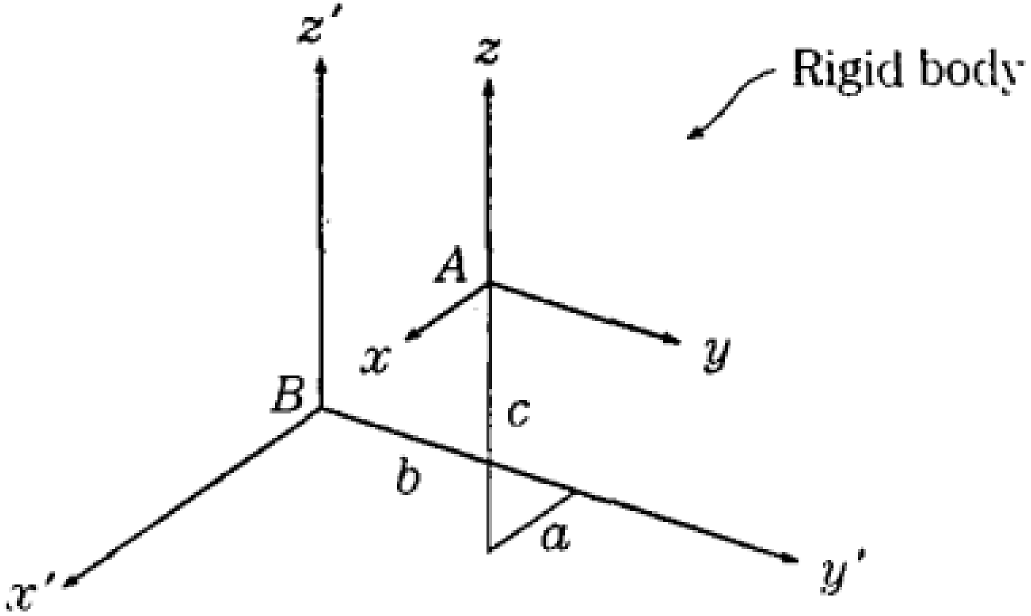
$$[I]_o = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}_o$$

Question

Find the inertia tensor for a uniform (that is, constant-density) rigid cube of mass M and edge length L about its center of mass for the central axes sketched in Figure 6-8.



Parallel Axis Theorem



$$x' = x + a \qquad y' = y + b \qquad z' = z + c$$

$$
\begin{aligned}
I_{z'z'} &= \sum_{i=1}^{N} m_i\left(x_i'^2 + y_i'^2\right) \\
&= \sum_{i=1}^{N} m_i\left[(x_i + a)^2 + (y_i + b)^2\right] \\
&= \sum_{i=1}^{N} m_i\left(x_i^2 + 2ax_i + a^2 + y_i^2 + 2by_i + b^2\right) \\
&= \sum_{i=1}^{N} m_i\left(x_i^2 + y_i^2\right) + \sum_{i=1}^{N} m_i\left(a^2 + b^2\right) + \sum_{i=1}^{N} 2m_i(ax_i + by_i)
\end{aligned}
$$

Recall from elementary mechanics that

$$M = \sum_{i=1}^{N} m_i$$

and

$$x_c = \frac{\sum_{i=1}^{N} m_i x_i}{M} \qquad y_c = \frac{\sum_{i=1}^{N} m_i y_i}{M} \qquad z_c = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

$$\boxed{I_{z'z'} = I_{zz} + M(a^2 + b^2) + 2M(ax_c + by_c)}$$

Product of Inertia Ext.

$$\begin{aligned} I_{x'y'} &= -\sum_{i=1}^{N} m_i x_i' y_i' \\ &= -\sum_{i=1}^{N} m_i (x_i y_i + ab + b x_i + a y_i) \\ &= I_{xy} - Mab - M(b x_c + a y_c) \end{aligned}$$

General form of inertia tensor

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}_B = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}_A + M \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & a^2 + c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix}$$

$$+ M \begin{bmatrix} 2(by_c + cz_c) & -(bx_c + ay_c) & -(cx_c + az_c) \\ -(bx_c + ay_c) & 2(cz_c + ax_c) & -(cy_c + bz_c) \\ -(cx_c + az_c) & -(cy_c + bz_c) & 2(ax_c + by_c) \end{bmatrix}$$

Find the moment of inertia of a structure consisting of a thin rod connecting two solid spheres about a vertical axis perpendicular to the rod and passing through its center.

Useful Formulas

- **Moment of inertia of a solid sphere about its own center:**

$$I_{\text{sphere, center}} = \frac{2}{5}MR^2$$

- **Parallel-axis theorem:**

$$I = I_{\text{center}} + Md^2$$

where $ d $ is the distance from the sphere's center to the axis.

- **Moment of inertia of a thin rod about its center (axis perpendicular to rod):**

$$I_{\text{rod, center}} = \frac{1}{12}ML^2$$