Kinematic analysis utilizing intermediate frames

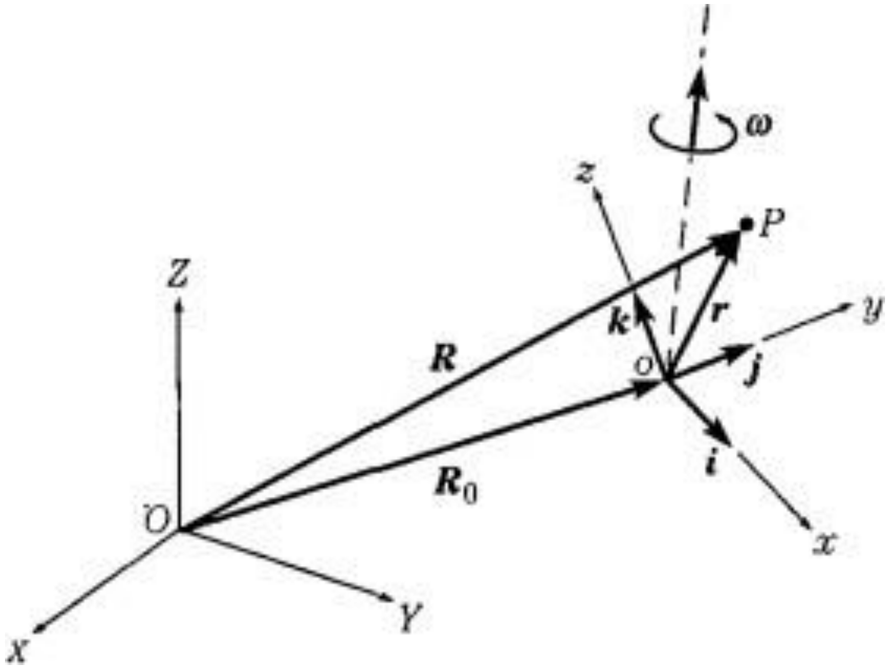
$OXYZ$ is the fixed reference frame.

$oxyz$ is the intermediate frame with unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

Frame $oxyz$ translates and rotates w.r.t the frame $OXYZ$.

Position vector of P w.r.t fixed frame is $\boldsymbol{R}$.

Relative position vector of P w.r.t intermediate frame is $\boldsymbol{r}$.



Goal:
Velocity ($\boldsymbol{v}$) and acceleration ($\boldsymbol{a}$) of point P w.r.t fixed frame ($OXYZ$).

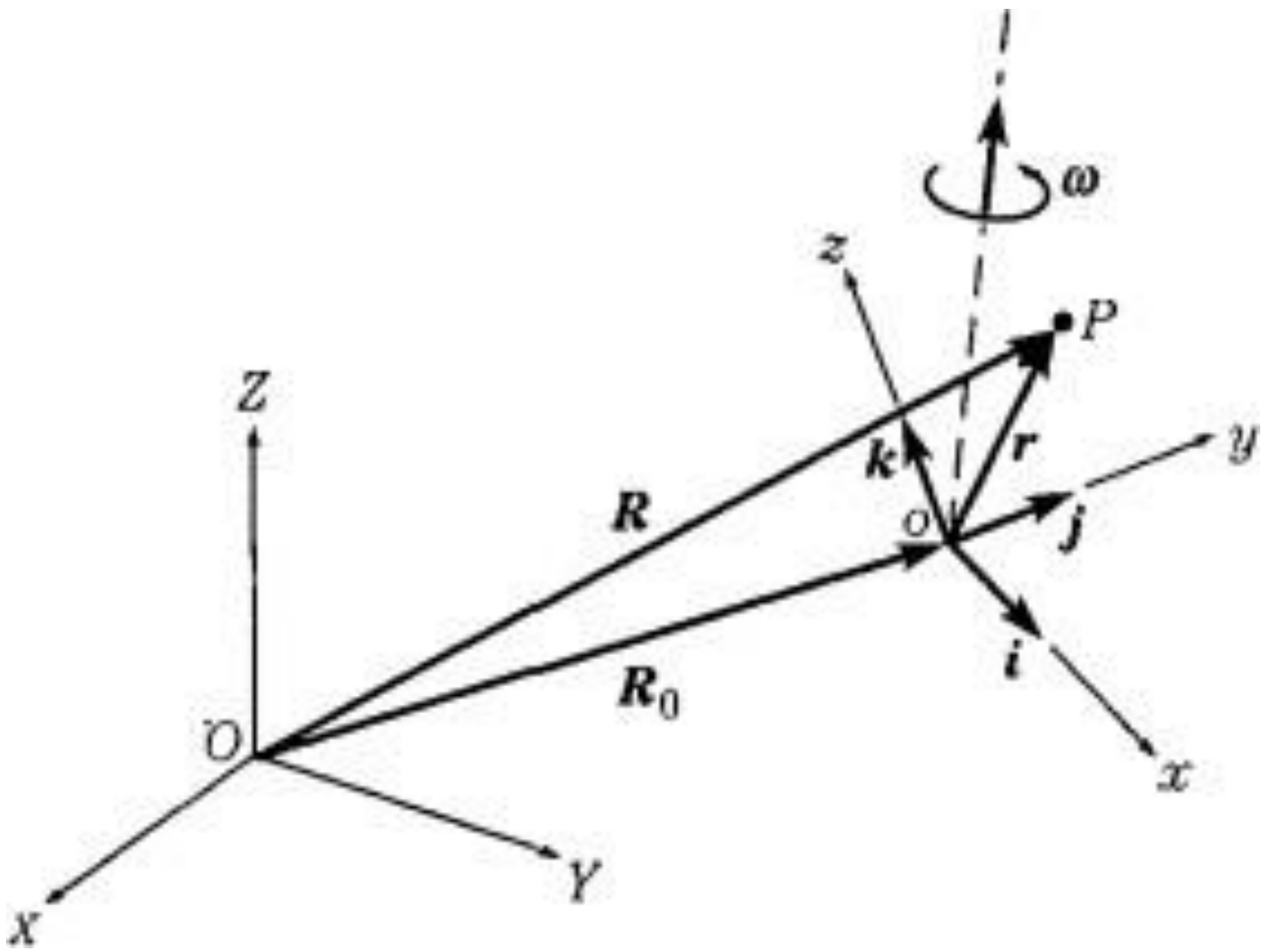
Relative position vector ‘$\boldsymbol{r}$’

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Velocity ($\boldsymbol{v}_{rel}$) and acceleration ($\boldsymbol{a}_{rel}$) of point P with respect (or relative) to intermediate frame ($oxyz$).

$$\boldsymbol{v}_{rel} = \left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_{rel} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$$

$$\boldsymbol{a}_{rel} = \left(\frac{\partial \boldsymbol{v}_{rel}}{\partial t}\right)_{rel} = \left(\frac{\partial^2 \boldsymbol{r}}{\partial t^2}\right)_{rel} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}$$


ω
z
P
Z
y
r
k
R
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i
R0
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Y

$$\boldsymbol{R} = \boldsymbol{R_0} + \boldsymbol{r}$$

Velocity of P w.r.t $OXYZ$ is

$$\boldsymbol{v} = \frac{d\boldsymbol{R}}{dt} = \frac{d\boldsymbol{R_0}}{dt} + \boxed{\frac{d\boldsymbol{r}}{dt}}$$

Time derivative of vector $\boldsymbol{r}$ that is defined with respect to oxyz.

In general

$$\frac{d(\quad)}{dt} = \left(\frac{\partial(\quad)}{\partial t}\right)_{rel} + \boldsymbol{\omega} \times (\quad)$$

$(\quad)$ - An arbitrary vector defined in intermediate frame.

This operational from provides a means for determining the time derivative w.r.t $OXYZ$ of a vector defined in $oxyz$, when $oxyz$ is rotating at $'\boldsymbol{\omega}'$ angular velocity w.r.t $OXYZ$.

$$\boldsymbol{v} = \frac{d\boldsymbol{R}_0}{dt} + \frac{d\boldsymbol{r}}{dt}$$

$$\boldsymbol{v} = \frac{d\boldsymbol{R}_0}{dt} + \left[\left(\frac{\partial \boldsymbol{r}}{\partial t} \right)_{rel} + \boldsymbol{\omega} \times \boldsymbol{r} \right]$$

$$\boldsymbol{v} = \frac{d\boldsymbol{R}_0}{dt} + \boldsymbol{v}_{rel} + (\boldsymbol{\omega} \times \boldsymbol{r})$$

- $\frac{d\boldsymbol{R}_0}{dt}$ - The velocity of the origin of the intermediate frame

- $\boldsymbol{v}_{rel}$ - The velocity of P w.r.t the intermediate frame

- $\boldsymbol{\omega} \times \boldsymbol{r}$ – Due to the rotation of the intermediate frame

Acceleration of P w.r.t $OXYZ$ is

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}\left[\frac{d\boldsymbol{R}_0}{dt} + \boldsymbol{v}_{rel} + (\boldsymbol{\omega} \times \boldsymbol{r})\right]$$

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{R}_0}{dt^2} + \frac{d\boldsymbol{v}_{rel}}{dt} + \frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{r})$$

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{R}_0}{dt^2} + \boxed{\frac{d\boldsymbol{v}_{rel}}{dt}} + \frac{d\,\boldsymbol{\omega}}{dt} \times \boldsymbol{r} + \boldsymbol{\omega} \times \boxed{\frac{d\boldsymbol{r}}{dt}}$$

$$\boldsymbol{a} = \frac{d^2\boldsymbol{R}_0}{dt^2} + \left[\left(\frac{\partial \boldsymbol{v}_{rel}}{\partial t}\right)_{rel} + \boldsymbol{\omega} \times \boldsymbol{v}_{rel}\right] + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \left[\left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_{rel} + \boldsymbol{\omega} \times \boldsymbol{r}\right]$$

$$\boldsymbol{a} = \frac{d^2\boldsymbol{R}_0}{dt^2} + \boldsymbol{a}_{rel} + \boldsymbol{\omega} \times \boldsymbol{v}_{rel} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{v}_{rel} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$\boldsymbol{a} = \frac{d^2\boldsymbol{R}_0}{dt^2} + \boldsymbol{a}_{rel} + 2\,\boldsymbol{\omega} \times \boldsymbol{v}_{rel} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$\frac{d^2 R_0}{dt^2}$ - The acceleration of the origin of the intermediate frame w.r.t $OXYZ$

$\boldsymbol{a}_{rel}$ - The relative acceleration of P w.r.t $oxyz$.

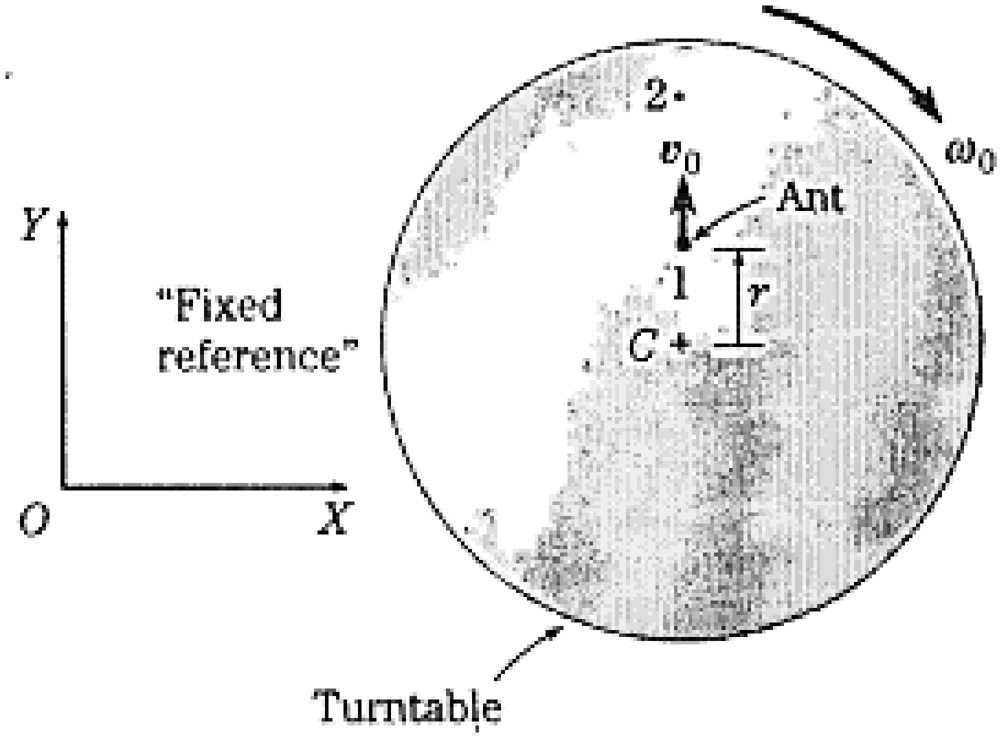
$2\boldsymbol{\omega} \times \boldsymbol{v}_{rel}$ - Coriolis acceleration

$\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$ - Euler acceleration, due to the angular acceleration $\dot{\omega}$

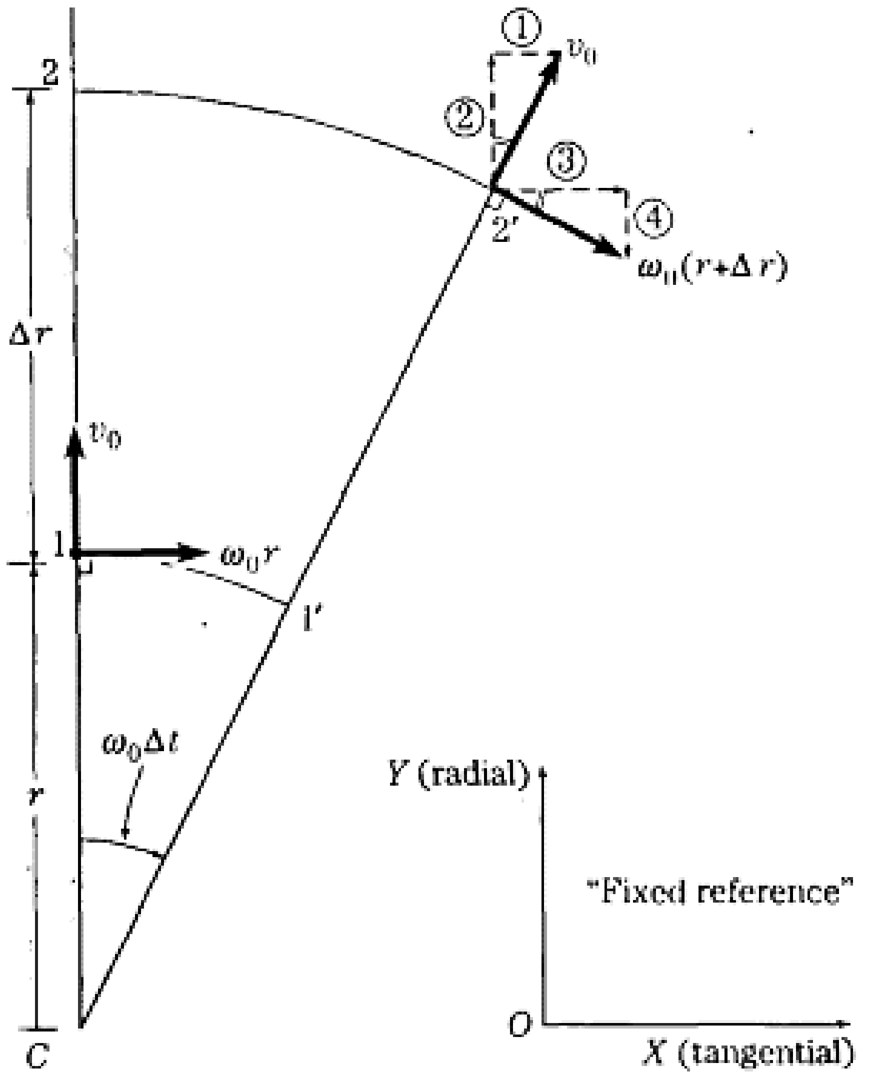
$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ - Centripetal acceleration

Centripetal and Coriolis acceleration

A turntable is rotating at a constant angular velocity $\boldsymbol{\omega}_0$ about its fixed center C. An ant (which model as a point) is walking along a radius of the turntable at a constant velocity $\boldsymbol{v}_0$, relative to the turntable. The ant is currently at point 1 and wants to get to the nearby point 2. Although $\boldsymbol{\omega}_0$ and $\boldsymbol{v}_0$ are constants, find the acceleration of the ant, relative to a fixed reference frame, which denote as $OXYZ$.



Ant moving at constant velocity $\boldsymbol{v}_0$, relative to turntable that is rotating at constant angular velocity $\boldsymbol{\omega}_0$



Velocity components of ant moving at constant velocity $\boldsymbol{v}_0$, relative to turntable that is rotating at constant $\boldsymbol{\omega}_0$.

Acceleration in radial direction (y – direction)

$$\boldsymbol{a}_{radial} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}_{radial}}{\Delta t}$$

$$\Delta \boldsymbol{v}_{radial} = \boldsymbol{v}_{final} - \boldsymbol{v}_{initial}$$

$$= [v_0 \cos \omega_0 \Delta t - \omega_0 (r + \Delta r) \sin \omega_0 \Delta t] - v_0$$

For small Δt, $\cos \omega_0 \Delta t = 1$, $\sin \omega_0 \Delta t = \omega_0 \Delta t$

$$\Delta r = v_0 \Delta t$$

$$\frac{\Delta \boldsymbol{v}_{radial}}{\Delta t} = \frac{v_0 - \omega_0 r . \omega_0 \, \Delta t - \omega_0 v_0 \Delta t . \omega_0 \, \Delta t - v_0}{\Delta t}$$

$$= -\omega_0^2 r - \omega_0^2 v_0 \Delta \mathrm{t}$$

$$\boxed{\boldsymbol{a}_{radial} = -\boldsymbol{\omega}_0^2 \boldsymbol{r}}$$

$\frac{d^2 R_0}{dt^2}$ - The acceleration of the origin of the intermediate frame w.r.t $OXYZ$

$\boldsymbol{a}_{rel}$ - The relative acceleration of P w.r.t $oxyz$.

$2\boldsymbol{\omega} \times \boldsymbol{v}_{rel}$ - Coriolis acceleration

$\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$ - Euler acceleration, due to the angular acceleration $\dot{\omega}$

$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ - Centripetal acceleration

$$\boldsymbol{a}_{radial} = -\boldsymbol{\omega}_0^2 \boldsymbol{r}$$

This is the centripetal acceleration.

The velocity $\boldsymbol{v}_0$ has no effect on $\boldsymbol{a}_{radial}$.

This acceleration always points radially “inward” toward the axis of rotation.

Depends on the position.

Acceleration in tangential direction (x – direction)

$$\boldsymbol{a}_{tangential} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}_{tangential}}{\Delta t}$$

$$\Delta \boldsymbol{v}_{tangential} = \boldsymbol{v}_{final} - \boldsymbol{v}_{initial}$$

$$= v_0 \sin \omega_0 \Delta t + \omega_0 (r + \Delta r) \cos \omega_0 \Delta t - \omega_0 r$$

For small Δt, $\cos \omega_0 \Delta t = 1$, $\sin \omega_0 \Delta t = \omega_0 \Delta t$

$$\Delta r = v_0 \Delta t$$

$$\frac{\Delta \boldsymbol{v}_{tangential}}{\Delta t} = \frac{v_0 . \omega_0 \, \Delta t + \omega_0 r + \omega_0 v_0 \Delta t - \omega_0 r}{\Delta t}$$

$$\boxed{\boldsymbol{a}_{tangential} = 2 \boldsymbol{\omega}_0 \boldsymbol{v}_0}$$

$\frac{d^2R_0}{dt^2}$ - The acceleration of the origin of the intermediate frame w.r.t $OXYZ$

$\boldsymbol{a}_{rel}$ - The relative acceleration of P w.r.t $oxyz$.

$2\boldsymbol{\omega} \times \boldsymbol{v}_{rel}$ - Coriolis acceleration

$\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$ - Euler acceleration, due to the angular acceleration $\dot{\omega}$

$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ - Centripetal acceleration

$$\boldsymbol{a}_{tangential} = 2\boldsymbol{\omega}_0\boldsymbol{v}_0$$

This is the Coriolis acceleration.

Depends on the velocity $\boldsymbol{v}_0$.

Does not depends on the position.

	Name	Variable or Equation
Motion of Point (defined in $OXYZ$) with respect to $OXYZ$	Position Vector	$\boldsymbol{R}(t)$
	Velocity Vector	$\boldsymbol{v} = \dfrac{d\boldsymbol{R}}{dt}$
	Acceleration Vector	$\boldsymbol{a} = \dfrac{d^2\boldsymbol{R}}{dt^2}$
Motion of Point (defined in $oxyz$) with respect to $OXYZ$	Position Vector	$\boldsymbol{R}(t) = \boldsymbol{R}_0(t) + \boldsymbol{r}(t)$
	Differential Operator	$\dfrac{d}{dt} = \left(\dfrac{\partial}{\partial t}\right)_{\text{rel}} + \boldsymbol{\omega}\times$
	Velocity Vector	$\boldsymbol{v} = \dfrac{d\boldsymbol{R}_0}{dt} + \boldsymbol{v}_{\text{rel}} + \boldsymbol{\omega} \times \boldsymbol{r}$
	Acceleration Vector	$\boldsymbol{a} = \dfrac{d^2\boldsymbol{R}_0}{dt^2} + \boldsymbol{a}_{\text{rel}} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{\text{rel}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$