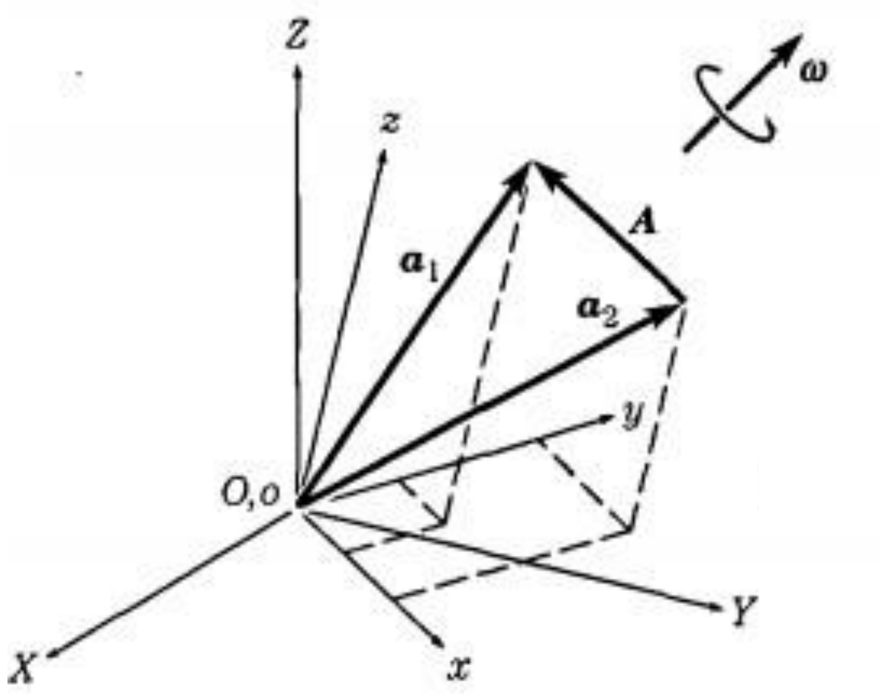
Time rate of change of vector in rotating frame.

This addresses the problem of the time rate of change of vector as seen by an observer in a moving frame to that seen by an observer in a fixed reference frame.

Frame oxyz has angular velocity $'\omega'$ w.r.t the fixed reference frame OXYZ. A vector **A** having a constant magnitude is fixed in oxyz and is carried around by the motion of this frame (oxyz).



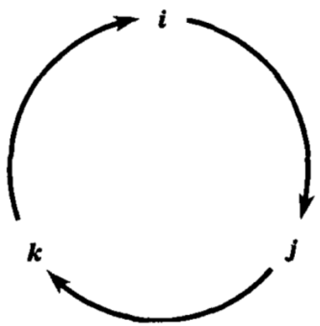
From the second reference frame OXYZ, the vector $\boldsymbol{A}$ appears to be changing its orientation, which is due to $\boldsymbol{\omega}$.

The rate of change of **A** observed from the fixed reference frame OXYZ is given by the ‘vector cross product’.

$$\frac{d\boldsymbol{A}}{dt} = \boldsymbol{\omega} \times \boldsymbol{A}$$

Vector cross product mnemonic

The use of the mnemonic expresses the fact that when traversing the circle in the direction of the arrows, the result is positive; and when traversing opposite to the direction of the arrows, the result is negative.



Vector cross product mnemonic

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$
$$\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$$
$$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

$$\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$$
$$\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$$
$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$$

Vector cross product

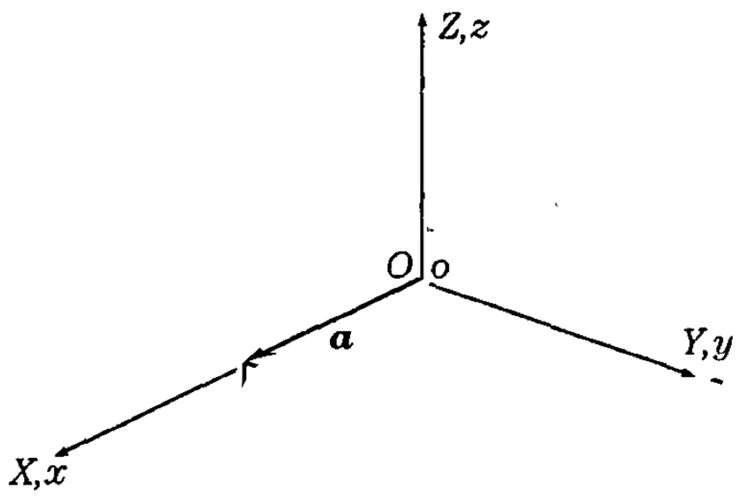
$$a = i + 3j + 4k$$
$$b = 2i + 7j - 5k$$

$$a \times b = (i + 3j + 4k) \times (2i + 7j - 5k)$$

$$a \times b = -43i + 13j + k$$

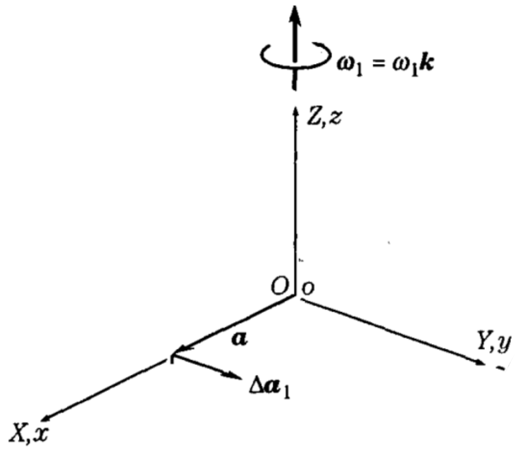
The time derivative of constant magnitude vector

The vector $\boldsymbol{a}$ of constant magnitude is defined in the oxyz reference frame as shown in Figure.



a. The oxyz reference frame is rotating at $\boldsymbol{\omega} = \boldsymbol{\omega_1} = \omega_1 \boldsymbol{k}$ w.r.t the fixed reference frame OXYZ as indicated.

Find the time derivative of $\boldsymbol{a}$ as seen from OXYZ.

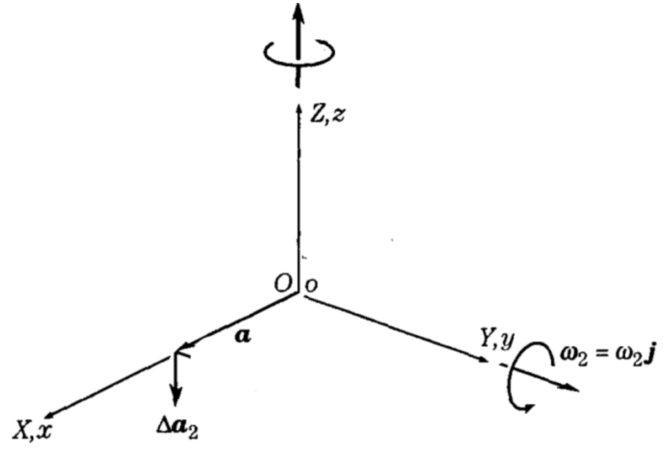


The change in $\boldsymbol{a}$ during a time interval due to $\boldsymbol{\omega_1}$.

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{\omega} \times \boldsymbol{a} = \omega_1 \boldsymbol{k} \times a\boldsymbol{i} = \omega_1 a\boldsymbol{j}$$

a. The oxyz reference frame is rotating at $\boldsymbol{\omega} = \boldsymbol{\omega_2} = \omega_2 \boldsymbol{j}$ w.r.t the fixed reference frame OXYZ as indicated.

Find the time derivative of $\boldsymbol{a}$ as seen from OXYZ.



The change in $\boldsymbol{a}$ during a time interval due to $\boldsymbol{\omega_2}$.

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{\omega} \times \boldsymbol{a} = \omega_2 \boldsymbol{j} \times a\boldsymbol{i} = -\omega_2 a\boldsymbol{k}$$

For a vector, characterised by a magnitude and a direction can be changed by changing its magnitude or its direction (or both).

But, if a vector's magnitude is constant, it can be changed only by changing its direction.

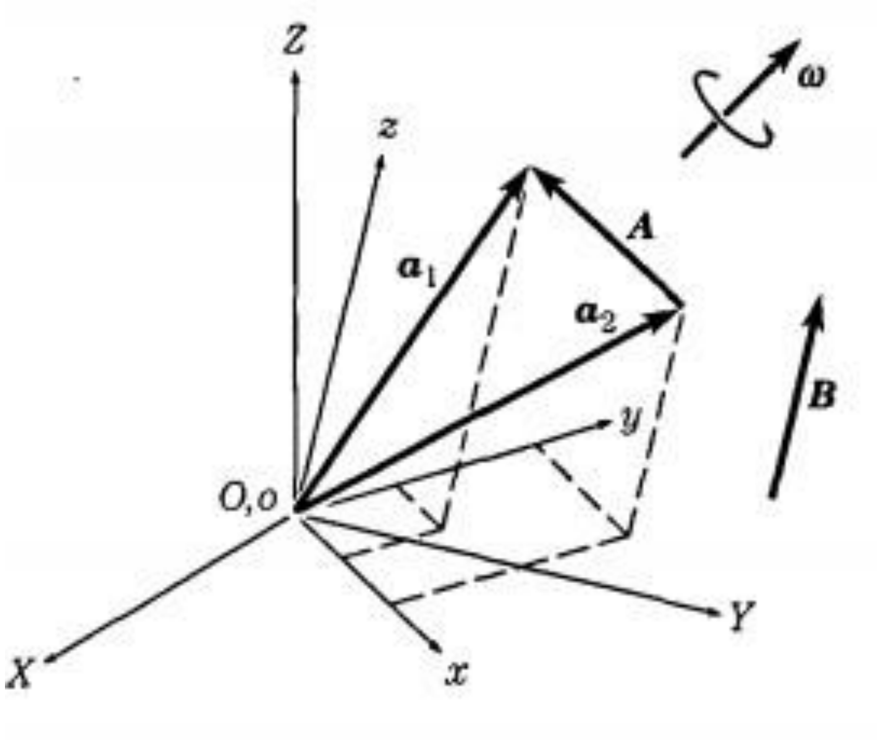
The time rate of change of such vector is given by

$$\frac{d\boldsymbol{A}}{dt} = \boldsymbol{\omega} \times \boldsymbol{A} \quad or \quad \frac{d\boldsymbol{a}}{dt}$$

For an arbitrary vector **B** defined in oxyz having changing magnitude and changing direction.

The instantaneous component representation of $\boldsymbol{B}$ referred to oxyz

$$\boldsymbol{B} = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$$



When viewed from the fixed reference ,OXYZ;

- Scalar components B_x, B_y, B_z will vary with time.
- Due to ω, the unit vectors attached to oxyz will vary in orientation.

For observers in rotating frame, oxyz;

- $B_x\,, B_y\,, B_z$ will change with time.
- The unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ will be <u>constant</u> in magnitude and direction.

$$\frac{dB}{dt} = \frac{d(B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k})}{dt}$$

$$= \frac{dB_x\boldsymbol{i}}{dt} + \frac{dB_y\boldsymbol{j}}{dt} + \frac{dB_z\boldsymbol{k}}{dt}$$

$$= B_x\frac{d\boldsymbol{i}}{dt} + \frac{dB_x}{dt}\boldsymbol{i} + B_y\frac{d\boldsymbol{j}}{dt} + \frac{dB_y}{dt}\boldsymbol{j} + B_z\frac{d\boldsymbol{k}}{dt} + \frac{dB_z}{dt}\boldsymbol{k}$$

$\boldsymbol{B}$ is the arbitrary vector with both translation and rotation.

Scalar magnitude (B_x, B_y, B_z) and direction $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ both are changing with time.

Time derivative of $\boldsymbol{B}$ as observed from the fixed reference frame OXYZ

$$\frac{d\boldsymbol{B}}{dt} = \frac{dB_x}{dt}\boldsymbol{i} + \frac{dB_y}{dt}\boldsymbol{j} + \frac{dB_z}{dt}\boldsymbol{k} + B_x\frac{d\boldsymbol{i}}{dt} + B_y\frac{d\boldsymbol{j}}{dt} + B_z\frac{d\boldsymbol{k}}{dt}$$

$$\frac{d\boldsymbol{B}}{dt} = \underbrace{\dot{B}_x\boldsymbol{i} + \dot{B}_y\boldsymbol{j} + \dot{B}_z\boldsymbol{k}} + \underbrace{B_x\frac{d\boldsymbol{i}}{dt} + B_y\frac{d\boldsymbol{j}}{dt} + B_z\frac{d\boldsymbol{k}}{dt}}$$

Rate of change of **B** as viewed from the oxyz frame.

$$\left(\frac{\partial\boldsymbol{B}}{\partial t}\right)_{rel}$$

rel – the time rate of change is w.r.t the oxyz frame.

Due to the rotation of oxyz w.r.t OXYZ

$$\dot{B}_x\boldsymbol{i} + \dot{B}_y\boldsymbol{j} + \dot{B}_z\boldsymbol{k}$$

The time derivatives of the scalar components as indicated by the superior dots are the same for observers in both oxyz and OXYZ frames.

$$B_x\frac{d\boldsymbol{i}}{dt} + B_y\frac{d\boldsymbol{j}}{dt} + B_z\frac{d\boldsymbol{k}}{dt}$$

To evaluate the time derivatives of the unit vectors, which are vectors of fixed magnitude but changing direction.

$$\boxed{\frac{d\boldsymbol{a}}{dt} = \boldsymbol{\omega} \times \boldsymbol{a}}$$

$$B_x \frac{d\boldsymbol{i}}{dt} + B_y \frac{d\boldsymbol{j}}{dt} + B_z \frac{d\boldsymbol{k}}{dt} = B_x \boldsymbol{\omega} \times \boldsymbol{i} + B_y \boldsymbol{\omega} \times \boldsymbol{j} + B_z \boldsymbol{\omega} \times \boldsymbol{k}$$

$$= \boldsymbol{\omega} \times (B_x \boldsymbol{i} + B_y \boldsymbol{j} + B_z \boldsymbol{k})$$

$$= \boldsymbol{\omega} \times \boldsymbol{B}$$

$$\frac{d\boldsymbol{B}}{dt} = \dot{B}_x\boldsymbol{i} + \dot{B}_y\boldsymbol{j} + \dot{B}_z\boldsymbol{k} + B_x\frac{d\boldsymbol{i}}{dt} + B_y\frac{d\boldsymbol{j}}{dt} + B_z\frac{d\boldsymbol{k}}{dt}$$

$$\frac{d\boldsymbol{B}}{dt} = \left(\frac{\partial\boldsymbol{B}}{\partial t}\right)_{rel} + \boldsymbol{\omega} \times \boldsymbol{B}$$

This equation relates the time rates of change of an arbitrary vector as viewed from the two reference frames oxyz and OXYZ.

The time derivative of $\boldsymbol{B}$ observed from the fixed reference frame OXYZ

$$\frac{d\boldsymbol{B}}{dt} = \left(\frac{\partial \boldsymbol{B}}{\partial t}\right)_{rel} + \boldsymbol{\omega} \times \boldsymbol{B}$$

The time derivative of $\boldsymbol{B}$ observed from the rotating reference frame oxyz

$$\left(\frac{\partial \boldsymbol{B}}{\partial t}\right)_{rel}$$

The angular velocity of the rotating reference frame, oxyz w.r.t the fixed reference frame,OXYZ

$$\boldsymbol{\omega} \times \boldsymbol{B}$$

In general

$$\frac{d(\quad)}{dt} = \left(\frac{\partial(\quad)}{\partial t}\right)_{rel} + \boldsymbol{\omega} \times (\quad)$$

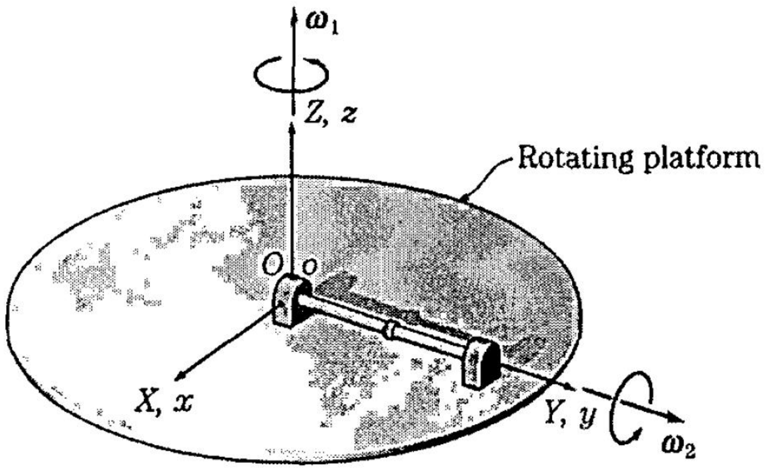
This operational from provides a means for determining the time derivative w.r.t OXYZ of a vector defined in oxyz, when oxyz is rotating at $'\omega'$ angular velocity w.r.t OXYZ.

The 1st term on the R.H.S of the equation refers to the change in scalar components in oxyz of the vector.

The 2nd term on the R.H.S of the equation refers to the direction change of the vector.

Time derivative of arbitrary vector

The shaft shown in Figure is rotating at $\boldsymbol{\omega_2}$ w.r.t its bearings, which are fixed on a platform. The platform is rotating at $\boldsymbol{\omega_1}$ w.r.t the fixed reference frame OXYZ. Find the first and second time derivatives of $\boldsymbol{\omega_2}$ w.r.t OXYZ.



Shaft rotating at $\boldsymbol{\omega_2}$ w.r.t platform which is rotating at $\boldsymbol{\omega_1}$ w.r.t fixed reference frame OXYZ.

The first time derivative of $\boldsymbol{\omega_2}$

$$\frac{d\boldsymbol{\omega_2}}{dt} = \left[\left(\frac{\partial}{\partial t}\right)_{rel} + \boldsymbol{\omega_1} \times\right] \boldsymbol{\omega_2}$$

$$= \left(\frac{\partial \boldsymbol{\omega_2}}{\partial t}\right)_{rel} + \boldsymbol{\omega_1} \times \boldsymbol{\omega_2}$$

$$\frac{d\boldsymbol{\omega_2}}{dt} = \dot{\boldsymbol{\omega}}_{\mathbf{2}} + \boldsymbol{\omega_1} \times \boldsymbol{\omega_2}$$

The 1st term on the R.H.S of the equation refers to the magnitude change of $\boldsymbol{\omega_2}$, that is the change of angular speed of the shaft relative to its bearings.

The 2nd term on the R.H.S of the equation refers to the change in direction of $\boldsymbol{\omega_2}$, is the fact that $\boldsymbol{\omega_1}$ is changing the direction of $\boldsymbol{\omega_2}$.

The second time derivative of $\boldsymbol{\omega_2}$

$$\frac{d^2\boldsymbol{\omega_2}}{dt^2} = \frac{d[\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega_1} \times \boldsymbol{\omega_2}]}{dt}$$

$$\frac{d^2\boldsymbol{\omega_2}}{dt^2} = \left[\left(\frac{\partial}{\partial t}\right)_{rel} + \boldsymbol{\omega_1} \times\right][\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega_1} \times \boldsymbol{\omega_2}]$$

$$\boxed{\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_{rel} + \boldsymbol{\omega} \times}$$

$$\frac{d^2\boldsymbol{\omega_2}}{dt^2} = \left[\left(\frac{\partial}{\partial t}\right)_{rel} + \boldsymbol{\omega_1} \times\right][\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega_1} \times \boldsymbol{\omega_2}]$$

$$= \left(\frac{\partial}{\partial t}\right)_{rel} [\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega_1} \times \boldsymbol{\omega_2}] + \boldsymbol{\omega_1} \times [\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega_1} \times \boldsymbol{\omega_2}]$$

$$= \ddot{\boldsymbol{\omega}}_2 + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{\omega_2} + \boldsymbol{\omega_1} \times \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega_1} \times \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{\omega_2})$$

$$\frac{d^2\boldsymbol{\omega_2}}{dt^2} = \ddot{\boldsymbol{\omega}}_2 + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{\omega_2} + 2\boldsymbol{\omega_1} \times \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{\omega_2})$$

The time derivative of the angular acceleration is known as angular jerk.