Momentum formulation for systems of particles

Particle

- A mass point with a fixed mass.
- A particle is an idealized concept.
- When the rotational causes and effects are negligible compared to the translational causes and effects, the body may be reasonably modeled as a particle.

Linear momentum and force

Linear momentum of a particle

$$\boldsymbol{p} = m\boldsymbol{v}$$

Resultant force

$$\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt}$$

Conservation of linear momentum of a particle:

When the resultant force is zero, $d\boldsymbol{p}/dt$ is zero. $\boldsymbol{p}$ remains constant, thus the particle's linear momentum $\boldsymbol{p}$ is conserved.

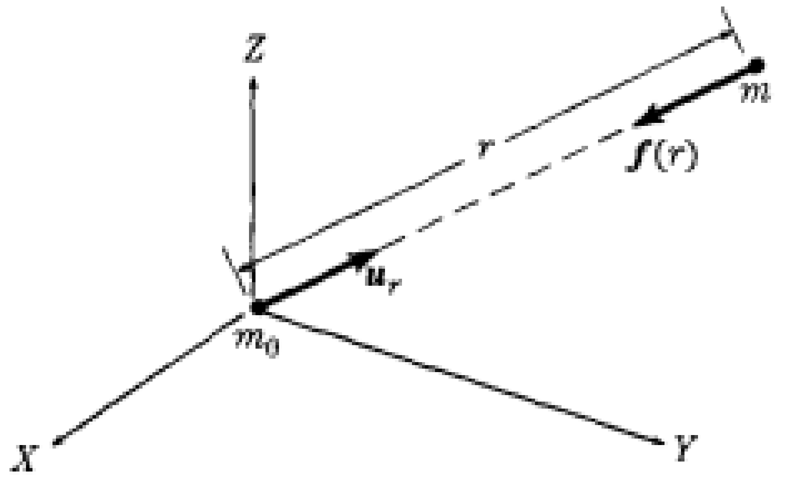
Inertial reference frame

An operational frame used to define motion.

The universal law of gravitation

Law of gravitation for particles,

$$\boldsymbol{f} = -\frac{G m_0 m}{r^2}\boldsymbol{u}_r$$



Gravitational field vector, $\boldsymbol{g}$

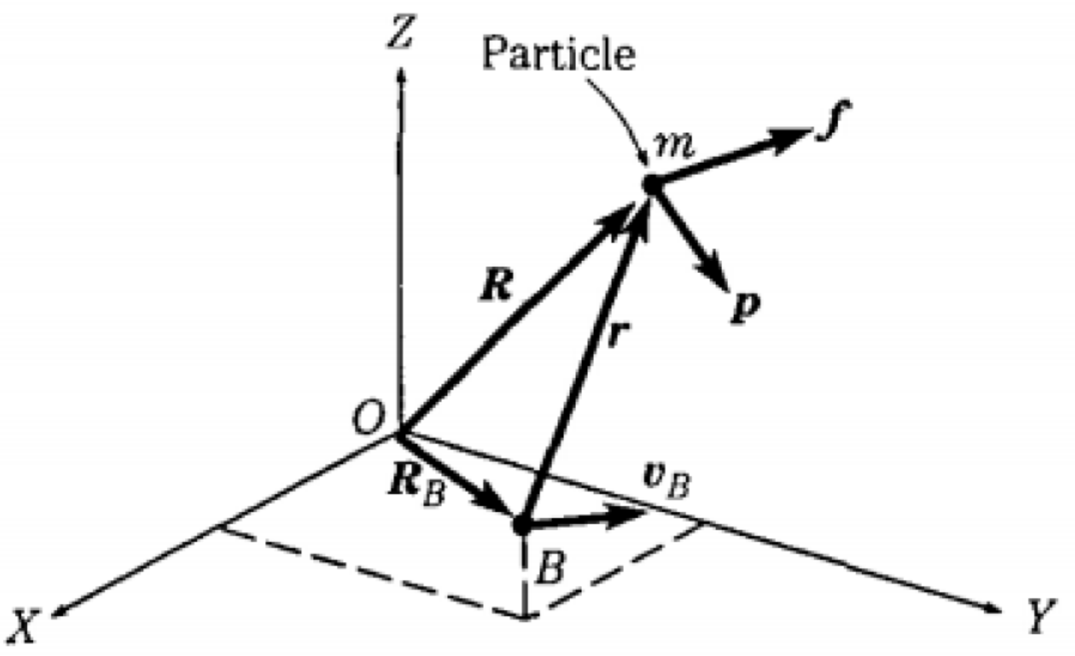
$$\boldsymbol{g} = \frac{\boldsymbol{f}}{m} = -G\frac{m_0}{r^2}\boldsymbol{u}_r$$

$$\boldsymbol{f}_g = \boldsymbol{W} = m\boldsymbol{g}$$

Torque and angular momentum for a particle

- Point B is an arbitrary point located by the position vector $\boldsymbol{R}_B$ and velocity $\frac{d\boldsymbol{R}_B}{dt}$ w.r.t inertial reference frame $OXYZ$.

- The particle m has a linear momentum $\boldsymbol{p}$ w.r.t inertial frame OXYZ, is subjected to the resultant force $\boldsymbol{f}$, and is located w.r.t B by $\boldsymbol{r}$.



Torque or moment, of the force $\boldsymbol{f}$ about B

$$\boldsymbol{\tau}_B = \boldsymbol{r} \times \boldsymbol{f}$$

Angular momentum or moment of momentum of the particle about B

$$\boldsymbol{h}_B = \boldsymbol{r} \times \boldsymbol{p}$$

$$\boldsymbol{\tau}_B = \frac{d\boldsymbol{h}_B}{dt} + \boldsymbol{v}_B \times \boldsymbol{p}$$

$\boldsymbol{v}_B \times \boldsymbol{p}$ will be zero, if

- B is fixed in the inertial reference frame; $\boldsymbol{v}_B = 0$

- The velocity of B is always parallel to the velocity of the particle m.

Angular momentum principle

$$\boldsymbol{\tau}_B = \frac{d\boldsymbol{h}_B}{dt}$$

The resultant torque applied to a particle is the time rate of change of the particle's angular momentum, if B is either fixed point or a point that moves parallel to the particle.

Conservation of angular momentum of a particle:

When the resultant torque about a point B of all the forces on a particle is zero, $\frac{d\boldsymbol{h}_B}{dt}$ is zero. $\boldsymbol{h}_B$ remains constant, thus the particle's angular momentum $\boldsymbol{h}_B$ is conserved.

Formulation of equations of motion

- Geometric requirements on the motions (kinematic requirements)
- Dynamic requirements on the forces (force –dynamic requirements)
- Constitutive requirements of all the system elements and fields

Antenna deployed by maneuvering airplane

An airplane moving with velocity v_1 and acceleration a_1, both w.r.t ground (fixed space), is rolling at ω_1 and pitching at ω_2, both w.r.t fixed space, as sketched in Figure 2. An antenna A (weight 2 lb) is being deployed from the airplane. The airplane is in a horizontal orientation (w.r.t gravity) and the antenna A is at a vertical distance of 10 ft from the centerline of the airplane, moving with velocity v_2 and acceleration a_2, both defined w.r.t the airplane.

Find the velocity and acceleration of the antenna A relative to fixed space at the instant shown.

Find the force applied by the massless mast on the antenna A.

Data:

$v_1 = 200\ ft/s$

$a_1 = 100\ ft/s^2$

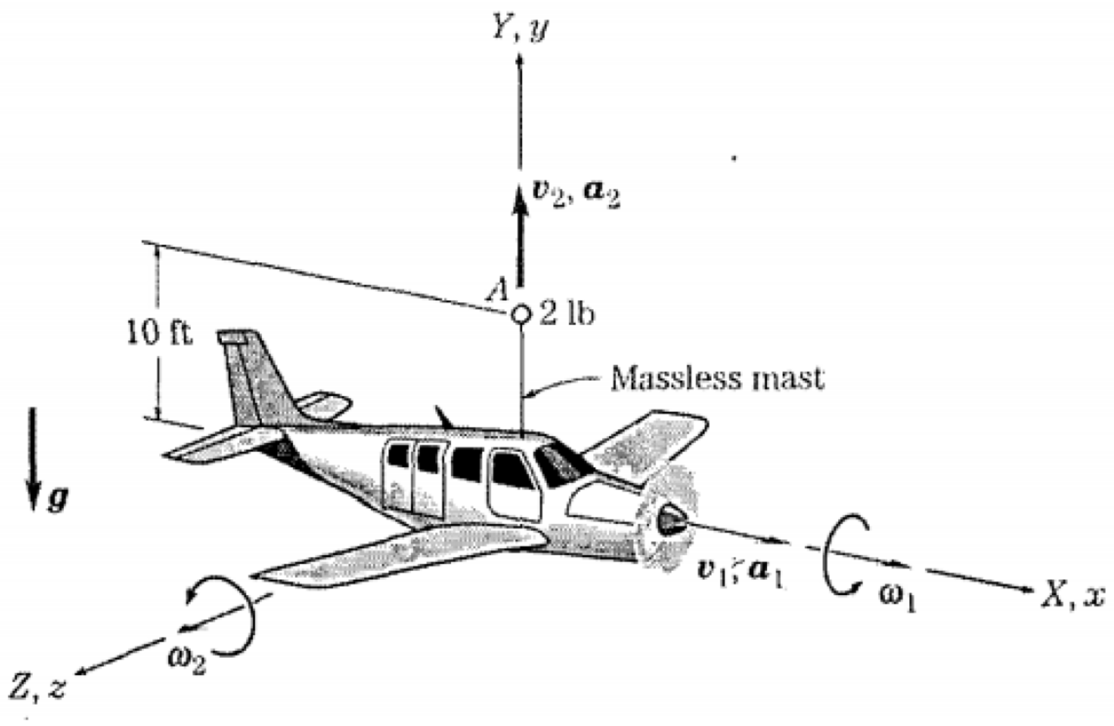
$v_2 = 6\ ft/s$

$a_2 = 0.1\ ft/s^2$

$\omega_1 = 3\ rad/min$

$\omega_2 = 2\ rad/min$

$g = 32.17\ ft/s^2$



OXYZ is a reference frame that is fixed in space and oxyz is an intermediate frame that is attached to the airplane.

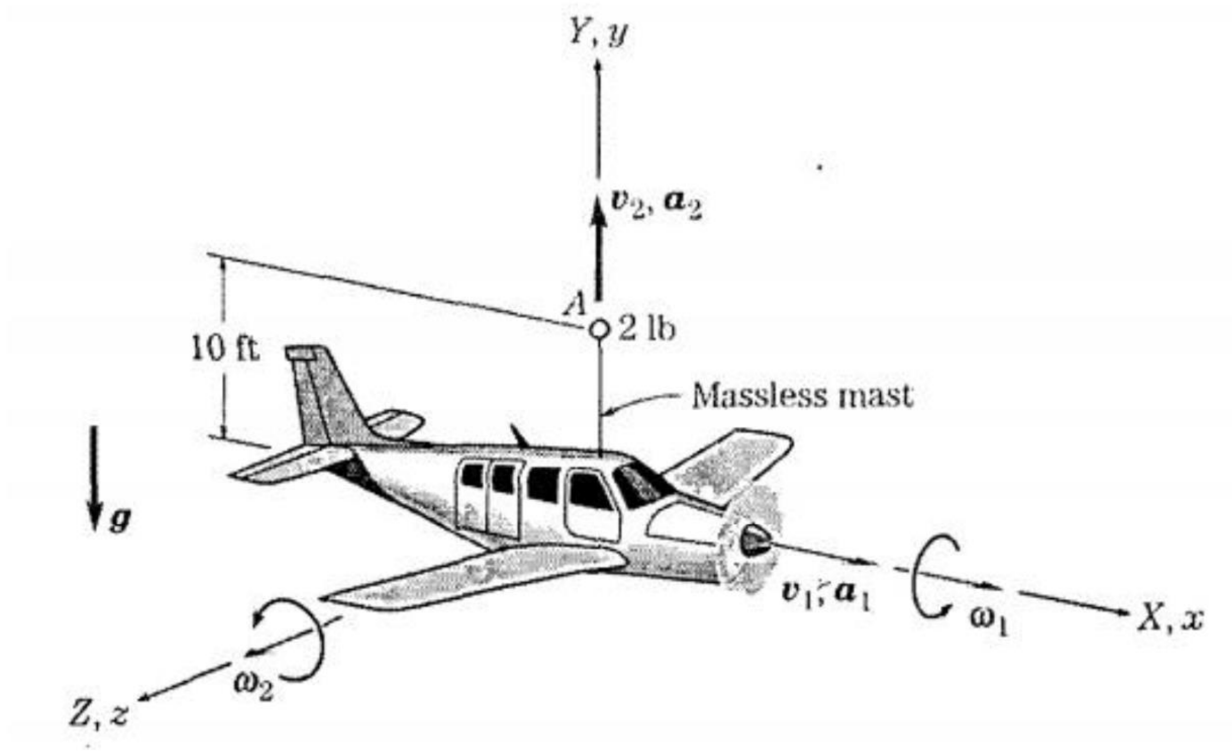


Figure 2: Sketch of airplane deploying antenna at tip of massless mast.

Kinematic requirements

$$v_A = 199.667\boldsymbol{i} + 6.000\boldsymbol{j} + 0.500\boldsymbol{k}\,\frac{ft}{s}$$

$$a_A = 99.600\boldsymbol{i} + 0.064\boldsymbol{j} + 0.600\boldsymbol{k}\,\frac{ft}{s^2}$$

Force – dynamic requirements

$$\sum_{i=1}^{q} \boldsymbol{f}_i = \frac{d\boldsymbol{p}}{dt}$$

$$\boldsymbol{f}_m + \boldsymbol{f}_g = \frac{d\boldsymbol{p}_{\boldsymbol{A}}}{dt}$$

Constitutive requirements

$$\boldsymbol{p}_A = m\boldsymbol{v}_A$$

$$\boldsymbol{f}_g = m\boldsymbol{g} = -mg\boldsymbol{j}$$

$$\boldsymbol{f}_m + \boldsymbol{f}_g = \mathrm{m}\mathbf{a}_{\mathrm{A}}$$

$$\boldsymbol{f}_m = m\boldsymbol{a}_A - \boldsymbol{f}_g$$

$$\boldsymbol{f}_m = [6.192\mathbf{i} + 2.004\mathbf{j} + 0.037\mathbf{k}]\ \mathrm{lb}$$

Angular momentum

Inertia tensor

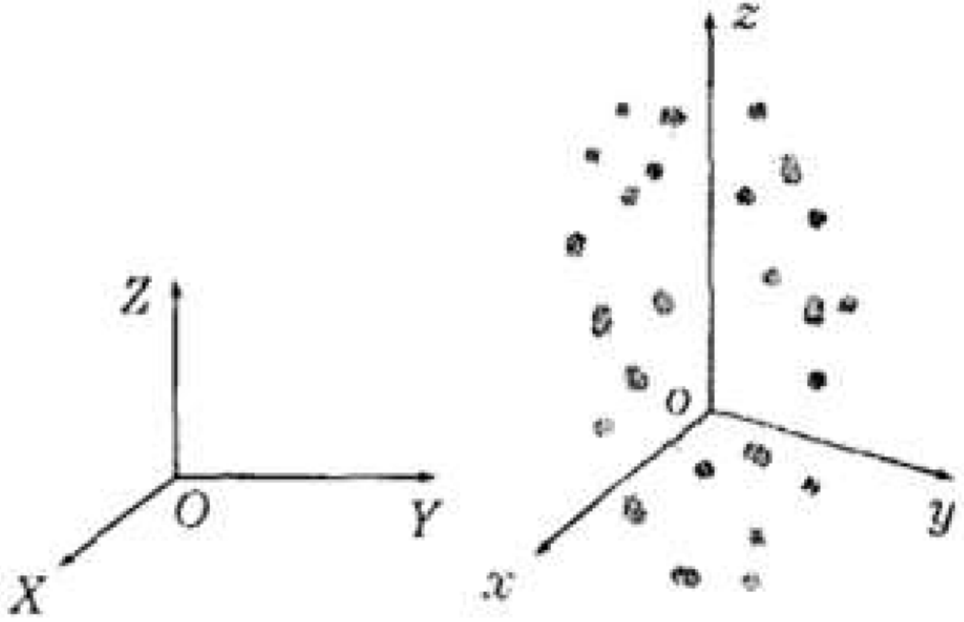
Models of rigid bodies

- A distribution of a very large number of discrete particles that are rigidly bound together

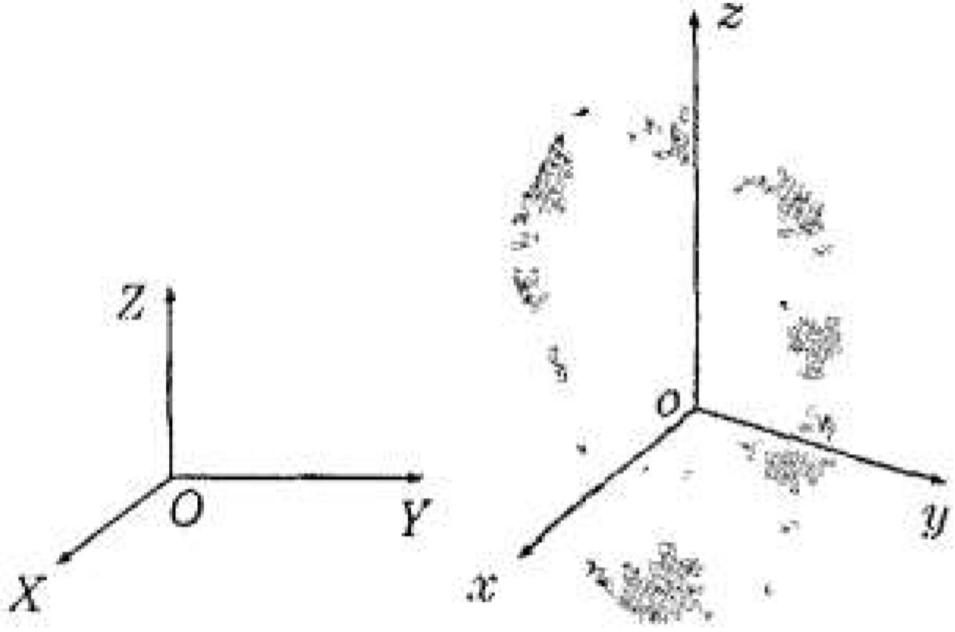
  System of particles

- A distribution of continuous rigid mass

  Continuous body



System of particles



Continuum model

$oxyz$ - Body coordinate frame

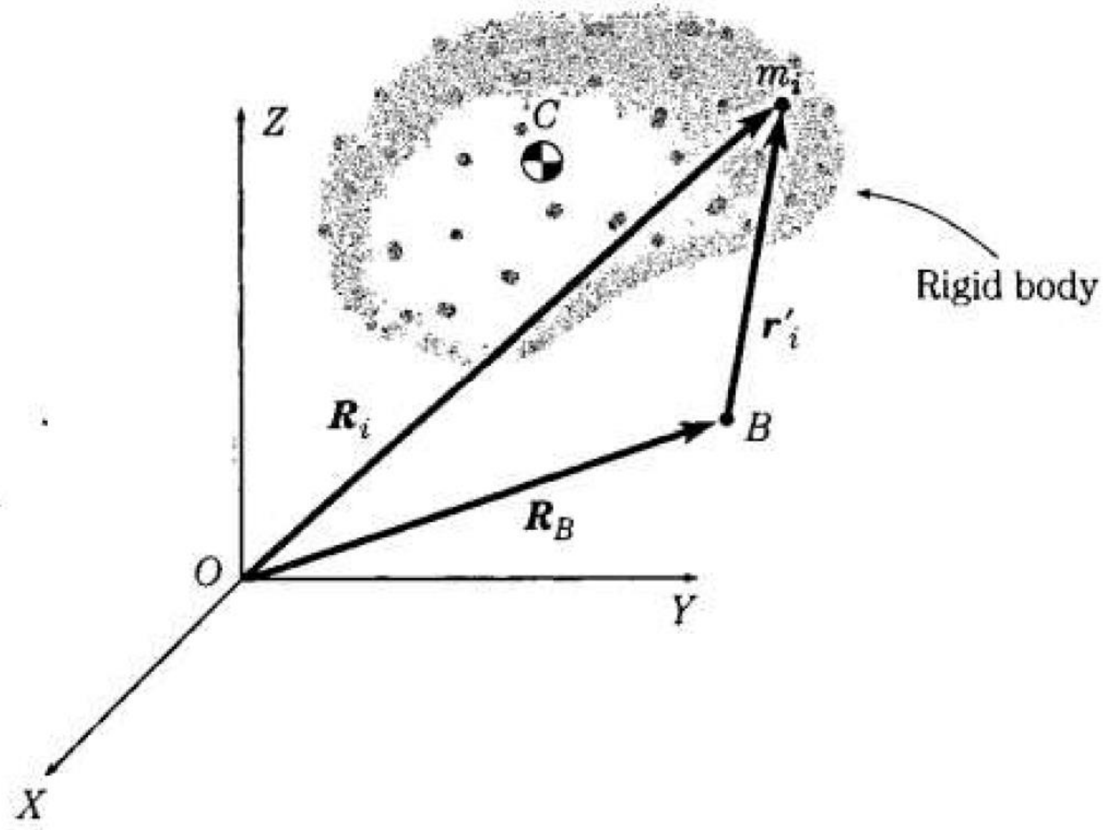
Momentum principles for rigid bodies modeled as system of particles

Linear momentum

$$P = \sum_{i=1}^{N} m_i v_i$$

Angular momentum

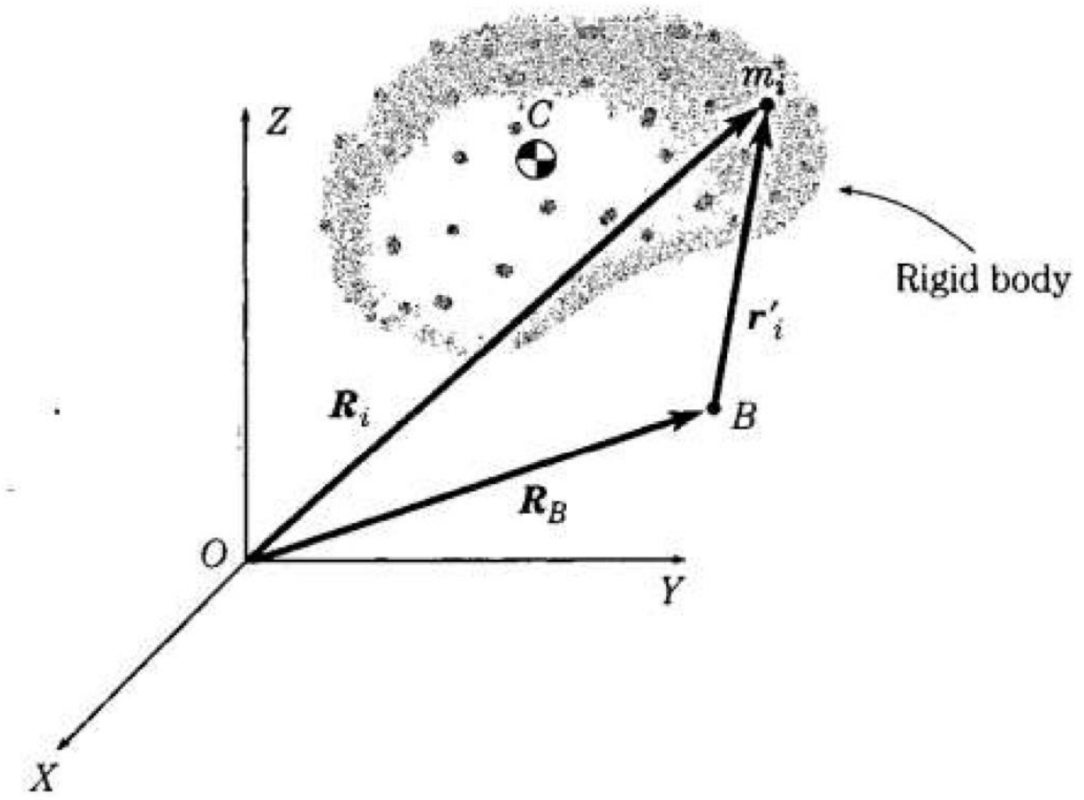
$$H_o = \sum_{i=1}^{N} R_i \times m_i v_i$$



w.r.t an arbitrary point, B

$$\tau_B = \frac{dH_B}{dt} + v_B \times P$$

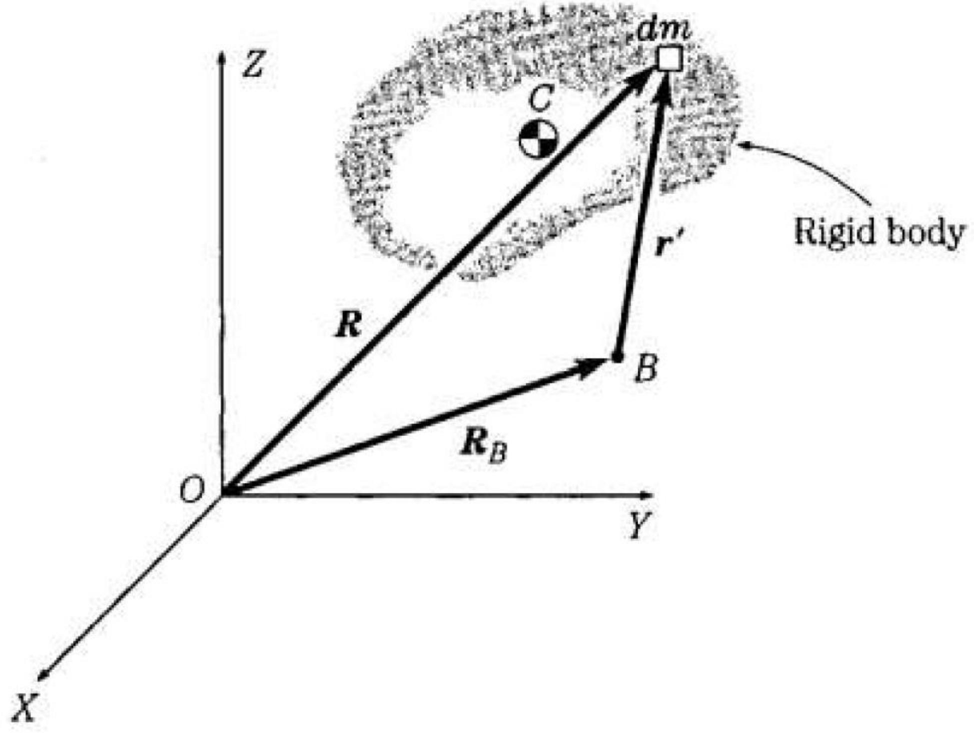
$$H_B = \sum_{i=1}^{N} r_i' \times m_i v_i$$



Momentum principles for rigid bodies modeled as continua

$$P = \int_M v dm$$

$$H_o = \int_M R \times v dm$$



w.r.t an arbitrary point, B

$$\tau_B = \frac{dH_B}{dt} + v_B \times P$$

$$H_B = \int_M r' \times v dm$$