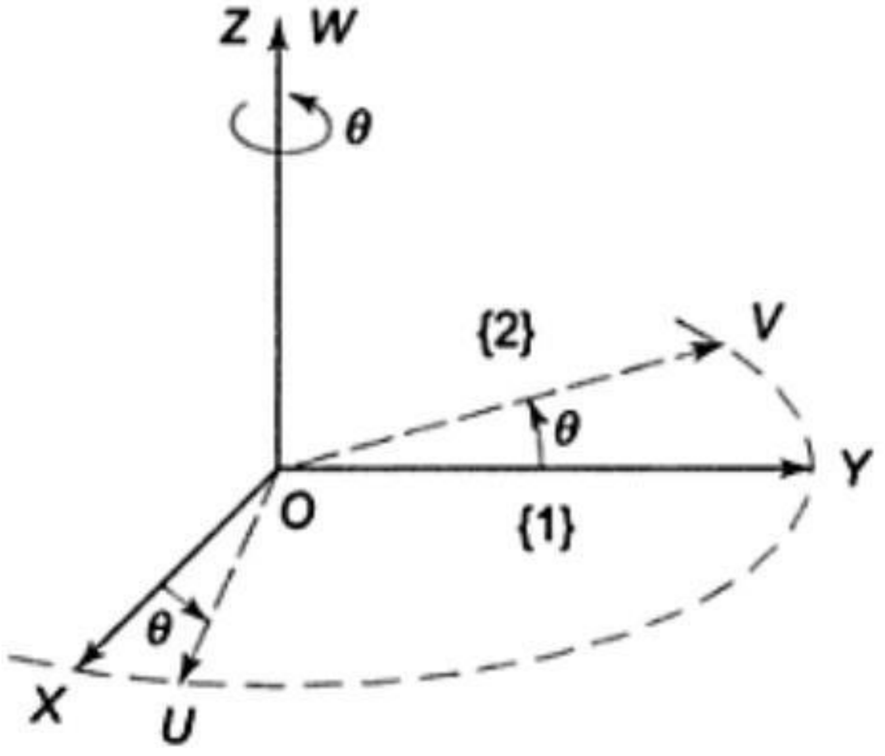
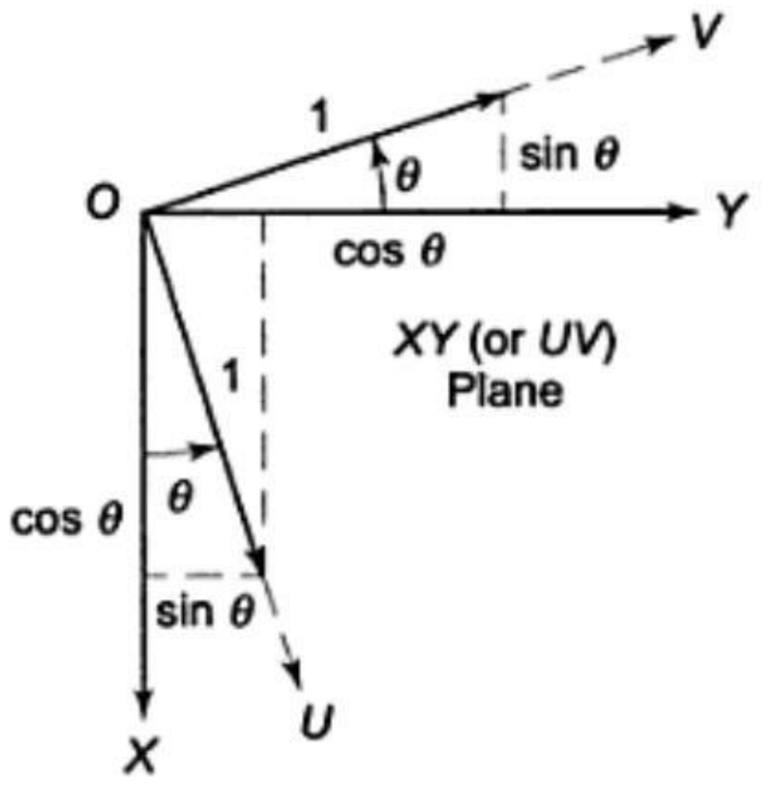
Fundamental rotation matrices

Principal axes rotation.

Determining the orientation of frame {2}, which is rotated about one of the three principal axes of frame {1}.

Frame {2} is rotated w.r.t frame {1} by angle θ about the z-axis of frame {1}.





The rotation matrix ${}^{1}R_2$, known as the *fundamental rotation matrix*, is denoted by the symbol $R_z(\theta)$ or $R(z,\theta)$ or $R_{z,\theta}$.

$R_z(\theta)$ is computed from the dot product of unit vectors along the principal axes.

The dot product of two unit vectors is the cosine of the angle between them.

$$\boldsymbol{x} \cdot \boldsymbol{u} = |x||u| \cos\theta$$

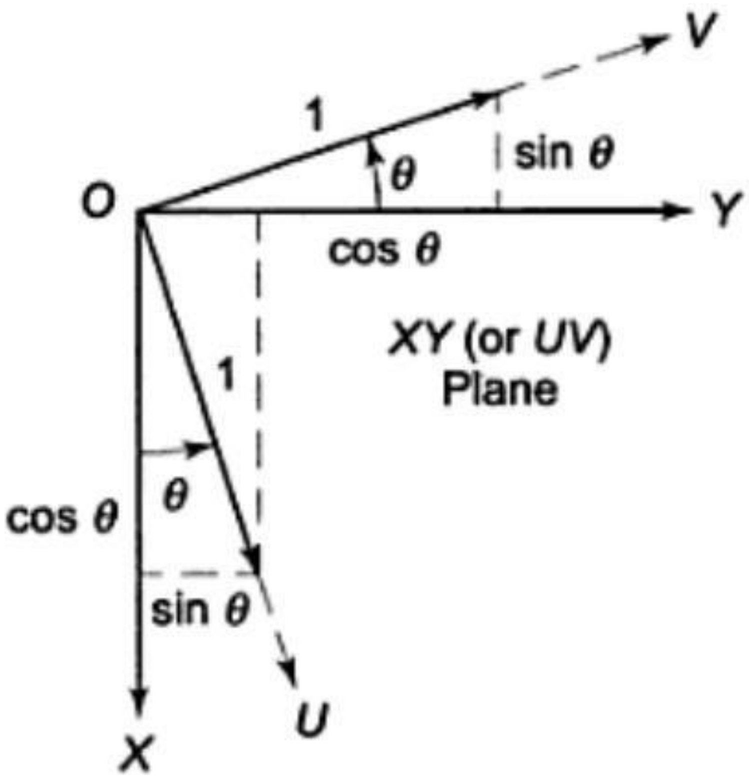
$$
{}^{1}\boldsymbol{R}_2 = \begin{bmatrix} \hat{x}\cdot\hat{u} & \hat{x}\cdot\hat{v} & \hat{x}\cdot\hat{w} \\ \hat{y}\cdot\hat{u} & \hat{y}\cdot\hat{v} & \hat{y}\cdot\hat{w} \\ \hat{z}\cdot\hat{u} & \hat{z}\cdot\hat{v} & \hat{z}\cdot\hat{w} \end{bmatrix}
$$

$$
\boldsymbol{R}_z(\theta) = \begin{bmatrix} \cos\theta & \cos(90^\circ+\theta) & \cos 90^\circ \\ \cos(90^\circ-\theta) & \cos\theta & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix}
$$

$$
\boldsymbol{R}_z(\theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$C\theta = \cos\theta$$

$$S\theta = \sin\theta$$



Fundamental rotation matrices for rotation about x-axis and y-axis are,

$$
{}^{1}\boldsymbol{R}_2 = \begin{bmatrix} \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{u}} & \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{v}} & \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{w}} \end{bmatrix}
$$

$$
\boldsymbol{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}
$$

$$
\boldsymbol{R}_y(\theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix}
$$

$C\theta = \cos\theta$ $\qquad$ $S\theta = \sin\theta$

Homogeneous transformation matrix with a rotation by an angle θ about z-axis and $D = [0 \quad 0 \quad 0]^T$.

$$
\boldsymbol{T}(z, \theta) = \left[\begin{array}{ccc:c} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
{}^{2}R_{1} = \begin{bmatrix} \boldsymbol{u}\cdot\boldsymbol{x} & \boldsymbol{u}\cdot\boldsymbol{y} & \boldsymbol{u}\cdot\boldsymbol{z} \\ \boldsymbol{v}\cdot\boldsymbol{x} & \boldsymbol{v}\cdot\boldsymbol{y} & \boldsymbol{v}\cdot\boldsymbol{z} \\ \boldsymbol{w}\cdot\boldsymbol{x} & \boldsymbol{w}\cdot\boldsymbol{y} & \boldsymbol{w}\cdot\boldsymbol{z} \end{bmatrix} = \left[\,{}^{1}R_{2}\right]^{T}
$$

$$
R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}
$$

$$
R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}
$$

$$
R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

The fundamental rotation matrices can be multiplied together to represent a sequence of finite rotations.

The overall rotation matrix representing a rotation of angle θ_1 about x-axis followed by a rotation of angle θ_2 about y-axis is given by,

$$^{1}R_2 = R_y(\theta_2) R_x(\theta_1)$$

$$R = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \sin\theta_1 & \sin\theta_2 \cos\theta_1 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ -\sin\theta_2 & \cos\theta_2 \sin\theta_1 & \cos\theta_2 \cos\theta_1 \end{bmatrix}$$

The sequence of multiplication of R matrices is very important.

A different sequence may not give the same result and obviously will not correspond to same orientation of the rotated frame. This is because the matrix product is not commutative.

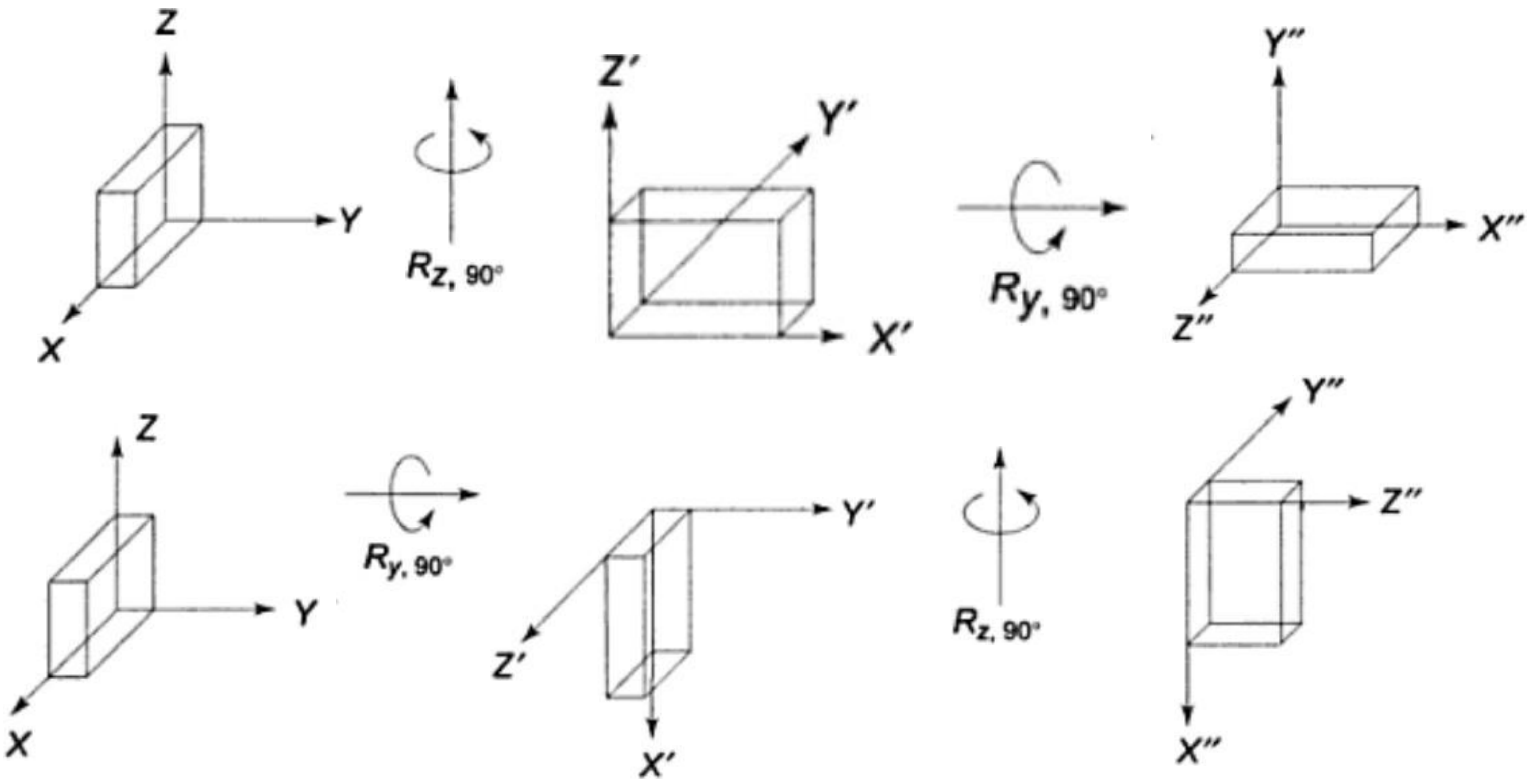
Two rotations in general do not result in same orientation and the resultant rotation matrix depends on the **order of rotations**.

Another significant factor is **how the rotations are performed**. There are two alternatives.

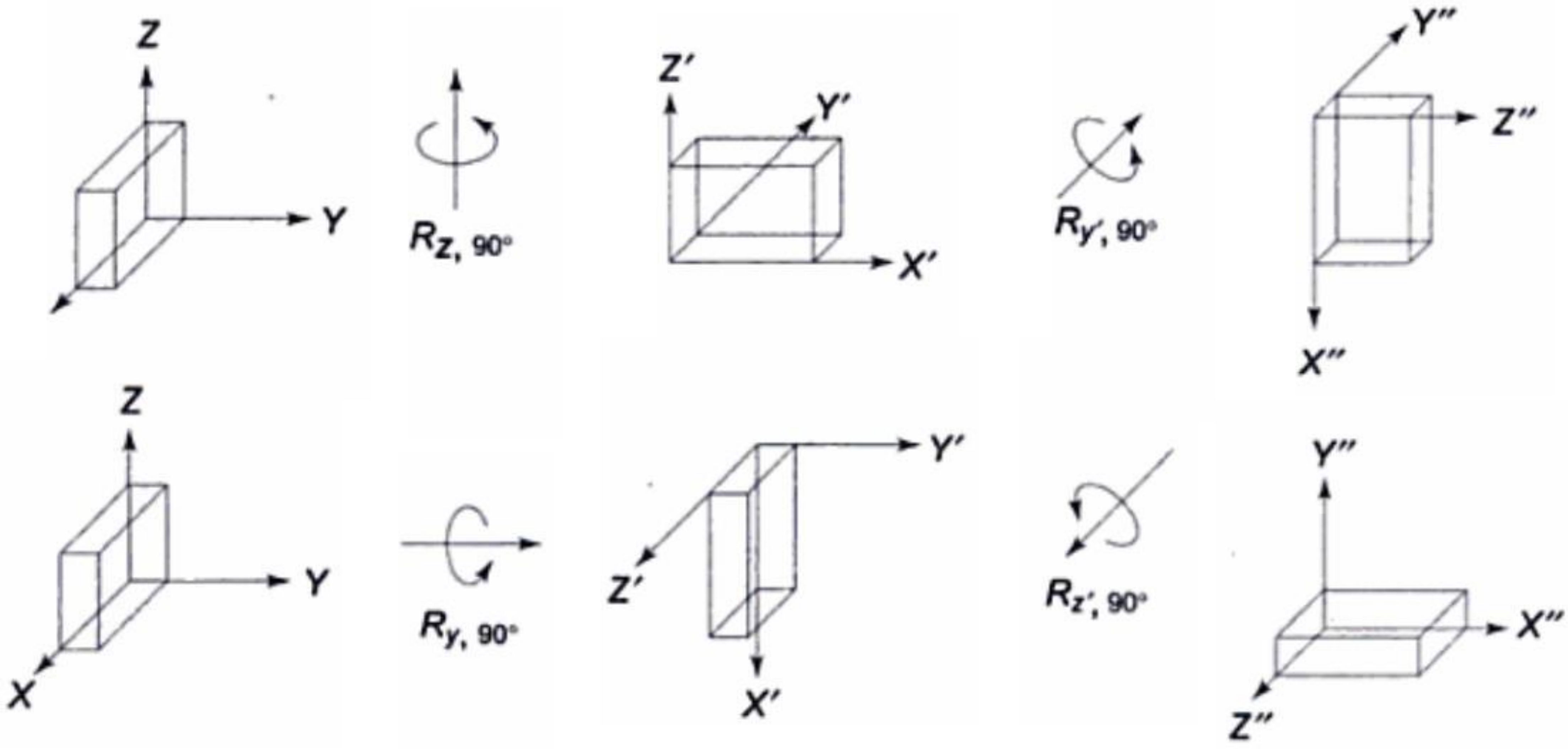
1. To perform successive rotations about the principal axes of the fixed frame.

2. To perform successive rotations about the current principal axes of a moving frame.

Effect of order of rotations of a cuboid about principal axes of a fixed frame.



Effect of order of rotations of a cuboid about axes of the moving frame.



Fixed angle representation

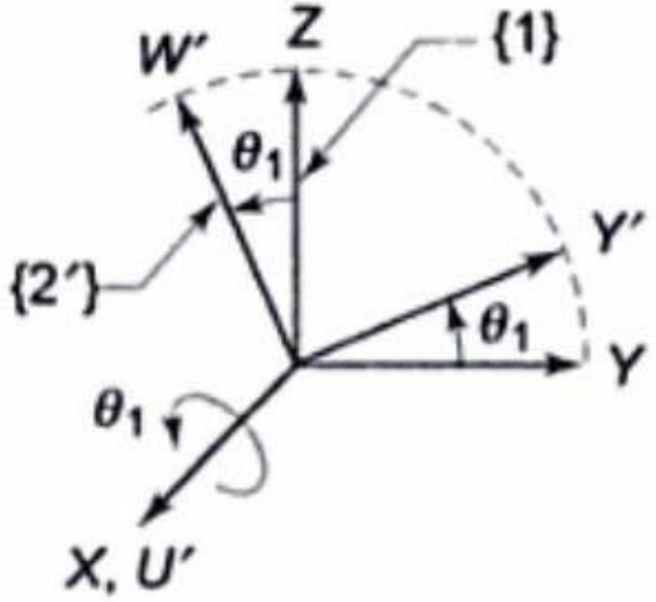
Each rotation is specified about an axis of fixed reference frame.

Let the fixed frame {1} and moving frame {2} be initially coincident.

Consider the following sequence of rotations about the three axes of fixed frame. These rotations are referred as XYZ-fixed angle rotations.

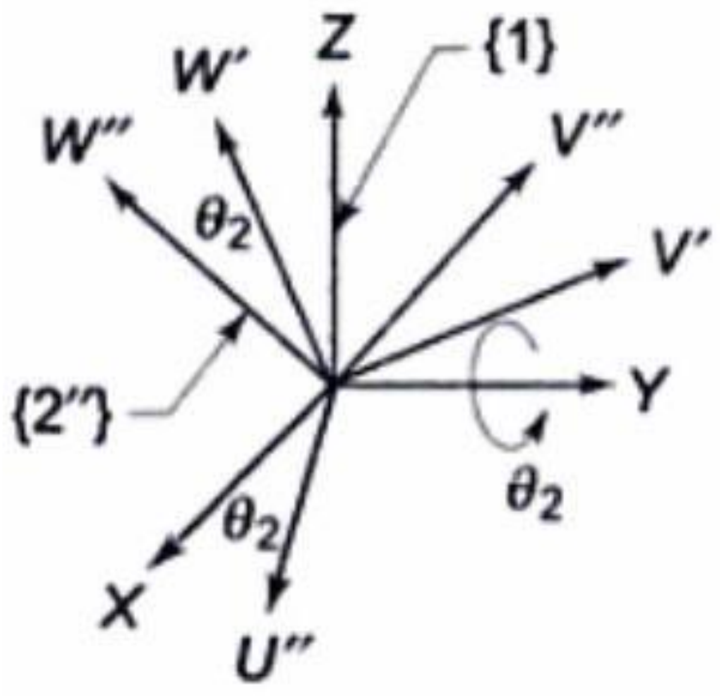
1. Moving frame {2} is rotated by an angle θ_1 about X-axis to frame {2'}.

This rotation is described by the rotation matrix $R_x(\theta_1)$.



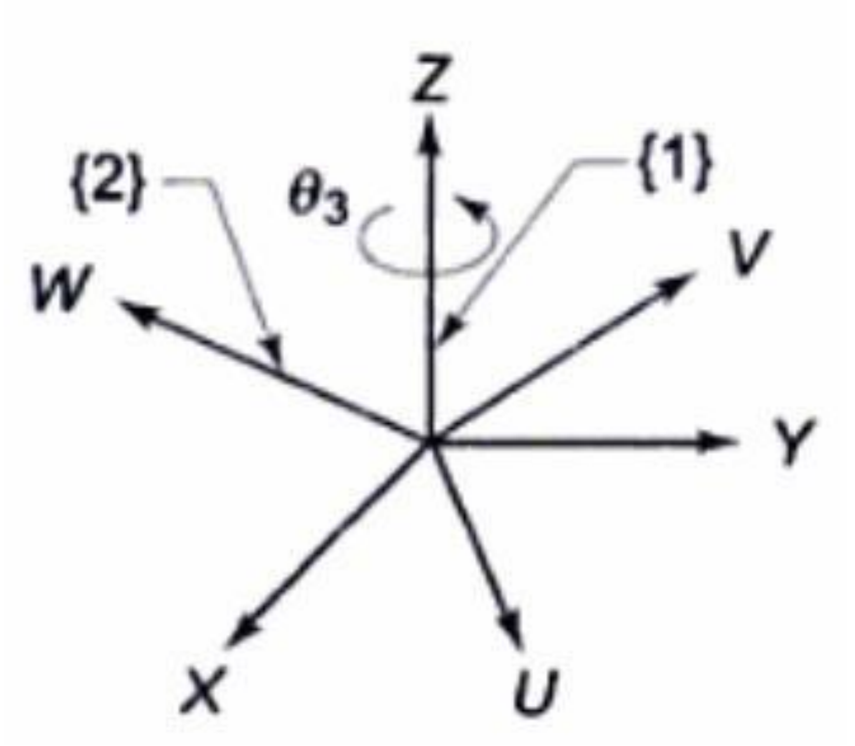
2. The frame {2’} is rotated by an angle θ_2 about Y-axis to give frame {2”}.

This rotation is described by the rotation matrix $R_y(\theta_2)$.



3. The frame {2"} is rotated by an angle θ_3 about Z-axis to give frame {2}.

This rotation is described by the rotation matrix $R_z(\theta_3)$.



The final frame orientation is obtained by composition of rotations w.r.t the fixed frame and overall rotation matrix ${}^{1}R_{2}$ is computed by pre-multiplication of the matrices of elementary rotations.

$$R_{xyz}(\theta_3\theta_2\theta_1) = {}^{1}R_2 = R_z(\theta_3)\mathrm{R_y}(\theta_2)R_x(\theta_1)$$

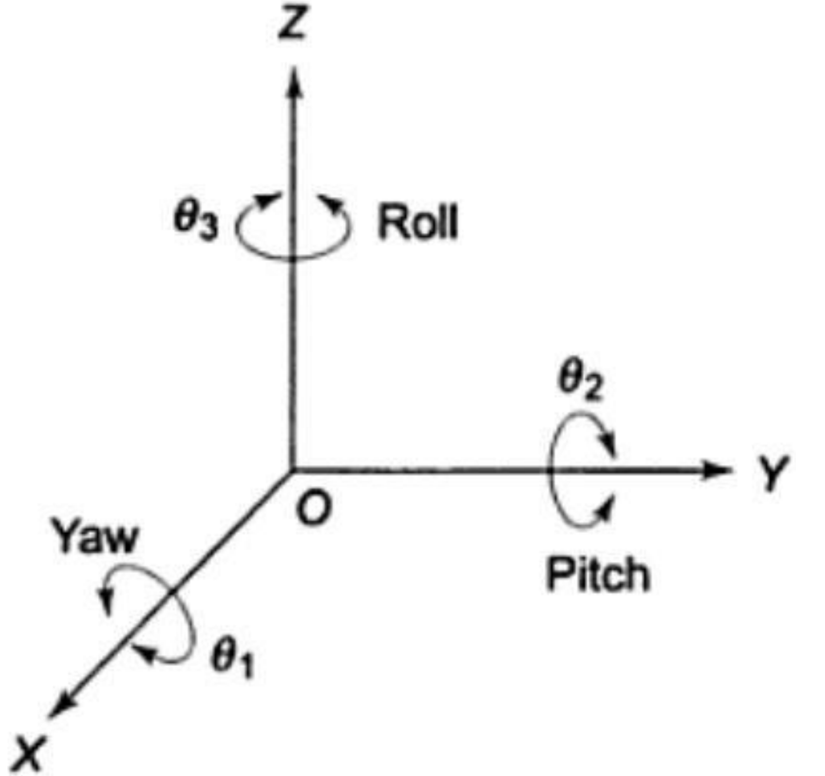
(rotation ordering right to left)

$$R_{xyz}(\theta_3\theta_2\theta_1)$$

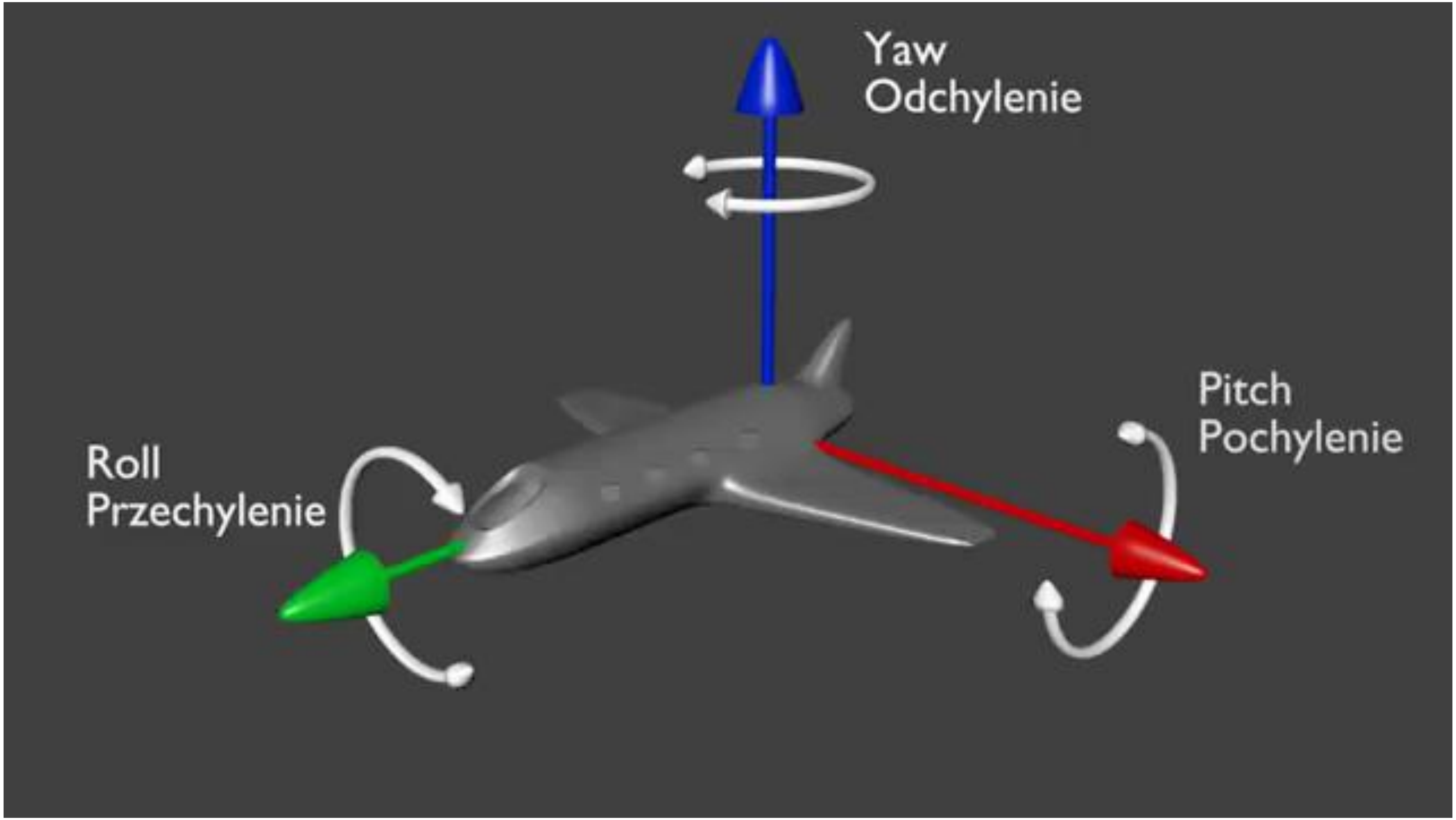
$$= \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$R_{xyz}(\theta_3\,\theta_2\,\theta_1) = \begin{bmatrix} C_2C_3 & S_1S_2C_3 - C_1S_3 & C_1S_2C_3 + S_1S_3 \\ C_2S_3 & S_1S_2S_3 + C_1C_3 & C_1S_2S_3 - S_1C_3 \\ -S_2 & S_1C_2 & C_1C_2 \end{bmatrix}$$

Where $C_i = \cos\theta_i$ and $S_i = \sin\theta_i$



Representation of roll, pitch and yaw (RPY) rotations


Yaw
Odchylenie
Pitch
Pochylenie
Roll
Przechylenie

Example 1

The coordinates of point P in frame {2} are $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T$. The position vector P is rotated about the z-axis by $45°$. Find the coordinates of point Q, the new position of point P.

Solution

The rotation about z-axis by $45°$

$$R_Z(45°) = \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q = R_Z(45°)P$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 5/\sqrt{2} \\ 1 \end{bmatrix}$$

Example 2

Frame {2} is rotated w.r.t frame {1} about the x-axis by an angle of 60°. The position of the origin of frame {2} as seen from frame {1} is ${}^{1}D_2 = [7 \quad 5 \quad 7]^T$. Obtain the transformation matrix ${}^{1}T_2$ which describes frame {2} relative to frame {1}.

Find the description of point P in frame {1} if ${}^{2}P = [2 \quad 4 \quad 6]^T$.

Solution

The homogeneous transformation matrix describing frame {2} w.r.t frame {1} is,

$$
{}^{1}\boldsymbol{T}_2 = \left[\begin{array}{ccc|c} & {}^{1}\boldsymbol{R}_2 & & {}^{1}\boldsymbol{D}_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Frame {2} is rotated relative to frame {1} about x-axis by 60°,

$$ {}^{1}\boldsymbol{R}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin 60^\circ \\ 0 & \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.500 & -0.866 \\ 0 & 0.866 & 0.500 \end{bmatrix} $$

$$ {}^{1}\boldsymbol{T}_2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7.000 \\ 0 & 0.500 & -0.866 & 5.000 \\ 0 & 0.866 & 0.500 & 7.000 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Point P in frame {1} is given by,

$$^{1}P = \ ^{1}R_2 \ ^{2}P$$

$$^{1}P = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 0.5 & -0.866 & 5 \\ 0 & 0.866 & 0.5 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 1.804 \\ 13.464 \\ 1 \end{bmatrix}$$

Example 3

Base of the robot is frame {1} and the end-effector is frame {2}. The end-effector of a robot holds a tool with tool tip which is denoted by point ^{2}P having co-ordinates of $[5\ 8\ 13]^{T}$. The end-effector is rotated about the base frame z-axis by 90°, then about the base frame x-axis by 120°.

i. Find the rotational matrices R_{OX} and R_{OZ}.
ii. Obtain the equivalent rotation matrix $^{1}R_{2}$.
iii. Find the co-ordinates of point ^{1}P in frame {1}.

Solution

$$R_{OX,120^\circ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & -0.866 \\ 0 & 0.866 & -0.5 \end{bmatrix}$$

$$R_{OZ,90^\circ} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{1}R_2 = R_{OX,120^\circ} \cdot R_{OZ,90^\circ}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & -0.866 \\ 0 & 0.866 & -0.5 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -0.5 & 0 & -0.866 \\ 0.866 & 0 & -0.5 \end{bmatrix}$$

$$^{1}P = \ ^{1}R_2 \ ^{2}P = \begin{bmatrix} 0 & -1 & 0 \\ -0.5 & 0 & -0.866 \\ 0.866 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 13 \end{bmatrix} = \begin{bmatrix} -8 \\ -13.76 \\ -2.17 \end{bmatrix}$$

Example 4

Frame {1} and frame {2} have coincident origins and differ only in orientation. Frame {2} is initially coincident with frame {1}. Certain rotations are carried out about the axis of the fixed frame {1}: first rotation about x-axis by 45° then about y-axis by 30° and finally about x-axis by 60°.

Obtain the equivalent rotation matrix ${}^{1}R_{2}$.

Solution

Rotations are in order X-Y-X about the fixed axes; hence , it is a case of fixed angle representation.

$$ {}^{1}R_{2} = R_{x}(60°)R_{y}(30°)R_{x}(45°) $$

$^{1}R_{2}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60° & -\sin 60° \\ 0 & \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} \cos 30° & 0 & \sin 30° \\ 0 & 1 & 0 \\ -\sin 30° & 0 & \cos 30° \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45° & -\sin 45° \\ 0 & \sin 45° & \cos 45° \end{bmatrix}$$

$$^{1}R_{2} = \begin{bmatrix} 0.866 & 0.354 & 0.354 \\ 0.433 & -0.177 & -0.884 \\ -0.25 & 0.919 & 0.306 \end{bmatrix}$$

Example 5

In example 2, the transformation matrix ${}^1T_2$ was obtained, which describes the position and orientation of frame {2} relative to frame {1}. Using this matrix, determine the description of frame {1} relative to frame {2}.

Solution

The homogeneous transformation for describing frame {1} relative to frame {2], ${}^2T_1$ is given by,

$$ {}^2T_1 = \left[\begin{array}{ccc|c} & {}^2R_1 & & {}^2D_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\,{}^1T_2\right]^{-1} $$

The inverse of ${}^1T_2$ is given by,

$$ {}^2\boldsymbol{T}_1 = \left[{}^1\boldsymbol{T}_2\right]^{-1} = \left[\begin{array}{ccc|c} & {}^1\boldsymbol{R}_2^T & & -{}^1\boldsymbol{R}_2^T\,{}^1\boldsymbol{D}_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

From ${}^{1}T_2$ in Example 2,

$$^{1}R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.866 \\ 0 & 0.866 & 0.5 \end{bmatrix}$$

$$^{1}D_2 = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$$

The orientation of frame {1} w.r.t frame {2} is given by,

$$^{2}R_1 = \left[\, {}^{1}R_2 \right]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.866 \\ 0 & -0.866 & 0.5 \end{bmatrix}$$

The position of the origin of frame {1} w.r.t frame {2} is given by,

$$ {}^{2}D_1 = -\,{}^{1}R_2^T\;{}^{1}D_2 $$

$$ {}^{2}D_1 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.866 \\ 0 & -0.866 & 0.5 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ -8.562 \\ 0.83 \end{bmatrix} $$

$$ {}^{2}T_1 = \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 0.5 & 0.866 & -8.562 \\ 0 & -0.866 & 0.5 & 0.83 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Euler angle representation

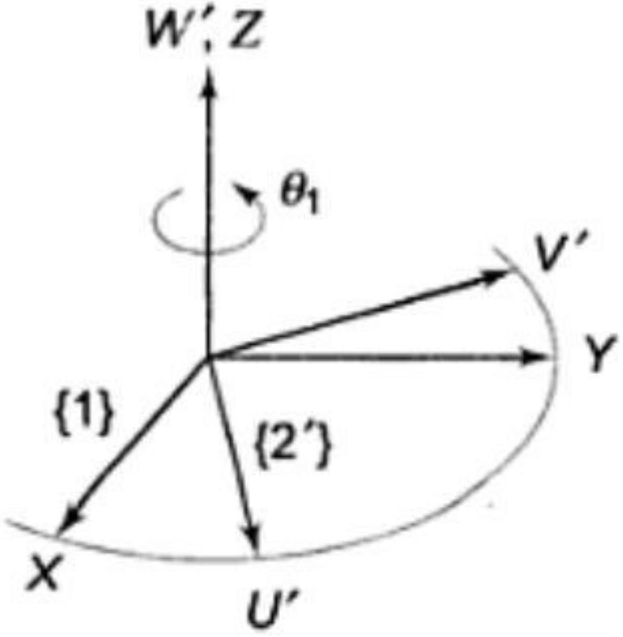
The moving frame, instead of rotating about the principal axes of the fixed frame, can rotate about its own principal axes.

Consider the rotations of frame {2} w.r.t frame {1}, starting from the position when the two frames are coincident.

This convention for specifying orientation is called WVU-Euler angle representation.

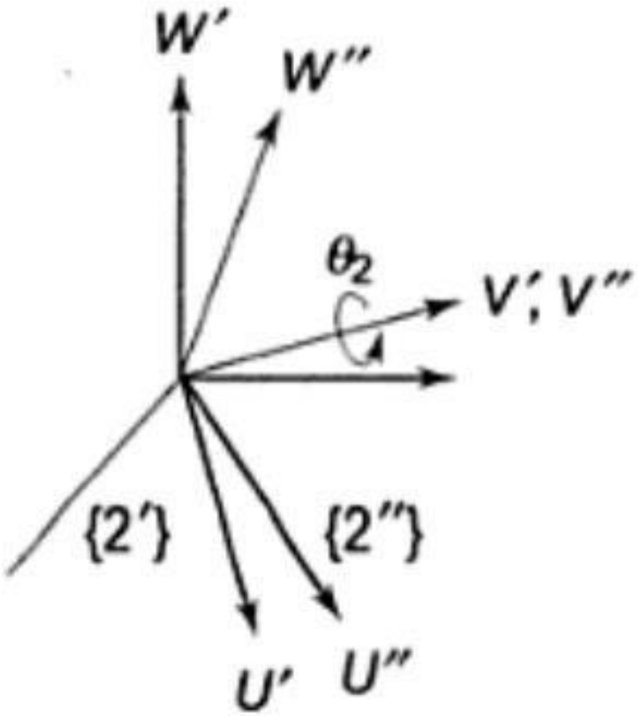
1. Frame {2} is rotated by an angle θ_1 about its w-axis coincident with z-axis of frame {1}. The rotated frame is now {2'}.

This rotation is described by the rotation matrix $R_w(\theta_1)$.



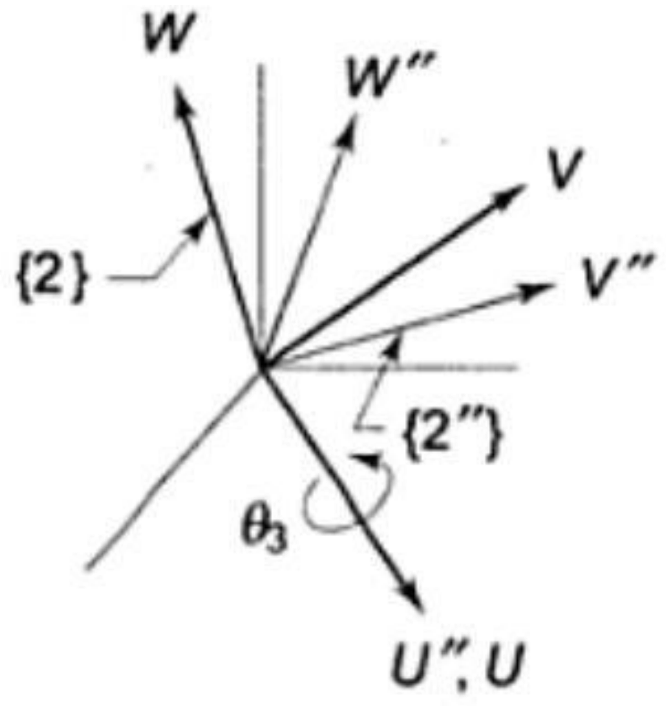
2. Moving frame {2’} is rotated by an angle θ_2 about v’-axis, the rotated v-axis to frame {2”}.

This rotation is described by the rotation matrix $R_{v'}(\theta_2)$.



3. Frame {2”} is rotated by an angle θ_3 about its u”-axis, the rotated u-axis to give frame {2}.

This rotation is described by the rotation matrix $R_{u''}(\theta_3)$.



The equivalent rotation matrix is computed by post multiplication of the matrices of the elementary rotations.

$$R_{wvu}(\theta_1\theta_2\theta_3) = {}^1R_2$$

$$= R_w(\theta_1)R_{v'}(\theta_2)R_{u''}(\theta_3)$$

(rotation ordering left to right)

The rotations are performed about the current axes of the moving frame {uvw}.

$$R_{wvu}(\theta_1\theta_2\theta_3) = R_w(\theta_1)R_{v'}(\theta_2)R_{u''}(\theta_3)$$

$$\boldsymbol{R}_{wvu}(\theta_1\theta_2\theta_3) = \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & 0 & S_2 \\ 0 & 1 & 0 \\ -S_2 & 0 & C_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_3 & -S_3 \\ 0 & S_3 & C_3 \end{bmatrix}$$

$$\boldsymbol{R}_{wvu}(\theta_1\theta_2\theta_3) = \begin{bmatrix} C_2C_3 & S_1S_2C_3 - C_1S_3 & C_1S_2C_3 + S_1S_3 \\ C_2S_3 & S_1S_2S_3 + C_1C_3 & C_1S_2S_3 - S_1C_3 \\ -S_2 & S_1C_2 & C_1C_2 \end{bmatrix}$$

Example 1

The end point of a link of a manipulator is at $P = [2 \quad 2 \quad 6]^T$. The link undergoes a rotation by 90° about x-axis, then by 180° about its own w-axis.

i. Find the resulting rotation matrix.

ii. Find the final location of end point.

Solution

$$ {}^{1}R_2 = R_u(90°) \cdot R_w(180°) $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} $$

$$^{1}P = \ ^{1}R_{2}\ ^{2}P$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -6 \\ -2 \end{bmatrix}$$

Example 2

Frame {B} is located as follows: initially coincident with frame {A}, then the origin of frame {B} is translated by ${}^{A}D_{B} = [5 \quad -4 \quad 3]^{T}$, then the translated frame is rotated about y_B axis by 30° and then the resulting frame is rotated about its own u_B axis by 45°.

i. Determine the rotational matrix, ${}^{A}R_{B}$.

ii. Find the description of point ${}^{A}P$ if ${}^{B}P$ is $[6 \quad 3 \quad 5 \quad 1]^{T}$.

Solution

$$^{A}R_{B} = R_{v}(30°) \cdot R_{u}(45°)$$

$$= \begin{bmatrix} \cos 30 & 0 & \sin 30 \\ 0 & 1 & 0 \\ -\sin 30 & 0 & \cos 30 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & \sin 45 & \cos 45 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & 0.3535 & 0.3535 \\ 0 & 0.707 & -0.707 \\ -0.5 & 0.6123 & 0.6123 \end{bmatrix}$$

$$ {}^{A}P = {}^{A}T_{B}\,{}^{B}P $$

$$ = \left[\begin{array}{ccc|c} 0.866 & 0.3535 & 0.3535 & 5 \\ 0 & 0.707 & -0.707 & -4 \\ -0.5 & 0.6123 & 0.6123 & 3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 6 \\ 3 \\ 5 \\ 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 13.024 \\ -5.414 \\ 4.8984 \\ 1 \end{bmatrix} $$