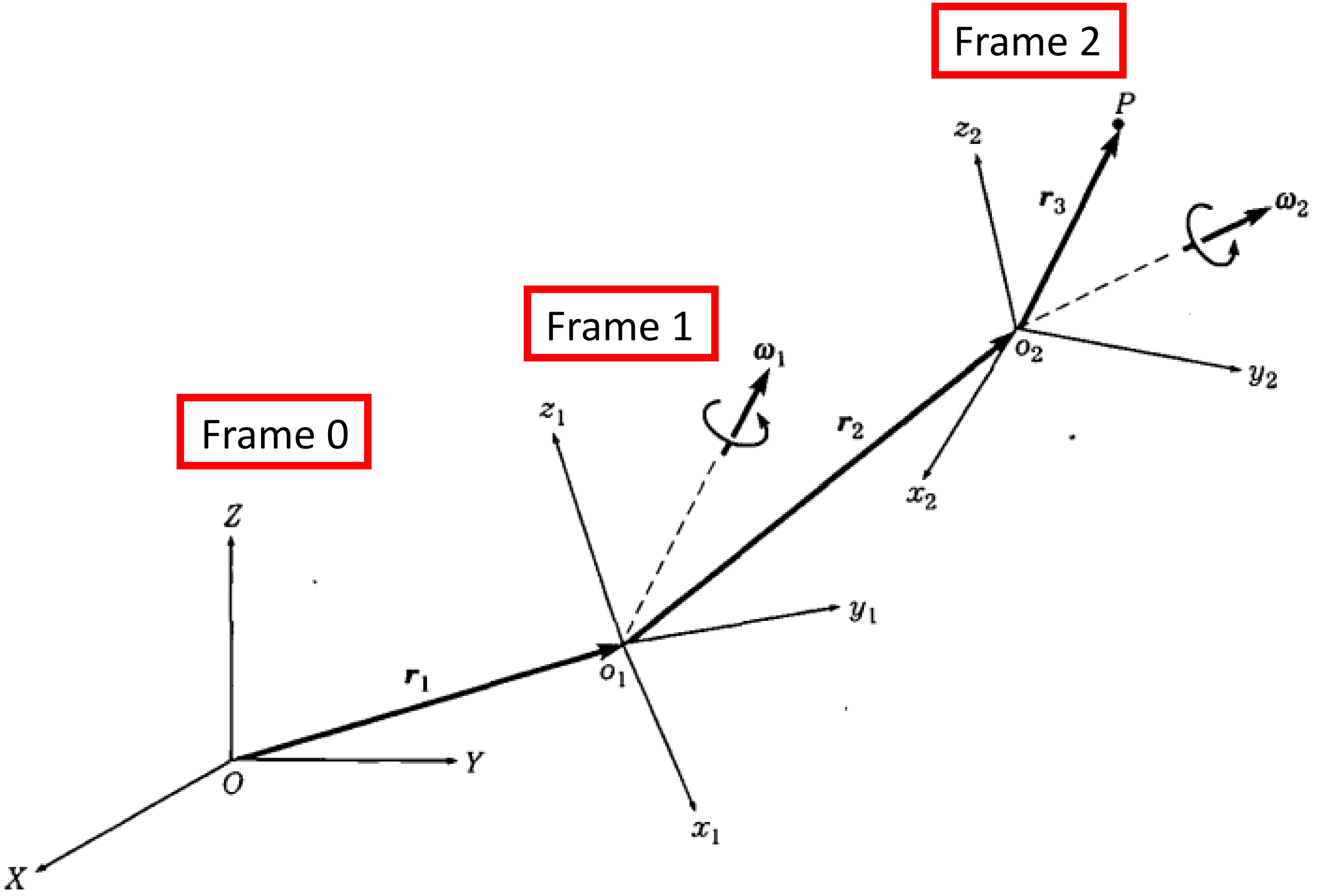
Generalizations of kinematic expressions

Calculation of velocity and acceleration when two intermediate frames are used.

Point P is defined in the intermediate reference frame $o_2x_2y_2z_2$, which is moving w.r.t another intermediate reference frame $o_1x_1y_1z_1$.

Frame $o_1x_1y_1z_1$ is rotating at the angular velocity $\boldsymbol{\omega}_1$ and angular acceleration $\dot{\boldsymbol{\omega}}_1$ w.r.t the fixed reference frame OXYZ.



Case – 1

The angular motions of $o_2x_2y_2z_2$, $\boldsymbol{\omega}_2$ and $\dot{\boldsymbol{\omega}}_2$, are defined w.r.t frame OXYZ.

Case – 2

The angular motions of $o_2x_2y_2z_2$, $\boldsymbol{\omega}_2$ and $\dot{\boldsymbol{\omega}}_2$, are defined w.r.t frame $o_1x_1y_1z_1$.

Case 1:

The angular motions of $o_2x_2y_2z_2$, $\boldsymbol{\omega}_2$ and $\dot{\boldsymbol{\omega}}_2$, are defined w.r.t frame OXYZ.

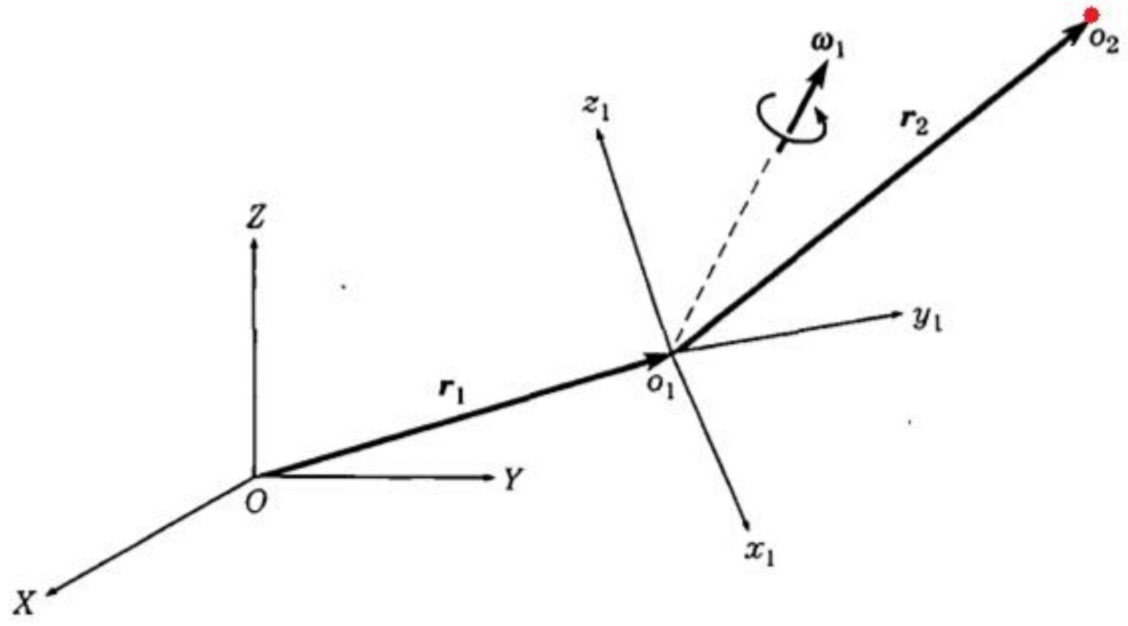
Question:

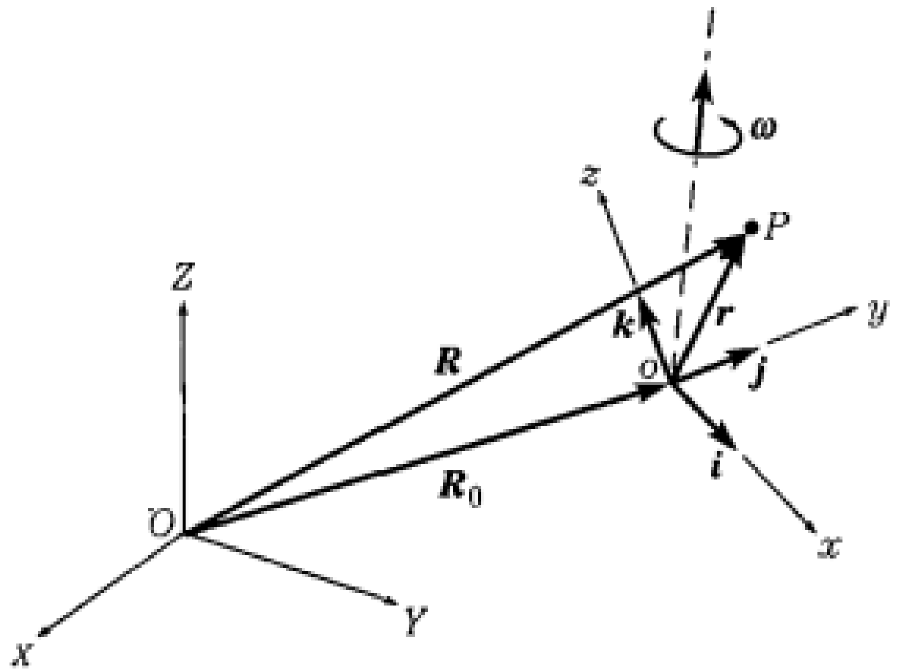
Find the motion of P w.r.t OXYZ.

Solution:

Find the motion of o_2 w.r.t OXYZ and then the motion of P w.r.t OXYZ.

Motion of o_2 (defined in $o_1x_1y_1z_1$) w.r.t OXYZ.





Velocity of P: $\boldsymbol{v} = \frac{d\boldsymbol{R}_0}{dt} + \boldsymbol{v}_{rel} + (\boldsymbol{\omega} \times \boldsymbol{r})$

Acceleration of P: $\boldsymbol{a} = \frac{d^2\boldsymbol{R}_0}{dt^2} + \boldsymbol{a}_{rel} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{rel} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$

$$\boldsymbol{R}_0 = \boldsymbol{r}_1$$

$$\boldsymbol{r} = \boldsymbol{r}_2$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1$$

The time derivatives:

$$\frac{d\boldsymbol{R}_0}{dt} = \dot{\boldsymbol{r}}_1 \qquad \frac{d^2 R_0}{dt^2} = \ddot{\boldsymbol{r}}_1$$

$$\boldsymbol{v}_{rel} = \dot{\boldsymbol{r}}_2 \qquad \boldsymbol{a}_{rel} = \ddot{\boldsymbol{r}}_2$$

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1$$

$$\boldsymbol{v}_{o_2(OXYZ)} = \frac{d\boldsymbol{R}_0}{dt} + \boldsymbol{v}_{rel} + (\boldsymbol{\omega} \times \boldsymbol{r})$$

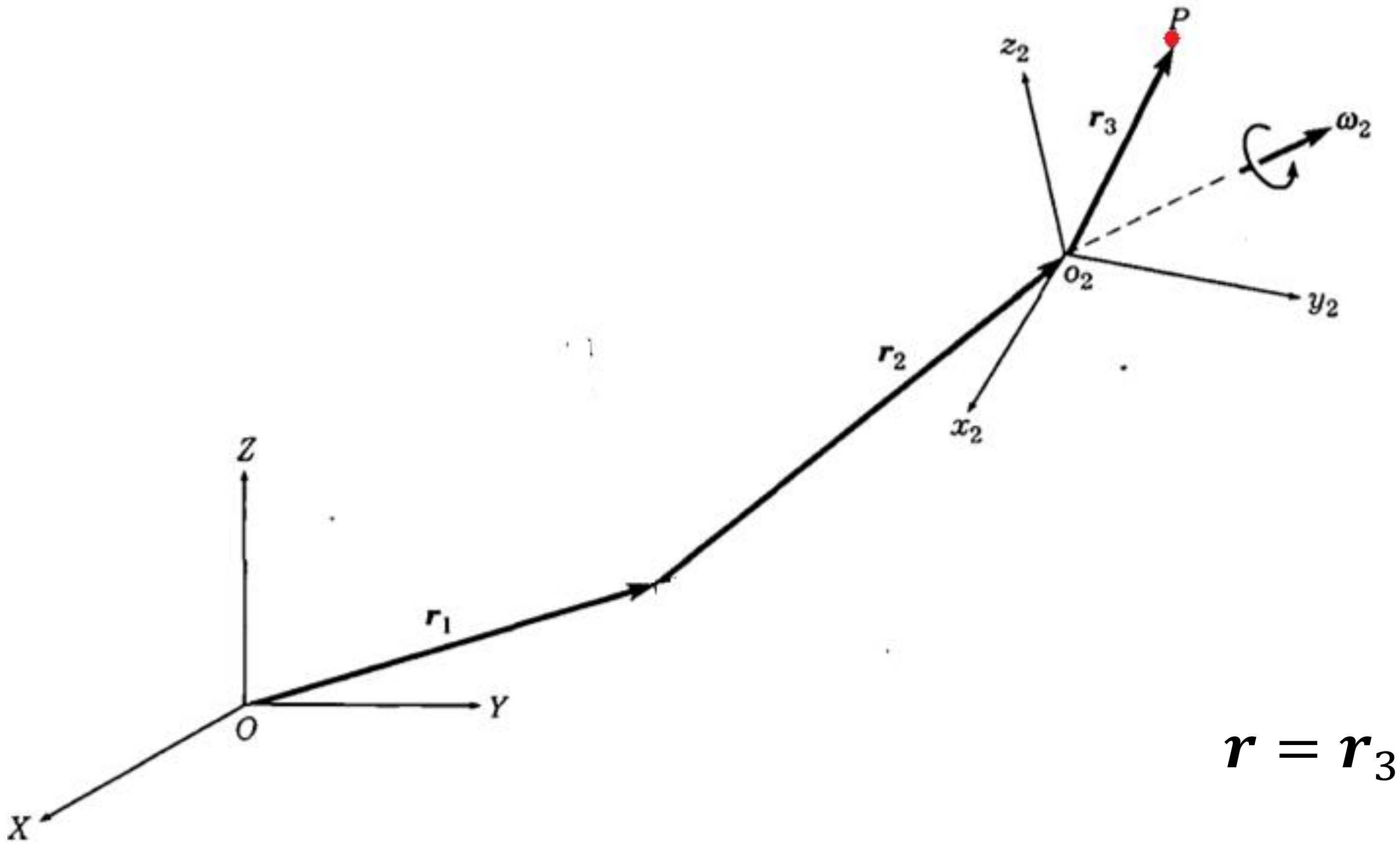
$$= \dot{\boldsymbol{r}}_1 + \dot{\boldsymbol{r}}_2 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_2$$

$$\boldsymbol{a}_{o_2(OXYZ)} = \frac{d^2\boldsymbol{R}_0}{dt^2} + \boldsymbol{a}_{rel} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{rel} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$= \ddot{\boldsymbol{r}}_1 + \ddot{\boldsymbol{r}}_2 + 2\boldsymbol{\omega}_1 \times \dot{\boldsymbol{r}}_2 + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_2 + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{r}_2)$$

Velocity and acceleration of point o_2 w.r.t OXYZ.

Motion of P (defined in $o_2x_2y_2z_2$) w.r.t OXYZ.



$$\boldsymbol{R}_0 = \boldsymbol{r}_1 + \boldsymbol{r}_2$$

$$\frac{d\boldsymbol{R}_0}{dt} = \boldsymbol{v}_{o_2(OXYZ)}$$

$$\frac{d^2\boldsymbol{R}_0}{dt^2} = \boldsymbol{a}_{o_2(OXYZ)}$$

$$\boldsymbol{r} = \boldsymbol{r}_3 \qquad \boldsymbol{v}_{rel} = \dot{\boldsymbol{r}}_3 \qquad \boldsymbol{a}_{rel} = \ddot{\boldsymbol{r}}_3$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_2 \qquad \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_2$$

$$\boldsymbol{v}_{P(OXYZ)} = \frac{d\boldsymbol{R}_0}{dt} + \boldsymbol{v}_{rel} + (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$= \boldsymbol{v}_{o_{2(OXYZ)}} + \dot{\boldsymbol{r}}_3 + \boldsymbol{\omega}_2 \times \boldsymbol{r}_3$$

$$= \dot{\boldsymbol{r}}_1 + \dot{\boldsymbol{r}}_2 + \dot{\boldsymbol{r}}_3 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_2 + \boldsymbol{\omega}_2 \times \boldsymbol{r}_3$$

Velocity of point P w.r.t OXYZ.

$$\boldsymbol{a}_{P(OXYZ)} = \frac{d^2\boldsymbol{R}_0}{dt^2} + \boldsymbol{a}_{rel} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{rel} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$= \boldsymbol{a}_{o_{2(OXYZ)}} + \ddot{\boldsymbol{r}}_3 + 2\boldsymbol{\omega}_2 \times \dot{\boldsymbol{r}}_3 + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_3 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_3)$$

$$= \ddot{\boldsymbol{r}}_1 + \ddot{\boldsymbol{r}}_2 + \ddot{\boldsymbol{r}}_3 + 2\boldsymbol{\omega}_1 \times \dot{\boldsymbol{r}}_2 + 2\boldsymbol{\omega}_2 \times \dot{\boldsymbol{r}}_3 + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_2 + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_3 + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{r}_2) + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_3)$$

Acceleration of point P w.r.t OXYZ.

Case – 2

The angular motions of $o_2x_2y_2z_2$, ω_2 and $\dot{\omega}_2$, are defined w.r.t frame $o_1x_1y_1z_1$.

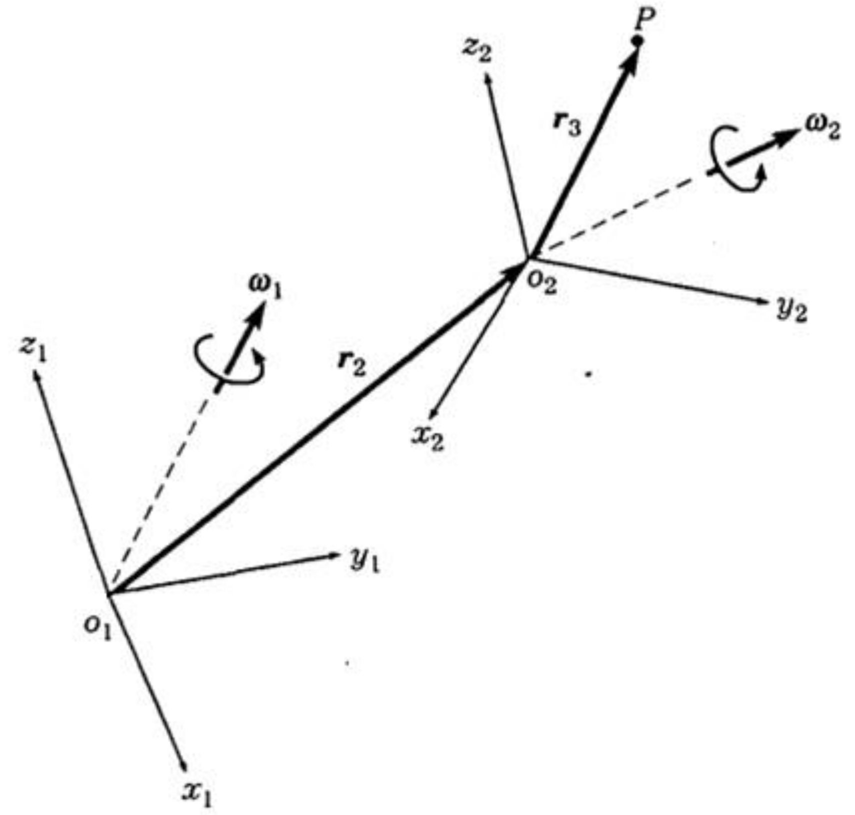
Question:

Find the motion of P w.r.t OXYZ.

Solution:

Find the motion of P w.r.t $o_1x_1y_1z_1$ and then the motion of P w.r.t OXYZ.

Motion of point P (defined in $o_2x_2y_2z_2$) w.r.t $o_1x_1y_1z_1$.



$$\boldsymbol{R}_0 = \boldsymbol{r}_2 \qquad \frac{d\boldsymbol{R}_0}{dt} = \dot{\boldsymbol{r}}_2 \qquad \frac{d^2\boldsymbol{R}_0}{dt^2} = \ddot{\boldsymbol{r}}_2$$

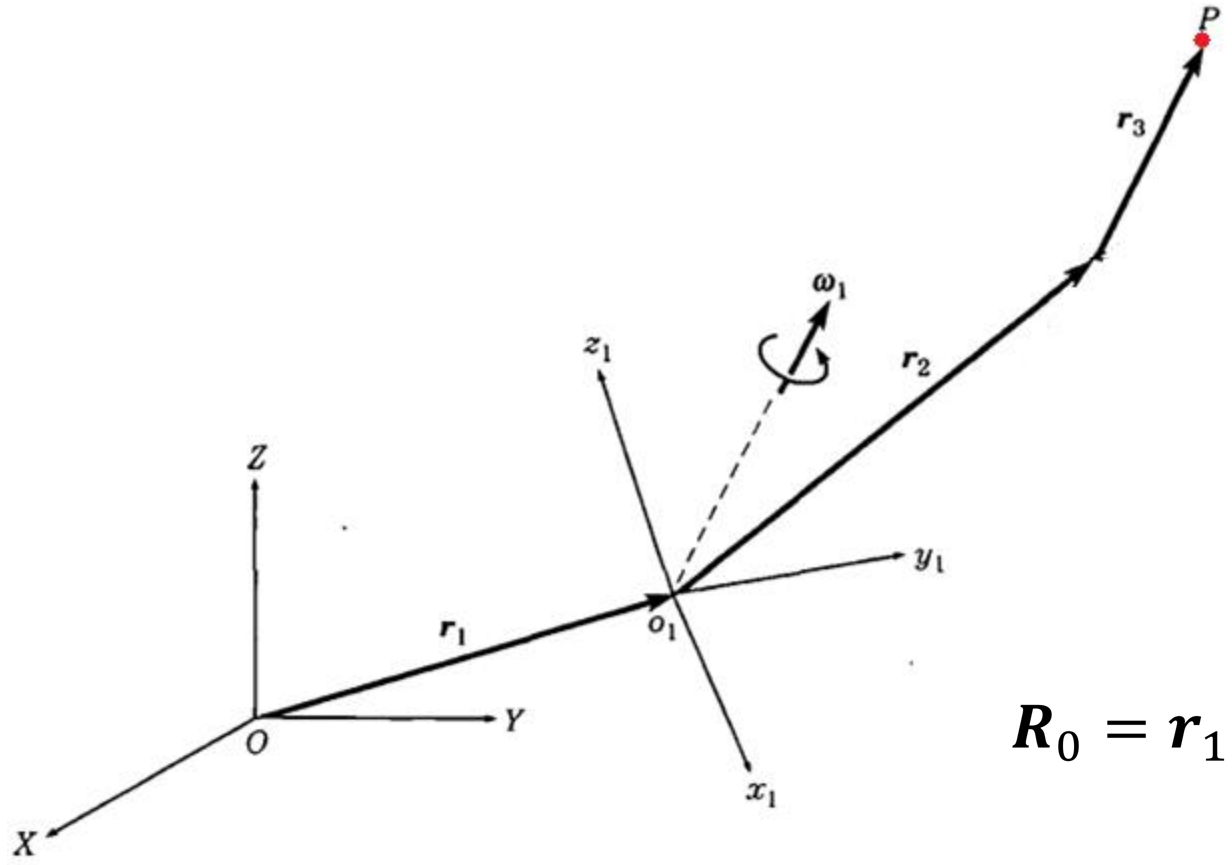
$$\boldsymbol{r} = \boldsymbol{r}_3 \qquad \boldsymbol{v}_{rel} = \dot{\boldsymbol{r}}_3 \qquad \boldsymbol{a}_{rel} = \ddot{\boldsymbol{r}}_3$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_2 \qquad \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_2$$

$$\boldsymbol{v}_{P(o_1x_1y_1z_1)} = \dot{\boldsymbol{r}}_2 + \dot{\boldsymbol{r}}_3 + \boldsymbol{\omega}_2 \times \boldsymbol{r}_3$$

$$\boldsymbol{a}_{P(o_1x_1y_1z_1)} = \ddot{\boldsymbol{r}}_2 + \ddot{\boldsymbol{r}}_3 + 2\boldsymbol{\omega}_2 \times \dot{\boldsymbol{r}}_3 + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_3 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_3)$$

Motion of point P (defined in $o_1x_1y_1z_1$) w.r.t OXYZ.



$$\boldsymbol{R}_0 = \boldsymbol{r}_1 \qquad \frac{d\boldsymbol{R}_0}{dt} = \dot{\boldsymbol{r}}_1 \qquad \frac{d^2\boldsymbol{R}_0}{dt^2} = \ddot{\boldsymbol{r}}_1$$

$$\boldsymbol{r} = \boldsymbol{r}_2 + \boldsymbol{r}_3 \qquad \boldsymbol{v}_{rel} = \boldsymbol{v}_{P(o_1x_1y_1z_1)} \qquad \boldsymbol{a}_{rel} = \boldsymbol{a}_{P(o_1x_1y_1z_1)}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 \qquad \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1$$

$$\boldsymbol{v}_{P(OXYZ)} = \dot{\boldsymbol{r}}_1 + \boldsymbol{v}_{P(o_1x_1y_1z_1)} + \boldsymbol{\omega}_1 \times (\boldsymbol{r}_2 + \boldsymbol{r}_3)$$

$$= \dot{\boldsymbol{r}}_1 + \dot{\boldsymbol{r}}_2 + \dot{\boldsymbol{r}}_3 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_2 + (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \boldsymbol{r}_3$$

$$\boldsymbol{a}_{P(OXYZ)}$$

$$= \ddot{\boldsymbol{r}}_1 + \boldsymbol{a}_{P(o_1x_1y_1z_1)} + 2\boldsymbol{\omega}_1 \times \boldsymbol{v}_{P(o_1x_1y_1z_1)} + \dot{\boldsymbol{\omega}}_1(\boldsymbol{r}_2 + \boldsymbol{r}_3) + \boldsymbol{\omega}_1$$

$$\times [\boldsymbol{\omega}_1 \times (\boldsymbol{r}_2 + \boldsymbol{r}_3)]$$

$$= \ddot{\boldsymbol{r}}_1 + \ddot{\boldsymbol{r}}_2 + \ddot{\boldsymbol{r}}_3 + 2\boldsymbol{\omega}_1 \times \dot{\boldsymbol{r}}_2 + 2(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \dot{\boldsymbol{r}}_3 + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_2$$

$$+ (\dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2) \times \boldsymbol{r}_3 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_3) + 2\boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_3) + \boldsymbol{\omega}_1$$

$$\times [\boldsymbol{\omega}_1 \times (\boldsymbol{r}_2 + \boldsymbol{r}_3)]$$

Bird on mobile.

An artistic mobile structure is modeled as sketched in Figure 1(a), consists of a large **+** and four smaller Y's, one each attached to a tip of the +. The + is horizontal at all times and rotates at a constant angular velocity ω_1 (w.r.t ground) about an axis through its center. Also, each four Y's remains at all times in a vertical plane and rotates at a constant angular velocity ω_2 (w.r.t its + tip) about an axis through its center. At the instant shown, a bird of mass m is on a leg of one of the Y's, which is oriented as indicated in Figure 1(b). Relative to the Y, the bird is running with a velocity v_0 and an acceleration a_0, at the instant shown. At the same instant, a gust of wind exerts a force F_w on the bird in the X direction.

Find the velocity and acceleration of the bird, which may be modeled as a point, at the instant shown.

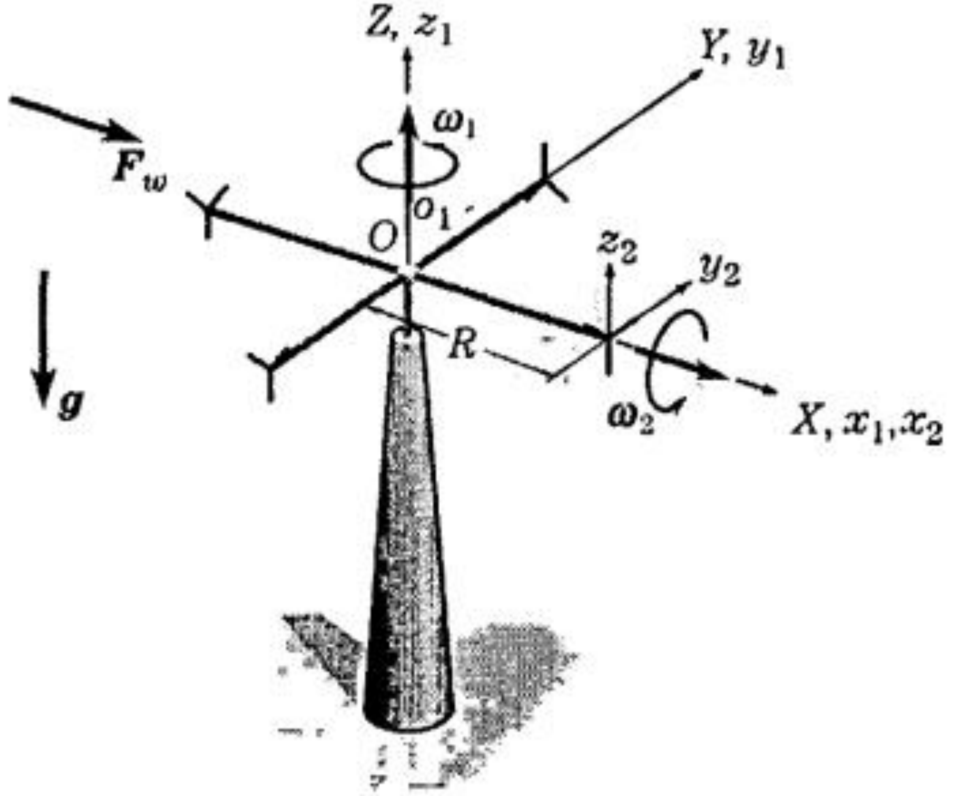


Figure 1(a): Mobile structure

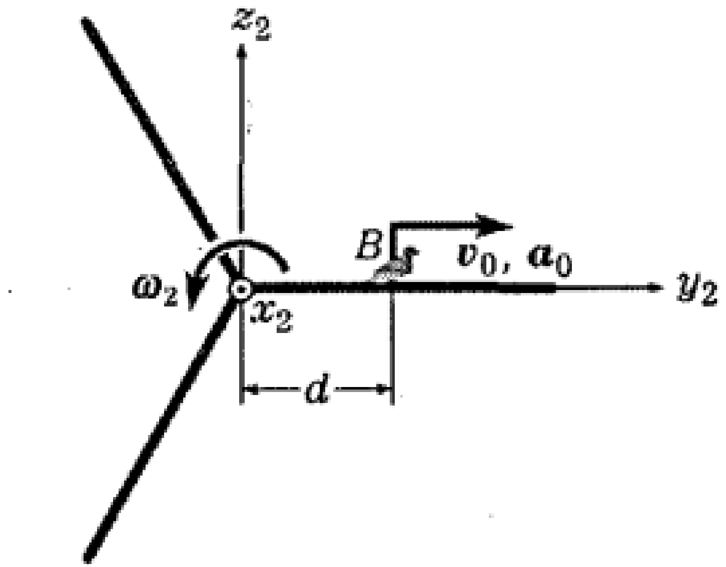


Figure 1(b): Detail of Y and bird.

Figure 1: Sketch of artistic mobile structure with bird running along horizontal leg of Y.

1. Motion of bird (defined in $o_2x_2y_2z_2$) w.r.t $o_1x_1y_1z_1$.

2. Motion of bird (defined in $o_1x_1y_1z_1$) w.r.t OXYZ.

Robot manipulating work piece.

A robot named JT is rolling w.r.t the shop floor at a constant speed of 0.5 m/s and carrying a work piece 1 m long, as sketched in Figure 2. each of the links of the robot arm is 0.75 m long, and the second link has an end gripper that holds the work piece which ,ay be considered as rigid. At the instant shown, the link AB is rotating a ω_1 (1 rev/3 s), and link BD is rotating at ω_2 (1 rev/2 s) and $\dot{\omega}_2$ (0.5 rad/s^2), all w.r.t the shop floor.

At the instant shown, find the velocity and acceleration of the center of the work piece, labeled point C, as sketched in Figure 2.

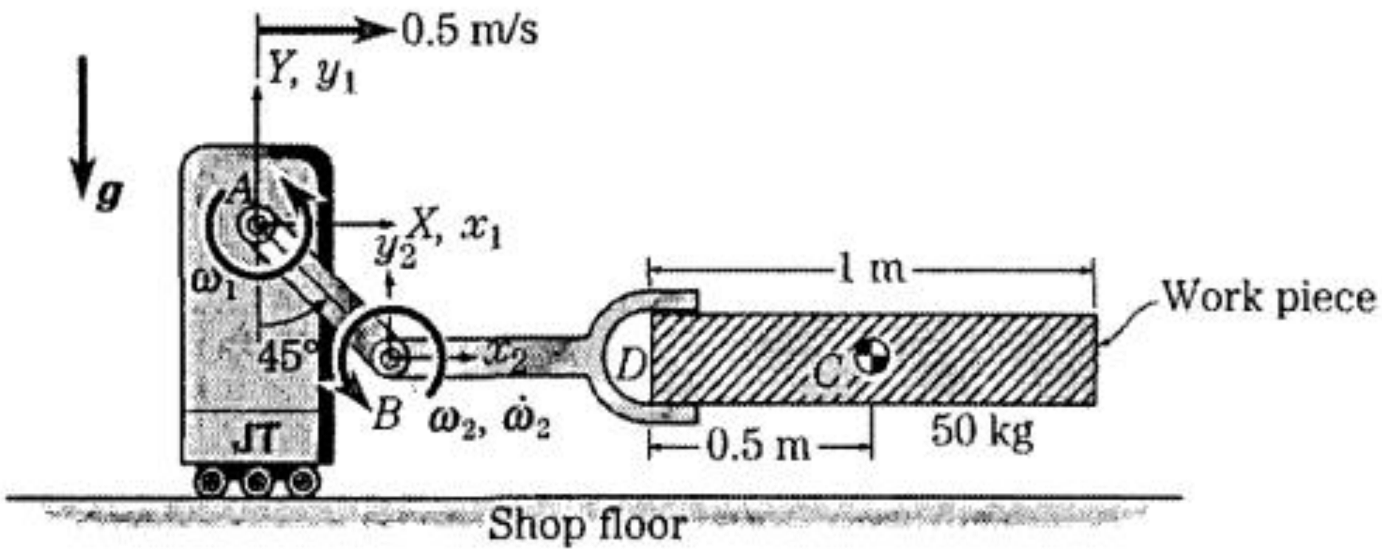


Figure 2: JT manipulating work piece.

1. Using the frame $Ax_1y_1z_1$ as an intermediate frame, find the motion of point B w.r.t the fixed reference frame $OXYZ$.

2. Using the frame $Bx_2y_2z_2$ as an intermediate frame, find the motion of point C w.r.t the fixed reference frame $OXYZ$.