Matrices types

1- Zero matrix => [all items are zeros]
2- Identity matrix => [main diagonal are ones]
3- Column matrix (Vector) => [m * 1]
4- Row matrix (Vector) => [1 * n]
5- Square matrix => [m = n]
6- Diagonal matrix => [main diagonal contains integers, other are zeros]
7- Upper triangular matrix => [main diagonal and upper is integers, other are zeros]
8- Lower triangular matrix => [main diagonal and lower is integers, other are zeros]
9- Symmetric matrix => [lower triangular equal upper triangular]

Addition & scalar multiplication properties

1- A + B = B + A
2- A + (B + C) = (A + B) + C
3- (s1 * s2) A = s1 (s2 * A) = s2 (s1 * A)
4- I * A = A
5- s1 (A + B) = s1 * A + s2 * B
6- (s1 + s2) A = s1 * A + s2 * A
7- A + 0 = A
8- A + (-A) = 0
9- If s1 * A = 0 => s1 = 0 or A = 0

** s1 & s2 => scalar

Multiplication properties

1- A (B * C) = (A * B) C
2- A (B + C) = AB + AC
3- (A + B) C = AC + BC
4- s1 (A * B) = (s1 * A) B = A (s1 * B)
5- AB ≠ BA
6- A * I = I * A = A
7- If AC = BC => A = B

** s1 & s2 => scalar

** I => identity matrix

Power properties

1- $A^r * A^s = A^{r+s}$
2- $(A^r)^s = A^{r*s}$
3- $A^0 = 1$

** r & s => non-negative values

Transpose properties

1- $(A + B + C)^T = A^T + B^T + C^T$
2- $(s1 * A)^T = s1 * A^T$
3- $(A * B * C)^T = C^T * B^T * A^T$
4- $(A^T)^T = A$

** where s1 => scalar

** T => transpose (convert columns to rows & convert rows to columns)

Trace properties

1- Matrix MUST be square.
2- tr (A) => [summation of main diagonal]
3- tr (A + B) = tr (A) + tr (B)
4- tr (AB) = tr (BA)
5- tr (s1 * A) = s1 tr (A)
6- tr (A^T) = tr (A)

** s1 => scalar

Adjoint VS cofactor matrices

1- adj(A) = cof(A)^T & cof(A) = adj(A)^T
2- Suppose matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

adj(A) = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

change Signs

Interchange items

cof(A) = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Interchange items & change Signs

Interchange items

3- Suppose matrix A = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$\text{cof(A)} = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} e & f \\ h & i \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} d & f \\ g & i \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} b & c \\ h & i \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} a & c \\ g & i \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} b & c \\ e & f \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} a & c \\ d & f \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

adj(A) = cof(A)^T

Inverse properties

1- Matrices MUST be square
2- If A * B = B * A = I => B is inverse of A or A is inverse of B
3- If A has no inverse => called singular
4- General rule: $A^{-1} = \frac{1}{\det(A)}$ * adj(A) or $\frac{1}{\det(A)}$ * cof(A)^T
5- Inverse rule of $A_{(2 \times 2)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ => $A^{-1} = \begin{bmatrix} \frac{d}{\det(A)} & \frac{-b}{\det(A)} \\ \frac{-c}{\det(A)} & \frac{a}{\det(A)} \end{bmatrix}$ & det(A) = ad – bc
6- Inverse rule of $A_{(3 \times 3)}$ = $\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right]$

Try to convert original matrix to identity matrix using row exchange, multiply row by factor, row athematic operation

If original matrix cannot convert to identity => its singular matrix & has no inverse.

7- $(A^{-1})^{-1}$ = A
8- $(A^k)^{-1}$ = $(A^{-1})^k = A^{-k}$
9- $(s1 * A)^{-1}$ = $\frac{1}{s1}$ (A^{-1})
10- $(A^T)^{-1}$ = $(A^{-1})^T$
11- $(AB)^{-1}$ = $B^{-1} * A^{-1}$

Linear system

1- The greatest power in all equations (not solution) must be 1
2- Linear system has:
   a. No solution => zero row has integer solution (lines parallel)
   b. One solution => columns number = solution elements (intersect between lines in one point)
   c. Many solution => columns number > solution elements / there is a zero row (lines are identical)

Methods of solving linear systems (3 Ways)

1- Gaussian elimination

1- Convert matrix to upper triangular matrix (called row echelon form) or closer.
2- Back substitution from bottom to top.
3- If there is a row all elements have been zeros, it must be the last row.
4- After back substitution, must substitute in original equations by result values.

2- Gauss-Jordan elimination

1- Convert matrix to upper & lower triangular matrix (called reduced row echelon form) or closer.
2- Back substitution from bottom to top.
3- If there is a row all elements have been zeros, it must be the last row.
4- After back substitution, must substitute in original equations by result values.

3- Crammer role

Suppose $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$ => where Ax = b

1- Calculate det (A)
2- Change column 1 with solution & calculate det (A_1)
3- Change column 2 with solution & calculate det (A_2)
4- Change column 3 with solution & calculate det (A_3)
5- Calculate $x_1 = \frac{\det(A1)}{\det(A)}$ & $x_2 = \frac{\det(A2)}{\det(A)}$ & $x_3 = \frac{\det(A3)}{\det(A)}$

Determinants

1- Calculated on square matrices only.
2- If $|A| \neq 0$ => A is invertible A^{-1} is existing A is non-singular matrix
3- If det (A) = 0 => A is non-invertible A^{-1} is not existing A is singular matrix
4- det (AB) = det (A) * det (B)
5- $|C A| = C^n * |A|$ => A is matrix & C is constant & n is multiple coefficient
6- $|A^T|$ = $|A|$
7- $|A^{-1}|$ = $\frac{1}{|A|}$

8- If matrix B obtain from exchange two rows or columns in matrix A => det (A) = -det (B)
9- If matrix B obtain from row operations in matrix A => det (A) = det (B)
10- If matrix B obtain from multiply rows in matrix A by non-zero constant => det (B) = C det (A)

11- If there is row or column consist of zeros => det (A) = 0
12- If there is rows or columns are equal => det (A) = 0
13- If there is one row or column is multiple of another => det (A) = 0

Determinants Calculation methods

1- det (A) = ad – bc => A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2- det (A) = $\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$ => (a * e * i) + (b * f * g) + (c * d * h) – (b * d * i) – (a * f * h) – (c * e * g)

3- Using general method (minor & cofactor)

A = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ => det (A) = +a * $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ – b * $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$ + c * $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Minor M_{11} = +a * $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$, Cofactor $C_{11} = (-1)^{1+1} * M_{11}$

Notes:

1- Signs of first row are (+ - +) & second row MUST start with (–) regardless sign of last item in previous row.
2- If didn't dedicate which row must use, it preferring use row that have maximum items of zeros.
3- If matrix upper or lower triangular, det = multiplication of main diagonal.
4- Coefficient take from each row only, if take more than one coefficient then total of them calculate by multiplication.