(ccc)

1 Introduction to Numerical analysis

Course contents

1. the Error analysis
2. Finding roots
3. System of linear equations (Decompostion methods)
4. Interpolation
5. Numerical Integration
6. Numerical differentiation
7. Non linear ~~of~~ system of equations

(2) Grades

75 . Final Exam

25 → project

midterm 15

+ Assignmet 10

Referne

(1) Numerial Techinques
J. S. Chitode

(2) Numerial Technique
for computer
Science.

(3) Introduction to
numurial anlyss
using matlab.

(3)

What is numerical ~~anal~~ analysis

Problems

$\llcorner$ mathematical model

solution

exact solution:

$2x+3=9$

$2x=9-3$

$2x=6$

$\boxed{\therefore x=3}$

Approximate solution

$2x+e^{t}-\sin t=0$

$\int e^{x^2}$

$\int e^{x} = e^{x}+c$

Numerical techniques is used mathematics to find an approximate solution to complex problems or problems that ~~has~~ hasn't exact solution.

(4)

- Since we have an approximate solution, it means we have error

error = True value − Approximate value

True (exact)

Ex $\frac{10}{6} = 1.66666\ldots$

solution: $\frac{10}{6} = 1.67$

$= 1.6667$

$= 2$

Significant digits

Definitions: significant digits the numbers that are those can be used with ~~[illegible]~~ confidence.

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Significant digits

total

decimals (after)

$\pi = 3.\underline{14159}265358979\ldots$

Approximate to 5 total digits

$\underbrace{3.1415}_{\text{5 total digits}}9$

- Approximate 5 digits after decimals

$3.\underbrace{14159}_{\text{5 digits}}$

Accuracy vs Precision

Accuracy is defined as the closeness of calculated value to the true value

6

Exact value

5.3168

Approximate value

5.2917 less accurate

5.3018 more accurate

5.3152 most accurate

Precision:- ~~m~~ means repetition-ness of the value

True value

5.3168

5.2917
5.2918 less accurate and high precise
5.2919

5.3146
5.3148 more accurate and high precise
5.3149

~~5.1189~~
~~5.1185~~
5.111
4.7118
4.7119
4.7111

7 Types of Errors

In general we have two types of error:

1. Truncation error
2. Rounding error

Truncation error are generated when only required significant are considered and remaining discarded.

$\frac{10}{6} = 1.66666|667$

1.66666

$\pi \equiv \boxed{3.1415}9265$

Truncate to 4 terms

3.1415

Truncation in general used in series: e^x, $\cos x$, $\sin x$

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$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Rounding Errors

Rounding errors take ~~plce~~ plce

rounding last digits to nearest value.

$$\frac{10}{6} = 1.66666.$$

Round off to 4 digits

1.6667

$\pi = 3.141592\ldots$

~~Pr~~ round to 3 digits

3.142

MS517

1 Lec (2)

Types of error

Truncation — rounding

Symantic — Symmet

Error = True value − Approximate value

we call $\in$ (epsilon)

Absolute Error = |True − Aprox

~~percentage~~

Relative error =

$$RE = \left| \frac{\text{True} - \text{Approximate}}{\text{True}} \right|$$

percentage $RE = RE \times 100$

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② Example

Calculate absolute error and relative error for the following cases and comment

① True value $= 1\times10^{-6}$
Approximate value $= 0.5\times10^{-6}$

② True value $= 1\times10^{6}$
Approximate value $= 0.99\times10^{6}$

$0.001 = 1\times10^{-3}$

$1000 = 1\times10^{3}$

$0.10 = 1\times10^{-1}$

$0.$

Sol:

① absolute error $= |1\times10^{-6} - 0.5\times10^{-6}|$

$= \boxed{0.5\times10^{-6}}$

$$RE = \left|\frac{1\times10^{-6} - 0.5\times10^{-6}}{1\times10^{-6}}\right|$$

$= 0.5$

② absolute error $= |1\times10^{6} - 0.99\times10^{6}|$

$= 10000$

$$RE = \left|\frac{1\times10^{6} - 0.99\times10^{6}}{1\times10^{6}}\right|$$

$= 0.01$

3

How to calculate error when True is not known.

Absolut error = current approximation − previous approximation.

Relative error = $\frac{\text{current} - \text{previous}}{\text{current}}$

Truncation error in series approximation.

Taylor series

$$e^{X} = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots$$

We can write any function as a polynomial series using Taylor series.

4 $\quad x \sim c$

$$f(x) = f(c) + \frac{f'(x)(x-\cancel{c})}{1!} + \frac{f''(x)(x-c)^2}{2!} + \cancel{f'''(x)} + \frac{f'''(x)(x-c)^3}{3!} + \cdots$$

If $c = 0$ (Maclurine series)

$$f(c) = f(0) + \frac{f'(\cancel{x})\,x}{1!} + \frac{f''(\cancel{x})}{2!}x^2 + \cdots$$

~~Using~~ Example: Use maclurine series to expand e^x

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$f(x) = e^x$

$f'(x) = e^x$

e^x

5

Example;
Calculate e^x up to
fine first term and
estimate truncation error
at $X = .1$

Sol;
A using calculator

$$e^1 = 2.7182818 \quad \text{True}$$

$$\therefore e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

using first five terms

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$X = 1$

$$= 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$$

$$= 2.708333$$

Approximate

(6)

Absolute error = 2.7182818 − 2.7083333

= 0.0099485

EX:- Find the value of e^x using expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

for $x = 0.5$ with absolute error less than 0.005

$e^{0.5}$

Sols

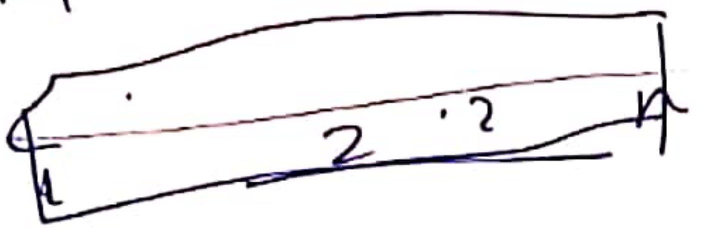
since error is given (upper bound is 0.005), we want to determine number of term that satisfy condition of error.

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

n th term is $\frac{x^n}{n!}$

7

$e \quad 1 \quad 2 \quad 7 \quad n$ error



$n+1$

$$\text{error } e^{X} = \frac{X^{n+1}}{(n+1)!}$$

$$\frac{X^{n+1}}{(n+1)!} < 0.005$$

$$\operatorname{Log}(X^{n+1}) - \operatorname{Log}(n+1)! < \operatorname{Log}(0.005)$$

$$(n+1)\operatorname{Log} X - \operatorname{Log}(n+1)! < -2.30103$$

$$\operatorname{Log}(X^a) \qquad a \operatorname{Log} X \qquad \operatorname{Log}\left(\frac{X}{y}\right) \qquad \operatorname{Log} X - \operatorname{Log} y$$

$$\operatorname{Log}(n+1)! - (n+1)\operatorname{Log}(X) > 2.30103$$

$X = 0.5$

$n = 2$

$$\operatorname{Log}(3)! - 3\operatorname{Log}(0.5) > 2.30103$$

$$2.5843312 > 2.3$$

Roots of equation

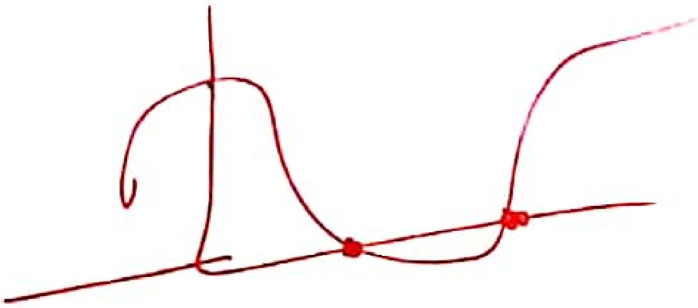
$\cancel{f(x)=5}$

$f(x)=0$

root

$3x+6=0$ $\quad x=\boxed{-2}$

$3x=-6$



$f(x)=x^2+3x+2$ → can find an exact solution

$f(x)=x^2+3x+2+e^{2x}$

$f(x)=x^{10}+2x^7+3x+2$

transcendental function

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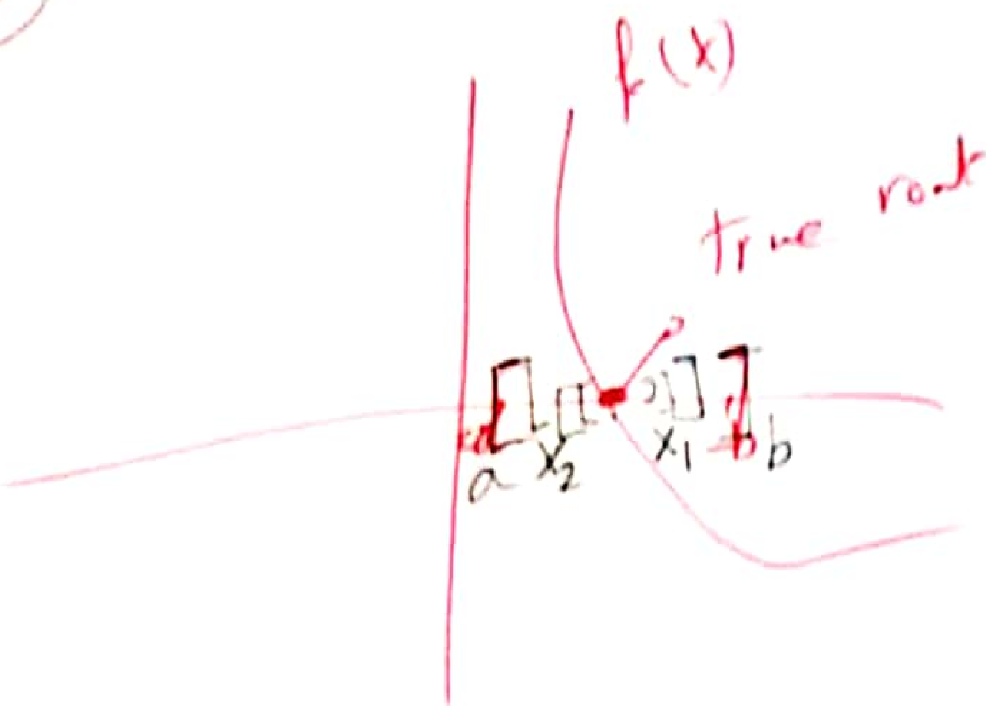
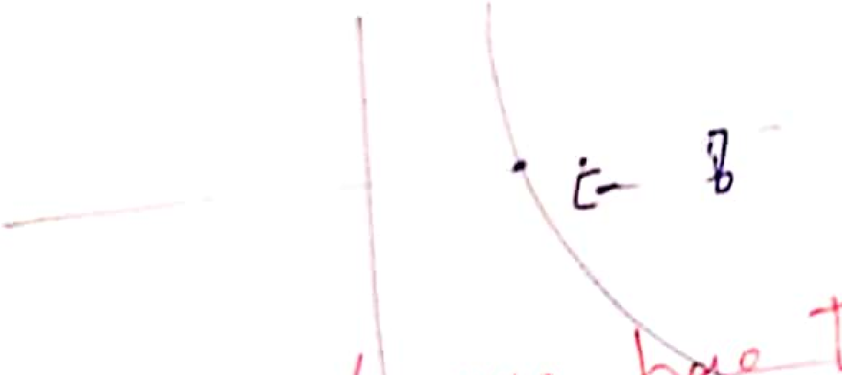
1)





In general we have two Common methods for finding roots

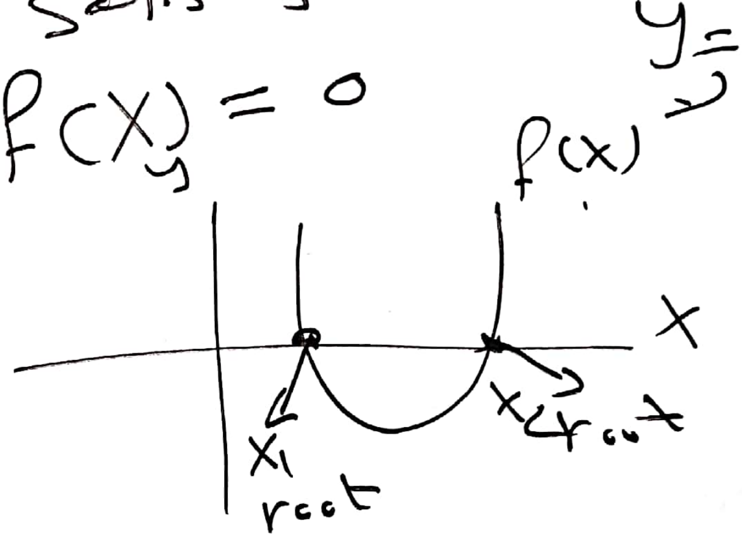
bracketing	open
- There are two Initial values	at Last one point
- are always Convergent	can be divergent (some times)
- More Iteration	- less iterations
bisection	secant, Newtons

Lec (3)

Roots of equation

The root is the value of X that satisfy $f(X) = 0$

$f(x)$ intersect X-axis.



we know ~~a, b~~ algebraic polynomial of degree, one, two, three, four can be solve analytically

→ $f(x) = X+2$ find root

$f(x) = 0$

$X+2 = 0 \Rightarrow \boxed{X = -2}$

→ $f(x) = X^2 + 5X + 6$

$(X+2)(X+3) = 0$

root

$X_1 = -2$

$X_2 = -3$

② Ex Find the roots of

$$X^7 + X^5 + 3X^4 + 2X \cancel{+1} = 0$$

In this cases we can't find algebraic solution. We use Numeric methods to find an approximate solution $f(x) = 0$

→ Find Solution $f(x) = 0$

$$f(x) = X^3 - \cos X + 3$$

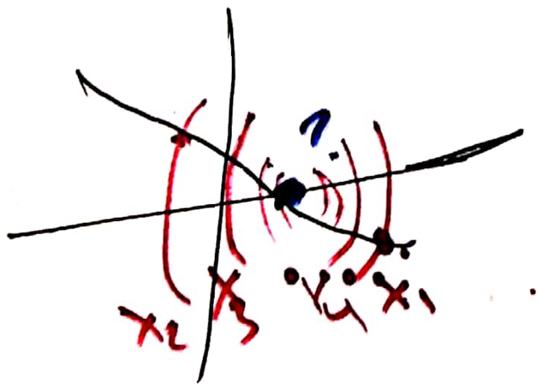
$$f(x) = \ln X - \sin 3X + X = 0$$

→ differat ~~the~~ Numerical methods can be used to find root for higher polynomial ~~and~~ ~~transcdantal~~ transcendental function.

3

→ Types of methods to obtain root.

Bracketing	open methods
- Two initial values to start	one or more initials values
- always convergent ~~convergent~~	- Sometimes can be divergent
- Need more number of iterations.	- less number of iterations.
EX: - Bisection method	- Newton method
- ~~secant method~~	- secant method
- Fixed point	
- regular false	



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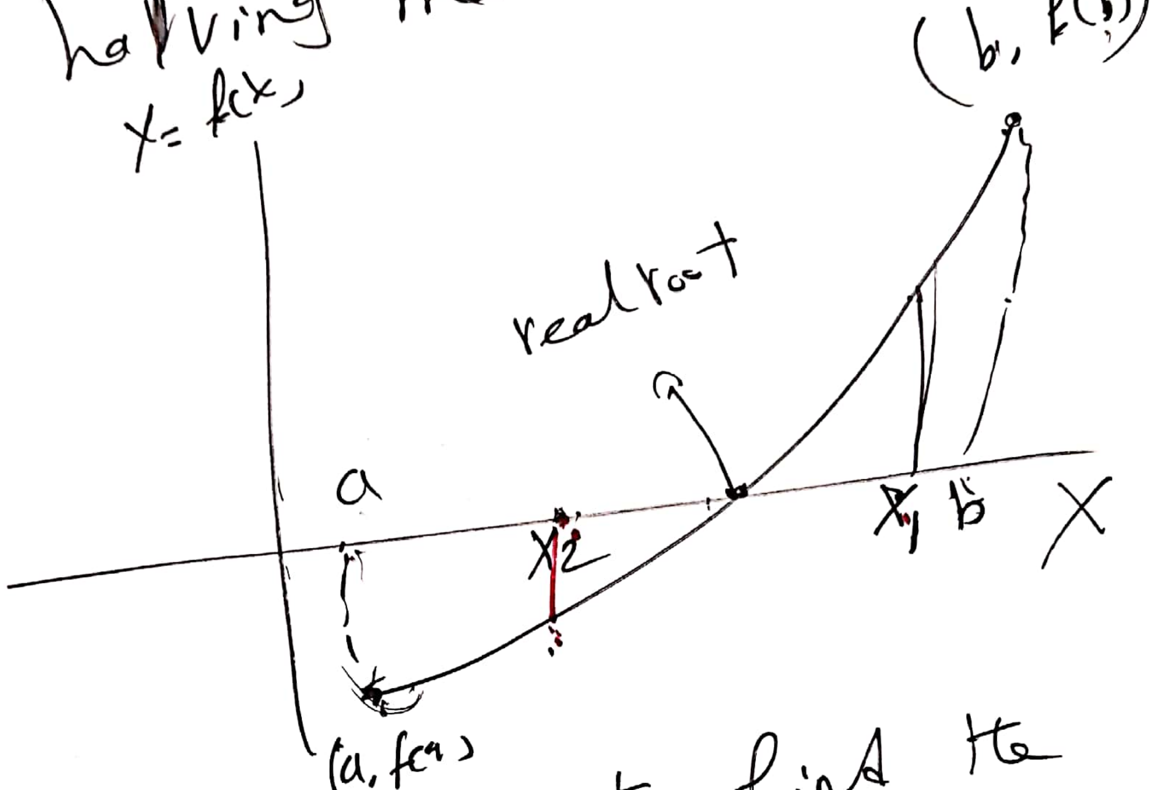
(4) Bisection method

This method is called interval halving method



suppose we want to find the solution of $f(x) = 0$, $f(x)$ is continuous in (a, b)

→ let $f(a)$ -ve and $f(b)$ +ve

$f(a) \cdot f(b) < 0$

$X_1 = \frac{a+b}{2}$ → $f(X_1)$ +ve

(5) $X_2 = \frac{X_1 + a}{2}$

$f(X) = 0$? yes → root
no → continue

$f(X_2)$ -ve

$X_3 = \frac{X_1 + X_2}{2}$

stop ~~wet~~ when
→ $X_n - X_{n-1} < \epsilon$

$\epsilon = 0.0001$

OR $f(X_n) \simeq 0$

(6) Find a root for the following polynomial using bisection method and error = ~~0.0001~~ 0.01

$$f(X) = X^3 - X - 2$$

using $[1, 2]$ with $a = 1$, $b = 2$

Solution

let $a = 1$ and $b = 2$

$f(a) = f(1) = 1^3 - 1 - 2 = -2$

$= \cancel{1 - 1} - 2 = -2$ -ve

$f(b) = f(2) = 2^3 - 2 - 2 = +4$ +ve

$f(a) \cdot f(b) = f(1) \cdot f(2) = -8 < 0$

we have root.

→ first iteration (iteration No. 1)

- $X_1 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = \boxed{1.5}$
- $f(X_1) = (1.5)^3 - (1.5) - 2 = -0.125$ -ve

(7)

Secont iteration

$\cancel{X_1} = 1.5$ $b = 2$

-ve +ve

$$X_2 = \frac{1.5+2}{2} = 1.75$$

$$f(1.75) = (1.75)^3 - 1.75 - 2$$

$$= 1.609375$$

+ve

$$\epsilon = |X_2 - X_1| = |1.75 - 1.5|$$

$$= 0.25$$

third iteration

$$X_3 = \frac{X_1 + X_2}{2} = \frac{1.5+1.75}{2}$$

$$= 1.625$$

$$f(X_3) = f(1.625)$$

$$= (1.625)^3 - (1.625) - 2$$

$$= 0.666015 6$$

$$\epsilon = |X_3 - X_2| = |1.625 - 1.75|$$

$$= 0.125$$

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fourth iteration $(1.5, 1.625)$

$-ve \quad +ve$

$$X_4 = \frac{1.5 + 1.625}{2}$$

$= 1.5625$

$f(1.5625) = 0.2521973$

$\epsilon = |X_4 - X_3| = |1.5625 - 1.625|$

$= 0.0625$

→ fifth iteration $(1.5, 1.5625)$

$$X_5 = \frac{1.5 + 1.5625}{2} = 1.53125$$

$f(1.53125) = 0.059113$ -

+ve

$\epsilon = |X_5 - X_4|$

$= |1.53125 - 1.5625|$

$= \boxed{0.03125}$

$X_6 = 1.515625$

1

Lec(4)

Newtons method

- "newton Raphson method"

How to find a root?

- suppose X_0 be an approximate root of $f(X) = 0$.

$f(X_0) \simeq 0$

- Let actual value of root $\boxed{X_1}$ is given as:

$$X_1 = X_0 + h \quad (1)$$

$\therefore$ X_1 is true root $f(X_1) = 0$

$$f(X|_{X=X_1}) = 0$$

$$f(X_1) = 0$$

$$f(X_0 + h) = 0$$

$\therefore$ We can use Taylor series to expand $f(X_0 + h)$

(3) $f(X_1) = f(X_0) + h f'(X_0) + \frac{h^2}{2!} \cancel{f''(X_0)} + \frac{h^3}{3!} f'''(X_0) \ldots$

$\approx 0 \quad = 0$

$\because$ h is very small, we can neglect all the terms above 2nd

i.e. $f(X_1) = \underbrace{f(X_0) + h f'(X_0)}$ $= 0$

$$f(X_0) + h f'(X_0) = 0$$

$$h f'(X_0) = -f(X_0)$$

$$h = \frac{-f(X_0)}{f'(X_0)}$$

In (1)

$$X_1 = X_0 - \frac{f(X_0)}{f'(X_0)}$$

wh X_1 approximate X_2 true valu

$$X_2 = X_1 + h$$

$$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$$

(3) we can generalize the rule to find root

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

~~Important~~ Important notes

(1) $f'(X_0)$ should not equal to zero

(2) $f'(X)$ exist

(3) For better convergence of Newton method select X_0 such that $f(X_0)$, $f''(X_0)$ is positive.

$f'(X_0) \cdot f''(X_0) > 0$

$\boxed{\text{Example}}$ Find the root $f(X) = X^3 - 5X + 3 = 0$ using Newton method using 4 iterations

(4) $X_0 = ?$,

$f(x) = x^3 - 5x + 3$

$f'(x) = 3x^2 - 5$

$f''(x) = 6x$

we want to find Initial value

X_0 ??

$X = 0$

$f(0) = 3$

$f(1) = -1$ we have root $(0, 1)$

$f''(0) = 0$, $f''(1) = 6$

$f(X_0) \cdot f''(X_0) > 0$

both 0, 1 ~~doest~~ doesn't satisfy the condition.

5

$[0, 1]$ $[0.3, 1]$

$f(0.3) = (0.3)^3 - 5(0.3) + 3$

$= 1.527$

$[0.3, 1]$

$f''(0.3) = 6(0.3) = 1.8$

$f(0.3) \cdot f''(0.3)$

$(1.527)(1.8) > 0$

$\therefore \boxed{X_0 = 0.3}$

general rule $X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$

iteration 1 $X_1 = X_0 - \frac{f(X_0)}{f'(X_0)}$

$= 0.3 - \frac{1.527}{(-4.73)}$

$\boxed{X_1 = 0.622833}$

(6) iteration 2

$$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$$

$$= 0.622833 - \frac{f(0.622833)}{f'(0.622833)}$$

$$= 0.622833 - \frac{0.127445}{(-3.836237)}$$

$$X_2 = 0.656054$$

iteration 3

$$X_3 = X_2 - \frac{f(X_2)}{f'(X_2)}$$

$$= 0.656054 - \frac{0.002099}{(-3.708778)}$$

$$X_3 = 0.656620$$

0.0001

iteration 4 $X_4 = X_3 - \frac{f(X_3)}{f'(X_3)}$

$$= 0.656620 - \frac{0.000001}{-3.706549}$$

$$= 0.6566243$$

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$X_3 = 0.656620$

$X_4 = 0.65662043$

The root 0.656620

$$\boxed{X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}}$$

- Secant method.
- Fixed point iterations.

Lecture 7

Numerical differentiation

Introduction

* differentiation gives a measure of the rate at which aquantity changes (Rate of change)

* differentiation is important in science, engineering and physics
* velocity, is the derivative of the position with respect to time

$$\text{السرعة} = \frac{\Delta \text{المسافة}}{\Delta \text{الزمن}}$$

* acceleration is the derivative of the velocity with respect to time

$$\text{العجلة} = \frac{\Delta \text{السرعة}}{\Delta \text{الزمن}}$$

* many models in physics & engineering are expressed in terms of rates — وهنا التغير يتم قياسه رياضياً عن طريق التفاضل

* differentiation is also used for finding the maximum & minimum values of a function.

The need for numerical differentiation:

- the function to be differentiated may be given in a table form (set of discrete points); in this case the analytical method can't be used, so we have to use numerical differentiation.

- numerical differentiation also plays an important role in some of the numerical methods used for solving differential equations.

Approaches to numerical differentiation depend on whether

points are equally spaced

Finite difference approximation of derivative at point x_i

2 points

forward;
backward;
central

3 points

forward
backward

difference formulas

points are not equally spaced

Lagrange polynomials

3 formules for 1st derivative at 3 points

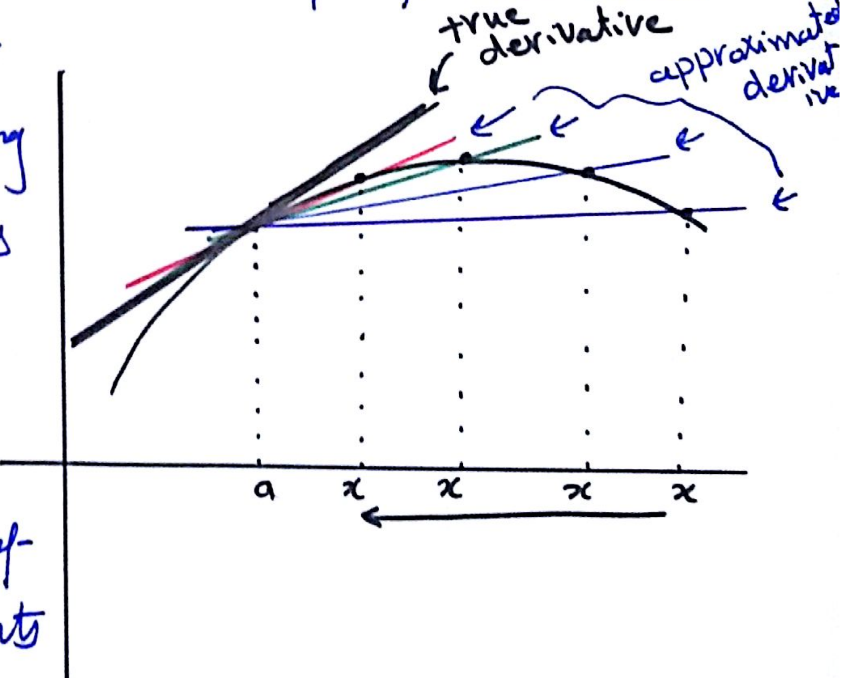
* finite difference approximation for the derivative at a point x_i is an approximate calculation based on the value of points in the neighborhood of x_i

* the accuracy of a finite difference approximation depends on the accuracy of the data points ; the spacing between the points & the formula used for approximation ~~the derivative~~

* the simplest formula approximates the derivative as the slop of line that connects 2 adjecent points.

finite difference approximation of the derivative

* the derivative $f'(x)$ of a function $f(x)$ at the point $x=a$ can be obtained graphically by taking a point x near $x=a$ & calculate the slope of the line connects the two points



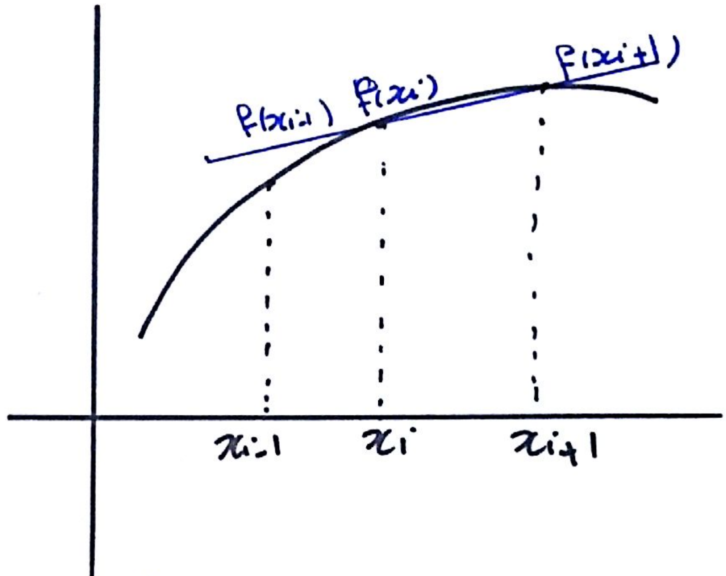
* the accuracy of calculating the derivative increases as point x is close to the point a

* in finite difference approx of the derivative, values of the function at different points in the neighborhood of the point $x=a$ are used for estimating the slope

- Two points

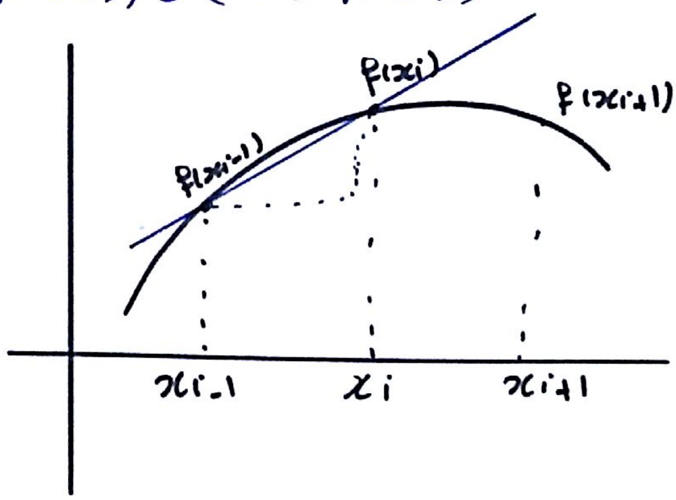
the finit difference approximation 3 formulas for the derivative is calculated from the value of 2 points as follows:

1- **forward difference**: is the slop of the line connects the points $(x_i, f(x_i))$ & $(x_{i+1}, f(x_{i+1}))$



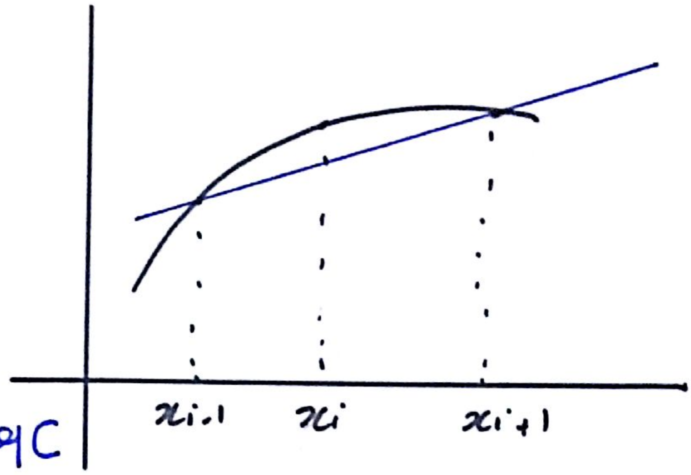
$$\left.\frac{df}{dx}\right|_{x=x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$ → eq A

2- **backward difference**: is the slope of the line connects the points $(x_{i-1}, f(x_{i-1}))$ & $(x_i, f(x_i))$



$$\left.\frac{df}{dx}\right|_{x=x_i} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$ → eq B

centeral difference: is the slope of the line that connects points $(x_{i-1}, f(x_{i-1}))$ & $(x_{i+1}, f(x_{i+1}))$



$$\boxed{\left.\frac{df}{dx}\right|_{x=x_i} = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}} \rightarrow \text{eq C}$$

* these formulas can be obtained by using Taylor series expansion; see Gilat & subramaniam P.240

ex

consider the function $f(x) = x^3$. calculate its 1st derivative at point $x=3$ numerically with the forward, backward & centeral finite difference formulas & using:

a) points $x=2$, $x=3$, $x=4$

b) points $x=2.75$, $x=3$ & $x=3.25$

compare the results with the exact (analytical) derivative.

sol

* analytical diff

$f'(x) = 3x^2$ & $f'(3) = 3(3)^2 = 27$

* numerical diff

a. $x =$ 2 3 4

$f(x) =$ 8 27 64

1- forward finite difference

$$\left.\frac{df}{dx}\right|_{x=3} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f(4) - f(3)}{4-3} = \frac{64-27}{1}$$

$$= 37 \longrightarrow \varepsilon = \left|\frac{37-27}{27}\right| \times 100 = 37.04\%$$

2. backward finit difference

$$\left.\frac{df}{dx}\right|_{x=3} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{f(3) - f(2)}{3-2} = \frac{27-8}{1}$$

$$= 19 \longrightarrow \varepsilon = \left|\frac{19-27}{27}\right| \times 100 = 29.63\%$$

3- centeral finit difference

$$\left.\frac{df}{dx}\right|_{x=3} = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} = \frac{f(4) - f(2)}{4-2} = \frac{64-8}{2}$$

$$= 28 \longrightarrow \varepsilon = \left|\frac{28-27}{27}\right| \times 100 = 3.704\%$$

b. $x = 2.75 \quad 3 \quad 3.25$

$f(x) = 20.7969 \quad 27 \quad 34.3281$

1- forward finit difference

$$\left.\frac{df}{dx}\right|_{x=3} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{34.3281 - 27}{3.25 - 3} = 29.3125$$

$$\varepsilon \longrightarrow = \left|\frac{29.3125 - 27}{27}\right| \times 100$$

$$\varepsilon = 8.565\%$$

2. backward finit difference

$$\left.\frac{df}{dx}\right|_{x=3} = \frac{f(3) - f(2.75)}{3 - 2.75} = \frac{27 - 20.7969}{0.25} = 24.8125$$

$$\rightarrow \varepsilon = \left|\frac{24.8125 - 27}{27}\right| \times 100 = 8.102\,\%$$

3. centeral finit difference

$$\left.\frac{df}{dx}\right|_{x=3} = \frac{f(3.25) - f(2.75)}{3.25 - 2.75} = \frac{34.3281 - 20.7969}{3.25 - 2.75} = 27.0625$$

$$\rightarrow \varepsilon = \left|\frac{27.0625 - 27}{27}\right| \times 100 = 0.2315\,\%$$

* the centeral finit differnce formula gives a more accurate approximation. Also smaller separation between points gives more accurate approximation

- Three points

The finit difference approximation 2 formulas for the derivative is calculated from the value of 3 points as follows:

1. 3 points forward difference

eq D

$$f'(x_i) = \frac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})}{2h} + \underbrace{O(h^2)}_{\text{truncation error}}$$

$h = x_{i+1} - x_i = x_{i+2} - x_{i+1}$

since the points are all equally spaced

is the distance between the points

* this formula can be used for calculating the derivative at the first point of a function that is given by a set of n discrete points

2. 3 points backward difference

$$f'(x_i) = \frac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h} + \underbrace{O(h^2)}_{\text{truncation error}}$$

eq E

ex

Consider the function $f(x)=x^3$. Calculate the 1st derivative at point $x=3$ numerically with the 3 points forward difference formula using

a. points $x=3$, $x=4$, $x=5$

b. points $x=3$, $x=3.25$, $x=3.5$

Compare the results with the exact value of the derivative obtained numerically.

sol

- analytical differentiation

$f'(x) = 3x^2$ & $f'(3) = 3(3)^2 = \boxed{27}$

- Numerical differentiation

a.

$x=$	3	4	5
$f(x)=$	27	64	125

- the 1st derivative using 3 points forward difference formula is

$$f'(x_i) = \frac{-3f(x_i)+4f(x_{i+1})-f(x_{i+2})}{2.h} = \frac{-3f(3)+4f(4)-f(5)}{2.1}$$

$$= \frac{-3(27)+4(64)-125}{2} = 25$$

$$\varepsilon = \left|\frac{25-27}{27}\right| \times 100$$

$$= 7.41\ \%$$

b. $x =$ 3 3.25 3.5

$f(x) =$ 27 34.3281 42.875

- the 1st derivative using 3 points forward formula is

$$f'(x_i) = \frac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})}{2h} = \frac{-3f(3) + 4f(3.25) - f(3.5)}{2(1)}$$

$$= \frac{-3(27) + 4(34.3281) - 42.875}{2} = 26.875$$

$$\varepsilon = \left|\frac{26.875 - 27}{27}\right| \times 100$$

$$\varepsilon = 0.46\ \%$$

Comparing the results for 3 points with 2 points we can see that:

the results show that the 3 points forward difference gives more accurate value than the 2 points forward difference for $h=1$ & for $h=0.25$

differentiation formulas using Lagrange polynomials

- this formula can be used when the distance between the points are not equally spaced.

formula → depending on 3 points (x_i, y_i), (x_{i+1}, y_{i+1}) & (x_{i+2}, y_{i+2}) is :-

$$f'(x) = \frac{2x - x_{i+1} - x_{i+2}}{(x_i - x_{i+1})(x_i - x_{i+2})} y_i + \frac{2x - x_i - x_{i+2}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} y_{i+1}$$

$$+ \frac{2x - x_i - x_{i+1}}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} \cdot y_{i+2} \longrightarrow \text{eq } 1$$

the 1st derivative at either one of the three points is calculated by substituting the corresponding value of x (x_i, x_{i+1}, x_{i+2}) in eq 1 & this gives the following three formulas for the first derivative at the three points x_i, x_{i+1}, x_{i+2} :→

① $$f'(x_i) = \frac{2x_i - x_{i+1} - x_{i+2}}{(x_i - x_{i+1})(x_i - x_{i+2})} \cdot y_i + \frac{x_i - x_{i+2}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} y_{i+1}$$

$$+ \frac{x_i - x_{i+1}}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} y_{i+2} \longrightarrow \text{eq 2}$$

② $$f'(x_{i+1}) = \frac{x_{i+1} - x_{i+2}}{(x_i - x_{i+1})(x_i - x_{i+2})} y_i + \frac{2x_{i+1} - x_i - x_{i+2}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} y_{i+1}$$

$$+ \frac{x_{i+1} - x_i}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} y_{i+2} \longrightarrow \text{eq 3}$$

③ $$f'(x_{i+2}) = \frac{x_{i+2} - x_{i+1}}{(x_i - x_{i+1})(x_i - x_{i+2})} y_i + \frac{x_{i+2} - x_i}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} y_{i+1}$$

$$+ \frac{2x_{i+2} - x_i - x_{i+1}}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} y_{i+2} \longrightarrow \text{eq 4}$$

H.W

proof that when the points are equally spaced then ① eq 2 will reduced to the 3 points forward difference

② eq 3 will reduced to the 2 points centeral diff

③ eq 4 " " " " 3 points backward "

hint: $x_i = (x_i)$

$x_{i+1} = (x_i + h)$

$x_{i+2} = (x_i + 2h)$

ex

Compute from the following table the value of the derivative of $f(x)$ at $x = 0.6$; where

	x_i	x_{i+1}	x_{i+2}
x	0.4	0.6	0.7
$f(x)$	3.3835	4.2442	4.7275

sol

since the distances between the points are not equally spaced so we have to use Lagrange polynomial

$$f'(x_{i+1}) = \frac{x_{i+1} - x_{i+2}}{(x_i - x_{i+1})(x_i - x_{i+2})} y_i + \frac{2x_{i+1} - x_i - x_{i+2}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} y_{i+1}$$

$$+ \frac{x_{i+1} - x_i}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} y_{i+2}$$

$$f'(0.6) = \frac{0.6 - 0.7}{(0.4 - 0.6)(0.4 - 0.7)} \times 3.3835 + \frac{2(0.6) - 0.4 - 0.7}{(0.6 - 0.4)(0.6 - 0.7)} \times 4.2442$$

$$+ \frac{0.6 - 0.4}{(0.7 - 0.4)(0.7 - 0.6)} 4.7225$$

$$f(0.6) = 4.6228$$

1- use the forward & backward 2 points difference formulas to determine the missing entry in the following table

a.

x	$f(x)$	$f'(x)$
0.5	0.4794	
0.6	0.5646	
0.7	0.6442	

b.

x	$f(x)$	$f'(x)$
0	0	
0.2	0.7414	
0.4	1.3718	

2. use the most accurate 3 point formula to determine each missing entry in the following tables

a.

x	$f(x)$	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

b.

x	$f(x)$	$f'(x)$
8.1	16.94410	
8.3	17.56492	
8.5	18.14056	
8.7	18.82091	

3- the following data was collected for the distance traveled versus time for a rocket

t, s	0	25	50	75	100	125
X, Km	0	32	58	78	92	100

use numerical differentiation to estimate the rocket's velocity (1st derivative) at each time

4- obtain the 1st derivative for the following function at $x = 1.5$ & $x = 1.6$

x	1	1.5	1.6	2.5	3.5
$f(x)$	0.6767	0.3734	0.3261	0.0842	0.0159

compare your results with the true derivatives where $f(x) = 5e^{-2x} . x$

Numerical integration

Introduction to numerical Integration [quadrature] :-

* in general integration is very important in engineering & science. Integration & Integrals are also used when solving differential equations.

* the general form of a definite integral (also called an antiderivative) is

$$I = \int_a^b f(x)\,dx$$

where $f(x)$ called the integrand

* the need for numerical integration

the integrand can be analytical function or a set of discrete points (tabulated data). Numerical integration is needed when analytical integration is difficult or not possible; and when the integrand is given as a set of discrete points.

Newton - cotes integration formulas

the goal is to approximate the difinit integral of f(x) over the interval [a,b] by evaluating f(x) at a finit number of sample points which called "quadrature" or numerical integration.

* the integral $I = \int_a^b f(x)\,dx \simeq \int_a^b F(x)\,dx$ is evaluated by direct analytical methods from calculus.

+ the simplest approximation for $\int_a^b f(x)\,dx$ is to take f(x) over the interval $x \in [a,b]$ as a constant equal to the value of f(x) at either one of the endpoints. so

$$I = \int_a^b f(x)\,dx \simeq \int_a^b f(a)\,dx = f(a)(b-a)$$

$$\text{or} \simeq \int_a^b f(b)\,dx = f(b)(b-a)$$

⇒ this called "Rectangle method"

* the error using this method is very Large.

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1- mid point method :-

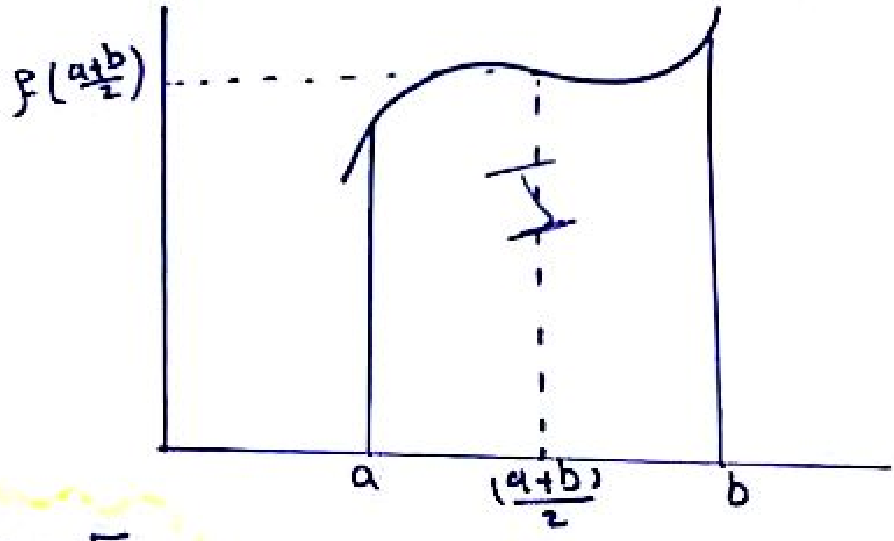
* We shall consider estimating $I = \int_a^b f(x)\,dx$ as the area under the curve $f(x)$ between $x=a$ & $x=b$
* instead of approximating the integral by the values of the function at $x=a$ or at $x=b$, the value of the integral at the middle of the interval, that is, $f\left(\frac{a+b}{2}\right)$ is used

so

$$I = \int_a^b f(x)\,dx$$

$$= \int_a^b f\left(\frac{a+b}{2}\right) dx$$

$$= f\left(\frac{a+b}{2}\right)(b-a)$$



$$I_M = f\left(\frac{a+b}{2}\right)(b-a) + E_M$$

where

$$E_M = \frac{(b-a)^3}{24} f''(\xi)$$

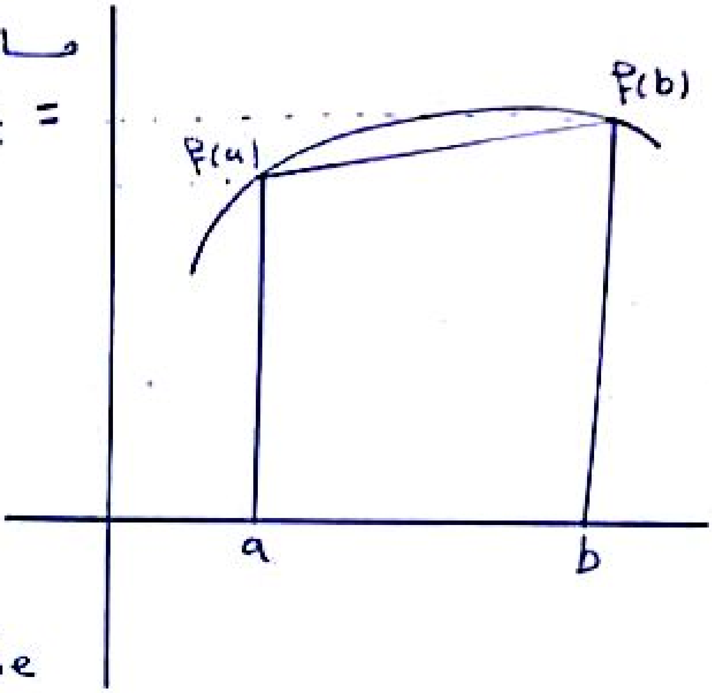
ξ is some points between a & b

2. Trapezoidal method

the trapezoidal method approximates I by the area of the trapezoidal $a\,b\,f(b)\,f(a)$

مساحة شبه المنحرف

= ½ (مجموع القاعدتين) X البعد بينهم



$$I_T = \frac{f(a)+f(b)}{2} * (b-a)$$

&

$$E_T = -\frac{(b-a)^3}{12} f''(\varepsilon)$$

the trapezoidal rule can be also derived by approximating $f(x)$ by a linear function say $L_1(x)$ at points $(a, f(a))$ & $(b, f(b))$

$$L_1(x) = \left[\frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)\right]$$

$$I_T = \int_a^b L_1(x)\,dx = \int_a^b \left[\left(\frac{x-b}{a-b}\right) f(a) + \left(\frac{x-a}{b-a}\right) f(b)\right] dx$$

$$= \frac{f(a)}{a-b} \int_a^b (x-b)\,dx + \frac{f(b)}{b-a} \int_a^b (x-a)\,dx$$

$$= \frac{f(a)}{a-b} \left[\frac{x^2}{2} - bx\right]_a^b + \frac{f(b)}{b-a} \left[\frac{x^2}{2} - ax\right]_a^b$$

$\Downarrow$ which will be

$$I_T = \frac{1}{2}(b-a)\,[f(a)+f(b)]$$

3- Simpson's method :-

the trapezoidal method approximate the integrand by a linear function. A better approximation can possibly be obtained by approximating the integrand with a non linear function that can be easily integrated.

* Consider the three points $x_0 = a$, $x_2 = b$ & $x_1 = \frac{a+b}{2}$

$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$L_2(x) = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0)\,dx + \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)\,dx$$

$$+ \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)\,dx$$

$$\Downarrow$$

which will be

$$\tilde{I}_S = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

where $h = \frac{b-a}{2}$

$$E_S = \frac{-1}{90}\left[\left(\frac{b-a}{2}\right)\right]^5 f^{(4)}(\xi)$$

ex

Approximate the integral

$$I = \int_0^1 \frac{1}{1+x^2}\,dx$$

by using the three methods of approximation

sol

1. mid point method

$$I_M = f\left(\frac{a+b}{2}\right)(b-a)$$

$$* \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

$$f(0.5) = \frac{1}{1+(0.5)^2}$$

$$I_M = \frac{1}{1+(0.5)^2}(1-0) = \frac{4}{5} = 0.8$$

2. Trapezoidal method

$$I_T = \left(\frac{b-a}{2}\right)[f(a) + f(b)]$$

$$= \left(\frac{1-0}{2}\right)\left[\frac{1}{1+(0)^2} + \frac{1}{1+(1)^2}\right] = \frac{3}{4} = 0.75$$

③ the simpson's method

$$I_s = \frac{h}{3}\left[f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$$

$$I_s = \frac{0.5}{3}\left[\frac{1}{1+(0)^2} + 4 \cdot \frac{1}{1+(0.5)^2} + \frac{1}{1+(1)^2}\right]$$

$$= \frac{1}{6}\left[1 + \frac{16}{5} + \frac{1}{2}\right] = \frac{1}{6}\left[\frac{47}{10}\right] = 0.7833$$

ex (H.W)

use the mid point, Trapezoidal, simpson's Rules to approximate $\int_0^2 f(x)\,dx$ when $f(x)$ is

a. x^2

b. x^4

c. $(x+1)^{-1}$

d. $\sqrt{1+x^2}$

e. $\sin(x)$

f. e^x

& compare the results with the analytical solution in each case

H.W Fairs & Burden p. 202

No, 1 & 2 & 5, 6, 9, 10