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#### **Lecture Week 02-03**

Probability

- The chance that something will happen
- Probability as a mathematical framework for reasoning about uncertainty
- Given infinite observations of an event, the proportion of observations where a given outcome happens
- Strength of belief that something is true
- "Mathematical language for quantifying uncertainty" Wasserman
- $\Omega$ : Sample Space, set of all outcomes of a random experiment
- A: Event ( $A\subseteq \Omega$ ), collection of possible outcomes of an experiment
- **P(A):** Probability of event **A, P** is a function: events $\rightarrow \mathbb{R}$

- $P(\Omega) = 1$
- $P(A) \ge 0$ , for all A
- If A1, A2, ... are disjoint events then:

$$P(\bigcup_{i}^{\infty} A_{i}) = \sum_{i}^{\infty} P(A_{i})$$

## **Some Properties:**

- ✓ If  $B \subseteq A$  then  $P(A) \ge P(B)$
- $\checkmark$  P(A  $\cup$  B)  $\leq$  P(A) + P(B)
- $\checkmark P(A \cap B) \leq \min(P(A), P(B))$
- $\checkmark P(\neg A) = P(\Omega / A) = 1 P(A)$
- / is set difference
- $P(A \cap B)$  will be notated as P(A, B)

### **Probabilistic models:**

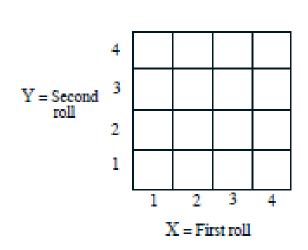
- sample space
- probability law
- Axioms of probability
- Simple examples

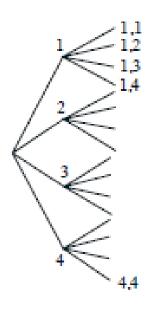
## Sample space $\Omega$ :

- "List" (set) of possible outcomes
- List must be:
  - Mutually exclusive
  - Collectively exhaustive

### Sample space: Discrete example

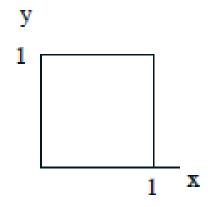
- Two rolls of a tetrahedral die
- Sample space vs. sequential description





#### Sample space: Continuous example

$$\Omega = \{(x, y) \mid 0 \le x, y \le 1\}$$



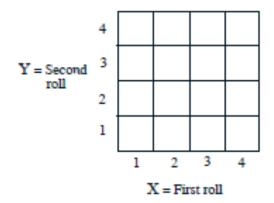
#### Probability axioms

- Event: a subset of the sample space
- Probability is assigned to events

#### Axioms:

- 1. Nonnegativity:  $P(A) \ge 0$
- Normalization: P(Ω) = 1
- 3. Additivity: If  $A \cap B = \emptyset$ , then  $P(A \cup B) = P(A) + P(B)$
- $P({s_1, s_2, ..., s_k}) = P({s_1}) + ... + P({s_k})$ =  $P(s_1) + ... + P(s_k)$
- Axiom 3 needs strengthening

#### Probability law: Example with finite sample space



- Let every possible outcome have probability 1/16
- P((X,Y) is (1,1) or (1,2)) =
- $P({X = 1}) =$
- P(X + Y is odd) =
- $P(\min(X, Y) = 2) =$

# PROBABILITY (REVIEW)

#### Discrete uniform law

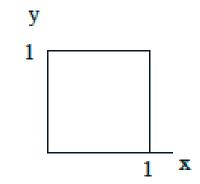
- Let all outcomes be equally likely
- Then,

$$P(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}}$$

- Computing probabilities 
   ≡ counting
- Defines fair coins, fair dice, well-shuffled decks

#### Continuous uniform law

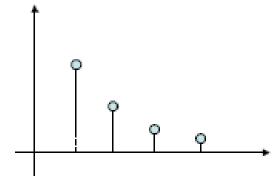
Two "random" numbers in [0,1].



- Uniform law: Probability = Area
- $P(X + Y \le 1/2) = ?$
- P((X,Y) = (0.5,0.3))

Probability law: Ex. w/countably infinite sample space

- Sample space: {1,2,...}
- We are given  $P(n) = 2^{-n}$ , n = 1, 2, ...
- Find P(outcome is even)



$$P({2,4,6,...}) = P(2) + P(4) + ... = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + ... = \frac{1}{3}$$

Countable additivity axiom (needed for this calculation):
 If A<sub>1</sub>, A<sub>2</sub>,... are disjoint events, then:

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

## Example (from the reading)

Experiment: toss a fair coin, report heads or tails.

Sample space:  $\Omega = \{H, T\}$ .

Probability function: P(H) = .5, P(T) = .5.

### Use tables:

Outcomes	Н	Τ
Probability	1/2	1/2

(Tables can really help in complicated examples)

### **Events**

### Events are sets:

- Can describe in words
- Can describe in notation
- Can describe with Venn diagrams

Experiment: toss a coin 3 times.

### Event:

You get 2 or more heads  $= \{ HHH, HHT, HTH, THH \}$ 

Events, sets and words

Experiment: toss a coin 3 times.

Which of following equals the event "exactly two heads"?

$$A = \{THH, HTH, HHT, HHH\}$$
  
 $B = \{THH, HTH, HHT\}$   
 $C = \{HTH, THH\}$ 

To keep the notation cleaner, let's use P(T) = (1 - p) = q. Since the flips are independent (we'll discuss this next week) the probabilities multiply. This gives the following  $2 \times 2$  table.

first flip 
$$H$$
  $p^2$   $pq$   $T$   $qp$   $q^2$ 

If probability of H is p and probability of T is 1-p, the write down possible mathematical expression for A, B & C.

Events, sets and words

Experiment: toss a coin 3 times.

Which of the following describes the event { THH, HTH, HHT }?

- (1) "exactly one head"
- (2) "exactly one tail"
- (3) "at most one tail"
- (4) none of the above

Events, sets and words

Experiment: toss a coin 3 times.

The events "exactly 2 heads" and "exactly 2 tails" are disjoint.

(1) True (2) False

**answer:** True:  $\{THH, HTH, HHT\} \cap \{TTH, THT, HTT\} = \emptyset$ .

Events, sets and words

Experiment: toss a coin 3 times.

The event "at least 2 heads" implies the event "exactly two heads".

(1) True (2) False

False. It's the other way around:  $\{THH, HTH, HHT\} \subset \{THH, HTH, HHT\}$ .

Probability rules in mathematical notation

Sample space: 
$$S = \{\omega_1, \omega_2, \dots, \omega_n\}$$

Outcome:  $\omega \in S$ 

Probability between 0 and 1:  $0 \le P(\omega) \le 1$ 

Total probability is 1: 
$$\sum_{j=1}^{n} P(\omega_j) = 1$$
,  $\sum_{\omega \in S} P(\omega) = 1$ 

Event A: 
$$P(A) = \sum_{\omega \in A} P(\omega)$$

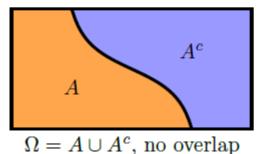
Probability and set operations on events

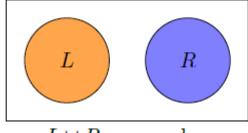
Events A, L, R

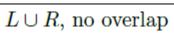
Rule 1. Complements:  $P(A^c) = 1 - P(A)$ .

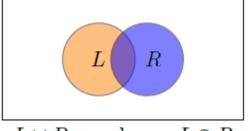
Rule 2. Disjoint events: If L and R are disjoint then  $P(L \cup R) = P(L) + P(R)$ .

Rule 3. Inclusion-exclusion principle: For any L and R:  $P(L \cup R) = P(L) + P(R) - P(L \cap R)$ .









 $L \cup R$ , overlap =  $L \cap R$ 

#### Permutations

```
Lining things up. How many ways can you do it? 
'abc' and 'cab' are different permutations of {a, b, c}
```

### Permutations of k from a set of n

Give all permutations of 3 things out of  $\{a, b, c, d\}$ 

```
abc abd acb acd adb adc
bac bad bca bcd bda bdc
cab cad cba cbd cda cdb
dab dac dba dbc dca dcb
```

Would you want to do this for 7 from a set of 10?

### Combinations

Choosing subsets – order doesn't matter. How many ways can you do it?

## Combinations of k from a set of n

Give all combinations of 3 things out of  $\{a, b, c, d\}$ 

**Answer:**  $\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$ 

### Permutations and Combinations

abcacbbacbcacba
$$\{a, b, c\}$$
abdadbbdadabdba $\{a, b, d\}$ acdadccadcdadacdca $\{a, c, d\}$ bcdbdccbdcdbdbcdcb $\{b, c, d\}$ 

Permutations:

$$_4P_3$$

$$\binom{4}{3} = {}_{4}C_{3}$$

$$\binom{4}{3} = {}_{4}C_{3} = \frac{{}_{4}P_{3}}{3!}$$

### **Board Question**

- (a) Count the number of ways to get exactly 3 heads in 10 flips of a coin.
- (b) For a fair coin, what is the probability of exactly 3 heads in 10 flips?
- <u>answer:</u> (a) We have to 'choose' 3 out of 10 flips for heads:  $\binom{10}{3}$ .
- (b) There are 2<sup>10</sup> possible outcomes from 10 flips (this is the rule of product). For a fair coin each outcome is equally probable so the probability of exactly 3 heads is

answer: 
$$\frac{\binom{10}{3}}{2^{10}} = \frac{120}{1024} = 0.117$$

- Variables with measured or count data might have thousands of distinct values.
- A basic step in exploring data is getting a "typical value" for each feature (variable): an estimate of where most of the data is located (i.e., its central tendency).

### **KEY TERM S FOR ESTIM ATES OF LOCATION**

### 1. Mean:

The sum of all values divided by the number of values.

Synonyms: average

Mean = 
$$\bar{x} = \frac{\sum_{i}^{n} x_{i}}{n}$$

## 2. Weighted mean

The sum of all values times a weight divided by the sum of the weights

There are two main motivations for using a weighted mean:

1. Some values are intrinsically more variable than others, and highly variable observations are given a lower weight. For example, if we are taking the average from multiple sensors and one of the sensors is less accurate, then we might down weight the data from that sensor.

2. The data collected does not equally represent the different groups that we are interested in measuring. For example, because of the way an online experiment was conducted, we may not have a set of data that accurately reflects all groups in the user base. To correct that, we can give a higher weight to the values from the groups that were underrepresented. Synonyms: weighted average

Weighted mean = 
$$\overline{x}_w = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}$$

### 3. Median

- The value such that one-half of the data lies above and below
- The median is the middle number on a sorted list of the data
- If there is an even number of data values, the middle value is one that is not actually in the data set, but rather the average of the two values that divide the sorted data into upper and lower halves

Synonyms: 50th percentile

## 4. Weighted median

- The value such that one-half of the sum of the weights lies above and below the sorted data
- For the same reasons that one uses a weighted mean, it is also possible to compute a weighted median. As with the median, we first sort the data, although each data value has an associated weight. Instead of the middle number, the weighted median is a value such that the sum of the weights is equal for the lower and upper halves of the sorted list.
- Like the median, the weighted median is robust to outliers.

### 5. Trimmed mean

- The average of all values after dropping a fixed number of extreme values
- A trimmed mean eliminates the influence of extreme values

Synonyms: truncated mean

Trimmed mean = 
$$\overline{x} = \frac{\sum_{i=p+1}^{n-p} x_{(i)}}{n-2p}$$

### 6. Robust

Not sensitive to extreme values.

Synonyms: resistant

### 7. Outlier

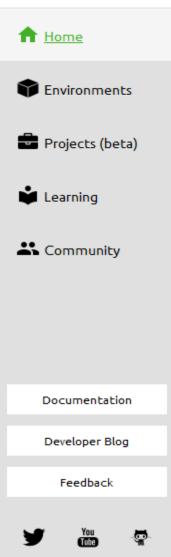
- A data value that is very different from most of the data
- An outlier is any value that is very distant from the other values in a data set

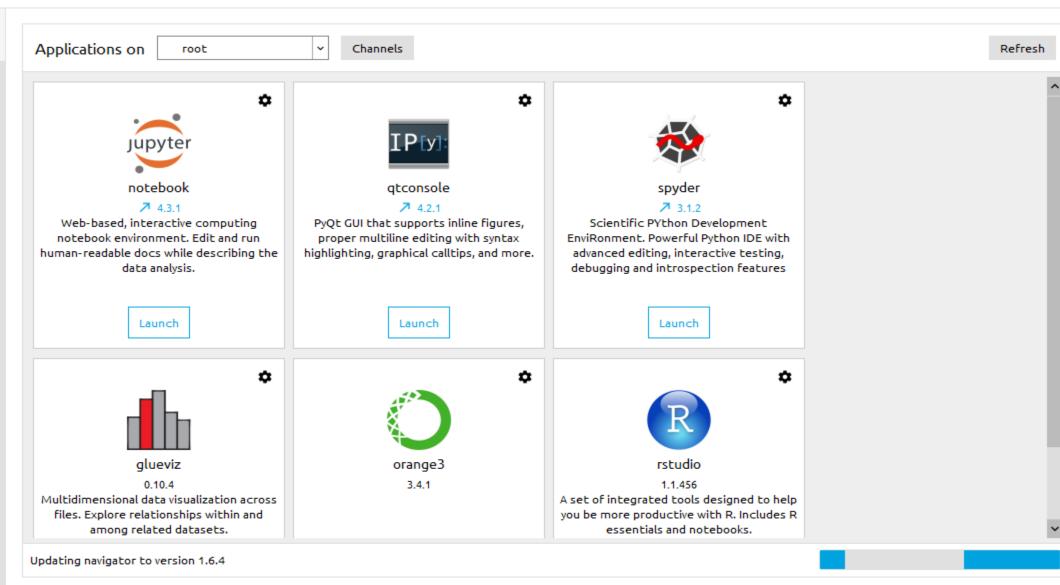
Synonyms: extreme value





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### EXAMPLE: LOCATION ESTIMATES OF POPULATION AND MURDER RATES

Compute the mean, trimmed mean, and median for the population using R:

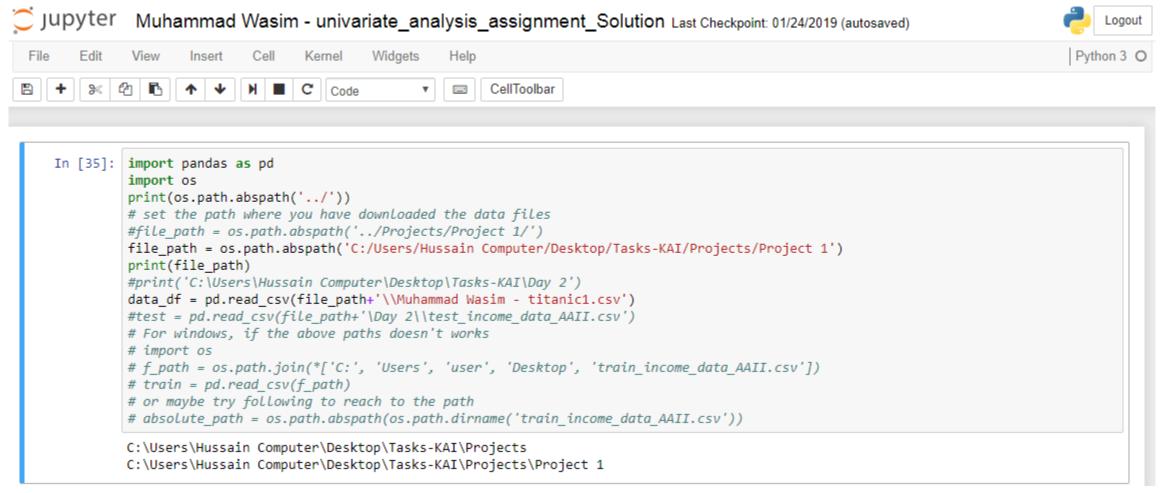
```
> state <- read.csv(file="/Users/andrewbruce1/book/state.csv")
> mean(state[["Population"]])
[1] 6162876
> mean(state[["Population"]], trim=0.1)
[1] 4783697
> median(state[["Population"]])
[1] 4436370
```

```
> weighted.mean(state[["Murder.Rate"]], w=state[["Population"]])
[1] 4.445834
> library("matrixStats")
> weightedMedian(state[["Murder.Rate"]], w=state[["Population"]])
[1] 4.4
```

		State	Population	Murder rate
)	1	Alabama	4,779,736	5.7
	2	Alaska	710,231	5.6
	3	Arizona	6,392,017	4.7
	4	Arkansas	2,915,918	5.6
)	5	California	37,253,956	4.4
	6	Colorado	5,029,196	2.8
	7	Connecticut	3,574,097	2.4
	8	Delaware	897,934	5.8

### EXAMPLE: LOCATION ESTIMATES OF POPULATION AND MURDER RATES

## Working with pandas in Python:



# ESTIMATES OF VARIABILITY

## **Estimates of Variability**

- Location is just one dimension in summarizing a feature. A second dimension, variability, also referred to as dispersion, measures whether the data values are tightly clustered or spread out.
- At the heart of statistics lies variability:
- Measuring it, reducing it, distinguishing random from real variability, identifying the various sources of real variability, and making decisions in the presence of it.

# KEY TERMS FOR VARIABILITY METRICS

### **Deviations**

The difference between the observed values and the estimate of location.

Synonyms: errors, residuals

### **Variance**

The sum of squared deviations from the mean divided by n-1 where n is the number of data values.

Synonyms: mean-squared-error

## KEY TERMS FOR VARIABILITY METRICS

### Standard deviation

The square root of the variance.

Synonyms: 12-norm, Euclidean norm

### Mean absolute deviation

The mean of the absolute value of the deviations from the mean.

Synonyms: 11-norm, Manhattan norm

### Median absolute deviation from the median

The median of the absolute value of the deviations from the median.

# KEY TERM S FOR VARIAB ILITY M ETRICS

## Range

The difference between the largest and the smallest value in a data set.

### **Order statistics**

Metrics based on the data values sorted from smallest to biggest.

Synonyms: ranks

### **Percentile**

The value such that P percent of the values take on this value or less and (100–P) percent take on this value or more.

Synonyms: quantile

## Interquartile range

The difference between the 75th percentile and the 25th percentile.

Synonyms: IQR

# KEY TERMS FOR VARIABILITY METRICS

- Just as there are different ways to measure location (mean, median, etc.) there are also different ways to measure variability.
- The most widely used estimates of variation are based on the differences, or deviations, between the estimate of location and the observed data. For a set of data {1, 4, 4}, the mean is 3 and the median is 4.
- The deviations from the mean are the differences: 1-3=-2, 4-3=1, 4-3=1.
- These deviations tell us how dispersed the data is around the central value.

# KEY TERMS FOR VARIABILITY METRICS

#### Mean absolute deviation:

- The sum of the deviations from the mean is precisely zero. Instead, a simple approach is to take the average of the absolute values of the deviations from the mean.
- In the preceding example, the absolute value of the deviations is  $\{2\ 1\ 1\}$  and their average is (2+1+1)/3=1.33.
- This is known as the mean absolute deviation and is computed with the formula:

Mean absolution deviation = 
$$\frac{\sum_{i=1}^{n} |x_i - \overline{x}|}{n}$$

## KEY TERMS FOR VARIABILITY METRICS

The best-known estimates for variability are the variance and the standard deviation, which are based on squared deviations. The variance is an average of the squared deviations, and the standard deviation is the square root of the variance.

Variance = 
$$s^2 = \frac{\sum (x - \overline{x})^2}{n - 1}$$
  
Standard deviation =  $s = \sqrt{\text{Variance}}$ 

# KEY TERMS FOR VARIABILITY METRICS

 A robust estimate of variability is the median absolute deviation from the median

or MAD:

Median absolute deviation = Median 
$$(|x_1 - m|, |x_2 - m|, ..., |x_N - m|)$$

• where *m* is the median. Like the median, the MAD is not influenced by extreme values. It is also possible to compute a trimmed standard deviation analogous to the trimmed mean.

### **EXAMPLE: VARIABILITY ESTIMATES OF STATE POPULATION**

```
> sd(state[["Population"]])
[1] 6848235
> IQR(state[["Population"]])
[1] 4847308
> mad(state[["Population"]])
[1] 3849870
```

	State	Population	Murder rate
1	Alabama	4,779,736	5.7
2	Alaska	710,231	5.6
3	Arizona	6,392,017	4.7
4	Arkansas	2,915,918	5.6
5	California	37,253,956	4.4
6	Colorado	5,029,196	2.8
7	Connecticut	3,574,097	2.4
8	Delaware	897,934	5.8

```
In [35]: import pandas as pd
         import os
         print(os.path.abspath('../'))
         # set the path where you have downloaded the data files
         #file path = os.path.abspath('../Projects/Project 1/')
         file path = os.path.abspath('C:/Users/Hussain Computer/Desktop/Tasks-KAI/Projects/Project 1')
         print(file path)
         #print('C:\Users\Hussain Computer\Desktop\Tasks-KAI\Day 2')
         data_df = pd.read_csv(file_path+'\\Muhammad Wasim - titanic1.csv')
         #test = pd.read csv(file path+'\Day 2\\test income data AAII.csv')
         # For windows, if the above paths doesn't works
         # import os
         # f_path = os.path.join(*['C:', 'Users', 'user', 'Desktop', 'train_income_data_AAII.csv'])
         # train = pd.read csv(f path)
         # or maybe try following to reach to the path
         # absolute path = os.path.abspath(os.path.dirname('train income data AAII.csv'))
         C:\Users\Hussain Computer\Desktop\Tasks-KAI\Projects
         C:\Users\Hussain Computer\Desktop\Tasks-KAI\Projects\Project 1
```

In [10]: print(type())
 data\_df.head()

<class 'pandas.core.frame.DataFrame'>

#### Out[10]:

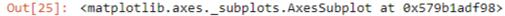
	Passengerid	Survived	Pclass	Name	Sex	Age	SibSp	Parch	Ticket	Fare	Cabin	Embarked
0	1	0	3	Braund, Mr. Owen Harris	male	22.0	1	0	A/5 21171	7.2500	NaN	S
1	2	1	1	Cumings, Mrs. John Bradley (Florence Briggs Th	female	38.0	1	0	PC 17599	71.2833	C85	С
2	3	1	3	Heikkinen, Miss. Laina	female	26.0	0	0	STON/O2. 3101282	7.9250	NaN	s
3	4	1	1	Futrelle, Mrs. Jacques Heath (Lily May Peel)	female	35.0	1	0	113803	53.1000	C123	s
4	5	0	3	Allen, Mr. William Henry	male	35.0	0	0	373450	8.0500	NaN	s

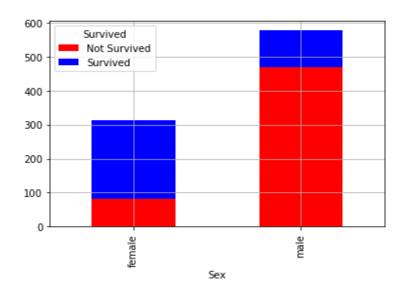
Out[11]: (891, 12)

```
In [12]:
         data_df.dtypes
Out[12]: PassengerId
                          int64
         Survived
                          int64
         Pclass
                          int64
                         object
         Name
                         object
         Sex
                        float64
         Age
                          int64
         SibSp
                          int64
         Parch
         Ticket
                         object
                        float64
         Fare
         Cabin
                         object
         Embarked
                         object
         dtype: object
In [13]: categorical_var = data_df.dtypes.loc[data_df.dtypes=='object'].index
         print(categorical_var)
         Index(['Name', 'Sex', 'Ticket', 'Cabin', 'Embarked'], dtype='object')
In [14]: data_df[categorical_var].apply(lambda x:len(x.unique()))
Out[14]: Name
                     891
         Sex
                       2
         Ticket
                     681
         Cabin
                     148
         Embarked
         dtype: int64
```

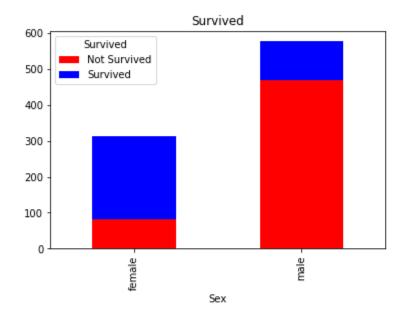
```
In [15]: # Here we will use .replace which we discussed in class topic "Data Preprocessing"
         data df['Survived'].replace({0:'Not Survived'}, inplace=True)
In [16]: data df['Survived'].replace({1:'Survived'}, inplace=True)
In [17]: data_df.dtypes
Out[17]: PassengerId
                          int64
         Survived
                         object
         Pclass
                          int64
                         object
         Name
                         object
         Sex
                        float64
         Age
         SibSp
                          int64
                          int64
         Parch
         Ticket
                         object
                        float64
         Fare
         Cabin
                         object
         Embarked
                         object
         dtype: object
In [18]: data df['Survived'].unique()
Out[18]: array(['Not Survived', 'Survived'], dtype=object)
```

```
In [19]: data df['Survived'].unique()
Out[19]: array(['Not Survived', 'Survived'], dtype=object)
In [20]: data df['Survived'].value counts()
Out[20]: Not Survived
                          549
         Survived
                          342
         Name: Survived, dtype: int64
In [21]: data df['Survived'].value counts()/data df.shape[0]
Out[21]: Not Survived
                          0.616162
         Survived
                         0.383838
         Name: Survived, dtype: float64
In [22]: # Here we observe that around 61.6 % are 'Not Survived' and around 38.3% are 'Survived'
In [23]: data_df.dtypes
Out[23]: PassengerId
                           int64
         Survived
                         object
         Pclass
                          int64
                         object
         Name
                         object
          Sex
                        float64
          Age
         SibSp
                           int64
         Parch
                          int64
         Ticket
                         object
                         float64
         Fare
         Cabin
                         object
          Embankod
                          abjact
```





```
In [26]: cross_tab.iloc[:-1,:-1].plot(kind='bar', stacked=True, color=['red','blue'], grid=False, title='Survived')
Out[26]: <matplotlib.axes._subplots.AxesSubplot at 0x579d288b70>
```



```
In [28]: df=data_df['Survived'].value_counts()/data_df.shape[0]
```

Out[28]: Not Survived 0.616162 Survived 0.383838

Name: Survived, dtype: float64

```
In [28]: df=data_df['Survived'].value_counts()/data_df.shape[0]
Out[28]: Not Survived
                         0.616162
                         0.383838
         Survived
         Name: Survived, dtype: float64
In [30]: df=data_df['Survived'].value_counts()/data_df.shape[0]
         %matplotlib inline
         df.plot(kind='bar',stacked=True, color=['red','blue'], grid=True)
Out[30]: <matplotlib.axes._subplots.AxesSubplot at 0x579d3476d8>
          0.6
          0.5
          0.4
          0.3
          0.2
          0.1
```



