1. ~~Just buy a pulley w/ 2.3 mm bore~~ — UNLIKELY
2. 2.3 mm to 5 mm shaft adapter and buy bigger pulley
3. Fit a sleeve on shaft, effectively increasing diameter
4. take a stock piece of metal,
   drill hole for shaft, and fit pulley on OD of
   make-shift sleeve
5. Take stock Aluminum rod & lathe it down to 5mm OD,
   drill 2.3 mm hole on one end.

Todo

1. Drill 2.3 mm hole into shaft
2. Fit pulley onto shaft using set screws & loctite
3. Drill holes for set screws into shaft
4. Mount motor, encoder, and belt/pulley system
5. Mount rod onto encoder (wood block)

→ Done

PARTS

1. LINEAR ENCODING STRIP - MEASURE POSITION of CART
2. ROTATIONAL ENCODER - MEASURE Angle of PENDELUM
3. RAILS & CART
4. PENDELUM (CAN be WOOD)
5. MOTOR driver - drives the MOTOR
6. DC MOTOR
7. MICROCONTROLLER

$$\frac{2\pi \, 8mm}{0.1mm} = 503 \text{ PPR}$$

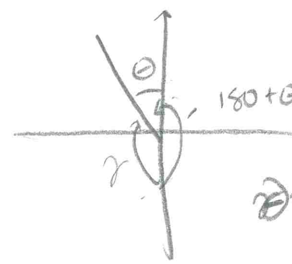
$\omega = 0.3$

2π

~~EPSO~~

$S = r\, d\theta$

	Epson	Hp			
dpi	2400	600	- PLENTY	$\frac{t_1}{\theta_1}$	$\frac{t_2}{\theta_2}$
V	120 V	110 V AC			
MAX PRINT length	8.5 in	8.5 in			
f	50 HZ	50 hz			
PPM	7.3/13	22/28			



$180 + \theta = \gamma$

$\theta = \gamma - 180$

$\gamma = 180 - \theta$

$\theta = 180 - \gamma$

Parts

✓ ① BELT — GT2, 2 mm PITCH, 6 mm wide, 530 mm long

✓ ② Timing pulley — GT2, 2mm PITCH, 6 mm wide,

DON'T need ③ IDLE PULLEY — 6 mm wide, teeth doesn't MATTER

✓ ④ LINEAR POSITION FINDER — ⓐ OPTICAL range finder
ⓑ rotary encoder

✓ ⑤ belt CLAMP TO CART —

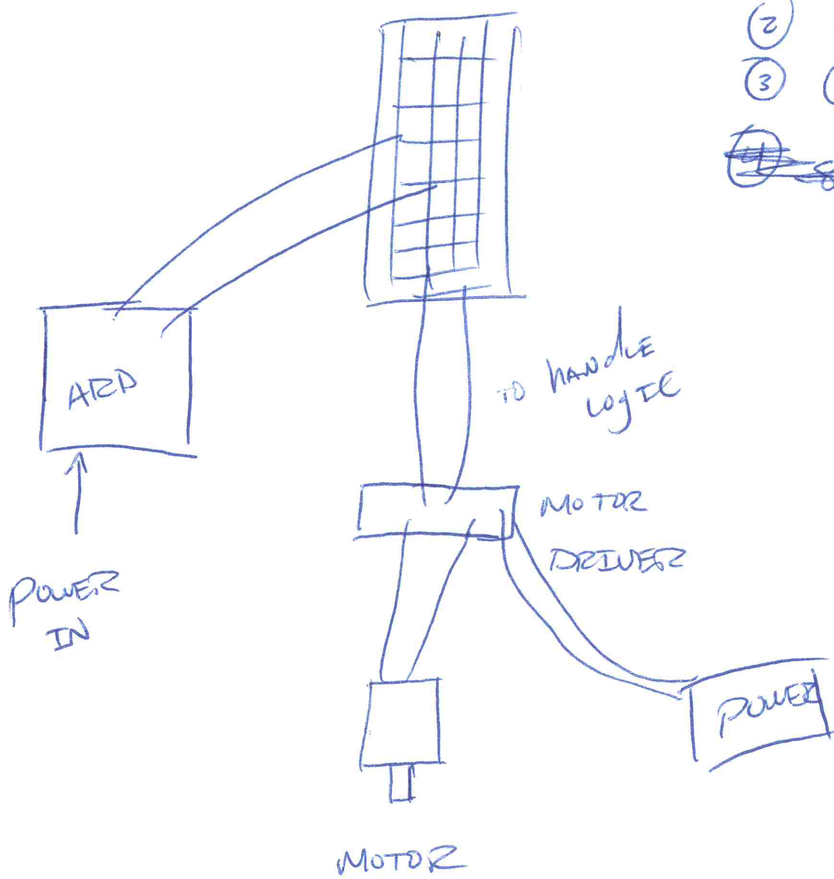
✓ ⑥ BELT TENSIONER —

$x = r_{gear}\, d\theta_{STEP}$

don't NEED ⑦ MOTOR — 12-18V MOTOR, Ideally we want < 0.5 mm POSITION

Parts

1. Multimeter
2. Battery pack
3. Clip on wires - alligator
~~4 8 gauge~~

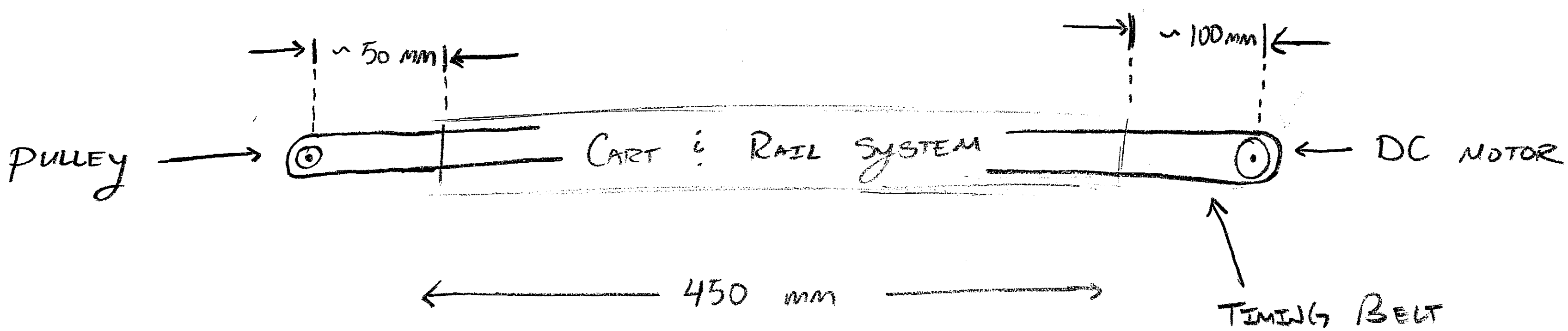


Todo

1. Measure voltage on Motor

V	I
3.2	0.3 A
	0.31

★ ORIGINAL BELT IS TOO SHORT, SO I NEED A NEW BELT THAT MEETS THE SPECS OF THE DC MOTOR HEAD



DC MOTOR HEAD:

WAS MADE TO FIT THE ORIGINAL BELT W/ SPECS:

~1 mm PITCH
~3 mm wide
? # of TEETH

Py CONTROL

1. AS A USER
   I WANT TO CREATE A PID CONTROLLER
   SO I CAN CONTROL MY HARDWARE

2. AS A USER
   I WANT TO SPECIFY MY GAINS ON THE PID CONTROLLER
   SO I CAN TUNE IT MYSELF

3. AS A USER
   I WANT TO SPECIFY SATURATION LIMITS
   SO MY PID CONTROLLER MEETS MY HARDWARE CONSTRAINTS

4. AS A USER
   I WANT TO SPECIFY AN ANTI-WINDUP METHOD
   SO MY INTEGRAL TERM DOESN'T ACCUMULATE
   UNREASONABLE COMMANDS

5. AS A USER,
   I WANT TO BE ABLE TO AUTOTUNE MY CONTROLLER
   SO IT PERFORMS OPTIMALLY TO THE TF I PROVIDE

6. AS A USER,
   I WANT TO TUNE MY PID BASED ON TIME-DOMAIN CONSTRAINTS
   SO I DON'T HAVE TO DO THE MATH MYSELF

(7) AS A USER,

I WANT TO TUNE MY PID BASED ON S-DOMAIN CONSTRAINTS

SO I DON'T HAVE TO DO THE MATH MYSELF

(8) AS A USER,

I WANT TO TUNE MY PID BASED ON FREQUENCY-DOMAIN CONSTRAINTS

SO I DONT HAVE TO DO THE MATH MYSELF

(9) AS A USER,

I WANT TO HAVE THE OPTION OF CHOOSING A DERIVATIVE FILTER

SO I CAN BLOCK HIGH-FREQUENCY NOISE

PyGA

---

(1) AS A USER

I WANT TO SPECIFY THE # OF POPS, # OF CHILDREN,

SO I CAN HAVE CONTROL OF WHAT THE ALGO IS PERFORMING

(2) AS A USER,

I WANT TO SPECIFY THE FITNESS FUNCTION TO EVALUATE RESULTS ON

SO THAT I CAN CUSTOMIZE THE RESULTS

(3) AS A USER,

I WANT TO BE RETURNED ALL OF THE DATA ABOUT THE

ALGORITHM SO I CAN DO WHAT I WANT W/ THE DATA.

Measurements

$d_p = 0.233''$

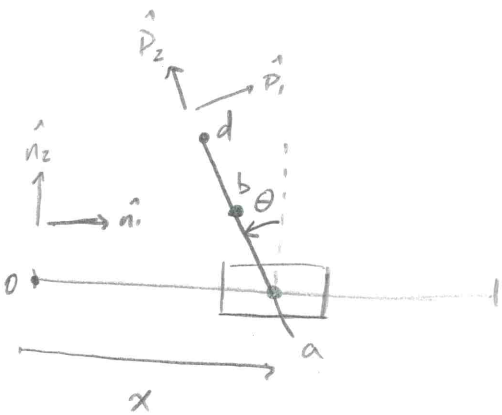
$d_e = 0.235''$

A

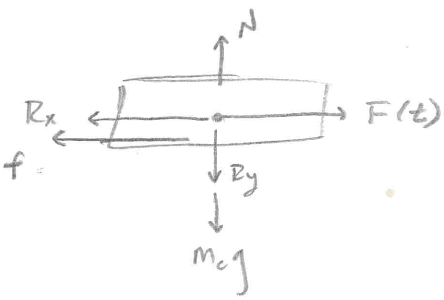
DYNAMIC

MODEL

1 IDENTIFY SYSTEM



$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{pmatrix}$$



POTENTIAL: $M_c g$

CONSTRAINT: N, R_y

IMPRESSED: R_x ?, $F(t)$, f

$$\vec{f}_c = (F(t) - R_x - f)\hat{n}_1 + (N - R_y - M_c g)\hat{n}_2$$

WHERE,

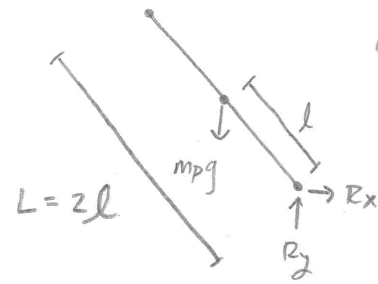
$$f = \varepsilon \dot{x}$$

SO,

$$\vec{f}_c = (F(t) - R_x - \varepsilon \dot{x})\hat{n}_1 + (N - R_y - M_c g)\hat{n}_2$$

$$\vec{l}_c^{\,c} = 0$$

2 FBD



POTENTIAL: $m_p g$

CONSTRAINT: R_y, R_x

$$\vec{f}_p = (R_x)\hat{n}_1 + (R_y - m_p g)\hat{n}_2$$

$$\vec{l}_c^{\,p} = \vec{r} \times \vec{F}$$

$$= \begin{pmatrix} 0 \\ -l \\ 0 \end{pmatrix}_P \times \begin{pmatrix} R_x \\ R_y \\ 0 \end{pmatrix}_N$$

$$= l \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}_N \times \begin{pmatrix} R_x \\ R_y \\ 0 \end{pmatrix}_N$$

$$= l\,(R_y \sin\theta + R_x \cos\theta)\,\hat{n}_3$$

3 KINEMATICS : CARDINAL VECTORS

PENDULUM

$$\vec{r}_{b/o} = \vec{r}_{a/o} + \vec{r}_{b/a}$$

$$= x\hat{n}_1 + l\hat{p}_2$$

CART

$$\vec{r}_{a/o} = x\hat{n}_1$$

$$\dot{\vec{r}}_{a/o} = \dot{x}\hat{n}_1$$

$$\dot{\vec{r}}_{b/o} = {}^N\frac{d}{dt}\vec{r}_{a/o} + {}^P\frac{d}{dt}\vec{r}_{b/a} + \vec{\omega}_{P/N} \times \vec{r}_{b/a}$$

$$= \dot{x}\hat{n}_1 + \cancel{{}^P\frac{d}{dt}(l\hat{p}_2)}^{\,0} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}_P \times \begin{pmatrix} 0 \\ l \\ 0 \end{pmatrix}_P$$

$$= \dot{x}\hat{n}_1 - l\dot{\theta}\hat{p}_1$$

$$= \dot{x}\hat{n}_1 - l\dot{\theta}(\cos\theta\,\hat{n}_1 + \sin\theta\,\hat{n}_2)$$

$$= (\dot{x} - l\dot{\theta}\cos\theta)\hat{n}_1 + (-l\dot{\theta}\sin\theta)\hat{n}_2$$

$$\vec{\omega}_{P/N} = \dot{\theta}\hat{p}_3 = \dot{\theta}\hat{n}_3$$

CARDINAL VECTORS

$$\tau_{\dot{x}}^{c} = \frac{\partial \vec{v}_{a/o}}{\partial \dot{x}} = \hat{n}_1$$

$$\tau_{\dot{\theta}}^{c} = 0$$

$$\rho_{\dot{x}}^{c} = \rho_{\dot{\theta}}^{c} = 0$$

$$\tau_{\dot{x}}^{p} = \frac{\partial \vec{v}_{b/o}}{\partial \dot{x}} = \hat{n}_1$$

$$\tau_{\dot{\theta}}^{p} = \frac{\partial \vec{v}_{b/o}}{\partial \dot{\theta}} = -l\hat{p}_1$$

$$\rho_{\dot{\theta}}^{p} = \frac{\partial \vec{\omega}_{P/N}}{\partial \dot{\theta}} = \hat{p}_3 = \hat{n}_3$$

④ GENERALIZED FORCES

WORK RATE PRINCIPAL:

$$\dot{W} = \sum_{body\ i} \left[\vec{f}_i \cdot \vec{v}_i + \vec{l}_c^{\,i} \cdot \omega_i \right]$$

$$\dot{W}_{pend} = \vec{f}_P \cdot \dot{\vec{r}}_{b/o} + \vec{l}_c^{\,P} \cdot \vec{\omega}_{P/N}$$

$$= \begin{pmatrix} R_x \\ R_y - m_P g \\ 0 \end{pmatrix}_N \cdot \begin{pmatrix} \dot{x} - l\dot{\theta}\cos\theta \\ -l\dot{\theta}\sin\theta \\ 0 \end{pmatrix}_N + \begin{pmatrix} 0 \\ 0 \\ l(R_y s\theta + R_x c\theta) \end{pmatrix}_N \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}_N$$

$$= R_x\dot{x} - (R_x l\cos\theta)\dot{\theta} - (R_y l\sin\theta)\dot{\theta} + (m_P g l\sin\theta)\dot{\theta}$$

$$+\ l(R_y\sin\theta + R_x\cos\theta)\,\dot{\theta}$$

$$= (R_x)\dot{x} + l\left(m_P g\sin\theta - \cancel{R_x\cos\theta} - \cancel{R_y\sin\theta} + \cancel{R_y\sin\theta} + \cancel{R_x\cos\theta}\right)\dot{\theta}$$

$$= (R_x)\dot{x} + (m_P g\, l\sin\theta)\,\dot{\theta}$$

$$\dot{W}_{CART} = \vec{f}_c \cdot \dot{\vec{r}}_{a/o} + \cancel{\vec{l}_c^{\,c}}^{\,0} \cdot \cancel{\omega_N}^{\,0}$$

$$= \begin{pmatrix} F(t) - R_x - \epsilon\dot{x} \\ N - R_y - m_c g \\ 0 \end{pmatrix}_N \cdot \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix}_N$$

$$= (F(t) - R_x - f)\,\dot{x}$$

$$\therefore\quad \dot{W} = \dot{W}_{pend} + \dot{W}_{CART}$$

$$= (F(t) - f)\dot{x} + (m_P g\, l\sin\theta)\,\dot{\theta}$$

$$= \underbrace{\left(F(t) - \epsilon\dot{x}\right)}_{Q_{\dot{x}}}\dot{x} + \underbrace{(m_P g\, l\sin\theta)}_{Q_{\dot{\theta}}}\dot{\theta}$$

(5) LAGRANGE'S EQ

$$L = \sum_{body\ i} (T_i - V_i)$$

$$= T^P_{trans} + T^P_{rot} + T^c_{trans}$$

$$= \frac{1}{2} m_p \vec{v}_p \cdot \vec{v}_p + \frac{1}{2} \vec{\omega}_{P/N} \cdot I^P_c \vec{\omega}_{P/N} + \frac{1}{2} m_c \vec{v}_c \cdot \vec{v}_c$$

ASSUME A PRINCIPAL AXIS frame on the pendulum, so I^P_c IS CONSTANT and we have $I^P_c = I^P_3 = \frac{1}{12} m_p L^2$, WHERE L IS the <u>total</u> length of the rod. SO,

$$L = \frac{1}{2} m_p \begin{pmatrix} \dot{x} - l\dot{\theta}\cos\theta \\ -l\dot{\theta}\sin\theta \\ 0 \end{pmatrix}_N \cdot \begin{pmatrix} \dot{x} - l\dot{\theta}\cos\theta \\ -l\dot{\theta}\sin\theta \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}_P \cdot I^P_c \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}_P + \frac{1}{2} m_c \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix}_N \cdot \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix}_N$$

$$= \frac{1}{2} m_p \left[(\dot{x} - l\dot{\theta}\cos\theta)^2 + l^2\dot{\theta}^2 \sin^2\theta \right] + \frac{1}{2} I^P_c \dot{\theta}^2 + \frac{1}{2} m_c \dot{x}^2$$

$$= \frac{1}{2} m_p \left[\dot{x}^2 - 2\dot{x} l\dot{\theta}\cos\theta + l^2\dot{\theta}^2 \cancel{\cos^2\theta} + l^2\dot{\theta}^2 \cancel{\sin^2\theta} \right] + \frac{1}{2} I^P_c \dot{\theta}^2 + \frac{1}{2} m_c \dot{x}^2$$

$$= \frac{1}{2} m_p \left[\dot{x}^2 - (2l\cos\theta)\dot{x}\dot{\theta} + l^2\dot{\theta}^2 \right] + \frac{1}{2} I^P_c \dot{\theta}^2 + \frac{1}{2} m_c \dot{x}^2$$

$$= \frac{1}{2}(m_p + m_c)\dot{x}^2 + \frac{1}{2}(m_p l^2 + I^P_c)\dot{\theta}^2 - (m_p l \cos\theta)\dot{x}\dot{\theta}$$

⑥ EQUATIONS OF MOTION

$$Q_{\dot{\theta}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

<u>derivatives</u>

$$\frac{\partial L}{\partial \dot{\theta}} = (m_p \ell^2 + I_c^p)\dot{\theta} - (m_p \ell \cos\theta)\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (m_p \ell^2 + I_c^p)\ddot{\theta} - m_p \ell\left(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta\right)$$

$$= (m_p \ell^2 + I_c^p)\ddot{\theta} + m_p \ell\left(\dot{x}\dot{\theta}\sin\theta - \ddot{x}\cos\theta\right)$$

$$\frac{\partial L}{\partial \theta} = -m_p \ell \dot{x}\dot{\theta}(-\sin\theta) = m_p \ell \dot{x}\dot{\theta}\sin\theta$$

$$\therefore \quad Q_{\dot{\theta}} = m_p g \ell \sin\theta = (m_p \ell^2 + I_c^p)\ddot{\theta} + m_p \ell\left(\cancel{\dot{x}\dot{\theta}\sin\theta} - \ddot{x}\cos\theta\right) - \cancel{m_p \ell \dot{x}\dot{\theta}\sin\theta}$$

$$\boxed{m_p g \ell \sin\theta = (m_p \ell^2 + I_c^p)\ddot{\theta} - m_p \ell \ddot{x}\cos\theta} \quad \text{— (1)}$$

I think this should be positive...

$$Q_{\dot{x}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

Derivatives

$$\frac{\partial L}{\partial \dot{x}} = (m_p + m_c)\dot{x} - m_p l \dot{\theta}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_p + m_c)\ddot{x} - m_p l\left(\ddot{\theta}\cos\theta + \dot{\theta}(-\sin\theta)\dot{\theta}\right)$$

$$= (m_p + m_c)\ddot{x} - m_p l\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)$$

$$= (m_p + m_c)\ddot{x} + m_p l\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right)$$

$$\frac{\partial L}{\partial x} = 0$$

$$\therefore \quad Q_{\dot{x}} = F(t) - \varepsilon\dot{x} = (m_p + m_c)\ddot{x} + m_p l\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right)$$

$$= (m_p + m_c)\ddot{x} + m_p l\dot{\theta}^2\sin\theta - m_p l\ddot{\theta}\cos\theta$$

$$\boxed{\rightarrow F(t) - \varepsilon\dot{x} = (m_p + m_c)\ddot{x} + m_p l\dot{\theta}^2\sin\theta - m_p l\ddot{\theta}\cos\theta} \quad - (2)$$

SHOULD be NEGATIVE

SHOULD be POSITIVE

Solving (1) for $\ddot{\theta}$ and (2) for $\ddot{x}$ yields

$$\ddot{\theta} = m_p l \left(m_p l^2 + I_c^p\right)^{-1}\left(\ddot{x}\cos\theta + g\sin\theta\right) \quad — \quad (1)$$

$$\ddot{x} = \left(m_p + m_c\right)^{-1}\left(F(t) - f + m_p l \ddot{\theta}\cos\theta - m_p l \dot{\theta}^2 \sin\theta\right) \quad — \quad (2)$$

Substituting (1) into (2) yields

$$(m_p + m_c)\ddot{x} = F(t) - f - m_p l \dot{\theta}^2 \sin\theta + m_p l \cos\theta\,(\beta)\left(\ddot{x}\cos\theta + g\sin\theta\right)$$

$$= F(t) - f - m_p l \dot{\theta}^2 \sin\theta + m_p l \beta\, \ddot{x}\cos^2\theta + m_p l g \beta \sin\theta\cos\theta$$

$$\left(m_p + m_c - m_p l \beta\cos^2\theta\right)\ddot{x} = F(t) - f - m_p l \dot{\theta}^2\sin\theta + m_p l g \beta \sin\theta\cos\theta$$

$$\therefore \quad \ddot{x} = \left[m_p + m_c - m_p l \beta\cos^2\theta\right]^{-1}\left[F(t) - f + m_p l \sin(\theta)\left(g\beta\cos\theta - \dot{\theta}^2\right)\right]$$

$$— \quad (3).$$

Plugging (3) into (1) yields

$$\ddot{\Theta} = \beta g \sin\theta + \beta \cos\theta \, (\ddot{x})$$

$$\ddot{\Theta} = \beta g \sin\theta + \beta\cos\theta\left[m_p + m_c - m_p l \beta \cos^2\theta\right]^{-1}\left[F(t) - f + m_p l \sin\theta\left(g\beta\cos\theta - \dot{\theta}^2\right)\right]$$

$$= \frac{\beta}{m_c + (1 - l\beta\cos^2\theta)m_p}\left[g\sin\theta\left(m_c + (1 - l\beta\cos^2\theta)m_p\right) + \cos\theta\left(F(t) - f + m_p l \sin\theta\left(g\beta\cos\theta - \dot{\theta}^2\right)\right)\right]$$

$$= \gamma\Big[m_c g \sin\theta + m_p g \sin\theta - m_p \cancel{l g \beta \sin\theta}\cos^2\theta$$
$$+ F(t)\cos\theta - f\cos\theta + m_p \cancel{l g \beta \sin\theta}\cos^2\theta - m_p l \dot{\theta}^2 \sin\theta\cos\theta\Big]$$

$$= \gamma\left[(m_c + m_p) g \sin\theta + (F(t) - f)\cos\theta - m_p l \dot{\theta}^2 \sin\theta\cos\theta\right]$$

SWAP SIGNS

$$\ddot{x} = \left(m_c + m_p(1 - l\beta\cos^2\theta)\right)^{-1}\left(F(t) - f + m_p l g \beta \sin\theta\cos\theta - m_p l \dot{\theta}^2 \sin\theta\right)$$

$$\therefore \quad \ddot{\Theta} = \beta\left(m_c + m_p(1 - l\beta\cos^2\theta)\right)^{-1}\left((m_c + m_p) g \sin\theta + (F(t) - f)\cos\theta - m_p l \dot{\theta}^2 \sin\theta\cos\theta\right)$$

SWAP SIGNS

$$\beta = \frac{m_p l}{m_p l^2 + I_c^p} = \frac{1}{l + I_c^p / m_p l}$$

We now convert this to a system of first order DEs by letting

$$x_1 = \theta \qquad x_3 = \dot{\theta}$$

$$x_2 = x \qquad x_4 = \dot{x},$$

so we obtain the system of first-order nonlinear DEs

$$
(5) \quad \begin{cases}
\dot{x}_1 = x_3 \\
\dot{x}_2 = x_4 \\
\dot{x}_3 = \dfrac{\beta}{m_c + m_p(1 - l\beta\cos^2 x_1)} \left[(m_c + m_p) g \sin x_1 + \left(u(t) - \varepsilon x_4\right)\cos x_1 - m_p l\, x_3^2 \sin x_1 \cos x_1 \right] \\
\dot{x}_4 = \dfrac{u(t) - \varepsilon x_4 + m_p l g \beta \sin x_1 \cos x_1 - m_p l\, x_3^2 \sin x_1}{m_c + m_p(1 - l\beta\cos^2 x_1)}
\end{cases}
$$

which has the form $\dot{\vec{x}} = f(\vec{x})$.

Next, we find the EQ points by solving $f(\vec{x}) = 0$. That is, $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0$. Consider the case w/ no control input, then

$$\dot{x}_1 = 0 = x_3 \implies \bar{x}_3 = 0 \implies \bar{x}_1 = \text{const.}$$

$$\dot{x}_2 = 0 = x_4 \implies \bar{x}_4 = 0 \implies \bar{x}_2 = \text{const.}$$

$$\dot{x}_3 = 0 = (M_c + M_p)\, g \sin(\bar{x}_1) + (\cancelto{0}{u(t)} - \cancelto{0}{\varepsilon \bar{x}_4}) \cos x_1 - 0$$

$$0 = \sin(\bar{x}_1)$$

$$\therefore\ \bar{x}_1 = \arcsin(0) = n\pi, \quad n \in \mathbb{Z}_{\geq 0}$$

$$\dot{x}_4 = 0 = \cancelto{0}{u(t)} - \cancelto{0}{\varepsilon \bar{x}_4} + m_p l g \beta \sin\bar{x}_1 \cos\bar{x}_1 - 0$$

$$0 = \sin\bar{x}_1 \cos\bar{x}_1$$

$$= (0) \cos(\bar{x}_1)$$

$$0 = 0.$$

Therefore,

$$\vec{x}_{eq} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{pmatrix} = \begin{pmatrix} n\pi \\ c \\ 0 \\ 0 \end{pmatrix} \quad \text{for } n \in \mathbb{Z}_{\geq 0} \text{ and } c \in \mathbb{R}$$

We now linearize 5 about $\vec{x}_{eq}$ using Taylor series expansion method. Note that,

$$\begin{aligned}
\dot{x}_1 &= f_1(x_3) \\
\dot{x}_2 &= f_2(x_4) \\
\dot{x}_3 &= f_3(x_1, x_3, x_4, u(t)) \\
\dot{x}_4 &= f_4(x_1, x_3, x_4, u(t)).
\end{aligned}$$

So,

$$\delta \dot{\vec{x}} = Df(\vec{x}_{eq})\, \delta\vec{x}$$

where $\delta \vec{x} = \vec{x} - \vec{x}_{eq}$, and $Df(\vec{x}_{eq})$ is the Jacobian of $f(\vec{x}_{eq})$.

Computing the Jacobian

$$Df(\vec{x}_{eq}) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\
\frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4}
\end{pmatrix}$$

yields

COMPUTATIONS

$$\partial \dot{x}_3 \simeq \cancel{f_3(x_{eq})}^{0} + \left.\frac{\partial f_3}{\partial x_1}\right|_{eq}(x_1-\bar{x}_1) + \left.\frac{\partial f_3}{\partial x_3}\right|_{eq}(x_3-\bar{x}_3) + \left.\frac{\partial f_3}{\partial x_4}\right|_{eq}(x_4-\bar{x}_4) + \left.\frac{\partial f_3}{\partial u(t)}\right|_{eq}(u(t)-\bar{u}(t))$$

WOLFRAM ALPHA

$$\left.\frac{\partial f_3}{\partial x_1}\right|_{eq} = \beta\frac{\partial}{\partial x_1}\left[\frac{(m_c+m_p)g\sin x_1 + (u(t)-\varepsilon x_4)\cos x_1 - m_p l x_3^2 \sin x_1 \cos x_1}{m_c + m_p(1-l\beta\cos^2 x_1)}\right]$$

$$= \frac{\beta}{(m_c+m_p(1-l\beta\cos^2\bar{x}_1))}\left((m_c+m_p)g\cos\bar{x}_1 - \cancel{(\bar{u}(t)-\varepsilon\bar{x}_4)}^{0}\cancel{\sin\bar{x}_1}^{0} - m_p l\cancel{\bar{x}_3^2}^{0}(\ \ )\right)$$

$$= \frac{(m_c+m_p)g\beta\cos\bar{x}_1}{m_c+m_p(1-l\beta\cos^2\bar{x}_1)}, \quad \text{but } \cos^2(\bar{x}_1=n\pi) \text{ IS ALWAYS } = 1$$

$$= \frac{(m_c+m_p)g\beta\cos(\bar{x}_1)}{m_c+m_p(1-l\beta)}.$$

$$\left.\frac{\partial f_3}{\partial x_3}\right|_{eq} = \frac{\beta}{m_c+m_p(1-l\beta\cos^2\bar{x}_1)}\left(-2m_p l\cancel{\bar{x}_3}^{0}\sin\bar{x}_1\cos\bar{x}_1\right) = 0$$

$$\left.\frac{\partial f_3}{\partial x_4}\right|_{eq} = \frac{\beta}{m_c+m_p(1-l\beta\cancel{\cos^2(\bar{x}_1)}^{1})}\left(-\varepsilon\cos\bar{x}_1\right) = \frac{-\varepsilon\beta\cos\bar{x}_1}{m_c+m_p(1-l\beta)}$$

$$\left.\frac{\partial f_3}{\partial u(t)}\right|_{eq} = \frac{\beta}{m_c+m_p(1-l\beta\cancel{\cos^2(\bar{x}_1)}^{1})}\left(\cos(\bar{x}_1)\right) = \frac{\beta\cos(\bar{x}_1)}{m_c+m_p(1-l\beta)}$$

$$\partial \dot{x}_4 \simeq \cancelto{0}{f_4(x_{eq})} + \left.\frac{\partial f_4}{\partial x_1}\right|_{eq}(x_1-\bar{x}_1) + \left.\frac{\partial f_4}{\partial x_3}\right|_{eq}(x_3-\bar{x}_3) + \left.\frac{\partial f_4}{\partial x_4}\right|_{eq}(x_4-\bar{x}_4) + \left.\frac{\partial f_4}{\partial u(t)}\right|_{eq}(u(t)-\bar{u}(t))$$

WOLFRAM ALPHA

$$\left.\frac{\partial f_4}{\partial x_1}\right|_{eq} = \frac{\left(m_p g l \beta \cancelto{1}{\cos^2 \bar{x}_1} - m_p l \cancelto{0}{\bar{x}_3^2} \sin(\bar{x}_1)\right)}{m_c + m_p(1 - l\beta \cancelto{1}{\cos^2 \bar{x}_1})} = \frac{m_p g l \beta}{m_c + m_p(1 - l\beta)}$$

$$\left.\frac{\partial f_4}{\partial x_3}\right|_{eq} = \frac{1}{\sim}\left(-2m_p l \cancelto{0}{\bar{x}_3} \sin(\bar{x}_1)\right) = 0$$

$$\left.\frac{\partial f_4}{\partial x_4}\right|_{eq} = \frac{1}{m_c + m_p(1 - l\beta \cancelto{1}{\cos^2 \bar{x}_1})}(-\varepsilon) = \frac{-\varepsilon}{m_c + m_p(1 - l\beta)}$$

$$\left.\frac{\partial f_4}{\partial u(t)}\right|_{eq} = \frac{1}{m_c + m_p(1 - l\beta \cancelto{1}{\cos^2(\bar{x}_1)})} = \frac{1}{m_c + m_p(1 - l\beta)}$$

$$A \equiv Df(\vec{x}_{eq}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{g\beta(m_c+m_p)\cos(\bar{x}_1)}{m_c+m_p(1-l\beta)} & 0 & 0 & \frac{-\varepsilon\beta\cos(\bar{x}_1)}{m_c+m_p(1-l\beta)} \\ \frac{m_p g l\beta}{m_c+m_p(1-l\beta)} & 0 & 0 & \frac{-\varepsilon}{m_c+m_p(1-l\beta)} \end{pmatrix}$$

and,

$$B \equiv \begin{pmatrix} 0 \\ 0 \\ \frac{\beta\cos(\bar{x}_1)}{m_c+m_p(1-l\beta)} \\ \frac{1}{m_c+m_p(1-l\beta)} \end{pmatrix}, \quad C \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D \equiv 0.$$

THEREFORE OUR STATE-SPACE MODEL IS

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$
$$\vec{y} = C\vec{x} \qquad (6)$$

(B)

INPUT FORCE

and FRICTION

AT 12 V, I'M MEASURING MAX CURRENT draw AROUND ~ 7A WITH SPIKES NORTH of 10A AROUND STALL.

V (V)	I (A)	
12	1.27	
2	1.2	
12	~7A – ~15A	NEAR STALLING CONDITIONS

INPUT FORCE

Let $u(t) = \alpha V(t) - \gamma \dot{x}$. FROM (McGILVRAY 2002), this can be MODELED AS

$$u(t) = \alpha V(t) - \gamma \dot{x}$$

$$= \frac{K_m K_g}{r R_A} V(t) - \frac{(K_m K_g)^2}{r^2 R_A} \dot{x}.$$

WHERE:

K_m : MOTOR TORQUE CONST.

K_g : gEAR RATIO of gEAR head

r : radius of OUTPUT gEAR

R_A : resistance of MOTOR

1. MEASURING R_a

   a. APPLY VOLTAGE TO MOTOR

   b. MEASURE CURRENT DRAW

   c. $R_a = \frac{V}{i}$

   d. REPEAT & AVERAGE

② Measuring K_m

The eq for K_m is defined as

$$K_m = \frac{T}{i} = \frac{60}{2\pi \cdot K_v} \quad \text{(wikipedia: Motor constants).}$$

Where K_v = motor velocity constant. But T is a little difficult to measure, so let's find K_v. K_v is defined as

$$K_v = \frac{\omega_{no\text{-}load}}{V}.$$

Technically V is supposed to be the V_{peak} not V_{RMS}. I think this only applies to AC voltages, so I'm just going to use the voltage supplied to the motor.

Two ways to do this:

① speed - torque curve:

- ⓐ Measure no load RPM, $(T=0)$
- ⓑ Measure stall torque, $(\omega=0)$
- ⓒ Measure current for ⓐ & ⓑ
- ⓓ plot speed vs. torque using ⓐ and ⓑ.
- ⓔ plot current vs. torque
- ⓕ The inverse of the slope of ⓔ is K_m

② From K_v:

- ⓐ Measure $\omega_{no\text{-}load}$
- ⓑ $K_v = \frac{\omega_{NL}}{V_{IN}}$, $K_m = \frac{60}{2\pi \cdot K_v}$

3 MEASURING K_g

- WE DON'T HAVE MULTIPLE GEARS SO OUR $K_g = 1$

THEREFORE,

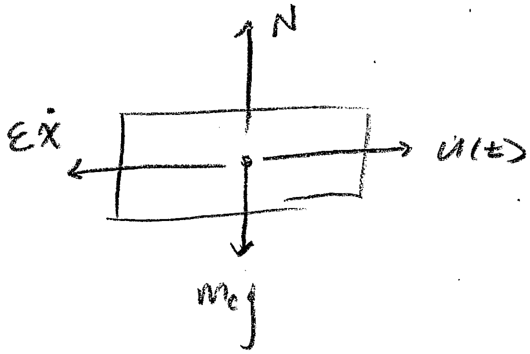
$$u(t) = \alpha V(t) - \gamma \dot{x}$$

$$\alpha = \frac{K_m K_g}{r R_A}$$

$$\gamma = \frac{(K_m K_g)^2}{r^2 R_A}$$

Determining friction

* At first I was going to just measure ε by moving cart w/o pendulum across track, but pendulum will introduce more coulomb friction & I hope to capture that in ε by including (m_c+m_p) in Newton's 2nd law



$$\varepsilon\dot{x} = [N]$$

$$\varepsilon\,[m/s] = [kg \cdot m/s^2]$$

$$\varepsilon = [kg/s]$$

Including m_p will allow us to find ε that encompasses coulomb friction.

$$u(t) - \varepsilon\dot{x} = (m_c+m_p)\ddot{x}$$

$$\alpha V(t) - \gamma\dot{x} - \varepsilon\dot{x} = (m_c+m_p)\ddot{x}$$

$$\ddot{x} + \frac{1}{m_c+m_p}(\varepsilon+\gamma)\dot{x} = \frac{\alpha}{m_c+m_p}V(t)$$

Assume $V(t) = V =$ const, and let $x(0) = \dot{x}(0) = 0$. Taking the laplace transform yields

$$s^2X(s) - s\cancel{x(0)}^{\,0} - \cancel{\dot{x}(0)}^{\,0} + a(sX(s) - \cancel{x(0)}^{\,0}) = \frac{b}{s}$$

$$(s^2+as)X(s) = \frac{b}{s}$$

$$X(s) = \frac{b}{s(s^2+as)}$$

$$= \frac{b}{s^2(s+a)}$$

* Doing it this way is easier than using the complicated $\ddot{x}$ EOM, and this should be good enough.

$$\left[\frac{b}{s^2(s+a)} = \frac{y_1}{s} + \frac{y_2}{s^2} + \frac{y_3}{s+a}\right] \cdot s^2(s+a)$$

$$b = y_1 s(s+a) + y_2(s+a) + y_3 s^2$$

$$= y_1 s^2 + y_1 sa + y_2 s + y_2 a + y_3 s^2$$

$$= (y_1 + y_3)s^2 + (ay_1 + y_2)s + ay_2$$

equating coeffs yields

$$b = a y_2 \quad \Rightarrow \quad y_2 = \frac{b}{a}$$

$$0 = ay_1 + y_2 \quad \Rightarrow \quad y_1 = \frac{-y_2}{a} = -b/a^2$$

$$0 = y_1 + y_3 \quad \Rightarrow \quad y_3 = -y_1 = b/a^2$$

SO,

$$X(s) = \frac{b}{s^2(s+a)} = \frac{b/a}{s^2} + \frac{b/a^2}{s+a} - \frac{b/a^2}{s}$$

WITH $b = \frac{1}{m_c+m_p}\alpha V$ and $a = \frac{1}{m_c+m_p}(\varepsilon + \gamma)$.

$$\boxed{\begin{aligned} \therefore\ x(t) &= \frac{b}{a}t + \frac{b}{a^2}e^{-at} - \frac{b}{a^2} \\ \dot{x}(t) &= \frac{b}{a}(1 - e^{-at}). \end{aligned}}$$

How we can estimate ε:

It would prob be easier to just use the $\dot{x}$ equation, but I don't have a reliable way to measure velocity, so I'm going to use the position eq.

We know the length of the track $(X(t_f))$, and we can measure how much time (t) it takes for the cart to get to the end track at the applied constant voltage V.

Let $X(t_f) =$ cm

Then,

Trial	Voltage (V)	time (s)
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

We will then numerically approximate ε in the eq

$$X(t) = \frac{b}{a^2}\left(e^{-at} + at - 1\right)$$

$$= \frac{\frac{1}{m_c+m_p}(\alpha V)}{\left(\frac{1}{m_c+m_p}\right)^2(\varepsilon+\gamma)^2}\left(e^{-\frac{\varepsilon+\gamma}{m_c+m_p}(t)} + \left(\frac{\varepsilon+\gamma}{m_c+m_p}\right)t - 1\right)$$

$$\boxed{X(t) = \frac{\alpha V(m_c+m_p)}{(\varepsilon+\gamma)^2}\left(e^{-\left(\frac{\varepsilon+\gamma}{m_c+m_p}\right)t} + \left(\frac{\varepsilon+\gamma}{m_c+m_p}\right)t - 1\right)}$$

* No load

Using Audacity to get Δt of 1 cycle, it's pretty consistent that

$$RPM = \frac{60\,s}{\Delta t_{cycle}} = \frac{60\,s}{0.005\,s} = 12{,}000$$

Measurements

V (v)	i (A)	Ra
20.4	1.51	
↓	1.54	
	1.87	
	1.53	
	1.561	
	1.54	
	1.475	

1.45 to 1.65 A ⇒ ~~14.06 to 12.36 Ω~~

~~$Ra \simeq 13\,\Omega$~~

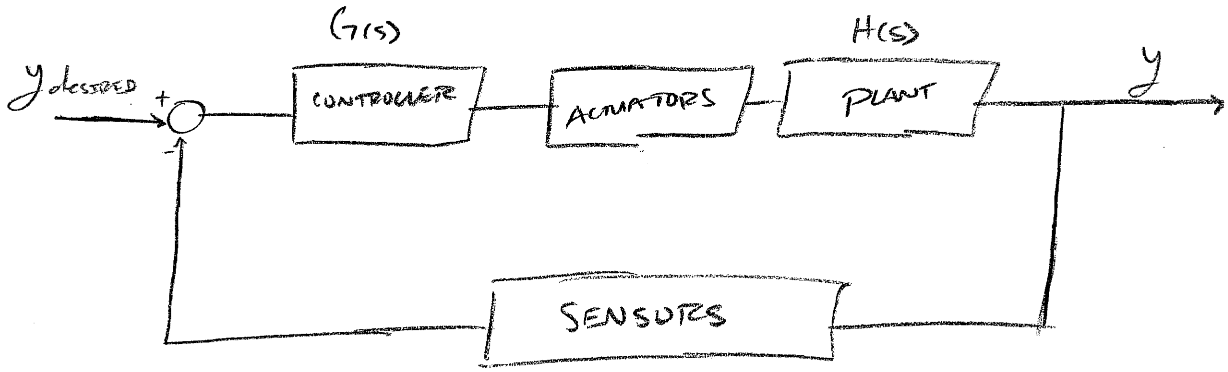
WRONG!

In order to find Ra, I supplied ~10% of rated voltage, locked the motor, and found current. I supplied 2 V but after jolting the motor it only was drawing ~0.340 V, the current was ~1.70 A, so

$$R_a = \frac{V}{I} \simeq 0.2\,\Omega.$$

The multimeter probe resistance is ~0.1 Ω, but I'm just going to use 0.2 Ω.

NOTES ON TRANSFER FUNCTIONS



$$T_{OL} = G(s)\,H(s)$$

$$T_{CL} = \frac{G(s)\,H(s)}{1 + G(s)\,H(s)}$$

WHEN TO USE WHAT :

- T_{CL} → USE to ANALYZE STABILITY (eg. ROUTH CRITERIA)
- T_{OL} → ① ANALYZE STEADY STATE error
  ② USE TO START the ROOT locus
  ③ BODE PLOT
  ④ Nyquist diagram

YOU CAN STILL USE T_{CL} TO DRAW the ROOT locus!

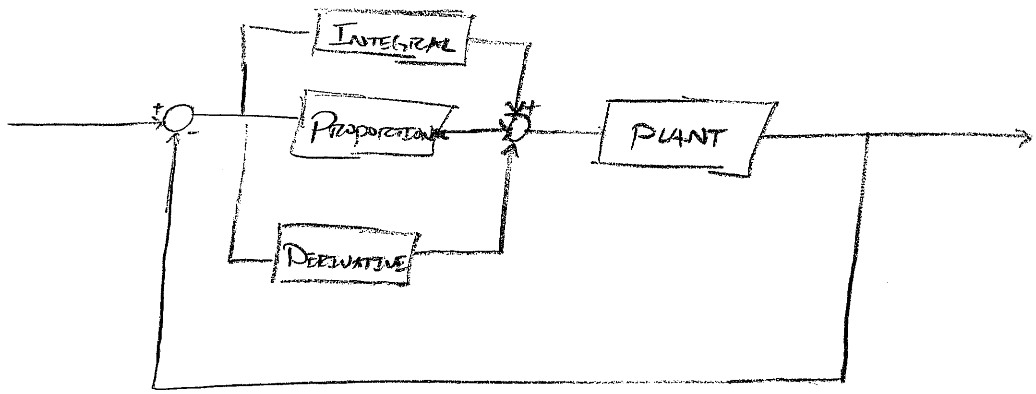
* Use T_{OL} to analyze & design CONTROLLERS for the CLOSED-loop SYSTEM

* CHARACTERISTIC EQ Always refers TO the DENOMINATOR of T_{CL}

C

PID TUNING

PID CONTROL



→ THE IDEA:

(a) PROPORTIONAL K_P, reduces steady state error

(b) INTEGRAL K_I, "remembers the past", reduces SS error

(c) DERIVATIVE K_D, "predicts the future", improves transient response.

(a) → Multiplies error by K_P. the output (eg. Voltage) to the plant scales proportionally by K_P. More error → more voltage

(b) → Integrates error. By integrating error, we can sum the error over time Δt and if error continues to persist we send higher and higher voltages.

(c) → Derivative of error. The derivative of the error lets us know how fast we're closing in on our position goal, allowing us to slow down & prevent overshoot (ie. Transient response).

* PID isn't perfect. We can have integrator wind-up, and high frequency noise can impact the derivative path a lot.

Summary of PID coeff. effects

	Rise Time	Overshoot	Settling Time	SS error	Stability
Increasing K_P	$\downarrow$	$\uparrow$	~$\uparrow$	$\downarrow$	degrades
Increasing K_I	~$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$!	degrades
Increasing K_D	~$\downarrow$	$\downarrow$	$\downarrow$	~0	Improves

~ : small change

! : big change

PID Controller Design

① Obtain a mathematical model

→ This can be done by doing physics or by fitting a model to data collected from your system

② Tune your PID controller to the system

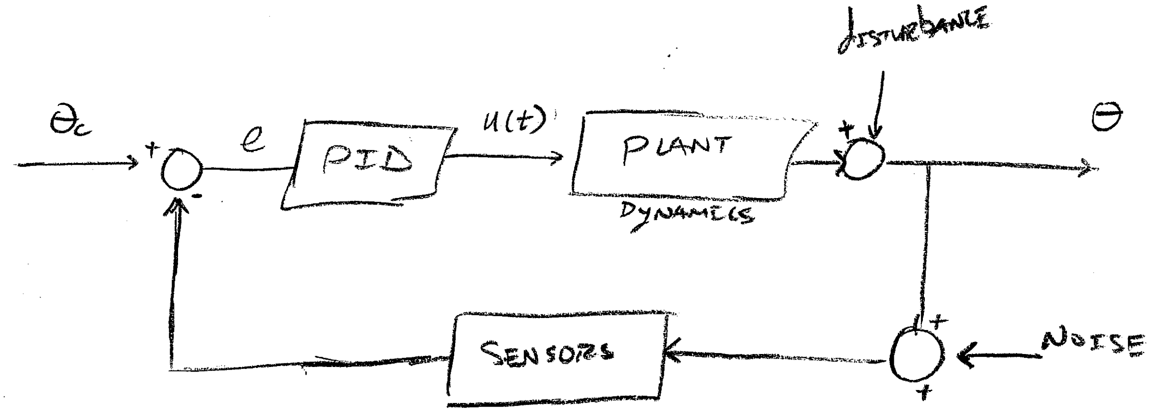
ⓐ Manual tuning → pole placement (root locus)
loop shaping (Bode plot : phase margin)

* This is done using the open-loop system

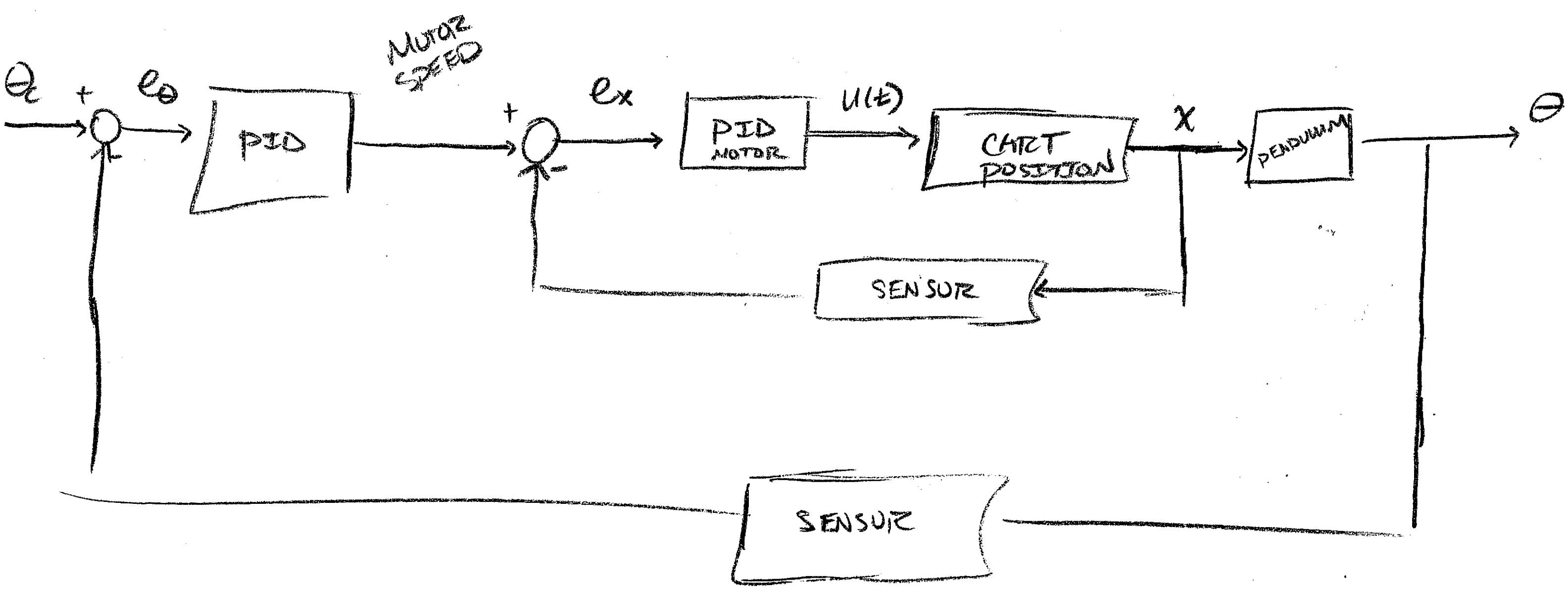
ⓑ Automatic Tuning

* For non-linear systems, look into Gain scheduling, LQR, H_∞, etc., look at wiki for non-linear control.

PID block diagram



if we add a PID controller to both the cart position & the pendulum angle, we get something like



We can analyze this system using MIMO pole placement to find PID values for each open-loop system that will help stabalize the closed loop system.

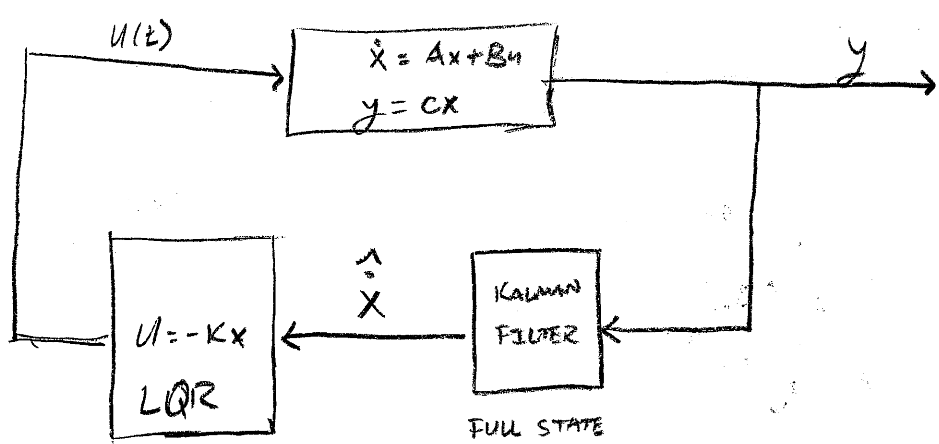
My system

① There's two ways to do this:
ⓐ 2 PID controllers
ⓑ Full state feedback
↳ LQR
↳ pole placement

ⓐ 2 PID controllers : See previous page

ⓑ Full state feedback: (LQR & pole placement)

→ Since we don't have sensors to measure $\dot{x}$ & $\dot{\theta}$, we can use a Kalman filter to estimate these states. This will be our observer.



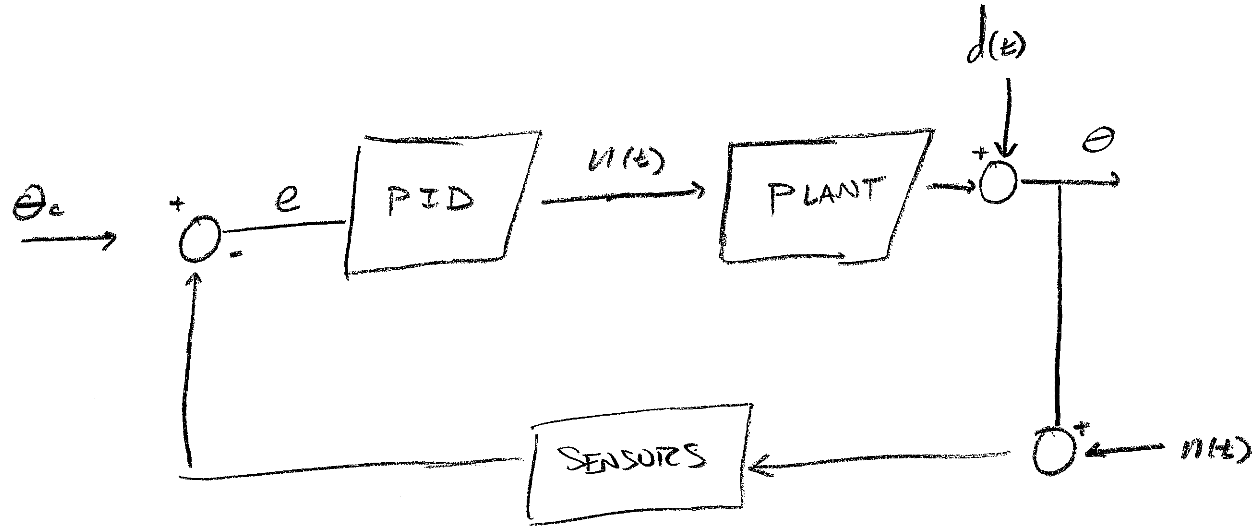
Full state estimator

★ Full-state feedback (eg. LQR & MIMO pole placement) requires ALL states. Since I don't have sensors to measure velocity, this requires a state estimator (ie. full-observer) like a Kalman filter

Applying PID control

→ Since full-state feedback is more involved (read: Kalman filter), let's do multi-loop PID control so we can get a working project. Then we will try to implement LQR.

Ultimately, here's what we want



We probably could implement this for just θ and only have one PID controller, but we also want to control the cart's position, so let's use cascade control. We need to choose the inner & outer loop.

★ Rule of thumb: Inner loop should be ~ 3 to 5x faster than outer loop.

My first intuition was to choose θ as outer loop & X as inner-loop, but this doesn't seem to be consistent w/ examples I've seen online. After some thought, I think it makes sense.

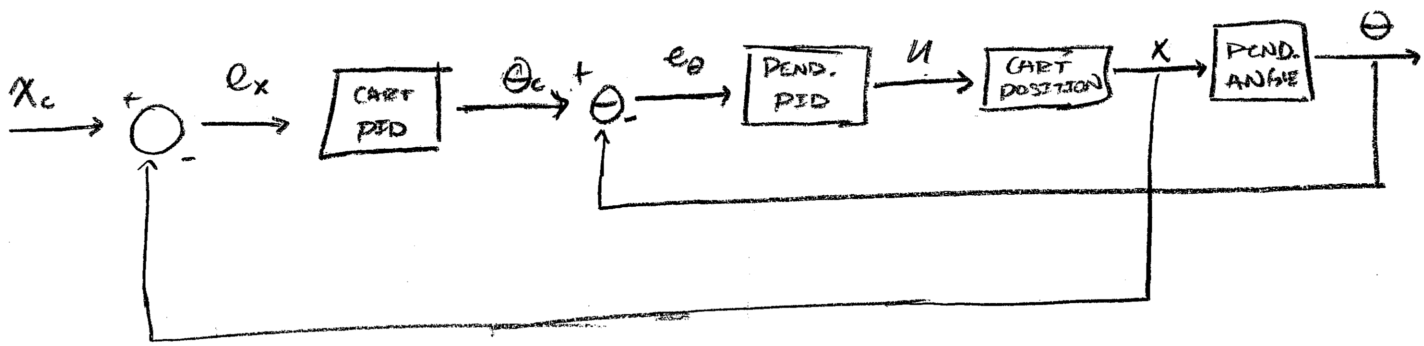
<u>Case 1:</u> θ outer loop, X inner loop

→ This means we're driving X closer to the set point faster than we're driving θ to its set point. This shouldn't happen, we should <u>first</u> drive theta to its setpoint then drive X to the set point. Plus, I believe it is faster to drive θ to its set point since driving X to its set point requires θ to be at a set point then we can drive X to a set point.

<u>Case 2:</u> X outer loop, θ inner loop

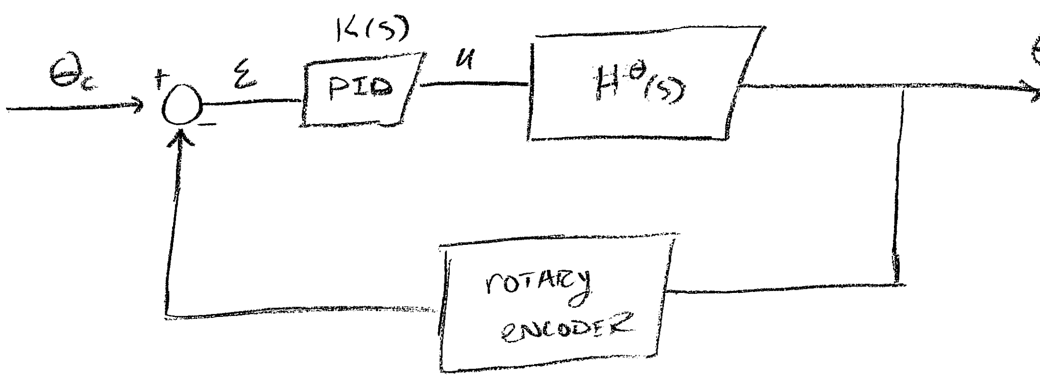
→ This means we're constantly moving X to drive θ to its setpoint, then this means X is at some distance away from the setpoint so the outer loop drives X to the set-point. However, while X is being driven to its setpoint, the inner loop is constantly correcting θ.

THIS SHOULD give US A CONTROL-block diagram that looks like:



* Is this right?

~~SCRATCH THIS~~ Let's JUST WORK ON TUNING the PID CONTROLLER for θ.



CONTROL LAW: $u(t) = K_P \varepsilon + K_I \int_0^t \varepsilon \, dt + K_D \frac{d\varepsilon}{dt}$

$$U(s) = K_P E(s) + \tfrac{1}{s} K_I E(s) + K_D s E(s)$$

$$\left(K_P + \tfrac{1}{s} K_I + s K_D\right) E(s)$$

Plugging in some sample values like

$$M_c = 0.320 \text{ kg} \qquad I = \tfrac{1}{3} M_P L_{TOT}^2$$

$$M_P = 0.25 \text{ kg} \qquad \varepsilon = 0.5 \text{ kg/s}$$

$$L_{TOT} = 0.3 \text{ m}$$

$$L_{CM} = \tfrac{1}{2} L_{TOT}$$

we get

$$H_{OL}^{\theta}(s) = \frac{3.086s}{s^3 + 1.08s^2 - 17.25s - 15.13}$$

and,

$$H_{CL}^{\theta}(s) = \frac{K(s)\,H_{OL}^{\theta}(s)}{1 + K(s)\,H_{OL}(s)} = \frac{K(s)\,\frac{N(s)}{D(s)}}{1 + K(s)\,\frac{N(s)}{D(s)}} = \frac{K(s)N(s)}{D(s) + K(s)N(s)}$$

$$= \frac{K(s)\;(3.086)s}{(s^3 + 1.08s^2 - 17.25s - 15.13) + K(s)\,(3.086)s}$$

Using $K(s) = \left(K_p + \frac{1}{s}K_I + K_d\left(\frac{sN}{s+N}\right)\right)$ where $\frac{N}{s+N}$ is a low pass filter.

THE CHARACTERISTIC EQ for the system IS

$$\Delta^{\theta} = (s^3 + as^2 + bs + c) + \left(K_p + \tfrac{1}{s}K_I + K_d \tfrac{sN}{s+N}\right) ds$$

$$= s^3 + \underline{as^2} + bs + c + dK_p s + dK_I + \underline{dK_d \tfrac{s^2N}{s+N}}$$

$$= s^3 + \left(a + dK_d \tfrac{N}{s+N}\right)s^2 + (b + dK_p)s + (c + dK_I)$$

WHERE,

$a = 1.08$ $\quad c = -15.13$

$b = -17.25$ $\quad d = 3.086$

THE OPEN loop POLES ARE located AT

$$0 = s^3 + as^2 + bs + c$$

these ARE ALSO the eigenVALUES of A.

WOLFRAM ALPHA $\Rightarrow$

$s_1 \simeq -4.2828$

$s_2 \simeq -0.8674$

$s_3 \simeq 4.0707$ $\longleftarrow$ SYSTEM IS UNSTABLE.

Let's say we wish to have % OS $\leq 5\%$ and $T_s \leq \frac{1}{2}$ s.

This implies

$$T_s \doteq \frac{4}{\zeta\omega_n} \leq \beta$$

$$\therefore \zeta\omega_n \geq \frac{4}{\beta}$$

$$PO = 100\, e^{-\zeta\pi(1-\zeta^2)^{-1/2}} \leq \gamma$$

$$-\zeta\pi(1-\zeta^2)^{-1/2} \leq \ln\left(\frac{\gamma}{100}\right)$$

$$\zeta\pi(1-\zeta^2)^{-1/2} \geq -\ln\left(\frac{\gamma}{100}\right) = \ln\left(\frac{100}{\gamma}\right)$$

$$\zeta\pi \geq \ln\left(\frac{100}{\gamma}\right)(1-\zeta^2)^{1/2}$$

$$\zeta^2\pi^2 \geq \ln\left(\frac{100}{\gamma}\right)^2(1-\zeta^2)$$

$$\zeta^2\pi^2 \geq \ln\left(\frac{100}{\gamma}\right)^2 - \ln\left(\frac{100}{\gamma}\right)^2\zeta^2$$

$$\left(\pi^2 + \ln\left(\frac{100}{\gamma}\right)^2\right)\zeta^2 \geq \ln\left(\frac{100}{\gamma}\right)^2$$

$$\therefore \zeta \geq \ln\left(\frac{100}{\gamma}\right)\left[\pi^2 + \ln\left(\frac{100}{\gamma}\right)^2\right]^{-1/2}$$

For $\gamma = 5$ and $\beta = \frac{1}{2}$, we get

$$\zeta \geq \sim 0.69$$

$$\omega_n \geq \sim 11.59.$$

* We now have an issue because these values only have meaning for 2^{nd} order systems and we're dealing w/ a 3^{rd} order system. So instead of thinking about specifying ζ and ω_n, we need to think in terms of where to place the poles and zeros to achieve the desired response.

Notes on Root Locus & PID Design

The workflow for root locus design in Matlab goes like this:

1. Define OL transfer function
2. Plot the poles & zeros in the s-domain
3. Look at how changing gain effects the pole placement
4. Look at how new pole placement effects closed-loop stability

There are several different ways to do each step:

1.
   a. `s = tf('s')`. Then $G = \frac{5s}{4+s}$, etc.
   b. Define numerator & denominator coeffs:

      `n = [1]`
      `d = [1 5 6]` → `G = tf(n,d)` → $G = \frac{1}{s^2+5s+6}$

   c. Define using poles, zeros, and gain.

      `z = [ ]`
      `p = [0, -4, -2+4i, -2-4i]` → `G = zpk(z, p, k)`
      `k = 1`

(2)

(a) pzmap(G) — shows poles & zeros of transfer function

→ doesn't show effects of gain!

(3)

(a)

```
for K = 1:10
    pzmap(feedback(G.K, 1))
end
```

— Plots closed-loop poles ~~poles~~ for varying gain

→ This is difficult!

(b) rlocus(G) — easy! shows effects of gain on closed-loop poles.

(4)

(a) Use new closed loop poles from (3) to analyze step response.

tf CL = tf(~)

step(tf CL).

★ All these steps are tedious!!

Practical Approach

1. Use MATLAB Control System Designer App
2. Open root locus and step plots

★ This is for manual tuning a PID.

Here is my current plan of attack now that I have a little more insight on how to implement this:

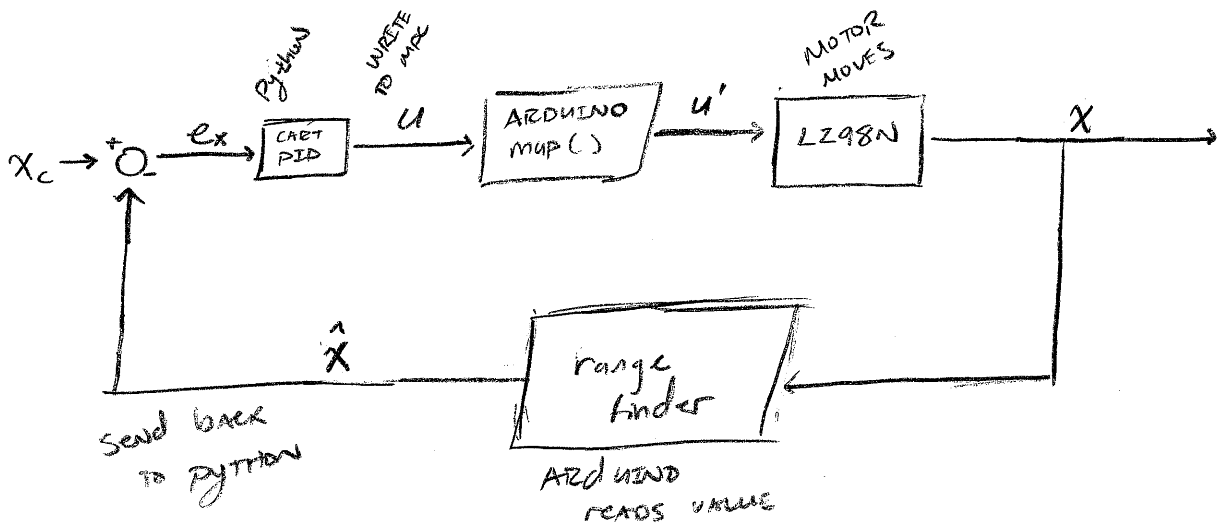
1. Play around with Simulink and learn about controlling & tuning PIDs

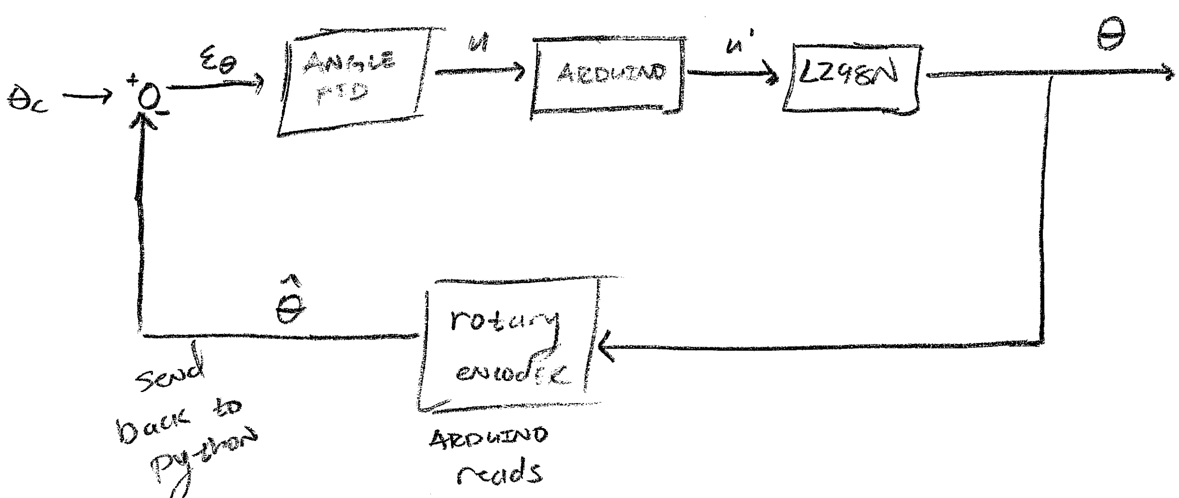
2. Once I feel comfortable doing this, move to Python and write a PID controller that takes inputs: reference, K_p, K_I, K_d and will calculate the output. We can tune this using a GA.

3. Control Arduino via Python by having Python write to the output to the Arduino. Look into how to make this fast & efficient.

4. If possible, write all this code on the Arduino MPU itself for optimal speed.

5. Move from PID control to LQR & LQG. This should honestly be easier. LQG will require a Kalman filter though which will be difficult.





* Technically, the output of a PID can be any physical value. The output is just an electrical signal $O(t)$, and by tuning the gains of the PID, you're making sure $O(t)$ is within the range of your desired command value. For example, the command value for my motor is voltage, so how does the PID know how to convert angle error to voltage? It doesn't. By tuning the gains you're making sure the PID output is in the correct voltage range that gives the desired response.

After learning that previous fact, we can definitely choose either cascaded PID or break it into two PIDs.

→ If choosing cascaded, since the θ loop affects the X_{cart} loop, we need to first tune the inner loop then tune the outer loop.

Python Simulation

Psuedocode:

```
ICs = ( ⋮ )
tspan = ( )

Integrate( non-linear-model, tspan, ICs ) = t, states

for i in len(tspan):
    state = states(i, :)
      x = state(1)
      θ = state(0)

      square w/ center at x
      rod w/ one end at x and other end at (Lcosθ, Lsinθ)
end
```

Redoing Model Computation using EOMs from paper

$$M_{TOTAL}\ddot{x} + \varepsilon\dot{x} + m_p l\ddot{\theta}\cos\theta - m_p l\dot{\theta}^2\sin\theta = u(t) \qquad \text{—— (1)}$$

$$m_p l\ddot{x}\cos\theta + (m_p l^2 + I_c^p)\ddot{\theta} - m_p l g\sin\theta = 0 \qquad \text{—— (2)}$$

Solving for $\ddot{\theta}$ in (2) gives

$$\ddot{\theta} = m_p l\,(m_p l^2 + I_c^p)^{-1}(g\sin\theta - \ddot{x}\cos\theta) \qquad \text{—— (3)}$$

Plugging into (1) yields and letting $\beta = m_p l(m_p l^2 + I_c^p)^{-1}$ gives

$$m_T\ddot{x} + \varepsilon\dot{x} - m_p l\dot{\theta}^2\sin\theta + m_p l\cos\theta \cdot \beta(g\sin\theta - \ddot{x}\cos\theta) = u(t)$$

$$m_T\ddot{x} + \varepsilon\dot{x} - m_p l\dot{\theta}^2\sin\theta + \beta g m_p l\sin\theta\cos\theta - \beta m_p l\ddot{x}\cos^2\theta = u(t)$$

$$m_T\ddot{x} - \beta m_p l\ddot{x}\cos^2\theta = u(t) - \varepsilon\dot{x} + m_p l\dot{\theta}^2\sin\theta - \beta g m_p l\sin\theta\cos\theta$$

$$(m_T - \beta m_p l\cos^2\theta)\ddot{x} = u(t) - \varepsilon\dot{x} + m_p l\sin\theta(\dot{\theta}^2 - g\beta\cos\theta)$$

$$\therefore\ \ddot{x} = (m_T - \beta m_p l\cos^2\theta)^{-1}\left[u(t) - \varepsilon\dot{x} + m_p l\sin\theta(\dot{\theta}^2 - g\beta\cos\theta)\right] \qquad \text{—— (4)}$$

Plugging 4 into 3 gives

$$\ddot{\theta} = \beta\left[g\sin\theta - \cos\theta \cdot (m_T - \beta m_p l\cos^2\theta)^{-1}\left(u(t) - \varepsilon\dot{x} + m_p l\sin\theta(\dot{\theta}^2 - g\beta\cos\theta)\right)\right]$$

$$= \frac{\beta}{m_T - \beta m_p l\cos^2\theta}\left[g\sin\theta(m_T - \beta m_p l\cos^2\theta) - \cos\theta\cdot\left(u(t) - \varepsilon\dot{x} + m_p l\dot{\theta}^2\sin\theta - m_p l g\beta\sin\theta\cos\theta\right)\right]$$

$$= \gamma\left[m_T g\sin\theta - \cancel{\beta m_p g l\sin\theta\cos^2\theta}\right.$$
$$\left. - u(t)\cos\theta + \varepsilon\dot{x}\cos\theta - m_p l\dot{\theta}^2\sin\theta\cos\theta + \cancel{\beta m_p g l\sin\theta\cos^2\theta}\right]$$

$$\therefore \ddot{\theta} = \gamma\left[m_T g\sin\theta + (\varepsilon\dot{x} - u(t))\cos\theta - m_p l\dot{\theta}^2\sin\theta\cos\theta\right] \quad \text{— (5).}$$

Thus, we have

$$\boxed{\begin{aligned}
\ddot{x} &= \frac{1}{d}\left(u(t) - \varepsilon\dot{x} + m_p l\sin\theta(\dot{\theta}^2 - g\beta\cos\theta)\right) \\
\ddot{\theta} &= \frac{\beta}{d}\left(M_T g\sin\theta + (\varepsilon\dot{x} - u(t))\cos\theta - m_p l\dot{\theta}^2\sin\theta\cos\theta\right) \\
&\text{with,} \\
\beta &= \frac{m_p l}{m_p l^2 + I_c^p} \\
d &= m_T - \beta m_p l\cos^2\theta \\
&= (m_c + m_p) - \beta m_p l\cos^2\theta \\
&= m_c + m_p(1 - \beta l\cos^2\theta).
\end{aligned}}$$

We convert this to a system of first order DEs by letting

$$X_1 = \theta \qquad X_3 = \dot{\theta}$$

$$X_2 = x \qquad X_4 = \dot{x}$$

then,

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = \dfrac{\beta}{m_T - \beta m_p l \cos^2 x_1}\left[m_T g \sin x_1 + (\varepsilon x_4 - u(t))\cos x_1 - m_p l x_3^2 \sin x_1 \cos x_1\right] \\ \dot{x}_4 = \dfrac{1}{m_T - \beta m_p l \cos^2 x_1}\left[u(t) - \varepsilon x_4 + m_p l \sin x_1 (x_3^2 - g\beta \cos x_1)\right] \end{cases}$$

ie. $\dot{\vec{x}} = f(\vec{x})$. Now we solve for the eq. points by setting

$0 = f(\vec{x})$, so

$$\dot{x}_1 = 0 = \bar{x}_3 \implies \bar{x}_3 = 0 \implies \bar{x}_1 = \theta = \text{const.}$$

$$\dot{x}_2 = 0 = \bar{x}_4 \implies \bar{x}_4 = 0 \implies \bar{x}_2 = x = \text{const.}$$

$$\dot{x}_3 = 0 = m_T g \sin(\bar{x}_1) - \overset{0}{\cancel{u(t)}} \cos(\bar{x}_1)$$

$$\implies \bar{x}_1 = \arcsin(0) = n\pi, \quad n \in \mathbb{Z}_{\geq 0}$$

$$\dot{x}_4 = 0 = 0$$

$$\therefore \vec{x}_{eq} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{pmatrix} = \begin{pmatrix} n\pi \\ c \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \quad \begin{matrix} n \in \mathbb{Z}_{\geq 0} \\ c \in \mathbb{R} \end{matrix}.$$

Refer to pg. 5. Computing the Jacobian yields

$$Df(\vec{x}_{eq}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}.$$

Note that the only thing that changed between systems is

	OLD		New
$\ddot{\theta}$:	$(u(t) - \varepsilon \dot{x})$	$\longrightarrow$	$(\varepsilon \dot{x} - u(t))$
$\ddot{x}$:	$(g\beta\cos\theta - \dot{\theta}^2)$	$\longrightarrow$	$(\dot{\theta}^2 - g\beta\cos\theta)$.

The only elements of $Df(\vec{x}_{eq})$ that changes are ... (I'm about to check).

Using wolfram,

$$\left.\frac{\partial f_3}{\partial x_1}\right|_{\vec{x}_{eq}} = \beta\left(\frac{m_T g\cos(\bar{x}_1)}{m_T - \beta m_p l \cancel{\cos^2 \bar{x}_1}}\right) = \frac{\beta m_T g \cos(\bar{x}_1)}{m_T - \beta m_p l} \checkmark$$

$$\frac{\partial f_3}{\partial x_2}, \frac{\partial f_3}{\partial x_3} \checkmark$$

$$\left.\frac{\partial f_3}{\partial x_4}\right|_{\vec{x}_{eq}} = \frac{\beta}{d(\bar{x}_1)}\left(\varepsilon\cos(\bar{x}_1)\right) = \frac{\beta\varepsilon\cos(\bar{x}_1)}{m_T - \beta m_p l}$$

$$\left.\frac{\partial f_3}{\partial u}\right|_{eq} = \frac{-\beta\cos(\bar{x}_1)}{m_T - \beta m_p l}$$

and, $a = b = d = 0$

$$\left.\frac{\partial f_4}{\partial x_1}\right|_{eq} = \frac{-m_p l g \beta \cancel{\cos^2(\bar{x}_1)}}{m_T - \beta M_p l} = \frac{-m_p l g \beta}{M_T - \beta M_p l} = \frac{-g}{\frac{M_c + M_p}{\beta M_p l} - 1}$$

$$\left.\frac{\partial f_4}{\partial x_2}\right|_{eq} \checkmark, \quad \left.\frac{\partial f_4}{\partial x_3}\right|_{eq} \checkmark, \quad \left.\frac{\partial f_4}{\partial u}\right|_{eq} \checkmark$$

$$\left.\frac{\partial f_4}{\partial x_4}\right|_{eq} = \frac{-\varepsilon}{d(\bar{x}_1)} = \frac{-\varepsilon}{M_T - \beta M_p l} \checkmark$$

* Coefficients that changed in state space model:

A_{41}, A_{34}, B_{31}

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\beta M_T g \cos(\bar{x}_1)}{M_T - \beta M_p l} & 0 & 0 & \frac{\beta \varepsilon \cos(\bar{x}_1)}{M_T - \beta M_p l} \\ -\frac{M_p l g \beta}{M_T - \beta M_p l} & 0 & 0 & \frac{-\varepsilon}{M_T - \beta M_p l} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ \frac{-\beta \cos(\bar{x}_1)}{M_T - \beta M_p l} \\ \frac{1}{M_T - \beta M_p l} \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$D = 0$$

Python PID control

ε_{i-1} =

θ_{i-1} =

ICs = [0,0,0,0]

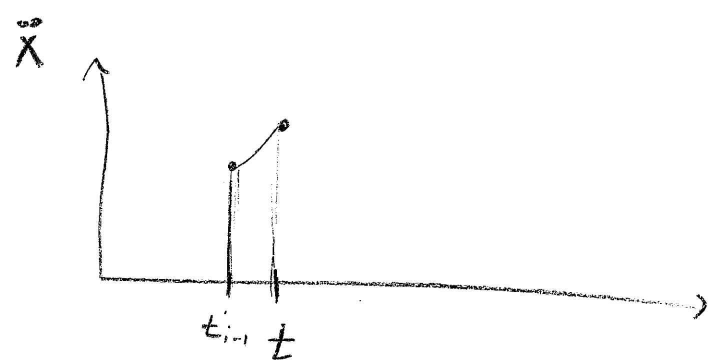
for t in $\bar{T}$:

Calculate $\theta, x, \dot{\theta}, \dot{x}$

error= $\theta_i - \theta_{i-1}$

Voltage = PID($K_p, K_i, K_d, N, \varepsilon_i, \varepsilon_{i-1}$)

system. control_law = Voltage



LQR CONTROL

→ OK SCRATCH PID. REASON'S why we're going w/ LQR:

1. LQR WILL WORK better (MORE Robust & better TRACKING)
2. I CAN ONLY CHOOSE ONE (ASSUMINGLY)
3. LQR WILL EASILY EXTEND TO LQG, WHICH WILL be really good EXPERIENCE IF I IMPLEMENT A KALMAN filter.
4. SINCE LQR MINIMIZES CONTROL Effort, this will be good CONSIDERING I Bought A PRETTY beefy MOTOR

MATH

1. CHOOSE Q, R
2. SOLVE RICCATI EQ for P

$$PA + A^T P + Q - PBR^{-1}B^T P = 0$$

3. Compute feedback gain MATRIX

$$K = R^{-1} B^T P$$

4. SIMULATE the CLOSED loop RESPONSE

$$\dot{x} = (A - BK)x = (A - BR^{-1}B^T P)x$$

→ TO DETERMINE HOW good the RESPONSE IS, LOOK AT SINGULAR VALUES

1 CHOOSING Q, R

Let $q = (q_1 \; q_2 \; q_3 \; q_4)$, then $Q = q I$.

$$q_1 = \frac{1}{T_{s_1} x_{1_{MAX}}^2} = \left(0.3s \cdot \left(\frac{\pi}{4}\right)^2\right)^{-1}$$

$$q_2 = \frac{1}{T_{s_2} x_{2_{MAX}}^2} = \left(5s \cdot (0.19\,m)^2\right)^{-1}$$

$$q_3 = q_4 = 1.$$

SIMILARLY for $R = \rho \vec{r} I$,

$$r_1 = \frac{1}{u_{1_{MAX}}^2} = (18v)^{-2}$$

$$\rho = 1$$

So,

$$Q = \begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{pmatrix}, \quad R = (r_1)$$

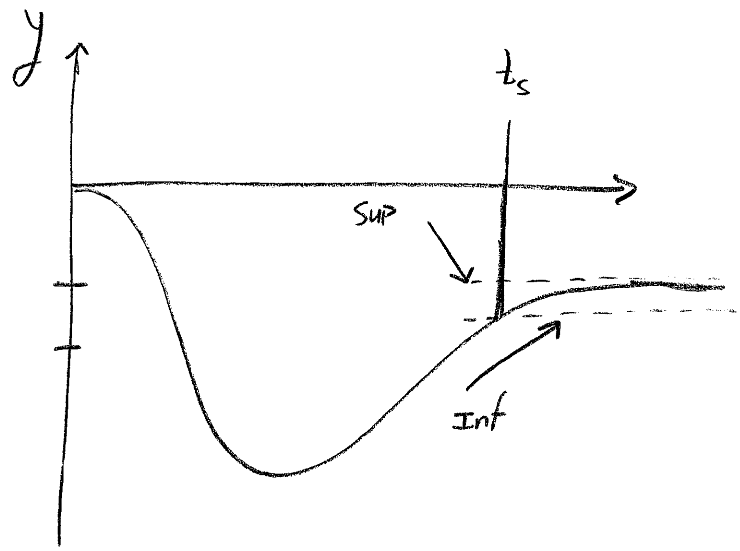
C

RANDOM

NOTES

Step Response Parameters

① Settling Time



for y_i from y_f to y_i

if $(y_i > \text{sup}$ or $y_i < \text{inf})$:

Settling time = time_{i+1}

② Rise Time