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CoSTAR

Website CoSTAR

The MATLAB Toolbox CoSTAR

Institute of Mechanics, Engineering Dynamics Group
Department of Mechanical Engineering (FB15)
University of Kassel (Germany)

INSTITUTE FOR MECHANICS

ENGINEERING DYNAMICS

Website Engineering Dynamics Group

Content

- Overview and Features
- Theoretical Basics / Publications
- Flow Chart and Code Structure
- Basic Use and Where To Start
- Outlook, Feedback and Download

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CoSTAR

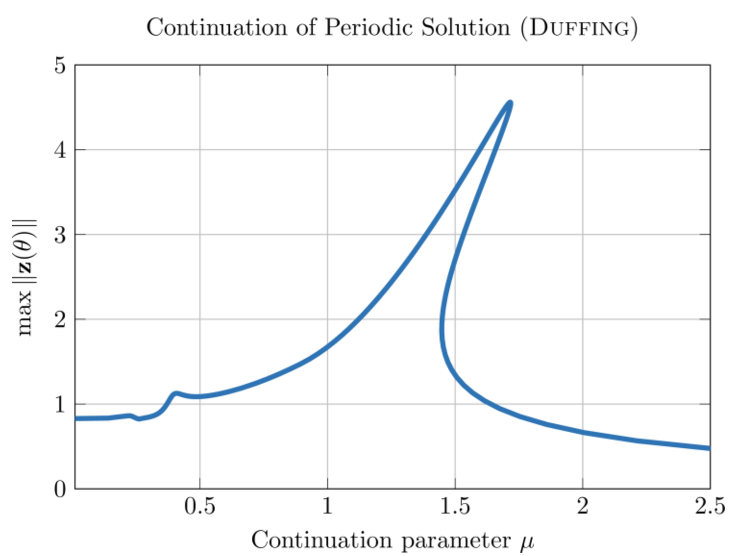
Continuation of **S**olution **T**orus **A**pp**R**oximations

$$\dot{\mathbf{z}} = \mathbf{f}(t, \mathbf{z}, \mu)$$
$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mu)$$

- **Computation of *stationary* solutions of dynamical systems**
  (*stationary* solution: solution type persists for infinite time interval)
  - Equilibrium solutions (**EQ**)
  - Periodic solutions (**PS**)
  - Quasi-periodic solutions (**QPS**)
    (2 base frequencies)

- **Approximation methods for (quasi-)periodic solutions**
  - Finite Difference Method (**FDM**)
  - FOURIER-GALERKIN Method (**FGM**)
  - (Multiple) Shooting Method (**SHM**)
    - **PS:** Multiple Shooting Method
    - **QPS:** Single Shooting Method

- **Continuation of solution branches**



ENGINEERING DYNAMICS

Theoretical basics can be found in following publications:

- FIEDLER, R., HETZLER, H. & BÄUERLE, S.
  Efficient numerical calculation of LYAPUNOV-exponents and stability assessment for quasi-periodic motions in nonlinear systems.
  Nonlinear Dyn **112**, 8299–8327 (2024). https://doi.org/10.1007/s11071-024-09497-9

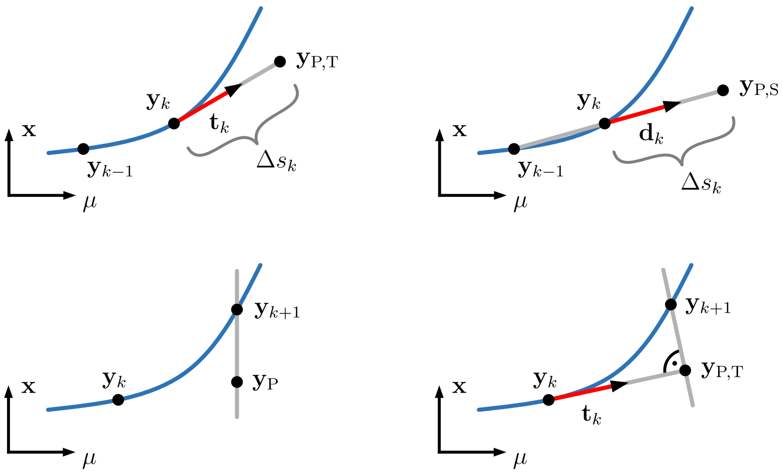
- HETZLER, H. & BÄUERLE, S.
  Stationary solutions in applied dynamics: A unified framework for the numerical calculation and stability assessment of periodic and quasi-periodic solutions based on invariant manifolds.
  GAMM-Mitteilungen **46** (2023), e202300006. https://doi.org/10.1002/gamm.202300006

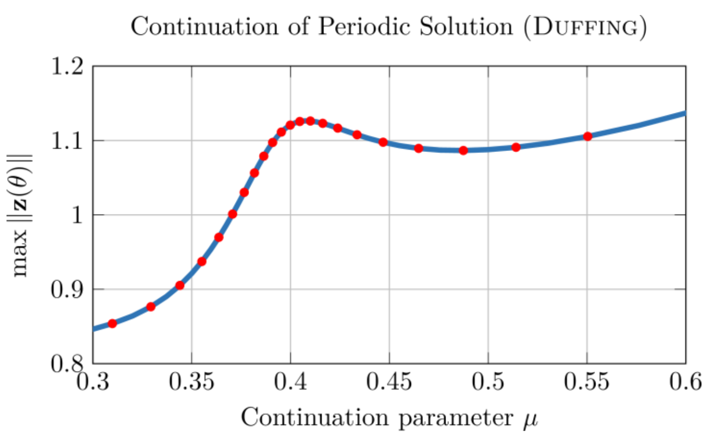
- BÄUERLE, S., SEIFERT, A., KAPPAUF, J. & HETZLER, H.
  A continuation framework for quasi-periodic solution branches based on different torus discretization strategies.
  Proceedings of ISMA Conference, Leuven, Belgien, 12.-14. September 2022.

- BÄUERLE, S., FIEDLER, R. and HETZLER, H.
  An engineering perspective on the numerics of quasi-periodic oscillations.
  Nonlinear Dyn **108** (2022), no. 4, 3927–3950. https://doi.org/10.1007/s11071-022-07407-5

All features can be used as required

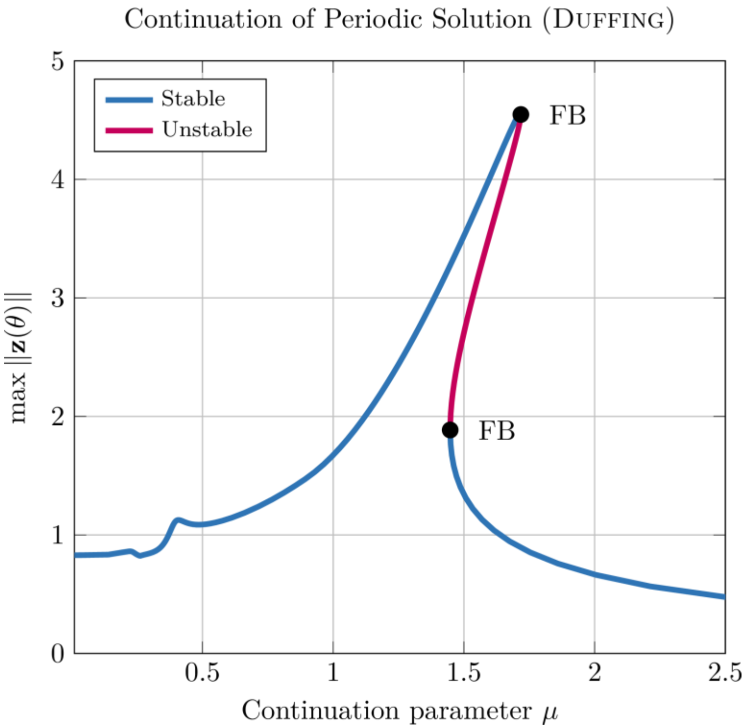
- **Continuation: Predictor-corrector algorithm**
  - Predictors
    - Tangent
    - Polynomials of order 1, 2 and 3
  - Parametrisations
    - Natural
    - Arclength and pseudo-arclength
    - 1-norm
  - Step control
    - Various algorithms based on geometrical information and solver iterations
  - Live plot
    - Creating continuation plot during computation





All features can be used as required

- **Stability computation of solutions**
  - Equilibrium solutions
  - Periodic solutions
    - **SHM**
    - **FDM** & **FGM** (algorithm resorts to **SHM**)
  - Quasi-periodic solutions
    - **SHM**
    - **FDM** & **FGM** (algorithm resorts to **SHM**)

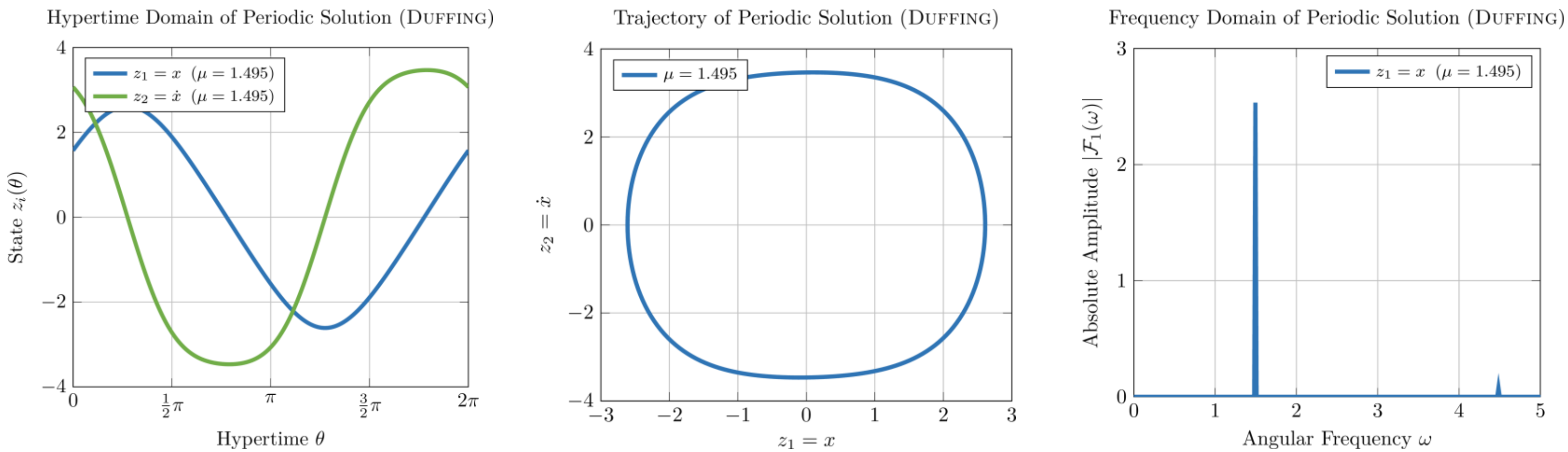


- **Detection of bifurcation points**
  - Fold / Pitchfork / Transcritical (**FB**)
  - Period Doubling (**PDB**)
  - HOPF (**HB**)
  - NEIMARK-SACKER (**NSB**)

- **Error Control**
  - **FGM:** (PS & QPS) Automatic adaption of number of harmonics based on residuum

All features can be used as required

- **Postprocessing methods**
  - `contplot`
    - Creates continuation / bifurcation diagrams (plots solution branches with respect to μ)
  - `solplot`
    - Plots individual solutions in different *solution spaces*
      (Available *solution spaces*: time, hypertime, trajectory and frequency domain)



  - `solget`
    - Returns solution data in different solution spaces

Help Features

- **Gatekeeper** ***cannot be bypassed***
  - ➢ Checks the input (the defined options) from the user
  - ➢ Reports errors in case of illogical or invalid input

You shall not pass!

- **Help**
  - – `costarhelp` function
    - ➢ Quick help in the command window
    - ➢ Overview of the available options with a short description
    - ➢ Type **`costarhelp.costar`** in the command window to start

```
>> costarhelp.costar
------------------------------------------------------------
---------------------- CoSTAR Help -------------------------
------------------------------------------------------------

CoSTAR is a Matlab path continuation toolbox written by the
Engineering Dynamics Group of the University of Kassel, Germany.

CoSTAR enables you to compute single stationary solutions or
to continue a  branch of equilibria, periodic or quasi-periodic
solutions with different numerical techniques, whilst tracking
the solution stability and identify occurring bifurcations.
```

  - – Examples
    - ➢ Short MATLAB scripts
    - ➢ Sample code showing usage of a certain **CoSTAR** module

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          CoSTAR                            %
%    Continuation of Solution Torus AppRoximations           %
%                                                            %
%                         Example:                           %
%                      Postprocessing                        %
%                       - contplot -                         %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%              Welcome to the CoSTAR Examples!               %
```

  - – Tutorials
    - ➢ MATLAB live scripts
    - ➢ There is one tutorial for each example (identical code)
    - ➢ Comprehensive explanations of a certain **CoSTAR** module

CoSTAR

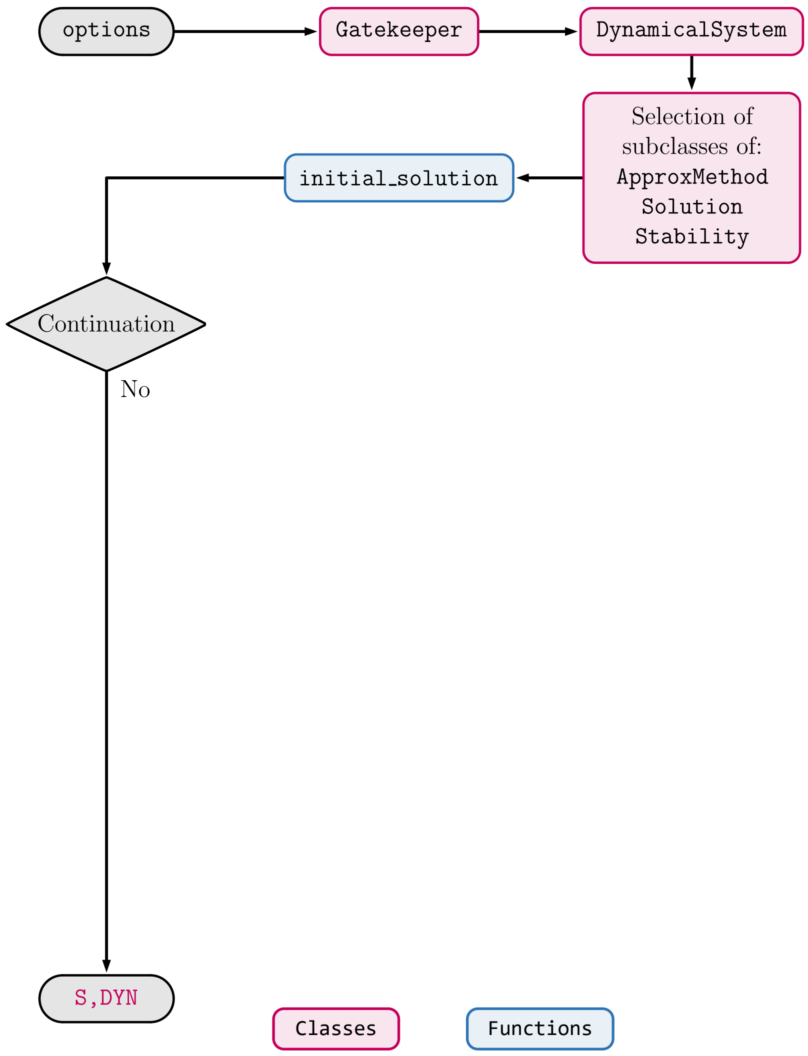
Continuation of Solution Torus AppRoximations

Engineering Dynamics Group, Institute for Mechanics, University of Kassel, Germany

Tutorial: Postprocessing Function contplot

Welcome to the CoSTAR Tutorials!

Source „*You shall not pass!*" picture:
https://knowyourmeme.com/photos/42374-you-shall-not-pass-2 (2024-11-25)

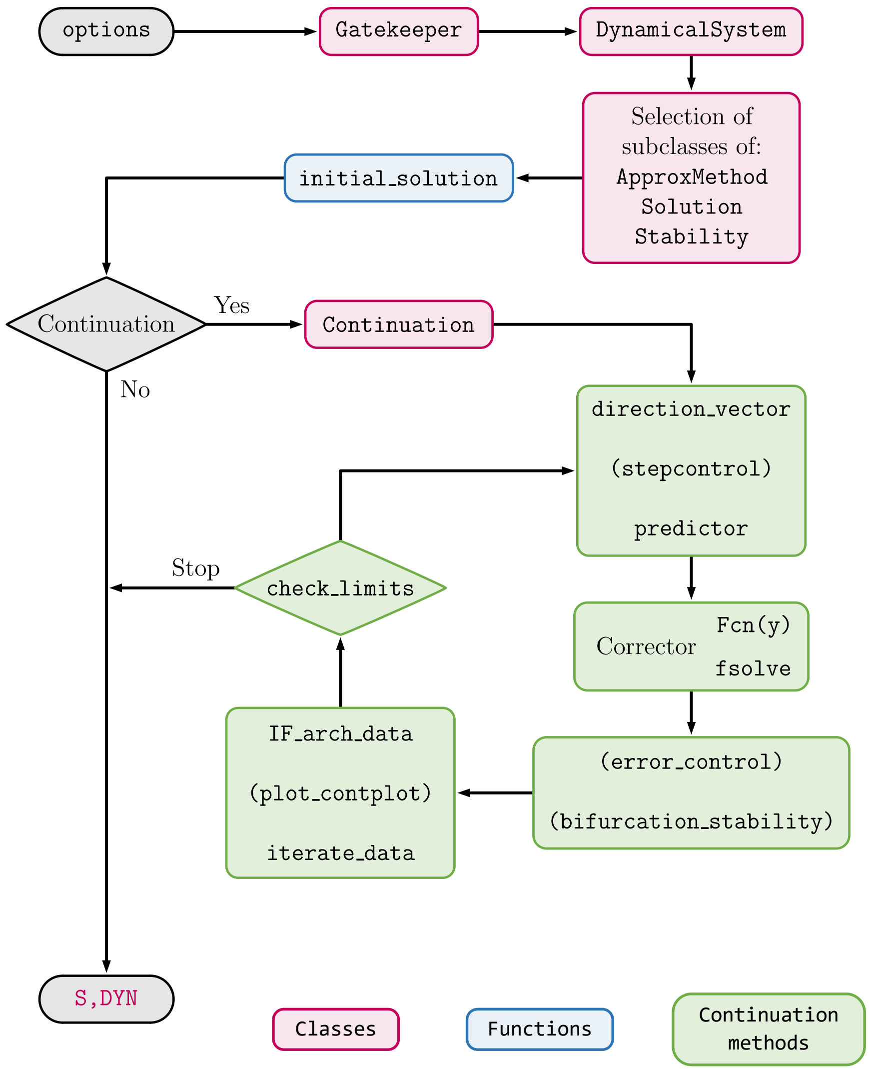


1. options
   - Structure array
   - Contains user-defined options for computation
2. Gatekeeper
   - Checks options
3. DynamicalSystem
   - Stores all options in object DYN
   - Can be used to restart the computation
4. Selection of subclasses
   - ApproxMethod
     - Applies approximation method
   - Solution
     - Stores solution data in object S
   - Stability
     - Used for stability computation
5. initial_solution
   - Computes the first *(initial)* solution

In case of no continuation:

6. Return S and DYN

Flow Chart



6. Continuation loop
   - 6.1. direction_vector
   - 6.2. (stepcontrol) (can be skipped)
   - 6.3. predictor
     - ➢ Computes direction vector, new step width and predictor point
   - 6.4. Corrector
     - ➢ fsolve solving `Fcn(y) = 0`
   - 6.5. (error_control) (can be skipped)
   - 6.6. (bifurcation_stability) (can be skipped)
     - ➢ Performs error control and computes stability as well as bifurcation point
   - 6.7. IF_arch_data
   - 6.8. (plot_contplot) (can be skipped)
   - 6.9. iterate_data
     - ➢ Stores data, updates live plot and performs iterations for next loop
   - 6.10. check_limits
     - ➢ Checks exit conditions

When exit condition is met:

7. Return S and DYN

ENGINEERING DYNAMICS

Code Structure

- Classes ➢ All classes and associated methods
- Functions ➢ Functions not belonging to any class
- RHS ➢ Functions defining right-hand side of $\dot{\mathbf{z}} = \mathbf{f}(t, \mathbf{z}, \mu)$ or $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mu)$
- test ➢ Scripts to test the code
- Tutorials ➢ Tutorial & example scripts
- Version_Log ➢ Version log files documenting code development
- costar ➢ Main **CoSTAR** function (to be called by the user)

Code Structure

Classes	All classes and associated methods
@Continuation	➢ Class and methods to perform the continuation
@costarhelp	➢ Class and methods for the **`costarhelp`** feature
@DynamicalSystem	➢ Class for storing the **`options`** structure
@Gatekeeper	➢ Class and methods for the **`Gatekeeper`** feature
ApproxMethod_SC	➢ Classes and methods to construct the residuum function
@AM_EQ @AM_PS_FDM @AM_PS_FGM @AM_PS_SHM @AM_QPS_FDM @AM_QPS_FGM @AM_QPS_SHM	➢ Subclasses and methods
@ApproxMethod	➢ Superclass
Solution_SC	
Stability_SC	

Code Structure

- Classes
  - @Continuation
  - @costarhelp
  - @DynamicalSystem
  - @Gatekeeper
  - ApproxMethod_SC
  - Solution_SC
    - @SOL_EQ
    - ...
    - @Solution
  - Stability_SC
    - @ST_EQ
    - @ST_PS_SHM
    - @ST_QPS_SHM
    - @Stability

All classes and associated methods

- Class and methods to perform the continuation
- Class and methods for the **`costarhelp`** feature
- Class for storing the **`options`** structure
- Class and methods for the **`Gatekeeper`** feature
- Classes and methods to construct the residuum function
- Classes and methods to save computed data
  - Subclasses and methods (analogous to ApproxMethod_SC)
  - Superclass
- Classes and methods to compute stability
  - Subclass and methods for equililibrium solutions
  - Subclass and methods for periodic shooting method
  - Subclass and methods for quasi-periodic shooting method
  - Superclass and methods

costar

```
function [S,DYN] = costar(options)

    %% Gatekeeper
    GC = Gatekeeper();
    options = GC.m_gatekeeper(options);
    clear GC;

    %% Dynamical System class
    DYN = DynamicalSystem(options);

    %% Approximation Method class
    AM = ApproxMethod.s_method_selection(DYN);

    %% Solution class
    S = Solution.s_solution_selection(DYN,AM);

    %% Stability class
    ST = Stability.s_stability_selection(DYN,AM);

    %% Calculate initial solution
    [S,AM,DYN] = initial_solution(DYN,S,AM,ST);

    %% Continuation
    if strcmpi(DYN.cont,'on')
        CON = Continuation(options.opt_cont);
        S = CON.m_continuation(DYN,S,AM,ST);
    end
```

- **Input:** options structure
  Output: Solution object S, DynamicalSystem object DYN
- **Gatekeeper** checks all input options
- **Save** all options in DynamicalSystem object DYN
- **Create** ApproxMethod object AM
  (methods construct the residuum function)
- **Create** Solution object S
  (stores all solution data)
- **Create** Stability object ST
  (methods compute the stability of a solution)
- **Compute** the initial (first) solution
- **Create** Continuation object CON
- **Do** the continuation

1. Define important parameters and functions
(not necessarily needed, but it helps to keep the overview)

```
%% 1. Define important parameters and functions (not necessarily needed, but it helps to keep the overview)
D = 0.05;     kappa = 0.3;     g = 1;               % Parameters needed for the Duffing differential equation
mu_limit = [0.01, 2.5];                              % Limits of the continuation
eta0 = mu_limit(1);                                  % Value of continuation parameter at start of continuation
param = {kappa, D, eta0, g};                         % Parameter array
active_parameter = 3;                                % Location of continuation parameter within the array
IC = [1; 0];                                         % Initial condition (point in state space) for fsolve

% Functions
non_auto_freq = @(mu) mu;                            % Non-autonomous excitation frequency
Fcn =  @(t,z,param) duffing_ap(t,z,param);           % Right-hand side of dz/dtau = f(tau,z,kappa,D,eta,g)
```

2. Define the options structure
(it comprises all information that CoSTAR needs)

```
%% 2. Define the options structure (it comprises all information that CoSTAR needs)
options.system = costaropts('order',1,'dim',2,'rhs',Fcn,'param',param,'info','continuation of Duffing equation');   % Properties of the system
options.opt_sol = costaropts('sol_type','periodic','approx_method','shooting','cont','on','stability','on', ...    % Properties of the solution
                             'non_auto_freq',non_auto_freq,'act_param',active_parameter);                         % Properties of the solution
options.opt_init = costaropts('ic',IC);                                                                           % Property for initial solution
options.opt_approx_method = costaropts('solver','ode45');                                                         % Properties of approximation method
options.opt_cont = costaropts('mu_limit',mu_limit);                                                               % Properties for continuation
```

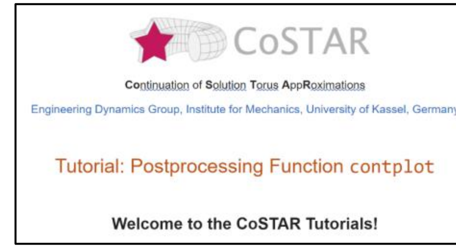
3. Call CoSTAR (and do the continuation)

```
%% 3. Call CoSTAR and do the continuation
[S,DYN] = costar(options);                           % CoSTAR is called by costar(options)
```

4. Individual postprocessing

```
%% 4. Individual postprocessing
% ...
```

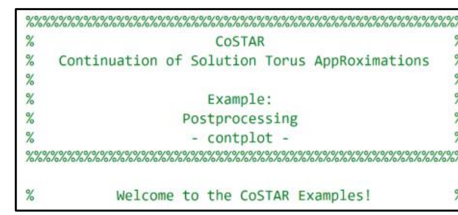
- **If you are new to CoSTAR or certain modules**
  - Tutorials
    - ➢ Comprehensive explanations of a certain **CoSTAR** module
    - ➢ Currently available:



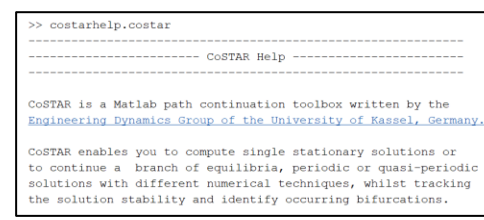
Start with one of these if you have not used CoSTAR yet

- ✓ Equilibrium solutions (`Tutorial_EQ`)
- ✓ Periodic and quasi-periodic solutions approximated by FDM, FGM and SHM (`Tutorial_PS_FDM`, `Tutorial_PS_FGM`, …, `Tutorial_QPS_SHM`)
- ✓ Postprocessing methods contplot, solplot and solget (`Tutorial_Postprocessing_contplot`, …)

- **If you already used CoSTAR**
  - Examples
    - ➢ Sample code showing usage of a certain **CoSTAR** module
    - ➢ Good rescue point to restart working with **CoSTAR**
    - ➢ There is one example for each tutorial (examples are labelled `Example_[…]`)



  - costarhelp feature
    - ➢ Overview of the available options with a short description
    - ➢ Quick help in the command window while using **CoSTAR**
    - ➢ Type **`costarhelp.costar`** in the command window to start



Outlook

Note: The following list may be incomplete and only lists ideas for future improvements to the toolbox. There is no guarantee for actual implementation.

- **Approximation methods**
  - Multiple Shooting Method for quasi-periodic solutions
- **Features**
  - **Stability computation**
    - **PS:** Directly from solution data when using FDM or FGM without JACOBIAN of SHM
  - **Error control**
    - Finite Difference Method (PS & QPS)
    - Handle different exit flags from `fsolve` when computing new solution with updated discretization
  - **Step control**
    - Algorithm(s) based on convergence of solver (when self-written solver is available)
  - **Postprocessing**
    - **FGM & SHM** (QPS hypertime plots): `[1x2]` array for `options` structure field `'resolution'`
  - **Tutorials & Examples**
    - Update default-script tutorials to live scripts
    - Tutorials and examples for continuation options, step control and stability computation

Outlook

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Note: The following list may be incomplete and only lists ideas for future improvements to the toolbox. There is no guarantee for actual implementation.

- **Initial value** (for the solver to compute the initial solution)
  - Standardise the parameters, which create an initial value, for all approximation methods
  - Use of a solution of a different approximation method as initial value
  - Homotopy methods
- **Continuation**
  - Predictor: Polynomials of order > 3
  - Additionally compute the solution at specified (desired) μ-values
- **Computational effort**
  - Make parallel computing available to enhance performance
- **Solver**
  - Self-written solver to remove the need of MATLAB's Optimization Toolbox
- **Dynamic System**
  - Computation of non-hyperbolic manifolds (solutions of Hamiltonian systems)

Feedback and Download

- **Download**
  - CoSTAR is available for free as [GitHub repository]
  - CoSTAR is licenced under the *Apache 2.0* licence

- **Report of bugs**
  - Please create a GitHub issue, labelled as bug, if you experience a new bug
  - If a GitHub issue already exists for your bug, no action is required

- **Suggestions for improvement, wishes and ideas for future releases**
  - Please create a GitHub issue for any wishes, improvements and ideas for future releases and label it accordingly

[Website CoSTAR]

[Website Engineering Dynamics Group]