Mechanics of Deformable Bodies

Table of contents

Introduction

Mechanics can be broadly categorized in three parts, of which this text covers one. **Statics** explores the effects of *external loads* applied to *rigid bodies* in *equilibrium.* **Dynamics** explores the effects of *external loads* applied to *rigid bodies* that are not in *equilibrium.* In **Mechanics of Deformable Bodies**, the subject of this text, we explore the effects of both *external* and *internal loads* on *non-rigid bodies* in *equilibrium.*

Since bodies in this text are in equilibrium, many of the principles from statics still apply. However, because these bodies are non-rigid, a much deeper analysis is required. Objects in this text will deform. When loads are applied, they will change their shapes. They will elongate and bend and twist and, if we're not careful, may even break. Real objects are deformable, and engineers must take great care to ensure that their designs do not break or deform so much that they are unfit for purpose.

Have you ever wondered why civil engineers tend to use certain materials for bridges and buildings, while automotive engineers use an entirely different set of materials in cars? Meanwhile, aerospace engineers don't make rockets out of the same materials that biomedical engineers use for prosthetics. There are many material properties that determine applications that a given material may, or may not, be suitable for. Two fundamental concepts explored in this text are stress, which helps us determine whether an object will break under a given load, and strain, which relates to the object's deformation.

This text begins with an overview of stress and strain. Depending on how external loads are applied, there are a variety of potential internal loads on our object. You are likely already familiar with these loads and their effects, even if you haven't explicitly though about them.

- **Axial forces** push or pull on a body. They cause elongation and compression. For example, imagine pulling on the ends of an elastic band. The band will stretch significantly. Objectd may be pulled apart or crushed by this load. Imagine doing the same with a piece of chalk. There will be a small amount of deformation and the chalk will break.
- **Bending moments** act to bend an object. Imagine propping a long piece of plywood up on two supports and standing in the middle. The wood will bend. If you jump and land hard enough, it may even break.
- **Shear forces** are created when one body (or part of a body) slides relative to another. They commonly occur in conjunction with bending moments.

- **Torsional moments** act to twist an object. Imagine holding the ends of a cable and rotating your wrists so that the cable twists around itself along its length. You will see the cable rotate through ever larger angles. Twist it too far and it will break.

This text will explore the effects of these loads and expand your understanding of these effects. It aims to describe these effects in a conceptual way, building on your existing understanding. Once these effects are understood conceptually, we'll learn how to calculate the exact amount of deformation and the exact size of the stress created. We'll learn to make sure that our designs stay within appropriate limits. There will necessarily be a significant amount of math involved in these calculations but it is just as important, if not more so, that we understand the principles behind those calculations. As such, this text leads with conceptual descriptions and realistic examples before working numerical problems.

The stress and strain resulting from these four types of load account for the majority of this text. Once they are thoroughly understood, more advanced topics necessary for a thorough understanding of these fundamental concepts are included. Upon completion of a course using this text, students will be able to explain the stress and strain observed under different types of load, calculate the amount of stress and strain, and design individual components to be within acceptable limits for both. Students will be prepared for advanced courses in a variety of engineering disciplines, and will be able to apply their knowledge to realistic scenarios.

1 Introduction and Statics Review

Learning Objectives

- Define external and internal force and moment reactions
- Calculate external reactions
- Determine internal pin and two-force member reactions for structures made up of connected members
- Determine internal reactions in continuous bodies
- Calculate reactions in three dimensions

Introduction

Click to expand

Like in Statics, all bodies and structures discussed in this text will be assumed to be in static equilibrium. Unlike Statics, which assumed that all bodies were rigid, bodies in this text are deformable. In order to determine how an applied loading situation affects any given body or structure, and potentially causes it to deform, we must start by applying statics to establish the distribution of forces and moments within the body. This will be the first step of many problems. This chapter will present a review of those aspects of statics concepts that will be prevalent throughout this course.

External forces and moments are the forces and moments that act on the boundaries of a system. They are the loads that are applied arbitrarily (weight, wind, pressure, etc.), as well as the reactions they induce in the supporting elements (pins, rollers, welds, etc.). In the case of external reactions, the word “reaction” refers to the forces and/or moments exerted by the supports on the body in reaction to the other loading in order to keep the body in equilibrium.

Finding the external reactions will be the first step of many of the types of problems that will be covered in this text. This process will entail first drawing a free body diagram (FBD) of the body, or a sketch of the body freed from its supports, which shows the applied and reaction forces and moments. The reactions that correspond to the most common supports are illustrated below in Table 1.1.

Table 1.1: Table 1.1: Free body diagrams for common supports.

Support	Reactions	Free body diagram
Pin	Force acting in an unknown direction. Since the direction is unknown, show as x and y components on the FBD	[figure]
Normal supports (including rollers, rockers, and smooth contact surface)	Force in the direction normal to the support. Since the direction is known (normal to the support), show as total force in the known direction.	[figure]
Cables	Force in the direction of the cable. Should always be drawn in tension.	[figure]
Fixed support	Force in an unknown direction (so draw x and y components of force) as well as reaction couple moment	[figure]

Notice that while pins and fixed supports react with a force in a specific direction, both the magnitude and direction of the force are unknown until equilibrium equations are applied to solve for them. Instead of expressing the components of the unknown F in terms of the unknown , the reactions are normally shown as the components Fx and Fy. Once they are found, the overall magnitude of the force in the pin and its direction can be calculated ([math]).

1.1 Equilibrium in Two Dimensions

Click to expand

Once the FBD is drawn, the next step is to apply the equilibrium equations. In two dimensions (x-y plane), these are:

[math]

Since there are three equations, a statically determinate problem should have no more than 3 unknowns.

Example 1.1 illustrates the process of finding external reactions.

Example 1.1

A 3 ft beam is supported by a pin connection at the wall at point A and a cable at point C as shown. A load is applied 2 ft away from point A. Find the force in pin A as well as the tensile force in the cable.
[figure]
Step 1: Draw the FBD
Note that a guess needs to be made for the positive or negative sense of Ax and Ay, but the tensile force from a cable should always be shown to pull away from the body. The correct sense of Ax and Ay will be determined by obtaining positive or negative answers for the values. A positive answer means the direction was correctly assumed and a negative answer means the force should be in the opposite direction.
[figure]
Step 2: Apply equilibrium equations
Starting with the moment about A will eliminate two unknowns from the equation so that tensile force T can be solved for. Then substitute the result from T into the other two equations to find Ax and Ay.
[math]
[math]
[math]
Since a negative answer was obtained for Ax, that force actually acts in the positive x direction.
The total force in pin A is then [math]
Answer: T = 1600 lbs, FA = 1441 lbs

1.1.1 Two Force Members

One special type of pin connection for which the direction of the reaction force is known is one in which the pin is connected to a **two-force member**. Contrary to the name, a two-force member is not necessarily a member on which only two forces are applied, but rather it is a member on which forces are applied at only two locations. A two-force member can be any shape, as is demonstrated in Figure 1.1. One easy way to recognize a two-force member is to note the presence of only two connection points (such as two pins) but no other locations at which a force or moment couple is applied. Once a member is recognized to be a two-force member, it can be concluded that the resultant force at both connection points will be equal in magnitude (so FA = FB in Figure 1.1) and opposite in direction and follow a line of action that goes through the connections. For a straight member, it can also be concluded that the force within the two-force member (internal reaction as will be discussed in Section 1.1) is equal to the reaction force in the pin.

[figure]

The presence and recognition of two force members can make some otherwise statically indeterminate problems become statically determinate. This is demonstrated in Example 1.2 below.

Example 1.2

Determine the force in pin C and in pin A.
[figure]
Step 1: Draw the FBD
[figure]
Based on this FBD, it appears that there are 4 unknowns and therefore not solvable by just 3 equilibrium equations. However, recognizing that bar AB is a two-force member (since there is a pin at A and a pin at B but no other forces acting on that bar), it can be taken as known that the line of action of the reaction force at A goes through points A and B. Thus, the FBD can be redrawn with just 3 unknowns:
[figure]
Step 2: Apply equilibrium equations
[math]
Solving (1) for FA yields FA = 1154.7 N
Subbing this result into (1) and (2) yields Cx = -577.4 N and Cy = 1000 N
[math]
Answer: FA = 1155 N and FC = 1155 N

1.2 Internal Reactions

Click to expand

Internal reactions can refer to forces and moments at connection points between members (such as a pin connecting multiple members of a frame, machine, or truss), as well as to reactions at any point in a continuous body (for example a point in the middle of a beam). These reactions are the forces and/or moments necessary to hold a structure or a body together and are ultimately the aspect of loading that is needed to determine if and how a body will deform or even break.

1.2.1 Internal reactions at a connection with two force members

Pins that connect members can be represented on an FBD in the same way as pins that connect the structure to external supports. That is, the reaction would be drawn as the two components of the overall force. However, the connected members would be considered as separate bodies, as if the connecting pin were pulled out and the members separated. The pin reactions would then be drawn on the FBD for each separated member. Since the pin would exert equal and opposite forces on the connected members, one needs to be careful to show the reaction forces in opposite directions on the FBD's of those members. All three equilibrium equations can be applied separately to each member so one could theoretically solve for 3 times the number of unknowns as separate FBD's drawn.

However, just as was discussed above, when one of the connected members is a two-force member, the reaction at the pin will be known to follow a line of action that goes through the points of application of the forces on the two-force member. Example 1.3 demonstrates these concepts.

Example 1.3

A plant hanger is secured to a wall with a pin and additionally supported by a brace that is pin connected to the hanger at B and to the wall at C. Determine the external reactions at A and C as well as the reaction in the internal pin B.
[figure]
If we do not notice that BC is a two-force member, we would approach the problem by separating the brace from the hanger and drawing an FBD of each part separately. Notice that Bx and By are drawn in opposite directions in the two different diagrams since the pin will exert an equal and opposite force on each bar.
[figure]
There are now 6 unknowns and 6 equilibrium equations (3 equations per body) available to use per bar, so the problem is technically solvable. However, since BC is a two force member (there is a pin force at B and a pin force at C but no other forces at any other point on the bar), it can be taken as known that the reaction force at B follows a line of action that goes through B and C. We can also conclude that the force in pin C is equal to the force in pin B. The FBD of bar AB can be redrawn:
[figure]
With the components Bx and By replaced with the resultant force FB with known direction, the number of unknowns on bar AB is reduced to 3. These unknowns can be solved for using the equilibrium equations:
[math]
Solving the equations (1)-(3) yields FB = 145.8 lb, Ax = 87.5 lb, and Ay = -66.7 lb. Therefore, the pin force in pin B is 145.8 lb and the pin force in A is A = [math]. Since BC is a two-force member, FC = FB.
Answer: FA = 110 lb, FB = FC = 146.8 lb

1.2.2 Internal reactions in truss structures

Truss structures are made up of only two force members. The two main methods of determining internal reactions in planar (2D) truss structures are Method of Joints and Method of Sections. In using Method of Joints, an FBD is drawn of the connecting pins (joints) within the truss.

Since all the members connected at any given pin will be two-force members, the reactions can be drawn in known directions. However, since the forces all pass through the same point on the body, the moment equilibrium equation is not useful, so only the forces equilibrium equations can be used for each joint. This means that only two unknowns can be solved for at each joint. This method is most useful when the forces of all the truss members are sought or if the only forces sought are attached to a joint with only two members.

To use Method of Sections, a cut is made through the truss structure and analysis is based on the FBD of the intact part of the structure that is to the left of the cut or the intact part of the structure to the right of the cut. The FBD of either given side will show the applied forces and the reaction forces from the members that were cut through. These reactions will be equal in magnitude but opposite in direction between the two sides of the cut. The side to examine is usually based on which one will not require having to find external reactions (if there is a free end to the truss) and/or which one is least complicated to deal with in terms of geometry or applied loads. All three equilibrium equations can generally be effectively applied with Method of Sections, so three unknowns can be solved for with any given cut.

Example 1.4 demonstrates the use of both Method of Joints and Method of Sections to solve for forces in truss members.

Example 1.4

Determine the forces in members ED and EF. Let P1 = 8 kN and P2 = 12 kN.
[figure]
Considering Joint E, it can be seen that since there are 4 members attached to the joint, there would be 4 unknown forces to account for. Therefore, starting with Method of Joints at E would not work. However, looking at joint D, there are only two unknowns, with one of them being member ED, so that would be a useful place to start.
FBD Joint D
Note: The triangle ECD is a right triangle, so the sin and cos of the angle at corner D can be expressed in terms of the ratios of the sides. If the angle at D is , then cos( )= [math] and sin( )= [math] where [math] is the hypotenuse of the triangle.
[math]
[figure]
Solving (1) and (2) gives FDE = 13.42 kN (tensile) and FDC = -6 kN.
Since the force is shown to be tensile in the FBD (the force is pointed away from the joint), the negative answer indicates the force is actually compressive.
Even with knowing FDE, joint E would have three unknown forces. Joint C only has two unknown forces, so we can draw that joint next. Note that we previously determined FCD = 6 kN in compression, so we will draw that force in compression at joint C. Forces FBC and FCE are unknown and may be drawn in either direction. For this example they've been drawn in tension.
[figure]

[math]
Then at joint E:
[figure]
[math]
Answer: FDC = 6 kN (Compressive), FEF = 18 kN (Tensile)
While Method of Joints can be used here, it is inefficient. We needed to draw and analyze free body diagrams of joints D, C, and E and keep track of the forces at these joints as well as whether each force was in tension or compression.
Method of Sections may be used as an alternative to find force FEF. To apply Method of Sections for this problem, a cut will be made through the truss that passes vertically through members EF, BE, and BC. Once that cut is made, a choice needs to be made to draw an FBD for the intact part of the truss to the left of the cut (everything to the left of BF in this case) or the intact part of the truss to the right of the cut (everything to the right of EC in this case). Either way, the external forces on the side would need to be shown on the FBD as well as the force of each member that was cut through. This is illustrated below. Notice that the forces are drawn in opposite directions on the different sides of the cut.
[figure]
Looking at the FBD of the two cut sections, the right side section would be easiest to work with since there are no external reactions on that side. If the left side section were picked, the reactions at A and G would first need to be determined (note that FGF = Gx). All three equilibrium equations can be used with the Method of Sections. In this particular case in which only EEF is left to solve for, only the moment about B is needed to solve since the other two unknown forces both pass through B.
[math]
Once again, the forces are drawn on the FBD in the tensile direction, so the negative answer indicates a compressive force.
Answer: FDC = 6 kN (Compressive), FEF = 18 kN (Tensile)

1.2.3 Internal reactions in continuous bodies

Internal reactions also exist within a body or structure. These reactions are necessary to hold the body together and will vary from point to point in a body depending on the distribution of external loading. As shown in Figure 1.2, the reactions at any given point can be examined by making a cut at the point of interest in the body. One can think of any point within a body as acting as a fixed support for the rest of the body. That is, every point must potentially exert a force parallel to the cross section where the cut is made which is the shear force V, a force perpendicular to the cross section where the cut is made which is the normal force N, and a reaction moment where the cut is made which is the bending moment M. Moreover, as was discussed for internal pin reactions and the cut made for Method of Sections for trusses, the reactions at a cut will be equal and opposite on the two sides of the cut.

[figure]

To determine the reactions, the FBD of the part of the beam to the left of the cut can be drawn and used, or the part of the beam to the right of the cut can be drawn and used. As was discussed with Method of Sections for trusses, the choice of which side of the cut to examine is based primarily on which side appears easiest and most efficient to analyze. Once the FBD of the cut section is drawn, the three equilibrium equations can be applied to determine the internal reactions. The determination of shear and bending moments in beams will be reviewed in more detail in Chapter 8. The focus of Example 1.5 is on the determination of the normal force, as this will be important in Chapters 2 and 3.

Example 1.5

Two solid bars make up the axial assembly loaded as shown. Determine the normal force in each bar. State whether the force is tensile or compressive.
[figure]
Though the assembly is not a beam, determining the internal reactions will work in the same way. In this particular case, all the forces are in the normal direction (no shear force) and due to the central placement of the 60 kN force and the symmetry of the 125 kN forces, there will be also be no bending moment. Consequently, only the normal reaction force will be drawn on the FBDs.
You may recall from shear and moment analysis of beams, a cut is made whenever there is a loading change. In this example, there is a loading change at B, so there will be two sections of the assembly with distinct normal reactions.
Making the cut in section AB and drawing the FBD allows us to determine the normal force in section AB. Note that drawing the left section of the cut for the FBD results in avoiding needing to know the external reactions at wall C.
[figure]
[math]
Making the cut in section BC and drawing the FBD allows us to determine the normal force in section BC. Once again, drawing the left section of the cut for the FBD results in avoiding needing to know the external reactions at wall C.
[figure]
[math]
For both AB and BC, the internal normal force was assumed tensile in the FBD and equilibrium equation. In the case of NAB, the positive answer confirms that it is tensile. In the case of NBC, the negative answer reveals that it is compressive.
[figure]
Answer: NAB = 60 kN (T), NBC = 190 kN (C)

1.3 Equilibrium and Reactions in Three Dimensions

Click to expand

Because real life structures will be subject to forces and moments in all directions, we will also see problems in which it will be necessary to consider 3 dimensional forces and 3 dimensional moments.

For 3D systems, there are 6 total scalar equilibrium equations:

[math]

Note that the moment vector describes the axis around which the body tends to rotate. Each individual component represents the tendency of the body to rotate around the specified axis. In 3D, there are three internal forces and three internal moments (Figure 1.3). There is one normal force (Nx) perpendicular to the cross-section and two shear forces (Vy and Vz) parallel to the cross-section. There are two bending moments (My and Mz) which act around the axes parallel to the cross-section and one torsional moment (Tx) which acts around the axis perpendicular to the cross-section. We will study each of these loads in detail over the next few chapters.

[figure] **Figure 1.3:** In 3D there are three internal forces (normal force Nx and two shear forces Vy and Vz) and three internal moments (torsional moment Tx and two bending moments My and Mz)

Note that the choice of coordinate system here is arbitrary. Loads are defined as normal, shear, bending, or torsion based on how they act relative to the cross-section. For example, the normal force may act in the y-direction depending on how the cross-section is cut, and the torsional moment would then act around the y-axis.

While summing the reaction forces in 3D is a straight-forward process of adding forces in each direction, summing moments in 3D can prove to be more complicated. To sum moments, there are generally two options. One option is to use the cross product to calculate moments: [math]

In the cross-product equation, **r** is the position vector from the point the moment is about to any point on the line of action of the force. Using the cross product to calculate moment will result in a vector expression for the moment equation that gives all three components at one time with the correct signs to indicate clockwise (negative) or counterclockwise (positive) rotation.

The second option to calculate moments is to perform scalar calculations in which the sum of the moments about the x, y, and z axis is calculated individually. To use this option, it might be helpful to recall:

1. The general scalar equation for moment is M = F*d, where d is the perpendicular distance from the axis of rotation at the point the moment is being taken about to anywhere on the line of action of the force. One can also use M = F r sin where r is the distance from the point to any point on the force and is the angle between the position vector **r** (that corresponds to the magnitude r used) and the force vector.
2. Forces do not cause moments about points they go through or axes they act through.
3. Forces do not cause moments about axes they are parallel to (ie, Fx wouldn't cause a moment around an x-axis no matter where the point is).
4. When taking the moment about a point, the origin of the coordinate axes should be moved to that point for the purpose of determining the distance between the axis and the force.

Given all the reminders above, one can apply the following equations:

[math] where z and y are the respective distances from the x-axis at the point in question to Fy and Fz components of the force respectively.

[math] where z and x are the respective distances from the y-axis at the point in question to Fx and Fz components of the force respectively.

[math] where y and x are the respective distances from the z-axis at the point in question to the Fx and Fy components of the force respectively.

The [math] is decided based on the right-hand rule (see text below for guidance) or visual inspection. When using visual inspection, the direction of rotation is judged by looking from the positive end of the axis towards the negative end. A counterclockwise rotation is considered to be positive and a clockwise rotation is considered to be negative.

To apply the right-hand rule:

1. Orient your right hand so that the fingers are aligned with the moment arm with the palm at the point and the fingertips extending towards the force. The moment arm is the axis along which the perpendicular distance to the axis would be determined. For example, if finding Mx due to Fy, the moment arm is in the z direction.
2. The thumb is aligned with the axis of rotation (axis around which the moment is being calculated).
3. Curl your fingers in the direction of the force. If your thumb must be pointed in the positive axis direction to perform this action, the moment is counterclockwise. If your thumb must be pointed in the negative axis direction, the moment is clockwise. Typical convention designates counterclockwise rotation to be positive and clockwise to be negative.

These concepts are further reviewed in Example 1.6.

Example 1.6

Determine the internal reactions at point P, located at the center of the cross-section of the rectangular bar and 1.75 ft from the fixed support, if **F** = -150 **i** - 225 **j** + Fz = 300 **k** lb.
[math]
To determine the internal reactions at P, a cut is made at P that is parallel to the cross section. Since the left side of the cut would include the wall but the right side of the cut would be free with no external reactions to determine, the right side section will be used.
[math]
The force reactions at P will just be equal (but opposite in direction) to the force shown since there is only one. The moment reactions can be found by taking the moments about point P.
The moment about the x axis at point P is:
[math]
The moment about the y axis at point P is:
[math]
The moment about the z axis at point P is:
[math]
The sign on the individual multiplicative terms in each equation are determined by right hand rule or visualization.

Summary

Click to expand

Key takeaways

Bodies in this text are in static equilibrium and subjected to external forces and moments. Reaction forces and moments at supports can be determined in 2D and 3D through equilibrium equations, just like in Statics.
Unlike Statics, bodies in this course are deformable. We must also determine internal loads. These can be found by cutting a cross-section at the point of interest and again applying equilibrium equations.
In 2D there are two internal forces and one internal moment:

- Normal force (N) perpendicular to the cross-section
- Shear force (V) parallel to the cross-section
- Bending moment (M)

In 3D there are three internal forces and three internal moments:

- Normal force (N) perpendicular to the cross-section
- Two shear forces (V) parallel to the cross-section
- Torsional moment (T) acting around the axis perpendicular to the cross-section
- Two bending moments (M) acting around the axes parallel to the cross-section

The effects of these internal loads on deformable bodies is the focus of this text.

Key equations

Static equilibrium:
[math]

2 Stress

> Learning Objectives
>
> - Insert text

2.1 Introduction

Click to expand

Insert text

3 Strain

Learning Objectives

- Insert text

3.1 Introduction

Click to expand

Insert text

4 Mechanical Properties of Materials

Learning Objectives

- Insert text

4.1 Introduction

Click to expand

Insert text

5 Axial Loading

Learning Objectives

- Explain stress concentrations and calculate maximum stress based on geometry
- Calculate axial deformation in a bar subjected to axial load
- Determine deformation in a series of parallel bars
- Define statically indeterminate problems and solve them using knowledge of deformation
- Calculate thermal deformation and solve statically indeterminate problems involving changes in temperature

Introduction

Click to expand

In this chapter we'll take a closer look at the effects of axial loading. Axial loads are those applied perpendicular to the cross-section of an object (Figure 5.1) and such loads affect the object in several ways.

[figure]

Figure 5.1: Axial loads act perpendicular to the cross-section. They cause normal stress on the cross-section and normal strains in both the axial (A) and transverse (T) directions.

Axial loads create axial stresses, which we have already seen in Section 2.1 and we'll revisit briefly in Section 5.1. We previously considered the average normal stress in a cross-section, but the geometry of the cross-section can cause large localized stresses much higher than the average. We'll discuss these stress concentrations in Section 5.2.

This text is concerned with both stress and deformation, so in Section 5.3 we'll explore the deformation caused by axial loads. We'll extend this to a special case of axial deformation involving parallel bars in Section 5.4, before using our knowledge of axial deformation to help us solve statically indeterminate problems in Section 5.5. These are problems where the equilibrium equations are not sufficient to determine the reaction and internal forces.

Finally we'll investigate the effects of temperature in Section 5.6. As the temperature of an object changes it will expand or contract. Thus there can be deformation even in the absence of any applied forces.

5.1 Axial Stress

Click to expand

Axial loads create axial stresses, also known as normal stresses. As discussed in Section 2.1, the average normal stress is calculated from

$$\sigma = \frac{N}{A}$$

where

= *Average normal stress [Pa, psi]*

N = *Internal normal force [N, lb]*

A = *Cross-sectional area [m^2, in.^{2}]*

5.2 Stress Concentrations

Click to expand

The normal stress discussed in the previous section is averaged over the cross-section and assumes that stresses (and therefore strains) at the cross-section are uniform. In reality stresses and strains can vary across a cross-section, especially if the cross-section is close to an applied load, a support, or a change in geometry. At these points localized stress concentrations occur, leading to large, localized stresses and strains. The intensity of these concentrations depends on the type of loading, support, or geometry (Figure 5.2).

[figure]

Figure 5.2: Each bar has a fixed support on the left, a cross-section of 30 in.^{2}, and is subjected to a force of 10,000 lb. The top bar is uniform and experiences a uniform average normal stress of $\sigma = \frac{10000}{30} = 333$ psi at all points, except those close to the support and load. The bottom bar experiences the same stress at most points, but significantly higher stress concentrations close to the hole.

However, these effects disappear a certain distance away from the force, support, or local geometry. It is therefore acceptable to assume uniform stress and strain (as we have been so far) provided that we also assume our cross-section is a sufficient distance away from these points. This is known as Saint-Venant's principle, which can be formally stated as "The stresses and strains created at a point in a body by two statically equivalent loads are equivalent at points sufficiently far removed from the applied load" (Figure 5.3).

[figure]

Figure 5.3: Two statically equivalent loads applied to a bar. Close to the point of application there will be localized stresses and deformations, but at a cross-section sufficiently far away from the applied loads the internal effects are equivalent.

More advanced courses will use the theory of elasticity to study the areas of variable stress and strain, but for now we'll continue to apply Saint-Venant's principle and assume that our cross-sections are sufficiently far away from the applied loads that we don't need to consider these stress concentrations. We will however address the question of stress concentrations around changes in geometry here. We'll study two specific examples; holes and fillets. Figure 5.2 shows an example of the stress concentrations around a hole. Figure 5.4 shows an example of the stress concentrations around a fillet. A fillet is a rounded corner used to help transition between two geometries.

[figure]

Figure 5.4: The fillet helps prevent a sharp corner but still causes stress concentrations. The bar has a thickness of 3 in. The height of the smaller section is 6 in. and the larger section is 10 in. The applied load is 10,000 kips, so the average normal stress in the smaller section is 556 psi and in the larger section is 333 psi. The maximum stress at the fillet however is 973 psi.

Fully modeling these stress concentrations is very complicated, but for our purposes it is sufficient to find only the maximum stress that occurs around these stress concentrations. These can be significantly larger than the average stress we have been calculating, and can cause localized failure even if the average stress is below the yield stress for the material. The maximum stress can be found simply by multiplying the average stress by a stress concentration factor, K.

$$\sigma_{\text{max}} = K\sigma_{\text{avg}}$$

where

$_{max}$ = *Maximum normal stress [Pa, psi]*

K = *Stress concentrtion factor [unitless]*

$_{avg}$ = *Average normal stress [Pa, psi]*

This stress concentration factor depends on the geometry at hand and can be found from curves in design handbooks (Figure 5.5). See Examples 5.1 and 5.2 to see how these curves can be used.

[figure]

Figure 5.5: Graphs showing how the stress concentration factor, K changes based on the geometry of the object.

Example 5.1: Stress concentration for hole

A hole is drilled through a steel plate to allow two components to be bolted together. The plate has the dimensions shown and is 15 mm thick. If the plate is subjected to an axial load of P = 50 kN, determine the maximum stress in the plate if the hole diameter d = 20 mm.
[figure]

Solution

Start by calculating the average normal stress in the plate at the location of the hole. The cross-section is a rectangle with a base of 15 mm (0.015 m) and a height of 100 mm (0.1 m), with a 20 mm (0.02 m) diameter hole cut out.
[figure]

$$\sigma_{avg} = \frac{N}{A} = \frac{50000}{0.015 * (0.1 - 0.02)} = 41.7\text{MPa}$$

Determine the ratio $\frac{d}{D}$ where d = hole diameter and D = Height of the cross-section. Use the appropriate stress concentration curve to read off the stress concentration factor K for this geometry.
[figure]

$$\frac{d}{D} = \frac{20}{100} = 0.2$$
$$K = 2.525$$

Calculate the maximum stress.

$$\sigma_{\max} = K\sigma_{avg} = 2.525 * 41.7 = 105MPa$$

Example 5.2: Stress concentration for fillet

A 1-inch-thick connecting rod in an engine assembly has the dimensions shown. If the maximum allowable stress in the rod is 40 ksi, determine the maximum axial load that may be applied to the rod.
[figure]

Solution

The connecting rod has two holes and a fillet. We can start by determining which of these geometries experiences the largest maximum stress, in terms of applied load P. For each geometry, calculate the average normal stress and then use the appropriate stress concentration curve to determine the maximum normal stress.

For the smaller circle:

[figure]

$$\begin{aligned}
\sigma_{\text{avg}} &= \frac{N}{A} = \frac{P}{1*(4-2)} = \frac{P}{2} = 0.5P \\
\frac{d}{D} &= \frac{2}{4} = 0.5 \\
\text{K} &= 2.15 \\
\sigma_{\max} &= K\sigma_{\text{avg}} = 2.15 * 0.5P = 1.075P
\end{aligned}$$

For the larger circle:

[figure]

$$\begin{aligned}
\sigma_{\text{avg}} &= \frac{N}{A} = \frac{P}{1*(8-6)} = \frac{P}{2} = 0.5P \\
\frac{d}{D} &= \frac{6}{8} = 0.75 \\
\text{K} &= 2.05 \\
\sigma_{\max} &= K\sigma_{\text{avg}} = 2.05 * 0.5P = 1.025P
\end{aligned}$$

For the fillet:

[figure]

$$\begin{aligned}
\sigma_{\text{avg}} &= \frac{N}{A} = \frac{P}{1*4} = \frac{P}{4} = 0.25P \\
\frac{D}{d} &= \frac{8}{4} = 2 \\
\frac{r}{d} &= \frac{1}{4} = 0.25 \\
\text{K} &= 1.82 \\
\sigma_{\max} &= K\sigma_{\text{avg}} = 1.82 * 0.25P = 0.455P
\end{aligned}$$

The largest stress is at the small hole, where ${}_{\max} = 1.075P$.
Since the maximum allowable stress is 40 ksi, $40 = 1.075P \rightarrow P = \frac{40}{1.075} = 37.2$ kips.

Step-by-step: Stress concentrations

1. Calculate the average normal stress using $\sigma_{\text{avg}} = \frac{N}{A}$. Do this at the hole or at the thinner section of the fillet.
2. Use the appropriate stress concentration curve to determine the stress concentration factor, K, for the given geometry.
3. Calculate the maximum stress from $\sigma_{\max} = K\sigma_{\text{avg}}$

5.3 Axial Deformation

Click to expand

Consider a simple bar of uniform cross-section subjected to an axial force (Figure 5.6).

[figure]

Figure 5.6: A bar of uniform cross-section, A, and length, L, subjected to axial load, F.

Assuming elastic behavior, we have equations for stress and strain, as well as Hooke's law which relates the two equations:

$$\sigma = \frac{F}{A}$$
$$\varepsilon_{\text{long}} = \frac{\Delta L}{L}$$
$$\sigma = E\varepsilon_{\text{long}}$$

Replacing the stress and strain terms in Hooke's law:

$$\frac{F}{A} = E\frac{\Delta L}{L}$$

Rearranging:

$$\Delta L = \frac{FL}{AE}$$

where

ΔL = Change in length [m, in.]

F = internal axial load [N, lb]

L = *Original length [m, in.]$*

A = *Cross-sectional area [m*^{2}*, in.*^{2}*]*

E = *Elastic modulus [Pa, psi]*

This equation can be used to directly find the change in length of an object subjected to an axial load.

In practice, there may be multiple axial loads applied to the bar. The cross-sectional area may change at different points along the bar, or perhaps there could be different materials with different elastic moduli connected in series to form the bar. In any of these cases we can split the bar into sections where each section has a constant F, A, and E. We can calculate the change in length of each section separately and sum them to find the total. See Example 5.3 for a problem involving multiple segments of a bar.

$$\Delta L = \sum \frac{FL}{AE}$$

Example 5.3: Axial change in length problem with different loads, areas, and materials

A component is made by welding together 2 circular rods. Rod (1) is made of steel (E = 29 x 10^6 psi) and is hollow, with an outer diameter of 4 in. and an inner diameter of 2 in. Rod (2) is made of copper (E = 17 x 10^6 psi) and is solid, with a diameter of 5 in. If the component is subjected to the axial loads shown, determine the total deformation of the component.
[figure]

Solution

We can find the change in length of the component using $\Delta L = \sum \frac{FL}{AE}$.
First, break the component into three segments and determine the internal load in each segment. The first segment covers the 9-inch steel rod. After the first 9 inches of the component, both the cross-sectional area and elastic modulus change so a second segment is needed. This segment covers the next 4 inches. At this point the loading changes, so a third segment is required to cover the final 10 inches of the component.
[figure]

$$N_1 = N_2 = 60\text{kips}$$
$$N_3 = -20\text{kips}$$

Now calculate the deformation of each segment. Be sure to use the appropriate dimensions and material properties for each segment.
Cross-sectional areas:

$$A_1 = \pi * (2^2 - 1^2) = 9.42 in.^2$$
$$A_2 = \pi * (2.5^2) = 19.6 in.^2$$

Deformation of segment 1:

$$\Delta L_1 = \frac{60000 * 9}{9.42 * 29 \times 10^6} = 0.00198\text{in.}$$

Deformation of segment 2:

$$\Delta L_2 = \frac{60000 * 4}{19.6 * 17 \times 10^6} = 0.000719\text{in.}$$

Deformation of segment 3:

$$\Delta L_3 = \frac{-20000 * 10}{19.6 * 17 \times 10^6} = -0.000599\text{in.}$$

Note that segment 3 is in compression and so gets shorter, while segments 1 and 2 are in tension and so get longer. The total deformation in the component is simply the sum of these three deformations.

$$\Delta L = 0.00198 + 0.000719 - 0.000599 = 0.0021\text{in.}$$

Step-by-step: Axial deformation

1. Split the bar into sections of constant F, A, and E. Any time any of these terms changes, begin a new section.
2. Determine the length of each section.
3. Calculate the change in length of each section using $\Delta L = \frac{FL}{AE}$.
4. Determine the total change in length by summing the change in length of each section. Remember some sections may elongate while others contract.

5.4 Deformation in Series of Bars

Click to expand

Sometimes a structure will have more than one axially loaded member. In such cases there are multiple bars that will experience a change in length and the amount of deformation won't necessarily be the same in each member. The deformable members are generally connected by a rigid (nondeformable) beam. These problems generally involve a little geometry alongside our deformation equation (Figure 5.6).

[figure]

Figure 5.6: Rigid beam AB is attached to deformable poles 1 and 2. The poles may deform different amounts so point A is displaced by amount ΔL_1, point B by amount ΔL_2, and point C by an amount in between these two values.

By identifying the force in each member through equilibrium, we may calculate the deformation of each member separately using:

$$\Delta L = \frac{FL}{AE}$$

Once the change in length of each member is known, we can find the displacement at different points on the rigid beam through simple geometry of the rigid beam (Figure 5.7). This process is demonstrated in Example 5.4.

[figure]

Figure 5.7: Beam deflection in the case where only bar 2 deforms. Point B will deflect downward by amount ΔL_2. The deflection at point C can be determined through geometry.

The deflection of point C will be somewhere between the deflection at A and the deflection at B. Assuming there is no deflection at A and assuming that the deflections (and therefore the angle at A) are small, we can use similar triangles to find:

$$\frac{\Delta L_C}{L_{AC}} = \frac{\Delta L_2}{L} \rightarrow \Delta L_C = \Delta L_2 \frac{L_{AC}}{L}$$

If there is a deflection at point A (Figure 5.8) this simply becomes:

$$\Delta L_C = \Delta L_1 + (\Delta L_2 - \Delta L_1)\,\frac{L_{AC}}{L}$$

[figure]

Figure 5.8: Similar triangles for calculating deflection at a point when both bars experience a change in length.

Example 5.4: Apply numbers to figure 5.6

A rigid beam is supported by two non-rigid poles and subjected to a distributed load w = 75 kN/m. Pole 1 is made of steel (E = 200 GPa) and has a diameter of 30 mm. Pole 2 is made of cast iron (E = 70 GPa) and has a diameter of 50 mm. Determine the deflection at point C of the rigid beam.
[figure]

Solution

Although the beam is rigid, poles 1 and 2 will both elongate. We can find the force in each pole by drawing a free body diagram and applying equilibrium equations.
[figure]

$$\sum M_A = 0 : -(600 * 8) + (F_2 * 12) = 0 \rightarrow F_2 = 400kN$$
$$\sum F_y = 0 : F_1 - 600 + 400 = 0 \rightarrow F_1 = 200kN$$

We can then calculate the change in length of each pole.

$$\Delta L_1 = \frac{F_1 L_1}{A_1 E_1} = \frac{200000 * 5}{\pi * 0.015^2 * 200 * 10^9} = 0.00707 \text{ m} = 7.07 \text{ mm}$$
$$\Delta L_2 = \frac{F_2 L_2}{A_2 E_2} = \frac{400000 * 3}{\pi * 0.025^2 * 70 * 10^9} = 0.00873 \text{ m} = 8.73 \text{ mm}$$

To determine the deflection at point C, first determine that point B has deflected (8.73 mm – 7.07 mm) = 1.66 mm more than point A.
[figure]
We can find how much more point C has deflected than point A by using similar triangles.
[figure]

$$\frac{1.66}{12} = \frac{\Delta L_{C/A}}{4} \rightarrow \Delta L_{C/A} = 0.553 \text{ mm}$$

So point C deflects 0.553 mm more than point A, which is a total deflection at point C of

$$\Delta L_C = 7.07 \text{ mm} + 0.553 \text{ mm} = 7.62 \text{ mm}$$

Step-by-step: Deformation in series of bars

1. Use equilibrium to determine the internal force in each bar.
2. Calculate the change in length of each bar using $\Delta L = \frac{FL}{AE}$.
3. Use similar triangles to find the deflection at any point between the parallel bars.

5.5 Statically Indeterminate Problems

Click to expand

A statically indeterminate problem is one which has more unknowns than we have equilibrium equations to solve for those unknowns. This is an issue because it prevents us from finding the internal loads and we therefore can't calculate stress or deformation.

We'll study two types of statically indeterminate problems. In the first type there will be additional supports beyond those needed to maintain equilibrium. These are known as redundant supports and they are quite common in practice (Figure 5.9).

[figure]

Figure 5.9: An example of redundancy on a suspension bridge. There are multiple supports and multiple cables at each support, such that if one cable fails the entire structure does not collapse.

In such problems, it is not possible to determine all of the reaction forces using equilibrium alone. However, we can use our knowledge of deformation to help. If a member is held between two supports then its total deformation must be zero. Since the change in length depends on the internal force in the member, this introduces an additional equation to use alongside the equilibrium equations. There are two possible approaches here.

Approach 1: To determine the reaction force at the redundant support, begin by removing the redundant support from the problem and determining the deformation that would occur if the support was not there. Then replace the reaction force at the support, which will cause the member to deform in the opposite direction (Figure 5.10).

[figure]

Figure 5.10: (a) A bar held between two rigid supports. (b) A free body diagram of the beam reveals that there are two unknowns but only one equilibrium equation. (c) Removing one of the supports allows the bar to elongate. (d) Replacing the support force causes the bar to contract.

The sum of these two deformations must equal the actual deformation of the member. If the member is held between two rigid supports, the total deformation will be zero. If there is a small gap or the supports allow a certain amount of movement, the total deformation will be equal to the size of this gap. Example 5.5 shows this process applied to a bar made of 2 materials.

Example 5.5: Bar between 2 supports, 2 materials, and areas

A 15-foot-tall concrete (E = 4,000 ksi) column has a square cross-section of 6 in. by 6 in. A 10-foot-tall-copper (E = 17,000 ksi) cylinder with an outer diameter of 4 in. and in inner diameter of 3 in. is attached to the top of the concrete. The structure is fixed between two supports. A force of 25 kips is applied as shown. Determine the average normal stress in each material.
[figure]

Solution

To determine the internal force in each material, we first need to find the reaction forces at the supports. We would usually draw a free body diagram and apply equilibrium equations to find these forces. However, this won't work here because the problem is statically indeterminate.
[figure]

$$\sum F_y = 0 : F_1 + F_2 - 25 = 0$$
$$F_1 + F_2 = 25 \text{ kips}$$

To solve the statically indeterminate problem, remove one of the supports and allow the structure to deform. Either support may be removed. Let's remove the top support. In this scenario there is no load in the copper cylinder and there is a compressive load of 25 kips in the concrete column.
[figure]
The total deformation of the structure in this scenario is:

$$\Delta L = \sum \frac{FL}{AE} = 0 - \frac{(25) * (15 * 12)}{(6 * 6) * (4,000)} = -0.03125\text{in}$$

Then replace force F_1 and calculate the deformation caused by this force. Since the force is applied at the top of the structure, both the concrete and the copper will elongate.
[figure]

$$\Delta L = \sum \frac{FL}{AE} = \frac{F_1 * (10 * 12)}{\pi\left(2^2 - 1.5^2\right) * (17,000)} + \frac{(F_1) * (15 * 12)}{(6 * 6) * (4,000)} = 0.001284F_1 + 0.00125F_1 = 0.002534F_1$$

Since the structure is fixed at both ends, the actual total deformation must be zero.

$$-0.03125 + 0.002534F_1 = 0 \rightarrow F_1 = 12.3\text{kips}$$

Returning to our equilibrium equation, we can also find force F_2.

$$F_1 + F_2 = 25 \text{ kips } \rightarrow 12.3 + F_2 = 25 \rightarrow F_2 = 12.7 \text{ kips}$$

The internal force in the copper cylinder will be 12.3 kips and the internal force in the concrete column will be 12.7 kips. Finally, calculate the stress in each material. Note from our original free body diagram that the copper cylinder is in tension while the concrete column is in compression. We'll introduce a negative sign here to indicate compressive stress in the concrete.

$$\sigma_{\text{copper}} = \frac{N_1}{A_1} = \frac{12.3}{\pi\left(2^2 - 1.5^2\right)} = 2.24\text{ksi}$$

$$\sigma_{\text{concrete}} = -\frac{N_2}{A_2} = -\frac{12.7}{(6 * 6)} = -0.353\text{ksi}$$

Approach 2: An alternate approach is to start by noting that the total deformation of the bar must be zero. In the case shown in Figure 5.11, we can say that the deformation of segment 1 plus the deformation of segment 2 must add up to zero.

[figure]

Figure 5.11: (a) A bar held between two rigid supports. (b) A free body diagram of the bar.

The internal load in segment 1 is FA and the internal load in segment 2 is FB, so there are currently two unknowns. We may use an equilibrium equation to solve for these two unknowns simultaneously.

$$\frac{F_A L_1}{A_1 E_1} + \frac{F_B L_2}{A_2 E_2} = 0$$
$$F - F_A - F_B = 0$$

Example 5.6 re-solves Example 5.5 using this method instead.

Example 5.6: Bar between 2 supports, 2 materials, and areas

A 15-foot-tall concrete (E = 4,000 ksi) column has a square cross-section of 6 in. by 6 in. A 10-foot-tall-copper (E = 17,000 ksi) cylinder with an outer diameter of 4 in. and in inner diameter of 3 in. is attached to the top of the concrete. The structure is fixed between two supports. A force of 25 kips is applied as shown. Determine the average normal stress in each material.
[figure]

Solution

As before, begin with an equilibrium equation that relates the support loads to the applied load. In this equation we use the convention that forces pointing upwards are positive and forces pointing downwards are negative.
[figure]

$$\sum F_y = 0 : F_1 + F_2 - 25 = 0$$
$$F_1 + F_2 = 25\text{kips}$$

This bar consists of 2 segments – the copper cylinder (segment 1) and the concrete column (segment 2). Since the bar is held between two rigid supports, the total deformation of these two segments must sum to zero. Note that in this diagram segment 1 is in tension while segment 2 is in compression. We must be consistent with the sign convention that tension is positive and compression is negative.

$$\frac{F_1 L_1}{A_1 E_1} - \frac{F_2 L_2}{A_2 E_2} = 0$$

Here F1 and F2 are the internal forces in segments 1 and 2 of the bar. These will be the same as the reaction loads at the supports that we are trying to solve for.

$$\frac{F_1 * (10 * 12)}{\pi \left(2^2 - 1.5^2\right) * (17,000)} - \frac{F_2 * (15 * 12)}{(6 * 6) * (4,000)} = 0$$

Rearrange the equilibrium equation and substitute into the deformation equation.

$$F_1 = 25 - F_2$$
$$\frac{(25 - F_2) * (10 * 12)}{\pi \left(2^2 - 1.5^2\right) * (17,000)} - \frac{F_2 * (15 * 12)}{(6 * 6) * (4,000)} = 0$$

Then simplify and solve for force F2.

$$0.0321 - 0.001284 F_2 - 0.00125 F_2 = 0$$
$$0.0321 = 0.002534 F_2$$
$$F_2 = 12.7\text{kips}$$

Then use the equilibrium equation again to find F1.

$$F_1 = 25 - 12.7 = 12.3\text{kips}$$

These are the same reactions that we found in Example 5.5 when we solved this problem using the other approach. From here we can find the stress in each material as before, noting again from our diagram that segment 2 is in compression.

$$\sigma_{\text{copper}} = \frac{N_1}{A_1} = \frac{12.3}{\pi \left(2^2 - 1.5^2\right)} = 2.24\text{ksi}$$
$$\sigma_{\text{concrete}} = -\frac{N_2}{A_2} = -\frac{12.7}{(6 * 6)} = -0.353\text{ksi}$$

The second type of indeterminate problem involves two materials bonded together in parallel. In these problems it's possible to find the reactions at the supports, but not possible to find the internal force in each material using only equilibrium (Figure 5.12).

[figure]

Figure 5.12: When two materials in parallel are subjected to force F, the force is split between the two materials (F1 and F2). By equilibrium $F_1 + F_2 = F$ but this is not sufficient to find the force in each material.

We have one equilibrium equation but two unknown internal forces. However, since the materials are bonded together we can say that they must deform by the same amount. By setting the deformation for each material equal to each other we can define a second equation that involves the two internal forces and, combined with the equilibrium equation, we can now solve for both internal forces. See Example 5.7 for a demonstration.

Example 5.7: 2 materials in parallel

A 20-ft-tall concrete (E = 4,000 ksi) column has a square cross-section 6 inches on each side. It is reinforced by six pieces of steel (E = 30,000 ksi) rebar that extend through the length of the column. Each has a diameter of 0.5 inches. The column is subjected to a compressive load of 70 kips. Determine the stress in each material.
[figure]

Solution

The 70 kips force will be split between the two materials. We need to find the force in each material before we can calculate the stress. Start by cutting a cross-section through the column, drawing a free body diagram, and writing an equilibrium equation.
[figure]

$$\sum F_y = 0 : F_C + F_S = 70\text{kips}$$

This problem is statically indeterminate, since we have two unknowns and only one equilibrium equation. However, since the two materials are bonded together we can say that they must both deform the same amount.

$$\Delta L_C = \Delta L_S$$

Since we know $\Delta L = \frac{FL}{AE}$

$$\frac{F_C L_C}{A_C E_C} = \frac{F_S L_S}{A_S E_S}$$

The length of both materials is 20 ft, so this term will cancel.
Although the steel is 6 individual pieces, it's fine to determine the total area of the steel and the total force in the steel. The total area of the steel is $A_S = 6 * \pi * 0.25^2 = 1.178\text{in}^2$
For the area of the concrete, calculate the area of the square and then remove the area of the 6 rebar rods.

$$A_C = (6 * 6) - 1.178 = 34.82\text{in.}^2$$

Substitute these into the deformation equation and rearrange.

$$\frac{F_C}{34.82 * 4,000} = \frac{F_S}{1.178 * 30,000}$$
$$F_C = F_S \left[\frac{34.82 * 4,000}{1.178 * 30,000}\right]$$
$$F_C = 3.941 F_S$$

Substitute this into the equilibrium equation.

$$F_C + F_S = 70\text{kips}$$
$$3.941 F_S + F_S = 70\text{kips}$$
$$4.941 F_S = 70\text{kips}$$
$$F_S = 14.2\text{kips}$$

Then since $F_C = 3.941 F_S \rightarrow F_C = 3.941 * 14.2 = 55.8\text{kips}$
Now that we know the force in each material, we can calculate the stress in each material.

$$\sigma_S = \frac{14.2}{1.178} = 12.0\text{ksi}$$
$$\sigma_C = \frac{55.8}{34.82} = 1.60\text{ksi}$$

Step-by-step: Statically indeterminate axial problems

Problems with redundant supports
Approach 1:

1. Write out equilibrium equations. There will be too many unknowns, so put these to one side for now.

2. Remove one of the supports and calculate the amount of deformation that would occur if the support wasn't there using $\Delta L_1 = \sum \frac{FL}{AE}$
3. Replace the force from the removed support and determine the deformation, ΔL_2, caused by this force in terms of the force itself.
4. Set $\Delta L_1 + \Delta L_2$ equal to the total allowed deformation for the bar and use this to solve for the unknown force at the redundant support.
5. Now that one force is known, use the equilibrium equations to determine the other unknown forces.

Approach 2:

1. Draw a free body diagram of the entire structure and write out the relevant equilibrium equations. For axial loads this will just be one sum of force equation, either horizontally or vertically depending on the applied loading.
2. Write an equation saying that the total deformation in each segment of the bar must sum to the total deformation of the bar (zero if the bar is held between two rigid supports). A new segment must be made any time the internal load, cross-sectional area, or material changes.
3. This deformation equation will involve the same two unknowns as the equilibrium equation. Solve these equations simultaneously to find the reaction loads at the supports

Problems with two materials in parallel

1. Cut a cross-section through the member and set up an equilibrium equation for the internal forces where the internal forces will sum to equal the external applied force.
2. Set the deformation of the two materials equal.
3. Solve the deformation equation and equilibrium equation simultaneously to determine the internal force in each material.

5.6 Thermal Deformation

Click to expand

So far, we've studied the effects of axial forces on an object and how they create stresses and deformations. Temperature changes will also cause an object to deform. The strain due to temperature can be predicted by:

$$\varepsilon_T = \alpha \Delta T$$

where

$_{T}$ = *Thermal strain*

= *Coefficient of thermal expansion* $[\frac{1}{^{\circ}C}, \frac{1}{^{\circ}F}]$

ΔT = *Change in temperature* [°C, °F]

As before, strain is dimensionless. The coefficient of thermal expansion is a material constant that can be looked up in handbooks or in Appendix C. Note that for a given change in temperature, the thermal strain will be the same in the axial and transverse directions.

Since strain is also defined as $\varepsilon = \frac{\Delta L}{L}$, we can also predict the deformation due to a change in temperature:

$$\varepsilon_T = \alpha \Delta T = \frac{\Delta L}{L} \rightarrow \Delta L = \alpha \Delta T L$$

If the object is free to expand or contract this doesn't cause any issues and is easy to account for. Many real applications will include a small gap to allow for changes in length due to temperature changes (Figure 5.13). It is even possible that an object is subjected to both a physical force and a temperature change and the total change in length is simply the sum of these effects:

$$\Delta L = \Delta L_F + \Delta L_T = \frac{FL}{AE} + \alpha \Delta T L$$

[figure]

Figure 5.13 – Expansion joint on a bridge, allowing for thermal expansion and contraction of the bridge as temperature changes through the year.

However if we do not design a gap, or if the gap is not large enough, then the object is not free to expand or contract. As the member pushes or pulls on its supports, a physical force is created. This in turn creates a stress in the object and these stresses can be very large. Such problems are statically indeterminate as the force applied on the member by the support is unknown and can't be found using only equilibrium. Solving these problems is very similar to solving the first type of statically indeterminate problem. First, remove a support and determine the amount of deformation that would occur due to the change in temperature if the object were free to deform. Then replace the force from the removed support, which will cause the object to deform in the other direction. The sum of these two deformations will equal the total deformation of the member as before. Example 5.8 works through an indeterminate thermal expansion problem.

Example 5.8: Indeterminate thermal expansion without gap. Then add gap and rework

Steel (E = 200 GPa,   = 11.7x10-6 /°C) train rails are laid end-to-end. Each rail is 20 meters long. A small section of this track is shown below. The rails are laid in winter when the temperature is 0° C. In summer, the maximum temperature is 40°C. Determine the compressive stress in the rail as a result of the temperature change if:
(a) In a particular section of track, the rails are laid with no gap between them.
(b) In another section, the rails have a 5-mm-gap between them.
[figure]

Solution

(a)

First determine the deformation of each rail that would occur if it was free to expand.

$$\Delta L_T = \alpha \Delta T L = 11.7 \times 10^{-6} * 40 * 20 = 0.00936 \text{ m}$$

Since each rail will contact the rail next to it as it tries to expand, each rail will experience a compressive force. This force causes compression in the rail.

$$\Delta L = -\frac{FL}{AE}$$

The actual deformation of each rail must be zero since they are in contact end-to-end. Thus, these deformations must sum to zero.

$$0.00936 - \frac{F * 20}{A * 200 * 10^9} = 0$$

Although we don't know the cross-sectional area of the rail, we can replace $\sigma = \frac{F}{A}$

$$0.00936 - \sigma * \frac{20}{200 * 10^9} = 0$$

$$\sigma = \frac{0.00936 * 200 * 10^9}{20} = 93.6\text{MPa}$$

(b)

In this case we again begin by determining the deformation of each rail that would occur if it was free to expand.

$$\Delta L_T = \alpha \Delta T L = 11.7 \times 10^{-6} * 40 * 20 = 0.00936 \text{ m}$$

Each rail would expand 9.36 mm, but the gap between rails is only 5 mm so the rails close this gap and make contact, which means there will again be some compressive stress. Set up the deformation equation as in part (a), but this time the deformation is 5 mm (0.005 m) instead of zero.

$$0.00936 - \frac{F * 20}{A * 200 * 10^9} = 0.005$$

Replace $\sigma = \frac{F}{A}$ as before and determine the stress.

$$0.00936 - \sigma * \frac{20}{200 * 10^9} = 0.005$$

$$0.00936 - 0.005 = \sigma * \frac{20}{200 * 10^9}$$

$$\sigma = \frac{0.00436 * 200 * 10^9}{20} = 43.6\text{MPa}$$

We can see that installing the rails with a small gap has significantly reduced the thermal stress. Accurately predicting the amount of deformation and leaving a suitably sized gap would reduce the stress even further.

Step-by-step: Thermal deformation

1. Remove one of the supports and calculate the total deformation that would occur if the support wasn't there using $\Delta L_T = \alpha \Delta T L$ for thermal deformation and

$\Delta L = \frac{FL}{AE}$ for any applied axial loads.

2. If the total deformation causes the member to contact a support that prevents some of the deformation, replace the force from the removed support and determine the deformation caused by this force in terms of the force itself using $\Delta L = \frac{FL}{AE}$.
3. Sum these deformations and set them equal to the total allowed deformation. Use this equation to solve for the force at the redundant support.
4. Use equilibrium to determine the force at any other supports.

Summary

Click to expand

Key takeaways

Axial loads cause normal stresses. While we generally calculate the average normal stress, stress concentrations occur close to applied loads, supports, and changes in geometry. These stress concentrations can be large and cause localized failure even if the average stress is within acceptable limits. Modeling stress concentration is beyond the scope of this course, but we can determine maximum stresses for different geometries using stress concentration curves.
It is often desirable to limit the total change in length of a bar subjected to axial loads. Change in length can be predicted directly.
In practice, it is useful to leave a gap to allow for change in length. If a change in length is prevented from happening, this creates a stress in the bar.
Changes in temperature will also cause axial deformation of a bar, known as thermal deformation. Thermal deformation can lead to a physical stress in the bar if the deformation is prevented from occurring by a support.

Key equations

Stress concentrations

$$\sigma_{\text{avg}} = \frac{N}{A}$$

$$\sigma_{\text{max}} = K\sigma_{\text{avg}}$$

Axial deformation

$$\Delta L = \frac{FL}{AE}$$

Thermal strain

$$\varepsilon_T = \alpha \Delta T$$

Thermal deformation

$$\Delta L = \alpha \Delta T L$$

6 Torsional Loading

Learning Objectives

- Insert text

6.1 Introduction

Click to expand

Insert text

7 Beams

Learning Objectives

- Apply equilibrium methods to determine internal shear forces and bending moments equations, and specific values in beams.
- Draw the shear and bending moment diagrams using integration and graphical methods.

Introduction

Click to expand

[Figure 7.1 Examples of Structural Beams]

Beams are structural members that support loads along their length. Typically, these loads are perpendicular to the axis of the beam and cause only shear and bending moments.

[Figure 7.2 Overarching book figure]

As we found in previous chapters, internal forces and moments are crucial to calculating stresses and deflections. The same is true for finding stresses and deflections in beams. Due to the nature of the loadings and constraints of beams, finding internal forces requires new methods in statics. This chapter describes these methods for finding a beam's internal shear force and bending moments.

Sign convention

Before we can present methods for finding internal shear and moments of beams, we need to establish a sign convention. This way, we provide consistency and know how to interpret our results. It should be noted that anytime you start a new project or subject you should ensure that a sign convention is established.

[Figure - will be 7.3]

For the purposes of this book, the shear is positive when the external forces shear off as depicted in the figure above. Another way to think about it is that a positive shear is when the internal shear forces cause a clockwise rotation of the beam segment.

The bending moment is positive when the external forces bend the beam in a concave up shape as indicated in the figure. This causes the top fibers of the beam to be in compression while the bottom fibers are in tension.

7.1 Internal Shear Force and Bending Moment by Equilibrium

Click to expand

One way to find internal shear and bending forces is to cut sections and analyze the free-body diagrams, as we did in previous chapters of this book. The first step for a statically determinate beam is to find the external reactions. To determine the internal forces at any point along the beam, cut the section at that point and draw the free-body diagram from that point to one end of the beam. We will replace the cut with an internal shear force, V, and an internal moment, M, and then use equilibrium equations to solve for those internal.

Example 7.1 Simple internal shear and moment problem

Find the internal shear and bending moment at points J, K, and L in the beam pictured below.
[figure]
The first step is to find the external reactions at the supports A and B.
[figure]
Next, we will draw a free-body diagram of the right end of the beam from point A to J by cutting the beam at section J. The internal forces VJ and MJ are placed at the point of the cut using the positive sign convention. Note: You can assume the direction of these forces; the statics will work out the correct direction.
To find the internal shear, we will sum forces in the y-direction. This results in an internal shear force of + 33kips.
[figure]
To find the internal moment, we will sum moments at point J. This results in an internal moment of +66 kip*ft.
[figure]
To find the internal shear, we will sum forces in the y-direction. This results in an internal shear force of -7 kips.
To find the internal moment, we will sum moments at point K. This results in an internal moment of +90 kip*ft.
Section L
[figure]

To find the internal shear, we will sum forces in the y-direction. This results in an internal shear force of -17 kips. To find the internal moment, we will sum moments at point L. This results in an internal moment of +17 kip*ft.
We can observe that the internal forces vary at different points along the length of the beam. This method of finding the internal stresses is convenient when the loading is simple or when you know a specific point along the length of the beam. However, as the loading gets more complex, we should consider one of the methods outlined in the following sections.

7.2 Relationship between Load, Shear, and Moment

Click to expand

There is a relationship between loading, shear, and moment (and as you will see later in this text, slope and deflection) in a beam. Consider the beam below, which is subjected to a distributed load, w per unit length.

[figure]

We will now look at a small section from that beam that has a width of Δx, shown below. We replaced the distributed load with a resultant force ΔF (this is indicated by a dashed arrow in the FBD).

Since we cut both sides of the beam, we replace each cut with internal shear and moment forces assuming the positive sign convention from the previous section in this chapter.

7.2.1 Relationship between Load and Shear

This FBD is in static equilibrium so we can use our equilibrium equation to sum forces in the y-direction and set it equal to zero.

[math]

Dividing both sides by Δx and then letting Δx approach zero gives us:

[math]

We could then rearrange this by multiplying both sides of the equation by dx.

[math]

Now we can integrate between any two points A and B on the beam:

[math]

Equation 11.1 is valid for distributed loads and not when there is a discontinuity in the shear diagram that is caused by concentrated loads. This relationship should only be used between concentrated loads.

7.2.2 Relationship between Shear and Moment

[figure]

Let's go back to the FBD of the small section of the beam subjected to a distributed load. We can apply another static equilibrium equation, summing moments about point C:

[math]

Dividing both sides by Δx and then letting Δx approach zero gives us:

[math]

We could then rearrange this by multiplying both sides of the equation by dx.

[math]

Now we can integrate between any two points A and B on the beam:

[math]

Similar to equation 11.1, equation 11.2 is not valid at points where a concentrated force or concentrated moment occurs. These concentrated loads cause discontinuities in the moment diagram however, equation 11.2 can be used in between these concentrated loads.

7.2.3 How can I use my calculus knowledge when building Shear and Moment Diagrams?

In a basic sense, the relationship between load, shear, and moment can be described in the figure below.

[figure]

We can combine these relationships with what we know about derivatives and integrals from our calculus course. Here are some items of note:

- The slope of the shear diagram at any point is the derivative of the shear function evaluated at that point. This is equal to the sign and magnitude of the distributed load. If there is no load on a section of the beam then the slope of the shear diagram would be zero.

- Similarly, the slope of the moment diagram at any point is the derivative of the moment function evaluated at that point. This is equal to the sign and magnitude of the shear at that point.
- The area under the distributed load between two points is equal to the change in the shear between those same two points.
- The area under the shear diagram between two points is equal to the change in the moment between those same two points.
- The maximum moment occurs where its derivative (shear) is equal to zero.
- At the points of concentrated forces, the shear diagram will jump up or down depending on the direction of the force. If the force is up then the shear diagram will jump up, and if the force is down, the shear diagram will drop down.
- At the points of concentrated moments, the moment diagram will jump up or down depending on the direction of the rotation. If the concentrated moment is clockwise, the moment diagram will jump up, and if it is counter-clockwise, the moment diagram will drop down. This "opposite" direction effect is for the internal bending moment is the reaction to the applied moment to stay consistent with the established sign convention. The shear diagram will not be impacted.
- If the load can be represented with a polynomial, then we can easily predict the degree of the subsequent shear and moment functions. For example, if the load is a uniformly distributed load (constant), the shear will be a linear function (n+1), and the moment will be a parabola (n+2).

Use all that you know about calculus and the relationships between functions when you derive and integrate to build, check, and analyze your shear and moment diagrams.

7.3 Determining Equations by Equilibrium and Equations

Click to expand

The easy relationship between load, shear, and moment allows us to build complete shear and moment diagrams using a few different methods. This section shows you a method that will work for both simple and complicated loading situations. Let's work through an example to illustrate this method.

Example 7.2

[figure]
The first step is to draw a FBD of the beam being sure to change the supports to the correct external reaction forces, as shown below.

[figure]
We will then use static equilibrium equations to solve for the magnitude of the support reactions.
[math]
[figure]
Now we will cut a cross-section within the loading region we are concerned with. When choosing sections be sure to cut within the uniform load and between loads and reactions. For this example, we will need to cut three sections to get the complete shear and moment diagram. These sections are depicted with the green lines. We will then use equilibrium equations to find the internal shear force as a function of x, V(x).

Section between A and C

[figure]
[math]
Good Gut Check Notice that you can give a quick check on your statics as V(x) is the derivative of M(x). For polynomials, this is a quick test to see if your equations make sense.
[figure]
We can plot our V(x) and M(x) equations on an axis from x = 0 to x = 6ft. Notice how we draw the shear diagram directly below the beam and the moment diagram directly below the shear. These three (load, shear, and moment) share the same x-axis, whereas the vertical axis for the Shear (V) and Moment (M) diagrams have different units and possibly different scales.
Drawing the shear and moment diagrams directly below the beam is good practice so that you can get a complete picture of what is going on along the length of the beam.

Section between C and D

[figure]
[math]
[figure]
Again, we can add the section from C to D using our V(x) and M(x) equations. You will notice that since there are no applied loads on the beam from C to D, the shear diagram is constant. This means that the moment diagram is linear between x = 6 and x = 8 ft.

Section between D and B

[figure]
[math]
[figure]
Finally, we will graph the V(x) and M(x) equations we obtained from point D to point B (x = 8 to x = 10 ft).
We now have complete shear and moment diagrams. Be sure that you label your axis, including units, and all pertinent values.
These diagrams are used in the design of the beam and its components.
[figure]

Good Gut Check We can see that in this example, the entire length of the beam is subjected to positive moment. Going back to our sign convention from the beginning of this chapter, a positive moment indicates a concave up behavior. When we look at the beam and the external loads it makes sense as the beam will want to bend concave up between supports, as shown in the figure.

7.4 Graphical Method Shear Force and Bending Moment Diagrams

Click to expand

When the loading is relatively simple, consisting of concentrated forces, concentrated moments, and uniformly distributed loads, we can use geometry to find the area of the load and shear diagrams since the shapes are simple. Let's work through an example to illustrate this method.

Example 7.3: Draw the shear and moment for the beam shown.

[figure]
[figure]
The first step is to draw a FBD of the beam, being sure to change the supports to the correct external reaction forces, as shown below.
We will then use static equilibrium equations to solve for the magnitude of the support reactions.
[math]
Now that we know the external forces, we can build the shear diagram. We will start at the leftmost point of the beam, point A.
[figure]
The first force we encounter is at point A, the reaction force of 11kN. The force is going up, so we will do that same thing on our shear diagram.
From that point, we look at the beam, and there are no forces acting between points A and C. This indicates that our shear diagram will remain constant at 11kN.
[figure]
The 4kN/m uniformly distributed load starts at point C and goes for 6 meters until point D. The force is going down for a total of 24kN (4kN/m * 6m). Since the load is constant from point C to D, the shear will be linear between those points. The slope of the shear diagram will be -4 and over those 6 meters will decrease the total of 24kN.
[figure]
At point D, there is a concentrated 10kN load going down. On the shear diagram, this load will be represented by a discontinuity, jumping down by 10kN to -23kN.
From that point, we look at the beam, and there are no forces acting between points D

and B. This indicates that our shear diagram will remain constant at -23kN.
[figure]
At point B, the roller support, there is an external reaction of 38kN going up. This concentrated force will cause a discontinuity in the shear diagram. From -23kN we will add 38kN to end at 15kN.
From that point, we look at the beam, and there are no forces acting between points B and E. This indicates that our shear diagram will remain constant at 15kN.
[figure]
At point E, there is a 15kN concentrated force going down. This concentrated force will cause a discontinuity in the shear diagram. From 15kN we subtract 15kN to end back at zero.
The shear diagram should start and end at zero. At the end of the beam, our shear diagram "closes," which means that we end back at zero. This is a good check that you are on the right track. If you round your reactions you might be slightly off at the end of your shear diagram.
Now that we have the shear diagram, we can build the moment diagram. Remember from the previous section that the internal moment is the area under the shear diagram. Our shear diagram consists of basic shapes (rectangles and triangles) so we can use geometry to find these areas.
[figure]
Just like for the shear diagram, we will start at zero at the leftmost point of the beam, point A. Our first section from A to C is a rectangle. We will calculate the area under the shear curve to find the change in the internal moment between A and C. We will keep in mind the following three things:

- **Magnitude of the Change -** This is the area under the curve. The height of the rectangle is 11kN, and the width is 2m, so the area is 22kN*m.
- **Direction of the Change -** This area is on the positive side of the shear diagram, which indicates that it will go up from A to C.
- **Shape of the Segment** – The shear diagram is constant, so the moment diagram will be linear with a slope of 11.

[figure]
From point C to D there are two triangles, one that is on the positive side of the shear diagram and the other that is on the negative side.
To calculate the area under the curve we will need to first calculate the distance from point C to where the shear diagram crosses the x-axis.
There are many ways to do this. We illustrate using similar triangles to find this distance. In the figure, we are comparing the larger yellow triangle and the smaller pink triangle, where our unknown distance x is the base. We set up the following proportion to accomplish this:

[math]

Magnitude of the Change

This is the area under the curve. The area of the triangle is 15.125 kN*m [A = ½ (2.75m)(11kN)]

Direction of the Change

This area is on the positive side of the shear diagram, which indicates that it will go up from C to the zero point on the shear diagram. So we will add the area, 15.125, to the internal moment at point C (22kN*m) to be at 37.125 kN*m at the point of zero shear.

Shape of the Segment

The shear diagram is linear, so the moment diagram will be parabolic that is concave down.

[figure]

We can now account for the second triangle from C to D.

Magnitude of the Change

This is the area under the curve. The area of the triangle is 21.125 kN*m [A = ½ (6 - 2.75m)(13kN)].

Direction of the Change

This area is on the negative side of the shear diagram, which indicates that it will go down from the point of zero shear to point D. So we will subtract the area, 21.125 to the internal moment at the zero point (37.125kN) to be at 16 kN*m at the point D.

Shape of the Segment

The shear diagram is linear, so the moment diagram will be parabolic that is concave down. The concavity of a parabolic function can be determined by examining whether the shear diagram is increasing or decreasing. In this example, the shear diagram is decreasing between points C and D so the parabola will be concave down.

[figure]

We are now able to account for the section from D to B.

Magnitude of the Change

This is the area under the curve. The height of the rectangle is 23kN, and the width is 2m, so the area is 46kN*m.

Direction of the Change

This area is on the negative side of the shear diagram, which indicates that it will go down from D to B. We will subtract the area from the internal moment at D. So we will do 16 − 46 to end at negative 30.

Shape of the Segment

The shear diagram is constant, so the moment diagram will be linear with a slope of -23.

[figure]

Finally, we can build our moment diagram from B to E.

Magnitude of the Change

This is the area under the curve. The height of the rectangle is 15kN, and the width is 2m, so the area is 30kN*m.

Direction of the Change
This area is on the positive side of the shear diagram, which indicates that it will go up from B to E. We will add the area from the internal moment at B. So we will do (-30 + 30) to end at zero.
Shape of the Segment
The shear diagram is constant, so the moment diagram will be linear with a slope of +15. At the end of the beam, our moment diagram "closes," which means that we end back at zero. Moment diagrams must start and end at zero. This is a good check that you are on the right track. If you round your reactions and areas, you might be slightly off at the end of your moment diagram.
[figure]
Our final product is in the figure on the left. For a complete shear and moment diagram your axis should be labeled, including units, and the pertinent values indicated on each diagram. As we mentioned earlier it is best to draw these right below the beam so it is easy to see what the internal shear and bending moment forces are in relation to a location on the beam.
[figure]
Good Gut Check We can see that in this example, this beam is subjected to both positive and negative moments. Remember from the beginning of this chapter that a positive moment indicates concave up bending behavior, and a negative is concave down. When we look at the beam and the external loads it makes sense as the beam will want to bend concave up between supports in the area of positive moment. The beam will then bend concave down over support B in the area of negative moment, as shown in the figure.

Example 7.4: Draw the shear and moment for the beam shown

[figure]
This beam a little different than other examples in this chapter as the support is a fixed end and includes a force-couple system at point C. However, the processes that we presented here still apply in this situation.
[figure]
The first step is to draw a FBD of the beam, being sure to change the supports to the correct external reaction forces, as shown to the left. The 2 kips that acts on the arm at point B will cause a force on beam AD at the connection point, C. Additionally there will be a concentrated moment from this force that also acts at point C. The magnitude of this concentrated moment is (2kips)*(1 ft) or 2 k*ft counter clockwise.
We will then use static equilibrium equations to solve for the magnitude of the support reactions.
[math]
[figure]

We can now build the shear diagram. Keep in mind that the concentrated moments (at point A and C) do not contribute to the shear diagram.
We start with the vertical reaction at point A. This is going up 7kN. Then between A and C there are no loads on the beam so the shear diagram remains constant.
The concentrated force at point C is 2 kips down. So we will subtract 2 kips from 7kips. There are no loads between points C and D so the shear diagram remains constant at 5 kips.
At point D there is a concentrated load of 5kN down. This will bring our shear diagram to a close at zero.
[figure]
We will build the moment diagram with the combination of the area of the shear diagram and the concentrated moments.
We start from zero and go down 34 k*ft since the reaction at a, MA, is counter-clockwise.
Between A and B
Magnitude of the Change
This is the area under the curve. The height of the rectangle is 7 kips, and the width is 3 ft, so the area is 21 k*ft.
Direction of the Change
This area is on the positive side of the shear diagram, which indicates that it will go up from A to C. We will add the area from the internal moment at A (-34 + 21 = -13).
Shape of the Segment
The shear diagram is constant, so the moment diagram will be linear with a slope of +7.
[figure]
At point C, there is an applied concentrated moment of 2 k*ft counter-clockwise. This will create a discontinuity in the moment diagram by dropping down 2 (-13 – 2 = -15).
[figure]
Between C and D
Magnitude of the Change
This is the area under the curve. The height of the rectangle is 5 kips, and the width is 3 ft, so the area is 15 k*ft.
Direction of the Change
This area is on the positive side of the shear diagram, which indicates that it will go up from C to D. We will add the area from the internal moment at A (-15 + 15 = 0).
Shape of the Segment
The shear diagram is constant, so the moment diagram will be linear with a slope of +5.
[figure]
The final shear and moment diagram is in the figure to the left. You will notice that this cantilever structure with downward loads will bend concave down. This indicates that the entire beam will be in negative moment—which is what our moment diagram indicates.

Case Study

Click to expand

Use Kurt Gramoll's Case Study with Semi Truck ??

https://www.ecoursesbook.com/cgi-bin/ebook.cgi?topic=me&chap_sec=03.2&page=case_intro

https://www.ecoursesbook.com/cgi-bin/ebook.cgi?topic=me&chap_sec=03.2&page=case_sol

Summary

Click to expand

Shear and moment diagrams allow us to calculate and visualize the internal forces of beams. These internal forces are used to calculate stresses and deformations (upcoming chapters). The general procedure for building the shear and moment diagrams is as follows:

1. Sketch the beam, replacing support conditions with equivalent force(s).
2. Find the support reactions using equilibrium.
3. Use the method of equations or geometry or a combination to build the shear diagram directly below your beam sketch.
4. Use the method of equations or geometry or a combination to build the moment diagram directly below your shear diagram.
5. Ensure that your diagrams are labeled, including units, and that all pertinent values are indicated.

8 Geometric Properties

Learning Objectives

- Insert text

8.1 Introduction

Click to expand

Insert text

9 Bending Loads

Learning Objectives

- Describe the bending behavior of beams.
- Calculate the bending stresses in beams.
- Select an adequate beam using the section modulus.
- Analyze beams subjected to unsymmetric bending.

Introduction

Click to expand

Beams are structural members that support loads along their length. Typically, these loads are perpendicular to the axis of the beam and cause only shear forces and bending moments.

Figure 9.1 Overarching book figure

As we found in previous chapters, internal forces and moments and section properties are crucial to calculating stresses and deflections. The same is true for finding stresses and deflections in beams. This chapter describes calculating bending stresses in beam with considerations for unsymmetric bending and beam design.

Figure 9.2 Looking up at bridge beams used to support a landscaped bicycle and pedestrian bridge in Seattle.

9.1 Bending Stress

Click to expand

This section will discuss bending behavior of straight, symmetric, homogeneous beams. This section will be limited to beams that have a cross section that is symmetric with an axis and the bending moment is around an axis that is perpendicular to the axis of symmetry. Examine the initially unaltered beam depicted in Figure 9.3, characterized by a rectangular cross-section, and annotated with both horizontal and vertical grid lines.

Figure 9.3 A unloaded beam with a rectangular cross-section.

Upon the application of a bending moment, the tendency emerges to deform these lines, conforming to the pattern illustrated in Figure 9.4. Observe the curvature of the horizontal lines, and note that the vertical lines, while maintaining straightness, undergo a rotation. The application of a bending moment induces stretching in the material at the bottom section of the bar and compression in the material at the top section. As a result, within the transitional zone between these two regions, there exists a surface known as the **neutral surface**, where the horizontal fibers of the material experience no change in length (no compression or tension). When viewed on a cross-section, the neutral surface appears as a horizontal line known as the **neutral axis**.

Figure 9.4 The deformation of the beam in Figure 9.3 when subjected to a symmetric bending moment.

We can make a similar observation when a kids toy undergoes bending as shown in Figure 9.5. This toy is subjected to a bending moment in the opposite direction of the beam in Figure 9.4, this will switch the behavior of the top and the bottom surfaces. Notice in Figure 9.5 the fibers at the top are pulled apart while the bottom surface fibers are pushed together.

Figure 9.5 The distortion of a kid's toy when undergoing bending.

We can take this knowledge about deformations due to bending and apply it to the failure of the column in Figure 9.6. Notice that the column was fixed at the base and was subjected to a transverse force that caused failure. The fibers on the right side of the column failure point were pulled apart while the fibers on the left side of the column failure point were compressed.

Figure 9.6 Column failure due to bending.

We will make the following assumptions about deformations due to bending:

- There must be a neutral surface parallel to both the upper and lower surfaces, where the length remains constant.
- Throughout the deformation, all cross sections of the beam remain plan and perpendicular to the longitudinal axis.
- The cross section will keep it's shape, we will ignore the Poisson effects discussed in Chapter 4 of this text.

Figure 9.7 Element in beam bending.

To understand bending stress in a beam subjected to arbitrary loads, examine a small element extracted from the beam in Figure 9.7. The derivation of the bending strain equation remains unaffected by the beam type or specific loads. Remember the fundamental definition of normal strain:

[math]

We can use this to calculate the normal strain along AB in our beam.

[math]

Distance y is measured relative to the neutral surface and is positive above the neutral surface and negative below. Prior to bending, the line AB is the same length at all values of y. However, when bending occurs, the length of A'B' varies. The length A'B' experiences a reduction in length as the section moves farther above the neutral surface, and conversely, it undergoes an increasing extension below the neutral surface. By definition, the length of the neutral surface doesn't change and remains the same as length AB. We can describe the lengths AB and A'B' using the radius of curvature ( ) and the differential angle (d) shown in Figure 9.7.

[math]

We can now substitute these lengths into our strain equation:

[math]

Simplifying

[nath]

This relationship shows us that the longitudinal strain, , varies lineraly with the distance, y, from the neutral surface. The maximum stress occurs at the outermost fibers, extreme top and bottom of the section. We call this maximum distance from the netrual surface, c. We can now write a relationship for the maximum absolute value of the strain, m.

[math]

Assuming that our material behaves in a linearly elastic manner, we can use Hooke's Law, [math], to rewrite the strain relationship above into a relationship of stresses:

[math]

Therefore, similar to the variation in normal strain, normal stress, , will fluctuate from zero at the neutral surface to a maximum value, max, at a distance c from the neutral surface as shown in Figure 9.8.

Figure 9.8 Bending stress variation.

To determine the position of the neutral surface, we necessitate the condition where the resultant force generated by the stress distribution across the cross-sectional area is equal to zero.

[math]

We can now substitute our previous relationship between stress and distance from neutral axis.

[math]

Since [math] does not equal zero we are left with:

[math]

This integral represents the first moment of area, as discussed in Section 8.1. This equation indicates that the first moment of the cross-section about its neutral axis must be zero. Recall that the location of the centroid was determined by [math]. Consequently, for a member experiencing pure bending and as long as the stresses remain within the elastic range, the neutral axis traverses through the centroid of the section since [math] (the distance from the neutral axis to the centroid) is zero.

We can ascertain the stress in the beam by setting the moment M equal to the moment produced by the stress distribution around the neutral axis.

[math]

Since

[math]

We can write

[math]

You might notice that the integral, [math], represents the moment of inertia, or second moment of area, of the cross-sectional area about the neutral axis. We will denote the moment of inertia with **I**. This topic was covered in Section 8.2 of this text if you need a review.

Rearranging the previous equation to obtain the **flexure formula**:

[math]

Where:

$_{max}$ = the maximum normal stress in the beam, note that a complete description includes magnitude, units, tension, or compression

M = the internal moment calculated about the neutral axis of the cross section, determined from method of sections or shear and moment diagrams

c = the perpendicular distance from the neutral axis to the point the farthest away from the neutral axis

I = the moment of inertial or second moment of area about the neutral axis

We know from previous derivation that:

[math]

we can rearrange to provide the following relationship:

[math]

We can determine a similar flexure formula to calculate the bending stress at any point along the cross section:

[math]

Similar to Chapter 2, when reporting normal stress you include if that stress is in tension or compression. During the derivation, we used a beam that was bending concave upwards. In Chapter 7 we indicated that this type of moment was considered positive. This leads to compression above the neutral axis and tension below. The inverse is true when the beam is subjected to negative bending moment as shown in Figure 9.9.

Figure 9.9 Bending stress sign convention.

Step-by-step: Calculating Bending Stress

1. Determine the **internal moment** at the point to which you want to calculate the bending stress. If you know the point along the length of the beam that you are investigating, then you can cut a section and apply equilibrium equations to determine this moment. If you need to find the maximum bending stress then draw the moment diagram to find the maximum internal moment. (Review of this is in Chapter 7 of this text).

2. Determine the section properties of the beam. This will require knowing the location of the **centroid**, **neutral axis**, and the **moment of inertia**. (Review of this is in Chapter 8 of this text or a table listing values of I for selected common shapes is in the Appendix.)

3. Identify the specific **location on the section** for which you are computing the normal stress, y (or c, if you are calculating the maximum normal stress.)

4. Ensure that the units of M, I, and y (or c) are consistent then use one of the flexure formulas to calculate bending stress. Ensure that the result contains the magnitude, units, and tension or compression.

Example 9.1: Simple bending stress problem

A beam with the shown cross section is subjected to a positive moment of 50 k-in. Calculate the normal stress at the top, bottom, and interface between flange and web.
[figure]
We first start with the internal moment, this is a given in the problem statement to be +50k-in. Positive moment indicates that the beam is bending concave up, compression above the neutral axis and tension below.
[figure]
We next need to determine the beam properties – centroid and moment of inertia. Since

this is not a standard shape and the shape is not symmetric in the y direction, we will need to calculate this by hand. We cover this shape in detail in Chapter 8.
[figure]
[table with math]
[math Y bar =]
There are three parts to this problem that change the y value. We will start with calculating the stress at the top of the section, using the flexure formula:
[math]
Before we plug in our values, we need to ensure that the units are consistent.

- M = 50k · in.
- y = 2.25 in.
- I = 86.0625 in.4

Since all units are in kips and inches, we are going to plug into our flexure formula:
[math]
For a complete answer we need to include magnitude, units, and tension or compression. The magnitude is 1.31, the units will be [math] or ksi. To determine the tension or compression we will look at the sign of the internal moment and where along the section we are calculating the bending stress. In our example the moment is positive, and we calculate the stress at the top of the section our answer will be in compression.
[figure]
[math]
Note that, as long as we are careful to use the correct signs for M and y (both positive in this case), the answer will also come out with the correct sign (negative in this case which indicates compression).
We will now calculate the normal stress at the bottom of the beam. The only quantity that changes is the y value – we will now use the distance from the neutral axis to the bottom of the beam (3.75 in).
[math]
The moment is positive, so the beam bends concave up, anything below the neutral axis is in tension.
[math]
This again works mathematically as long as we use the correct positive sign for the M and negative sign for y (since the point of interest is below the neutral axis). The answer comes out positive, indicating tension and matching our expectation.
Finally, we can calculate the normal stress at the junction of the flange and the web. We will need to find the distance from the neutral axis the flange web junction.
[figure]
[figure]
[math]

Since the flange web junction is above the neutral axis and our beam is subjected to positive moment, our bending stress will be in compression.
[math]

Example 9.2: Bending stress problem

Calculate the maximum tensile and maximum compressive bending stress in the beam shown below.
[figure]
[figure]
[figure]
The first step is to find the internal forces in the beam so that we can find the maximum positive and negative moments. The best way to do this is to draw the moment diagram. We did this in detail in an example problem [which one?] in Chapter 7 of this text.
The moment diagram is to the left and we can see that the maximum positive moment is 37.125 kN*m and the maximum negative moment is 30 kN*m.
We next need to determine the beam properties – centroid and moment of inertia. Since this is not a standard shape and the shape is not symmetric in the y direction, we will need to calculate this by hand. We cover this shape in detail in Chapter 8 [link to centroid and moment of inertia example in ch 8].
[figure]
[figure]
[table with math]
Y bar =
[table with math]
Ix =
The positive bending moment will cause compression at the top of the cross-section and compression at the bottom. The negative bending moment will cause tension at the top and compression at the bottom. It's not obvious by inspection which of these two tensile stresses will be larger, nor which of the two compressive stresses will be larger. We should calculate all four to compare.
Before we plug in our values, we need to ensure that the units are consistent so we will change everything to Newtons and meters.

- M_{pos} = 37.125 kN · m = 37.125 x 103 N · m
- M_{neg} = 30 kN · m = 30 x 103 N · m
- y_{top} = 89.43 mm = 0.08943 m
- y_{btm} = 130.57 mm = 0.13057 m
- I_x = 90,862,095 mm4 = 90.862095 x 10-6 m4

We will start with calculating the stress at the top and bottom of the section due to the maximum positive moment, using the flexure formula:
[figure]
[math]
Now we can do something similar for the maximum negative bending stress.
[figure]
[math]
The overall maximum tensile stress for this beam occurs at the bottom of the section due to positive moment, 53.35 MPa (T). The overall maximum compressive stress (43.11 MPa (C)) also occurs at the bottom of the section in a different part of the beam, but is due to the negative moment.
For many situations the overall largest stress would be sufficient. However, there are materials that behave differently depending on whether they are subjected to tensile or compressive stress. Concrete is a good example of this, as it is very good with compressive stresses but poor with tensile stresses. Steel rebar is placed inside concrete members to support the tensile stresses as shown in Figure 9. 10. At this point in your engineering career it is good practice to report both the maximum tensile and compressive stresses at each critical point on the beam.
Figure 9.10 The end of the prestressed concrete beam showing the prestressing steel and rebar.

9.2 Beam Design for Bending

Click to expand

Frequently, beam design is governed by the bending moment. In this section, we will leverage our understanding of bending stress to design beams capable of withstanding their applied internal moments. We will design two common categories of beams. We'll start with rolled steel beams of varying cross sections. You can find a sampling of these standard shapes in Appendix A of this text. We'll then discuss rectangular cross-sections, which are common for timber beams.

A safe design requires that the allowable stress of the material used is greater than the stress due to the loading. When we design a structure, we need to know what the material properties are so that we can produce safe designs. In the flexure equation for maximum bending stress, [math], there are four variables: m, Mmax, c, and I. We will start with the allowable stress of the material, all, and use the [Equation] derived from the loading condition. What is left are section properties, I, and c. For standard shapes these are combined into one variable, section modulus (S):

[math]

We can substitute this value into the flexure formula and rearrange for Smin – the minimum allowable value of the section modulus for the beam:

[math]

9.2.1 Design of Standard Steel Sections

As long as we design our beam such that its section modulus is larger than Smin, the beam is guaranteed not to fail due to bending stress. Rolled steel beams tend to have relatively complex cross-sections with multiple dimensions that can be varied in order to alter its section modulus and therefore its resistance to bending stress. Designing each dimension individually would be time consuming and it would be costly to manufacture a unique beam for every loading condition. Instead, beams are mass produced in standard sizes and engineers simply select the most appropriate beam for their specific need. There will be many standard steel shapes that will satisfy this equation, meaning they will be safe. Engineering is determining which of the shapes that work you will use. Often there are many criteria that you will need to consider. In this section we will use cost to determine the shape. Steel beams are priced by their weight, the higher the weight the more expensive. To choose the most economical shape, we will choose the shape with the smallest weight (smallest cross-sectional area).

There are basic structural steel shapes that have standardized cross sectional dimensions. W shapes (standing for Wide Flange) are often used for beam design as they have an efficient cross section. Figure 9.11 is a sample of the beam table in the Appendix.

Figure 9.11 Portion of the W Section table from Appendix A

Each beam has a designation that gives us quick information about the section, see figure 9.12.

Figure 9.12 Description of the Shape designation in U.S. Customary Units.

Notice in the sample beam section table in Figure 9.11 they are all W shapes and grouped by approximate height (first number). Within each height grouping the shapes are arranged from lightest to heaviest (second number) when going from the bottom to top of the height grouping. For each standard shape in this table the cross-sectional area, depth (height), flange and web dimensions, moment of inertia in the x and y directions, section modulus, in the x and y direction, and the radius of gyration in the x and y directions (this is a geometric property related to buckling, we will use this in Chapter 15 of this text) are given.

Step-by-step: Most Economical Beam Design (standard shapes)

1. Establish the allowable stress, $_{all}$, value of sigma for the chosen material by referring to a table of material properties or consulting design specifications. Proceed with the following steps assuming that $_{all}$ is the same for tension and compression.

2. Determine the maximum absolute value of the bending moment [Equation] **i**n the beam by drawing the shear force and bending moment diagrams. (Review of this is in Chapter 7 of this text).

3. Calculate the minimum allowable value of the section modulus using this equation: [math]

4. For a rolled steel beam, refer to the relevant table in the Appendix. Among the accessible beam sections, focus solely on those with a section modulus surpassing the minimum value calculated in step 3, [math]. Start at the bottom of the table and select one beam in each height section (if applicable). Then from this short list, choose the section with the smallest weight per unit length, as it represents the most economical option for which [math]. It's essential to note that this choice may not necessarily correspond to the section with the smallest S value.

In certain instances, the selection of a section might be constrained by factors such as the permissible depth of the cross-section or the allowable deflection of the beam. This discussion is limited to materials that behave the same in tension and compression. If this is not the case (such as when you use concrete), then you may need to check multiple points along the length of the beam. Additionally, if the section is not symmetric about the neutral axis, the maximum tensile and compressive stresses may not occur at the point where M is max or min. Lastly, this procedure only considers bending stresses. Although bending stresses do control the design of most beams, there are instances where shear or deflection will control. We will discuss this more in Chapter 11 in this text.

Example 9.3: Design of a Standard Steel Section

Knowing that the allowable normal stress for the steel used is 24 ksi, select the most economical wide-flange beam to support the loading shown.
[figure]
The first step is to find the internal forces in the beam, we will start by calculating the reactions.
[figure]
[math]
Now that we have the reactions, we can build the shear diagram. We will start on the left side and go up the reaction at support A, 6.45 kips. We will continue along the beam following the loads to finish the shear diagram.
[figure]
To build the moment diagram we will calculate the area under the curve from the shear diagram using geometry. We will calculate the area of three rectangles and two triangles.
[small figure?]
[math]

From the moment diagram we can see that [math]. We will use this value and all = 24 ksi (given in the problem statement) to calculate the minimum section modulus needed, S_{min}.
[math]
Now that we know the minimum section modulus needed for this beam, we will go to the standard beam table in Appendix A. Focusing on the Sx column, we will start at the bottom of the table and choose one shape from each height section that is applicable. For our beam, the following sections will be adequate for bending:

- W8 x 21
- W10 x 19
- W12 x 22
- W14 x 22

We didn't choose any from the W4, W5, and W6 groupings because none of these beams have a section modulus greater than 17.64 in^3. We also didn't continue with groupings greater than W14 because those sections are far larger than what we need as noted by the weight (second number in the designation).
Now we will choose the most economical beam by comparing the last number in the designation, as this represents the weight per foot of the beam. The most economical section for this material and loading is the **W10 x 19**. This beam weighs 19 lb/ft and is the lightest of the beams we identified.

9.2.2 Design of Rectangular Sections

Rectangular cross-sections are common for timber beams. There are only two dimensions that need to be specified, base and height. While there are standard rectangular cross-sections too (e.g. 2" x 4") we'll approach the design of rectangular cross-sections by assuming that one dimension scales with the other and so we only need to specify a single dimension. Rather than select from a list of standard cross-sections, we'll specify this required dimension exactly.

We can calculate the minimum section modulus as before

[math]

Now set this value

[math]

For a rectangle

[math]

So

[math]

Problems will be set up such that there is a relationship between b and h, which leaves only one unknown that can be solved for directly. This will be the minimum acceptable size for this dimension—anything larger than the calculated value will not fail due to bending stress.

Example 9.4: Design of rectangular cross-section

A simply supported timber beam spans a gap of 10 m. It is subjected to a uniform distributed load w = 15 kN/m as shown. The beam has a rectangular cross-section and will be manufactured such that the height is 2x the base. Determine the minimum acceptable dimensions for the cross-section of this beam if the maximum allowable bending stress of the timber is 12 MPa.
[figure]
The allowable stress is given as all = 12 MPa. Determine the magnitude of the maximum internal bending moment by drawing the shear force and bending moment diagrams. Since the loading is symmetric, the total applied load of 15 * 10 = 150 kN will be split evenly between the two supports. The reaction force at each support will be 75 kN.
[figure]
The magnitude of the maximum internal bending moment [math]
Now we can determine the required section modulus. Be careful with the order of magnitude of the bending moment and allowable stress.
[math]
We can use this value to determine the required dimensions of the cross-section. Recall that h = 2b for this beam.
[math]
The minimum required dimensions to ensure this beam doesn't fail due to bending stress are b = 153 mm and h = 306 mm.

For the purposes of this text, it is appropriate to give exact answers to 3 significant figures. In practice we wouldn't generally specify the required dimension down to the mm since this is unnecessarily precise to manufacture. It is more practical to design to perhaps the nearest 10 mm, remembering to always round up. For example, we calculated the required dimension b = 153 mm. We must not round this down to 150 mm as this are below the required value, but rounding up to 160 mm would be acceptable. While not included in this problem, there would also be a factor of safety (Section 4.8) included in all designs in practice.

Step-by-step: Title?

1. Establish the allowable stress, $_{all}$, value of sigma for the chosen material by referring to a table of material properties or consulting design specifications. Proceed with

the following steps assuming that all is the same for tension and compression.

2. Determine the maximum absolute value of the bending moment [Equation]in the beam by drawing the shear force and bending moment diagrams. (Review of this is in Chapter 7 of this text).

3. Calculate the minimum allowable value of the section modulus using this equation: [math]

4. Set this [math], substitute in the given relationship between b and h, and solve for the unknown dimension.

9.3 Unsymmetric Bending—Moment Arbitrarily Applied

Click to expand

Occasionally, a member may experience loading where the bending moment, M, does not align with one of the principal axes of the cross-section. In such cases, it is advisable to initially resolve the moment into components along the principal axes. Subsequently, the flexure formula can be applied to ascertain the normal stress induced by each moment component. Finally, by employing the principle of superposition, the resultant normal stress at the specific point can be determined.

Figure 9.13 Moment arbitrarily applied to a rectangular cross section.

To systematize this process, envision a beam with a rectangular cross-section subjected to a moment, **M**, as depicted in Figure 9.13(a). Here, **M** forms an angle, , with the maximum principal z-axis, which is the axis of maximum moment of inertia for the cross-section. It is assumed that is positive when directed from the positive z-axis towards the positive y-axis. We can resolve **M** into y and z components:

[math]

We can see in Figure 9.13 (b) the components of M acting on the rectangular cross section. It is important to keep track of which side of the beam is in compression and tension when subjected to both Mz and My. We will use tensile stress as positive and compressive stress as negative. When we look at a +Mz we can see that this moment causes the top of the section to be in compression and the bottom of the section to be in tension. When we inspect the section when subjected to +My, we can see that the left side will be in tension while the right side will be in compression. Combining what we know about the sign convention with the flexure formula we can calculate the resultant normal stress at any point on the cross section using the following equation:

[math]

Note that for the signs to work correctly we must assign the appropriate sign to the y and z values and to the components of the bending moment. These values are measured from the coordinate system that originates at the centroid where the x axis is coming out of the page. You can also ignore all of the signs and use your spatial awareness to determine if each term is in tension or compression due to the applied moment.

Previously we identified the neutral axis—a line marking points of zero stress, When the bending moment acted around the horizontal axis, the neutral axis was also horizontal and passed through the centroid of the cross-section. When the bending moment acts at an angle ( ) from the horizontal axis, the neutral axis will still pass through the centroid but it will not be horizontal. It too will be at an angle ( ). We can now determine the orientation of the neutral axis, as this will commonly not coincide with the axis of bending moment. We can derive the orientation of the neutral axis by setting the previous normal stress equation equal to zero:

[math]

Now we can rearrange this to solve for y.

[math]

Now we can put this equation in terms of M by using [math] and [math] to obtain:

[math]

Since the slope of the line is [math] where the angle represents the angle the neutral axis forms with the z axis and can be written as:

[math]

Example 9.5: Unsymmetric Bending

A bending moment M = 400 kip in. is applied around an axis 65° above the negative z-axis as shown. Determine the bending stress at corners A, B, C, and D. Also determine the orientation of the neutral axis with respect to the z-axis.
[figure]
Solution
We first determine the components of the bending moment around the y- and z-axes.
[math]
We'll also need the area moment of inertia around both axes.
[math]
Now we can calculate the stress at each corner using.
[math]
Pay close attention to the signs used for each term.
A:

[math]
B:
[math]
C:
[math]
D:
[math]
Note that due to the symmetry of the cross-section, the stresses at diagonally opposite corners have the same magnitude but opposite signs. This can be used to shorten the problem by only calculating the stresses at two corners (say A and B) and identifying the stresses at the opposite corners.
[figure]
The orientation of the neutral axis is found from:
[math]

Example 9.6: Unsymmetric Bending

A bending moment M = 25 kip ft is applied to a W14 x 43 beam around an axis 40° above the positive z-axis as shown. Determine the maximum tensile bending stress in the section and the orientation of the neutral axis with respect to the z-axis.
[figure]
Solution
[figure]
The dimensions of the W14 x 43 beam can be found in Appendix A. We can also look up the area moment of inertia about both axes.
[math]
Next determine the components of the bending moment around the y- and z-axes. Since the bending moment is given in kip ft and all of the dimensions are given in inches, we'll convert the bending moment to kip in. at this stage.
[math]
The bending stress at any point can be calculated using:
[math]
The maximum tensile bending stress will occur at the corner where both of these terms are positive. Since both Mz and My are positive, it should be apparent that the y-distance should be negative and the z-distance should be positive, so the maximum tensile bending stress will occur at the bottom left corner. At this point, [math] and [math].
[math]
[figure]
The orientation of the neutral axis is found from:
[math]

9.4 Summary

Click to expand

Key takeaways

Beams under load will experience normal stresses that vary linearly along the height of the cross section. Assuming that the material is homogenous and linear elastic, the flexure formula can be used to calculate the normal stresses. Care must be taken to ensure that bending stresses are reported with magnitude units and tension/compression.

Engineers designing beams must consider the response of the beam to bending stress, amongst other criteria. We start the design process in this chapter by using the allowable stress and maximum moment to determine the minimum section modulus. This minimum section modulus is used to choose an adequate, economical beam shape. We will discuss future chapters additional criteria to design a beam.

When a member experiences loading where the bending moment, M, does not align with one of the principal axes of the cross-section, a method of super position is used to determine the resultant normal stress at a specific point. We can also calculate the orientation of the of the neutral axis due to this unsymmetric loading.

Key equations

Flexure formulas:
[math]
Unsymmetric bending—moment arbitrarily applied:
[math]
Orientation of the neutral axis:
[math]

10 Shear Loads

Learning Objectives

- Insert text

10.1 Introduction

Click to expand

Insert text

11 Beam Deflection

Learning Objectives

- Derive an equation for the elastic curve of a loaded beam
- Calculate the slope and deflection at any point of a loaded beam
- Use standard solutions and the method of superposition to determine deflection in more complex problems
- Use knowledge of deflection to solve statically indeterminate problems
- Design beam cross-sections that meet specifications for bending stress, shear stress, and deflection

Introduction

Click to expand

We now know how to calculate both the bending stress (Chapter 9) and shear stress (Chapter 10) in a beam. In this chapter we'll consider the deformation of a beam subjected to bending and shear stresses. Under load, beams deflect from their original position (@figure11.1). Some amount of deflection is unavoidable, but it is usually desirable to limit deflection as much as possible. Although deflection often doesn't pose a safety risk (unless the allowable stress of the beam is also exceeded) too much deflection can render a beam unfit for purpose.

Idealized sketches showing the shape of the deflected beam are a helpful visual aid. This deflected shape is known as the elastic curve (@figure11.2). Sketches of the elastic curve provide a quick visual reference of how the beam will deflect and can be used to check that our numerical answers are realistic.

In section 5.3 we learned to calculate deformation due to axial load $\left(\Delta L = \frac{FL}{AE}\right)$. In section 6.2 we learned to calculate deformation due to torsional load $\left(\emptyset = \frac{TL}{JG}\right)$. Note that both types of deformation depend on:

- Applied/internal load (F, T)
- Length of the object (L)
- Cross-section geometry (A, J)

Figure 11.1: A model bridge showing severe deflection along its span (left). Although the bridge hasn't broken, there is too much deflection for the application. The deflection in a hockey stick during a shot (right). In this application the deflection is helpful as it stores and releases energy during the shot.

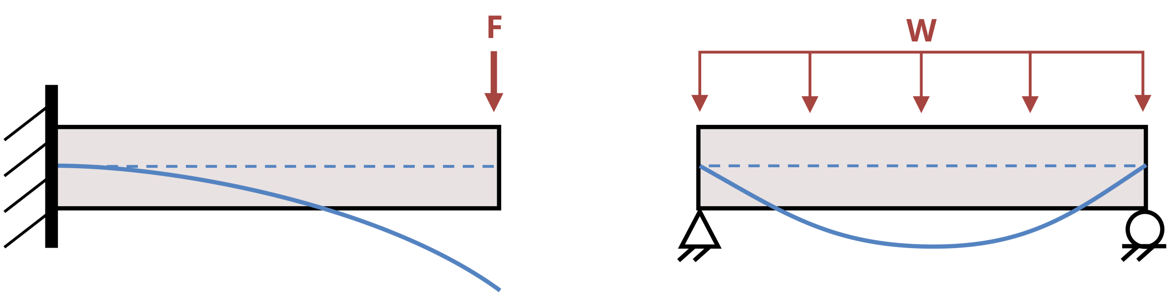


Figure 11.2: Two example beams showing the undeflected position of the centerline (dotted line) and the deflected position of the elastic curve (solid line).

- Material properties (E, G)

It is probably not surprising then that beam deflection also depends on these four things. We'll study two common techniques for calculating deflection: The method of integration (sections 11.1 and 11.2) and the method of superposition (section 11.3). Being able to calculate deflection is vitally important, as it allows us to predict how much a beam will deflect before we build a structure. Once we're able to calculate deflection, we'll use this to help solve some more statically indeterminate problems (Section 11.4). Finally, we'll build on our beam design work of section 9.2 by expanding our specifications to include limitations on not just bending stress, but shear stress and deflection as well.

11.1 Integration of the Moment Equation

Click to expand

We'll begin by deriving an equation for calculating the deflection at any point in a loaded beam. In Chapter 9, we derived two equations relating to bending stress.

$$\sigma = -\frac{Ey}{\rho}$$

$$\sigma = -\frac{My}{I}$$

Set these equations equal:

$$\frac{Ey}{\rho} = \frac{My}{I}$$

Rearrange:

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

The radius of curvature ( ) can be related to the deflection (y) by the equation

$$\frac{1}{\rho} = \frac{\frac{\partial^2 y}{\partial x^2}}{\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{\frac{3}{2}}}$$

This equation describes the exact deflection (y) at any distance (x) along the beam. In most engineering applications the deflection (y) is very small. We can simplify this equation significantly by assuming that our deflection (y) will be small and thus that $\frac{dy}{dx}$ is very small and $\left(\frac{dy}{dx}\right)^2$ is negligible. This simplifies the equation to

$$\frac{1}{\rho} = \frac{\frac{\partial^2 y}{\partial x^2}}{[1+0]^{\frac{3}{2}}} = \frac{\partial^2 y}{\partial x^2}$$

Finally

$$\frac{1}{\rho} = \frac{\partial^2 y}{\partial x^2} = \frac{M(x)}{EI}$$

To solve for deflection we can first find an equation for the internal bending moment as a function of (x), using equilibrium just as we did in section 7.2 for drawing the bending moment diagram. We can then set this equation equal to $EI\frac{d^2 y}{dx^2}$ and integrate twice.

$$\begin{gathered} M(x) = EI\frac{\partial^2 y}{\partial x^2} \\ \int M(x)dx + C1 = EI\frac{\partial y}{\partial x} \\ \iint M(x)dx + C1x + C2 = EIy \end{gathered}$$

In these equations, y represents the deflection of the beam and $\frac{dy}{dx}$ represents the slope of the deflected beam. Since the equations are functions of x, we can find the slope and deflection at any point along the beam.

Note that each successive integral introduces a constant of integration which we must solve for. We do this through the use of boundary conditions—that is, points on the beam where we already know the slope and/or deflection. The most common boundary conditions for our applications will be:

- At any support the deflection (y) is zero
- At a fixed support the slope $\left(\frac{\partial y}{\partial x}\right)$ is zero

@figure11.3 illustrates these boundary conditions. Once the constants of integration are known, we can define equations for the slope and deflection of the beam in terms of distance (x) along the beam and calculate the slope and deflection at any value of x. See Example 11.1 and Example 11.2 to see this process applied to a simply supported beam and a cantilever beam respectively.

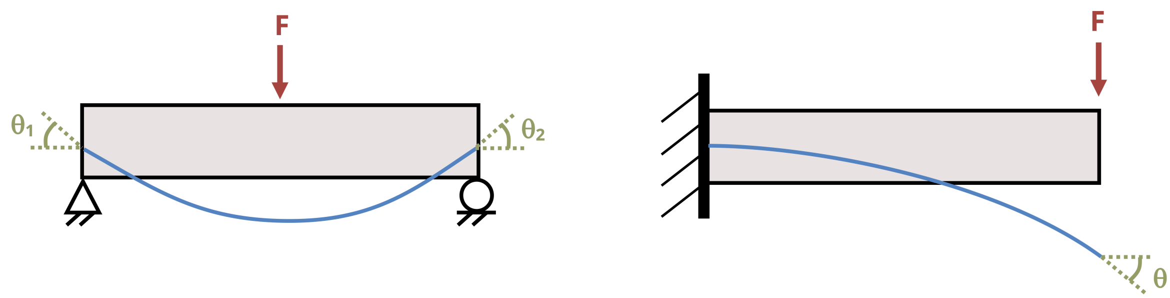
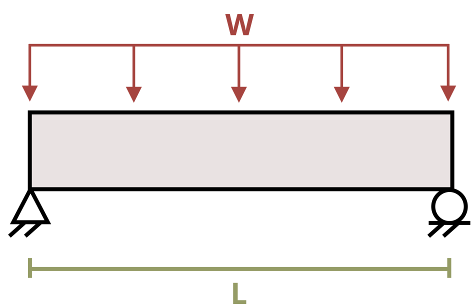


Figure 11.3: At pin and roller supports, the deflection is zero but the slope is not. At a fixed support, both the slope and deflection are zero. For a statically determinate beam in equilibrium we will always have two boundary conditions.

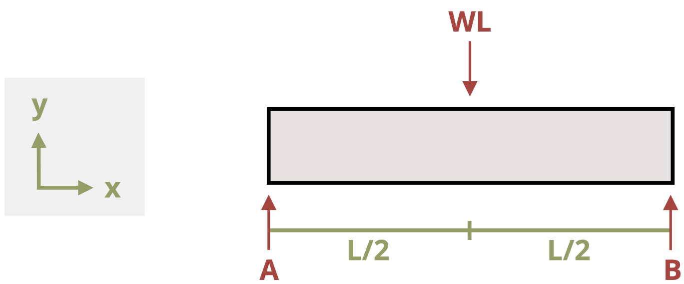
Example 11.1

A simply supported beam of length L = 10 m is subjected to a uniform distributed load of w = 20 kN/m. Determine the equation of the elastic curve and use this to find the deflection of the beam at the midpoint, x = 5 m. Assume E = 200 GPa and I = 350 x 10^6 mm^4.



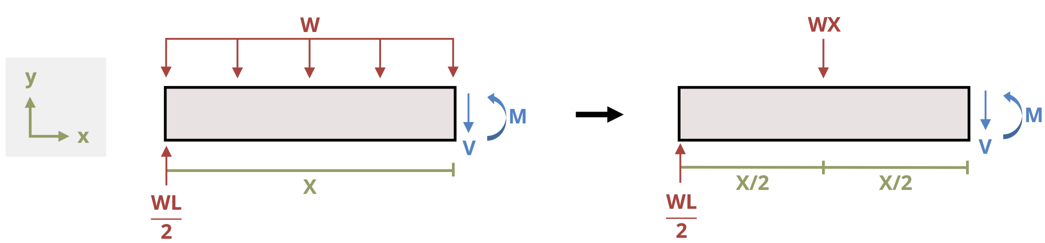
Solution

Sketch a free body diagram of the beam and use equilibrium equations to solve for the reaction forces at the supports:



$$\sum M_A = 0: \quad -\omega L\left(\frac{L}{2}\right) + BL = 0 \quad \rightarrow \quad B = \frac{\omega L}{2}$$
$$\sum Fy = 0: \quad A - \omega L + \frac{\omega L}{2} = 0 \rightarrow A = \frac{\omega L}{2}$$

Cut a cross-section through the beam at distance x and draw a free body diagram of everything to the left of the cut, including the internal loads. Use equilibrium to determine an equation for the internal bending moment, M, as a function of x:



$$\sum M_A = 0: M + wx\left(\frac{x}{2}\right) - \frac{wL}{2}(x) = 0$$
$$M = \frac{wLx}{2} - \frac{wx^2}{2} = EI\frac{\partial^2 y}{\partial x^2}$$

Integrate this equation twice:

$$\frac{wLx^2}{4} - \frac{wx^3}{6} + c_1 = EI\frac{\partial y}{\partial x}$$
$$\frac{wLx^3}{12} - \frac{wx^4}{24} + c_1 x + c_2 = EIy$$

Apply boundary conditions at the supports:

$$\text{At } x = 0, y = 0 \rightarrow C_2 = 0$$
$$\text{At } x = L, y = 0 \rightarrow \frac{wL(L)^3}{12} - \frac{w(L)^4}{24} + C_1 L = 0$$
$$C_1 = \frac{wL^3}{24} - \frac{wL^3}{12} = -\frac{wL^3}{24}$$

Substitute these constants into the deflection equation and rearrange:

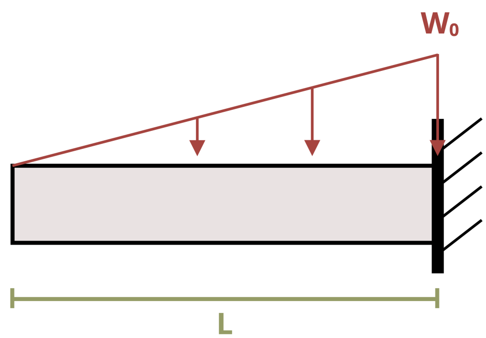
Answer only

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3}{24}x\right]$$

$$y = -37.2\ mm$$

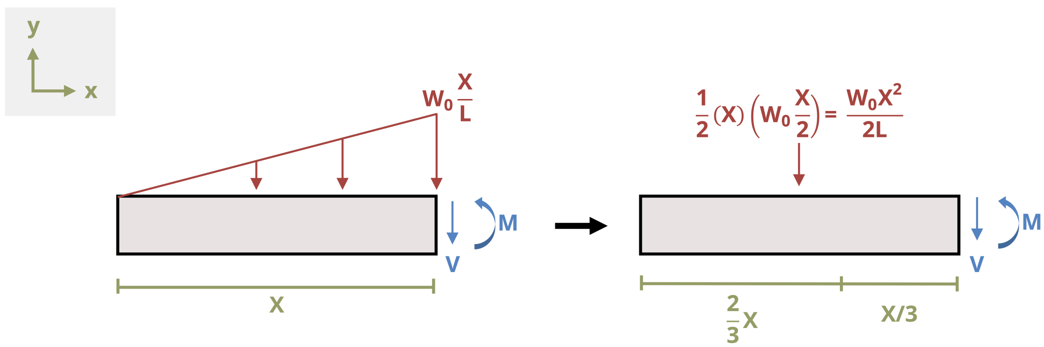
Example 11.2

A cantilever beam of length L = 8 ft is subjected to a linear distributed load where w_0 = 30 kip/ft. Determine the equation of the elastic curve and use this to find the deflection of the beam at the free end. Assume E = 29 x 10^6 psi and I = 375 in.^{4}.



Solution

Cut a cross-section through the beam at distance x and draw a free body diagram of everything to the left of the cut, including the internal loads. This diagram won't include any of the reactions at the fixed support so there's no need to solve for these first. Use equilibrium to determine an equation for the internal bending moment, M, as a function of x. Then integrate this equation twice:



$$\sum M_x = 0: \quad M + \frac{w_0 x^2}{2L}\left(\frac{x}{3}\right) = 0$$

$$M = -\frac{w_0 x^3}{6L} = EI\frac{\partial^2 y}{\partial x^2}$$

$$-\frac{w_0 x^4}{24L} + C_1 = EI\frac{\partial y}{\partial x}$$

$$-\frac{w_0 x^5}{120L} + C_1 x + C_2 = EIy$$

Apply boundary conditions at the fixed support:

$$\text{At } x = L, \frac{\partial y}{\partial x} = 0 \quad \rightarrow \quad \frac{w_0 L^4}{24L} + C_1 = 0$$

$$c_1 = \frac{w_0 L^3}{24}$$

$$\text{At } x = L, y = 0 \quad \rightarrow \quad \frac{w_0 L^5}{120L} + \frac{w_0 L^3}{24}(L) + C_2 = 0$$

$$C_2 = \frac{w_0 L^4}{120} - \frac{w_0 L^4}{24} = -\frac{4 w_0 L^4}{120} = -\frac{w_0 L^4}{30}$$

Substitute these constants into the deflection equation and rearrange:

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

Substitute in the given values and solve this equation at the free end (x = 0). Remember to convert feet to inches in both the distributed load and the length:

$$y = \frac{1}{29 \times 10^6 \times 375}\left[-\frac{\frac{30000}{12} \times (8 \times 12)^4}{30}\right]$$

$$y = -0.651 \text{ in.}$$

Answer only

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

$$y = -0.651\ in.$$

The above examples have single continuous loads. If the loading is discontinuous we must modify our approach as there will not be one single moment equation that describes the entire beam. We saw in section 7.2 when drawing shear force & bending moment diagrams that we can find a different equation for the internal bending moment in each loading region (i.e., a piecewise function over the length of the beam). We must find separate moment equations for each loading region and integrate each separately. This must be done each time the loading changes (a discontinuity in the moment diagram), such as:

- Both sides of a concentrated load
- Whenever a distributed load begins or ends

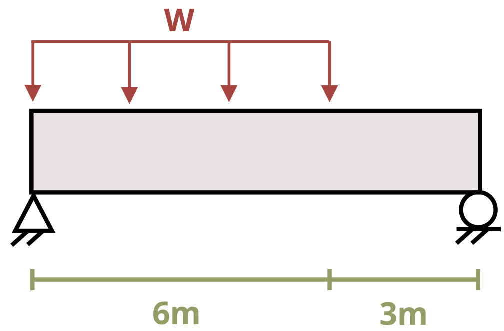
See Example 11.3 for an example of a beam with 2 loading regions. Note that in this example we will need to determine 2 internal bending moment equations and integrate each equation twice, thereby introducing four constants of integration. Two of these can still be determined by using boundary conditions, but we still need to determine the other two. This is done with continuity conditions.

Equation set 1 describes the internal moment, slope, and deflection of the beam from point A to point B, and equation set 2 describes the internal moment, slope, and deflection from point B to point C. Note that both sets of equations describe point B, where the loading changes. They must therefore return the same results at point B. We can therefore say that

$$\left(\frac{\partial y}{\partial x}\right)_1 = \left(\frac{\partial y}{\partial x}\right)_2$$
$$y_1 = y_2$$

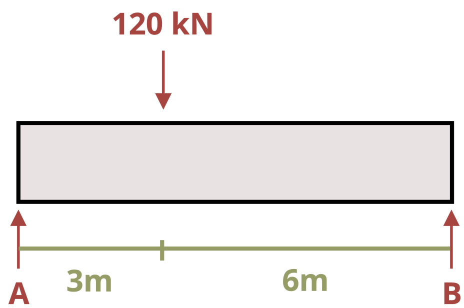
Example 11.3

A simply supported beam is subjected to a distributed load of w = 20 kN/m as shown. Determine the equation of the elastic curve between 0 x 6 m and between 6 m x 9m.
Then determine the deflection at x = 5m. Assume E = 200 GPa and I = 394 x 10^6 mm^4.


w
6m
3m

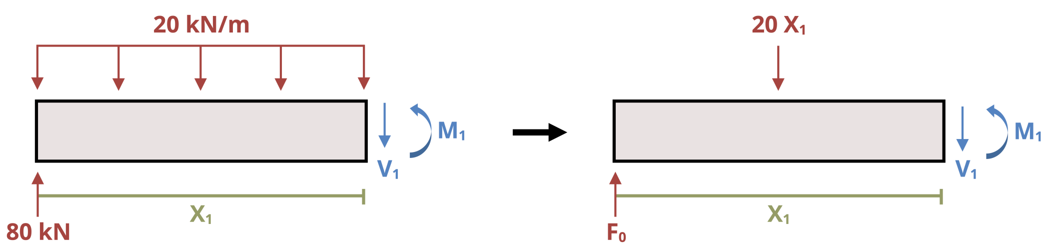
Solution

Sketch a free body diagram of the beam, and use equilibrium equations to solve for the reaction forces at the supports:



$$\sum M_A = 0: \quad -(120 \times 3) + (B + 9) = 0 \quad \rightarrow \quad B = 40 \text{ kN}$$
$$\sum Fy = 0: \quad A - 120 + 40 = 0 \quad \rightarrow \quad A = 80 \text{ kN}$$

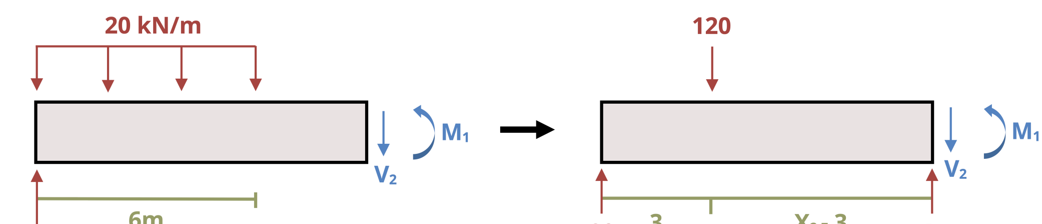
Cut a cross-section in the first loading region at distance x_1. Draw a free body diagram of everything to the left of this cut, including the internal loads. Use equilibrium to determine an equation for the internal bending moment, M_1, as a function of x_1. Then integrate this equation twice:



$$\sum M_x = 0: \quad M_1 + 20x_1\left(\frac{x_1}{2}\right) - 80x_1 = 0$$
$$M_1 = 80x_1 - 10x_1^2 = EI\frac{\partial^2 y_1}{\partial x_1^2}$$
$$\frac{80x_1^2}{2} - \frac{10x_1^3}{3} + c_1 = EI\frac{\partial y_1}{\partial x_1}$$
$$\frac{80x_1^3}{6} - \frac{10x_1^4}{12} + C_1x_1 + c_2 = EIy_1$$

These equations are valid from 0 x_1 6m.

Set these equations aside and cut a cross-section in the second loading region at distance x_2. Draw a free body diagram of everything to the left of this cut, including the internal loads. Use equilibrium to determine an equation for the internal bending moment, M_2, as a function of x_2. Then integrate this equation twice:



Answer only

From 0 x 6 m:

$$y_1 = \frac{1}{EI}\left[\frac{80x_1^3}{6} - \frac{10x_1^4}{12} - 480x_1\right]$$

From 6 x 9 m:

$$y_2 = \frac{1}{EI}\left[\frac{360x_2^2}{2} - \frac{40x_2^3}{6} + 1200x_3 + 1080\right]$$

At x = 5 m:

$$y_1 = -15.9\ mm$$

It is not required that we make both cuts from the left. In some cases it may be easier to make one cut from the left and the other from the right. Consider the simple beam in @figure11.4. When making a cut in the region not under the distributed load, a free body diagram to the left of the cut has many more forces than a free body diagram of the right side.

The moment equation will be much simpler if we draw the right hand side of the cut. This in turn simplifies the integrations and the math involved in finding the constants of integration. We must however make one change to our continuity conditions if we measure distance x_2 from the right. When making both cuts from the left we said that at point C the slope and deflection from calculated from equation 1 must be the same as those calculated from equation 2.

$$\left(\frac{dy}{dx}\right)_1 = \left(\frac{dy}{dx}\right)_2$$
$$y_1 = y_2$$

When measuring distance x_2 from the right, it is still true at point C that [math] as the deflection is still downwards regardless of whether we measure our horizontal distance from the left or right. The magnitude of the slope will also be the same at point C regardless of which equation we use to calculate it. However, the direction of rotation of the slope will change from counter-clockwise to clockwise due to us changing the coordinate system (@figure11.5). Thus our second boundary condition must become

$$\left(\frac{dy}{dx}\right)_1 = -\left(\frac{dy}{dx}\right)_2$$

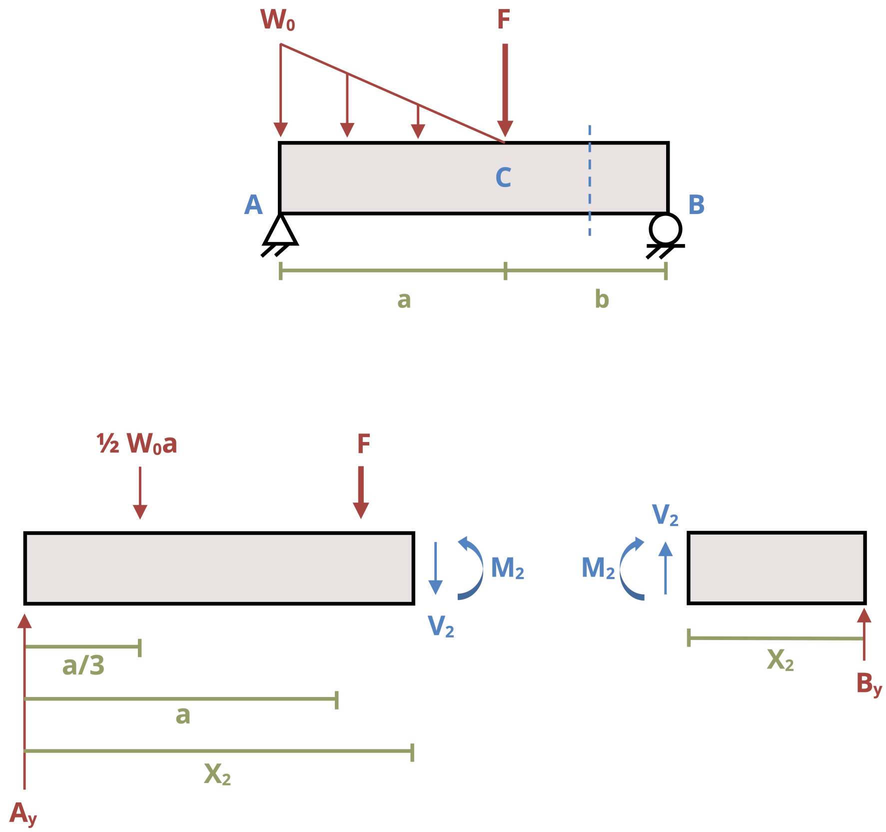


Figure 11.4: Simple beam showing two options for finding the internal bending moment. Note that the moment equation will be much simpler if we draw the right hand side of the cut.

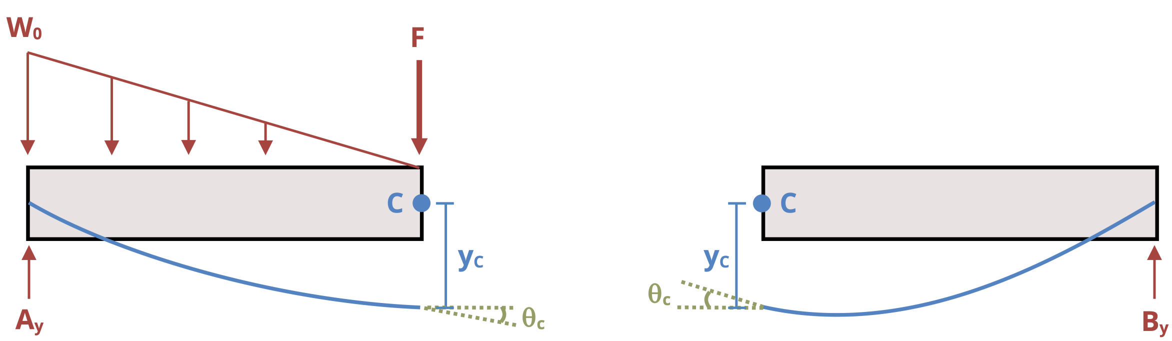
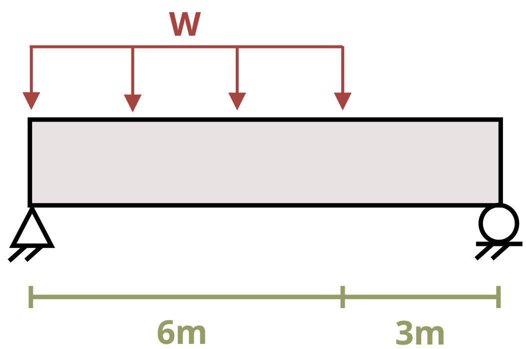


Figure 11.5: Depending on which side of the cut we draw, the deflection at point C will have the same magnitude and direction. The slope at point C will have the same magnitude but opposite directions if we make one cut from the left and the other from the right.

This negative sign is vitally important, and we will not get the correct answer if we forget to include it. Example 11.3 has been reworked below, this time with one cut made from the right hand side instead. Compare to the previous solution to see the differences.

Example 11.3 (reworked)

A simply supported beam is subjected to a distributed load of w = 20 kN/m as shown. Determine the equation of the elastic curve between 0   x   6 m and between 6 m   x 9m.
Then determine the deflection at x = 5m. Assume E = 200 GPa and I = 394 x 10^6 mm4.

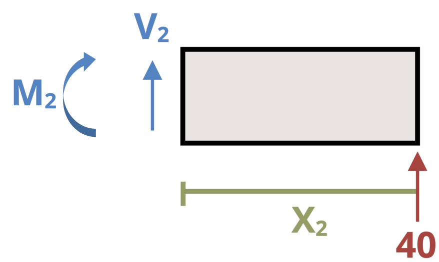


Solution

Proceed through first cut as before and find:

$$\frac{80x_1^2}{2} - \frac{10x_1^3}{3} + c_1 = EI\frac{\partial y_1}{\partial x_1}$$
$$\frac{80x_1^3}{6} - \frac{10x_1^4}{12} + c_1x_1 + c_2 = -EIy_1$$

Make second cut from right-hand-side. Draw a free body diagram and use equilibrium to determine an equation for the internal bending moment, M_2, as a function of x_2. Then integrate this equation twice.



$$\sum M_x = 0: \quad 40x_2 - M_2 = 0$$
$$M_2 = 40x_2 = EI\frac{\partial^2 y_2}{\partial {x_2}^2}$$
$$\frac{40x_2^2}{2} + c_3 = EI\frac{\partial y_2}{\partial x_2}$$
$$\frac{40x_2^3}{6} + c_3x_2 + c_4 = EIy_2$$

Apply boundary conditions as before, although this time the roller support is at $x_2 = 0$ since we're measuring x_2 from the right side:

$$\text{At } x_1 = 0, y_1 = 0 \quad \rightarrow \quad c_2 = 0 \text{ (as before)}$$
$$\text{At } x_2 = 0, y_2 = 0 \quad \rightarrow \quad c_4 = 0$$

Apply continuity conditions at the point where the loading changes (x_1 = 6 m, x_2 = 3 m):

$$\text{At } x_1 = 6 \text{ m}, x_2 = 3 \text{ m}, y_1 = y_2 \rightarrow \frac{80(6)^3}{6} - \frac{10(6)^4}{12} + c_1(6) = \frac{40(3)^3}{6} + c_3(3)$$
$$2880 - 1080 + 6c_1 = 180 + 3c_3$$
$$540 + 2c_1 = c_3$$

$$\text{At } x_1 = 6 \text{ m}, x_2 = 3\text{m}, \frac{dy_1}{dx_1} = -\frac{dy_2}{dx_2} \rightarrow 90\frac{80(6)^2}{2} - \frac{10(6)^3}{3} + c_1 = -\frac{40(3)^2}{2} - c_3$$
$$1440 - 720 + c_1 = -180 - c_3$$
$$900 + c_1 = -c_3$$

Substitute in $540 + 2c_1 = c_3$:

$$540 + 2c_1 = -900 - c_1$$

Answer only

From 0 x 6 m:

$$y_1 = \frac{1}{EI}\left[\frac{80x_1^3}{6} - \frac{10x_1^4}{12} - 480x_1\right]$$

From 6 x 9 m:

$$y_2 = \frac{1}{EI}\left[\frac{40x_2^3}{6} - 420x_2\right]$$

At x = 5 m:

$$y_1 = -15.9\ mm$$

Step-by-step: Deflection by Integration of Moment Equation

1. Solve for reaction loads using equilibrium.
2. Cut a cross-section through the beam and find the internal bending moment as a function of x. If the external loading changes, repeat this in each segment of the beam.
3. Set the internal bending moment $M(x) = EI\frac{d^2y}{dx^2}$ and integrate twice.
4. Use boundary conditions to determine the two constants of integration.
   a. At a fixed support, slope $\left(\frac{dy}{dx}\right) = 0$
   b. At any support, deflection (y) = 0
5. Determine the slope and deflection at any value of x as required.

11.2 Integration of the Load Equation

Click to expand

In section 11.1 we used equilibrium equations to determine the internal bending moment in a beam as a function of x. We related this bending moment to the deflection by

$$M(x) = EI\frac{d^2y}{dx^2}$$

In section 7.3 we learned that there is a relationship between the internal bending moment, the internal shear force, and the external load. For an external distributed load, w, we found

$$V = \int_0^L w(x)dx$$

$$M = \int_0^L V(x)dx$$

We may therefore relate the deflection of a beam to not only the internal bending moment, but also the internal shear force and external distributed load through successive integrations. For some loading configurations it can be much simpler to integrate the distributed load equation 4 times instead of finding the equation of the internal bending moment through equilibrium and integrating twice. When doing this, remember that loads acting downwards (as most loads on beams do) are negative.

$$\begin{aligned}
w(x) &= EI\frac{d^4y}{dx^4} \\
V(x) = EI\frac{d^3y}{dx^3} &= \int w(x)dx + C1 \\
M(x) = EI\frac{d^2y}{dx^2} &= \int V(x)dx + C2 \\
\theta(x) = EI\frac{dy}{dx} &= \int M(x)dx + C3 \\
y(x) &= \int \theta(x)dx + C4
\end{aligned}$$

Note that each successive integral will introduce a new constant of integration. We must again apply boundary conditions in order to solve for these constants. There are now four constants so we require four boundary conditions. Two of these are the same as in section 11.1:

- At any support the deflection is 0
- At a fixed support, both the slope and deflection are zero

The other two boundary conditions come from knowing the value of the shear force and bending moment at a point in the beam. We are able to find these from our work in chapter 7. While we can use known values at any point in the beam, it is generally easiest to use the shear force and bending moment at x = 0. At this point:

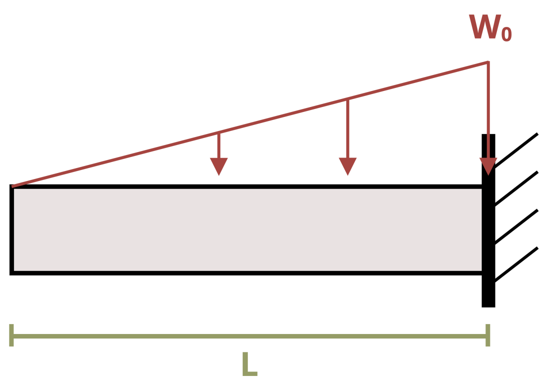
- The shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.

- The bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.

See Example 11.4 for a demonstration of how to find the deflection of a beam by integrating the load equation. Note that while this method can be used for any beam, it is best applied to beams with a single continuous distributed load acting over the length of the beam. Beams with discontinuous loading will require integration of multiple load equations.

Example 11.4

A cantilever beam from Example 11.2 is repeated here. Length L = 8 ft is subjected to a linear distributed load where $w_0 = 30$ kip/ft. Determine the equation of the elastic curve and use this to find the deflection of the beam at the free end.



Solution

First define the equation of the load, which increases linearly from zero to w_0:

$$\omega = -\omega_0 \frac{x}{L} = EI\frac{\partial^4 y}{\partial x^4}$$

Integrate this equation 4 times:

$$\begin{aligned}
&-\frac{w_0 x^2}{2L} + c_1 = EI\frac{\partial^3 y}{\partial x^3} \\
&-\frac{w_0 x^3}{6L} + c_1 x + c_2 = EI\frac{\partial^3 y}{\partial x^2} \\
&-\frac{w_0 x^4}{24L} + \frac{c_1 x^2}{2} + c_2 x + c_3 = EI\frac{\partial y}{\partial x} \\
&-\frac{w_0 x^5}{120L} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 = EIy
\end{aligned}$$

Apply boundary conditions at the fixed support. At the free end the internal shear force and bending moment will be zero. At the fixed support (at x = L) the slope and deflection will be zero.

$$\begin{aligned}
&\text{At } x = 0, v = 0 \quad \rightarrow \quad c_1 = 0 \\
&\text{At } x = 0, M = 0 \quad \rightarrow \quad c_2 = 0 \\
&\text{At } x = L, \frac{dy}{dx} = 0 \quad \rightarrow \quad c_3 = \frac{w_0 L^3}{24} \\
&\text{At } x = L, y = 0 \quad \rightarrow \quad c_4 = -\frac{w_0 L^4}{30}
\end{aligned}$$

This is the same result found in Example 11.2 and the problem proceeds in the same way to find the deflection at the free end.
Substitute these constants into the deflection equation and rearrange:

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

Substitute in the given values and solve this equation at the free end (x = 0). Remember to convert feet to inches in both the distributed load and the length:

$$\begin{aligned}
&y = \frac{1}{29 \times 10^6 \times 375}\left[-\frac{\frac{30000}{12} \times (8 \times 12)^4}{30}\right] \\
&y = -0.651 \text{ in.}
\end{aligned}$$

Answer only

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

$$y = -0.651\ in.$$

Step-by-step: Deflection by Integration of Load Equation

1. Define an equation that describes the external distributed load as a function of x. Remember that loads acting downwards are negative.
2. Integrate this equation four times to determine equations for internal shear force, internal bending moment, slope, and deflection respectively.
3. Apply boundary conditions to determine the four constants of integration
   a. At $x = 0$, the shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.
   b. At $x = 0$, the bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.
   c. At a fixed support, slope $\left(\frac{dy}{dx}\right) = 0$
   d. At any support, deflection (y) = 0
4. Determine the slope and deflection at any value of x as required.

11.3 Superposition

Click to expand

It should be apparent from the previous sections that finding deflection using the method of integration can become quite complicated and time-consuming if there are multiple loads. In such cases, the method of superposition is a good alternative.

Beams are typically loaded in a relatively small number of standard configurations, and the deflection behavior of a beam subjected to one of these standard loads is very well understood. A subset of beam deflections have been included in Appendix C. @figure11.6 shows an example of the information provided for one such beam.

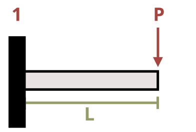
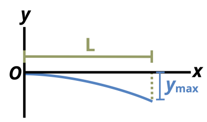
Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = -\frac{P}{6EI}(x^3 - 3Lx^2)$

Figure 11.6: Appendix C contains some standard loading configurations along with a sketch of the elastic curve, equations for the maximum deflection and maximum slope, and the general equation of the elastic curve.

This can be useful even for more complex loading. When multiple loads act on a beam, the effect of each load on the deflection of the beam may be considered independently. Thus, even for loading more complex than the loads in Appendix C, it may be possible to simplify the problem by considering each load separately and calculating the deflection caused at a point by each load (@figure11.7). These deflections can then be added together to find the overall deflection at that point. We must of course calculate the deflection at the same point for each load in order to add them together. See Example 11.5 for a demonstration.

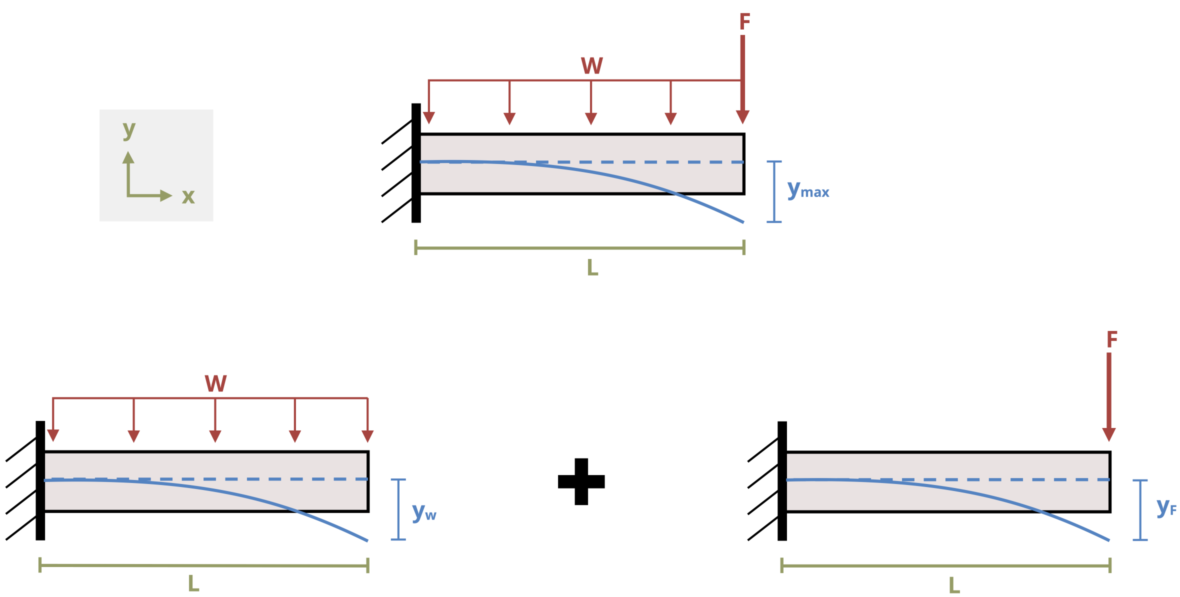


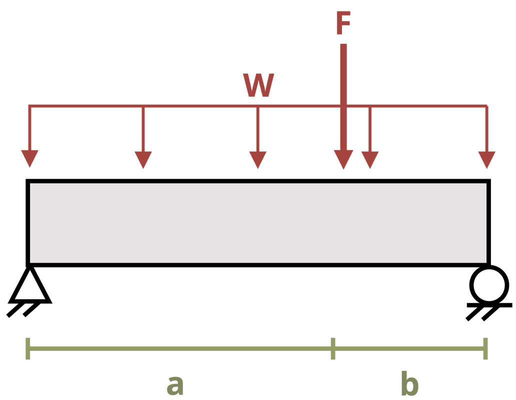
Figure 11.7: Although Appendix C doesn't contain a cantilever beam with both a distributed load and a concentrated load, it does contain cantilever beams with each load separately. We may find the combined deflection by finding the deflection of each load separately and adding them together.

$$y_w = -\frac{wL^4}{8EI} \qquad\qquad y_F = -\frac{FL^3}{3EI}$$

$$y_{max} = -\frac{wL^4}{8EI} - \frac{FL^3}{3EI}$$

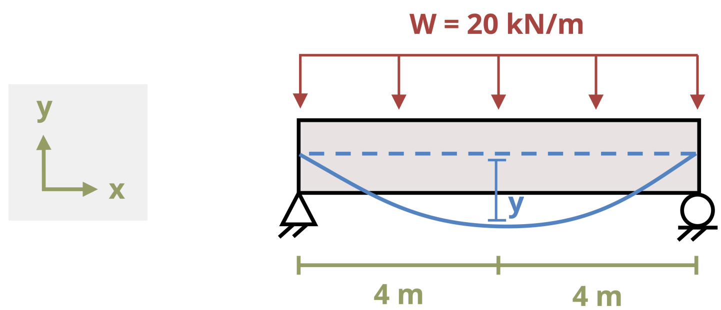
Example 11.5

A simply supported beam is subjected to a distributed load w = 10 kN/m and a concentrated load F = 50 kN. If distance a = 6 m and b = 2 m, determine the deflection at the midpoint of the beam (x = 4 m).
Assume E = 210 GPa and I = 275 x 10^6 mm^4.



Solution

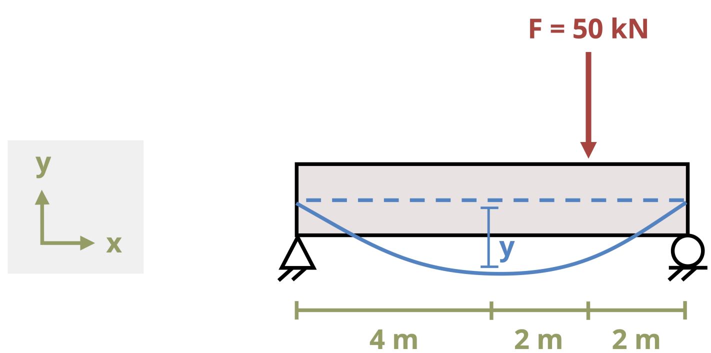
Consider each load separately. We'll start with the distributed load (although this can be done in any order).



From Appendix C we can see that the maximum deflection for this load occurs at the center of the beam, which is the point we're interested in.

$$y = -\frac{5wL^4}{384EI} = -\frac{5 \times 20000 \times 8^4}{384 \times 210 \times 10^9 \times 275 \times 10^{-6}} = -0.01847 \text{ m}$$
$$= -18.47 \text{ mm}$$

Now consider the concentrated load



Here the maximum deflection does not occur at the center of the beam. We can use the elastic curve equation for x < a to find the deflection at x = 4 m when the load is applied at a = 6 m.

$$y = \frac{Fb}{6EIL}\left[x^3 - \left(L^2 - b^2\right)x\right] = \frac{50000 \times 2}{6 \times 210 \times 10^9 \times 275 \times 10^{-6} \times 8}\left[4^3 - \left(8^2 - 2^2\right)4\right]$$
$$= -0.00635 \text{ m}$$
$$= -6.35 \text{ mm}$$

The total deflection at the midpoint of the beam is

$$y = -18.47\cancel{98}\ 6.35 \text{ mm}$$
$$y = -24.8 \text{ mm}$$

Answer only

$$y = -24.8 \ mm$$

Step-by-step: Deflection by Superposition

1. Sketch the beam with just one load acting on it.
2. Use Appendix C to find a beam with the same supports and loading.
3. Use the relevant equation in appendix C to find the deflection at the point of interest.
4. Repeat this process until all loads have been considered. Add the individual deflections at the point of interest to find the overall deflection at this point.

11.4 Statically Indeterminate Deflection

Click to expand

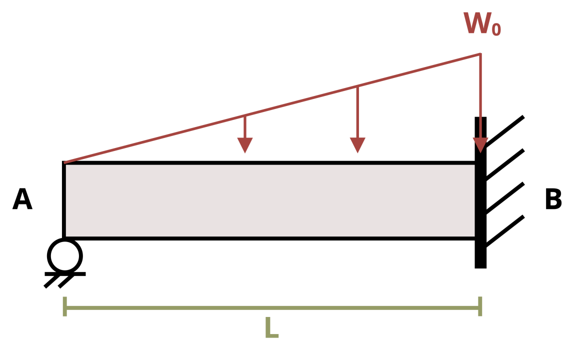
A problem is statically indeterminate if there are more unknown support reactions than there are equilibrium equations to solve for them. We have seen statically indeterminate problems in section 5.5 and section 6.4. In both sections we used a similar method to solve statically indeterminate problems where we took advantage of our knowledge of deformation in order to solve for one of the reaction forces before using equilibrium equations to find the others. We'll do something similar here. We may use either an integration method similar to that of section 11.1 or a superposition method similar to that of section 11.3.

In practice, engineers design structures with redundancies so that if one part fails it can be replaced without the entire structure collapsing. As such, static indeterminacy is the norm.

In section 11.1 we cut a cross-section and found the internal bending moment as a function of x, M = f(x). We integrated this twice to determine the equation of the elastic curve. For statically indeterminate problems, we will write our moment equation as a function of both x and A, where A is the unknown force at the redundant support. After integrating this equation twice we will have three unknowns, but also three boundary conditions. We will therefore be able to solve for the redundant reaction force and then use equilibrium equations to solve for the other reactions. See Example 11.6.

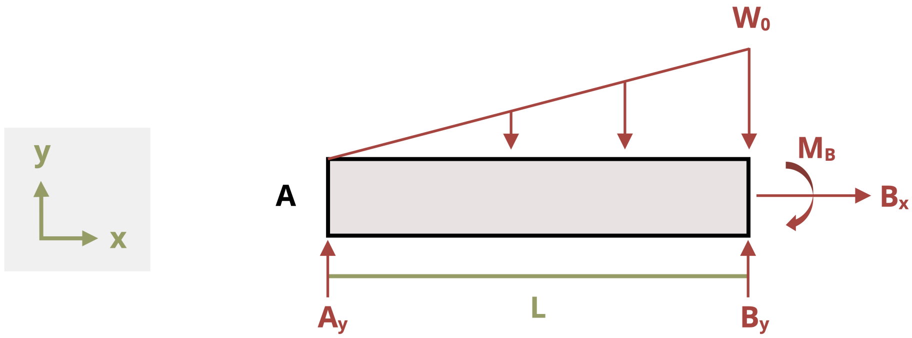
Example 11.6

A propped cantilever beam of length L = 6 ft is subjected to a linear distributed load where w_0 = 20 kips/ft. Determine the reactions at supports A and B.

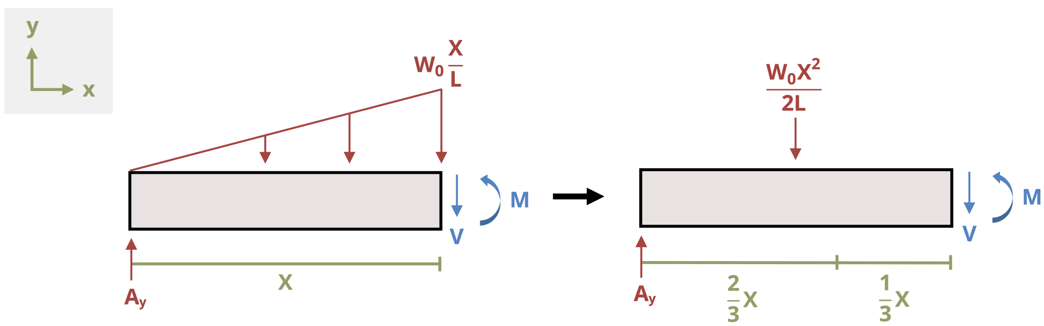


Solution

A free body diagram of the beam shows that we have 4 unknowns and only 3 equilibrium equations. We can solve $\sum F_x = 0$ to find $B_x = 0$, but that still leaves us with 3 unknowns and 2 equations.



Cut a cross-section and determine the internal bending moment in terms of distance x and redundant force A_y. Set this equation equal to $EI\frac{\partial^2 y}{\partial x^2}$ and integrate twice:



$$\sum M_x = 0: \quad M + \frac{w_0 x^2}{2L}\left(\frac{x}{3}\right) - A_y x = 0$$

$$M = A_y x - \frac{w_0 x^3}{6L} = EI\frac{\partial^2 y}{\partial x^2}$$

$$\frac{A_y x^2}{2} - \frac{w_0 x^4}{24L} + c_1 = EI\frac{\partial y}{\partial x}$$

$$\frac{A_y x^3}{6} - \frac{w_0 x^5}{120L} + c_1 x + c_2 = EIy$$

We have 3 unknowns (A_y, c_1, c_2) and 3 boundary conditions (at the roller, deflection equals zero. At the fixed support, both slope and deflection equal zero). Apply the boundary conditions:

$$\text{At } x = 0, y = 0 \quad \rightarrow \quad c_2 = 0$$

$$\text{At } x = L, y = 0 \quad \rightarrow \quad \frac{A_y L^3}{6} - \frac{w_0 L^4}{120} + c_1 L = 0$$

$$\text{At } x = L, \frac{\partial y}{\partial x} = 0 \quad \rightarrow \quad \frac{AyL^2}{2} - \frac{w_0 L^3}{24} + c_1 = 0$$

Rearrange and substitute:

$$c_1 = \frac{w_{0L}{}^3}{24} - \frac{A_y L^2}{2}$$

Answer only

$$A_y = 12 \text{ kips}$$
$$B_y = 48 \text{ kips}$$
$$M_B = 48 \text{ kip} \cdot \text{ft}$$

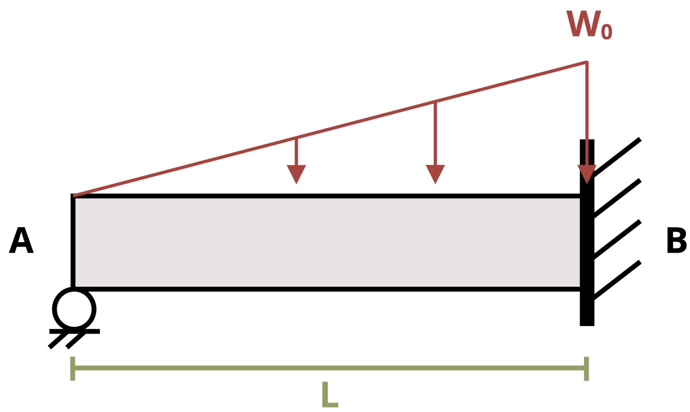
This method works well for beams with a single continuous load. While it can be used for more complex loads, it quickly becomes complicated and time consuming. For these problems it is generally better to use superposition.

The method is very similar to the methods used to solve other statically indeterminate problems in section 5.5 and section 6.4. To solve a statically indeterminate deflection problem using superposition, first identify the redundant support. We have 3 equilibrium equations and so can solve for three unknown support reactions. Any support reactions beyond this are redundant. Remove the redundant reaction and determine the deflection due to the applied loads at the point that the reaction was removed. It is best to remove a support such that the remaining beam is either simply supported or cantilever as these are the only support configuration in Appendix C.

Then replace the redundant reaction and determine the deflection due to this reaction and the point that the reaction is applied. Since there is actually a support here, the total deflection must be zero. Sum both deflections and set them equal to zero. This will allow you to calculate the value of the redundant support. Once this is known the other reaction loads can be determined using equilibrium. See Example 11.7 for a demonstration.

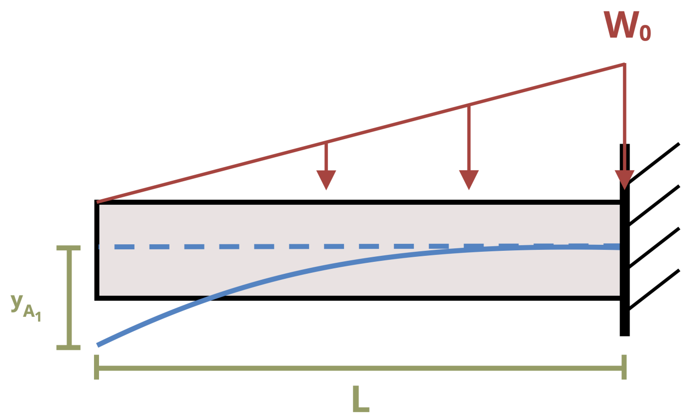
Example 11.7

The propped cantilever beam of Example 11.6 is shown again here. If length L = 6 ft is subjected to a linear distributed load where w_0 = 20 kips/ft. Determine the reactions at supports A and B.


W0
A
B
L

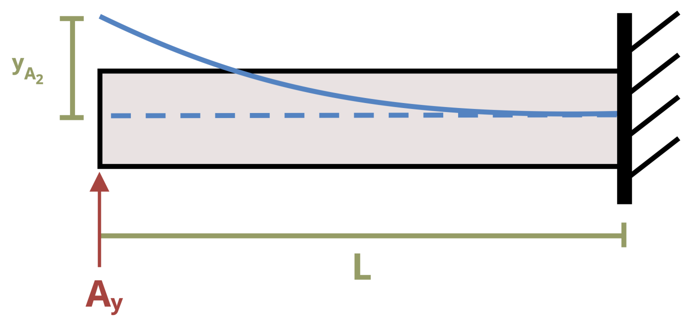
Solution

As demonstrated in Example 11.6, this problem is statically indeterminate.
This time, let's remove redundant support A_y and use Appendix C to find the deflection at point A.



$$y_{A_1} = -\frac{w_0 L^4}{30EI}$$

Now replace A_y and use Appendix C to find the deflection at point A caused by A_y.



$$y_{A_2} = \frac{{A_y L}^3}{3EI}$$

Since there is actually a support at A, the deflection at A must be zero.

$$-\frac{w_0 L^4}{30EI} + \frac{A_y L^3}{3EI} = 0 \quad \rightarrow \quad A_y = \frac{W_0 L}{10}$$

This is the same answer we found in Example 11.6, and the rest of the problem proceeds in the same way.

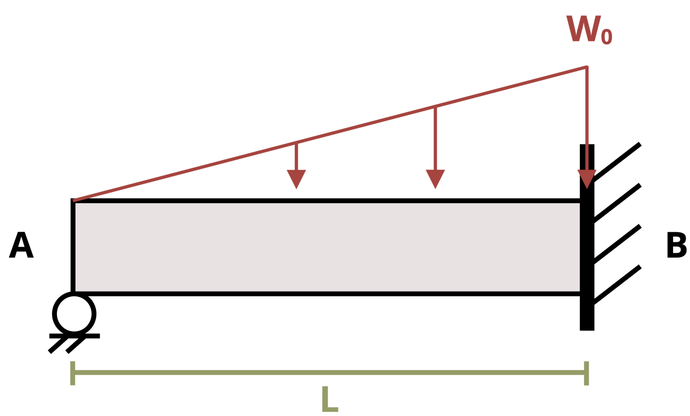
Answer only

$$A_y = 12 \text{ kips}$$
$$B_y = 48 \text{ kips}$$
$$M_B = 48 \text{ kip} \cdot \text{ft}$$

It is possible to remove a moment reaction in order to leave a simply supported beam. In this case, determining the deflection at the point that the load was removed isn't helpful as removing the moment reaction from a fixed support effectively leaves a pin support and the deflection with the moment removed will still be zero. In this situation we can instead look at the slope at the point that the reaction was removed. Example 11.7 is repeated below with the moment at B removed instead of the force at A.

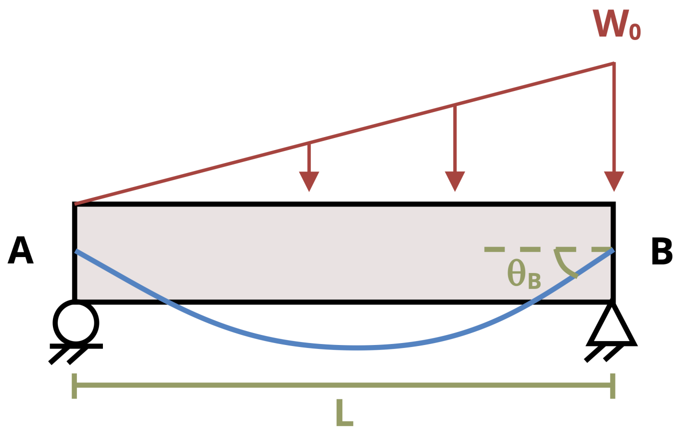
Example 11.7 (reworked)

The propped cantilever beam of Example 11.6 is shown again here. If length L = 6 ft is subjected to a linear distributed load where w_0 = 20 kips/ft. Determine the reactions at supports A and B.



Solution

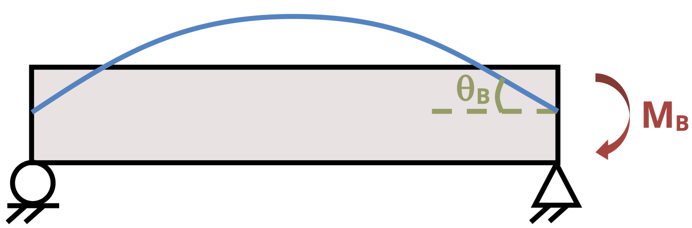
This time we'll remove the moment reaction at B to leave a simply supported beam.



Finding the deflection at B would not help as it would simply be zero. Instead, use Appendix C to find the slope at B.

$$\theta_B = \frac{W_0 L^3}{45EI}$$

Now replace the moment reaction at B and use Appendix C to find the slope at B due to this moment.



$$\theta_B = -\frac{M_B L}{3EI}$$

Since there is actually a fixed support at B, the slope here must be zero.

$$\frac{W_0 L^3}{45EI} - \frac{M_{BL}}{3EI} = 0$$

$$M_B = \frac{W_0 L^2}{15}$$

Adding numbers we get:

$$M_B = \frac{20 \times 6^2}{15} = 48 \text{ kip} \cdot \text{ft}$$

Which is the same answer we found in Example 11.6. Reactions A_y and B_y can be found by equilibrium and will also give the same answers as Example 11.6.

Answer only

$$A_y = 12 \text{ kips}$$
$$B_y = 48 \text{ kips}$$
$$M_B = 48 \text{ kip} \cdot \text{ft}$$

Note that some problems may involve multiple redundant reactions. In such cases, remove all redundant reactions and determine the deflection due to the external loads at every point that a reaction was removed. Then replace the redundant reactions one at a time and determine the deflection due to each load at every point that a reaction was removed. As before, the total deflection at each of these points must be zero. Sum the deflection at each point and set it equal to zero. There will be one equation for every redundant reaction and so all redundant reactions can be calculated.

Step-by-step: Statically Indeterminate Deflection

1. Identify and remove redundant reaction(s).
2. Determine the deflection at the point that the reaction was removed. Use superposition if possible, or integration otherwise.
    1. If the removed reaction was a couple, determine the slope at this point instead.
3. Replace the redundant reaction and determine the deflection caused by this load at the point that the load acts.
    1. If the removed reaction was a couple, determine the slope at this point instead.
4. These deflections (or slopes) must add up to the total deflection at this point. This is typically zero due to the support unless indicated otherwise. Use this to determine the magnitude of the redundant reaction.
5. Use equilibrium to solve for the other reactions. Then solve for anything else required as normal.

11.5 Intermediate Beam design

Click to expand

In Section 9.2 we learned how to use the section modulus to design a beam to meet bending stress specifications. In practice, beams must also meet specifications for shear stress and deflection. Limitations will be placed on all three of these criteria and we must design our beams to meet all three simultaneously. We'll begin with rectangular cross-sections and then move on to W beams.

As in section 9.2, problems involving rectangular cross-sections will typically relate the base and height of the cross-section such that we only need to determine one dimension. Previously we designed the cross-section to resist bending stress using:

$$S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}} = \frac{I}{c}$$

Since $I = \frac{bh^3}{12}$ and $c = \frac{h}{2}$ we were able to determine the required dimensions to meet the minimum required section modulus. The equation $S = \frac{I}{c}$ will simplify to $S = \frac{\frac{bh^3}{12}}{\frac{h}{2}} = \frac{2bh^3}{12h} = \frac{bh^2}{6}$ for a rectangle.

To meet the minimum required shear stress we will use $\tau = \frac{VQ}{It}$. Using the method of section 10.1, it can be shown for a rectangle (**?@fig-11.8**) that $Q = yA = \frac{h}{4}\frac{bh}{2} = \frac{bh^2}{8}$. Since $I = \frac{bh^3}{12}$ and t = b, we can re-write the shear stress equation to find the maximum shear stress in a rectangular cross-section as $\tau_{\text{max}} = \frac{V\frac{bh^2}{8}}{\frac{bh^3}{12}b} = \frac{12Vbh^2}{8b^2h^3} = \frac{12V}{8bh} = \frac{3}{2}\frac{V}{A}$.

Thus we may set $\tau_{\text{max}} = \frac{3}{2}\frac{V}{A}$ and solve for the required dimensions to not exceed the maximum allowable shear stress.

The maximum deflection of the beam under a given loading configuration can be found using the methods of this chapter. Wherever possible, use superposition and Appendix C. If it is not possible to use Appendix C then the method of integration can be used instead. In either case the equation for the maximum deflection of the beam will include the area moment of inertia, I. Set the maximum allowable deflection equal to this equation and replace $I = \frac{bh^3}{12}$. We can then solve for the required dimensions of the cross-section to not exceed the maximum allowable deflection of the beam.

We now have three different dimensions—one required to not exceed the maximum bending stress, one required to not exceed the maximum shear stress, and one required to not exceed the maximum deflection. Since we must not exceed any of these limitation, we select the largest of our three potential answers. See Example 11.8 for a demonstration.

Example 11.8

The simply supported wooden beam (E = 1700 ksi) of length L = 12 ft is subjected to a uniform distributed load of w = 1.5 kips/ft.
Assume the beam has an allowable bending stress of 900 psi, an allowable shear stress of 180 psi, and the deflection is limited to beam span / 240.
The beam has a rectangular cross-section where h = 1.5b. Determine the minimum

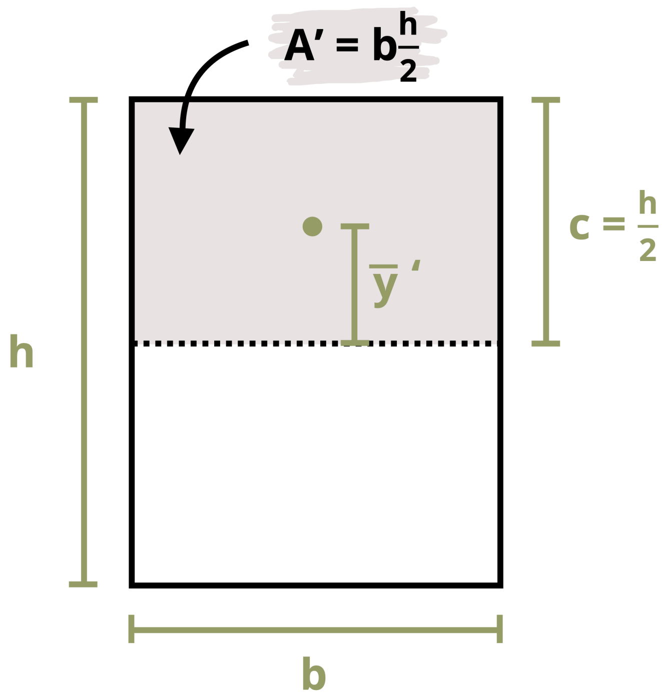
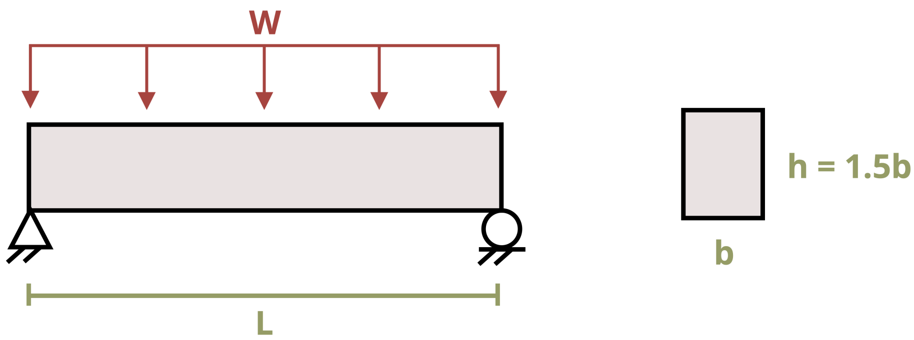


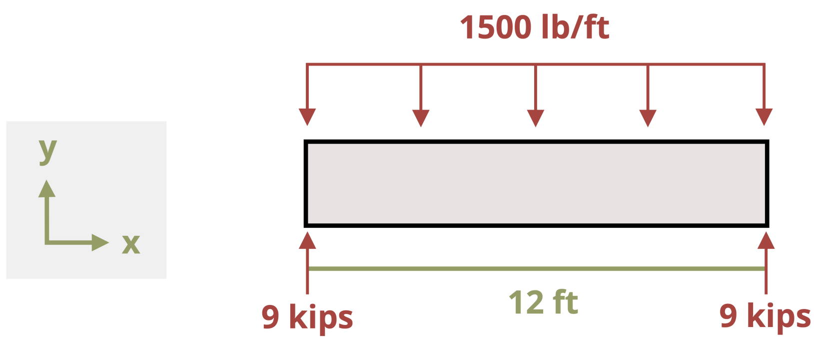
Figure 11.8: Rectangular cross-section showing the dimensions used for calculating the section modulus (S) and first moment of area (Q).

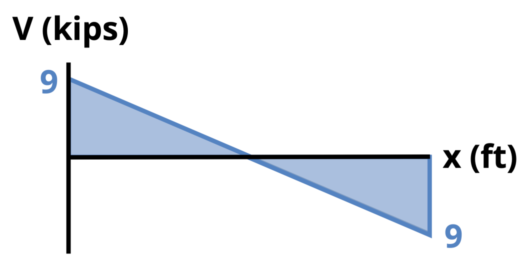
required dimensions of the beam's cross-section to meet these specifications. Give your answer rounded up to the nearest inch to allow for realistic manufacturing tolerances.

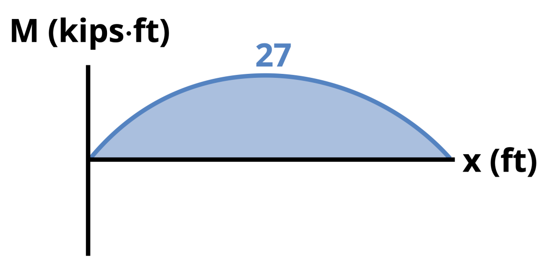


Solution

Start by drawing shear force and bending moment diagrams to determine the maximum internal loads:







$\mathrm{V_{max}} = 9$ kips
$\mathrm{M_{max}} = 27$ kip $\cdot$ ft
Design for bending by using the minimum required section modulus, $\mathrm{S_{min}}$:

$$S_{\mathrm{min}} = \frac{M_{\mathrm{max}}}{\sigma_{\mathrm{allow}}} = \frac{27000 \times 12}{900} = 360 \text{ in.}^3$$
$$360 = \frac{bh^2}{6} = \frac{b(1.5b)^2}{6} = \frac{2.25\ \mathrm{b}^3}{6}$$
$$b = 9.805 \text{ in.}$$

Design for shear:

$$\tau_{\mathrm{max}} = \frac{3}{2}\frac{V}{A}$$
$$180 = \frac{3}{2} \times \frac{9000}{A}$$
$$A = \frac{3}{2} \times \frac{9000}{180} = bh = b(1.5b) = 1.5b^2$$
$$b = 7.071 \text{ in.}$$

Answer only

$$b_{\text{req}} = 10 \text{ in.}$$
$$h_{req} = 15 \text{ in.}$$

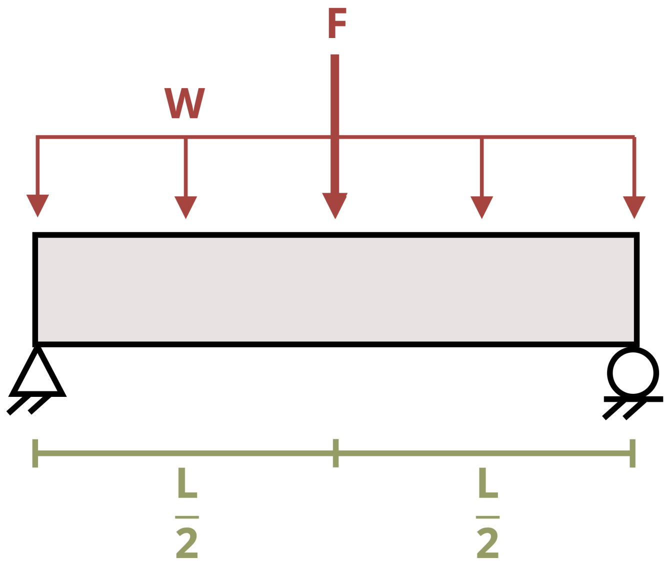
For W beams we can again use the method of section 9.2 to select a beam from Appendix A that meets the specifications for bending stress. Use $S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}}$ and select a beam with $S > S_{\text{min}}$.

Next determine the equation for the maximum deflection of the beam under the given loading conditions. As before, use superposition and Appendix C whenever possible. Set the maximum allowable deflection equal to this equation and solve for the minimum required area moment of inertia, I_{min}. Now select the lightest beam from Appendix A that meets both S_{min} and I_{min}.

Steel W beams rarely fail due to shear stress, but shear must still be checked. An exact value for the maximum shear stress in the beam can be found from $\tau_{\text{max}} = \frac{VQ}{It}$. However, most of the shear stress is resisted by the web of the beam. A reasonable estimate of the maximum shear stress can be found using $\tau_{\text{estimate}} = \frac{V}{A_{web}}$. This calculation is much simpler, but it is important to note that it is only an estimate. In most cases the maximum shear stress will be significantly less than the allowable shear stress and this estimate will suffice to demonstrate that. In the rare case that the maximum shear stress is within 20% of the allowable shear stress, an exact value must be determined using $\tau_{\text{max}} = \frac{VQ}{It}$. The beam was already selected to meet the bending stress and deflection requirements. As long as the maximum shear stress is less than the allowable shear stress, the beam can be used. See Example 11.9 for a demonstration.

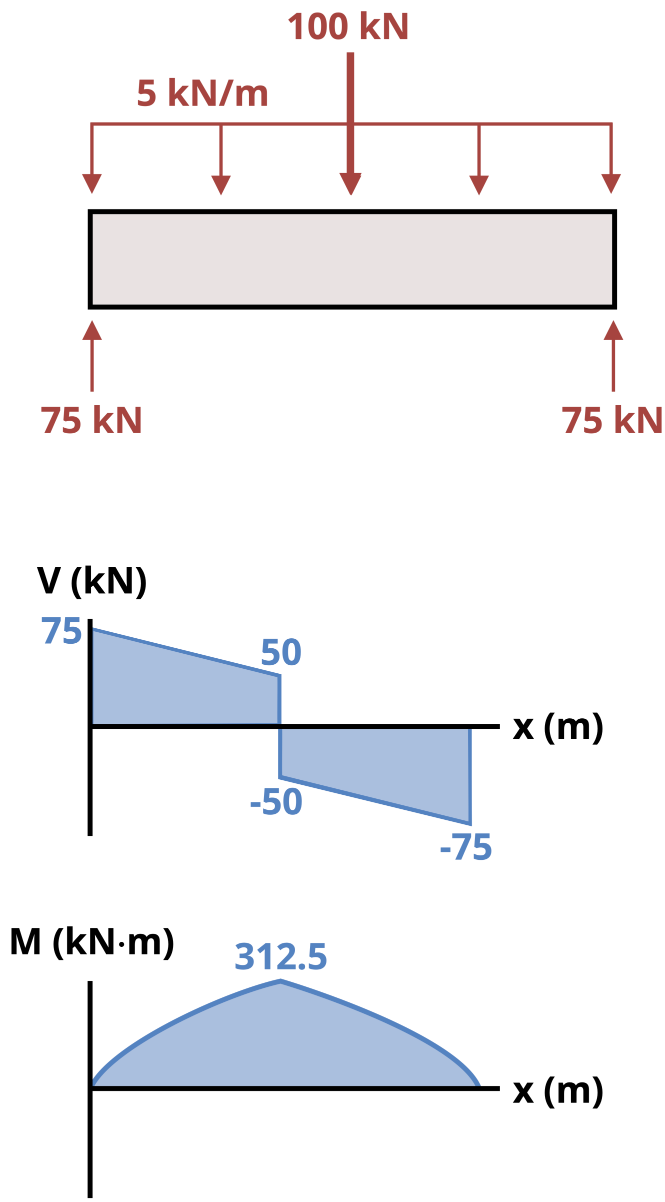
Example 11.9

A steel (E = 210 GPa) W-beam of length L = 10 m will need to support the load shown, where w = 5 kN/m and F = 100 kN.
Assume the beam has an allowable bending stress of 150 MPa, an allowable shear stress of 100 MPa, and the maximum deflection is limited to 40 mm.
Select the lightest weight W-beam from Appendix A that meets these specifications.


F
w
L
2
L
2

Solution

Load is symmetric, so each support will hold half the load. Draw shear force and bending moment diagrams to find the maximum internal loads.



$\mathrm{V_{max}} = 75$ kN
$\mathrm{M_{max}} = 312.5\ \mathrm{kN \cdot m}$
Design for bending by using the minimum required section modulus $\mathrm{S_{min}}$:

$$S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{Allow}}} = \frac{312.5 \times 10^3}{150 \times 10^6} = 0.002083\ \text{m}^3$$
$$= 2083 \times 10^3\ \text{mm}^3$$

Design for deflection:
We know the maximum deflection is limited to 40 mm (0.04 m) and the beam will deflect downward. From Appendix C we can see the maximum deflection for both the concentrated load and the distributed load occur in the same place. We can

Answer only

W610 x 101

Step-by-step: Intermediate Beam Design

Rectangular cross-sections

1. Determine minimum required section modulus, $S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}} = \frac{bh^2}{6}$ and determine the minimum dimensions to meet the required S_{min}.
2. Set the maximum allowable shear stress, $\tau_{\text{max}} = \frac{3}{2}\frac{\text{V}}{\text{A}}$ and determine the minimum dimensions to not exceed τ_{max}.
3. Determine the maximum deflection for the applied load(s) and set this equal to the maximum allowable deflection. This equation will contain I, which depends on dimensions b and h. Determine the minimum dimensions to not exceed the maximum allowable deflection.
4. Use the largest of the three calculated dimensions as these will be the smallest dimensions that meet all three requirements.

W-beams

1. Determine minimum required section modulus, $S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}}$.
2. Determine the maximum deflection for the applied load(s) and set this equal to the maximum allowable deflection. Determine the minimum required area moment of inertia, I_{min}, to not exceed the maximum allowable deflection.
3. Select the lightest beam from Appendix A that meets both the S_{min} and I_{min} requirements.
4. Estimate the maximum shear stress in the selected beam using $\tau = \frac{V}{A_{web}}$. If this is within ~20% of the maximum allowable shear stress, calculate the maximum shear stress using $\tau_{\text{max}} = \frac{VQ}{It}$.
5. If the maximum shear stress is less than the maximum allowable value, use this beam. If it's greater than the maximum allowable value, return to step 3 and select the next lightest beam that meets both the S_{min} and I_{min} requirements.

Summary

Click to expand

Key takeaways

- Beams under load will experience a type of deformation known as deflection. It is typically required to limit deflection to some small amount.
- Deflection can be determined using either the method of integration or the method of superposition.
- Both methods can be used for both statically determinate and statically indeterminate beams, though superposition is recommended whenever possible as it is generally quicker.
- Engineers designing beams must consider the response of the beam to bending stress, shear stress, and deflection simultaneously.

Key equations

Deflection by integration:

$$w(x) = EI\frac{d^4y}{dx^4}$$
$$V(x) = EI\frac{d^3y}{dx^3} = \int w(x)dx + C1$$
$$M(x) = EI\frac{d^2y}{dx^2} = \int V(x)dx + C2$$
$$\theta(x) = EI\frac{dy}{dx} = \int M(x)dx + C3$$
$$y(x) = \int \theta(x)dx + C4$$

Boundary conditions:

- At any support the deflection is 0
- At a fixed support, both the slope and deflection are zero
- The shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.
- The bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.

Beam Design for rectangular cross-sections:

$$S = \frac{bh^2}{6}$$
$$\tau_{\max} = \frac{3}{2}\frac{V}{A}$$

Beam design for W beams:

$$\begin{aligned} S_{\min} &= \frac{M_{\max}}{\sigma_{\text{allow}}} \\ \tau_{\max} &= \frac{VQ}{It} \\ \tau_{\text{estimate}} &|= \frac{V}{A_{\text{web}}} \end{aligned}$$

Chapter questions

References

Show references

12 Stress Transformation

Learning Objectives

- Insert text

12.1 Introduction

Click to expand

Insert text

13 Thin-Walled Pressure Vessels

Learning Objectives

- Insert text

13.1 Introduction

Click to expand

Insert text

14 Combined Loads

Learning Objectives

- Insert text

14.1 Introduction

Click to expand

Insert text

15 Columns

Learning Objectives

- Insert text

15.1 Introduction

Click to expand

Insert text

A Geometric Properties of Standard Beams

Insert text

B Slope and Deflection Tables

Insert text

C Material Properties

Insert text

D Units and Order of Magnitude

Insert text