Mechanics of Deformable Bodies

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Introduction

Mechanics can be broadly categorized in three parts, of which this text covers one. **Statics** explores the effects of *external loads* applied to *rigid bodies* in *equilibrium.* **Dynamics** explores the effects of *external loads* applied to *rigid bodies* that are not in *equilibrium.* In **Mechanics of Deformable Bodies**, the subject of this text, we explore the effects of both *external* and *internal loads* on *non-rigid bodies* in *equilibrium.*

Since bodies in this text are in equilibrium, many of the principles from statics still apply. However, because these bodies are non-rigid, a much deeper analysis is required. Objects in this text will deform. When loads are applied, they will change their shapes. They will elongate and bend and twist and, if we're not careful, may even break. Real objects are deformable, and engineers must take great care to ensure that their designs do not break or deform so much that they are unfit for purpose.

Have you ever wondered why civil engineers tend to use certain materials for bridges and buildings, while automotive engineers use an entirely different set of materials in cars? Meanwhile, aerospace engineers don't make rockets out of the same materials that biomedical engineers use for prosthetics. There are many material properties that determine applications that a given material may, or may not, be suitable for. Two fundamental concepts explored in this text are stress, which helps us determine whether an object will break under a given load, and strain, which relates to the object's deformation.

This text begins with an overview of stress and strain. Depending on how external loads are applied, there are a variety of potential internal loads on our object. You are likely already familiar with these loads and their effects, even if you haven't explicitly though about them.

- **Axial forces** push or pull on a body. They cause elongation and compression. For example, imagine pulling on the ends of an elastic band. The band will stretch significantly. Objectd may be pulled apart or crushed by this load. Imagine doing the same with a piece of chalk. There will be a small amount of deformation and the chalk will break.
- **Bending moments** act to bend an object. Imagine propping a long piece of plywood up on two supports and standing in the middle. The wood will bend. If you jump and land hard enough, it may even break.
- **Shear forces** are created when one body (or part of a body) slides relative to another. They commonly occur in conjunction with bending moments.

- **Torsional moments** act to twist an object. Imagine holding the ends of a cable and rotating your wrists so that the cable twists around itself along its length. You will see the cable rotate through ever larger angles. Twist it too far and it will break.

This text will explore the effects of these loads and expand your understanding of these effects. It aims to describe these effects in a conceptual way, building on your existing understanding. Once these effects are understood conceptually, we'll learn how to calculate the exact amount of deformation and the exact size of the stress created. We'll learn to make sure that our designs stay within appropriate limits. There will necessarily be a significant amount of math involved in these calculations but it is just as important, if not more so, that we understand the principles behind those calculations. As such, this text leads with conceptual descriptions and realistic examples before working numerical problems.

The stress and strain resulting from these four types of load account for the majority of this text. Once they are thoroughly understood, more advanced topics necessary for a thorough understanding of these fundamental concepts are included. Upon completion of a course using this text, students will be able to explain the stress and strain observed under different types of load, calculate the amount of stress and strain, and design individual components to be within acceptable limits for both. Students will be prepared for advanced courses in a variety of engineering disciplines, and will be able to apply their knowledge to realistic scenarios.

1 Introduction and Statics Review

Learning Objectives

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2 Stress

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11 Beam Deflection

Learning Objectives

- Derive an equation for the elastic curve of a loaded beam
- Calculate the slope and deflection at any point of a loaded beam
- Use standard solutions and the method of superposition to determine deflection in more complex problems
- Use knowledge of deflection to solve statically indeterminate problems
- Design beam cross-sections that meet specifications for bending stress, shear stress, and deflection

Introduction

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We now know how to calculate both the bending stress (Chapter 9) and shear stress (Chapter 10) in a beam. In this chapter we'll consider the deformation of a beam subjected to bending and shear stresses. Under load, beams deflect from their original position (Figure 11.1). Some amount of deflection is unavoidable, but it is usually desirable to limit deflection as much as possible. Although deflection often doesn't pose a safety risk (unless the allowable stress of the beam is also exceeded) too much deflection can render a beam unfit for purpose.

Idealized sketches showing the shape of the deflected beam are a helpful visual aid. This deflected shape is known as the elastic curve (Figure 11.2). Sketches of the elastic curve provide a quick visual reference of how the beam will deflect and can be used to check that our numerical answers are realistic.

In section 5.3 we learned to calculate deformation due to axial load $\left(\Delta L = \frac{FL}{AE}\right)$. In section 6.2 we learned to calculate deformation due to torsional load $\left(\emptyset = \frac{TL}{JG}\right)$. Note that both types of deformation depend on:

- Applied/internal load (F, T)
- Length of the object (L)
- Cross-section geometry (A, J)

Figure 11.1: A model bridge showing severe deflection along its span (left). Although the bridge hasn't broken, there is too much deflection for the application. The deflection in a hockey stick during a shot (right). In this application the deflection is helpful as it stores and releases energy during the shot.

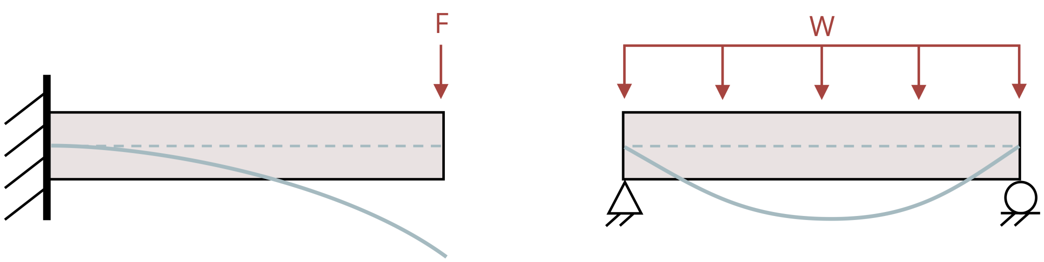

Figure 11.2: Two example beams showing the undeflected position of the centerline (dotted line) and the deflected position of the elastic curve (solid line).

- Material properties (E, G)

It is probably not surprising then that beam deflection also depends on these four things. We'll study two common techniques for calculating deflection: The method of integration (sections 11.1 and 11.2) and the method of superposition (section 11.3). Being able to calculate deflection is vitally important, as it allows us to predict how much a beam will deflect before we build a structure. Once we're able to calculate deflection, we'll use this to help solve some more statically indeterminate problems (Section 11.4). Finally, we'll build on our beam design work of section 9.2 by expanding our specifications to include limitations on not just bending stress, but shear stress and deflection as well.

11.1 Integration of the Moment Equation

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We'll begin by deriving an equation for calculating the deflection at any point in a loaded beam. In Chapter 9, we derived two equations relating to bending stress.

$$\sigma = -\frac{Ey}{\rho}$$

$$\sigma = -\frac{My}{I}$$

Set these equations equal:

$$\frac{Ey}{\rho} = \frac{My}{I}$$

Rearrange:

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

The radius of curvature ( ) can be related to the deflection (y) by the equation

$$\frac{1}{\rho} = \frac{\frac{\partial^2 y}{\partial x^2}}{\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{\frac{3}{2}}}$$

This equation describes the exact deflection (y) at any distance (x) along the beam. In most engineering applications the deflection (y) is very small. We can simplify this equation significantly by assuming that our deflection (y) will be small and thus that $\frac{dy}{dx}$ is very small and $\left(\frac{dy}{dx}\right)^2$ is negligible. This simplifies the equation to

$$\frac{1}{\rho} = \frac{\frac{\partial^2 y}{\partial x^2}}{[1+0]^{\frac{3}{2}}} = \frac{\partial^2 y}{\partial x^2}$$

Finally

$$\frac{1}{\rho} = \frac{\partial^2 y}{\partial x^2} = \frac{M(x)}{EI}$$

To solve for deflection we can first find an equation for the internal bending moment as a function of (x), using equilibrium just as we did in section 7.2 for drawing the bending moment diagram. We can then set this equation equal to $EI\frac{d^2y}{dx^2}$ and integrate twice.

$$M(x) = EI\frac{\partial^2 y}{\partial x^2}$$
$$\int M(x)dx + C1 = EI\frac{\partial y}{\partial x}$$
$$\iint M(x)dx + C1x + C2 = EIy$$

In these equations, y represents the deflection of the beam and $\frac{dy}{dx}$ represents the slope of the deflected beam. Since the equations are functions of x, we can find the slope and deflection at any point along the beam.

Note that each successive integral introduces a constant of integration which we must solve for. We do this through the use of boundary conditions—that is, points on the beam where we already know the slope and/or deflection. The most common boundary conditions for our applications will be:

- At any support the deflection (y) is zero
- At a fixed support the slope $\left(\frac{\partial y}{\partial x}\right)$ is zero

Figure 11.3 illustrates these boundary conditions. Once the constants of integration are known, we can define equations for the slope and deflection of the beam in terms of distance (x) along the beam and calculate the slope and deflection at any value of x. See Example 11.1 and Example 11.2 to see this process applied to a simply supported beam and a cantilever beam respectively.

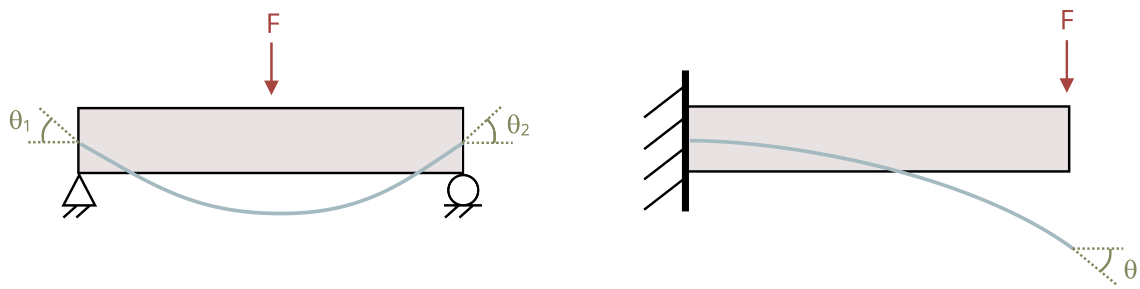
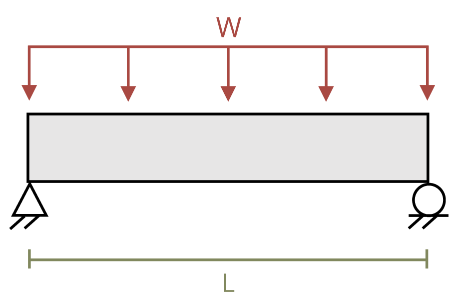


Figure 11.3: At pin and roller supports, the deflection is zero but the slope is not. At a fixed support, both the slope and deflection are zero. For a statically determinate beam in equilibrium we will always have two boundary conditions.

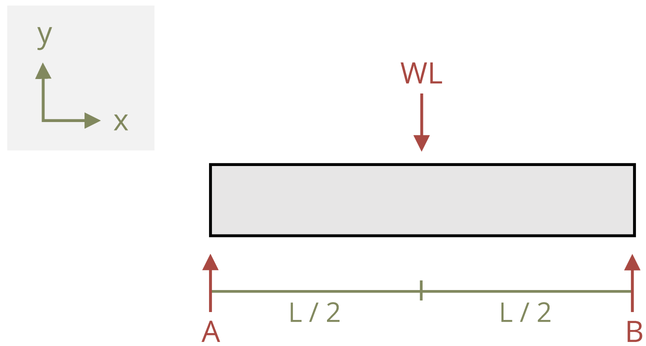
Example 11.1

A simply supported beam of length L = 10 m is subjected to a uniform distributed load of w = 20 kN/m. Determine the equation of the elastic curve and use this to find the deflection of the beam at the midpoint, x = 5 m. Assume E = 200 GPa and I = 350 x 10^6 mm^4.



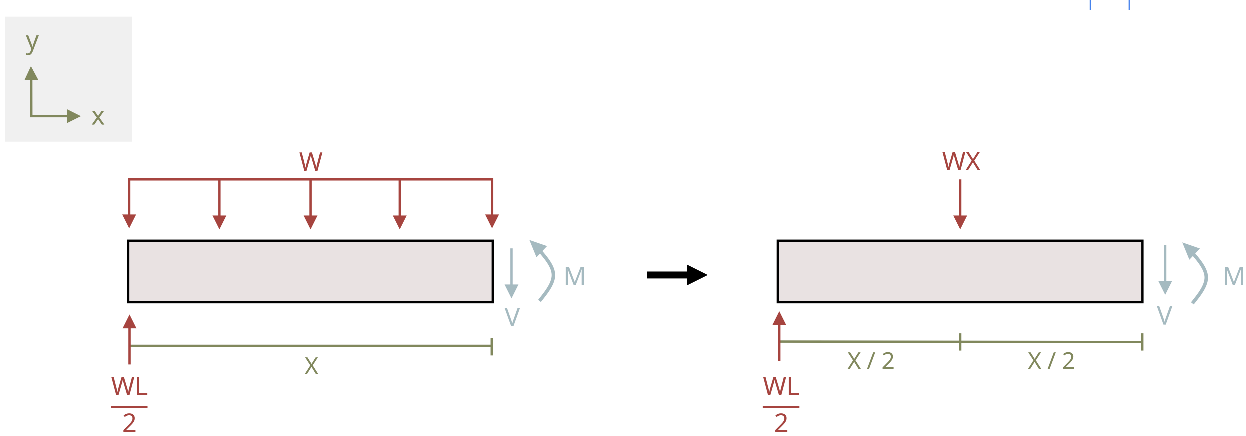
Solution

Sketch a free body diagram of the beam and use equilibrium equations to solve for the reaction forces at the supports:



$$\sum M_A = 0: \quad -\omega L\left(\frac{L}{2}\right) + BL = 0 \quad \rightarrow \quad B = \frac{\omega L}{2}$$
$$\sum Fy = 0: \quad A - \omega L + \frac{\omega L}{2} = 0 \rightarrow A = \frac{\omega L}{2}$$

Cut a cross-section through the beam at distance x and draw a free body diagram of everything to the left of the cut, including the internal loads. Use equilibrium to determine an equation for the internal bending moment, M, as a function of x:



$$\sum M_A = 0: M + wx\left(\frac{x}{2}\right) - \frac{wL}{2}(x) = 0$$
$$M = \frac{wLx}{2} - \frac{wx^2}{2} = EI\frac{\partial^2 y}{\partial x^2}$$

Integrate this equation twice:

$$\frac{wLx^2}{4} - \frac{wx^3}{6} + c_1 = EI\frac{\partial y}{\partial x}$$
$$\frac{wLx^3}{12} - \frac{wx^4}{24} + c_1 x + c_2 = EIy$$

Apply boundary conditions at the supports:

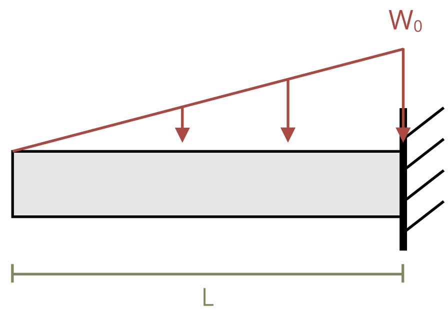
Answer only

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3}{24}x\right]$$

$$y = -37.2\ mm$$

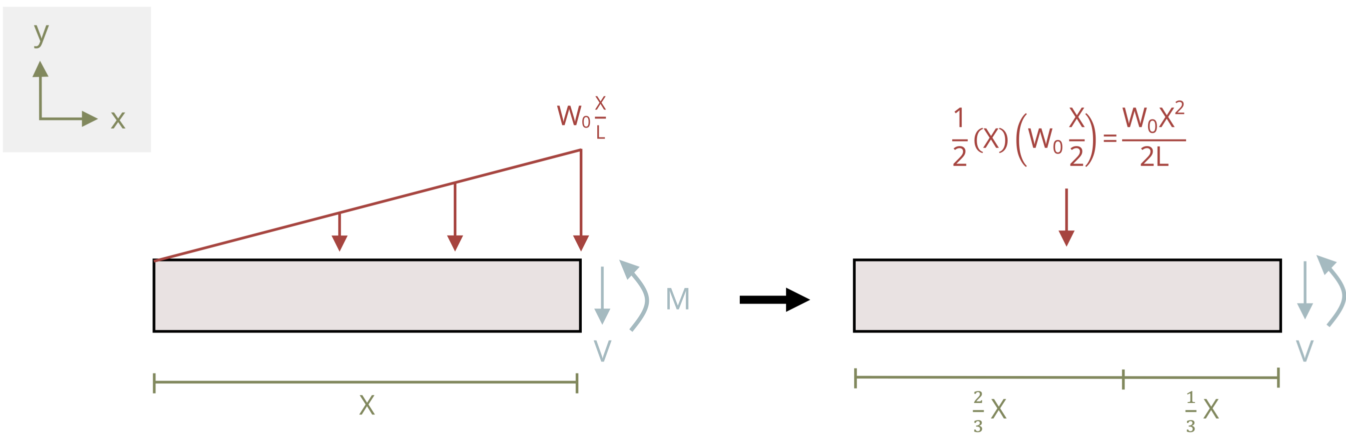
Example 11.2

A cantilever beam of length L = 8 ft is subjected to a linear distributed load where w_0 = 30 kip/ft. Determine the equation of the elastic curve and use this to find the deflection of the beam at the free end. Assume E = 29 x 10^6 psi and I = 375 in.^{4}.



Solution

Cut a cross-section through the beam at distance x and draw a free body diagram of everything to the left of the cut, including the internal loads. This diagram won't include any of the reactions at the fixed support so there's no need to solve for these first. Use equilibrium to determine an equation for the internal bending moment, M, as a function of x. Then integrate this equation twice:



$$\sum M_x = 0: \quad M + \frac{w_0 x^2}{2L}\left(\frac{x}{3}\right) = 0$$

$$M = -\frac{w_0 x^3}{6L} = EI\frac{\partial^2 y}{\partial x^2}$$

$$-\frac{w_0 x^4}{24L} + C_1 = EI\frac{\partial y}{\partial x}$$

$$-\frac{w_0 x^5}{120L} + C_1 x + C_2 = EIy$$

Apply boundary conditions at the fixed support:

$$\text{At } x = L, \frac{\partial y}{\partial x} = 0 \quad \rightarrow \quad \frac{w_0 L^4}{24L} + C_1 = 0$$

$$c_1 = \frac{w_0 L^3}{24}$$

$$\text{At } x = L, y = 0 \quad \rightarrow \quad \frac{w_0 L^5}{120L} + \frac{w_0 L^3}{24}(L) + C_2 = 0$$

$$C_2 = \frac{w_0 L^4}{120} - \frac{w_0 L^4}{24} = -\frac{4w_0 L^4}{120} = -\frac{w_0 L^4}{30}$$

Substitute these constants into the deflection equation and rearrange:

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

Substitute in the given values and solve this equation at the free end (x = 0). Remember to convert feet to inches in both the distributed load and the length:

$$y = \frac{1}{29 \times 10^6 \times 375}\left[-\frac{\frac{30000}{12} \times (8 \times 12)^4}{30}\right]$$

$$y = -0.651 \text{ in.}$$

Answer only

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

$$y = -0.651\ in.$$

The above examples have single continuous loads. If the loading is discontinuous we must modify our approach as there will not be one single moment equation that describes the entire beam. We saw in section 7.2 when drawing shear force & bending moment diagrams that we can find a different equation for the internal bending moment in each loading region (i.e., a piecewise function over the length of the beam). We must find separate moment equations for each loading region and integrate each separately. This must be done each time the loading changes (a discontinuity in the moment diagram), such as:

- Both sides of a concentrated load
- Whenever a distributed load begins or ends

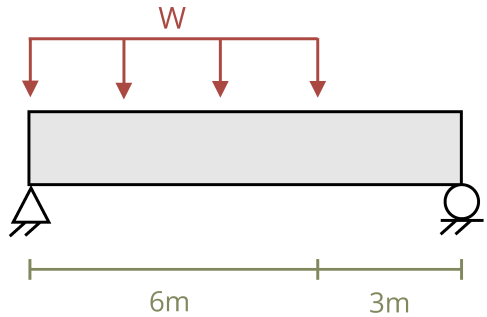
See Example 11.3 for an example of a beam with 2 loading regions. Note that in this example we will need to determine 2 internal bending moment equations and integrate each equation twice, thereby introducing four constants of integration. Two of these can still be determined by using boundary conditions, but we still need to determine the other two. This is done with continuity conditions.

Equation set 1 describes the internal moment, slope, and deflection of the beam from point A to point B, and equation set 2 describes the internal moment, slope, and deflection from point B to point C. Note that both sets of equations describe point B, where the loading changes. They must therefore return the same results at point B. We can therefore say that

$$\left(\frac{\partial y}{\partial x}\right)_1 = \left(\frac{\partial y}{\partial x}\right)_2$$
$$y_1 = y_2$$

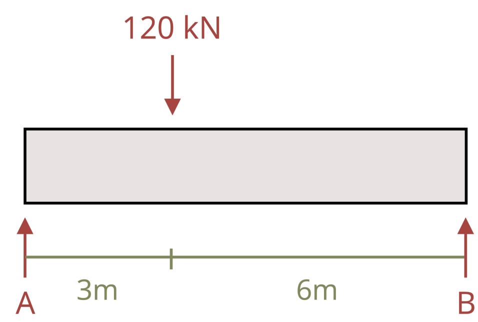
Example 11.3

A simply supported beam is subjected to a distributed load of w = 20 kN/m as shown. Determine the equation of the elastic curve between 0   x   6 m and between 6 m   x   9m.
Then determine the deflection at x = 5m. Assume E = 200 GPa and I = 394 x 10^6 mm^4.


W
6m
3m

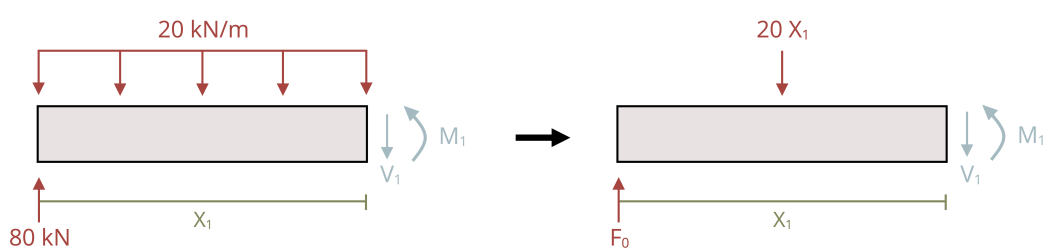
Solution

Sketch a free body diagram of the beam, and use equilibrium equations to solve for the reaction forces at the supports:



$$\sum M_A = 0: \quad -(120 \times 3) + (B + 9) = 0 \quad \rightarrow \quad B = 40 \text{ kN}$$
$$\sum Fy = 0: \quad A - 120 + 40 = 0 \quad \rightarrow \quad A = 80 \text{ kN}$$

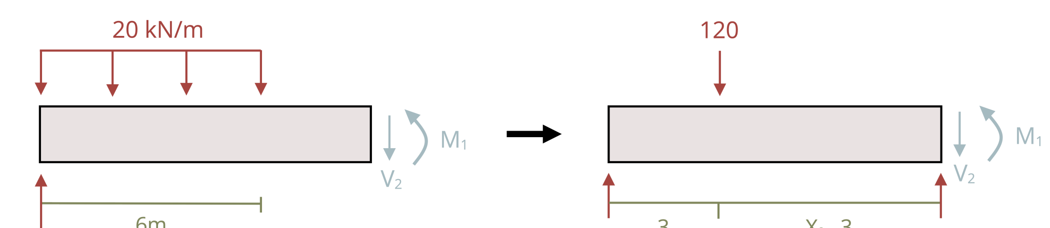
Cut a cross-section in the first loading region at distance x_1. Draw a free body diagram of everything to the left of this cut, including the internal loads. Use equilibrium to determine an equation for the internal bending moment, M_1, as a function of x_1. Then integrate this equation twice:



$$\sum M_x = 0: \quad M_1 + 20x_1\left(\frac{x_1}{2}\right) - 80x_1 = 0$$
$$M_1 = 80x_1 - 10x_1^2 = EI\frac{\partial^2 y_1}{\partial x_1^2}$$
$$\frac{80x_1^2}{2} - \frac{10x_1^3}{3} + c_1 = EI\frac{\partial y_1}{\partial x_1}$$
$$\frac{80x_1^3}{6} - \frac{10x_1^4}{12} + C_1x_1 + c_2 = EIy_1$$

These equations are valid from 0 x_1 6m.

Set these equations aside and cut a cross-section in the second loading region at distance x_2. Draw a free body diagram of everything to the left of this cut, including the internal loads. Use equilibrium to determine an equation for the internal bending moment, M_2, as a function of x_2. Then integrate this equation twice:



Answer only

From 0 x 6 m:

$$y_1 = \frac{1}{EI}\left[\frac{80x_1^3}{6} - \frac{10x_1^4}{12} - 480x_1\right]$$

From 6 x 9 m:

$$y_2 = \frac{1}{EI}\left[\frac{360x_2^2}{2} - \frac{40x_2^3}{6} + 1200x_3 + 1080\right]$$

At x = 5 m:

$$y_1 = -15.9\ mm$$

It is not required that we make both cuts from the left. In some cases it may be easier to make one cut from the left and the other from the right. Consider the simple beam in Figure 11.4. When making a cut in the region not under the distributed load, a free body diagram to the left of the cut has many more forces than a free body diagram of the right side.

The moment equation will be much simpler if we draw the right hand side of the cut. This in turn simplifies the integrations and the math involved in finding the constants of integration. We must however make one change to our continuity conditions if we measure distance x_2 from the right. When making both cuts from the left we said that at point C the slope and deflection from calculated from equation 1 must be the same as those calculated from equation 2.

$$\left(\frac{dy}{dx}\right)_1 = \left(\frac{dy}{dx}\right)_2$$
$$y_1 = y_2$$

When measuring distance x_2 from the right, it is still true at point C that [math] as the deflection is still downwards regardless of whether we measure our horizontal distance from the left or right. The magnitude of the slope will also be the same at point C regardless of which equation we use to calculate it. However, the direction of rotation of the slope will change from counter-clockwise to clockwise due to us changing the coordinate system (Figure 11.5). Thus our second boundary condition must become

$$\left(\frac{dy}{dx}\right)_1 = -\left(\frac{dy}{dx}\right)_2$$

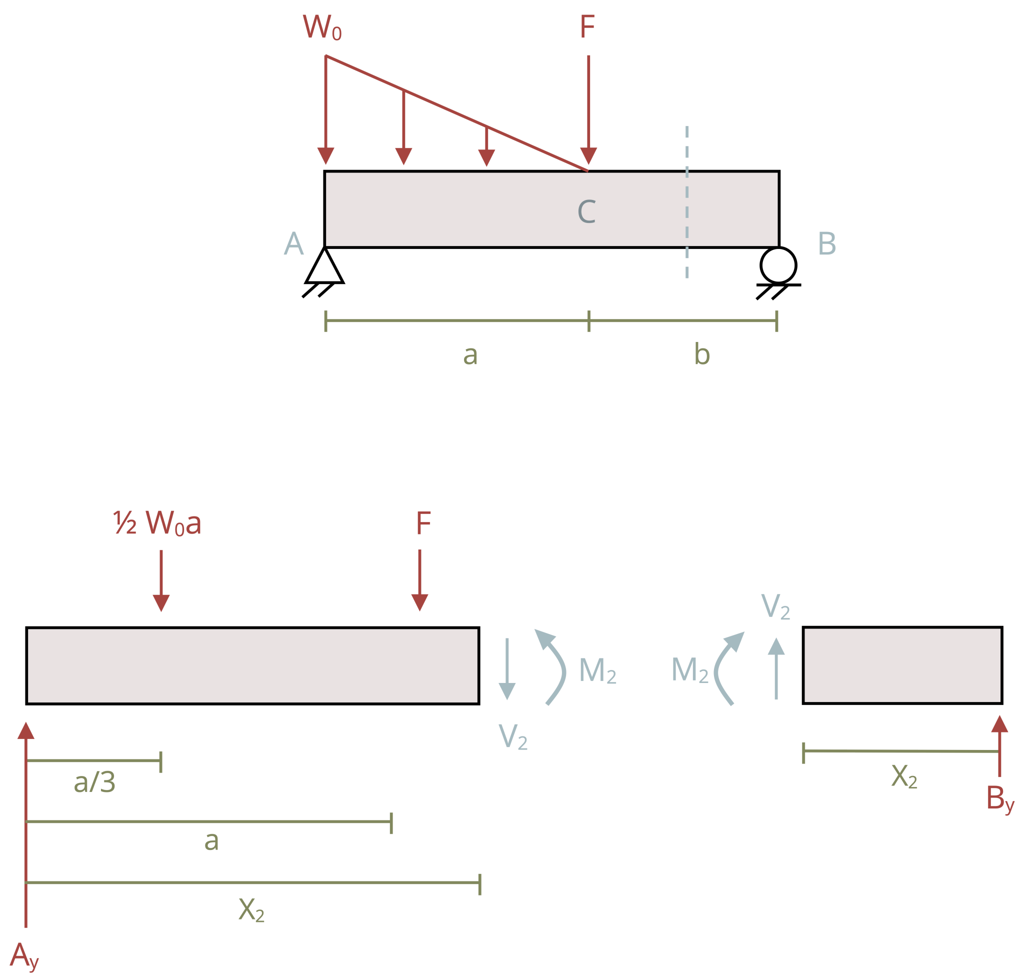


Figure 11.4: Simple beam showing two options for finding the internal bending moment. Note that the moment equation will be much simpler if we draw the right hand side of the cut.

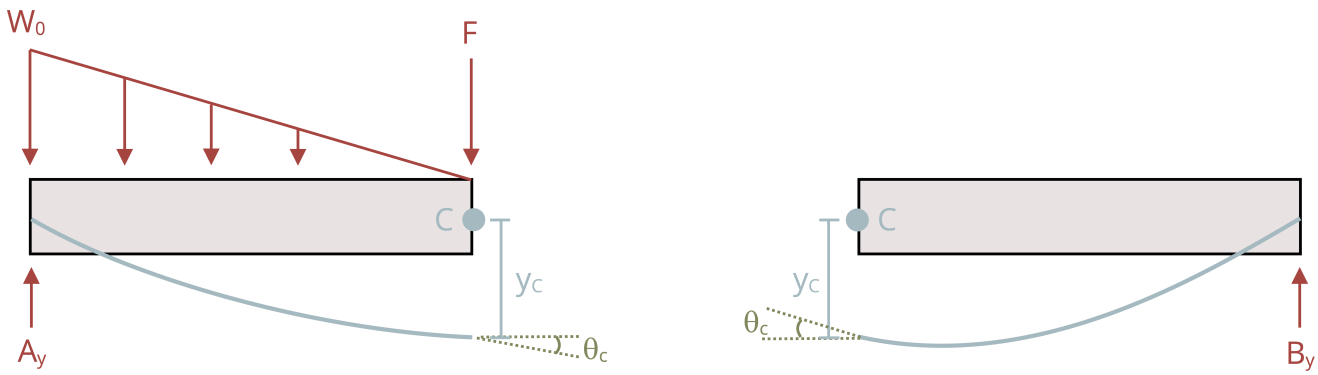
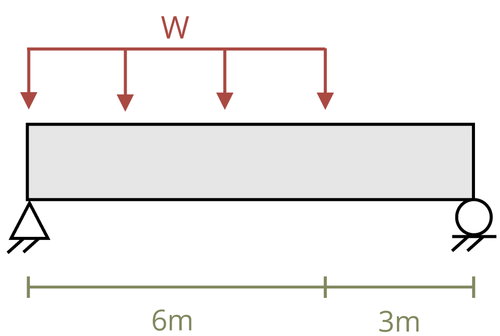


Figure 11.5: Depending on which side of the cut we draw, the deflection at point C will have the same magnitude and direction. The slope at point C will have the same magnitude but opposite directions if we make one cut from the left and the other from the right.

This negative sign is vitally important, and we will not get the correct answer if we forget to include it. Example 11.3 has been reworked below, this time with one cut made from the right hand side instead. Compare to the previous solution to see the differences.

Example 11.3 (reworked)

A simply supported beam is subjected to a distributed load of w = 20 kN/m as shown. Determine the equation of the elastic curve between 0   x   6 m and between 6 m   x   9m.
Then determine the deflection at x = 5m. Assume E = 200 GPa and I = 394 x 10^6 mm4.

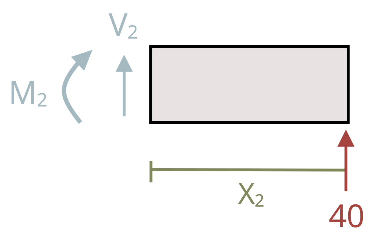


Solution

Proceed through first cut as before and find:

$$\frac{80x_1^2}{2} - \frac{10x_1^3}{3} + c_1 = EI\frac{\partial y_1}{\partial x_1}$$
$$\frac{80x_1^3}{6} - \frac{10x_1^4}{12} + c_1x_1 + c_2 = -EIy_1$$

Make second cut from right-hand-side. Draw a free body diagram and use equilibrium to determine an equation for the internal bending moment, M_2, as a function of x_2. Then integrate this equation twice.



$$\sum M_x = 0: \quad 40x_2 - M_2 = 0$$
$$M_2 = 40x_2 = EI\frac{\partial^2 y_2}{\partial {x_2}^2}$$
$$\frac{40x_2^2}{2} + c_3 = EI\frac{\partial y_2}{\partial x_2}$$
$$\frac{40x_2^3}{6} + c_3x_2 + c_4 = EIy_2$$

Apply boundary conditions as before, although this time the roller support is at $x_2 = 0$ since we're measuring x_2 from the right side:

$$\text{At } x_1 = 0, y_1 = 0 \quad \rightarrow \quad c_2 = 0 \text{ (as before)}$$
$$\text{At } x_2 = 0, y_2 = 0 \quad \rightarrow \quad c_4 = 0$$

Apply continuity conditions at the point where the loading changes (x_1 = 6 m, x_2 = 3 m):

$$\text{At } x_1 = 6 \text{ m}, x_2 = 3 \text{ m}, y_1 = y_2 \rightarrow \frac{80(6)^3}{6} - \frac{10(6)^4}{12} + c_1(6) = \frac{40(3)^3}{6} + c_3(3)$$
$$2880 - 1080 + 6c_1 = 180 + 3c_3$$
$$540 + 2c_1 = c_3$$

$$\text{At } x_1 = 6 \text{ m}, x_2 = 3\text{m}, \frac{dy_1}{dx_1} = -\frac{dy_2}{dx_2} \rightarrow \frac{80(6)^2}{2} - \frac{10(6)^3}{3} + c_1 = -\frac{40(3)^2}{2} - c_3$$
$$1440 - 720 + c_1 = -180 - c_3$$
$$900 + c_1 = -c_3$$

Substitute in $540 + 2c_1 = c_3$:

$$540 + 2c_1 = -900 - c_1$$
$$3c_1 = -1440$$

Answer only

From 0   x   6 m:

$$y_1 = \frac{1}{EI}\left[\frac{80x_1^3}{6} - \frac{10x_1^4}{12} - 480x_1\right]$$

From 6   x   9 m:

$$y_2 = \frac{1}{EI}\left[\frac{40x_2^3}{6} - 420x_2\right]$$

At x = 5 m:

$$y_1 = -15.9\ mm$$

Step-by-step: Deflection by Integration of Moment Equation

1. Solve for reaction loads using equilibrium.
2. Cut a cross-section through the beam and find the internal bending moment as a function of x. If the external loading changes, repeat this in each segment of the beam.
3. Set the internal bending moment $M(x) = EI\frac{d^2y}{dx^2}$ and integrate twice.
4. Use boundary conditions to determine the two constants of integration.
   a. At a fixed support, slope $\left(\frac{dy}{dx}\right) = 0$
   b. At any support, deflection (y) = 0
5. Determine the slope and deflection at any value of x as required.

11.2 Integration of the Load Equation

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In section 11.1 we used equilibrium equations to determine the internal bending moment in a beam as a function of x. We related this bending moment to the deflection by

$$M(x) = EI\frac{d^2y}{dx^2}$$

In section 7.3 we learned that there is a relationship between the internal bending moment, the internal shear force, and the external load. For an external distributed load, w, we found

$$V = \int_0^L w(x)dx$$

$$M = \int_0^L V(x)dx$$

We may therefore relate the deflection of a beam to not only the internal bending moment, but also the internal shear force and external distributed load through successive integrations. For some loading configurations it can be much simpler to integrate the distributed load equation 4 times instead of finding the equation of the internal bending moment through equilibrium and integrating twice. When doing this, remember that loads acting downwards (as most loads on beams do) are negative.

$$\begin{aligned} w(x) &= EI\frac{d^4y}{dx^4} \\ V(x) = EI\frac{d^3y}{dx^3} &= \int w(x)dx + C1 \\ M(x) = EI\frac{d^2y}{dx^2} &= \int V(x)dx + C2 \\ \theta(x) = EI\frac{dy}{dx} &= \int M(x)dx + C3 \\ y(x) &= \int \theta(x)dx + C4 \end{aligned}$$

Note that each successive integral will introduce a new constant of integration. We must again apply boundary conditions in order to solve for these constants. There are now four constants so we require four boundary conditions. Two of these are the same as in section 11.1:

- At any support the deflection is 0
- At a fixed support, both the slope and deflection are zero

The other two boundary conditions come from knowing the value of the shear force and bending moment at a point in the beam. We are able to find these from our work in chapter 7. While we can use known values at any point in the beam, it is generally easiest to use the shear force and bending moment at x = 0. At this point:

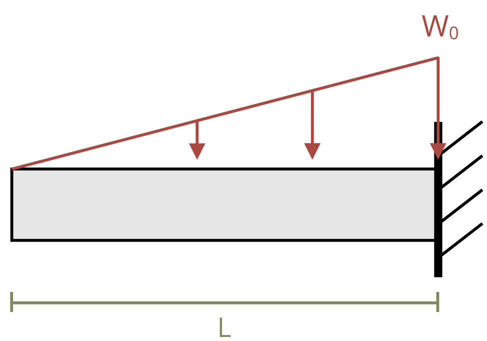
- The shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.

- The bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.

See Example 11.4 for a demonstration of how to find the deflection of a beam by integrating the load equation. Note that while this method can be used for any beam, it is best applied to beams with a single continuous distributed load acting over the length of the beam. Beams with discontinuous loading will require integration of multiple load equations.

Example 11.4

A cantilever beam from Example 11.2 is repeated here. Length L = 8 ft is subjected to a linear distributed load where $w_0 = 30$ kip/ft. Determine the equation of the elastic curve and use this to find the deflection of the beam at the free end.



Solution

First define the equation of the load, which increases linearly from zero to w_0:

$$\omega = -\omega_0 \frac{x}{L} = EI \frac{\partial^4 y}{\partial x^4}$$

Integrate this equation 4 times:

$$\begin{aligned}
&-\frac{w_0 x^2}{2L} + c_1 = EI \frac{\partial^3 y}{\partial x^3} \\
&-\frac{w_0 x^3}{6L} + c_1 x + c_2 = EI \frac{\partial^3 y}{\partial x^2} \\
&-\frac{w_0 x^4}{24L} + \frac{c_1 x^2}{2} + c_2 x + c_3 = EI \frac{\partial y}{\partial x} \\
&-\frac{w_0 x^5}{120L} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 = EIy
\end{aligned}$$

Apply boundary conditions at the fixed support. At the free end the internal shear force and bending moment will be zero. At the fixed support (at x = L) the slope and deflection will be zero.

$$\begin{aligned}
&\text{At } x = 0, v = 0 \quad \rightarrow \quad c_1 = 0 \\
&\text{At } x = 0, M = 0 \quad \rightarrow \quad c_2 = 0 \\
&\text{At } x = L, \frac{dy}{dx} = 0 \quad \rightarrow \quad c_3 = \frac{w_0 L^3}{24} \\
&\text{At } x = L, y = 0 \quad \rightarrow \quad c_4 = -\frac{w_0 L^4}{30}
\end{aligned}$$

This is the same result found in Example 11.2 and the problem proceeds in the same way to find the deflection at the free end.
Substitute these constants into the deflection equation and rearrange:

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

Substitute in the given values and solve this equation at the free end (x = 0). Remember to convert feet to inches in both the distributed load and the length:

$$\begin{aligned}
y &= \frac{1}{29 \times 10^6 \times 375}\left[-\frac{\frac{30000}{12} \times (8 \times 12)^4}{30}\right] \\
y &= -0.651 \text{ in.}
\end{aligned}$$

Answer only

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

$$y = -0.651\ in.$$

Step-by-step: Deflection by Integration of Load Equation

1. Define an equation that describes the external distributed load as a function of x. Remember that loads acting downwards are negative.
2. Integrate this equation four times to determine equations for internal shear force, internal bending moment, slope, and deflection respectively.
3. Apply boundary conditions to determine the four constants of integration
   a. At $x = 0$, the shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.
   b. At $x = 0$, the bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.
   c. At a fixed support, slope $\left(\frac{dy}{dx}\right) = 0$
   d. At any support, deflection (y) = 0
4. Determine the slope and deflection at any value of x as required.

11.3 Superposition

Click to expand

It should be apparent from the previous sections that finding deflection using the method of integration can become quite complicated and time-consuming if there are multiple loads. In such cases, the method of superposition is a good alternative.

Beams are typically loaded in a relatively small number of standard configurations, and the deflection behavior of a beam subjected to one of these standard loads is very well understood. A subset of beam deflections have been included in Appendix C. Figure 11.6 shows an example of the information provided for one such beam.

This can be useful even for more complex loading. When multiple loads act on a beam, the effect of each load on the deflection of the beam may be considered independently. Thus, even

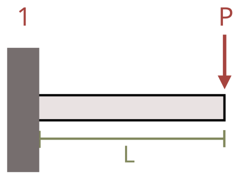
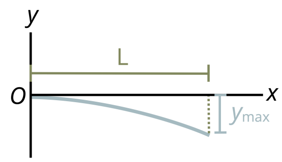
Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equatio
		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = -$

Figure 11.6: Appendix C contains some standard loading configurations along with a sketch of the elastic curve, equations for the maximum deflection and maximum slope, and the general equation of the elastic curve.

for loading more complex than the loads in Appendix C, it may be possible to simplify the problem by considering each load separately and calculating the deflection caused at a point by each load (Figure 11.7). These deflections can then be added together to find the overall deflection at that point. We must of course calculate the deflection at the same point for each load in order to add them together. See Example 11.5 for a demonstration.

$$y_w = -\frac{wL^4}{8EI} \qquad\qquad y_F = -\frac{FL^3}{3EI}$$

$$y_{max} = -\frac{wL^4}{8EI} - \frac{FL^3}{3EI}$$

Example 11.5

A simply supported beam is subjected to a distributed load w = 10 kN/m and a concentrated load F = 50 kN. If distance a = 6 m and b = 2 m, determine the deflection at the midpoint of the beam (x = 4 m).
Assume E = 210 GPa and I = 275 x 10^6 mm^4.

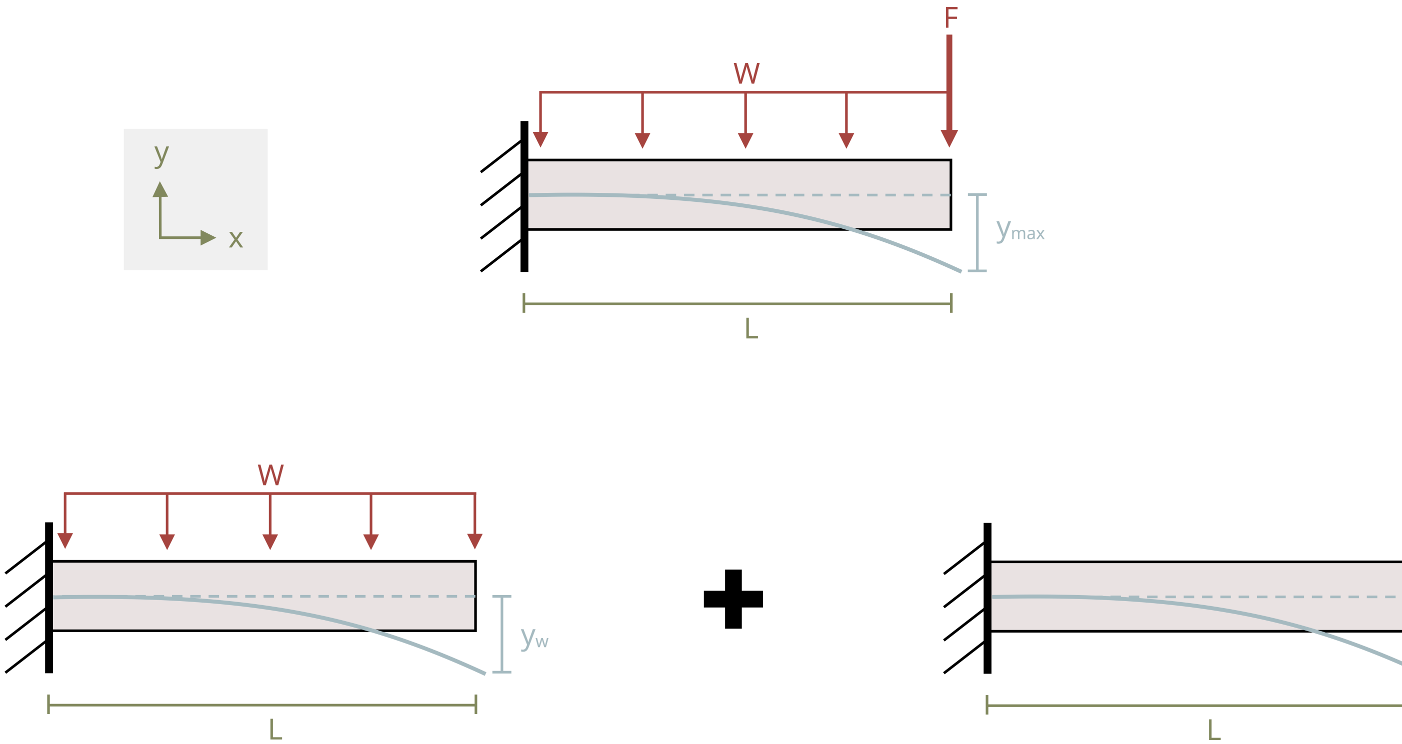
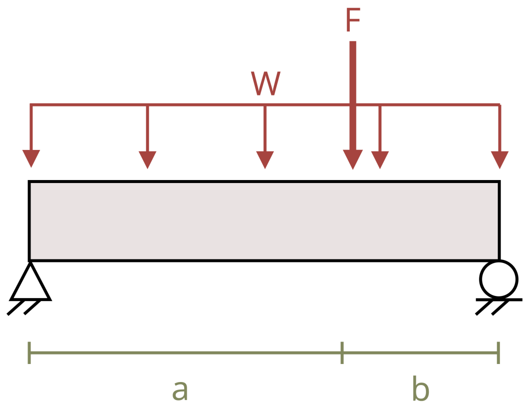
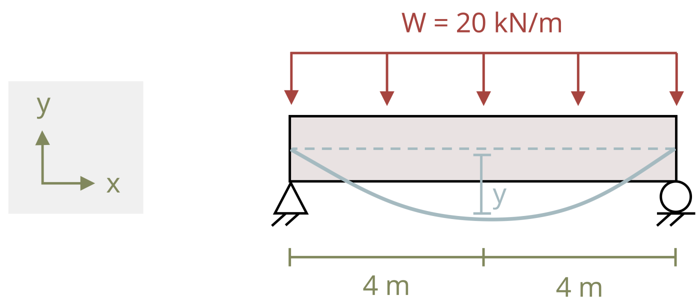


Figure 11.7: Although Appendix C doesn't contain a cantilever beam with both a distributed load and a concentrated load, it does contain cantilever beams with each load separately. We may find the combined deflection by finding the deflection of each load separately and adding them together.


F
w
a
b

Solution

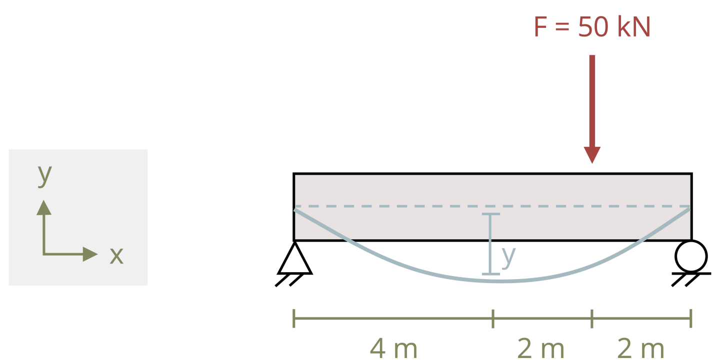
Consider each load separately. We'll start with the distributed load (although this can be done in any order).



From Appendix C we can see that the maximum deflection for this load occurs at the center of the beam, which is the point we're interested in.

$$y = -\frac{5wL^4}{384EI} = -\frac{5 \times 20000 \times 8^4}{384 \times 210 \times 10^9 \times 275 \times 10^{-6}} = -0.01847 \text{ m}$$
$$= -18.47 \text{ mm}$$

Now consider the concentrated load



Here the maximum deflection does not occur at the center of the beam. We can use the elastic curve equation for x < a to find the deflection at x = 4 m when the load is applied at a = 6 m.

$$y = \frac{Fb}{6EIL}\left[x^3 - \left(L^2 - b^2\right)x\right] = \frac{50000 \times 2}{6 \times 210 \times 10^9 \times 275 \times 10^{-6} \times 8}\left[4^3 - \left(8^2 - 2^2\right)4\right]$$
$$= -0.00635 \text{ m}$$
$$= -6.35 \text{ mm}$$

The total deflection at the midpoint of the beam is

$$y = -18.47 - 6.35 \text{ mm}$$
$$y = -24.8 \text{ mm}$$

Answer only

$$y = -24.8\ mm$$

Step-by-step: Deflection by Superposition

1. Sketch the beam with just one load acting on it.
2. Use Appendix C to find a beam with the same supports and loading.
3. Use the relevant equation in appendix C to find the deflection at the point of interest.
4. Repeat this process until all loads have been considered. Add the individual deflections at the point of interest to find the overall deflection at this point.

11.4 Statically Indeterminate Deflection

Click to expand

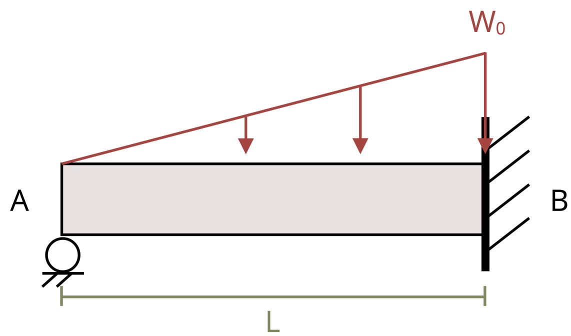
A problem is statically indeterminate if there are more unknown support reactions than there are equilibrium equations to solve for them. We have seen statically indeterminate problems in section 5.5 and section 6.4. In both sections we used a similar method to solve statically indeterminate problems where we took advantage of our knowledge of deformation in order to solve for one of the reaction forces before using equilibrium equations to find the others. We'll do something similar here. We may use either an integration method similar to that of section 11.1 or a superposition method similar to that of section 11.3.

In practice, engineers design structures with redundancies so that if one part fails it can be replaced without the entire structure collapsing. As such, static indeterminacy is the norm.

In section 11.1 we cut a cross-section and found the internal bending moment as a function of x, M = f(x). We integrated this twice to determine the equation of the elastic curve. For statically indeterminate problems, we will write our moment equation as a function of both x and A, where A is the unknown force at the redundant support. After integrating this equation twice we will have three unknowns, but also three boundary conditions. We will therefore be able to solve for the redundant reaction force and then use equilibrium equations to solve for the other reactions. See Example 11.6.

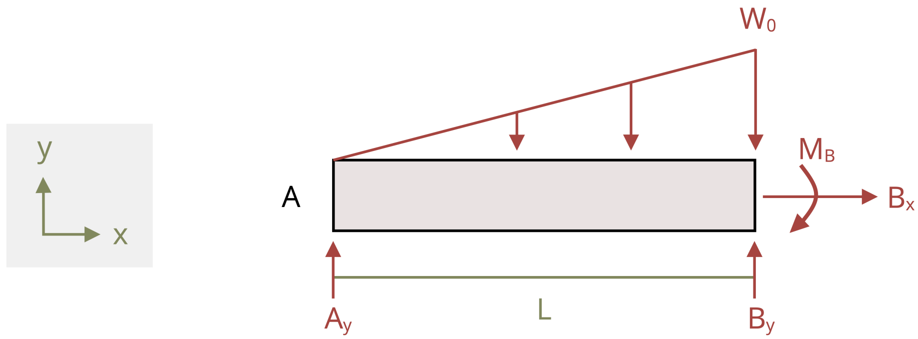
Example 11.6

A propped cantilever beam of length L = 6 ft is subjected to a linear distributed load where w_0 = 20 kips/ft. Determine the reactions at supports A and B.

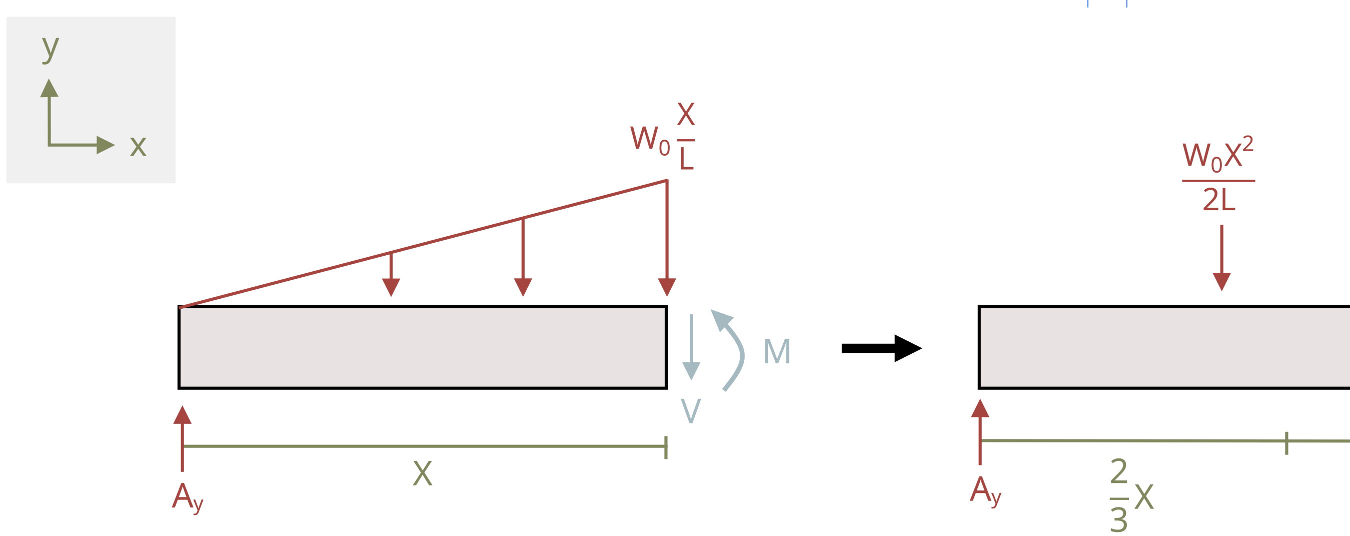


Solution

A free body diagram of the beam shows that we have 4 unknowns and only 3 equilibrium equations. We can solve $\sum F_x = 0$ to find $B_x = 0$, but that still leaves us with 3 unknowns and 2 equations.



Cut a cross-section and determine the internal bending moment in terms of distance x and redundant force A_y. Set this equation equal to $EI\frac{\partial^2 y}{\partial x^2}$ and integrate twice:



$$\sum M_x = 0: \quad M + \frac{w_0 x^2}{2L}\left(\frac{x}{3}\right) - A_y x = 0$$
$$M = A_y x - \frac{w_0 x^3}{6L} = EI\frac{\partial^2 y}{\partial x^2}$$
$$\frac{A_y x^2}{2} - \frac{w_0 x^4}{24L} + c_1 = EI\frac{\partial y}{\partial x}$$
$$\frac{A_y x^3}{6} - \frac{w_0 x^5}{120L} + c_1 x + c_2 = EIy$$

We have 3 unknowns (A_y, c_1, c_2) and 3 boundary conditions (at the roller, deflection equals zero. At the fixed support, both slope and deflection equal zero). Apply the boundary conditions:

$$\text{At } x = 0, y = 0 \quad \rightarrow \quad c_2 = 0$$
$$\text{At } x = L, y = 0 \quad \rightarrow \quad \frac{A_y L^3}{6} - \frac{w_0 L^4}{120} + c_1 L = 0$$

Answer only

$$A_y = 12 \text{ kips}$$
$$B_y = 48 \text{ kips}$$
$$M_B = 48 \text{ kip} \cdot \text{ft}$$

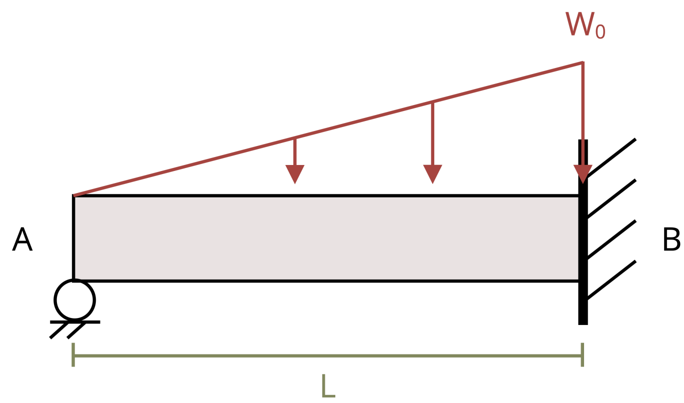
This method works well for beams with a single continuous load. While it can be used for more complex loads, it quickly becomes complicated and time consuming. For these problems it is generally better to use superposition.

The method is very similar to the methods used to solve other statically indeterminate problems in section 5.5 and section 6.4. To solve a statically indeterminate deflection problem using superposition, first identify the redundant support. We have 3 equilibrium equations and so can solve for three unknown support reactions. Any support reactions beyond this are redundant. Remove the redundant reaction and determine the deflection due to the applied loads at the point that the reaction was removed. It is best to remove a support such that the remaining beam is either simply supported or cantilever as these are the only support configuration in Appendix C.

Then replace the redundant reaction and determine the deflection due to this reaction and the point that the reaction is applied. Since there is actually a support here, the total deflection must be zero. Sum both deflections and set them equal to zero. This will allow you to calculate the value of the redundant support. Once this is known the other reaction loads can be determined using equilibrium. See Example 11.7 for a demonstration.

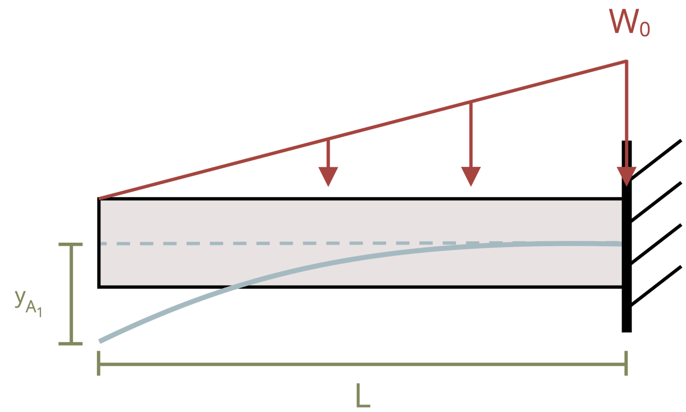
Example 11.7

The propped cantilever beam of Example 11.6 is shown again here. If length L = 6 ft is subjected to a linear distributed load where w_0 = 20 kips/ft. Determine the reactions at supports A and B.


W0
A
B
L

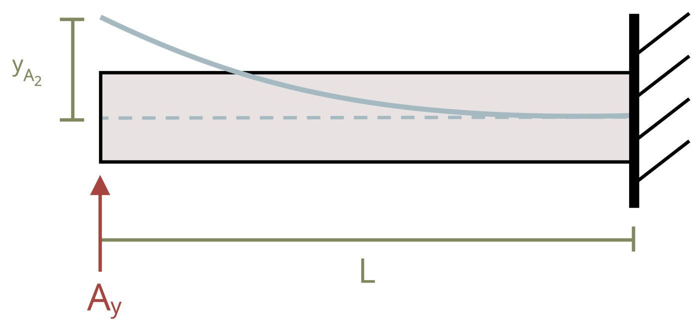
Solution

As demonstrated in Example 11.6, this problem is statically indeterminate.
This time, let's remove redundant support A_y and use Appendix C to find the deflection at point A.



$$y_{A_1} = -\frac{w_0 L^4}{30EI}$$

Now replace A_y and use Appendix C to find the deflection at point A caused by A_y.



$$y_{A_2} = \frac{A_{yL}{}^3}{3EI}$$

Since there is actually a support at A, the deflection at A must be zero.

$$-\frac{w_0 L^4}{30EI} + \frac{A_y L^3}{3EI} = 0 \quad \rightarrow \quad A_y = \frac{W_0 L}{10}$$

This is the same answer we found in Example 11.6, and the rest of the problem proceeds in the same way.

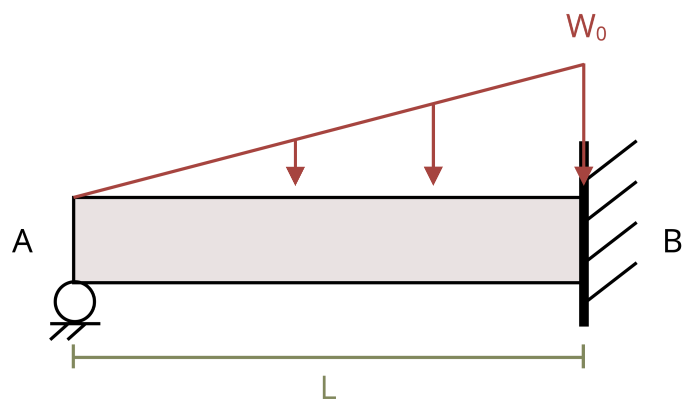
Answer only

$$A_y = 12 \text{ kips}$$
$$B_y = 48 \text{ kips}$$
$$M_B = 48 \text{ kip} \cdot \text{ft}$$

It is possible to remove a moment reaction in order to leave a simply supported beam. In this case, determining the deflection at the point that the load was removed isn't helpful as removing the moment reaction from a fixed support effectively leaves a pin support and the deflection with the moment removed will still be zero. In this situation we can instead look at the slope at the point that the reaction was removed. Example 11.7 is repeated below with the moment at B removed instead of the force at A.

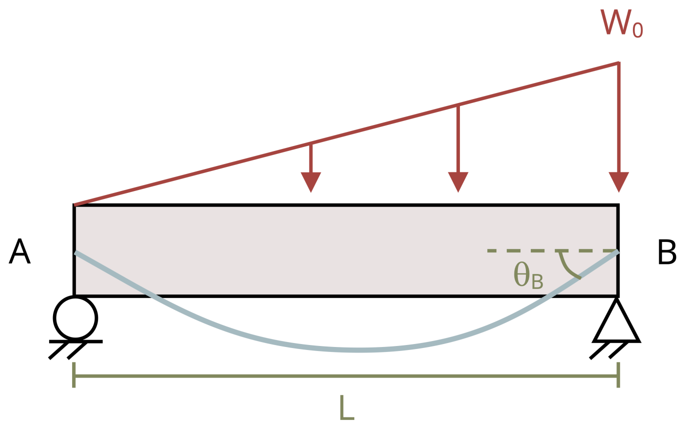
Example 11.7 (reworked)

The propped cantilever beam of Example 11.6 is shown again here. If length L = 6 ft is subjected to a linear distributed load where w_0 = 20 kips/ft. Determine the reactions at supports A and B.



Solution

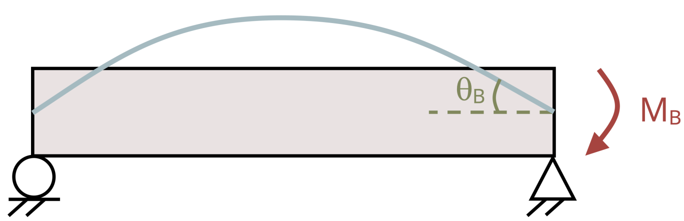
This time we'll remove the moment reaction at B to leave a simply supported beam.



Finding the deflection at B would not help as it would simply be zero. Instead, use Appendix C to find the slope at B.

$$\theta_B = \frac{W_0 L^3}{45EI}$$

Now replace the moment reaction at B and use Appendix C to find the slope at B due to this moment.



$$\theta_B = -\frac{M_B L}{3EI}$$

Since there is actually a fixed support at B, the slope here must be zero.

$$\frac{W_0 L^3}{45EI} - \frac{M_{BL}}{3EI} = 0$$

$$M_B = \frac{W_0 L^2}{15}$$

Adding numbers we get:

$$M_B = \frac{20 \times 6^2}{15} = 48 \text{ kip} \cdot \text{ft}$$

Which is the same answer we found in Example 11.6. Reactions A_y and B_y can be found by equilibrium and will also give the same answers as Example 11.6.

Answer only

$$A_y = 12 \text{ kips}$$
$$B_y = 48 \text{ kips}$$
$$M_B = 48 \text{ kip} \cdot \text{ft}$$

Note that some problems may involve multiple redundant reactions. In such cases, remove all redundant reactions and determine the deflection due to the external loads at every point that a reaction was removed. Then replace the redundant reactions one at a time and determine the deflection due to each load at every point that a reaction was removed. As before, the total deflection at each of these points must be zero. Sum the deflection at each point and set it equal to zero. There will be one equation for every redundant reaction and so all redundant reactions can be calculated.

Step-by-step: Statically Indeterminate Deflection

1. Identify and remove redundant reaction(s).
2. Determine the deflection at the point that the reaction was removed. Use superposition if possible, or integration otherwise.
    1. If the removed reaction was a couple, determine the slope at this point instead.
3. Replace the redundant reaction and determine the deflection caused by this load at the point that the load acts.
    1. If the removed reaction was a couple, determine the slope at this point instead.
4. These deflections (or slopes) must add up to the total deflection at this point. This is typically zero due to the support unless indicated otherwise. Use this to determine the magnitude of the redundant reaction.
5. Use equilibrium to solve for the other reactions. Then solve for anything else required as normal.

11.5 Intermediate Beam design

Click to expand

In Section 9.2 we learned how to use the section modulus to design a beam to meet bending stress specifications. In practice, beams must also meet specifications for shear stress and deflection. Limitations will be placed on all three of these criteria and we must design our beams to meet all three simultaneously. We'll begin with rectangular cross-sections and then move on to W beams.

As in section 9.2, problems involving rectangular cross-sections will typically relate the base and height of the cross-section such that we only need to determine one dimension. Previously we designed the cross-section to resist bending stress using:

$$S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}} = \frac{I}{c}$$

Since $I = \frac{bh^3}{12}$ and $c = \frac{h}{2}$ we were able to determine the required dimensions to meet the minimum required section modulus. The equation $S = \frac{I}{c}$ will simplify to $S = \frac{\frac{bh^3}{12}}{\frac{h}{2}} = \frac{2bh^3}{12h} = \frac{bh^2}{6}$ for a rectangle.

To meet the minimum required shear stress we will use $\tau = \frac{VQ}{It}$. Using the method of section 10.1, it can be shown for a rectangle (Figure 11.8) that $Q = yA = \frac{h}{4}\frac{bh}{2} = \frac{bh^2}{8}$. Since $I = \frac{bh^3}{12}$ and t = b, we can re-write the shear stress equation to find the maximum shear stress in a rectangular cross-section as $\tau_{\text{max}} = \frac{V\frac{bh^2}{8}}{\frac{bh^3}{12}b} = \frac{12Vbh^2}{8b^2h^3} = \frac{12V}{8bh} = \frac{3}{2}\frac{V}{A}$.

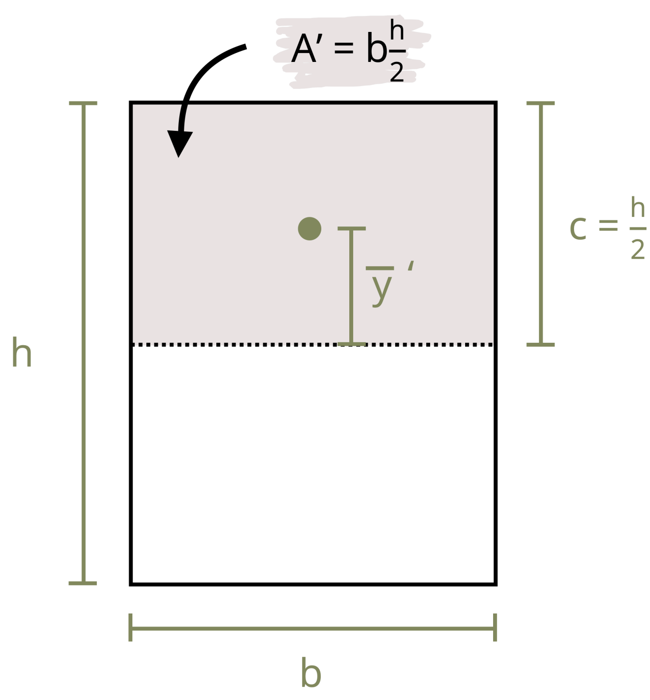


Figure 11.8: Rectangular cross-section showing the dimensions used for calculating the section modulus (S) and first moment of area (Q).

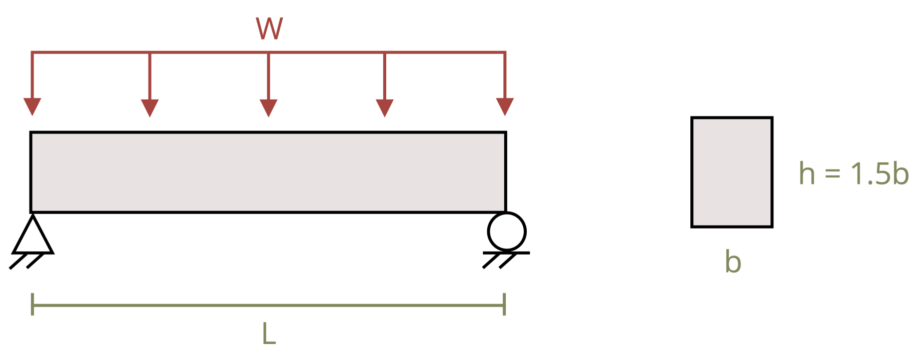
Thus we may set $\tau_{\max} = \frac{3}{2}\frac{V}{A}$ and solve for the required dimensions to not exceed the maximum allowable shear stress.

The maximum deflection of the beam under a given loading configuration can be found using the methods of this chapter. Wherever possible, use superposition and Appendix C. If it is not possible to use Appendix C then the method of integration can be used instead. In either case the equation for the maximum deflection of the beam will include the area moment of inertia, I. Set the maximum allowable deflection equal to this equation and replace $I = \frac{bh^3}{12}$. We can then solve for the required dimensions of the cross-section to not exceed the maximum allowable deflection of the beam.

We now have three different dimensions—one required to not exceed the maximum bending stress, one required to not exceed the maximum shear stress, and one required to not exceed the maximum deflection. Since we must not exceed any of these limitation, we select the largest of our three potential answers. See Example 11.8 for a demonstration.

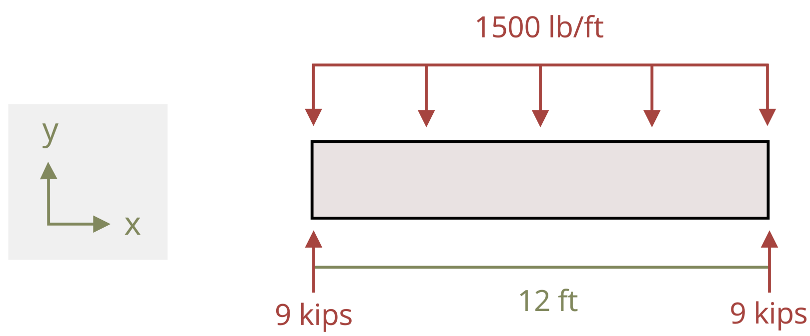
Example 11.8

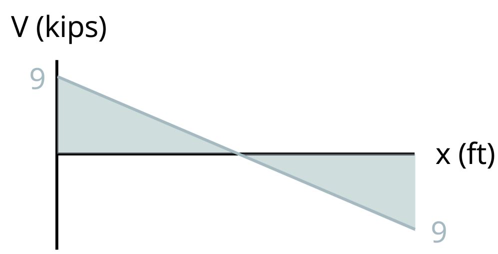
The simply supported wooden beam (E = 1700 ksi) of length L = 12 ft is subjected to a uniform distributed load of w = 1.5 kips/ft.
Assume the beam has an allowable bending stress of 900 psi, an allowable shear stress of 180 psi, and the deflection is limited to beam span / 240.
The beam has a rectangular cross-section where h = 1.5b. Determine the minimum required dimensions of the beam's cross-section to meet these specifications. Give your answer rounded up to the nearest inch to allow for realistic manufacturing tolerances.

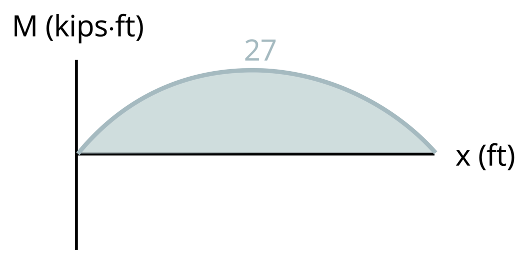


Solution

Start by drawing shear force and bending moment diagrams to determine the maximum internal loads:







$\mathrm{V_{max}} = 9$ kips
$\mathrm{M_{max}} = 27$ kip $\cdot$ ft
Design for bending by using the minimum required section modulus, $\mathrm{S_{min}}$:

$$S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}} = \frac{27000 \times 12}{900} = 360 \text{ in.}^3$$
$$360 = \frac{bh^2}{6} = \frac{b(1.5b)^2}{6} = \frac{2.25\ \mathrm{b}^3}{6}$$
$$b = 9.805 \text{ in.}$$

Design for shear:

$$\tau_{\text{max}} = \frac{3}{2}\frac{V}{A}$$
$$180 = \frac{3}{2} \times \frac{9000}{A}$$
$$A = \frac{3}{2} \times \frac{9000}{180} = bh = b(1.5b) = 1.5b^2$$
$$b = 7.071 \text{ in.}$$

Answer only

$$\begin{aligned} b_{\text{req}} &= 10 \text{ in.} \\ h_{req} &= 15 \text{ in.} \end{aligned}$$

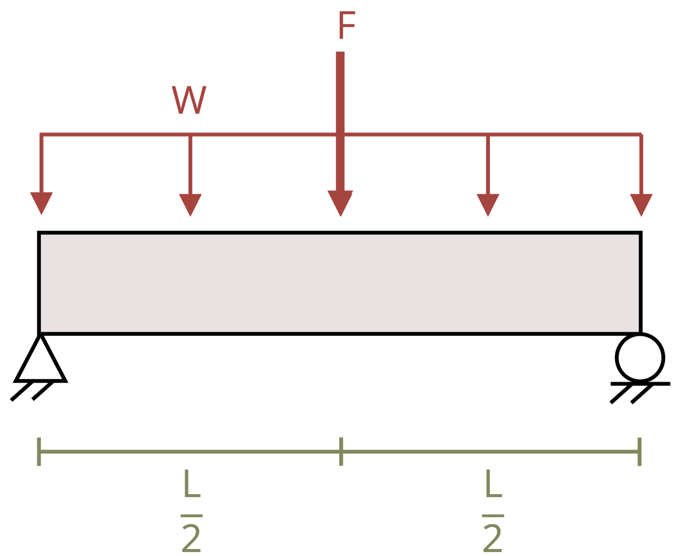
For W beams we can again use the method of section 9.2 to select a beam from Appendix A that meets the specifications for bending stress. Use $S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}}$ and select a beam with $\text{S} > \text{S}_{\text{min}}$.

Next determine the equation for the maximum deflection of the beam under the given loading conditions. As before, use superposition and Appendix C whenever possible. Set the maximum allowable deflection equal to this equation and solve for the minimum required area moment of inertia, I_{min}. Now select the lightest beam from Appendix A that meets both S_{min} and I_{min}.

Steel W beams rarely fail due to shear stress, but shear must still be checked. An exact value for the maximum shear stress in the beam can be found from $\tau_{\text{max}} = \frac{VQ}{It}$. However, most of the shear stress is resisted by the web of the beam. A reasonable estimate of the maximum shear stress can be found using $\tau_{\text{estimate}} = \frac{V}{A_{web}}$. This calculation is much simpler, but it is important to note that it is only an estimate. In most cases the maximum shear stress will be significantly less than the allowable shear stress and this estimate will suffice to demonstrate that. In the rare case that the maximum shear stress is within 20% of the allowable shear stress, an exact value must be determined using $\tau_{\text{max}} = \frac{VQ}{It}$. The beam was already selected to meet the bending stress and deflection requirements. As long as the maximum shear stress is less than the allowable shear stress, the beam can be used. See Example 11.9 for a demonstration.

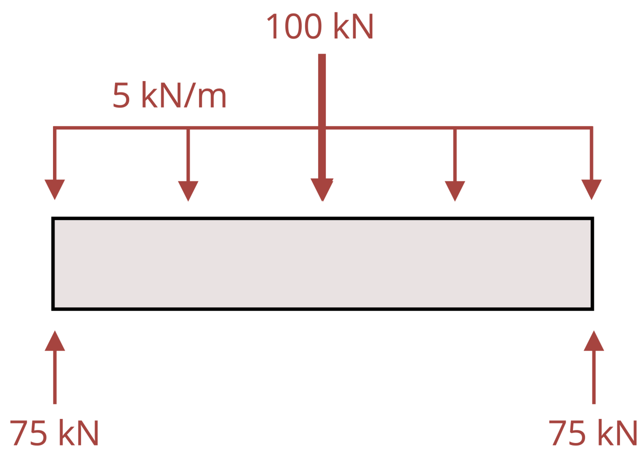
Example 11.9

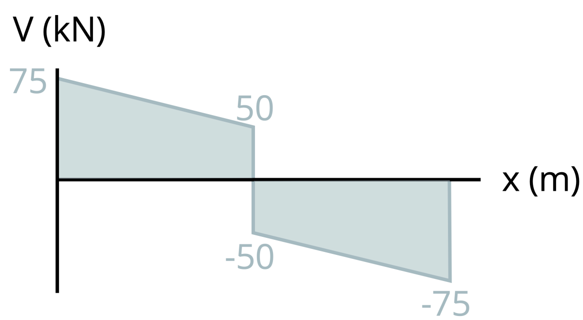
A steel (E = 210 GPa) W-beam of length L = 10 m will need to support the load shown, where w = 5 kN/m and F = 100 kN.
Assume the beam has an allowable bending stress of 150 MPa, an allowable shear stress of 100 MPa, and the maximum deflection is limited to 40 mm.
Select the lightest weight W-beam from Appendix A that meets these specifications.

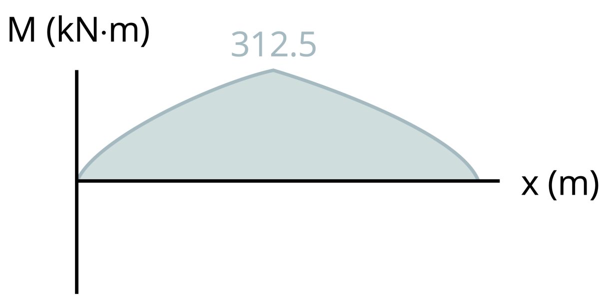

F
W
L
2
L
2

Solution

Load is symmetric, so each support will hold half the load. Draw shear force and bending moment diagrams to find the maximum internal loads.







$\mathrm{V_{max}} = 75\ \mathrm{kN}$
$\mathrm{M_{max}} = 312.5\ \mathrm{kN \cdot m}$
Design for bending by using the minimum required section modulus $\mathrm{S_{min}}$:

$$S_{\mathrm{min}} = \frac{M_{\mathrm{max}}}{\sigma_{\mathrm{Allow}}} = \frac{312.5 \times 10^3}{150 \times 10^6} = 0.002083\ \mathrm{m}^3$$
$$= 2083 \times 10^3\ \mathrm{mm}^3$$

Design for deflection:
We know the maximum deflection is limited to 40 mm (0.04 m) and the beam will deflect downward. From Appendix C we can see the maximum deflection for both

Answer only

W610 x 101

Step-by-step: Intermediate Beam Design

Rectangular cross-sections

1. Determine minimum required section modulus, $S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}} = \frac{bh^2}{6}$ and determine the minimum dimensions to meet the required S_{min}.
2. Set the maximum allowable shear stress, $\tau_{\text{max}} = \frac{3}{2}\frac{V}{A}$ and determine the minimum dimensions to not exceed τ_{max}.
3. Determine the maximum deflection for the applied load(s) and set this equal to the maximum allowable deflection. This equation will contain I, which depends on dimensions b and h. Determine the minimum dimensions to not exceed the maximum allowable deflection.
4. Use the largest of the three calculated dimensions as these will be the smallest dimensions that meet all three requirements.

W-beams

1. Determine minimum required section modulus, $S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{allow}}}$.
2. Determine the maximum deflection for the applied load(s) and set this equal to the maximum allowable deflection. Determine the minimum required area moment of inertia, I_{min}, to not exceed the maximum allowable deflection.
3. Select the lightest beam from Appendix A that meets both the S_{min} and I_{min} requirements.
4. Estimate the maximum shear stress in the selected beam using $\tau = \frac{V}{A_{web}}$. If this is within ~20% of the maximum allowable shear stress, calculate the maximum shear stress using $\tau_{\text{max}} = \frac{VQ}{It}$.
5. If the maximum shear stress is less than the maximum allowable value, use this beam. If it's greater than the maximum allowable value, return to step 3 and select the next lightest beam that meets both the S_{min} and I_{min} requirements.

Section wrap-up

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Key takeaways

- Beams under load will experience a type of deformation known as deflection. It is typically required to limit deflection to some small amount.
- Deflection can be determined using either the method of integration or the method of superposition.
- Both methods can be used for both statically determinate and statically indeterminate beams, though superposition is recommended whenever possible as it is generally quicker.
- Engineers designing beams must consider the response of the beam to bending stress, shear stress, and deflection simultaneously.

Key equations

Deflection by integration:

$$w(x) = EI\frac{d^4y}{dx^4}$$
$$V(x) = EI\frac{d^3y}{dx^3} = \int w(x)dx + C1$$
$$M(x) = EI\frac{d^2y}{dx^2} = \int V(x)dx + C2$$
$$\theta(x) = EI\frac{dy}{dx} = \int M(x)dx + C3$$
$$y(x) = \int \theta(x)dx + C4$$

Boundary conditions:

- At any support the deflection is 0
- At a fixed support, both the slope and deflection are zero
- The shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.
- The bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.

Beam Design for rectangular cross-sections:

$$S = \frac{bh^2}{6}$$
$$\tau_{\max} = \frac{3}{2}\frac{V}{A}$$

Beam design for W beams:

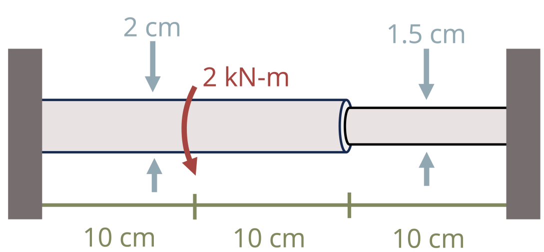
$$S_{\min} = \frac{M_{\max}}{\sigma_{\text{allow}}}$$
$$\tau_{\max} = \frac{VQ}{It}$$
$$\tau_{\text{estimate}} \; |= \frac{V}{A_{\text{web}}}$$

Chapter questions

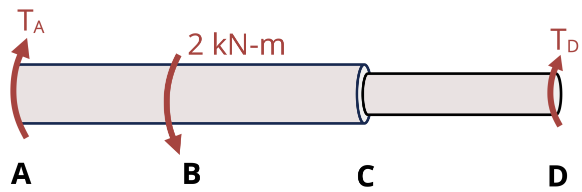
*Note: The following chapter questions do not correspond with the contents of chapter 11. They are included as a way of visually representing what a finalized chapter ***could*** look like.

Question 1

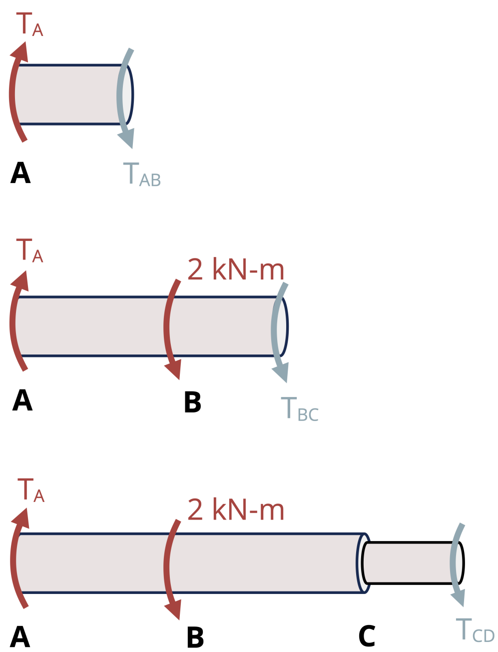
Two steel circular rods are firmly welded together and attached between two walls. A single moment of 2 kN-m is applied in the middle of the first rod. What is the moment reaction at the left wall? Assume $G_{\text{steel}} = 77$ GPa.
[Kurt Gramoll example problem below]



Solution



INTERNAL MOMENTS



Equilibrium
$T_A + T_D = 2$
Internal moments for each segment:
$T_{AB} = T_A$
$T_{BC} = T_A - 2$
$T_{CD} = T_A - 2$
Compatibility
The rotation angle at point C is the same but in an opposite direction for segments AC and CD. (Note that you could also use point B.)

$$\theta_{AC} = -\theta_{CD}$$

$$\frac{T_{AB} * L}{G * J_{AB}} + \frac{T_{BC} * L}{G * J_{BC}} = -\frac{T_{CD} * L}{G * J_{CD}}$$

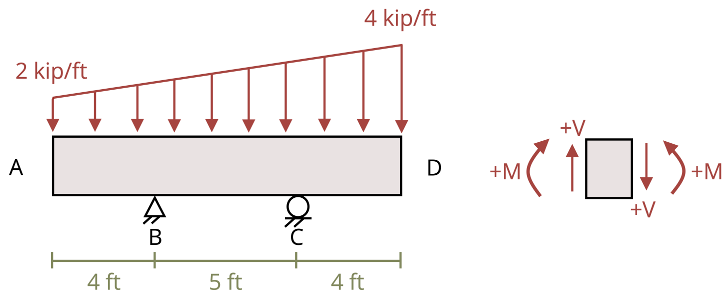
$$\frac{T_{AB}}{J_{AB}} + \frac{T_{BC}}{J_{BC}} = -\frac{T_{CD}}{J_{CD}}$$

Polar moment of inertia for each bar

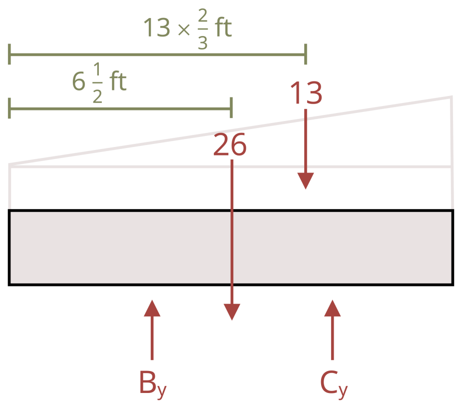
$$J_{AB} = J_{BC} = \frac{\pi r^4}{2} = \frac{\pi 1^4}{2} = 1.571 \text{ cm}^4$$

Question 2

What is the maximum moment for the loading shown?
[Kurt Gramoll example problem below]



Solution

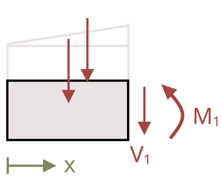


Full structure

$$\Sigma M_C = 0$$
$$26(9 - 6.5) - B_y(5) + 13(9 - (13 * 2/3)) = 0$$
$$B_y = 13.87 \text{ kip}$$
$$\Sigma F_y = 0$$

$$C_y = 26 + 13 - 13.87 = 25.13 \text{ kip}$$

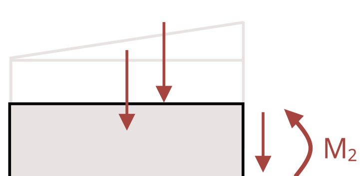
First beam section



$$V_1 = -2x - 0.5x\frac{2x}{13} = -2x - \frac{x^2}{13}$$

$$M_1 = -2x\frac{x}{2} - (0.5x\frac{2x}{13})\frac{x}{3} = -x^2 - \frac{x^3}{39}$$

Second beam section



References

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Learning Objectives

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12.1 Introduction

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13 Thin-Walled Pressure Vessels

Learning Objectives

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13.1 Introduction

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14 Combined Loads

Learning Objectives

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14.1 Introduction

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15 Columns

Learning Objectives

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15.1 Introduction

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A Geometric Properties of Standard Beams

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B Slope and Deflection Tables

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C Material Properties

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D Units and Order of Magnitude

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