Derivatives

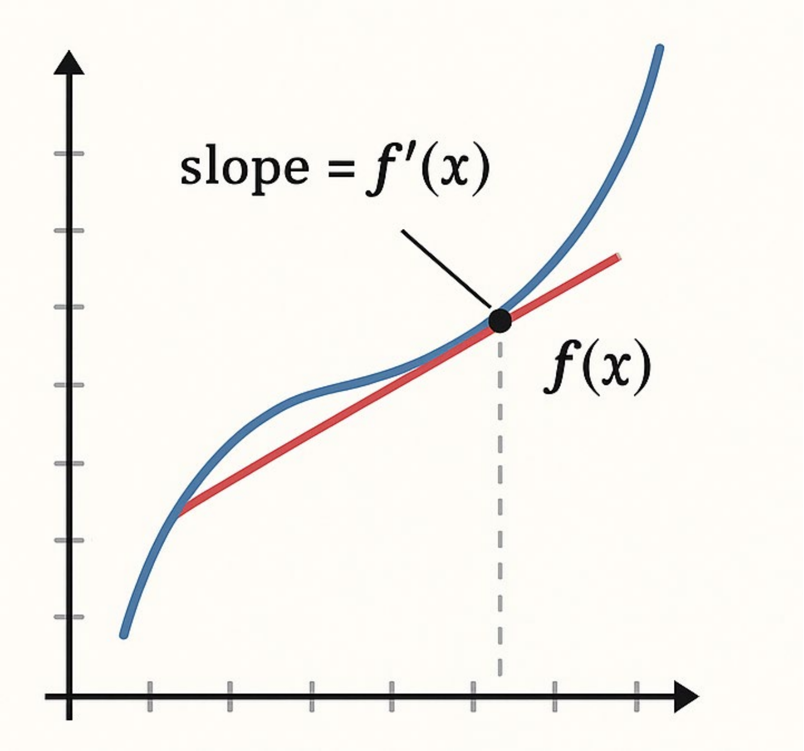
York College & University Centre

What are derivatives?

- **Derivatives measure change:** They represent how a function's output changes in response to a small change in its input—essentially the rate of change or slope at a specific point.

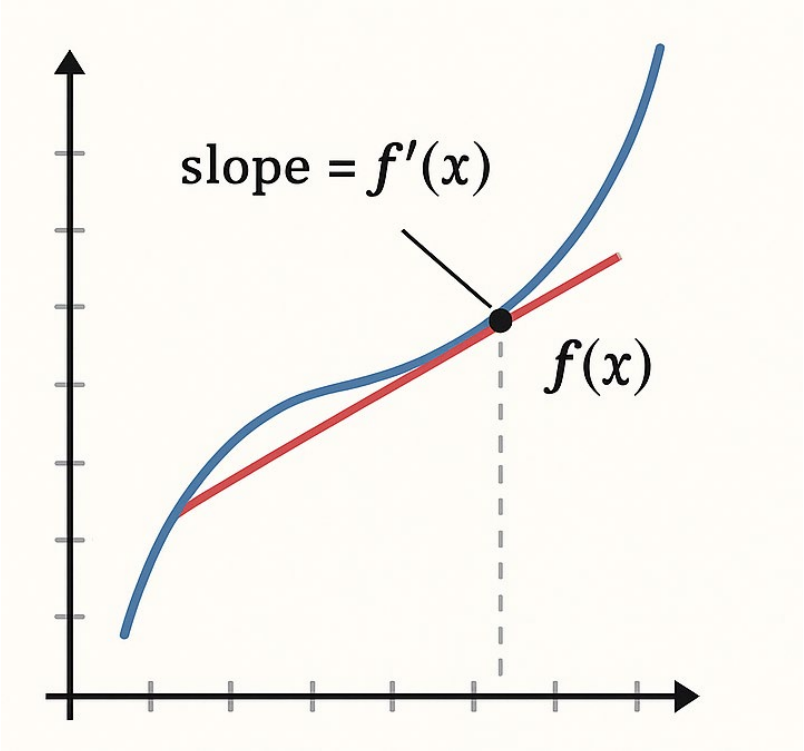
- **Used to find slopes and trends:** In graphs, a derivative tells you the slope of the tangent line to a curve at a point, indicating whether the function is increasing or decreasing.

- **Applied in many fields:** Derivatives are widely used in physics, engineering, and economics to model motion, optimize systems, and analyze change.



Applications of Derivatives

- **Physics:** Derivatives describe motion — velocity is the derivative of position, and acceleration is the derivative of velocity.
- **Engineering:** Used to analyse changing currents, voltages, or stresses in materials.
- **Economics:** Help determine marginal cost and revenue — how cost or profit changes with production level.



Derivative Symbol

- When we work out derivatives, we use special notation so we can read it back easily

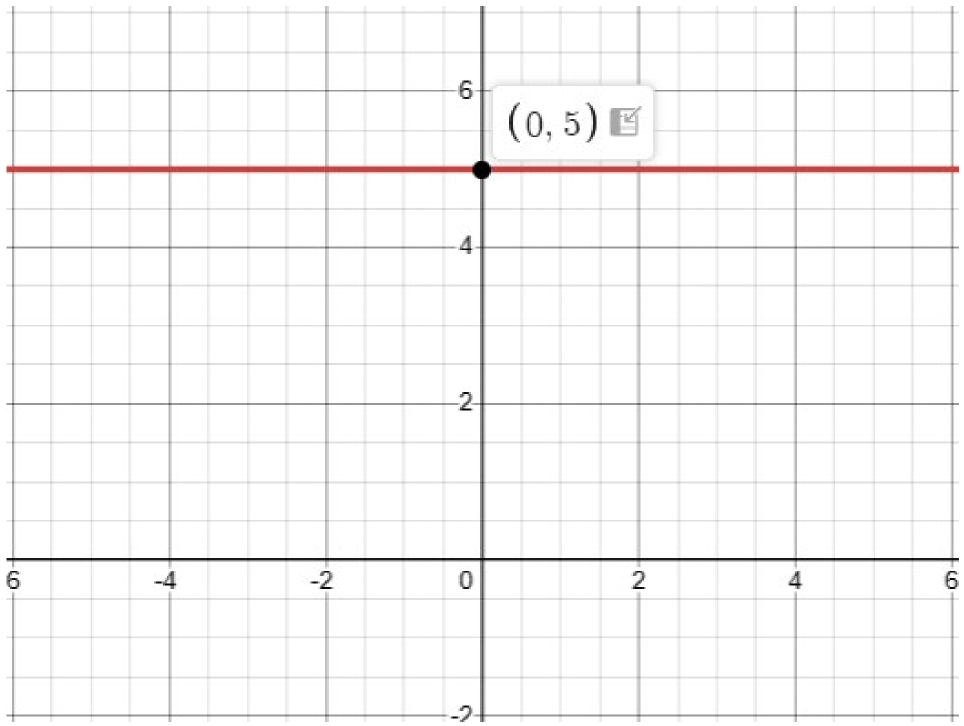
- The most common symbol is $\frac{dy}{dx}$ which means the derivative of y with respect to x (y is the output/function, x is the input/variable) this is the Leibniz notation

- We can also use the prime notation $f'(x)$ which is read as “f prime of x”

$$\frac{dy}{dx}$$

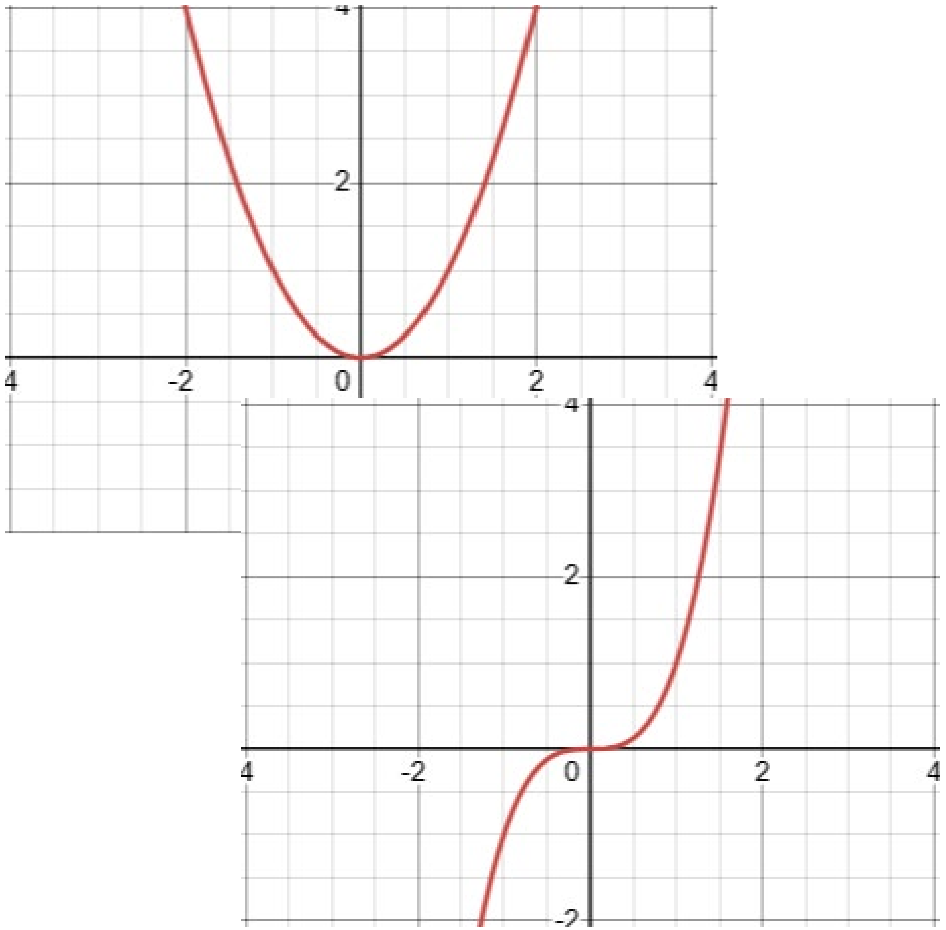
Derivatives of constants

- The derivative of any constant will always be 0
- This is because constants don't change
- $\frac{d}{dx}[c] = 0$
- $\frac{d}{dx}[5] = 0$
- $\frac{d}{dx}[-7] = 0$



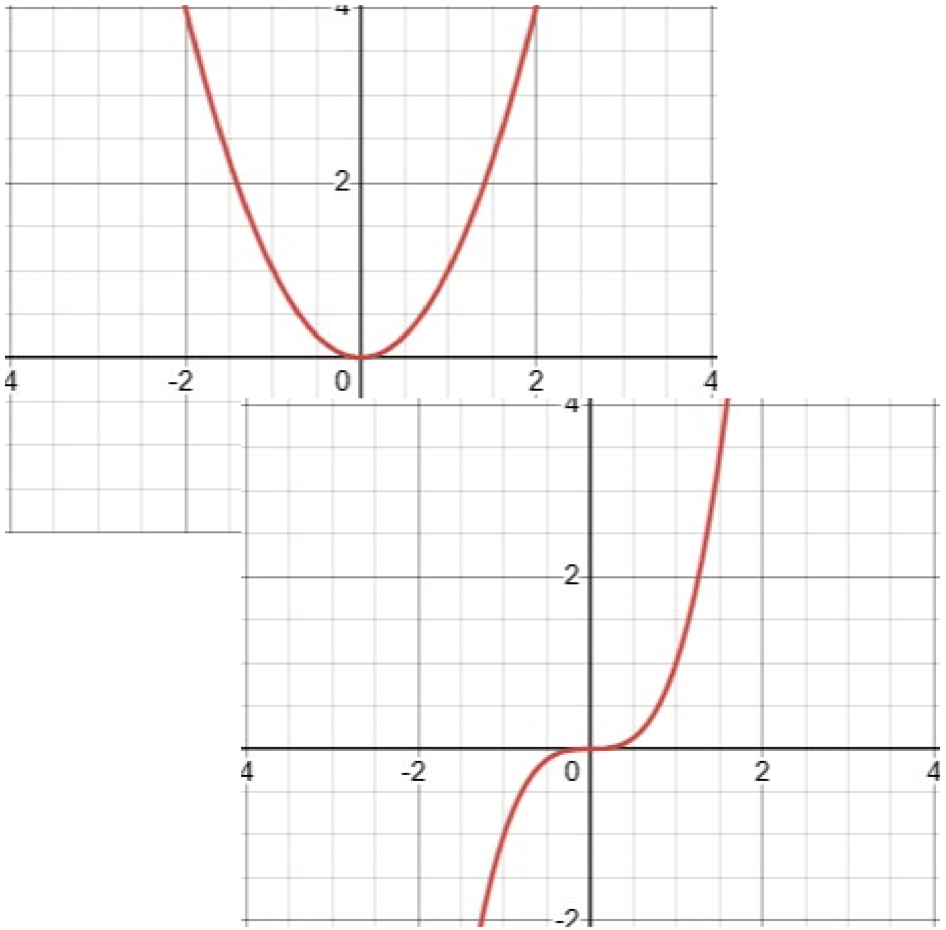
Derivatives of monomials

- A monomial is a polynomial which only has 1 term
- We use the power rule to work out monomials
- The power rule dictates:
- $\frac{d}{dx}(x^n) = nx^{n-1}$



Examples of derivatives of monomials

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x^1 = 2x$
- $\frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3$

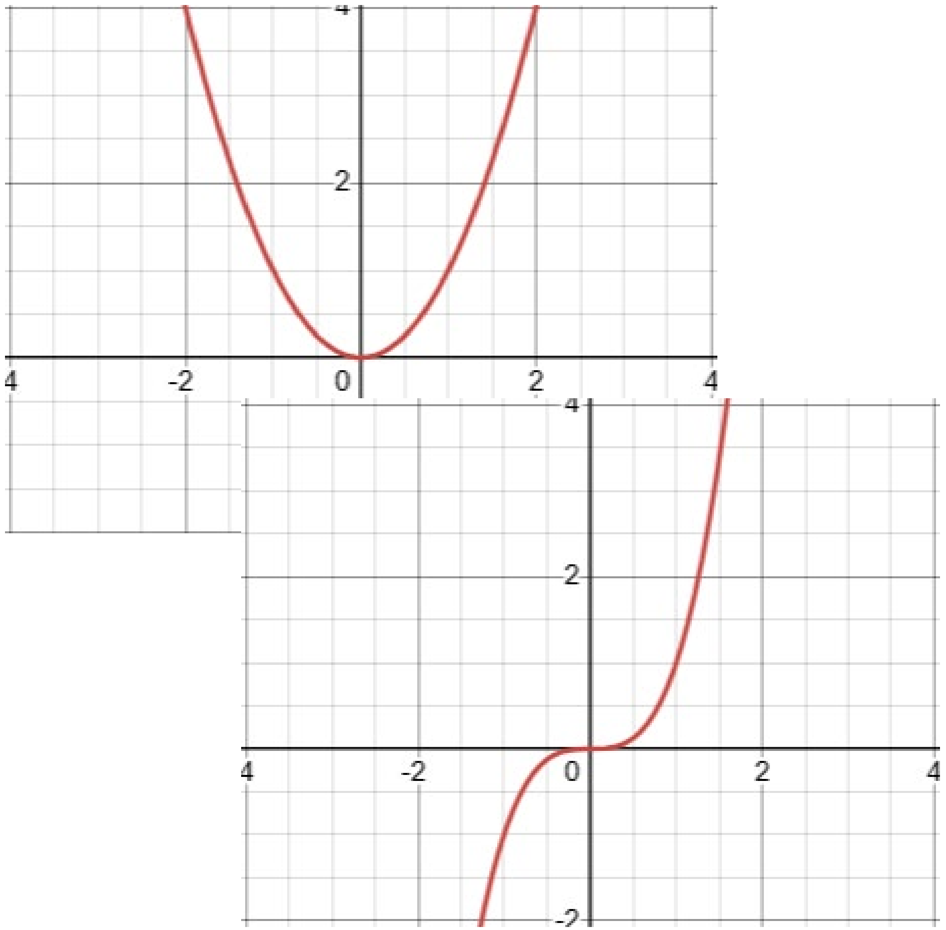


The constant multiple rule

- When we add constants in front of monomials we don’t change much about our equation

- $\frac{d}{dx}[c * f(x)] = c * \frac{d}{dx}[f(x)]$

- We can then use the power rule to work out $\frac{d}{dx}[f(x)]$

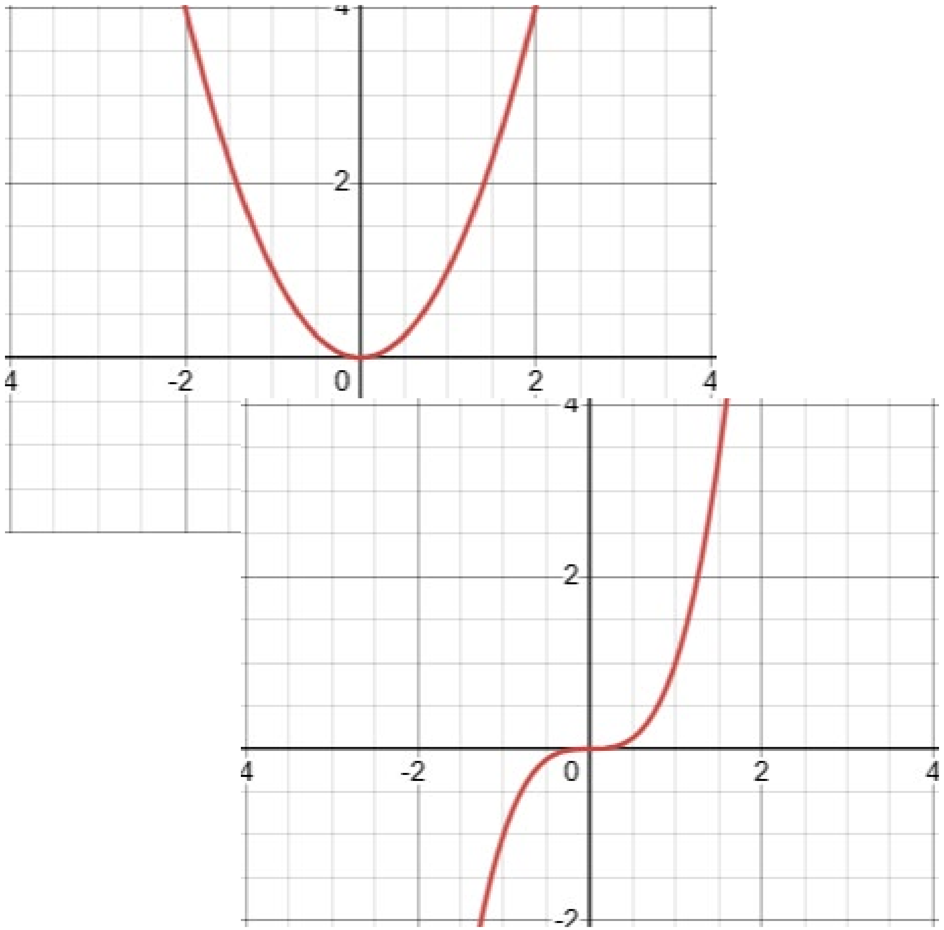


The constant multiple rule

- $\frac{d}{dx}[3x^6] = 3 * \frac{d}{dx}[x^6]$

- $\frac{d}{dx}[x^6] = 6x^{6-1} = 6x^5$

- $\frac{d}{dx}[3x^6] = 3 * 6x^5 = 18x^5$



Your Turn

- Can you work out the derivative of these equations

- $f(x) = x^4$

- $f(x) = x^{11}$

- $f(x) = 4x^5$

- $f(x) = 1.5x^3$

Your Turn - Answers

- Can you work out the derivative of these equations
- $f'(x) = 4x^3$
- $f'(x) = 11x^{10}$
- $f'(x) = 20x^4$
- $f'(x) = 4.5x^2$

Definition of a Derivative

- **We know that if $\boldsymbol{f(x) = x^2}$ then $\boldsymbol{f'(x) = 2x}$**

- This follows the mathematical definition of a derivative which follows this function:

- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

- We can prove his thus proving the definition

Definition of a Derivative

- **We know that if $\boldsymbol{f(x) = x^2}$ then $\boldsymbol{f'(x) = 2x}$**
- If we plug $f(x)$ into our equation $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ we get:
- $f'(x) = \lim_{h \to 0} \frac{(x+h)^2-x^2}{h}$
- Next, we use our polynomial multiplication to work out $(x+h)^2$
- $f'(x) = \lim_{h \to 0} \frac{(x+h)(x+h)-x^2}{h}$

Definition of a Derivative

- **We know that if $\boldsymbol{f(x) = x^2}$ then $\boldsymbol{f'(x) = 2x}$**

- $f'(x) = \lim\limits_{h \to 0} \frac{(x+h)(x+h)-x^2}{h} = \lim\limits_{h \to 0} \frac{\left(x^2+xh+xh+h^2\right)-x^2}{h} = \lim\limits_{h \to 0} \frac{2xh+h^2}{h}$

- Next we take out the greatest common factor

- $f'(x) = \lim\limits_{h \to 0} \frac{2xh+h^2}{h} = \lim\limits_{h \to 0} \frac{h(2x+h)}{h} = \lim\limits_{h \to 0} 2x + h$

Definition of a Derivative

- **We know that if $\boldsymbol{f(x) = x^2}$ then $\boldsymbol{f'(x) = 2x}$**
- $f'(x) = \lim_{h \to 0} 2x + h$
- As h approaches 0 we put in 0 as h
- So $f'x = 2x + 0 = 2x$
- Thus we have proven the derivative equation and thus defined it

Breather

That was some complicated maths lets take a second to let it sink in

Finding the slope at an x value

- Once we have worked out our derivative, we can then substitute in an x value to find the slope

- So, if we have $f(x) = x^3$ then we know $f'(x) = 3x^2$

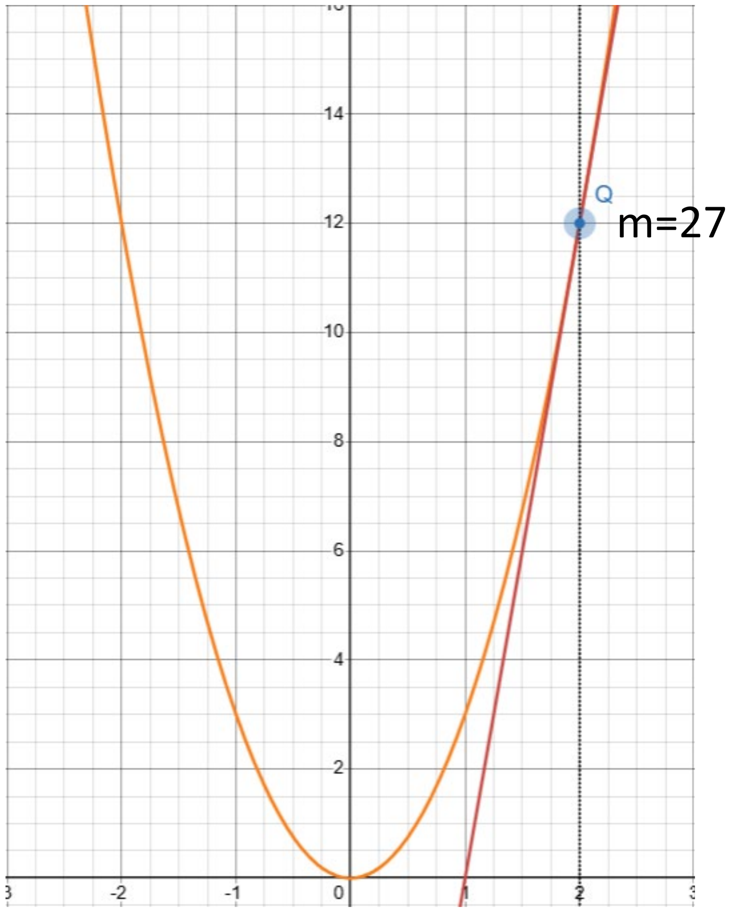
- And if we want to find the slope when x = 2 then we can substitute it in

- $f'(3) = 3(3^2) = 12$

- **So, the slope of the tangent line @ x =3 is equal to 27**

Finding the slope at an x value

- **So, the slope of the tangent line @ x =3 is equal to 27**
- So, if we draw out the graph, we can draw the tangent line with a gradient of 27
- For simplicity when drawing out the graph and tangent you can just sketch on both and just ensure values are given



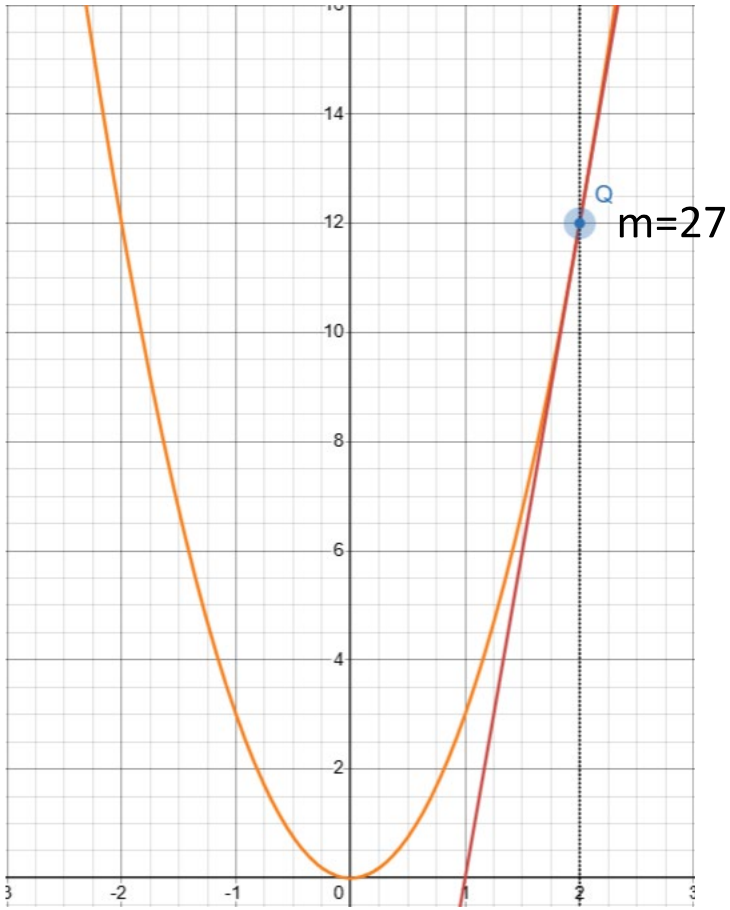
Proving the tangent gradient

- You may be asked to prove that the tangent has that gradient

- To do this we use another rule which is the equation of a straight line from 2 points

- $m = \frac{y_2 - y_1}{x_2 - x_1}$

- To use this rule, we plug in two points which average out to our x value



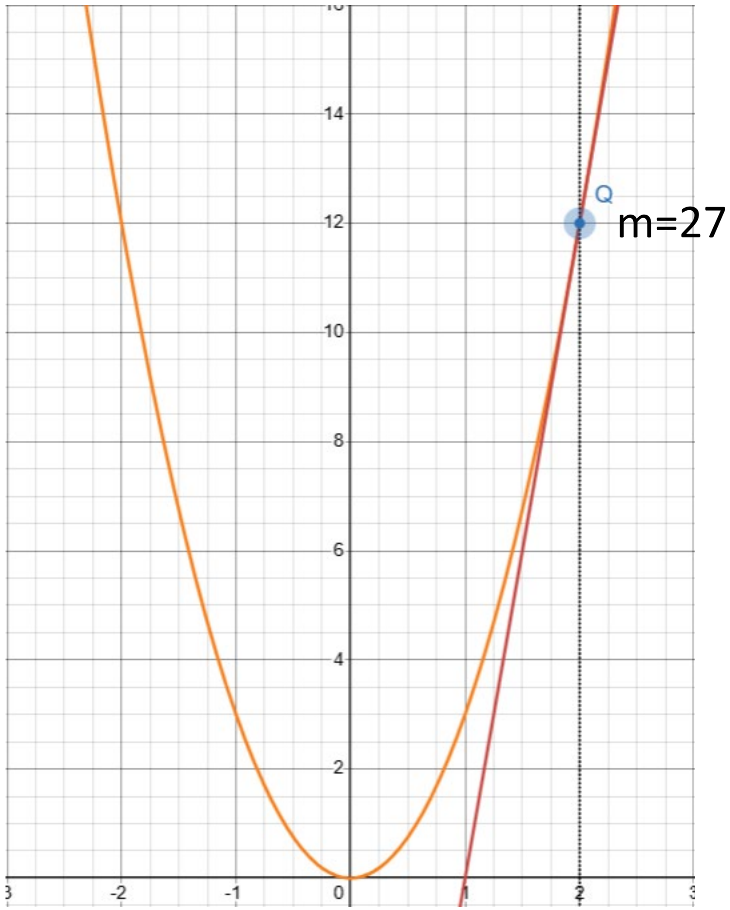
Proving the tangent gradient

- **So, the slope of the tangent line @ x =3 is equal to 27 when $\boldsymbol{f(x) = x^3}$**

- We could use the values x=4 and x=2 because they average to x=3

- But we want much closer values to be more accurate so we will use x=2.99 and x=3.01, we then use the $f(x)$ for y values

- So point
- $(x_1, y_1) = (2.99, 2.99^3)$
- and
- $(x_2, y_2) = (3.01, 3.01^3)$



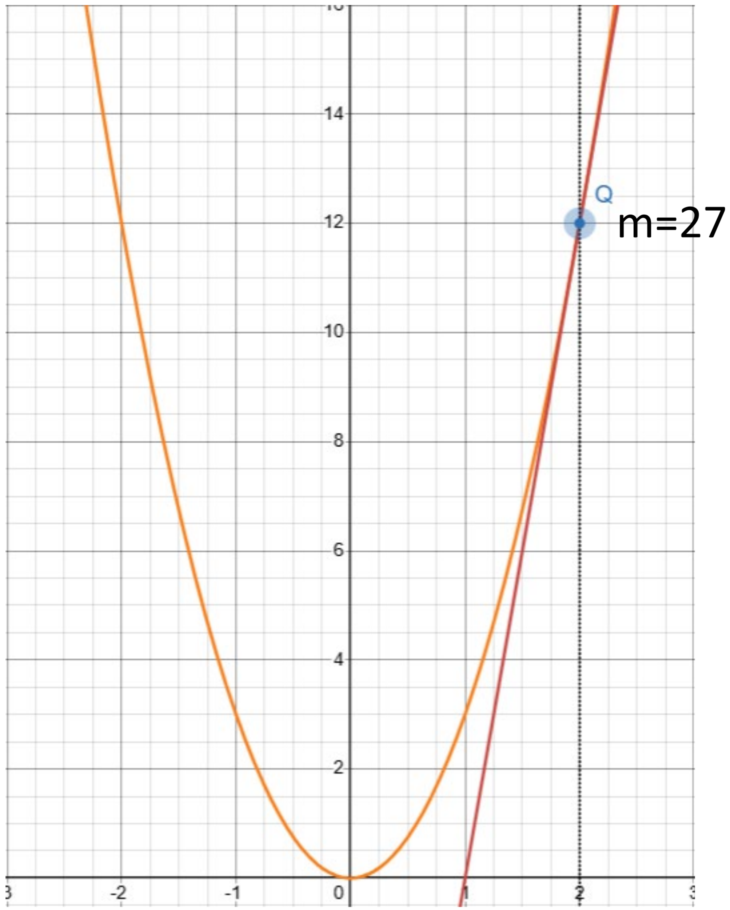
Proving the tangent gradient

- $(x_1, y_1) = (2.99, 2.99^3)$
- $(x_2, y_2) = (3.01, 3.01^3)$

- We then plug that into our equation:

- $m = \frac{3.01^3 - 2.99^3}{3.01 - 2.99} = 27.0001 \approx 27$

- Meaning our answer is correct



Example of harder derivative

- When doing larger derivatives don't panic, just break it down

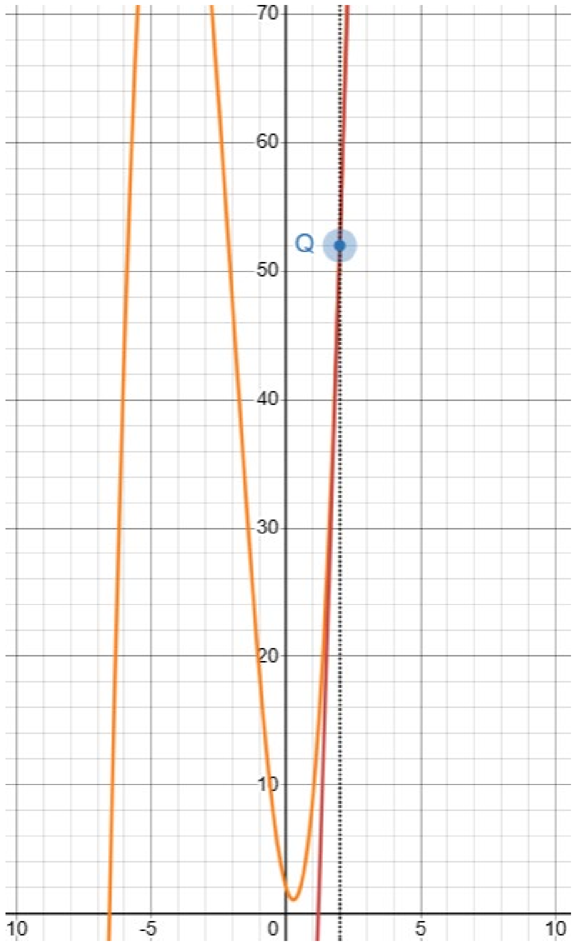
- $f(x) = 2x^3 + 12x^2 - 7x + 2$

- $f(x) = 2x^3 \rightarrow f'(x) = 3(2x^2) = 6x^2$
- $f(x) = 12x^2 \rightarrow f'(x) = 2(12x) = 24x$
- $f(x) = -7x \rightarrow f'(x) = 1(-7x^0) = -7$
- $f(x) = 2 \rightarrow f'(x) = 0$

- $f'(x) = 6x^2 + 24x - 7$

Example of harder derivative - gradient

- Now we have:
- $f'(x) = 6x^2 + 24x - 7$
- We want to find m when $x = 2$
- $f'(2) = 6(2)^2 + 24(2) - 7 = 65$
- $m = 65$

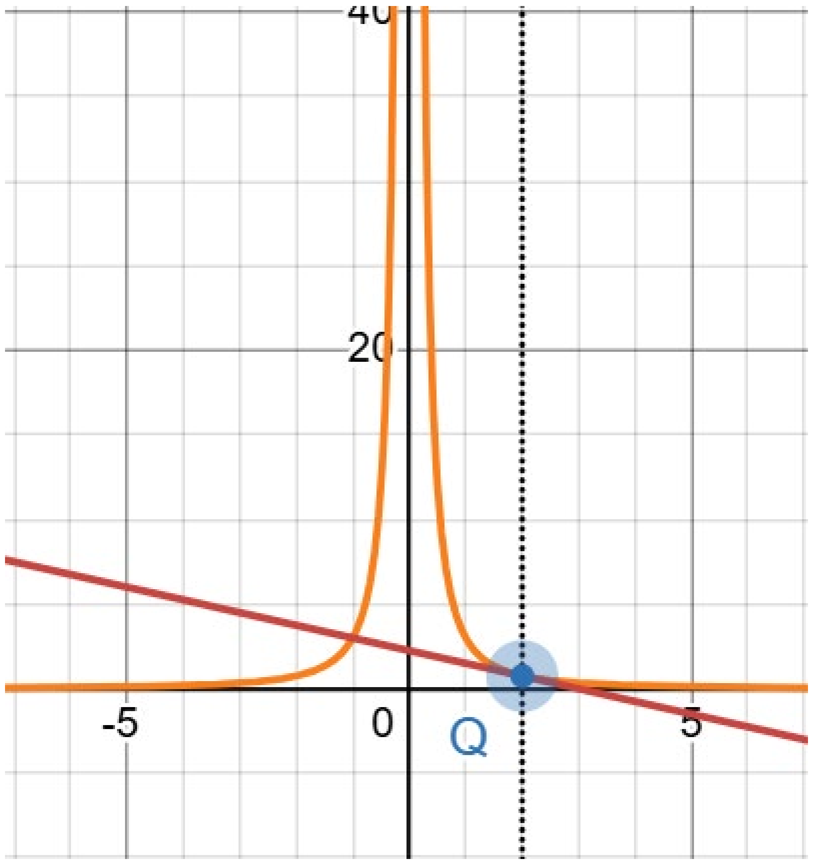


Example of harder derivative #2

- When doing division derivatives don’t panic, just use negative exponentials
- $f(x) = \frac{3}{x^2}$
- $f(x) = \frac{3}{x^2} = 3(x^{-2})$
- $f'(x) = 3(-2x^{-2-1}) = 3(-2x^{-3}) = -6x^{-3} = -\frac{6}{x^3}$

Example of harder derivative #2 - gradient

- Now we have:
- $f'(x) = -6x^{-3}$
- We want to find m when $x = 2$
- $f'(2) = -6(2^{-3}) = -0.75$
- $m = -0.75$

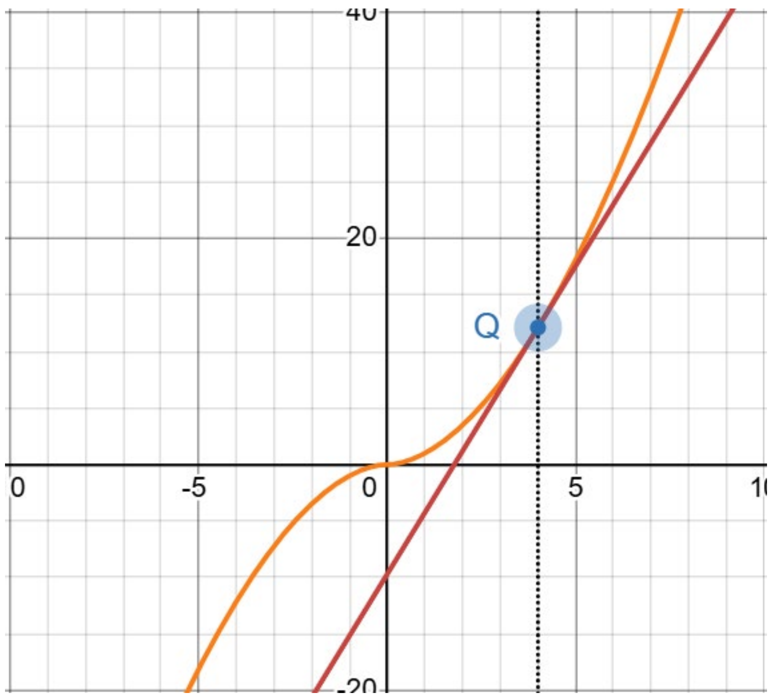


Example of harder derivative #3

- When doing division derivatives don't panic, just use negative exponentials
- $f(x) = \sqrt[5]{x^9}$
- $f(x) = x^{9/5}$
- $f'(x) = \frac{9}{5}\left(x^{\frac{9}{5}-1}\right) = \frac{9}{5}\left(x^{\frac{4}{5}}\right) = \frac{9x^{\frac{4}{5}}}{5} = \frac{9\sqrt[5]{x^4}}{5}$

Example of harder derivative #3 - gradient

- Now we have:
- $f'(x) = \frac{9\sqrt[5]{x^4}}{5}$
- We want to find m when $x = 4$
- $f'(4) = \frac{9\sqrt[5]{4^4}}{5} = 5.456579639$
- $m \approx 5.46$



Your Turn

- Can you find the derivative of these functions and their gradient when x = 4

- $f(x) = 7x^2 + 15x + 9$

- $f(x) = 19x^3 + 10x^2 + 9x + 27$

- $f(x) = \frac{8}{x^4}$

- $f(x) = \sqrt[3]{x^7}$

Your Turn - Results

- Can you find the derivative of these functions and their gradient when x = 4
- $f'(x) = 14x + 15$
- $f'(x) = 57x^2 + 20x + 9$
- $f'(x) = \frac{-32}{x^5}$
- $f'(x) = \frac{7\sqrt[3]{x^4}}{3}$

Derivative of Trigonometric Functions

- The trigonometric functions follow simple rules:

- $\frac{d}{dx}[\sin(x)] = \cos(x)$
- $\frac{d}{dx}[cos(x)] = -\sin(x)$
- $\frac{d}{dx}[\sec(x)] = \sec(x) * \tan(x)$
- $\frac{d}{dx}[\csc(x)] = -csc(x) * \cot(x)$
- $\frac{d}{dx}[\tan(x)] = sec^2(x)$
- $\frac{d}{dx}[\cot(x)] = -csc^2(x)$

Product Rule

- We use the product rule when we are multiplying two functions together

- It follows the rule:

- $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$

Product Rule - Example

- $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$
- $f(x) = 3x + 2$
- $g(x) = 5x^2 + 2x + 1$
- $f'(x) = 3$
- $g'(x) = 10x + 2$

Product Rule - Example

- $\frac{\boldsymbol{d}}{\boldsymbol{dx}}[\boldsymbol{f(x)g(x)}] = \boldsymbol{f(x)g'(x)} + \boldsymbol{f'(x)g(x)}$

- $f'(x) = 3$

- $g'(x) = 10x + 2$

- $\frac{d}{dx}[(3x + 2\ )(5x^2 + 2x + 1\ )] = (3x + 2)(10x + 2)\ \ + 3(5x^2 + 2x + 1)$

Product Rule - Example

- $\frac{d}{dx}[(3x+2)(5x^2+2x+1)] = (3x+2)(10x+2) \ +3(5x^2+2x+1)$
- $\frac{d}{dx}[(3x+2)(5x^2+2x+1)] = (30x^2+20x+6x+4) \ +(15x^2+6x+3)$
- $\frac{d}{dx}[(3x+2)(5x^2+2x+1)] = (45x^2+32x+7)$

Product Rule - Example

- If we want to find m when x=3 we just plug it in
- $\frac{d}{dx}[(3x+2\ )(5x^2+2x+1\ )] = (45x^2+32x+7)$
- $\frac{d}{dx}[(3x+2\ )(5x^2+2x+1\ )] = (45(3)^2+32(3)+7) = 508$
- $m = 508$

Quotient Rule

- Used when dividing one function by another

- It follows the rule:

- $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Quotient Rule - Example

- $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
- $f(x) = 3x + 2$
- $g(x) = 5x^2 + 2x + 1$
- $f'(x) = 3$
- $g'(x) = 10x + 2$

Quotient Rule - Example

- $\frac{\boldsymbol{d}}{\boldsymbol{dx}}\left[\frac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}}\right] = \frac{\boldsymbol{g(x)f'(x) - f(x)g'(x)}}{\boldsymbol{(g(x))^2}}$

- $f'(x) = 3$

- $g'(x) = 10x + 2$

- $\frac{d}{dx}\left[\frac{3x+2}{5x^2+2x+1}\right] = \frac{3(5x^2+2x+1) - (3x+2)(10x+2)}{(5x^2+2x+1)^2}$

Quotient Rule - Example

- $\frac{d}{dx}\left[\frac{3x+2}{5x^2+2x+1}\right] = \frac{3(5x^2+2x+1) - (3x+2)(10x+2)}{(5x^2+2x+1)^2}$

- $\frac{(15x^2+6x+3) - (30x^2+26x+4)}{(5x^2+2x+1)(5x^2+2x+1)}$

- $\frac{-15x^2-20x-1}{25x^4+15x^3+5x^2+15x^3+4x^2+2x+5x^2+2x+1} = \frac{-15x^2-20x-1}{25x^4+30x^3+14x^2+4x+1}$