Advanced Mathematics

First order Differential Equations

1. Find the plane curve through the point $(0, 2)$ whose tangent at any of its points has a slope equal to three times the value of its ordinate value.

2. Parachutist's descent. Assume that the air resistance force acting against the motion of a falling parachutist is a function that depends solely on its velocity v; taking into account that a reasonable expression for that friction force is Kv^2, where K is a constant, find the differential equation that links altitude and time. (**Hint:** Newton's second law states that the total force acting on a body of mass m is equal to the product of its mass and its acceleration: $\overrightarrow{F}(t) = m\overrightarrow{x}''(t)$).

3. A toxic substance in some medium destroys a bacterial species at a rate that is proportional to the product of the number of bacteria and the toxin concentration. On the other hand, in the absence of toxin, bacteria would grow at a rate proportional to its population. We assume that the toxin increases at a constant rate and the production started at $t = 0$. If $y(t)$ represents the number of bacteria at time t:

   (a) Find the first order differential equation for $y(t)$.

   (b) Solve the resulting equation. What is the behaviour of the solution for $t \to \infty$?

4. **Variable separation.**

   (a) Integrate the following differential equations:
   I) $\frac{dy}{dx} = \frac{x}{y}$ II) $\frac{dy}{dx} = y - y^2$ III) $\frac{dy}{dx} = y^2 \sin(x)$
   IV) $(1 + e^t)\, x(t)\, x'(t) = e^t$, V) $y' = 2y^2 - 2y$.

   (b) Solve the following initial and boundary value problems:
   I) $\frac{dy}{dt} = 2y^2 - 2y,\ y(0) = 2$ II) $\frac{dy}{dt} = (1 - 2t)y^2,\ y(0) = -\frac{1}{6}$
   III) $y' = y^2 \sin x,\ y(0) = \frac{1}{2}$ IV) $y'x^3 \sin y = 2,\quad \lim_{x\to\infty} y(x) = \frac{\pi}{3}$.

5. **First order linear ODE's**

   (a) Solve the following equations:
   1) $\frac{dy}{dx} + 2y = x$ 2) $\frac{dy}{dx} + 2y = \cos x$ 3) $\frac{dy}{dx} - 3y = -2e^{-2x}$
   4) $\frac{dy}{dx} = \frac{x^3 - 2y}{x}$ 5) $x' + 5x = t^2$ 6) $tx' + \frac{t}{1+t^2} = x$

   (b) Solve the following initial value problems:
   I) $\frac{dy}{dt} = 2y,\ y(0) = 4$; II) $\frac{dy}{dt} - 3y = -2e^{-2t},\ y(0) = 5$; III) $\frac{dy}{dt} = 3y + \cos t,\ y(\pi) = 4$
   IV) $x' = (\tan t)\, x + \cos t\,,\quad x(0) = 1$ V) $x' + 2tx = t^3,\quad x(0) = 1$

6. The population of a city grows at a constant rate. If the number of individuals has doubled in 3 years, and in 5 years reached the value of 40.000, how many people lived in the city at the begining of this five year period?

7. **(*)** a) If f_1 and f_2 are two solutions of the equation $y' + p(x)\, y = 0$, prove that $c_1 f_1(x) + c_2 f_2(x)$ is also a solution of the same equation for every $c_1, c_2 \in \mathbb{R}$.
   b) If f is a solution of the differential equation $y' + p(x)y = g(x)$, prove that $f(x) + c_1 f_1(x) + c_2 f_2(x)$ is also a solution of the same equation for $c_1, c_2 \in \mathbb{R}$.