Advanced Mathematics

Convergence of sequences and series of functions

1. Consider the following sequences of functions on the corresponding interval: $f_n(x) = x^n$ for $x \in [0,1]$ and $f_n(x) = (\cos \pi x)^{2n}$ for $x \in \mathbb{R}$.

   a) Make a graphical representation of $f_1(x), f_2(x)$ and $f_3(x)$ for each case.

   b) Study pointwise and uniform convergence for each function sequence.

2. Study pointwise and uniform convergence in the interval $[0,1]$ of the function sequences:

$$f_n(x) = \frac{x}{1+nx} \quad \text{and} \quad g_n(x) = \frac{1}{1+nx}.$$

3. Study pointwise and uniform convergence of the following sequences:

   a) $f_n(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{1}{n} \\ \frac{-x}{n-1} + \frac{1}{n-1} & \text{for } \frac{1}{n} \le x \le 1 \end{cases}$

   b) $f_n(x) = \dfrac{1-x^n}{1+x^n}$ for $1 \le x < \infty$

   c) $f_n(x) = x - x^n$ for $x \in [0,1]$

   d) $f_n(x) = (1-x)^n$ for $0 \le x \le 1$.

4. a) Let $f_n(x) = xe^{-nx}$, $x \ge 0$. Show that this sequence converges uniformly over $[0,\infty)$.
   b) Let $f_n(x) = \dfrac{\sin nx}{1+nx}$, $x \ge 0$. Show that for every $a > 0$ the sequence is uniformly convergent on $[a,\infty)$, but not on $[0,\infty)$.
   c) Let $f_n(x) = \dfrac{nx}{1+nx}$, $x \ge 0$. Show that for all $a > 0$ the sequence is uniformly convergent on $[a,\infty)$, but not on $[0,a]$.

5. Show that the sequence $\frac{x^n}{1+x^n}$ does not converge uniformly on $[0,2]$.

6. Study pointwise and uniform convergence of the sequence $f_n(x) = n^2xe^{-nx^2}$ on the interval $[0,1]$.

7. Find $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{ne^x}{n+x} dx$.

8. Study pointwise and uniform convergence of the following series of functions:

   a) $\displaystyle\sum_{n=0}^{\infty} x^n$ for $x \in [0,1]$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^2}$, $x \in \mathbb{R}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{x^2+1}$, $x \in \mathbb{R}$.

9. Write down in terms of series the following integrals:

$$\int_1^a \frac{\sin t}{t} \mathrm{d}t \quad \text{and} \quad \int_1^a \frac{e^{-x^2}}{x} \mathrm{d}x$$