FINAL EXAM: ADVANCED MATHEMATICS

February 3 2016

1.- Study pointwise and uniform convergence of the sequence $f_n(x) = 3 + \dfrac{\sin(n\,x)}{n}$

Solution: $|\sin nx| \leq 1$ so,

$$\left|\frac{\sin(nx)}{n}\right| \leq \frac{1}{n}$$

Then

$$|f_n(x) - 3| \leq \frac{1}{n}$$

For any $\varepsilon > 0$ we may choose any $N_0 > \frac{1}{\varepsilon}$, and then for any $n > N_0$ we have

$$|f_n(x) - 3| < \varepsilon \quad \forall x \in \mathbb{R}$$

Then, we have pointwise and uniform convergence in the whole set of real numbers.

2.- Find the Fourier series of $f(t) = \cos(3t + \frac{\pi}{4})$, $t \in [-\pi, \pi]$.

Solution: By using the trigonometric identity:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

we have that:

$$\cos(3t + \pi/4) = \cos(3t)\underbrace{\cos(\pi/4)}_{1/\sqrt{2}} - \sin(3t)\underbrace{\sin(\pi/4)}_{1/\sqrt{2}}$$

Then, the Fourier expansion of $f(t)$ has only two terms:

$$f(t) = \frac{1}{\sqrt{2}}\cos(3t) - \frac{1}{\sqrt{2}}\sin(3t)\ .$$

3.- Given that $K_1e^{3t} + K_2te^{3t} + 2t + 1$ is a solution of the Ordinary Differential Equation

$$x'' + a_1x' + a_2x = At + B.$$

Find the values of a_1, a_2, A and B.

Solution: We check the given expression in the differential equation. We know that the terms with exponential functions must form the solution of the homogeneous and the rest, ie $2t + 1$ is a particular solution. If we replace $2t + 1$ in the differential equation we obtain

$$2a_1 + a_2(2t + 1) = At + B$$

and we obtain the relationships:

$$2a_2 = A$$
$$2a_1 + a_2 = B$$

We can find a_1 and a_2 assuming that e^{3t} and te^{3t} are solutions of the homogeneous equation:

$$9e^{3t} + 3a_1e^{3t} + a_2e^{3t} = 0 \Rightarrow 9 + 3a_1 + a_2 = 0$$

and then, replacing $x(t)$ by te^{3t}:

$$(9t+6)e^{3t} + a_1(1+3t)e^{3t} + a_2te^{3t} = 0 \Rightarrow (9+3a_1+a_2)t + (6+a_1) = 0$$

and we must have:

$$a_1 = -6$$
$$9 + 3a_1 + a_2 = 0 \Rightarrow a_2 = 9$$

So the differential equation is:

$$x'' - 6x' + 9x = 18t - 3$$

4.- a) Find the remainder of the division of 3^{2016} by 14.

b) Show that for every integer a such that $0 < a < 17$, the remainder of the division of a^{2016} by 17 is the same, independently of a.

Solution:

a) 3 and 14 are relatively primes, so we know that

$$3^{\varphi(14)} = 1 \mod 14$$

We compute the Euler's function of 14 as

$$\varphi(14) = \varphi(2)\varphi(7) = 1 \times 6 = 6$$

Then

$$3^6 = 1 \mod 14$$

Taking integer division between 2016 and 6, we obtain that the remainder is 0. Then:

$$3^{2016} = 3^{q\times 6} = \left(3^6\right)^q \equiv_{14} 1\,.$$

So, the remainder is 1.

b) 17 is prime, so $a < 17$ is relatively prime with 17. We have that $\varphi(17) = 16$ and 2016 is divisible by 16, so:

$$a^{16} = 1 \mod 17$$

and

$$a^{2016} = a^{q\times 16} \equiv_{17} 1$$

so, the remainder is always 1.

5.- a) Is it possible to find and isomorphism from $(\mathbb{Z}_8^*, \times)$ over $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$? Explain the answer.

b) Show if $p = x^5 + 2x^4 + 2x + 1$ has multiple roots in $\mathbb{Z}_7$.

Solution:

a) Both sets have 4 elements, since $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$. And

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}\ .$$

We have elements of the same order on both groups, 1 and $(0,0)$ have order 1, and the other elements have order 2. For example in $\mathbb{Z}_8^*$, $3*3 = 9 = 1$ modulo 8, $5*5 = 25 = 1$ modulo 8, etc. To find an isomorphism, first we must send the neutral element to the neutral element:

$$\varphi(1) = (0,0)$$

Then we must take care that:

$$\varphi(3) + \varphi(5) = \varphi(15 \equiv_8 7)$$

There is at least one way to satisfy the equation with the sum of two different elements. We can put:

$$\varphi(3) = (1,0)\ , \varphi(5) = (0,1)\, , \varphi(7) = (1,1)\ .$$

Now we can check:

$$\varphi(3) + \varphi(7) = \varphi(21 \equiv_8 5) \Rightarrow (1,0) + (1,1) = (0,1)$$

and

$$\varphi(5) + \varphi(7) = \varphi(35 \equiv_8 3) \Rightarrow (0,1) + (1,1) = (1,0)$$

b) We check the roots. 6 is the only root, so we can check if it is a double root by checking the derivative polinomial:

$$5x^4 + 8x^3 + 2 \equiv 5x^4 + x^3 + 2$$

6 is not a root of this polynomial, so there are no multiple roots.

6.- Consider the polynomials $p(x) = x^2 - x - 1$ and $q(x) = x^2 + x + 1$ in $\mathbb{Z}_3[x]$.

a) Show if $\mathbb{K}_1 = \mathbb{Z}_3[x]/\langle p\rangle$ or $\mathbb{K}_2 = \mathbb{Z}_3[x]/\langle q\rangle$ are fields

b) Find, if possible, $[x]^{-1}$ in $\mathbb{K}_1$ and in $\mathbb{K}_2$.

Solution:

p has no roots in $\mathbb{Z}_3$, is irreducible, then the quotient is a field. q has root 1, so it is not a field. The inverse of x in the field can be computed as:

$$x^2 - x - 1 \equiv 0 \Rightarrow x(x-1) \equiv 1$$

Then the inverse of x is $(x-1)$ (or $x+2$).

The inverse in the other field also exists since x is relatively prime with $q(x)$:

$$x(x+1) = -1 \Rightarrow x(-x-1) = 1$$

then $2x + 2$ is the inverse of x in the second field.

The review will be next wednesday 10th of February at 16:15 hs in room 13. Attendance is not required.

Remarks: You can use only paper and pen/pencil during the examination.

The duration of the exam is **3 hours.** Once the examination has started, you can not leave the room within 40 minutes.

LAPLACE TRANSFORM TABLE

The following functions of t have Laplace transforms given by the functions of s on the right:

1. If $f(t) = 1$, $\forall t$ then $\quad Lf(s) = \frac{1}{s}$
2. If $f(t) = \sin(\alpha t)$, then $\quad Lf(s) = \frac{\alpha}{s^2+\alpha^2}$
3. If $f(t) = \cos(\alpha t)$, then $\quad Lf(s) = \frac{s}{s^2+\alpha^2}$
4. If $f(t) = e^{-\alpha t}$, then $\quad Lf(s) = \frac{1}{s+\alpha}$
5. If $f(t) = \sinh(\alpha t)$, then $\quad Lf(s) = \frac{\alpha}{s^2-\alpha^2}$
6. If $f(t) = \cosh(\alpha t)$, then $\quad Lf(s) = \frac{s}{s^2-\alpha^2}$
7. If $f(t) = e^{-\alpha t}\sin(\beta t)$, then $\quad Lf(s) = \frac{\beta}{(s+\alpha)^2+\beta^2}$
8. If $f(t) = e^{-\alpha t}\cos(\beta t)$, then $\quad Lf(s) = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$
9. In general, given $f(t)$, then $\quad L[e^{-\alpha t} f(t)](s) = Lf(s+\alpha)$
10. If $f(t) = t^n$, then $\quad Lf(s) = \frac{\Gamma(n+1)}{s^{n+1}}$, ($\Gamma$ is the Euler's Gamma function).
11. If $f(t) = te^{-\alpha t}$, then $\quad Lf(s) = \frac{1}{(s+\alpha)^2}$
12. If $f(t) = t\,\sin(\alpha t)$, then $\quad Lf(s) = \frac{2\alpha s}{(s^2+\alpha^2)^2}$
13. If $f(t) = t\cos(\alpha t)$, then $\quad Lf(s) = \frac{s^2-\alpha^2}{(s^2+\alpha^2)^2}$
14. In general, given $f(t)$, then $\quad L[t^n f(t)](s) = (-1)^n \frac{\partial^n Lf(s)}{\partial s^n}$