科学计算作业 练习 4a

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1 习题

8. 对权函数 $\rho(x) = 1 + x^2 \cdots\cdots$

解 设 $\psi_n = x^n$，$\eta_n = \varphi_n/\|\varphi_n\|$，则

$$\begin{aligned}
&\psi_0 = 1, && \varphi_0 = 1, && \eta_0 = \frac{3}{8}, \\
&\psi_1 = x, && \varphi_1 = x, && \eta_1 = \frac{15}{16}x, \\
&\psi_2 = x^2, && \varphi_2 = x^2 - \frac{2}{5}, && \eta_2 = \frac{525}{136}\left(x^2 - \frac{2}{5}\right), \\
&\psi_3 = x^3, && \varphi_3 = x^3 - \frac{9}{14}x.
\end{aligned}$$

10. 证明对每一个 Chebyshev 多项式$\cdots\cdots$

证明

$$P = \int_{-1}^{1} \frac{T_n^2(x)}{\sqrt{1-x^2}}\mathrm{d}x = \int_{-1}^{1} \cos^2(n \arccos x)\mathrm{d}\arccos x$$

另 $x = \cos t$，$u = nt$，根据 $\cos^2 x$ 的周期性，有

$$P = \int_0^{\pi} \cos(nt)\mathrm{d}t = \int_0^{n\pi} \cos^2 u\mathrm{d}\frac{u}{n} = \int_0^{\pi} \cos^2 u\mathrm{d}u = \frac{\pi}{2}. \quad \blacksquare$$

11. 用 $T_3(x)$ 的零点做插值点$\cdots\cdots$

解 T_3 的零点为

$$x_1 = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad x_2 = \cos\frac{\pi}{2} = 0, \quad x_3 = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}.$$

在零点处的函数值分别为

$$f_1 = e^{\sqrt{3}/2}, \quad f_2 = 1, \quad f_3 = e^{-\sqrt{3}/2}.$$

则二次插值多项式为

$$
\begin{aligned}
P_2(x) &= f[x_1] + f[x_1,x_2](x-x_1) + f[x_1,x_2,x_3](x-x_1)(x-x_2) \\
&= e^{\sqrt{3}/2} + \frac{e^{\sqrt{3}/2}-1}{\sqrt{3}/2}(x-\frac{\sqrt{3}}{2}) + \frac{2}{3}\left(e^{\sqrt{3}/2} - 2 + e^{-\sqrt{3}/2}\right)(x-\frac{\sqrt{3}}{2})x \\
&= \frac{2}{3}\left(e^{\frac{\sqrt{3}}{2}} + e^{-\frac{\sqrt{3}}{2}} - 2\right)x^2 + \left(\frac{e^{\frac{\sqrt{3}}{2}} - e^{-\frac{\sqrt{3}}{2}}}{\sqrt{3}}\right)x + 1
\end{aligned}
$$

对于它的误差，满足

$$
\varepsilon \le \frac{1}{2^2(2+1)!}\|(e^x)^{(3)}\|_\infty = \frac{e}{24} \quad \blacksquare
$$

12. 设 $f(x) = x^2 + 3x + 2 \cdots\cdots$

解 另 $x = \frac{1}{2}(t+1)$，则

$$
f(x) = g(t) = \frac{1}{4}t^2 + 2t + \frac{15}{4}, \quad t \in [-1,1].
$$

设 $\widetilde{P}_n$ 是首项为 1 的 Legendre 多项式，g^* 为 g 的在 $S_2 = \mathrm{span}\{1,x\}$ 上的最佳平方逼近函数，则

$$
g - g^* = \frac{1}{4}\widetilde{P}_2 \quad \Rightarrow \quad g^* = g - \frac{1}{4}\widetilde{P}_2 = \frac{1}{4}t^2 + 2t + \frac{15}{4} - \frac{1}{4}(t^2 - \frac{1}{3}) = 2t + \frac{23}{6}
$$

又 $t = 2x - 1$，所以 f 的在 S_2 上的最佳平方逼近函数为

$$
f_2^* = 4x + \frac{11}{6}.
$$

易知，f 在 $S_3 = \mathrm{span}\{1,x,x^2\}$ 上的最佳平方逼近函数为

$$
f_3^* = f = x^2 + 3x + 2. \quad \blacksquare
$$