Matrix Analysis

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Contents

1 Eigenvalues, eigenvectors, and similarity

1.1 Introduction

1.

Proof. Let $S = \{x \in \mathbb{R}^n : x^T x = 1\}$, which is clearly a compact subset of $\mathbb{R}^n$. Consider the function $f : x \mapsto x^T Ax$. Since,

$$\|f(x+\delta) - f(x)\| = \|(x^T A)\delta + \delta^T (Ax) + \delta^T A\delta\| \leq K\|\delta\|$$

for every $x \in \mathbb{R}$ and some fixed K, f is continuous. Hence, by Weierstrass's theorem, f attains its maximum value at some point $x \in S$. Namely, (1.0.3) has a solution x. Therefore, there exists some $\lambda \in \mathbb{R}$ such that $2(Ax - \lambda x) = 0$, implying that every real symmetric matrix has at least one real eigenvalue. $\square$

2.

Proof. Let $S = \{x \in \mathbb{R}^n : x^T x = 1\}$ and m be the maximum value of $x \mapsto x^T Ax$ in S. Suppose λ is an eigenvalue of A and $u \neq 0$ is its associated eigenvector, then

$$Au = \lambda u \quad \Rightarrow \quad u^T Au = \lambda\|u\|^2 \quad \Rightarrow \quad (u/\|u\|)^T A(u/\|u\|) = \lambda \quad \Rightarrow \quad m \geq \lambda.$$

Meanwhile, by the previous discussion, m itself is a eigenvalue of A. Hence, it is the largest real eigenvalue of A. $\square$

1.2 The eigenvalue-eigenvector equation

1.

Proof. It follows from

$$(A^{-1} - \lambda^{-1}I)x = (A^{-1} - \lambda^{-1}A^{-1}A)x = \lambda^{-1}A^{-1}(\lambda I - A)x = 0.$$

$\square$

3.

Proof. Since $A \in M_n(\mathbb{R})$, $u, v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$Ax = \lambda x \quad \Rightarrow \quad Au + iAv = \lambda u + i\lambda v$$

implies $Au = \lambda u$ and $Av = \lambda v$. As $x \neq 0$, at least one of u and v is nonzero and therefore A has a real eigenvector associated with λ. It can happen that only one of u and v is an eigenvector of A, because if $x \in \mathbb{R}^n$, which may happen as we discussed above, the imaginary part of x is 0. Finally, if x is a real eigenvector of A, then the eigenvalue λ it associated with must be real. Otherwise, at least one entry of λx is not real as $x \neq 0$, contradicting with the fact that Ax is real. $\square$

5.

Proof. Let $p(t) = t^2 - t$. Since A is idempotent, $p(A) = A^2 - A = 0$. Hence, 0 is the only eigenvalue of $p(A)$. By Theorem 1.1.6, the only values the eigenvalues of A can be are the zeros of p, namely, 0 and 1.

Suppose A is nonsingular, then multiplying A^{-1} on the both sides of $A^2 = A$ yields $A = I$. $\square$

7.

Proof. Suppose $\lambda \in \sigma(A)$ and x is its associated eigenvector, then

$$\begin{aligned} & 0 = (A - \lambda I)x = x^*(A^* - \bar{\lambda} I) = x^*(A - \bar{\lambda} I) \\ \Rightarrow \quad & 0 = x^*(A - \bar{\lambda} I)x = x^* A x - \bar{\lambda} x^* x = (\lambda - \bar{\lambda})\|x\|^2. \end{aligned}$$

Hence, $\lambda = \bar{\lambda}$, implying all eigenvalues of A are real. □

9.

Solution. Solve the equation $\det(A - \lambda I) = 0$ and we get $\lambda = \pm i$. □

11.

Proof. If $\operatorname{rank}(A - \lambda I) < n - 1$, then $\operatorname{adj}(A - \lambda I) = 0$ by (0.8.2) and therefore we can always choose y to be the 0 and the other parts of the proposition clearly hold. Hence, in the following discussion, we assume that $\operatorname{rank}(A - \lambda I) = n - 1$.

Apply the full-rank factorization and we get $\operatorname{adj}(A - \lambda I) = \alpha x y^*$ for some nonzero $\alpha \in \mathbb{C}$ and $x, y \in \mathbb{C}^n$. Replacing x with αx and α with 1 proves the first part.

Suppose $\operatorname{adj}(A - \lambda I) = [\beta_1, \ldots, \beta_n]$, then

$$(A - \lambda I) \operatorname{adj}(A - \lambda I) \quad \Rightarrow \quad (A - \lambda I)\beta_k = 0 \quad (k = 1, 2, \ldots, n),$$

implying that β_k is an eigenvector of A associated with λ as long as it is nonzero. □

13.

Proof. If $\operatorname{rank} A < n - 1$, then x is always an eigenvector of $\operatorname{adj} A$ associated with 0 as $\operatorname{adj} A = 0$. Hence, we may assume that rank $= n - 1$. Then $\operatorname{adj} A = (\det A) A^{-1}$. By Exercise 1, x is an eigenvector of A^{-1} and therefore an eigenvector of $\operatorname{adj} A$. □