Real Analysis

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Contents

3 Lebesgue Measure

3.1 Introduction

1.

Proof. Since $\mathfrak{M}$ is an σ-algebra, $B \setminus A \in \mathfrak{M}$ as long as $A, B \in \mathfrak{M}$. Since $B \setminus A$ and A are disjoint, $mB = mA + m(B \setminus A) \geq mA$ since m is nonnegative. $\square$

2.

Proof. Let $A_0 = E_0$ and $E_k = A_k \setminus A_{k-1}$ for $k \geq 1$. Clear that E_i and E_j are disjoint for distinct i and j, $\bigcup A_n = \bigcup E_n$ and $A_i \subset E_i$ for every i. Hence,

$$m\left(\bigcup E_n\right) = m\left(\bigcup A_n\right) = \sum mA_n \leq \sum mE_n,$$

where the last inequality comes from Exercise 1. $\square$

3.

Proof. Suppose that $mA < \infty$. Then $mA = m(A \cup \varnothing) = mA + m\varnothing$, implying that $m\varnothing = 0$. $\square$

3.2 Outer Measure

5.

Proof. We show that $\{I_n\}$ must cover the entire $[0, 1]$ by contradiction. Assume that $x \notin I_k$ for $k = 1, 2, \ldots, n$. Then, as I_k are open and n is finite, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ and I_k are disjoint for every k. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some rational number in $(x - \varepsilon, x + \varepsilon)$, contradicting with the hypothesis that $\{I_k\}$ covers all rational numbers between 0 and 1. $\square$

6.

Proof. By the definition of the outer measure, for every $\varepsilon > 0$, there exists some collection $\{I_n\}$ of open intervals that covers A and $\sum l(I_n) \leq m^*A + \varepsilon$. Let $O = \bigcup I_n$. O is a countable union of open sets and therefore is also open. And by Proposition 2, $m^*O \leq \sum l(I_n)$. Thus, $m^*O \leq m^*A + \varepsilon$.

Let $\varepsilon_n = 1/n$ and for each n, by the previous discussion, we can always get an open set O_k such that $A \subset O_k$ and $m^*O \leq m^*A + \varepsilon_m$. Let G be the countable intersection of these open sets. Clear that G is a G_δ set covering A and $m^*A = m^*G$. $\square$

7.

Proof. If $m^*E = \infty$, it is trivial. Suppose that $m^*E \leq \infty$. For any $x \in \mathbb{R}$, collection $\{I_n\}$ of open intervals covers $E + x$ iff $\{I_n - x\}$ covers E. Since the length of intervals is translation invariant, this implies $m^*(E + x) = m^*E$. $\square$

8.

Proof. Clear that $m^*A \leq m^*(A \cup B)$. Meanwhile, $m^*(A \cup B) = m^*A + m^*B = m^*B$. Hence, $m^*(A \cup B) = m^*B$. $\square$

3.3 Measurable Sets and Lebesgue Measure

10.

Proof.

$$\begin{aligned} mE_1 + mE_2 &= mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2). \end{aligned}$$

□

11.

Proof. $E_n = (n, \infty)$. □

12. This is the countable version of Lemma 9.

Proof. It suffices to prove $m^*(A \cap \bigcup E_i) \geq \sum m^*(A \cap E_i)$. Since $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i$ for every n,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of n, we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i),$$

completing the proof. □

13.

Proof. First we suppose that $m^*E < \infty$. By Proposition 5, there exists some open set $O \supset E$ such that $m^*O \leq m^*E + \varepsilon$. If E is measurable, then by the definition,

$$m^*(O \setminus E) = m^*O - m^*E \leq \varepsilon.$$

Namely, (ii) holds. Meanwhile, $O \subset \mathbb{R}$ is a countable union of disjoint open intervals $\{I_n\}$. Since $mO = m^*O$ is bounded and $mO = \sum l(I_n)$, there exists some integer $N > 0$ such that $mO - \sum_{n=1}^{N} l(I_n) < \varepsilon$. Let $U = \bigcup_{n=1}^{N} I_n$.

$$\begin{aligned} m^*(U \triangle E) &= m^*((U \cup E) \setminus (U \cap E)) \\ &\leq m^*(O \setminus (U \cap E)) \\ &= m^*((O \setminus U) \cup (O \setminus E)) \\ &\leq m^*(O \setminus U) + m^*(O \setminus E) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, (ii) implies (vi). Now we show that (vi) implies (ii). If $m^*(U \triangle E) < \varepsilon$, then there exists some countable collection $\{J_n\}$ of open interval such that

$$\sum l(J_n) \leq m^*(U \triangle E) + \varepsilon < 2\varepsilon.$$

Let $J = \bigcup J_n$ and $O = U \cup J$. $m^*J < 2\varepsilon$. And O is open and covers E. Meanwhile,

$$m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(J \setminus E) < 3\varepsilon.$$

Hence, (ii) holds.

Now, let E be an arbitrary set and $E_n = E \cap (-n, n)$, which is a set with finite measure. Then by the previous discussion, there exists some open set $O_n \supset E_n$ with $m^*(O_n \setminus E_n) < \varepsilon/2^n$. Let $O = \bigcup O_n$, an open set covering E and

$$m^*(O \setminus E) \leq \sum m^*(O_n \setminus E_n) < 2\varepsilon.$$

Hence, (i) implies (ii). Now we suppose (ii) holds and let $\varepsilon_n = 1/n$, then there exists a sequence of open sets $< O_n >$ such that $m^*(O_n \setminus E) < 1/n$. Let $G = \bigcap O_n \in G_\delta$. $m^*(G \setminus E) \leq m^*(O_n \setminus E) \leq 1/n$. Since the left hand side is independent of n, $m^*(G \setminus E) = 0$. If (iv) holds, then by Lemma 6, $G \setminus E$ is measurable. Since $G \in G_\delta$ is also measurable, E is measurable. Hence, (iv) implies (i).

By the previous result, for any measurable E, there exists some closed set $F \subset E$ such that $\bar{F}$, which is open, contains $barE$ and $m^*(\bar{F} \setminus \bar{E}) < \varepsilon$. Hence, $m^*(E \setminus F) < \varepsilon$. We can proceed in a similar manner as we did in the last paragraph to prove that (iii) $\Rightarrow$ (v) $\Rightarrow$ (i), leading to the final conclusion. $\square$