Solutions to *Topology*

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2 Topological Spaces and Continuous Functions

13 Basis for a Topology

1.

Proof. Let $\mathcal{T}$ be the topology of X. Since $\mathcal{T}$ is a basis for itself and the hypothesis implies that A is a set in the topology generated by $\mathcal{T}$, $A \in \mathcal{T}$, i.e., A is open. $\square$

4.

Proof.

(a) Put $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$. Since $\varnothing$ and X are contained in all $\mathcal{T}_\alpha$, they are also contained in $\mathcal{T}$. Let $\{U_\beta\}_{\beta\in J}$ be an indexed family of elements of $\mathcal{T}$ and put $U = \bigcup_{\beta\in J} U_\beta$. For every β, since U_β is open with respect to each $\mathcal{T}_\alpha$, by definition, so is $\bigcup_{\beta\in J}$. Similarly, we can show that $\mathcal{T}$ is closed under finite intersection. Thus, $\mathcal{T}$ is a topology.

The union $\bigcup \mathcal{T}_\alpha$, however, may not be a topology. Take $X = \{a, b, c\}$ for example. $\mathcal{T}_a = \{\varnothing, a, X\}$ and $\mathcal{T}_b = \{\varnothing, b, X\}$ are two topologies, but their union is not.

(b) Let $\mathcal{T}$ be the intersection of all topologies containing all $\mathcal{T}_\alpha$. By (a), $\mathcal{T}$ is a topology and clear that it is the unique smallest one. Now, let $\mathcal{T}' = \bigcap T_\alpha$, which is again a topology and is contained in all T_α. It can be verified that $\mathcal{T}'$ is the unique largest one.

(c) $\{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$; $\{\varnothing, X, \{a\}\}$. $\square$

5.

Proof. Let $\mathcal{A}$ be a basis, $\mathcal{T}$ the topology generated by $\mathcal{A}$, $\{\mathcal{T}_\alpha\}$ the collection of all topologies containing $\mathcal{A}$ and $\mathcal{T}' = \bigcap \mathcal{T}_\alpha$. For every union U of elements of $\mathcal{A}$, since, for every α, $\mathcal{A} \subset \mathcal{T}_\alpha$ and $\mathcal{T}_\alpha$ is closed under arbitrary union, $U \in \mathcal{T}_\alpha$. Hence, $\mathcal{T} \subset \mathcal{T}'$. Consequently, $\mathcal{T}'$ is also the intersection of all topologies containing $\mathcal{T}$. Since $\mathcal{T}$ contains itself as a subset, $\mathcal{T}' \subset \mathcal{T}$. Thus, $\mathcal{T} = \mathcal{T}'$.

Consider the collection of all finite intersections of $\mathcal{A}$, which is a basis, and apply the previous result to complete the proof. $\square$

6.

Proof. Let $\mathcal{T}_l$ and $\mathcal{T}_K$ be the topology of $\mathbb{R}_l$ and $\mathbb{R}_k$ respectively. $B = (-1, 1) - K$ is a basis element of $\mathcal{T}_k$ and $0 \in B$. However, no half-open interval containing 0 is in B. Hence, $\mathcal{T}_l$ is no finer than $\mathcal{T}_K$. Conversely, $C = [1, 2)$ is a basis element of $\mathcal{T}_l$ and $1 \in C$, but as $1 \in K$, there is no basis element of $\mathcal{T}_K$ containing 1. Hence, $\mathcal{T}_K$ is no finer than $\mathcal{T}_l$. Thus, they are not comparable. $\square$

8.

Proof.

(a) First clear that $\mathcal{B} \subset \mathcal{T}$. For every $U \in \mathcal{T}$ and $x \in U$, since U is open, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subset U$. Hence, there exists some rational a and b such that $x-\delta < a < x < b < x+\delta$. Thus, by Lemma 13.2, $\mathcal{B}$ generates the standard topology on $\mathbb{R}$.

(b) Since $x \in [\lfloor x\rfloor, \lfloor x\rfloor + 1) \in \mathcal{C}$ for every $x \in \mathbb{R}$, the first condition for a basis is satisfied. Meanwhile, for every $B_1 = [a, b)$ and $B_2 = [c, d)$ in $\mathcal{C}$, if they are not disjoint, $[c, b) = B_1 \cap B_2$ is also in $\mathcal{C}$. Hence, the second condition is satisfied. Thus, $\mathcal{C}$ is a basis.

Since $[\sqrt{2}, 2)$ can not be represented by union of elements in $\mathcal{C}$, $\mathcal{C}$ does not generate the lower limit topology. $\square$

16 The Subspace Topology

1.

Proof. Denote the topologies inherited from X and Y by $\mathcal{T}$ and $\mathcal{T}'$ respectively. For every $E = H \in \mathcal{T}$, supposing that $E = H \cap A$ where H is open in X, then, since $E \subset A \subset Y$, $E = (Y \cap H) \cap A$. Namely, $E \in \mathcal{T}'$. For the converse, suppose that $F = K \cap A$ where K is open in Y, then, for some H open in X, $F = (H \cap Y) \cap A = H \cap A$. Namely, $F \in \mathcal{T}$. Thus, $\mathcal{T} = \mathcal{T}'$. $\square$

2.

Proof. Denote the corresponding subspace topologies by $\mathcal{S}$ and $\mathcal{S}'$ respectively. Clear that $\mathcal{S}'$ is finer than $\mathcal{S}$. The relation, however, may not be strict. As an example, put $Y = \{y\}$. Then both $\mathcal{S}$ and $\mathcal{S}'$ are $\{\varnothing, Y\}$. $\square$

4.

Proof. By Lemma 13.1, (U, V) is open in $X \times Y$ iff $U = \bigcup U_\alpha$ and $V = \bigcup V_\beta$ where all U_α and V_β are open in X and Y respectively. Hence, $\pi_1(U, V) = \bigcup U_\alpha$ and $\pi_2(U, V) = \bigcup V_\beta$ are also open. Thus, π_1 and π_2 are open maps. $\square$

6.

Proof. By Prob. 8(a), Sec. 13, $\{(a, b) : a < b,\ a, b \in \mathbb{Q}\}$ is a basis for $\mathbb{R}$. The result then follows immediately from Theorem 15.1. $\square$

7.

Proof. No. Let $X = \mathbb{Q}$ with the usual order and $Y = \{x : 0 \le x^2 \le 2\}$. Y is a proper subset of X and is convex in X but not an interval or a ray. $\square$

9.

Proof. $\mathcal{B}_d = \mathcal{P}(\mathbb{R}) \times \{(b, d) : b < d,\ b, d \in \mathbb{R}\}$ is a basis for $\mathbb{R}_d \times \mathbb{R}$ and by Example 2, Sec. 14, $\mathcal{B}_o = \{\{a\} \times (b, d) : a, b, d \in \mathbb{R},\ b < d\}$ is a basis for the dictionary order topology on $\mathbb{R} \times \mathbb{R}$. Clear that $\mathcal{B}_0 \subset \mathcal{B}_d$. Meanwhile, for every $E \in \mathcal{P}(\mathbb{R})$, $E = \bigcup_{x \in E} \{x\}$. Hence, $\mathcal{B}_d \subset \mathcal{B}_o$. Thus, these two topologies are the same.

The collection $\mathcal{B}$ of all products of open intervals is a basis for the standard topology on $\mathbb{R}^2$. Clear that $\mathcal{B} \subset \mathcal{B}_d$. Meanwhile, $\{0\} \times \mathbb{R}$ is open in $\mathbb{R}_d \times \mathbb{R}$ but not in the standard topological space. Thus, the previous two topologies are strictly finer than the standard topology. $\square$

10.

Proof. Denote these topologies by $\mathcal{T}_i$, $i = 1, 2, 3$, respectively. $[0,1] \times (1/2, 1] \in \mathcal{T}_1 \setminus \mathcal{T}_2$. Hence, $\mathcal{T}_2$ is no finer than $\mathcal{T}_1$. Meanwhile, since $\{1/2\} \times (1/2, 1) \in \mathcal{T}_2 \setminus \mathcal{T}_1$, $\mathcal{T}_1$ is no finer than $\mathcal{T}_2$. Thus, $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable.

Now we show that $\mathcal{T}_3$ is finer than both $\mathcal{T}_1$ and $\mathcal{T}_2$ and since $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable, this relation is strict. Let $\mathcal{B}_1$ be the collection of all products of open intervals in I and $\mathcal{B}_3$ the collection of all sets of form $\{a\} \times ((b, d) \cap [0, 1])$ where $a \in [0, 1]$. They are bases of $\mathcal{T}_1$ and $\mathcal{T}_3$. respectively. Since every element in $\mathcal{B}_1$ can be represented by an arbitrary union of elements in $\mathcal{B}_3$, $\mathcal{T}_3$ is finer than $\mathcal{T}_1$. Similarly, we assert that $\mathcal{T}_3$ is also finer than $\mathcal{T}_2$. □

17 Closed Sets and Limit Points

2.

Proof. Since A is a subset of $Y \subset X$, $X \setminus A = (Y \setminus A) \cup (X \setminus Y)$. Since A is closed in Y and Y in closed in X, this implies that $X \setminus A$ is open. Thus, A is closed in X. □

4.

Proof. Since A is closed in X, A^c is open in X. Hence, $U \setminus A = U \cap A^c$ is open. Similarly, $A \setminus U$ is closed. □

6.

Proof.

(a) For any $x \in X$, if the neighborhood U of x intersects A, then it intersects B since $A \subset B$. Thus, $\bar{A} \subset \bar{B}$.

(b) Since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$ and $\bar{A} \cup \bar{B}$ is closed set containing $A \cup B$, $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. For the reverse inclusion, suppose that $x \in \bar{A} \cup \bar{B}$. If $x \in \bar{A}$, then all its neighborhood intersects $A \cup B \supset A$. Hence, $x \in \overline{A \cup B}$. Similarly for the case $x \in \bar{B}$. Therefore, $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Thus, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(c) The previous argument, *mutatis mutandis*, yields the inclusion. Let $X = \mathbb{R}$ and $A_n = [0, 1/n]$. Then, $\overline{\bigcup A_n} = [0, 1]$ and $\bigcup \bar{A}_n = [0, 1)$, which do not coincide. □

8.

Proof.

(a) We show that the equality holds. Since $\overline{A \cap B}$ is the smallest closed set containing $A \cap B$ and clear that $\bar{A} \cap \bar{B}$ is closed set containing $A \cap B$, $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. For the reverse inclusion, suppose that $x \in \bar{A} \cap \bar{B}$, then every neighborhood of x intersects both A and B. Hence, $x \in \overline{A \cap B}$. Thus, $\overline{A \cap B} = \bar{A} \cap \bar{B}$.

(b) The previous argument, *mutatis mutandis*, shows that $\overline{\bigcap A_\alpha} \subset \bigcap \bar{A}_\alpha$. The reverse inclusion does not hold in general. For example, let $X = \mathbb{R}$ and $A_n = (0, 1/n)$. Then $\overline{\bigcap A_n} = \varnothing$ but $\bigcap \bar{A}_n = \{0\}$.

(c) We show that $\overline{A \setminus B} \supset \bar{A} \setminus \bar{B}$.

$$\bar{A} \setminus \bar{B} \subset \bar{A} \setminus B = \bar{A} \cap B^c \subset \bar{A} \cap \overline{(B^c)} = \overline{A \cap B^c} = \overline{A \setminus B}$$

where the second equality comes from part (a). The reverse inclusion does not hold in general. For example, let $X = \mathbb{R}$, $A = [0,1]$ and $B = (0,1)$. Then $\overline{A \setminus B} = \{0,1\}$ but $\bar{A} \setminus \bar{B} = \varnothing$. $\square$

10.

Proof. Let X be a simply ordered set and $\mathcal{T}$ the order topology for X. If X contains only one single point, then, vacuously, $\mathcal{T}$ is Hausdorff. Otherwise, let $a < b$ be two distinct point in X. If $(a,b) \neq \varnothing$, that is, there exists some c with $a < c < b$, then $[-\infty, c)$ and $(c, \infty]$ are two disjoint open sets containing a and b respectively. If $(a,b) = \varnothing$, then $[-\infty, b) \setminus (a,b)$ and $(a, \infty] \setminus (a,b)$ are two such sets. Thus, we conclude that $\mathcal{T}$ is Hausdorff. (The case where the $\pm\infty$ can not be attained is similar.) $\square$

12.

Proof. Let Y be a subspace of the Hausdorff space X. Let a, b be two distinct points of Y. Since X is Hausdorff, there are two disjoint sets U and V which contain a and b respectively and are open in X. Hence, $U \cap Y$ and $V \cap Y$ are two disjoint open sets in Y containing a and b respectively. Thus, Y is also Hausdorff. $\square$

14.

Proof. Let $\mathcal{T}$ be the finite complement topology on $\mathbb{R}$. We show that for the sequence (x_n) converges to every point of $\mathbb{R}$. Let x be an arbitrary point of $\mathbb{R}$ and U an neighborhood of x. Since U is open and nonempty, U^c is finite. Hence, U^c contains at most finitely many points in (x_n). Thus, $x_n \in U$ for all sufficiently large n. Namely, $x_n \to x$. $\square$

21.b

Solution. $A = (-\infty, 1) \cup (1,2] \cup \{3\} \cup ([4,5] \cap \mathbb{Q}) \cup (6,7) \cup (7,8]$. $\square$

18 Continuous Functions

1.

Proof. For every open set V in $\mathbb{R}$, put $U = f^{-1}(V)$. Since the collection of open intervals forms a basis for the topology on $\mathbb{R}$, it suffices to show that for each $x \in U$, there is an open interval I containing x such that $I \subset U$ to conclude that U is open. Put $y = f(x)$. Since V is open, there is a $\varepsilon > 0$ such that $J = (y - \varepsilon, y + \varepsilon) \subset V$. By the $\varepsilon - \delta$ condition, there is a $\delta > 0$ such that for all $x' \in (x - \delta, x + \delta) = I$, $f(x') \in J \subset V$. Namely, $I \subset U$. Thus, f is continuous in the sense of the open set definition. $\square$

3.

Proof.

(a) i is continuous iff every open set in X is open in X' iff $\mathcal{T} \subset \mathcal{T}'$, i.e., $\mathcal{T}'$ is finer than $\mathcal{T}$.

(b) By part (a), $\mathcal{T}$ is finer than $\mathcal{T}'$ and *vice versa*. Hence, $\mathcal{T} = \mathcal{T}'$. $\square$

5.

Proof. Clear that $f(x) = a + t(b - a)$ is a homeomorphism in both cases. $\square$

7.a

Proof. Let x be a point in $\mathbb{R}_l$ and put $y = f(x) \in \mathbb{R}$. For every open interval J containing y, since f is continuous from the right, there exists a $\delta > 0$ such that $f(I) \subset J$ where $I = [x, x + \delta)$. Since I is open in $\mathbb{R}_l$, this implies that f is continuous when considered as a function from $\mathbb{R}_l$ to $\mathbb{R}$. $\square$

9.

Proof.
(a) It follows from the pasting lemma and the fact that the finite union of closed sets is still closed.
(b) Put $A_0 = \{0\}$ and $A_n = (-\infty, -1/n] \cup [1/n, \infty)$. Clear that $\{A_n\}_{n=0}^{\infty}$ is a sequence of closed set whose union is $\mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(0) = 0$ and $f(x) = 1$ for all $x \neq 0$. Clear that f is not continuous but $f|_{A_n}$ are all continous.
(c) By Theorem 18.1.4, it suffices to show that f is continuous at every $x \in X$. For each x, let U be the neighborhood of x that only intersects finitely many A_α. Let $\{A_k\}_{k \in K}$ denote the collection of such A_α. Since U is open, each $U \cap A_k$ is closed in U. Hence, part (a), $f|_U$ is continuous and, therefore, f is continuous at $x \in U$. Thus, f is continuous on X. $\square$

11.

Proof. Let $y_0 \in Y$ be fixed, we show that $h(x) = F(x \times y_0)$ is continuous. Let $x \in X$ be fixed and let V be a neighborhood of $h(x) = F(x \times y_0)$. Since F is continuous, by Theorem 18.1.4, there is a basis neighborhood $A \times B$ of $x \times y_0$ such that $F(A \times B) \subset V$. Note that $h(A) = F(A \times \{y_0\}) \subset F(A \times B) \subset V$. Hence, h is continuous. Since the roles of X and Y are interchangeable, this implies that F is continuous in each variable separately. $\square$

13.

Proof. Let g_1 and g_2 be continuous extensions of f. Clear that $g_1 = g_2$ on A. For every limit point x of A, let V_1 and V_2 be neighborhoods of $g_1(x)$ and $g_2(x)$ respectively. Since both g_1 and g_2 are continuous, by Theorem 18.1.4, there exists neighborhoods U_1 and U_2 of x such that $g_1(U_1) \subset V_1$ and $g_2(U_2) \subset V_2$. Since x is a limit point of A. A intersects both U_1 and U_2. Hence, there exists some $x^* \in A$ such that $g_i(x^*) \in V_i$ for $i = 1, 2$. Since $g_1(x^*) = g_2(x^*)$, this implies that V_1 and V_2 intersect. As the choice of neighborhoods V_1 and V_2 are arbitrary, this, by the definition of a Hausdorff space, implies that $g_1(x) = g_2(x)$. Namely, if the extension exists, it is unique. $\square$

19 The Product Topology

1.

Proof. We show the result for the box topology $\mathcal{T}_B$. For every $U \in \mathcal{T}_B$ and $x = (x_\alpha) \in U$, let $\prod_{\alpha \in J} U_\alpha \subset U$ be a basis neighborhood of x, where each U_α is open in X_α. Again, let $B_\alpha \in \mathcal{B}_\alpha$ be a basis neighborhood of x_α which is contained by U_α. Then, $x \in \prod B_\alpha \subset U$. Thus, the collection described is a basis for the box topology. Similarly for the product topology. $\square$

5.

Proof. Since the box topology is finer than the product topology. Hence, if f is continuous with respect to the box topology, it is continuous with respect to the product topology and, therefore, each f_α is continuous by Theorem 19.6. $\square$

7.

Proof. We show that, in the box topology, $\mathbb{R}^\infty$ is closed and, as a result, its closure is $\mathbb{R}^\infty$ itself. Suppose that $(y_n) \notin \mathbb{R}^\infty$, that is, (y_n) has subsequence whose elements are all nonzero. Define (ε_n)

$$\varepsilon_n = \begin{cases} 1, & y_n = 0, \\ y_n/2, & y_n \neq 0. \end{cases}$$

Then $\prod(y_n - \varepsilon_n, y_n + \varepsilon_n)$ is a neighborhood of (y_n) in the box topology. And it does not intersect $\mathbb{R}^\infty$. Hence, for (y_n) to be a limit point of $\mathbb{R}^\infty$, it must belong to $\mathbb{R}^\infty$. Thus, $\mathbb{R}^\infty$ is closed in the box topology and its closure is $\mathbb{R}^\infty$.

We show that, in the product topology, the closure of $\mathbb{R}^\infty$ is $\mathbb{R}^\omega$. Clear that $\mathbb{R}^\omega$ is closed and contains $\mathbb{R}^\infty$. For every $(y_n) \in \mathbb{R}^\omega$, let $(\prod_{n=1}^N (y_n - \varepsilon, y_n + \varepsilon_n)) \times \mathbb{R}^\omega$ be a basis neighborhood of (y_n), where $0 < \varepsilon_n \leq \infty$. Then the sequence (x_n) defined by $x_k = y_k$ for $k = 1, \ldots, N$ and $x_k = 0$ for $k > N$ belongs to both $\mathbb{R}^\infty$ and the neighborhood of (y_n). Thus, $\mathbb{R}^\omega \subset \overline{\mathbb{R}^\infty}$ in the product topology, which implies that $\mathbb{R}^\omega = \overline{\mathbb{R}^\infty}$. $\square$

10.

Proof.

(a) Let

$$\mathcal{B} = \{f_\alpha^{-1}(U_\alpha) \mid \forall \alpha \in J, \forall U_\alpha \text{ open in } X_\alpha\}.$$

Clear that $\mathcal{B}$ contains X. Hence, $\mathcal{B}$ serves as a subbasis for a topology $\mathcal{T}$ on A. By the definition of $\mathcal{B}$, each f_α is continuous with respect to $\mathcal{T}$. Note that for every topology $\mathcal{T}'$ on A that makes every f_α continuous, $\mathcal{B} \subset \mathcal{T}'$. As a result, $\mathcal{T} \subset \mathcal{T}'$. Thus, $\mathcal{T}$ is the unique coarsest topology on A relative to which each f_α is continuous.

(b) It follows immediately from the previous discussion.

(c) The "only if" part comes from Theorem 18.2(c) immediately. For the "if" part, suppose that all $f_\alpha \circ g$ are continuous. Assume, to obtain a contradiction, that there is an subbasis element W in A whose inverse image $g^{-1}(W)$ is not open. Suppose $W = f_\alpha^{-1}(U_\alpha)$ where U_α is open in X_α. Then $g^{-1}(W) = g^{-1}(f_\alpha^{-1}(U_\alpha)) = (f_\alpha \circ g)^{-1}(U_\alpha)$, which is open as $f_\alpha \circ g$ is continuous. Contradiction. Thus, g is continuous.

(d) Let E be an open set in A. We show that $f(E)$ is open in Z. For every $x \in f(E)$, suppose $f(a) = x$. Since E is open, there is a basis neighborhood B of a with $B \subset E$. Note that $x \in f(B) \subset f(E)$. Now we show that $f(B)$ is open in Z to complete the proof.

Since B is a basis element, we may write B as $B = \cap_{\beta \in I} B_\beta$ where I is a finite subset of J and $B_\beta = f_\beta^{-1}(U_\beta)$ for some open set U_β in X_β. Hence,

$$f(B) = f\left(\bigcap_{\beta \in I} f_\beta^{-1}(U_\beta)\right) = Z \cap \left(\bigcap_{\beta \in I} U_\beta\right).$$

Namely, $f(B)$ is open in Z. Thus, the image under f of each open set in A is open in Z. $\square$