Solutions to
Introductory Functional Analysis with Applications

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2 Normed Spaces. Banach Spaces

2.3 Further Properties of Normed Spaces

4. cf. Prob. 13, Sec 1.2

Proof. The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1|\|x_1 - x_2\| + |\alpha_1 - \alpha_2|\|x_2\|.$$

□

7.

Proof. Let Y and y_n be defined as in the hint. Then $\|y_n\| = 1/n^2$, constituting a convergent number series. However,

$$\sum_{n=1}^{N} y_n = (1, 1/4, \dots, 1/N^2, 0, \dots),$$

which is divergent as $N \to \infty$. □

8.

Proof. Let (x_n) be a Cauchy sequence in X. Hence, for every $n > 0$, there exists some $K_n > 0$ such that for all $p, q > K_n$, $\|x_p - x_q\| < 1/n^2$. Without loss of generality, we may assume that (K_n) is increasing. Since the series $\|x_{K_{n+1}} - x_{K_n}\|$ is bounded by $1/n^2$, it converges. By the hypothesis, the series $(x_{K_{n+1}} - x_{K_n})$ also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1} (x_{K_{i+1}} - x_{K_i}) \to x \quad \text{as } n \to \infty.$$

Now we show that (x_n) converges to x. For every $\varepsilon > 0$, since (x_n) is a Cauchy sequence, there exists some N_1 such that for all $p, q > N_1$, $\|x_p - x_q\| < \varepsilon$. Meanwhile, since $x_{K_n} \to x$, once K_n is large enough, $\|x - x_{K_n}\| < \varepsilon$. Let $K_n > N_1$. Then for every $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus, X is complete. □

9.

Proof. Let (x_n) be an absolutely convergent series in Banach space X. Let $s_n = \sum_{i=1}^{n} x_n$. Now we show that s_n is a Cauchy sequence and therefore convergent. Since $\sum_{i=1}^{\infty} \|x_i\| < \infty$, for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n > N$, $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$. Hence, for every $N < p \leq q$,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^{q} x_i \right\| \leq \sum_{i=p+1}^{q} \|x_i\| < \varepsilon,$$

completing the proof. □

10.

Proof. Let (e_n) be Schauder basis of X. Denote the underlying field of X by $\mathbb{K}$ and let $\mathbb{W} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$ if $\mathbb{K} = \mathbb{C}$. Now we show that

$$S = \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \dots \right\},$$

a countable subset of X, is dense in X to derive the separability.

For every $x \in X$ and $\varepsilon > 0$, by the definition of Schauder basis, there exists $\beta_1, \dots, \beta_n \in \mathbb{K}$ such that $\|x - (\beta_1 e_1 + \cdots + \beta_n e_n)\| < \varepsilon$. Let $M = \max_i \|e_i\|$. If $M = 0$, then there is nothing to prove. Otherwise, since $\mathbb{W}$ is dense in $\mathbb{K}$, for $i = 1, \dots, n$, there exists $\alpha_i \in \mathbb{W}$ with $|\alpha_i - \beta_i| < \varepsilon / 2^i M$. Hence,

$$\begin{aligned} \left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| \leq& \left\| x - \sum_{i=1}^{n} \beta_i e_i \right\| + \left\| \sum_{i=1}^{n} (\beta_i - \alpha_i) e_i \right\| \\ \leq& \varepsilon + \sum_{i=1}^{n} |\alpha_i - \beta_i| \|e_i\| \\ \leq& 2\varepsilon. \end{aligned}$$

Thus, S is dense in X and therefore X is separable. □

14.

Proof. Clear that $\|\cdot\|_0$ is nonnegative. And $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$. Meanwhile, $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$. Finally, we show that $\|\hat{x}\|_0 = 0$ implies $\hat{x} = Y$ and invoke Prob. 4, Sec 2.2 to complete the proof. Since $\|\hat{x}\|_0 = 0$, there exists $(x_n) \subset \hat{x}$ which converges to 0. Since Y is closed, Y is complete and so is its cosets. Therefore, $0 \in \hat{x}$, enforcing $\hat{x}$ to be Y. □

2.4 Finite Dimensional Normed Spaces

3.

Proof. The reflexive property clearly holds. If there are positive a and b such that $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$ for all $x \in X$, then $\|x\|_1 / b \leq \|x\|_0 \leq \|x\|/a$. Hence the relation is symmetric. Next we further suppose there exists positive c and d such that that $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$. Then $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$, giving the transitive property. Thus, the axioms of an equivalence relation hold. □

4.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Let $E \subset X$ be any open set with respect to $\|\cdot\|$, i.e., for every $x_0 \in E$, there exists some $\delta > 0$ such that $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$. Since $\|\cdot\| \sim \|\cdot\|_0$, there exists some positive c such that $\|x - x_0\| \leq c\|x - x_0\|_0$. Hence, $B = \{x \in X : \|x - x_0\| < \delta / c\} \subset A \subset E$. Namely, E is also open with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. □

5.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Then for every $x \in X$, there exists some $c > 0$ such that $\|x\|_0 \leq c\|x\|$. Let (x_n) be a Cauchy sequence with respect to $\|\cdot\|$, i.e., for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n, m > N$, $\|x_n - x_m\| < \varepsilon/c$. Hence, $\|x_n - x_m\|_0 < c\|x_n - x_m\| \leq \varepsilon$. Thus, (x_n) is also a Cauchy with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. □

2.5 Compactness and Finite Dimension

5.

Proof. Clear that every point in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a closed bounded, and therefore compact, neighborhood. Hence, $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact. □

6.

Proof. Let X be a compact metric space and x any point in X. Let E be a closed neighborhood of x. By Prob 10, E is compact. Thus, X is locally compact. □

7.

Proof. It suffices to show that $a = \inf_{y \in Y} \|v - y\|$ can actually be obtained. Let $\{b_1, \ldots, b_n\}$ be a basis of Y and $y_k = y_{k,1}b_1 + \cdots + y_{k,n}b_n$ a sequence in Y with $\|v - y_k\| \to a$. We may assume without loss of generality that $\|v - y_k\|$ is bounded.

Since Y is a proper subset of Z, $v, b_1, \ldots, b_n$ are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar $c > 0$ such that for every k,

$$\|v - y_{k,1}b_1 - \cdots - y_{k,n}b_n\| \geq c(1 + |y_{k,1}| + \cdots + |y_{k,n}|).$$

Hence, the sequence $(y_{k,1}, \ldots, y_{k,n})$ of n-tuples is bounded and therefore has a convergent subsequence. Consequently, (y_k) also has a convergent subsequence. Suppose that it converges to $z \in Z$. Note that $\|v - z\| = a$ and as Y is closed, $z \in Y$. Thus, a can be attained in Y. □

8.

Proof. Since the unit ball B with respect to $\|\cdot\|_2$ in $\mathbb{R}^n$ and $\mathbb{C}^n$ is compact and $\|\cdot\|$ is continuous, by 2.5-7, $x \mapsto \|x\|$ can attains its minimum, denoted by a, on B. Due to the positive definite property of a norm, a is positive. Hence, $0 < a \leq \|x/\|x\|_2\|$. Namely, $a\|x\|_2 \leq \|x\|$. □

9.

Proof. For every $(x_n) \subset M \subset X$, since X is compact, there exists a subsequence (x_{n_k}) of (x_n) which converges to some $y \in X$. Since M is closed, $y \in M$. Hence, M is compact. □

10.

Proof. From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of T is also continuous, it suffices to show that the image of a closed set $A \subset X$ under T is again a closed set. Since X is compact and A is closed, A is compact. Since T is continuous, by 2.5-6, $T(A)$ is compact and therefore closed. Hence, T is a homeomorphism. $\square$