# Real Analysis

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# 3 Lebesgue Measure

## 3.1 Introduction

**1.**

*Proof.* Since $\mathfrak{M}$ is an $\sigma$-algebra, $B \setminus A \in \mathfrak{M}$ as long as $A, B \in \mathfrak{M}$. Since $B \setminus A$ and $A$ are disjoint, $mB = mA + m(B \setminus A) \geq mA$ since $m$ is nonnegative. $\square$

**2.**

*Proof.* Let $A_0 = E_0$ and $E_k = A_k \setminus A_{k-1}$ for $k \geq 1$. Clear that $E_i$ and $E_j$ are disjoint for distinct $i$ and $j$, $\bigcup A_n = \bigcup E_n$ and $A_i \subset E_i$ for every $i$. Hence,

$$m\left(\bigcup E_n\right) = m\left(\bigcup A_n\right) = \sum mA_n \leq \sum mE_n,$$

where the last inequality comes from Exercise 1. $\square$

**3.**

*Proof.* Suppose that $mA < \infty$. Then $mA = m(A \cup \varnothing) = mA + m\varnothing$, implying that $m\varnothing = 0$. $\square$

## 3.2 Outer Measure

**5.**

*Proof.* We show that $\{I_n\}$ must cover the entire $[0,1]$ by contradiction. Assume that $x \notin I_k$ for $k = 1, 2, \ldots, n$. Then, as $I_k$ are open and $n$ is finite, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ and $I_k$ are disjoint for every $k$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some rational number in $(x - \varepsilon, x + \varepsilon)$, contradicting with the hypothesis that $\{I_k\}$ covers all rational numbers between 0 and 1. $\square$

**6.**

*Proof.* By the definition of the outer measure, for every $\varepsilon > 0$, there exists some collection $\{I_n\}$ of open intervals that covers $A$ and $\sum l(I_n) \leq m^*A + \varepsilon$. Let $O = \bigcup I_n$. $O$ is a countable union of open sets and therefore is also open. And by Proposition 2, $m^*O \leq \sum l(I_n)$. Thus, $m^*O \leq m^*A + \varepsilon$.

Let $\varepsilon_n = 1/n$ and for each $n$, by the previous discussion, we can always get an open set $O_k$ such that $A \subset O_k$ and $m^*O \leq m^*A + \varepsilon_m$. Let $G$ be the countable intersection of these open sets. Clear that $G$ is a $G_\delta$ set covering $A$ and $m^*A = m^*G$. $\square$

**7.**

*Proof.* If $m^*E = \infty$, it is trivial. Suppose that $m^*E \leq \infty$. For any $x \in \mathbb{R}$, collection $\{I_n\}$ of open intervals covers $E + x$ iff $\{I_n - x\}$ covers $E$. Since the length of intervals is translation invariant, this implies $m^*(E + x) = m^*E$. $\square$

**8.**

*Proof.* Clear that $m^*A \leq m^*(A \cup B)$. Meanwhile, $m^*(A \cup B) = m^*A + m^*B = m^*B$. Hence, $m^*(A \cup B) = m^*B$. $\square$

## 3.3 Measurable Sets and Lebesgue Measure

**10.**

*Proof.*

$$\begin{aligned} mE_1 + mE_2 &= mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2). \end{aligned}$$

□

**11.**

*Proof.* $E_n = (n, \infty)$. □

**12.** This is the countable version of Lemma 9.

*Proof.* It suffices to prove $m^*(A \cap \bigcup E_i) \geq \sum m^*(A \cap E_i)$. Since $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i$ for every $n$,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of $n$, we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i),$$

completing the proof. □

**13.**

*Proof.* First we suppose that $m^*E < \infty$. By Proposition 5, there exists some open set $O \supset E$ such that $m^*O \leq m^*E + \varepsilon$. If $E$ is measurable, then by the definition,

$$m^*(O \setminus E) = m^*O - m^*E \leq \varepsilon.$$

Namely, (ii) holds. Meanwhile, $O \subset \mathbb{R}$ is a countable union of disjoint open intervals $\{I_n\}$. Since $mO = m^*O$ is bounded and $mO = \sum l(I_n)$, there exists some integer $N > 0$ such that $mO - \sum_{n=1}^{N} l(I_n) < \varepsilon$. Let $U = \bigcup_{n=1}^{N} I_n$.

$$\begin{aligned} m^*(U \triangle E) &= m^*((U \cup E) \setminus (U \cap E)) \\ &\leq m^*(O \setminus (U \cap E)) \\ &= m^*((O \setminus U) \cup (O \setminus E)) \\ &\leq m^*(O \setminus U) + m^*(O \setminus E) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, (ii) implies (vi). Now we show that (vi) implies (ii). If $m^*(U \triangle E) < \varepsilon$, then there exists some countable collection $\{J_n\}$ of open interval such that

$$\sum l(J_n) \leq m^*(U \triangle E) + \varepsilon < 2\varepsilon.$$

Let $J = \bigcup J_n$ and $O = U \cup J$. $m^*J < 2\varepsilon$. And $O$ is open and covers $E$. Meanwhile,

$$m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(J \setminus E) < 3\varepsilon.$$

Hence, (ii) holds.

Now, let $E$ be an arbitrary set and $E_n = E \cap (-n, n)$, which is a set with finite measure. Then by the previous discussion, there exists some open set $O_n \supset E_n$ with $m^*(O_n \setminus E_n) < \varepsilon/2^n$. Let $O = \bigcup O_n$, an open set covering $E$ and

$$m^*(O \setminus E) \leq \sum m^*(O_n \setminus E_n) < 2\varepsilon.$$

Hence, (i) implies (ii). Now we suppose (ii) holds and let $\varepsilon_n = 1/n$, then there exists a sequence of open sets $< O_n >$ such that $m^*(O_n \setminus E) < 1/n$. Let $G = \bigcap O_n \in G_\delta$. $m^*(G \setminus E) \leq m^*(O_n \setminus E) \leq 1/n$. Since the left hand side is independent of $n$, $m^*(G \setminus E) = 0$. If (iv) holds, then by Lemma 6, $G \setminus E$ is measurable. Since $G \in G_\delta$ is also measurable, $E$ is measurable. Hence, (iv) implies (i).

By the previous result, for any measurable $E$, there exists some closed set $F \subset E$ such that $\bar{F}$, which is open, contains $barE$ and $m^*(\bar{F} \setminus \bar{E}) < \varepsilon$. Hence, $m^*(E \setminus F) < \varepsilon$. We can proceed in a similar manner as we did in the last paragraph to prove that (iii) $\Rightarrow$ (v) $\Rightarrow$ (i), leading to the final conclusion. □

## 3.5 Measurable Functions

**19.**

*Proof.* For every $\beta \in \mathbb{R}$, since $D$ is measurable, there exists a sequence of $\alpha_n \in D \cap (\beta - 1/n, \beta)$. As

$$\{x : f(x) > r\} \quad \Leftrightarrow \quad \bigcup_{n=1}^{\infty} \{x : f(x) > r - 1/n\} \quad \Leftrightarrow \quad \bigcup_{n=1}^{\infty} \{x : f(x) > \alpha_n\}$$

and $\{x : f(x) > \alpha_n\}$ are measurable, so is $\{x : f(x) > r\}$. Hence, $f$ is measurable. □

**21.**

*Proof.*

(a) It follows immediately from $\{x : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$.

(b) For $\alpha \geq 0$, the sets $\{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$ are the same. And for $\alpha < 0$,

$$\{x : f(x) > \alpha\} = \{x : g(x) > \alpha\} \setminus \bar{D} \quad \text{and} \quad \{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \bar{D}.$$

Hence, $f$ is measurable iff $g$ is measurable. □

**22.(d)**

*Proof.* Since $f$ and $g$ are finite almost everywhere, the set $A$ consisting of points where $f + g$ is of the form $\infty - \infty$ or $-\infty + \infty$ is of measure zero (and hence measurable). Therefore no matter how it is defined, $\{x \in A : f + g > \alpha\}$ is measurable for every $\alpha$. Namely, the restriction of $f + g$ to $A$ is measurable. Meanwhile, clear that the restriction to $D \setminus A$ is measurable where $D$ is the domain of $f$. Hence, by Exercise 21, $f$ is measurable. □

**23.**

*Proof.*

(a) Let $A_n = \{x : |f(x)| > n\}$, a sequence of measurable sets. As $A_{n+1} \subset A_n$, $mA_{n+1} \leq mA_n$. Since $A = \bigcap A_n = \{x : |f(x)| = \infty\}$, $mA_1 \leq m[a,b]$ is finite and $mA = 0$, by Proposition 14, there exists some $N$ such that for all $n \geq N$, $mA_n < \varepsilon/3$. Set $M = N$ to complete the proof.

(b) We consider the restriction of $f$ on to the set $E = [a,b] \setminus \{x : |f(x)| \geq M\}$, which is also a measurable real-valued function. To keep our notation simple, we denote the restriction by $f$ still. For every $\varepsilon > 0$, there exists some integer $N$ with $0 < 2M/N < \varepsilon$. Let $E_n = \{x : x \in [-M + (n-1)\varepsilon, -M + n\varepsilon]\}$ $(n = 1, 2, \ldots, N)$ and define

$$\varphi(x) = \sum_{i=1}^{N} f(x_i)\chi_{E_i},$$

where $x_n \in E_n$ is arbitrary. Clear that $\varphi$ is a simple function and satisfy all the requirements.

(c) Suppose that $\varphi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$. For each $i = 1, \ldots, N$, $E_i$ is measurable and therefore by Proposition 15, there exists a finite union $U_i$ of open intervals such that $m(U_i \triangle E_i) < \varepsilon$. Let

$$g(x) = \sum_{i=1}^{N} \alpha_i \chi_{U_i}.$$

Clear that $g$ and $\varphi$ only may differ on a set with measure $N\varepsilon$. (d) Suppose that $g(x) = \sum_{i=1}^{N} \alpha_i \chi_{U_i}$ is a step function. We may assume without loss of generality that $U_i$ are disjoint and $\bigcup U_i = [a,b]$. And suppose that $\{x_0 = a < x_1 < \cdots < x_N = b\}$ are the endpoints of the intervals. For each $i = 1, \ldots, N-1$, define

$$f(x) = (x - x_i + \varepsilon)g(x_i - \varepsilon) + (x_i + \varepsilon - x)g(x_i + \varepsilon), \quad x \in (x_i - \varepsilon, x_i + \varepsilon),$$

and $f(x) = g(x)$ for the other points. (We assume that $\varepsilon$ is small enough so that $f$ is well-defined.) Clear that $f$ is continuous and equals $g$ except on a set of measure less then $2N\varepsilon$. □

**24.**

*Proof.* For measurable $f$, we show that $\mathcal{A} = \{E : f^{-1}[E] \text{ is measurable}\}$ is a $\sigma$-algebra first. As the domain, denoted by $D$, of a measurable function is measurable, $\mathbb{R} \in \mathcal{A}$. If $E \in \mathcal{A}$, then since $f^{-1}[\bar{E}] = D \cap \overline{f^{-1}[E]}$, $f^{-1}[\bar{E}]$ is also measurable and therefore $\bar{E} \in \mathcal{A}$. Suppose that $< E_n >$ is a sequence of sets of $\mathcal{A}$. Then, as

$$f^{-1}\left[\bigcup_{n=1}^{\infty} E_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[E_n],$$

$\bigcup E_n \in \mathcal{A}$. Hence, $\mathcal{A}$ is a $\sigma$-algebra.

By the definition of a measurable function, every open interval belongs to $\mathcal{A}$. Since the collection of all Borel sets $\mathcal{B}$ is the $\sigma$-algebra generated by all open intervals, $\mathcal{B} \subset \mathcal{A}$. Namely, $f^{-1}[B]$ is measurable as long as $B \in \mathcal{B}$. □

## 3.6 Littlewood's Three Principles

### 30.

*Proof.* Let $\varepsilon_n = 1/n$ and $\delta_n = \eta/2^n$ $(n = 0, 1, \dots)$. By Proposition 24, for each $n$, there exists some $A_n$ with measure less than $\delta_n$ such that for all $x \in E_n \setminus A_n$, $|f_m(x) - f(x)| < \varepsilon_n$ for $m$ large enough. Let $A = \bigcup_{n=1}^{\infty} A_n$, the measure of which is less than $\sum \eta/2^n = \delta$. Meanwhile, for any $\varepsilon > 0$, by construction, for all $x \in E \setminus A$, $|f_m(x) - f(x)| < \varepsilon$ for $m$ large enough. Namely, $f_n$ converges to $f$ uniformly on $E \setminus A$. □

### 31.

*Proof.* Let $\varepsilon_n = \delta/2^n$ $(n \geq 0)$, then by Proposition 22, there exists continuous $g_n$ such that $E_n = \{x : |f(x) - g_n(x)| \geq \varepsilon_n\}$ is of measure less than $\varepsilon_n$. Let $E = \bigcup E_n$, the measure of which is less than $\delta$ and $g_n$ converges to $f$ on $[a, b] \setminus E$.

By Egoroff's Theorem, there exists some $A \subset [a, b] \setminus E$ with $mA < \delta$ such that $g_n$ converges to $f$ uniformly on $[a, b] \setminus (E \cup A)]$. Since $E \cup A$ is measurable, by Proposition 15, there exists some open set $O \supset E \cup A$ such that $m(O \setminus (E \cup A)) < \delta$. Let $F = [a, b] \setminus O$. We know that

1. $F$ is a closed set.
2. $mF < 3\delta$.
3. $g_n$ converges to $f$ uniformly on $F$.

Hence, $f$ is continuous on $F$ And by Problem 2.40, there exists some continuous function on $\mathbb{R}$ such that $\varphi(x) = f(x)$ for $x \in F$.

If $f$ is defined on $(-\infty, \infty)$, we can apply the previous result on each $[n, n+1]$ and "stick" the functions together as we did in Problem 23(c) to get the function required. □

# 4 The Lebesgue Integral

## 4.2 The Lebesgue Integral of a Bounded Function

**2.**

*Proof.*
(a) By Problem 2.51, $h$ is upper semicontinuous as $f$ is bounded and by Problem 2.50, $x : h(x) < \lambda$ is open and hence measurable for every $\lambda \in \mathbb{R}$. Thus, $h$ is measurable.

Let $\varphi(x) \geq f(x)$ be a step function and $x_0$ any point other than the endpoints of the intervals occurring in $\varphi$. Then there exists some $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$, $\varphi(x_0) = \varphi(x) \geq f(x)$. Hence,

$$h(x_0) = \inf_{\delta<0} \sup_{|x-x_0|<\delta} f(x) \leq \varphi(x_0).$$

Namely, $\varphi \geq h$ except at a finite number of points. Hence, $\int_a^b \varphi \geq \int_a^b h$ and therefore

$$R \overline{\int_a^b} f = \inf_{\varphi \geq f} \int_a^b \varphi(x) \mathrm{d}x \geq \int_a^b h.$$

We can also derive from the previous discussion that there is a sequence of $\langle \varphi_n \rangle$ of step functions satisfying $\varphi \downarrow h$. By Proposition 6,

$$\int_a^b h = \lim \int_a^b \varphi_n \geq R \overline{\int_a^b} f.$$

Hence, $R \overline{\int_a^b} f = \int_a^b h$.

(b) First suppose that $f$ is Riemann integrable and let $h$ and $g$ be the upper and lower envelope of $f$ respectively. By part (a), $f$ is Riemann integrable implies $\int_a^b (h - g) = 0$. Together with the fact that $h \geq g$, we conclude that $h = g$ a.e.. Therefore, by Problem 2.50, $f$ is continuous except on a set of measure zero.

Note that the argument remains true if we reverse the order, verifying the converse part. Hence, the proposition holds. □

## 4.3 The Integral of a Nonnegative Function

**3.**

*Proof.* Suppose that $E_n = \{x : f(x) > 1/n\}$. Then,

$$0 = \int f \geq \int_{E_n} f \geq \frac{mE_n}{n}$$

implies $mE_n = 0$. Hence, $m\{x : f(x) > 0\} = m(\bigcup E_n) \leq \sum mE_n = 0$. Namely, $f = 0$ a.e. □

**5.**

*Proof.* For any fixed $x_0 \in \mathbb{R}$, let $f_n(x) = f \cdot \chi_{(-\infty,x_0-1/n]}$, which is a increasing sequence of nonnegative measurable function whose limit is $f \cdot \chi_{-\infty,x_0}$. Then by Theorem 10,

$$F(x_0) = \int_{-\infty}^{x_0} f = \int f \cdot \chi_{-\infty,x_0} = \lim \int f \cdot \chi_{(-\infty,x_0-1/n]} = \lim F(x_0 - 1/n).$$

Meanwhile, since

$$|F(x_0) - F(x_0 + 1/n)| = \left| \int_{x_0}^{x_0+1/n} f(x)\mathrm{d}x \right| = \left| \int_{-1/n}^{0} g(x)\mathrm{d}x \right|,$$

where $g(x) = f(x_0 - x)$, arguing on $g$ in a similar manner yields $F(x_0) = \lim F(x_0 + 1/n)$. Thus, $F$ is continuous. □

**6.**

*Proof.* By Theorem 9, $\int f \leq \underline{\lim} \int f_n$. Meanwhile, $f_n \leq f$ implies $\int f_n \leq \int f$ and therefore $\overline{\lim} \int f_n \leq \int f$. Hence, $\int f = \lim \int f_n$. □

**7.**

*Solution.*
(a) Let $f_n(x) = n \cdot \chi_{[0,1/n]}$. $f_n$ converges to $f = 0$ except on $x = 0$. For each $n$, $\int f_n = 1$ but $\int f = 0$. Hence, the inequality could be strict.
(b) Let $f_n(x) = \chi_{[n,\infty)}$. Then $\langle f_n \rangle$ is a decreasing sequence which converges to $f = 0$, the integral of which is 0. However, for every $n$, $\int f_n = \infty$. □

**8.**

*Proof.* Let $g_n = \inf\{f_n, f_{n+1}, \dots\}$. Clear that

$$\int g_n \leq \int f_n. \tag{1}$$

Meanwhile $\langle g_n \rangle$ is a increasing sequence converging to $\underline{\lim}\, f_n$. Hence, by the Monotone Convergence Theorem and (1)

$$\int \underline{\lim}\, f_n = \int \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int g_n \leq \underline{\lim} \int f_n.$$

□

**9.**

*Proof.* By Fatou's Lemma,

$$\int_E f \leq \underline{\lim} \int_E f_n. \tag{2}$$

Similarly, $\int_{\bar{E}} f \leq \underline{\lim} \int_{\bar{E}} f_n$ and therefore

$$\int_E f_n = \int f_n - \int_{\bar{E}} f_n \quad \Rightarrow \quad \overline{\lim} \int_E f_n \leq \int f - \int_{\bar{E}} f = \int_{\bar{E}} f.$$

(2) and the inequality above together implies $\int_E f_n \to \int f$. □