Real Analysis

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Contents

3 Lebesgue Measure

3.1 Introduction

1.

Proof. Since $\mathfrak{M}$ is an σ-algebra, $B \setminus A \in \mathfrak{M}$ as long as $A, B \in \mathfrak{M}$. Since $B \setminus A$ and A are disjoint, $mB = mA + m(B \setminus A) \geq mA$ since m is nonnegative. $\square$

2.

Proof. Let $A_0 = E_0$ and $E_k = A_k \setminus A_{k-1}$ for $k \geq 1$. Clear that E_i and E_j are disjoint for distinct i and j, $\bigcup A_n = \bigcup E_n$ and $A_i \subset E_i$ for every i. Hence,

$$m\left(\bigcup E_n\right) = m\left(\bigcup A_n\right) = \sum mA_n \leq \sum mE_n,$$

where the last inequality comes from Exercise 1. $\square$

3.

Proof. Suppose that $mA < \infty$. Then $mA = m(A \cup \varnothing) = mA + m\varnothing$, implying that $m\varnothing = 0$. $\square$

3.2 Outer Measure

5.

Proof. We show that $\{I_n\}$ must cover the entire $[0,1]$ by contradiction. Assume that $x \notin I_k$ for $k = 1, 2, \ldots, n$. Then, as I_k are open and n is finite, there exists some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon)$ and I_k are disjoint for every k. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some rational number in $(x-\varepsilon, x+\varepsilon)$, contradicting with the hypothesis that $\{I_k\}$ covers all rational numbers between 0 and 1. $\square$

6.

Proof. By the definition of the outer measure, for every $\varepsilon > 0$, there exists some collection $\{I_n\}$ of open intervals that covers A and $\sum l(I_n) \leq m^*A + \varepsilon$. Let $O = \bigcup I_n$. O is a countable union of open sets and therefore is also open. And by Proposition 2, $m^*O \leq \sum l(I_n)$. Thus, $m^*O \leq m^*A + \varepsilon$.

Let $\varepsilon_n = 1/n$ and for each n, by the previous discussion, we can always get an open set O_k such that $A \subset O_k$ and $m^*O \leq m^*A + \varepsilon_m$. Let G be the countable intersection of these open sets. Clear that G is a G_δ set covering A and $m^*A = m^*G$. $\square$

7.

Proof. If $m^*E = \infty$, it is trivial. Suppose that $m^*E \leq \infty$. For any $x \in \mathbb{R}$, collection $\{I_n\}$ of open intervals covers $E + x$ iff $\{I_n - x\}$ covers E. Since the length of intervals is translation invariant, this implies $m^*(E + x) = m^*E$. $\square$

8.

Proof. Clear that $m^*A \leq m^*(A \cup B)$. Meanwhile, $m^*(A \cup B) = m^*A + m^*B = m^*B$. Hence, $m^*(A \cup B) = m^*B$. $\square$

3.3 Measurable Sets and Lebesgue Measure

10.

Proof.

$$\begin{aligned} mE_1 + mE_2 &= mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2). \end{aligned}$$

□

11.

Proof. $E_n = (n, \infty)$. □

12. This is the countable version of Lemma 9.

Proof. It suffices to prove $m^*(A \cap \bigcup E_i) \geq \sum m^*(A \cap E_i)$. Since $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i$ for every n,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of n, we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i),$$

completing the proof. □

13.

Proof. First we suppose that $m^*E < \infty$. By Proposition 5, there exists some open set $O \supset E$ such that $m^*O \leq m^*E + \varepsilon$. If E is measurable, then by the definition,

$$m^*(O \setminus E) = m^*O - m^*E \leq \varepsilon.$$

Namely, (ii) holds. Meanwhile, $O \subset \mathbb{R}$ is a countable union of disjoint open intervals $\{I_n\}$. Since $mO = m^*O$ is bounded and $mO = \sum l(I_n)$, there exists some integer $N > 0$ such that $mO - \sum_{n=1}^{N} l(I_n) < \varepsilon$. Let $U = \bigcup_{n=1}^{N} I_n$.

$$\begin{aligned} m^*(U \triangle E) &= m^*((U \cup E) \setminus (U \cap E)) \\ &\leq m^*(O \setminus (U \cap E)) \\ &= m^*((O \setminus U) \cup (O \setminus E)) \\ &\leq m^*(O \setminus U) + m^*(O \setminus E) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, (ii) implies (vi). Now we show that (vi) implies (ii). If $m^*(U \triangle E) < \varepsilon$, then there exists some countable collection $\{J_n\}$ of open interval such that

$$\sum l(J_n) \leq m^*(U \triangle E) + \varepsilon < 2\varepsilon.$$

Let $J = \bigcup J_n$ and $O = U \cup J$. $m^*J < 2\varepsilon$. And O is open and covers E. Meanwhile,

$$m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(J \setminus E) < 3\varepsilon.$$

Hence, (ii) holds.

Now, let E be an arbitrary set and $E_n = E \cap (-n, n)$, which is a set with finite measure. Then by the previous discussion, there exists some open set $O_n \supset E_n$ with $m^*(O_n \setminus E_n) < \varepsilon/2^n$. Let $O = \bigcup O_n$, an open set covering E and

$$m^*(O \setminus E) \leq \sum m^*(O_n \setminus E_n) < 2\varepsilon.$$

Hence, (i) implies (ii). Now we suppose (ii) holds and let $\varepsilon_n = 1/n$, then there exists a sequence of open sets $< O_n >$ such that $m^*(O_n \setminus E) < 1/n$. Let $G = \bigcap O_n \in G_\delta$. $m^*(G \setminus E) \leq m^*(O_n \setminus E) \leq 1/n$. Since the left hand side is independent of n, $m^*(G \setminus E) = 0$. If (iv) holds, then by Lemma 6, $G \setminus E$ is measurable. Since $G \in G_\delta$ is also measurable, E is measurable. Hence, (iv) implies (i).

By the previous result, for any measurable E, there exists some closed set $F \subset E$ such that $\bar{F}$, which is open, contains $barE$ and $m^*(\bar{F} \setminus \bar{E}) < \varepsilon$. Hence, $m^*(E \setminus F) < \varepsilon$. We can proceed in a similar manner as we did in the last paragraph to prove that (iii) $\Rightarrow$ (v) $\Rightarrow$ (i), leading to the final conclusion. □

3.5 Measurable Functions

19.

Proof. For every $\beta \in \mathbb{R}$, since D is measurable, there exists a sequence of $\alpha_n \in D \cap (\beta - 1/n, \beta)$. As

$$\{x : f(x) > r\} \quad \Leftrightarrow \quad \bigcup_{n=1}^{\infty} \{x : f(x) > r - 1/n\} \quad \Leftrightarrow \quad \bigcup_{n=1}^{\infty} \{x : f(x) > \alpha_n\}$$

and $\{x : f(x) > \alpha_n\}$ are measurable, so is $\{x : f(x) > r\}$. Hence, f is measurable. □

21.

Proof.

(a) It follows immediately from $\{x : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$.

(b) For $\alpha \geq 0$, the sets $\{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$ are the same. And for $\alpha < 0$,

$$\{x : f(x) > \alpha\} = \{x : g(x) > \alpha\} \setminus \bar{D} \quad \text{and} \quad \{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \bar{D}.$$

Hence, f is measurable iff g is measurable. □

22.(d)

Proof. Since f and g are finite almost everywhere, the set A consisting of points where $f + g$ is of the form $\infty - \infty$ or $-\infty + \infty$ is of measure zero (and hence measurable). Therefore no matter how it is defined, $\{x \in A : f + g > \alpha\}$ is measurable for every α. Namely, the restriction of $f + g$ to A is measurable. Meanwhile, clear that the restriction to $D \setminus A$ is measurable where D is the domain of f. Hence, by Exercise 21, f is measurable. □

23.

Proof.

(a) Let $A_n = \{x : |f(x)| > n\}$, a sequence of measurable sets. As $A_{n+1} \subset A_n$, $mA_{n+1} \le mA_n$. Since $A = \bigcap A_n = \{x : |f(x)| = \infty\}$, $mA_1 \le m[a,b]$ is finite and $mA = 0$, by Proposition 14, there exists some N such that for all $n \ge N$, $mA_n < \varepsilon/3$. Set $M = N$ to complete the proof.

(b) We consider the restriction of f on to the set $E = [a,b] \setminus \{x : |f(x)| \ge M\}$, which is also a measurable real-valued function. To keep our notation simple, we denote the restriction by f still. For every $\varepsilon > 0$, there exists some integer N with $0 < 2M/N < \varepsilon$. Let $E_n = \{x : x \in [-M + (n-1)\varepsilon, -M + n\varepsilon]\}$ $(n = 1, 2, \dots, N)$ and define

$$\varphi(x) = \sum_{i=1}^{N} f(x_i)\chi_{E_i},$$

where $x_n \in E_n$ is arbitrary. Clear that φ is a simple function and satisfy all the requirements.

(c) Suppose that $\varphi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$. For each $i = 1, \dots, N$, E_i is measurable and therefore by Proposition 15, there exists a finite union U_i of open intervals such that $m(U_i \triangle E_i) < \varepsilon$. Let

$$g(x) = \sum_{i=1}^{N} \alpha_i \chi_{U_i}.$$

Clear that g and φ only may differ on a set with measure $N\varepsilon$. (d) Suppose that $g(x) = \sum_{i=1}^{N} \alpha_i \chi_{U_i}$ is a step function. We may assume without loss of generality that U_i are disjoint and $\bigcup U_i = [a,b]$. And suppose that $\{x_0 = a < x_1 < \dots < x_N = b\}$ are the endpoints of the intervals. For each $i = 1, \dots, N-1$, define

$$f(x) = (x - x_i + \varepsilon)g(x_i - \varepsilon) + (x_i + \varepsilon - x)g(x_i + \varepsilon), \quad x \in (x_i - \varepsilon, x_i + \varepsilon),$$

and $f(x) = g(x)$ for the other points. (We assume that ε is small enough so that f is well-defined.) Clear that f is continuous and equals g except on a set of measure less then $2N\varepsilon$. $\square$

24.

Proof. For measurable f, we show that $\mathcal{A} = \{E : f^{-1}[E] \text{ is measurable}\}$ is a σ-algebra first. As the domain, denoted by D, of a measurable function is measurable, $\mathbb{R} \in \mathcal{A}$. If $E \in \mathcal{A}$, then since $f^{-1}[\bar{E}] = D \cap \overline{f^{-1}[E]}$, $f^{-1}[\bar{E}]$ is also measurable and therefore $\bar{E} \in \mathcal{A}$. Suppose that $< E_n >$ is a sequence of sets of $\mathcal{A}$. Then, as

$$f^{-1}\left[\bigcup_{n=1}^{\infty} E_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[E_n],$$

$\bigcup E_n \in \mathcal{A}$. Hence, $\mathcal{A}$ is a σ-algebra.

By the definition of a measurable function, every open interval belongs to $\mathcal{A}$. Since the collection of all Borel sets $\mathcal{B}$ is the σ-algebra generated by all open intervals, $\mathcal{B} \subset \mathcal{A}$. Namely, $f^{-1}[B]$ is measurable as long as $B \in \mathcal{B}$. $\square$

3.6 Littlewood's Three Principles

30.

Proof. Let $\varepsilon_n = 1/n$ and $\delta_n = \eta/2^n$ ($n = 0, 1, \ldots$). By Proposition 24, for each n, there exists some A_n with measure less than δ_n such that for all $x \in E_n \setminus A_n$, $|f_m(x) - f(x)| < \varepsilon_n$ for m large enough. Let $A = \bigcup_{n=1}^{\infty} A_n$, the measure of which is less than $\sum \eta/2^n = \delta$. Meanwhile, for any $\varepsilon > 0$, by construction, for all $x \in E \setminus A$, $|f_m(x) - f(x)| < \varepsilon$ for m large enough. Namely, f_n converges to f uniformly on $E \setminus A$. $\square$

31.

Proof. Let $\varepsilon_n = \delta/2^n$ ($n \geq 0$), then by Proposition 22, there exists continuous g_n such that $E_n = \{x : |f(x) - g_n(x)| \geq \varepsilon_n\}$ is of measure less than ε_n. Let $E = \bigcup E_n$, the measure of which is less than δ and g_n converges to f on $[a, b] \setminus E$.

By Egoroff's Theorem, there exists some $A \subset [a, b] \setminus E$ with $mA < \delta$ such that g_n converges to f uniformly on $[a, b] \setminus (E \cup A)]$. Since $E \cup A$ is measurable, by Proposition 15, there exists some open set $O \supset E \cup A$ such that $m(O \setminus (E \cup A)) < \delta$. Let $F = [a, b] \setminus O$. We know that

1. F is a closed set.

2. $mF < 3\delta$.

3. g_n converges to f uniformly on F.

Hence, f is continuous on F And by Problem 2.40, there exists some continuous function on $\mathbb{R}$ such that $\varphi(x) = f(x)$ for $x \in F$.

If f is defined on $(-\infty, \infty)$, we can apply the previous result on each $[n, n + 1]$ and "stick" the functions together as we did in Problem 23(c) to get the function required. $\square$

4 The Lebesgue Integral

4.2 The Lebesgue Integral of a Bounded Function

2.

Proof.
(a) By Problem 2.51, h is upper semicontinuous as f is bounded and by Problem 2.50, $x: h(x) < \lambda$ is open and hence measurable for every $\lambda \in \mathbb{R}$. Thus, h is measurable.

Let $\varphi(x) \geq f(x)$ be a step function and x_0 any point other than the endpoints of the intervals occurring in φ. Then there exists some $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$, $\varphi(x_0) = \varphi(x) \geq f(x)$. Hence,

$$h(x_0) = \inf_{\delta < 0} \sup_{|x - x_0| < \delta} f(x) \leq \varphi(x_0).$$

Namely, $\varphi \geq h$ except at a finite number of points. Hence, $\int_a^b \varphi \geq \int_a^b h$ and therefore

$$R \overline{\int_a^b} f = \inf_{\varphi \geq f} \int_a^b \varphi(x) \mathrm{d}x \geq \int_a^b h.$$

We can also derive from the previous discussion that there is a sequence of $\langle \varphi_n \rangle$ of step functions satisfying $\varphi \downarrow h$. By Proposition 6,

$$\int_a^b h = \lim \int_a^b \varphi_n \geq R \overline{\int_a^b} f.$$

Hence, $R \overline{\int_a^b} f = \int_a^b h$.

(b) First suppose that f is Riemann integrable and let h and g be the upper and lower envelope of f respectively. By part (a), f is Riemann integrable implies $\int_a^b (h - g) = 0$. Together with the fact that $h \geq g$, we conclude that $h = g$ a.e.. Therefore, by Problem 2.50, f is continuous except on a set of measure zero.

Note that the argument remains true if we reverse the order, verifying the converse part. Hence, the proposition holds. □

4.3 The Integral of a Nonnegative Function

3.

Proof. Suppose that $E_n = \{x : f(x) > 1/n\}$. Then,

$$0 = \int f \geq \int_{E_n} f \geq \frac{mE_n}{n}$$

implies $mE_n = 0$. Hence, $m\{x : f(x) > 0\} = m(\bigcup E_n) \leq \sum mE_n = 0$. Namely, $f = 0$ a.e. □

5.

Proof. For any fixed $x_0 \in \mathbb{R}$, let $f_n(x) = f \cdot \chi_{(-\infty, x_0 - 1/n]}$, which is a increasing sequence of nonnegative measurable function whose limit is $f \cdot \chi_{-\infty, x_0}$. Then by Theorem 10,

$$F(x_0) = \int_{-\infty}^{x_0} f = \int f \cdot \chi_{-\infty, x_0} = \lim \int f \cdot \chi_{(-\infty, x_0 - 1/n]} = \lim F(x_0 - 1/n).$$

Meanwhile, since

$$|F(x_0) - F(x_0 + 1/n)| = \left| \int_{x_0}^{x_0 + 1/n} f(x) \mathrm{d}x \right| = \left| \int_{-1/n}^{0} g(x) \mathrm{d}x \right|,$$

where $g(x) = f(x_0 - x)$, arguing on g in a similar manner yields $F(x_0) = \lim F(x_0 + 1/n)$. Thus, F is continuous. □

6.

Proof. By Theorem 9, $\int f \leq \underline{\lim} \int f_n$. Meanwhile, $f_n \leq f$ implies $\int f_n \leq \int f$ and therefore $\overline{\lim} \int f_n \leq \int f$. Hence, $\int f = \lim \int f_n$. □

7.

Solution.

(a) Let $f_n(x) = n \cdot \chi_{[0,1/n]}$. f_n converges to $f = 0$ except on $x = 0$. For each n, $\int f_n = 1$ but $\int f = 0$. Hence, the inequality could be strict.

(b) Let $f_n(x) = \chi_{[n,\infty)}$. Then $\langle f_n \rangle$ is a decreasing sequence which converges to $f = 0$, the integral of which is 0. However, for every n, $\int f_n = \infty$. □

8.

Proof. Let $g_n = \inf\{f_n, f_{n+1}, \ldots\}$. Clear that

$$\int g_n \leq \int f_n. \tag{1}$$

Meanwhile $\langle g_n \rangle$ is a increasing sequence converging to $\underline{\lim} f_n$. Hence, by the Monotone Convergence Theorem and (1)

$$\int \underline{\lim} f_n = \int \lim_{n \to \infty} g_n = \lim_{n \to \infty} \int g_n \leq \underline{\lim} \int f_n.$$

□

9.

Proof. By Fatou's Lemma,

$$\int_E f \leq \underline{\lim} \int_E f_n. \tag{2}$$

Similarly, $\int_{\bar{E}} f \leq \underline{\lim} \int_{\bar{E}} f_n$ and therefore

$$\int_E f_n = \int f_n - \int_{\bar{E}} f_n \quad \Rightarrow \quad \overline{\lim} \int_E f_n \leq \int f - \int_{\bar{E}} f = \int_{\bar{E}} f.$$

(2) and the inequality above together implies $\int_E f_n \to \int f$. □

4.4 The General Lebesgue Integral

12.

Proof. Note that $\langle g + f_n \rangle$ is a sequence of nonnegative measurable functions. Hence by Problem 8,

$$\int_E \underline{\lim}(g + f_n) \le \underline{\lim} \int_E (g + f_n) \quad \Rightarrow \quad \int_E \underline{\lim}\, f_n \le \underline{\lim} \int_E f_n.$$

The second inequality follows immediately from the definition of lower and upper limit. Replacing $g+f_n$ with $g-f_n$ and arguing in a similar manner gives the last inequality. $\square$

13.

Proof. $f_n \ge -h$ implies $f_n + h \ge 0$. Hence, $\int (f_n + h)$ always has a meaning. And since g is integrable, $\int f_n = \int (f_n + h) - \int h$ also has a meaning. Similarly, $\int f$ has a meaning. Meanwhile,

$$\int f = \int (f + h) - \int h \le \underline{\lim} \int (f_n + h) - \int h = \underline{\lim} \int f_n.$$

$\square$

15.

Proof.

(a) By Problem 4, for every $\varepsilon > 0$, there exists some simple functions $\varphi_1 \le f^+$ and $\varphi_2 \le f^-$ such that

$$\int_E f^+ - \int_E \varphi_1 < \varepsilon \quad \text{and} \quad \int_E f^- - \int_E \varphi_2 < \varepsilon.$$

Let $\varphi = \varphi_1 - \varphi_2$, which is also a simple function. Meanwhile,

$$\int_E |f - \varphi| \le \int_E (f^+ - \varphi_1) + \int_E (f^- - \varphi_2) < 2\varepsilon.$$

$\square$

16.

Proof. For every integrable f, by Problem 15, there exists some step function $\psi = \sum_{k=1}^N c_k \chi_{E_k}$ such that $\int |f - \psi| < \varepsilon$. Note that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \psi(x) \cos nx \mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{N} c_k \int_{E_k} \cos nx \mathrm{d}x = 0. \tag{3}$$

Hence,

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} f(x) \cos nx \mathrm{d}x \right| &= \left| \int_{-\infty}^{\infty} (f(x) - \psi(x) + \psi(x)) \cos nx \mathrm{d}x \right| \\
&\le \int_{-\infty}^{\infty} |f(x) - \psi(x)| |\cos nx| \mathrm{d}x + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx \mathrm{d}x \right| \\
&\le \varepsilon + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx \mathrm{d}x \right| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

□

18.

Proof. Let $\langle t_n \rangle$ be any sequence with $t_n \neq 0$ and tending to 0. Then $\langle f(x, t_n) \rangle$ is sequence of functions satisfying the hypotheses of Lebesgue Convergence Theorem. Meanwhile, $f(x, t_n) \to f$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \int f(x, t_n)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

Since the choice of $\langle t_n \rangle$ is arbitrary, by Problem 2.49f,

$$\lim_{t\to 0} \int f(x, t)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

If f is continuous in t for each x, then $\lim_{\Delta t\to 0} f(x, t + \Delta t) = f(x, t)$ holds for every t. Therefore, replacing t with Δt in the previous result yields

$$\lim_{\Delta t\to 0} \int f(x, t + \Delta t)\mathrm{d}x = \int f(x, t)\mathrm{d}x.$$

Namely, $h(t)$ is continuous. □

5 Differentiation and Integration

5.1 Differentiation of Monotone Functions

3. ”maximum” needs to be changed to ”minimum” in both (a) and (b).

Proof.
(a) We may assume without loss of generality that $c = 0$. Since f attains a local minimum at $x = 0$, $f(h) \geq f(0)$ for every h sufficiently small. Hence, for every small $h > 0$, $(f(c+h) - f(c))/h > 0$ and therefore $D_+f(c) \geq 0$. Meanwhile, by Problem 2.b,

$$-D_-f(0) = D^+f(0) \geq 0 \quad \Rightarrow \quad D_-f(0) \leq 0.$$

The other two inequalities follow immediately from the definitions of upper and lower limits.
(b) If f has a local minimum at a or b, then we only have the right or left half of the inequalities. ☐

4.

Proof. We first show this for g with $D^+g \geq \varepsilon > 0$. For every $a \leq x < y \leq b$, as g is continuous on $[a, b]$, g has a maximum in $[a, b]$ and by Problem 2 and 3, g can not attain the maximum in $[a, b)$. Namely, the restrict of f to $[x, y]$ attains the maximum at y. Hence, $g(x) \leq g(y)$.
For every f with nonnegative D^+, let $g(x) = f(x) + \varepsilon x$ where $\varepsilon > 0$. Then $D^+g \geq \varepsilon > 0$. Hence g is nondecreasing. Therefore, for every $a \leq x < y \leq b$,

$$g(x) \leq g(y) \quad \Rightarrow \quad f(x) + \varepsilon x \leq f(y) + \varepsilon y.$$

Since the choice of ε is arbitrary, this implies $f(x) \leq f(y)$. ☐

5.a

Proof.

$$\begin{aligned}\sup_{t\in(0,h)} \frac{(f+g)(x+t) - (f+g)(x)}{t} &= \sup_{t\in(0,h)} \left(\frac{f(x+t) - f(x)}{t} + \frac{g(x+t) - g(x)}{t}\right) \\ &\leq \sup_{t\in(0,h)} \frac{f(x+t) - f(x)}{t} + \sup_{t\in(0,h)} \frac{g(x+t) - g(x)}{t}.\end{aligned}$$

Letting $h \to 0$ yields $D^+(f+g) \leq D^+f + D^+g$. ☐

5.2 Functions of Bounded Variation

7.

Proof.
(a) It suffices to show this for monotone functions as each function of bounded variation is the difference of two monotone functions. Suppose that f is nondecreasing. Then the set $E = \{f(x) : x > c\}$ is bounded below and hence $A = \inf E$ is finite. For every $\varepsilon > 0$, there exists some $y > c$ such that $A \leq f(c) < A + \varepsilon$. Hence, as f is nondecreasing,

for every $x \in (c, y)$, $|f(x) - A| < \varepsilon$. Namely, $\lim_{x\to c+} f(x) = A$. Similarly, $\lim_{x\to c-} f(x)$ exists.

Let $D_n = \{x : |f(x+) - f(x-)| > 1/n\}$. Since f is nondecreasing, $|f(x) - f(y)| \le f(b) - f(a) < \infty$ for every $x, y \in [a, b]$. Hence, D_n is finite, otherwise we can choose a sequence $x_1 < \cdots < x_N$ with $N > (f(b) - f(a))/n$ such that $f(X_N) - f(x_1) > f(b) - f(a)$. Therefore, $\bigcup_{n=1}^{\infty} E_n$, the set of discontinuities, is countable.

(b) Suppose $\{x_1, \ldots, x_n, \ldots\} = \mathbb{Q} \cap [0, 1]$ and define $f(x) = \sum_{x_n<x} 2^{-n}$. Clear that f is monotone and continuous at every irrational point. For each rational $x = x_k$, $f(x+) - f(x-) = 2^{-n}$. Hence, f is discontinuous at each rational point. $\square$

8.

Proof.

(a) For every $\varepsilon > 0$, there exists some subdivision $a = x_0 < \cdots < x_p = c$ and $c = x_p < \cdots < x_q = b$ of $[a, c]$ and $[c, b]$ such that $T_a^c < t_a^c + \varepsilon$ and $T_c^b < t_c^b + \varepsilon$. Hence, $T_a^c + T_c^b - 2\varepsilon < t_a^c + t_c^b$. Meanwhile, as $a = x_0 < \cdots < x_q = b$ forms a subdivision of $[a, b]$, $T_a^b \ge t_a^b = t_a^c + t_c^b$. Therefore, $T_a^c + T_c^b - 2\varepsilon < T_a^b$. Since the choice of ε is arbitrary, $T_a^b + T_c^b \le T_a^b$.

To show that $T_a^b + T_c^b \ge T_a^b$, let $a = x_0 < \cdots < x_q = b$ be any subdivision of $[a, b]$ and by adding c into it, we get subdivisions of $[a, c]$ and $[c, b]$. Suppose that $c \in (x_k, x_{k+1}]$, then

$$|f(x_k) - f(c)| + |f(c) - f(x_{k+1})| + t_a^b = t_a^c + t_c^b + |f(x_k) - f(x_{k+1})|,$$

which implies $t_a^b \le t_a^c + t_c^b$. Hence,

$$T_a^b = \sup t_a^b \le \sup(t_a^c + t_c^b) \le T_a^c + T_c^b.$$

Thus, $T_a^b = T_a^c + T_c^b$ and therefore $T_a^c \le T_a^b$.

(b)

$$\begin{aligned}
T_a^b(f+g) &= \sup \sum_{i=1}^{k} |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\
&\le \sup \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \sup \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| \\
&\le T_a^b(f) + T_a^b(g).
\end{aligned}$$

$$T_a^b(cf) = \sup \sum_{i=1}^{k} |cf(x_i - cf(x_{i-1})| = |c| \sup \sum_{i=1}^{k} |f(x_i - f(x_{i-1})| = |c| T_a^b(f).$$

$\square$

9.

Proof. For every $\varepsilon > 0$, there exists a subdivision $a = x_0 < \cdots < x_k = b$ such that $t_a^b(f) \geq T_a^b(f) - \varepsilon$. Meanwhile, as f_n converges to f pointwisely

$$\begin{aligned}
t_a^b(f) &= t_a^b(\lim f_n) \\
&= \sum_{i=1}^{k} |(\lim f_n)(x_i) - (\lim f_n)(x_{i-1})| \\
&= \lim \sum_{i=1}^{k} |f_n(x_i) - f_n(x_{i-1})| \\
&\leq \underline{\lim} \sup \sum_{i=1}^{k} |f_n(x_i) - f_n(x_{i-1})| = \underline{\lim}\, T_a^b(f_n).
\end{aligned}$$

Hence, $T_a^b(f) - \varepsilon \leq \underline{\lim}\, T_a^b(f_n)$. Since the choice of ε is arbitrary, $T_a^b(f) \leq \underline{\lim}\, T_a^b(f_n)$. □

10.a

Solution. No. Let $x_k = (k\pi + \pi/2)^{-1/2}$, $k = 0, 1, \ldots$ and consider the subdivision $-1 < 0 < x_n < \cdots < x_0 < 1$. Then

$$t_n \geq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \geq \sum_{k=1}^{n} \frac{2}{(k + 1/2)\pi}.$$

$t_n \to \infty$ as $n \to \infty$ and therefore f is not of bounded variation on $[-1, 1]$. □

11.

Proof. By Lemma 4, $f(x) = f(a) + P_a^x - N_a^x$. Since P_a^x and N_a^x are monotone, by Theorem 3, they are differentiable almost everywhere as f, a function of bounded variation, does. Hence, for almost every $x \in [a, b]$,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} P_a^x - \frac{\mathrm{d}}{\mathrm{d}x} N_a^x \quad \Rightarrow \quad |f'(x)| \leq \frac{\mathrm{d}}{\mathrm{d}x} P_a^x + \frac{\mathrm{d}}{\mathrm{d}x} N_a^x = \frac{\mathrm{d}}{\mathrm{d}x} T_a^x.$$

Integrating on the both sides yields $\int_a^b |f'| \leq T_a^b(f)$. □

5.4 Absolute Continuity

12.

Solution. The continuous extension of $x^2 \sin(1/x^2)$ to $[0, 1]$ is absolutely continuous for all $[\varepsilon, 1]$ but is not of bounded variation on $[0, 1]$ and therefore is not absolutely continuous on $[0, 1]$.

Suppose that f is also of bounded variation on $[0, 1]$. Then f is differentiable almost everywhere. Hence $g(x) = \int_0^x f'(t)\mathrm{d}t + f(a)$ is well-defined. For every $\varepsilon > 0$, we have

$$g(x) = \int_0^\varepsilon f'(t)\mathrm{d}t + \int_\varepsilon^x f'(t)\mathrm{d}t + f(0) = \int_0^\varepsilon f'(t)\mathrm{d}t + f(x) - f(\varepsilon) + f(0),$$

where the second equality comes from the absolute continuity on $[\varepsilon, 1]$. By the continuity of f at $x = 0$, $f(\varepsilon) \to f(0)$. Hence, letting $\varepsilon \to 0$ yields $g(x) = f(x)$. Namely, f is an indefinite integral. Thus, by Theorem 14, it is absolutely continuous. □

13.

Proof. Since absolute continuity implies bounded variation, $\int_a^b |f'| \leq T_a^b(f)$ by Problem 11. By the definiton of T, for every $\varepsilon > 0$, there exists some subdivision $a = x_0 < \cdots < x_n = b$ such that $t_a^b(f) > T_a^b(f) - \varepsilon$. Meanwhile, for every $i = 1, \ldots, n$,

$$\int_{x_{i-1}}^{x_i} |f'| \geq \left| \int_{x_{i-1}}^{x_i} f' \right| = |f(x_i) - f(x_{i-1})|,$$

where the second equality is guaranteed by the absolute continuity. Hence, $\int_a^b |f'| > T_a^b(f) - \varepsilon$ for every $\varepsilon > 0$. Thus, $T_a^b(f) = \int_a^b |f'|$.

By Lemma 4, $2P_a^b(f) = T_a^b(f) + f(b) - f(a)$. Hence,

$$P_a^b(f) = \frac{1}{2}\left(\int_a^b |f'| + f(b) - f(a)\right) = \frac{1}{2}\int_a^b (|f'| + f') = \int_a^b [f']^+.$$

□

14.

Proof.

(a) Suppose that f and g are absolutely continuous. Then for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for all finite nonoverlapping $\langle (x_n, y_n) \rangle$ with $|x_n - y_n| < \varepsilon$,

$$\sum |f(x_n) + g(x_n) - f(y_n) - g(y_n)| \leq \sum |f(x_n) - f(y_n)| + |g(x_n) - g(y_n)| \leq 2\varepsilon.$$

Hence, $f + g$ is also absolutely continuous. Since $-g$ is absolutely continuous as long as g is, so is $f - g$.

(b) Suppose that f and g are absolutely continuous. Then they are bounded, by M for example. Hence for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for all finite nonoverlapping $\langle (x_n, y_n) \rangle$ with $|x_n - y_n| < \varepsilon$,

$$\begin{aligned}
&\sum |f(x_n)g(x_n) - f(y_n)g(y_n)| \\
=&\sum |f(x_n)g(x_n) - f(x_n)g(y_n) + f(x_n)g(y_n) - f(y_n)g(y_n)| \\
\leq&\sum \{|f(x_n)||g(x_n) - g(y_n)| + |f(x_n) - f(y_n)||g(y_n)|\} \\
\leq& M\varepsilon.
\end{aligned}$$

Thus, fg is also absolutely continuous.

(c) Since f is continuous on $[a, b]$, f can achieve its minimum in $[a, b]$. Hence, $|f(x)| \geq m > 0$ as f is never zero. Therefore for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for all finite nonoverlapping $\langle (x_n, y_n) \rangle$ with $|x_n - y_n| < \varepsilon$,

$$\sum \left| \frac{1}{f(x_n)} - \frac{1}{f(y_n)} \right| = \sum \left| \frac{f(x_n) - f(y_n)}{f(x_n)f(y_n)} \right| \leq \frac{1}{m^2} \sum |f(x_n) - f(y_n)| \leq \frac{\varepsilon}{m^2}.$$

□

17. Part (a) is wrong. It can be fixed if we further require g to be monotone increasing.

Proof.

(a) For every $\varepsilon > 0$, let δ_1 be the number in the definition of F corresponding to ε and δ_2 the number in the definition of g corresponding to δ_1. Then for every finite nonoverlapping $\langle (x_n, y_n) \rangle$ with $|x_n - y_n| < \delta_2$, $\sum |g(x_n) - g(y_n)| < \delta_1$. Since g is monotone increasing, $(g(x_n), g(y_n))$ are nonoverlapping. Therefore, $\sum |F(g(x_n)) - F(g(y_n))| < \varepsilon$. Hence, $F \circ g$ is absolutely continuous. □

18.

Proof. Without loss of generality, we assume that g is nondecreasing. Since $mE = 0$, for every $\varepsilon > 0$, by Proposition 3.15, there exists an open set $O \supset E$ with $mO < \varepsilon$. Meanwhile, there exists a sequence of disjoint open intervals $\langle I_n = (a_n, b_n) \rangle$ such that $\bigcup_{n=1}^{\infty} I_n = O$ and $l(I_n) < \delta$ where δ is the number in the definition of absolute continuity. Then $g[E] \subset \bigcup_{n=1}^{\infty} g[I_n \cap [0,1]]$. Since g is continuous, the image of an interval is still an interval and since g is also nondecreasing, $g[I_n \cap [0,1]] = (g(a_n'), g(b_n'))$, where $a_n' = \max\{a_n, 0\}$ and $b_n' = \min\{b_n, 1\}$. Finally,

$$m(g[E]) \leq \sum_{n=1}^{\infty} m(g[I_n]) = \sum_{n=1}^{\infty} |g(b_n') - g(a_n')| \leq \varepsilon,$$

where the last inequality comes from the absolute continuity of g. Since the choice of ε is arbitrary, $m(g[E]) = 0$. □

20.

Proof.

(a) For every $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then for every $\langle x_n \rangle_{i=1}^{n}$ and $\langle y_n \rangle_{i=1}^{n}$ with $|x_n - y_n| \leq \delta$,

$$\sum_{i=1}^{n} |f(x_n) - f(y_n)| \leq M \sum_{i=1}^{n} |x_n - y_n| \leq \varepsilon,$$

as f satisfies the Lipschitz condition.

(b) Suppose that f is absolute continuous and $|f'|$ is bounded by M. Then for every x and y in the interval,

$$|f(x) - f(y)| = \left| \int_x^y f'(t) \mathrm{d}t \right| \leq M|x - y|.$$

Hence, f satisfies the Lipschitz condition. The converse part has been proved in (a).

(c) It is wrong. A counterexample is $f(x) = \chi_{[0,1]}$, $x \in (-1, 1)$ □

21.

Proof.

(a) Suppose that $O = \bigcup_{n=1}^{\infty} (c_n, d_n)$ where (c_n, d_n) are disjoint. Since g is continuous and increasing, $g^{-1}(c_n, d_n)$ is still an open interval, denoting it by (a_n, b_n), and (a_n, b_n) are also disjoint. Meanwhile, $d_n - c_n = f(a_n) - f(b_n) = \int_{a_n}^{b_n} g'$. Hence,

$$mO = m\left(\bigcup_{n=1}^{\infty} (c_n, d_n) \right) = \sum_{n=1}^{\infty} (d_n - c_n) = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} g' = \int_{g^{-1}[O]} g'.$$

(b) Without loss of generality, we assume that $d \notin E$. For every $\varepsilon > 0$, there exists an open set $O \supset E$ with $mO < \varepsilon$. By Part (a),

$$\int_{g^{-1}[O]\cap H} g' = \int_{g^{-1}[O]} g' = mO < \varepsilon.$$

Since the choice of ε is arbitrary, $\int_{g^{-1}[O]\cap H} g' = 0$. Since $g' > 0$ on H, $g^{-1}[O] \cap H$ has measure zero.

(c) Since E is measurable, so is $g^{-1}[E]$. Meanwhile, by Theorem 3, g' is measurable, hence H is also measurable. Therefore, F is measurable.

We may assume without loss of generality that $c, d \notin E$. By Proposition 3.15, there exists some $G \in G_\delta$ such that $E \subset G \subset (c, d)$ and $m(G \setminus E) = 0$. Since g is increasing, $g^{-1}[G] \cap H = F \cup (g^{-1}[G \setminus E] \cap H)$ and by (b), $g[G \setminus E] \cap H$ is of measure zero. Therefore, $\int_F g' = \int_{g^{-1}[G]\cap H} g'$. Namely, it suffices to show the result for $G \in G_\delta$.

Suppose that $G = \bigcap_{n=1}^\infty O_n$ where each $O_n \subset (c, d)$ is open and $mO_1 < \infty$. Without loss of generality, we may assume that $\langle O_n \rangle$ is decreasing. Then $mG = \lim_{n\to\infty} mO_n$. By (a),

$$mO_n = \int_{g^{-1}[O_n]} g' = \int_a^b \chi_{O_n}(g(x))g'(x)\mathrm{d}x.$$

As $\chi_{O_n}(g(x))g'(x)$ is bounded by $|g'|$,

$$\lim_{n\to\infty} \int_a^b \chi_{O_n}(g(x))g'(x)\mathrm{d}x = \int_a^b \chi_G(g(x))g'(x)\mathrm{d}x.$$

Hence, $mG = \int_{g^{-1}[G]\cap H} g'$, completing the proof.

(d) By Problem 3.25, $f \circ g$ is measurable. And since g' is measurable by Theorem 3, $(f \circ g)g'$ is also measurable.

Let $\langle \varphi_n \rangle$ be an increasing sequence of nonnegative simple functions which converges to f, the existence of which is guaranteed by Problem 4.4. By the monotone convergence theorem, $\int_c^d f = \lim \int_c^d \varphi_n$.

For each n, suppose that $\varphi_n(y) = \sum_{k=1}^m a_k^{(n)}(y)\chi_{E_k^{(n)}}(y)$. Then

$$\int_c^d \varphi_n = \sum_{k=1}^m a_k^{(n)} mE_k^{(n)} = \sum_{k=1}^m a_k^{(n)} \int_a^b \chi_{E_k^{(n)}}(g(x))g'(x)\mathrm{d}x = \int_a^b \varphi_n(g(x))g'(x)\mathrm{d}x,$$

where the second equality comes from (c). Since g is increasing, $\langle \varphi_n(g(x))g'(x) \rangle$ is an increasing sequence. Hence,

$$\int_a^b f(g(x))g'(x)\mathrm{d}x = \lim_{n\to\infty} \int_a^b \varphi_n(g(x))g'(x)\mathrm{d}x.$$

Thus,

$$\int_c^d f(y)\mathrm{d}y = \lim_{n\to\infty} \int_c^d \varphi_n(y)\mathrm{d}y = \lim_{n\to\infty} \int_a^b \varphi_n(g(x))g'(x)\mathrm{d}x = \int_a^b f(g(x))g'(x)\mathrm{d}x.$$

□

5.5 Convex Functions

23.

Proof.
(a) Suppose that $x_0 \in (a,b)$ and $y(x) = m(x - x_0) + \varphi(x_0)$ is a supporting line. As $[a,b)$ is finite, $\varphi \geq \min\{\varphi(a), y(a), y(b)\}$.

(b) If φ is monotone, then the limits exists. If φ is not monotone, then since $D^+\varphi$ is nondecreasing, there exists some $[c,d] \subset (a,b)$ such that $D^+\varphi \leq 0$ on (a,c) and $D^+\varphi \geq 0$ on (d,b). Namely, φ is monotone on the (a,c) and (d,b). Therefore, the limits also exist.

Consider a finite interval near the finite endpoint. By (a), the limit can not be $-\infty$ as φ is bounded from below.

(c) If x and y are in the interior of I, the inequality holds by definition. By the continuity of φ, the statement holds for all $x, y \in I$. □

24.

Proof. Note that the existence of φ'' implies φ is continuously differentiable. Suppose that φ is convex on (a,b). Then $D^+\varphi$ is nondecreasing by Proposition 17, hence $\varphi''(x) \geq 0$ for each $x \in (a,b)$. And the converse of the statement follows from Proposition 18 immediately. □

25.

Proof.
(a) $\varphi''(t) = b^2p(p-1)(a+bt)^{p-2}$ which ≥ 0 on $[0,\infty)$ if $p \geq 1$ and ≤ 0 if $0 < p \leq 1$. □

26. TODO

27.

Proof. Note that $\log x$ is concave. Denote $A_N = \sum_{n=1}^{N} \alpha_n$ and $R_N = 1 - A_N$. The situition where $\langle \alpha_n \rangle$ is finite is simple. Hence we assume that $R_N \geq 0$ for all N. Then for every N,

$$\begin{aligned}
\log\left(\sum_{n=1}^{\infty} \alpha_n \xi_n\right) &= \log\left(A_N \sum_{n=1}^{N} \frac{\alpha_n}{A_N}\xi_n + R_N \sum_{n=N+1}^{\infty} \frac{\alpha_n}{R_N}\xi_n\right) \\
&\geq A_N \log\left(\sum_{n=1}^{N} \frac{\alpha_n}{A_N}\xi_n\right) + R_N \log\left(\sum_{n=N+1}^{\infty} \frac{\alpha_n}{R_N}\xi_n\right) \\
&\geq A_N \log\left(\sum_{n=1}^{N} \frac{\alpha_n}{A_N}\xi_n\right) \\
&\geq A_N \log\left(\prod_{n=1}^{N} \xi_n^{\alpha_n/A_N}\right)
\end{aligned}$$

Taking exp on the both sides yields

$$\sum_{n=1}^{\infty} \alpha_n \xi_n \geq \left(\prod_{n=1}^{N} \xi_n^{\alpha_n/A_N}\right)^{A_N} = \prod_{n=1}^{N} \xi_n^{\alpha_n} \to \prod_{n=1}^{\infty} \xi_n^{\alpha_n}.$$

□

28.

Proof. It follows immediately from the Jensen inequality and the fact that log is concave. $\square$

6 The Classical Banach Spaces

6.1 The L^p Spaces

1.

Proof. Put $S = \|f\|_\infty$ and $T = \|g\|_\infty$. Then $|f(t)| \leq S$ and $|g(t)| \leq T$ a.e. Hence, $S + T \geq |f(t)| + |g(t)| \geq |f(t) + g(t)|$ a.e. Namely, $m\{t : |f(t) + g(t)| > S + T\} = 0$. Thus, $S + T \geq \|f + g\|_\infty$ by the definition of $\operatorname{ess\,sup}$. $\square$

2.

Proof. Put $S = \|f\|_\infty$. Since $S \geq |f|$ a.e.,

$$\|f\|_p = \left\{ \int_0^1 |f|^p \right\}^{1/p} \leq \left\{ \int_0^1 S^p \right\}^{1/p} = S.$$

Therefore, $\overline{\lim}_{p\to\infty} \|f\|_p \leq S$. For the converse part, let ε be any positive number. Then the measure δ of $E = \{t : |f(t)| > S - \varepsilon\}$ is positive. Hence,

$$\left\{ \int_0^1 |f|^p \right\}^{1/p} \geq \left\{ \int_E |f|^p \right\}^{1/p} \geq \delta^{1/p}(S - \varepsilon) \to S - \varepsilon \quad \text{as } p \to \infty.$$

Hence, $\underline{\lim}_{p\to\infty} \geq S$, completing the proof. $\square$

3.

Proof.

$$\|f + g\|_1 = \int |f + g| \leq \int |f| + \int |g| = \|f\|_1 + \|g\|_1.$$

$\square$

4.

Proof. For every $M > \|g\|_\infty$, $|g| \leq M$ a.e. Hence,

$$\int |fg| \leq M \int |f| = \|f\|_1 M.$$

Since the choice of M is arbitrary, $\int |fg| \leq \|f\|_1 \|g\|_\infty$. $\square$

6.2 The Minkowski and Hölder Inequalities

8

Proof.
(a) The logarithm function is concave, so

$$\log(a^p/p + b^q/q) \geq \frac{1}{p} \log a^p + \frac{1}{q} \log b^q = \log ab.$$

Taking exp on the both sides yields the inequality. The equality holds iff $a^p = b^q$.

(b) The case where $p = \infty$ has been proved in Problem 4 and the case where $\|f\|_p = 0$ or $\|g\| = 0$ is straightforward. Hence, we assume that $1 < p, q < \infty$ and $\|f\|_p \|g\|_q \neq 0$.

Suppose $\alpha = \|f\|_p$ and $\beta = \|g\|_q$. By Young's inequality,

$$\left| \frac{fg}{\alpha\beta} \right| \le \frac{1}{p} \left(\frac{|f|}{\alpha} \right)^p + \frac{1}{q} \left(\frac{|g|}{\beta} \right)^q$$

for every x. Therefore,

$$\int |fg| = \alpha\beta \int \left| \frac{fg}{\alpha\beta} \right| \le \alpha\beta \int \left\{ \frac{1}{p} \left(\frac{|f|}{\alpha} \right)^p + \frac{1}{q} \left(\frac{|g|}{\beta} \right)^q \right\} = \alpha\beta. \tag{4}$$

The equality holds iff the equality in Young's inequality holds a.e. iff $\beta|f|^p = \alpha|g|^q$ a.e.

(c) Let $p' = 1/p$ and $q' = 1 - p' = -q/p$. Then for any nonnegative c and d, by Young's inequality,

$$cd \le \frac{c^{p'}}{p'} + \frac{d^{q'}}{q'} = pc^{1/p} - \frac{p}{q} d^{-q/p} \quad \Rightarrow \quad c^{1/p} \ge \frac{cd}{p} + \frac{d^{-q/p}}{q}.$$

Putting $c = (ab)^p$ and $d = b^{-p}$ yields the desired inequality.

(d) Just reverse the inequality in (4). □

6.3 Convergence and Completeness

9.

Proof. Suppose $\langle f_n \rangle \subset X$ converges to $f \in X$. Namely, for every $\varepsilon > 0$, there exists some N such that for all $n > N$, $\|f_n - f\| < \varepsilon$. Hence, for every $n, m > N$, by Minkowski inequality,

$$\|f_n - f_m\| \le \|f_n - f\| + \|f - f_m\| < 2\varepsilon.$$

Hence, $\langle f_n \rangle$ is a Cauchy sequence. □

10.

Proof. Suppose $f_n \to f$. Then $M_n = \|f_n - f\|_\infty = \operatorname{ess\,sup} |f_n - f| \to 0$. Let $E_n = \{x : |f_n(x) - f(x)| > M_m\}$, each of which is with measure zero. And therefore $E = \bigcup_{n=1}^\infty E_n$ is with measure zero. Note that $\tilde{E} = \{x : |f_n(x) - f(x)| < M_n, \forall n\}$, which implies the uniform convergence of f_n since $M_n \to 0$.

For the converse part, the uniform convergence on $\tilde{E}$ implies that for every $\varepsilon > 0$, there exists some N such that for every $n > N$ and $x \in \tilde{E}$, $|f_n(x) - f(x)| < \varepsilon$. Since $mE = 0$, this implies $\|f_n - f\|_\infty = \operatorname{ess\,sup} |f_n(x) - f(x)| < \varepsilon$. Hence, $f_n \to f$ in L^∞. □

11.

Proof. Let $\langle f_n \rangle \subset L^\infty$ be absolutely summable. Put $M_n = \|f_n\|_\infty$ and $A_n = \{t : |f_n(t)| > M_n\}$. By the definition of $\| \cdot \|_\infty$, $mA_n = 0$. Hence, $A = \bigcup_{n=1}^\infty A_n$ is of measure zero.

Note that $|f_n(x)| \le M_n$ for every n and $x \in E \setminus A$. Thus, by the Weierstrass M-test, $\sum_{n=1}^\infty f_n$ converges uniformly. Hence, on $E \setminus A$, $\sup |\sum_{n=1}^\infty f_n - \sum_{n=1}^N f_n| \to 0$ as $N \to \infty$. Since $mA = 0$, this implies the summability of $\langle f_n \rangle$. □

13.

Proof. Suppose $\langle f_n \rangle \subset C$ be absolutely summable. Since for every x, $0 \le |f_n(x)| \le \|f_n\|$, $\langle f_n \rangle$ is uniformly convergent on $[0,1]$. Put $s = \sum_{n=1}^{\infty} f_n$. Since each f_n is continuous, so is s. Therefore, $s \in C$.

For every $\varepsilon > 0$, there exists some N such that for every $n > N$ and $x \in [0,1]$, $|s(x) - \sum_{k=1}^{n} f_k(x)| < \varepsilon$. Hence, $\|s - \sum_{k=1}^{n} f_k\| < \varepsilon$. Thus, $\langle f_n \rangle$ is summable and therefore C is a Banach space. □

16.

Proof. Since $\|f_n - f\| \ge |\|f_n\| - \|f\||$, $f_n \to f$ in L^p implies $\|f_n\| \to \|f\|$. For the converse part, note that $2^p(|f_n|^p + |f|^p) - |f_n - f|^p \ge 0$ and for almost every x,

$$2^p(|f_n|^p + |f|^p) - |f_n - f|^p \to 2^{p+1}|f|^p.$$

By Fatou's Lemma,

$$\begin{aligned} 2^{p+1}\|f\|^p = 2^{p+1}\int |f|^p &\le \underline{\lim} \int \{2^p(|f_n|^p + |f|^p) - |f_n - f|^p\} \\ &= 2^{p+1}\|f\|^p - \overline{\lim}\, \|f_n - f\|^p. \end{aligned}$$

Hence, $\overline{\lim}\,\|f_n - f\|^p \le 0$. Since clear that $\underline{\lim}\,\|f_n - f\|^p \ge 0$, $\lim \|f_n - f\| = 0$, i.e., $f_n \to f$ in L^p. □

17. I assume that $1/p + 1/q = 1$.

Proof. Since $g \in L^p$, $|g|^q$ is integrable on $E = [0,1]$ and therefore for every $\varepsilon > 0$, there exists some δ such that for every $A \subset E$ with $mA < \delta$, $\int_A |g|^q < \varepsilon$. Meanwhile, since $f_n(x) \to f(x)$ for almost every x, by Egoroff's Theorem, there exists some $A \subset E$ with $mA < \delta$ such that $f_n g$ converges to fg uniformly on $E \setminus A$.

From the uniform convergence we conclude

$$\int_{E\setminus A} fg = \lim_{n\to\infty} \int_{E\setminus A} f_n g. \tag{5}$$

Meanwhile, by Hölder inequality,

$$\left|\int_A (f - f_n)g\right| \le \int_A |(f - f_n)g| \le \left\{\int_A |f_n - f|^p\right\}^{1/p} \left\{\int_A |g|^q\right\}^{1/q} \le M\varepsilon^{1/q}.$$

Hence, (5) can be extended to E.

For $p = 1$, this is not true. $f_n = n\chi_{[0,1/n]}$ and $g = \chi_{[0,1]}$ gives a counterexample. □

18.

Proof. By Minkowski inequality,

$$\|g_n f_n - gf\| = \|g_n(f_n - f) + (g_n - g)f\| \le \|g_n(f_n - f)\| + \|(g_n - g)f\|.$$

Fix $\varepsilon > 0$. Since $f, g_n, g \in L^p$, $|g_n - g|^p|f|^p$ is integrable and therefore there exists some $\delta > 0$ such that for all subsets with measure $< \delta$, the integral of over it $< \varepsilon$. Meanwhile, since $g_n \to g$ a.e., by Egoroff's Theorem, there exists some $A \subset E = [0,1]$ with $mA < \delta$ such that $g_n \to g$ uniformly on $E \setminus A$ and therefore there exists some $N_1 > 0$ such that for all $n > N_1$, $|g_n(x) - g(x)|^p < \varepsilon$ for $x \in E \setminus A$. Thus, for every $n > N_1$,

$$\begin{aligned}\|(g_n - g)f\| &= \left\{\int_{E\setminus A} |g_n - g|^p |f|^p\right\}^{1/p} + \left\{\int_A |g_n - g|^p |f|^p\right\}^{1/p} \\ &\leq \sqrt[p]{\varepsilon}\|f\| + \sqrt[p]{\varepsilon} \leq (\|f\| + 1)\varepsilon.\end{aligned}$$

Since $|g_n| \leq M$, $\|g_n(f_n - f)\| \leq M\|f_n - f\|$. And since $f_n \to f$ in L^p, there exists some $N_2 > 0$ such that for all $n > N_2$, $\|f_n - f\| < \varepsilon$. Put $N = \max(N_1, N_2)$, then for every $n > N$,

$$\|g_n f_n - gf\| \leq (\|f\| + 1 + M)\varepsilon.$$

Hence, $g_n f_n \to gf$ in L^p. □

6.4 Approximation in L^p

19.

Proof. Since $\|T_\Delta f\| \leq \|T_\Delta |f|\|$ and $\|f\| = \||f|\|$, we may assume without loss of generality that $f \geq 0$. For $p > 1$, by Jensen's inequality,

$$\begin{aligned}\|T_\Delta f\|_p^p &= \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} \left(\frac{1}{\xi_k - \xi_{k-1}} \int_{\xi_{k=1}}^{\xi_k} f\right)^p \\ &\leq \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} \frac{1}{\xi_k - \xi_{k-1}} \int_{\xi_{k-1}}^{\xi_k} f^p \\ &= \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} f^p \\ &= \int_0^1 f^p = \|f\|_p^p.\end{aligned}$$

□

11 Measure and Integration

11.1 Measure Spaces

1.

Proof. Put $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. (B_n) is a sequence of disjoint measurable sets. By the countable additivity of μ,

$$\mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} B_k\right).$$

Since $\bigcup_{k=1}^n B_k = \bigcup_{k=1}^n A_k$ for $k = 1, \ldots, n, \ldots, \infty$, this implies $\mu(\bigcup A_k) = \lim \mu(\bigcup_{k=1}^n A_k)$. □

3.

Proof.
(a) First,

$$0 = \mu(E_1 \triangle E_2) = \mu(E_1 \setminus E_2 \cup E_2 \setminus E_1) = \mu(E_1 \setminus E_2) + \mu(E_2 \setminus E_1).$$

Together with the nonnegativity of μ, we conclude that $\mu(E_1 \setminus E_2) = \mu(E_2 \setminus E_1) = 0$. Note that

$$\mu(E_1 \cup E_2) = \mu(E_1 \setminus E_2 \cup E_2) = \mu(E_1 \setminus E_2) + \mu(E_2).$$

Hence, $\mu(E_1 \cup E_2) = \mu(E_2)$. Similarly, $\mu(E_1 \cup E_2) = \mu(E_1)$. Thus, $\mu(E_1) = \mu(E_2)$.

(b) Since $\mu(E_1 \triangle E_2) = 0$ and $E_2 \setminus E_1 \subset E_1 \triangle E_2$, by the completeness of μ, $E_2 \setminus E_1 \in \mathcal{B}$. Similarly, $E_1 \setminus E_2 \in \mathcal{B}$. In consequence, $E_1 \cap E_2 = E_1 \setminus (E_1 \setminus E_2) \in \mathcal{B}$ and, therefore, $E_2 = (E_1 \cap E_2) \cup (E_2 \setminus E_1) \in \mathcal{B}$. □

7.

Proof. Let $\mathcal{B}_0$ be the collection of all sets $E = A \cup B$ where $B \in \mathcal{B}$ and $A \subset C$, $C \in \mathcal{B}$, $\mu C = 0$. Clear that $\mathcal{B} \subset \mathcal{B}_0$. Now we show that it is a σ-algebra. Since $X \in \mathcal{B}$, $X \in \mathcal{B}_0$. Let $E_n = A_n \cup B_n$ be a sequence of elements of $\mathcal{B}_0$. Then, $\bigcup E_n = (\bigcup A_n) \cup (\bigcup B_n)$ also belongs to $\mathcal{B}_0$ since $\bigcup B_n \in \mathcal{B}$ and $\bigcup A_n \subset \bigcup C_n$, which is a countable union of sets of measure zero. Hence, $\mathcal{B}_0$ is closed under countable union. Now, let $E = A \cup B \in \mathcal{B}_0$. Note that

$$E^c = A^c \cap B^c = (C \setminus A) \cup (B^c \setminus C),$$

where $C \setminus A \subset C$ and $B^c \setminus C \in \mathcal{B}$. Hence, $\mathcal{B}_0$ is closed under complement. Thus, it is a σ-algebra.

We define $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ by $\mu_0 E = \mu_0(A \cup B) = \mu B$. First, we show that it is well-defined, that is, if $E = A' \cup B'$, then $\mu B = \mu B'$. Since $C \in \mathcal{B}$ contains A, $(A \cup B) \setminus C \in \mathcal{B}$. Meanwhile, since $\mu C = 0$,

$$\mu B = \mu((A \cup B) \setminus C) = \mu(E \setminus C). \tag{6}$$

Since $E \setminus C \subset E \cup C'$,

$$\mu(E \setminus C) \leq \mu(E \cup C') = \mu((A' \cup B') \cup C') = \mu B', \tag{7}$$

where the measurability of $E \cup C'$ and the last equality both comes from the fact that $A' \subset C' \in \mathcal{B}$ and $\mu C' = 0$. Combine (6) and (7) and we get $\mu B \leq \mu B'$. Interchanging the role of $A \cup B$ and $A' \cup B'$ yields $\mu B \geq \mu B'$. Hence, $\mu B = \mu B'$ and, in consequence, μ_0 is well-defined. Meanwhile, clear that for $E \in \mathcal{B}$, $\mu E = \mu_0 E$.

Finally, we show that μ_0 is a measure. Clear that μ_0 is nonnegative and $\mu_0 \varnothing = 0$. Let $\langle E_n \rangle \subset \mathcal{B}_0$ be a sequence of disjoint sets. Then

$$\mu_0 \left(\bigcup E_n \right) = \mu_0 \left(\bigcup A_n \cup \bigcup B_n \right) = \mu \left(\bigcup B_n \right) = \sum \mu B_n = \sum \mu_0 E_n.$$

Namely, μ_0 is countably additive. Thus, μ_0 is a measure. □

9.

Proof.

(a) First, we argue by contradiction to show that $\mathcal{R}$ and $\mathcal{R}$. Assume that there exists some $E \in \mathcal{R} \cap \mathcal{R}'$, that is, $E \in \mathcal{R}$ and $E^c \in \mathcal{R}$. Then $X = E \cup E^c \in \mathcal{R}$, which contradicts the assumption that $\mathcal{R}$ is not a σ-algebra. Thus, $\mathcal{R} \cap \mathcal{R}' = \varnothing$.

Clear that $\mathcal{R} \cup \mathcal{R}'$ is a σ-algebra containing $\mathcal{R}$. Hence, $\mathcal{R} \cup \mathcal{R}' \supset \mathcal{B}$. Meanwhile, since $\mathcal{B} = \sigma(\mathcal{R})$, $\mathcal{R} \cup \mathcal{R}' \subset \mathcal{B}$. Thus, $\mathcal{R} \cup \mathcal{R}' = \mathcal{B}$.

(b) Since $\varnothing \in \mathcal{R}$, $\bar{\mu}\varnothing = \mu\varnothing = 0$. Meanwhile, clear that $\bar{\mu}$ is nonnegative. Let $(E_n) \subset \mathcal{B}$ be a sequence of disjoint sets. By part (a), each E_n is either an element of $\mathcal{R}$ or $\mathcal{R}'$. If all $E_n \in \mathcal{R}$, then by the countable additivity of μ, $\mu(\bigcup E_n) = \sum \mu E_n$. Suppose there exists some E_n in $\mathcal{R}$ and some E_m in $\mathcal{R}'$. Let F_1 and F_2 be the union of theses sets respectively. Since σ-ring is closed under union, $F_1 \in \mathcal{R}$, and since $(\bigcup E_m)^c = \bigcap E_m^c$, $F_2 \in \mathcal{R}'$. Hence, $F_1 \cup F_2 \in \mathcal{R}'$, otherwise, $F_2 = (F_1 \cup F_2) \setminus F_1$ would be an element of $\mathcal{R}$. Therefore, $\mu(\bigcup E_n) = \infty = \sum \mu E_n$. Thus, $\bar{\mu}$ is a measure on $\mathcal{B}$.

(c) Clear that $\underline{\mu}$ is nonnegative and $\underline{\mu}\varnothing = 0$. Let $(E_n) \subset \mathcal{B}$ be disjoint. Note that for $E \in \mathcal{R}$, $\mu E = \sup\{\mu A : A \subset E, A \in \mathcal{R}\}$. Hence, it suffices to show that

$$M = \sup \left\{ \mu A : A \subset \bigcup_n E_n, A \in \mathcal{R} \right\} = \sum_n \sup\{\mu A : A \subset E_n, A \in \mathcal{R}\} = \sum_n M_n.$$

By definition, for all $\varepsilon > 0$, there exists a sequence $(A_n) \subset \mathcal{R}$ such that $A_n \subset E_n$ and $M_n < \mu A_n + \varepsilon/2^n$. Put $A = \bigcup A_n$. Since (A_n) are disjoint as (E_n) are,

$$\sum M_n < \varepsilon + \sum \mu A_n = \varepsilon + \mu A.$$

Meanwhile, since $A \subset \bigcup E_n$ and $A \in \mathcal{R}$, $\mu A \leq M$. Therefore, $\sum M_n < \varepsilon + M$. Thus, $\sum M_n \leq M$.

For the converse, similarly, for every $\varepsilon > 0$, there exists an $A \in \mathcal{R}$ such that $A \subset \bigcup E_n$ and $M - \varepsilon > \mu A$. Put $A_n = E_n \cap A$. If $E_n \in \mathcal{R}$, $A_n \in \mathcal{R}$ by definition. If $E_n \in \mathcal{R}'$, $A_n = A \setminus E_n^c \in \mathcal{R}$. Hence, $A_n \in \mathcal{R}$ for each n. Thus,

$$M - \varepsilon < \mu A = \sum_n \mu A_n \leq \sum_n M_n,$$

implying that $M \leq \sum M_n$. Therefore, $M = \sum M_n$, i.e., $\underline{\mu}$ is countably additive. Thus, we conclude that $\underline{\mu}$ is a measure on $\mathcal{B}$.

(d) Clear that μ_β is nonnegative and $\mu_beta\varnothing = 0$. The preceding discussion, *mutatis mutandis*, yields the countable additivity. □

11.2 Measurable Functions

10.

Proof. For every integers n and k, let

$$\begin{aligned}
E_{n,k} &= \{x : k2^{-n} \le f(x) < (k+1)2^{-n}\}, (k \le 2^{2n}) \\
E_{n,2^{2n}+1} &= \{x : f(x) \ge (2^{2n}+1)2^{-n})\}, \\
\varphi_n &= 2^{-n} \sum_{k=0}^{2^{2n}+1} k\chi_{E_{n,k}}
\end{aligned}$$

Since f is measurable, all $E_{n,k}$ are measurable. Thus, $\langle \varphi_n \rangle$ is a sequence of nonnegative simple functions. Clear that for fixed n, $\langle E_{n,k} \rangle_k$ are disjoint. Let $x \in X$ be fixed. If $x \in E_{n,k}$ for some $k \le 2^{2n}$, then $x \in E_{n+1,2k} \cup E_{n+1,2k+1}$. Hence, $\varphi_{n+1}(x) \ge 2k/2^{-(n+1)} = \varphi_n(x)$. If $x \in E_{n,2^{2n}+1}$, then $x \in E_{n+1,k'}$ for some $k' \ge 2^{2n+2}$. Hence, $\varphi_{n+1}(x) \ge 2k/2^{-(n+1)} = \varphi_n(x)$. Thus, $\varphi_{n+1} \ge \varphi_n$ for all n.

Now, we show that φ_n converges to f pointwisely. Let $x \in X$ be fixed. If $f(x) = \infty$, then $\varphi_n(x) = 2^{-n}(2^{2n}+1) \to \infty$ as $n \to \infty$. If $f(x) < \infty$, then $f(x) < 2^N$ for some integer N. For all $n > N$, $x \in E_{n,k_n}$ where $k_n = \lfloor 2^n f(x) \rfloor$. Thus,

$$f(x) - \varphi_n(x) = f(x) - 2^{-n}\lfloor 2^n f(x) \rfloor \to 0$$

as $n \to \infty$. Namely, $\varphi_n(x) \to f(x)$.

If the measure space is σ-finite, then let $(X_n) \subset X$ be a sequence of measurable sets such that $X_n \subset X_{n+1}$, $\mu X_n < \infty$ and $X = \bigcup X_n$. Replacing $E_{n,k}$ with $E_{n,k} \cap X_n$ yields a sequence $\langle \varphi_n \rangle$ satisfying all previous requirements and vanishing outside X_n for each n. □

11.

Proof. Put $F_\alpha = \{x : f(x) \le \alpha\}$, $G_\alpha = \{x : g(x) \le \alpha\}$, $E = \{x : f(x) \ne g(x)\}$ and $E_\alpha = \{x \in E : g(x) \le \alpha\}$. Then $G_\alpha = (F_\alpha \setminus E) \cup E_\alpha$. Since F is measurable, all F_α are measurable. Since $f = g$ a.e., E is of measure zero. Meanwhile, since μ is complete, $E_\alpha \subset E$ is measurable. Thus, G_α is measurable. Namely, g is measurable. □

13.

Proof. Note that f_n converges to f in measure iff for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\{x \in X : |f_n(x) - f(x)| \ge \varepsilon\} = 0.$$

(a) By definition, for every $\varepsilon_m = 2^{-m}$, there exists some integer N_m such that for all $n \ge N_m$, $\mu\{x : |f_n(x) - f(x)| \ge \varepsilon_m\} < \varepsilon_m$. Consider the subsequence $\langle f_{N_m} \rangle_m$. We show that it converges to f almost everywhere. Put $E_m = \{x : |f_{N_m} - f(x)| \ge \varepsilon_m\}$ and $E = \limsup E_m$. Then, for each k,

$$\mu E \le \bigcup_{m=k}^{\infty} E_m \le \sum_{m=k}^{\infty} 2^{-m+1} \to 0, \quad \text{as} \quad k \to \infty.$$

For every $x \notin E$, $x \notin \bigcup_{m=k}^{\infty} E_m$ for some k. Then for all $m > k$, $|f_{N_m}(x) - f(x)| < \varepsilon_{N_m}$. Hence, $f_{N_m}(x) \to f(x)$. Namely, $f_{N_m} \to f$ almost everywhere.

(b) First we prove a lemma: Let $\langle E_n \rangle$ be a sequence of measurable subset of A. Then $\limsup \mu E_n \le \mu(\limsup E_n)$. Let $F_N = \bigcup_{n=N}^{\infty} E_n$. Clear that $F_{n+1} \subset F_n$ and $\mu F_1 < \infty$. Hence, by Prop. 2,

$$\limsup \mu E_n \le \lim \mu F_n = \mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \mu(\limsup E_n).$$

Thus, the lemma holds.

For fixed $\varepsilon > 0$, let $E_n = \{x \in A : |f_n(x) - f(x)| \ge \varepsilon\}$. We show that $\lim \mu E_n = 0$. First, clear that $0 \le \limsup \mu E_n$. Meanwhile, if $x \in \limsup E_n$, then x belongs to infinitely many E_n. As a consequence, f_n does not converges to f at x. Since f_n converges to f a.e., $\mu(\limsup E_n) = 0$. Note that all $E_n \subset A$ are of finite measure. Hence, by the preceding lemma, $\limsup \mu E_n \le 0$. Thus, $\lim \mu E_n = 0$. Let F_n denote $\{x \in X : |f_n(x) - f(x)| \ge \varepsilon\}$ and G the collection of points at which f_n does not converge to f. Since all f_n vanishes outside A, for a point outside to belong to F_n, it has to belong to G, a set of measure zero. Therefore, $E_n \subset F_n \subset E_n \cup G$, implying that $\mu F_n = \mu E_n$. Thus, f_n converges to f in measure.

(c) By definition, for each positive integer k, there is an integer N_k such that for all $n, m \ge N_k$, $\mu\{x \in X : |f_n(x) - f_m(x)| \ge 2^{-k}\} < 2^{-k}$. We may assume without loss of generality that N_k is increasing. Put $E_k = \{x : |f_{N_{k+1}} - f_{N_k}| \ge 2^{-k}\}$ and $E = \limsup E_k$. By our construction, $\mu E = 0$. For $x \notin E$, $|f_{N_{k+1}}(x) - f_{N_k}(x))| < 2^{-k}$ for large k and, therefore, the number series $\sum(f_{N_{k+1}}(x) - f_{N_k}(x))$ converges to some point, say, $g(x)$. Hence, f_{N_k} converges to $f = f_{N_1} + g$ almost everywhere. Since all f_{N_k} are measurable, f is measurable.

Now we show that f_n converges to f in measure. Let D be the set of points at which f_{N_k} does not converge to f. For every $\varepsilon > 0$, let $F_n = \{x \in X \setminus D : |f_n(x) - f(x)| \ge \varepsilon\}$. Note that for all sufficiently large N_k,

$$\begin{aligned} F_n &\subset \{x \in X \setminus D : |f_n(x) - f_{N_k}(x)| + |f_{N_k}(x) - f(x)| \ge \varepsilon\} \\ &\subset \{x \in X \setminus D : |f_n(x) - f_{N_k}(x)| \ge \varepsilon/2\}, \end{aligned}$$

where the measure of the last set can be less than ε for sufficiently large n and N_k as $\langle f \rangle$ is Cauchy in measure. Since D is of measure zero, we conclude that $\langle f_n \rangle$ converges to f in measure. □

16.

Proof. Egoroff: Let $(X, \mathcal{B}, \mu)$ be a measure space and $E \subset X$ is of finite measure. Let $\langle f_n \rangle$ be a sequence of measurable functions which converge to some function f a.e. on E. Then for every $\eta > 0$, there is a subset $A \subset E$ with $\mu A < \eta$ such that f_n converges to f uniformly on $E \setminus A$.

We may assume without loss of generality that all f_n vanish outside E. Then, by Prob. 13(b), f_n converges to f in measure over E. Fix $\eta > 0$. First, we construct A. Put $\delta_m = \delta/2^m$. For every m, there exists some integer N_m and a measurable set A_m with $\mu A_m < \delta_m$ such that for all $n > N_m$ and $x \notin A_m$, $|f_n(x) - f(x)| < \delta_m$. Put $A = \bigcup A_m$. Clear that $\mu A < \delta$.

Now we show that f_n converges to f uniformly on $E \setminus A$. Fix $x \in E \setminus A$. For every $\varepsilon > 0$, suppose there is an m such that $0 < \delta_m < \varepsilon$. For all $n > N_m$, since $x \notin A$, $|f_n(x) - f(x)| < \delta_m < \varepsilon$. Thus, $f_n \to f$ uniformly on $E \setminus A$. □

11.3 Integration

19.

Proof. Since $|\int_E f| \leq \int_E |f|$, it suffices to show the result for nonnegative f. Fix $\varepsilon > 0$. By definition, there is a nonnegative simple function $\varphi = \sum_{i=1}^n c_i \chi_{E_i}$ such that $\int f < \int \varphi + \varepsilon/2$. Put $M = \max_i c_i$ and $\delta = \varepsilon/2Mn$. Then, for every measurable E with $\mu E < \delta$, we have

$$\int_E f < \int_E \varphi + \varepsilon/2 = \sum_{i=1}^n c_i \mu(E_i \cap E) + \varepsilon/2 \leq Mn\delta + \varepsilon/2 = \varepsilon.$$

□

20.

Proof. We show here Fatou's Lemma: Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions which converges to a function f in measure on a measurable set E. Then $\int_E f \leq \underline{\lim} \int_E f_n$.

Since the collection of limits point of $\int_E f_n$ forms a closed set, there exists a subsequence $\langle f_{n_k} \rangle_k$ such that $\lim \int_E f_{n_k} = \liminf \int_E f_n$. Since f_{n_k} also converges to f in measure, by Prob. 13(a), there is a subsequence $\langle f_{n_{k_j}} \rangle$ which converges to f a.e. on E. Hence, by Theorem 10,

$$\int_E f \leq \liminf_j \int_E f_{n_{k_j}} = \lim_j \int_E f_{n_{k_j}} = \liminf_n \int_E f_n.$$

□

21.

Proof.

(a) We may assume without loss of generality that f is nonnegative since replacing f by $|f|$ dose not change the integrability and the set $E = \{x : f(x) \neq 0\}$. For every positive integer n, since $\int f < \infty$, the set $E_n = \{x : f(x) \geq 1/n\}$ is of finite measure. Thus, $E = \bigcup_{n=1}^\infty E_n$ is of σ-finite measure.

(b) It follows immediately from part (a) and Prop. 7.

(c) If $f \geq 0$, then the existence of such a φ comes directly from the definition. For general cases, let $f = f^+ - f^-$ and φ^+, φ^- two simple functions such that

$$\int |f^+ - \varphi^+| < \varepsilon/2 \quad \text{and} \quad \int |f^- - \varphi^-| < \varepsilon/2.$$

Note that $\varphi = \varphi^+ - \varphi^-$ is also a simple function and

$$\int |f - \varphi| \leq \int |f^+ - \varphi^+| + \int |f^- - \varphi^-| < \varepsilon.$$

□

22.

Proof.

(a) Clear that ν is nonnegative and $\nu\varnothing = 0$. Let $\langle E_n\rangle$ be a sequence of disjoint measurable sets and $E = \bigcup_n E_n$. By Corollary 14, we have

$$\nu E = \int_E g\mathrm{d}\mu = \int_E \sum g\chi_{E_n}\mathrm{d}\mu = \sum \int_E g\chi_{E_n}\mathrm{d}\mu = \sum \int_{E_n} g\mathrm{d}\mu = \sum \nu E_n.$$

Thus, ν is a measure.

(b) First, we show the identity for an arbitrary simple function $\varphi = \sum_{k=1}^n c_k\chi_{E_k}$ where E_k are disjoint.

$$\int \varphi\mathrm{d}\nu = \sum_{k=1}^n c_k\nu E_k = \sum_{k=1}^n c_k \int g\chi_{E_k}\mathrm{d}\mu = \int \varphi g\mathrm{d}\mu.$$

Let f be a nonnegative measurable function and $\langle\varphi_n\rangle$ a increasing sequence of simple functions converging to f, the existence of which is guaranteed by Prop. 7. Then, By the monotone convergence theorem,

$$\int f\mathrm{d}\nu = \lim \int \varphi_n\mathrm{d}\nu = \lim \int \varphi_n g\mathrm{d}\mu.$$

Note that $\langle\varphi_n g\rangle$ is a increasing sequence of functions converging to fg and with $\varphi_n g \leq fg$. Hence, again by the monotone convergence theorem,

$$\lim \int \varphi_n g\mathrm{d}\mu = \int fg\mathrm{d}\mu.$$

Thus, $\int f\mathrm{d}\nu = \int fg\mathrm{d}\mu$. □

11.4 General Convergence Theorems

24.

Proof. Since $\mu_n E$ is increasing for every E, such limits do exists. Clear that μ is nonnegative and $\mu\varnothing = 0$. Let $\langle E_k\rangle$ be a sequence of disjoint measurable sets. Then

$$\mu\left(\bigcup_{k=1}^\infty E_k\right) = \lim_{n\to\infty} \mu_n\left(\bigcup_{k=1}^n E_k\right) = \lim_{n\to\infty} \sum_{k=1}^\infty \mu_n(E_k).$$

Since for fixed k, $\mu_n(E_k) \leq \mu_{n+1}(E_k)$, it is valid to change the order of the limit and the summation, which implies that $\mu(\bigcup E_k) = \sum \mu E_k$. Thus, μ is a measure. □

11.5 Signed Measures

27.

Proof.

(a) Consider the usual Lebesgue measure on $\mathbb{R}$. Let A be any countable subset of $\mathbb{R}$ and $B = \mathbb{R}\setminus A$. Clear that A is negative set while B is a positive set. Namely, A and B form a Hahn decomposition of $\mathbb{R}$ for μ.

(b) Let $\{A_1, B_1\}$ and $\{A_2, B_2\}$ be two Hahn decomposition of X for ν and A_1 and A_2 are two positive sets. We show that $A_1 \triangle A_2$ is a null set. Since the roles of A_1 and A_2 are interchangeable, it suffices to show that $A_1 \setminus A_2$ is a null set. Since A_1 is positive, every subset $E \subset A_1 \setminus A_2 \subset A_1$ is of nonnegative measure. Meanwhile, $A_1 \setminus A_2$ is also contained in B_2, a negative set. Hence, $\nu E \leq 0$. Thus, $\nu E = 0$, implying that $A_1 \triangle A_2$ is a null set. □

28.

Proof. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν and A and B be such that $X = A \cup B$ and $\nu^+(A) = \nu^-(B) = 0$. For every $E \subset A$,

$$\nu E = \nu^+ E - \nu^- E = -\nu^- E \leq 0.$$

Hence, A is a negative set. Similarly, B is positive set. Thus, $\{A, B\}$ is a Hahn decomposition of X.

Let $\nu = \nu_1 + \nu_2$ be another Jordan decomposition of ν and $\{C, D\}$ be the corresponding Hahn decomposition. By Prob. 27(b), $\{A, B\}$ and $\{C, D\}$ only differ by two null sets. Thus, $\nu_1 = \nu^+$ and $\nu_2 = \nu^-$. Namely, the decomposition is unique. □

31.

Proof. Clear that

$$\left| \int_E f \mathrm{d}\nu \right| \leq \left| \int_E f \mathrm{d}\nu^+ \right| + \left| \int_E f \mathrm{d}\nu^- \right| \leq M\nu^+ E + M\nu^- E = M|\nu|(E).$$

Let $\{A, B\}$ be the corresponding Hahn decomposition of X and A is the positive set. Then define f by

$$f(x) = \begin{cases} 1, & x \in A, \\ -1, & x \notin A. \end{cases}$$

Clear that $|f| \leq 1$ and

$$\int_E f \mathrm{d}\nu = \int_E f \mathrm{d}\nu^+ - \int_E f \mathrm{d}\nu^- = \mu^+(A \cap E) + \nu^-(A \cap B) = |\nu|(E).$$

□

32.

Proof.

(a) Put $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$, which can be verified to be a measure. For every $E \subset X$, suppose $\mu E \leq \nu E$. Then

$$(\mu \wedge \nu)(E) = \frac{1}{2}(\mu E + \nu E - |\mu - \nu|(E)) = \frac{1}{2}(\mu E + \nu E - \nu E + \mu E) = \mu E.$$

Similarly, $(\mu \wedge \nu)(E) = \nu E$ if $\nu E \leq \mu E$. Hence, $\mu \wedge \nu$ is smaller than both μ and ν. Note that $(\mu \wedge \nu)(E) = \min\{\mu E, \nu E\}$. Thus, clear that it is larger than any other signed measure smaller than μ and ν.

(b) Put $\mu \vee \nu = \frac{1}{2}(|\mu - \nu| + \mu + \nu)$. The previous argument, *mutatis mutandis*, shows that $(\mu \vee \nu)(E) = \max\{\mu E, \nu E\}$. Thus, it is the smallest measure larger than μ and ν. Meanwhile, clear that $\mu \wedge \nu + \mu \vee \nu = \mu + \nu$.

(c) Suppose that μ and ν are mutually singular and let $\{A, B\}$ be such that $A \cup B = X$ $\mu A = \nu B = 0$. Then

$$(\mu \wedge \nu)(E) \leq (\mu \wedge \nu)(E \cap A) + (\mu \wedge \nu)(E \cap B) \leq \mu A + \nu B = 0.$$

For the converse, suppose that $\mu \wedge \nu = 0$. If $\mu = 0$ or $\nu = 0$, then $\mu \perp \nu$ holds vacuously. Suppose that both μ and ν are nonzero. Since the roles of μ and ν are interchangeable, we may assume without loss of generality that $\mu E = 0$ and $\nu E > 0$ for some measurable E. Then, $\mu E^c \neq 0$, forcing νE^c to be zero. Therefore, $\mu E = \nu E^c = 0$, implying that $\mu \perp \nu$. $\square$