Statistical Inference

Yunwei Ren

Contents

1 Probability Theory **2**

2 Transformations and Expectations **4**

3 Common Families of Distributions **7**

1 Probability Theory

1.5(a)

Solution. A U.S. birth results in female identical twins. □

1.5(b)

Solution.

$$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C) = P(A|B)P(B|C)P(C) = \frac{1}{2}\frac{1}{3}\frac{1}{90} = \frac{1}{540}.$$

□

1.24(b)

Solution. Suppose $E_i = \{\text{head first appears on } i\text{th toss}\}$, then

$$P(\text{A wins}) = P(\bigcup_{i=k}^{\infty} E_{2k+1}) = \sum_{k=0}^{\infty} P(E_{2k+1}) = \sum_{k=0}^{\infty} p(1-p)^{2k+1} = \frac{p}{1-(1-p)^2}.$$

□

1.31(a)

Proof. To get the average $(x_1+\cdots+x_n)/n$, we need the unordered sample to be $\{x_1, x_2, \ldots, x_n\}$. The number of ordered samples which results in it is $n!$ and there are n^n ordered samples in total. Hence, the probability is $n!/n^n$.

For any other resulted average, there will exist some double counting when counting the ordered samples. Therefore, the outcome with average $(x_1 + \cdots + x_n)/n$ is most likely. □

1.33

Solution.

$$\begin{aligned} P(\text{male}|\text{color-blind}) &= P(\text{color-blind}|\text{male})\frac{P(\text{male})}{P(\text{color-blind})} \\ &= 0.05 \times \frac{0.5}{0.5 \times 0.05 + 0.5 \times 0.0025} \\ &= \frac{20}{21} = 0.9524. \end{aligned}$$

□

1.36

Solution. The probabilities of all shots being missed and the target being hit exactly once are $(4/5)^5 = 0.32768$ and $5 \times (1/5)(4/5)^4 = 0.4096$ respectively. Hence,

$$P(\text{being hit at least twice}) = 1 - 0.32768 - 0.4096 = 0.26272.$$

And

$$\begin{aligned}&P(\text{being hit at least twice}|\text{being hit at least once})\\=&\frac{P(\text{being hit at least twice})}{P(\text{being hit at least once})} = \frac{0.26272}{0.4096} = 0.6414.\end{aligned}$$

□

1.39(a)

Proof. A and B are mutually exclusive means that $A \cap B = \varnothing$. Hence, $P(A \cap B) = 0$. However, $P(A), P(B) > 0$. Therefore, $P(A \cap B) \neq P(A)P(B)$. □

1.39(b)

Proof. As A and B are independent, $P(A \cap B) = P(A)P(B) > 0$, which implies that $A \cap B \neq \varnothing$. □

Notes on 1.39 An intuitive proof: Since A and B are mutually exclusive, if we know that A did not happen, then the possibility that B happened will increase. Hence, they are not independent.

1.52

Proof. Clear that $g(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\mathrm{d}x &= \int_{x_0}^{\infty} \frac{f(x)}{1 - F(x_0)}\mathrm{d}x\\&= \frac{1}{1 - F(x_0)}\left(\int_{-\infty}^{\infty} f(x)\mathrm{d}x - \int_{-\infty}^{x_0} f(x)\mathrm{d}x\right)\\&= \frac{1}{1 - F(x_0)}(1 - F(x_0)) = 1.\end{aligned}$$

Hence, by Theorem 1.6.5, $g(x)$ is a pdf. □

2 Transformations and Expectations

1.(b)

Solution. f_X is continuous and $g^{-1}(y) = (y-3)/4$ is continuously differentiable, therefore

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| = \frac{7}{4} e^{-7(y-3)/4}.$$

$\mathcal{Y} = (3, \infty)$ and

$$\int_{\mathcal{Y}} f_Y(y) = -\int_3^{\infty} \exp\left(-\frac{7}{4}y + \frac{21}{4}\right) \mathrm{d}\left(-\frac{7}{4}y + \frac{21}{4}\right) = 1.$$

□

3.

Solution.

$$f_Y(y) = P(Y = y) = P\left(\frac{X}{X+1} = y\right) = P\left(X = \frac{1}{1-y} - 1\right) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{1}{1-y}-1},$$

where $y = 0, 1/2, 2/3, \ldots$ □

5.

Solution. $g(x) = \sin^2 x$ is monotone on $(0, \pi/2]$, $[\pi/2, \pi)$, $(\pi, 3\pi/2]$ and $[3\pi/2, 2\pi)$ respectively and the ranges of g on the intervals are the same. Futhermore,

$$\begin{aligned} g_0^{-1}(y) &= \arcsin\sqrt{y}, & g_1^{-1}(y) &= -\arcsin(-\sqrt{y}) \\ g_2^{-1}(y) &= \pi + \arcsin\sqrt{y}, & g_3^{-1}(y) &= \pi - \arcsin(-\sqrt{y}), \end{aligned}$$

all of which are continuously differentiable. Hence,

$$f_Y(y) = \sum_{i=0}^{3} \frac{1}{2\pi} \left| \frac{\mathrm{d}}{\mathrm{d}y} g_i^{-1}(y) \right| = \frac{1}{\pi\sqrt{y(1-y)}}.$$

and vanishes elsewhere.

Next we show that the same answer can be obtained by differentiating (2.1.6). Note that $\mathrm{d}(x + c) = \mathrm{d}x$ and $x_2 - x_1 = 2(\pi - x)1$, then we get

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} P(Y \le y) = \frac{2}{\pi} \frac{\mathrm{d}}{\mathrm{d}y} x_1 = \frac{2}{\pi} \frac{\mathrm{d}}{\mathrm{d}y} \arcsin\sqrt{y} = \frac{1}{\pi\sqrt{y(1-y)}}.$$

□

9.

Solution. By Theorem 2.1.10, we only need to set $u = F_X$, that is,

$$u(x) = \int_{-\infty}^{x} f(x)\mathrm{d}x = \begin{cases} 0, & x \le 1, \\ (x-1)^2/4, & 1 < x < 3, \\ 1, & x \ge 3, \end{cases}$$

and the monotonicity is obvious. □

11.(a)

Solution. Direct calculating yields

$$\mathrm{E}X^2 = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \mathrm{d}x = 1.$$

And by Example 2.1.17,

$$\begin{aligned}\mathrm{E}Y &= \int_0^{\infty} y \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y})) \mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{y} e^{-y/2} \mathrm{d}y = \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^2 e^{-y^2/2} \mathrm{d}y = 1.\end{aligned}$$

□

13.

Solution. $f_X(x) = P(X = x) = p^x(1-p) + (1-p)^x p$ $(n = 1, 2, 3, \ldots)$ and

$$\mathrm{E}X = \sum_{x=1}^{\infty} x f_X(x) = (1-p) \sum_{x=1}^{\infty} x p^x + p \sum_{x=1}^{\infty} x(1-p)^x = \frac{p}{1-p} + \frac{1-p}{p}.$$

□

15.

Proof. Clear that $X + Y = (X \vee Y) + (X \wedge Y)$. Hence,

$$\mathrm{E}(X \vee Y) = \mathrm{E}(X + Y - (X \wedge Y)) = \mathrm{E}X + \mathrm{E}Y - \mathrm{E}(X \wedge Y).$$

□

18.

Proof.

$$\begin{aligned}\mathrm{E}|x-a| &= \int_{-\infty}^{\infty} |x-a| f(x) \mathrm{d}x \\ &= \int_{-\infty}^{a} (a-x) f(x) \mathrm{d}x + \int_a^{\infty} (x-a) f(x) \mathrm{d}x \\ &= a \int_{-\infty}^{a} f(x) \mathrm{d}x - \int_{-\infty}^{a} x f(x) \mathrm{d}x + \int_a^{\infty} x f(x) \mathrm{d}x - a \int_a^{\infty} f(x) \mathrm{d}x.\end{aligned}$$

Differentiating the both sides yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a} \mathrm{E}|x-a| &= \left(\int_{-\infty}^{a} f(x) \mathrm{d}x + a f(a) \right) - a f(a) - a f(a) - \left(\int_a^{\infty} f(x) \mathrm{d}x - a f(a) \right) \\ &= \int_{-\infty}^{a} f(x) \mathrm{d}x - \int_a^{\infty} f(x) \mathrm{d}x.\end{aligned}$$

Meanwhile, the value at m of the twice derivative is greater than 0. Therefore, $x = m$ is the minimum point. □

24.(a)

Solution.

$$\begin{aligned}\mathrm{E}X &= \int_0^1 x a x^{a-1}\mathrm{d}x = \frac{a}{a+1}.\\ \mathrm{E}(X^2) &= \int_0^1 x^2 a x^{a-1}\mathrm{d}x = \frac{a}{a+2}.\\ \mathrm{Var}X &= (\mathrm{E}X)^2 - \mathrm{E}(X^2) = \frac{a}{(a+1)(a+2)}.\end{aligned}$$

□

32

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\log\int_{-\infty}^{\infty} e^{tx}f(x)\mathrm{d}x\right) = \frac{1}{\int_{-\infty}^{\infty} e^{tx}f(x)\mathrm{d}x}\int_{-\infty}^{\infty} x e^{tx}f(x)\mathrm{d}x.$$

Therefore,

$$S'(0) = \frac{1}{\int_{-\infty}^{\infty} f(x)\mathrm{d}x}\int_{-\infty}^{\infty} x f(x)\mathrm{d}x = \mathrm{E}X.$$

□

3 Common Families of Distributions

5.

Solution. Assuming that the two kinds of drugs are equally effective and let X = the number of effective cases. Then with the binomial distribution

$$P(X \geq 85 | n = 100, p = 0.8) = \sum_{k=85}^{100} \binom{100}{k} 0.8^k 0.2^{100-k} = 0.1285.$$

Hence, there is still some possibility that the new drugs is no better than the old ones. □

7.

Solution.

$$P(X \geq 2) > 0.99 \quad \Rightarrow \quad e^{-\lambda}(1+\lambda) < 0.01 \quad \Rightarrow \quad \lambda \geq 6.6384.$$

Hence, the mean of the distribution is $\lambda = 6.6484$. □

13.(a)

Solution. $P(X = 0) = e^{-\lambda}$ and therefore $P(X > 0) = 1 - e^{-\lambda}$. Hence,

$$\begin{aligned} P(X_T = x) &= \frac{P(X = x)}{1 - P(X = 0)} = \frac{\lambda^x}{x!(e^\lambda - 1)}, \\ \mathrm{E}X_T &= \frac{1}{P(X > 0)} \sum_{x=1}^{\infty} xP(X = x) = \frac{\lambda}{1 - e^{-\lambda}}, \\ \mathrm{Var}X_T &= \frac{1}{1 - e^{-\lambda}} \lambda(\lambda + 1) - \left(\frac{\lambda}{1 - e^{-\lambda}} \right)^2. \end{aligned}$$

□