Differential Equations, Dynamical Systems and Linear Algebra

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2.

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(b) $x_1(t) = k_1 e^t$, $x_2(t) = k_2 e^{-2t}$ and $x_3(t) = k_3 e^{2t}$. $\square$

4.

Solution.

$$x'(t) = \begin{bmatrix} e^t \\ 4e^{2t} \\ 8e^{2t} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix} \begin{bmatrix} e^t \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 2 & \\ & 2 & 1 \end{bmatrix} \begin{bmatrix} e^t \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix}.$$

$\square$

6.

Solution. Suppose that $A = \mathrm{diag}(a_1, \dots, a_n)$, then $x_i(t) = K_i \exp(a_i t)$. Hence, $x(t) \to 0$ as $t \to \infty$ for all solutions iff $a_i < 0$ for each $i = 1, 2, \dots, n$. $\square$

8.

Proof.
(a) It follows immediately that both the differential operator and A are linear operators.
(b) Since every solution is of form $x(t) = (K_1 e^t, K_2 e^{-2t})$, $u(t) = (e^t, 0)$ and $v(t) = (0, e^{-2t})$ are two suitable solutions. $\square$

2 Newton's Equation and Kepler's Law

2.

Solution.
(a) Yes. $V(x, y) = (x^3 + 2y^3)/3$.
(b) No. If there exists some $V : \mathbb{R}^2 \to \mathbb{R}$ such that $F = -\nabla V$, then from

$$\frac{\partial V}{\partial x} = x^2 - y^2 \quad \text{and} \quad \frac{\partial V}{\partial y} = 2xy$$

we can respectively derive

$$V(x, y) = x^3/3 - xy^2 + C_1 \quad \text{and} \quad V(x, y) = xy^2 + C_2,$$

which is impossible.
(c) Yes. $V(x, y) = x^2/2$. □

4. I think that there should be a minus before the right hand side of the definition equation of work. And I assume that F is at least continuous.

Proof. Suppose that F is conservative and let $y(s)$ be any path from x_0 to x_1. Note that

$$\frac{\mathrm{d}}{\mathrm{d}s} V(y(s)) = \langle \nabla_y V(y(s), y'(s)) \rangle.$$

Hence,

$$-\int_{s_0}^{s_1} \langle F(y(s)), y'(s) \rangle \mathrm{d}s = \int_{s_0}^{s_1} \langle \nabla_y V(y(s)), y'(s) \rangle \mathrm{d}s = V(x_1) - V(x_0).$$

Since the choice of $y(s)$ is arbitrary, the work is independent of the path.

Now we suppose the work is independent of the path. Let x_0 be a fixed point and

$$V(x) = -\int_{s_0}^{s} \langle F(y(s)), y'(s) \rangle \mathrm{d}s.$$

where $y(s)$ is any path connecting $x_0 = y(s_0)$ and $x = y(s)$. And by the fundamental theorem of calculus, V is continuously differentiable and $F = -\nabla V$. Hence, F is conservative. □

6.

Proof. Let the notations have the same meanings. Note that $i = (\cos\theta, \sin\theta)$ and $j = (-\sin\theta, \cos\theta)$. Hence,

$$\frac{\mathrm{d}i}{\mathrm{d}t} = (-\theta' \sin\theta, \theta' \cos\theta) = \theta' j \quad \text{and} \quad \frac{\mathrm{d}j}{\mathrm{d}t} = -\theta' i.$$

Therefore,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(ri) = r'i + ri' = r'i + r\theta' j.$$

Differentiate again and we get

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} &= r''i + r'i' + (r\theta')'j + r\theta'j' \\ &= r''i + r'\theta'j + (r\theta')'j - r(\theta')^2 i \\ &= (r'' - r(\theta')^2)i + (r'\theta' + (r\theta')')j \\ &= (r'' - r(\theta')^2)i + \frac{1}{r}(r^2\theta')'j.\end{aligned}$$

Since F is central, $\langle F, j\rangle = 0$ and so does x''. Thus, the coefficient of j is zero, completing the proof. □