

# Real Analysis

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## Contents

<b>3</b>	<b>Lebesgue Measure</b>	<b>2</b>
3.1	Introduction . . . . .	2
3.2	Outer Measure . . . . .	2
3.3	Measurable Sets and Lebesgue Measure . . . . .	3
3.5	Measurable Functions . . . . .	4
3.6	Littlewood's Three Principles . . . . .	6
<b>4</b>	<b>The Lebesgue Integral</b>	<b>7</b>
4.2	The Lebesgue Integral of a Bounded Function . . . . .	7
4.3	The Integral of a Nonnegative Function . . . . .	7
4.4	The General Lebesgue Integral . . . . .	9
<b>5</b>	<b>Differentiation and Integration</b>	<b>11</b>
5.1	Differentiation of Monotone Functions . . . . .	11
5.2	Functions of Bounded Variation . . . . .	11
5.4	Absolute Continuity . . . . .	13
5.5	Convex Functions . . . . .	17
<b>6</b>	<b>The Classical Banach Spaces</b>	<b>19</b>
6.1	The $L^p$ Spaces . . . . .	19
6.2	The Minkowski and Hölder Inequalities . . . . .	19
6.3	Convergence and Completeness . . . . .	20
6.4	Approximation in $L^p$ . . . . .	22
<b>11</b>	<b>Measure and Integration</b>	<b>23</b>
11.1	Measure Spaces . . . . .	23

## 3 Lebesgue Measure

### 3.1 Introduction

1.

*Proof.* Since  $\mathfrak{M}$  is an  $\sigma$ -algebra,  $B \setminus A \in \mathfrak{M}$  as long as  $A, B \in \mathfrak{M}$ . Since  $B \setminus A$  and  $A$  are disjoint,  $mB = mA + m(B \setminus A) \geq mA$  since  $m$  is nonnegative.  $\square$

2.

*Proof.* Let  $A_0 = E_0$  and  $E_k = A_k \setminus A_{k-1}$  for  $k \geq 1$ . Clear that  $E_i$  and  $E_j$  are disjoint for distinct  $i$  and  $j$ ,  $\bigcup A_n = \bigcup E_n$  and  $A_i \subset E_i$  for every  $i$ . Hence,

$$m\left(\bigcup E_n\right) = m\left(\bigcup A_n\right) = \sum mA_n \leq \sum mE_n,$$

where the last inequality comes from Exercise 1.  $\square$

3.

*Proof.* Suppose that  $mA < \infty$ . Then  $mA = m(A \cup \emptyset) = mA + m\emptyset$ , implying that  $m\emptyset = 0$ .  $\square$

### 3.2 Outer Measure

5.

*Proof.* We show that  $\{I_n\}$  must cover the entire  $[0, 1]$  by contradiction. Assume that  $x \notin I_k$  for  $k = 1, 2, \dots, n$ . Then, as  $I_k$  are open and  $n$  is finite, there exists some  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon)$  and  $I_k$  are disjoint for every  $k$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exists some rational number in  $(x - \varepsilon, x + \varepsilon)$ , contradicting with the hypothesis that  $\{I_k\}$  covers all rational numbers between 0 and 1.  $\square$

6.

*Proof.* By the definition of the outer measure, for every  $\varepsilon > 0$ , there exists some collection  $\{I_n\}$  of open intervals that covers  $A$  and  $\sum l(I_n) \leq m^*A + \varepsilon$ . Let  $O = \bigcup I_n$ .  $O$  is a countable union of open sets and therefore is also open. And by Proposition 2,  $m^*O \leq \sum l(I_n)$ . Thus,  $m^*O \leq m^*A + \varepsilon$ .

Let  $\varepsilon_n = 1/n$  and for each  $n$ , by the previous discussion, we can always get an open set  $O_k$  such that  $A \subset O_k$  and  $m^*O \leq m^*A + \varepsilon_m$ . Let  $G$  be the countable intersection of these open sets. Clear that  $G$  is a  $G_\delta$  set covering  $A$  and  $m^*A = m^*G$ .  $\square$

7.

*Proof.* If  $m^*E = \infty$ , it is trivial. Suppose that  $m^*E \leq \infty$ . For any  $x \in \mathbb{R}$ , collection  $\{I_n\}$  of open intervals covers  $E + x$  iff  $\{I_n - x\}$  covers  $E$ . Since the length of intervals is translation invariant, this implies  $m^*(E + x) = m^*E$ .  $\square$

8.

*Proof.* Clear that  $m^*A \leq m^*(A \cup B)$ . Meanwhile,  $m^*(A \cup B) = m^*A + m^*B = m^*B$ . Hence,  $m^*(A \cup B) = m^*B$ .  $\square$

### 3.3 Measurable Sets and Lebesgue Measure

10.

*Proof.*

$$\begin{aligned} mE_1 + mE_2 &= mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2). \end{aligned}$$

□

11.

*Proof.*  $E_n = (n, \infty)$ .

□

12. This is the countable version of Lemma 9.

*Proof.* It suffices to prove  $m^*(A \cap \bigcup E_i) \geq \sum m^*(A \cap E_i)$ . Since  $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^n E_i$  for every  $n$ ,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of  $n$ , we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i),$$

completing the proof.

□

13.

*Proof.* First we suppose that  $m^*E < \infty$ . By Proposition 5, there exists some open set  $O \supset E$  such that  $m^*O \leq m^*E + \varepsilon$ . If  $E$  is measurable, then by the definition,

$$m^*(O \setminus E) = m^*O - m^*E \leq \varepsilon.$$

Namely, (ii) holds. Meanwhile,  $O \subset \mathbb{R}$  is a countable union of disjoint open intervals  $\{I_n\}$ . Since  $mO = m^*O$  is bounded and  $mO = \sum l(I_n)$ , there exists some integer  $N > 0$  such that  $mO - \sum_{n=1}^N l(I_n) < \varepsilon$ . Let  $U = \bigcup_{n=1}^N I_n$ .

$$\begin{aligned} m^*(U \triangle E) &= m^*((U \cup E) \setminus (U \cap E)) \\ &\leq m^*(O \setminus (U \cap E)) \\ &= m^*((O \setminus U) \cup (O \setminus E)) \\ &\leq m^*(O \setminus U) + m^*(O \setminus E) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, (ii) implies (vi). Now we show that (vi) implies (ii). If  $m^*(U \triangle E) < \varepsilon$ , then there exists some countable collection  $\{J_n\}$  of open interval such that

$$\sum l(J_n) \leq m^*(U \triangle E) + \varepsilon < 2\varepsilon.$$

Let  $J = \bigcup J_n$  and  $O = U \cup J$ .  $m^*J < 2\varepsilon$ . And  $O$  is open and covers  $E$ . Meanwhile,

$$m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(J \setminus E) < 3\varepsilon.$$

Hence, (ii) holds.

Now, let  $E$  be an arbitrary set and  $E_n = E \cap (-n, n)$ , which is a set with finite measure. Then by the previous discussion, there exists some open set  $O_n \supset E_n$  with  $m^*(O_n \setminus E_n) < \varepsilon/2^n$ . Let  $O = \bigcup O_n$ , an open set covering  $E$  and

$$m^*(O \setminus E) \leq \sum m^*(O_n \setminus E_n) < 2\varepsilon.$$

Hence, (i) implies (ii). Now we suppose (ii) holds and let  $\varepsilon_n = 1/n$ , then there exists a sequence of open sets  $\langle O_n \rangle$  such that  $m^*(O_n \setminus E) < 1/n$ . Let  $G = \bigcap O_n \in G_\delta$ .  $m^*(G \setminus E) \leq m^*(O_n \setminus E) \leq 1/n$ . Since the left hand side is independent of  $n$ ,  $m^*(G \setminus E) = 0$ . If (iv) holds, then by Lemma 6,  $G \setminus E$  is measurable. Since  $G \in G_\delta$  is also measurable,  $E$  is measurable. Hence, (iv) implies (i).

By the previous result, for any measurable  $E$ , there exists some closed set  $F \subset E$  such that  $\bar{F}$ , which is open, contains  $\text{bar}E$  and  $m^*(\bar{F} \setminus \bar{E}) < \varepsilon$ . Hence,  $m^*(E \setminus F) < \varepsilon$ . We can proceed in a similar manner as we did in the last paragraph to prove that (iii)  $\Rightarrow$  (v)  $\Rightarrow$  (i), leading to the final conclusion.  $\square$

### 3.5 Measurable Functions

19.

*Proof.* For every  $\beta \in \mathbb{R}$ , since  $D$  is measurable, there exists a sequence of  $\alpha_n \in D \cap (\beta - 1/n, \beta)$ . As

$$\{x : f(x) > r\} \Leftrightarrow \bigcup_{n=1}^{\infty} \{x : f(x) > r - 1/n\} \Leftrightarrow \bigcup_{n=1}^{\infty} \{x : f(x) > \alpha_n\}$$

and  $\{x : f(x) > \alpha_n\}$  are measurable, so is  $\{x : f(x) > r\}$ . Hence,  $f$  is measurable.  $\square$

21.

*Proof.*

(a) It follows immediately from  $\{x : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$ .

(b) For  $\alpha \geq 0$ , the sets  $\{x : f(x) > \alpha\}$  and  $\{x : g(x) > \alpha\}$  are the same. And for  $\alpha < 0$ ,

$$\{x : f(x) > \alpha\} = \{x : g(x) > \alpha\} \setminus \bar{D} \quad \text{and} \quad \{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \bar{D}.$$

Hence,  $f$  is measurable iff  $g$  is measurable.  $\square$

22.(d)

*Proof.* Since  $f$  and  $g$  are finite almost everywhere, the set  $A$  consisting of points where  $f + g$  is of the form  $\infty - \infty$  or  $-\infty + \infty$  is of measure zero (and hence measurable). Therefore no matter how it is defined,  $\{x \in A : f + g > \alpha\}$  is measurable for every  $\alpha$ . Namely, the restriction of  $f + g$  to  $A$  is measurable. Meanwhile, clear that the restriction to  $D \setminus A$  is measurable where  $D$  is the domain of  $f$ . Hence, by Exercise 21,  $f$  is measurable.  $\square$

**23.**

*Proof.*

(a) Let  $A_n = \{x : |f(x)| > n\}$ , a sequence of measurable sets. As  $A_{n+1} \subset A_n$ ,  $mA_{n+1} \leq mA_n$ . Since  $A = \bigcap A_n = \{x : |f(x)| = \infty\}$ ,  $mA_1 \leq m[a, b]$  is finite and  $mA = 0$ , by Proposition 14, there exists some  $N$  such that for all  $n \geq N$ ,  $mA_n < \varepsilon/3$ . Set  $M = N$  to complete the proof.

(b) We consider the restriction of  $f$  on to the set  $E = [a, b] \setminus \{x : |f(x)| \geq M\}$ , which is also a measurable real-valued function. To keep our notation simple, we denote the restriction by  $f$  still. For every  $\varepsilon > 0$ , there exists some integer  $N$  with  $0 < 2M/N < \varepsilon$ . Let  $E_n = \{x : x \in [-M + (n-1)\varepsilon, -M + n\varepsilon]\}$  ( $n = 1, 2, \dots, N$ ) and define

$$\varphi(x) = \sum_{i=1}^N f(x_i) \chi_{E_i},$$

where  $x_n \in E_n$  is arbitrary. Clear that  $\varphi$  is a simple function and satisfy all the requirements.

(c) Suppose that  $\varphi(x) = \sum_{i=1}^n \alpha_i \chi_{E_i}$ . For each  $i = 1, \dots, N$ ,  $E_i$  is measurable and therefore by Proposition 15, there exists a finite union  $U_i$  of open intervals such that  $m(U_i \triangle E_i) < \varepsilon$ . Let

$$g(x) = \sum_{i=1}^N \alpha_i \chi_{U_i}.$$

Clear that  $g$  and  $\varphi$  only may differ on a set with measure  $N\varepsilon$ . (d) Suppose that  $g(x) = \sum_{i=1}^N \alpha_i \chi_{U_i}$  is a step function. We may assume without loss of generality that  $U_i$  are disjoint and  $\bigcup U_i = [a, b]$ . And suppose that  $\{x_0 = a < x_1 < \dots < x_N = b\}$  are the endpoints of the intervals. For each  $i = 1, \dots, N-1$ , define

$$f(x) = (x - x_i + \varepsilon)g(x_i - \varepsilon) + (x_i + \varepsilon - x)g(x_i + \varepsilon), \quad x \in (x_i - \varepsilon, x_i + \varepsilon),$$

and  $f(x) = g(x)$  for the other points. (We assume that  $\varepsilon$  is small enough so that  $f$  is well-defined.) Clear that  $f$  is continuous and equals  $g$  except on a set of measure less than  $2N\varepsilon$ .  $\square$

**24.**

*Proof.* For measurable  $f$ , we show that  $\mathcal{A} = \{E : f^{-1}[E] \text{ is measurable}\}$  is a  $\sigma$ -algebra first. As the domain, denoted by  $D$ , of a measurable function is measurable,  $\mathbb{R} \in \mathcal{A}$ . If  $E \in \mathcal{A}$ , then since  $f^{-1}[\bar{E}] = D \cap \overline{f^{-1}[E]}$ ,  $f^{-1}[\bar{E}]$  is also measurable and therefore  $\bar{E} \in \mathcal{A}$ . Suppose that  $\langle E_n \rangle$  is a sequence of sets of  $\mathcal{A}$ . Then, as

$$f^{-1}\left[\bigcup_{n=1}^{\infty} E_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[E_n],$$

$\bigcup E_n \in \mathcal{A}$ . Hence,  $\mathcal{A}$  is a  $\sigma$ -algebra.

By the definition of a measurable function, every open interval belongs to  $\mathcal{A}$ . Since the collection of all Borel sets  $\mathcal{B}$  is the  $\sigma$ -algebra generated by all open intervals,  $\mathcal{B} \subset \mathcal{A}$ . Namely,  $f^{-1}[B]$  is measurable as long as  $B \in \mathcal{B}$ .  $\square$

### 3.6 Littlewood's Three Principles

30.

*Proof.* Let  $\varepsilon_n = 1/n$  and  $\delta_n = \eta/2^n$  ( $n = 0, 1, \dots$ ). By Proposition 24, for each  $n$ , there exists some  $A_n$  with measure less than  $\delta_n$  such that for all  $x \in E_n \setminus A_n$ ,  $|f_m(x) - f(x)| < \varepsilon_n$  for  $m$  large enough. Let  $A = \bigcup_{n=1}^{\infty} A_n$ , the measure of which is less than  $\sum \eta/2^n = \delta$ . Meanwhile, for any  $\varepsilon > 0$ , by construction, for all  $x \in E \setminus A$ ,  $|f_m(x) - f(x)| < \varepsilon$  for  $m$  large enough. Namely,  $f_n$  converges to  $f$  uniformly on  $E \setminus A$ .  $\square$

31.

*Proof.* Let  $\varepsilon_n = \delta/2^n$  ( $n \geq 0$ ), then by Proposition 22, there exists continuous  $g_n$  such that  $E_n = \{x : |f(x) - g_n(x)| \geq \varepsilon_n\}$  is of measure less than  $\varepsilon_n$ . Let  $E = \bigcup E_n$ , the measure of which is less than  $\delta$  and  $g_n$  converges to  $f$  on  $[a, b] \setminus E$ .

By Egoroff's Theorem, there exists some  $A \subset [a, b] \setminus E$  with  $mA < \delta$  such that  $g_n$  converges to  $f$  uniformly on  $[a, b] \setminus (E \cup A)$ . Since  $E \cup A$  is measurable, by Proposition 15, there exists some open set  $O \supset E \cup A$  such that  $m(O \setminus (E \cup A)) < \delta$ . Let  $F = [a, b] \setminus O$ . We know that

1.  $F$  is a closed set.
2.  $mF < 3\delta$ .
3.  $g_n$  converges to  $f$  uniformly on  $F$ .

Hence,  $f$  is continuous on  $F$ . And by Problem 2.40, there exists some continuous function on  $\mathbb{R}$  such that  $\varphi(x) = f(x)$  for  $x \in F$ .

If  $f$  is defined on  $(-\infty, \infty)$ , we can apply the previous result on each  $[n, n+1]$  and "stick" the functions together as we did in Problem 23(c) to get the function required.  $\square$

## 4 The Lebesgue Integral

### 4.2 The Lebesgue Integral of a Bounded Function

2.

*Proof.*

(a) By Problem 2.51,  $h$  is upper semicontinuous as  $f$  is bounded and by Problem 2.50,  $x : h(x) < \lambda$  is open and hence measurable for every  $\lambda \in \mathbb{R}$ . Thus,  $h$  is measurable.

Let  $\varphi(x) \geq f(x)$  be a step function and  $x_0$  any point other than the endpoints of the intervals occurring in  $\varphi$ . Then there exists some  $\delta > 0$  such that for all  $x \in (x_0 - \delta, x_0 + \delta)$ ,  $\varphi(x_0) = \varphi(x) \geq f(x)$ . Hence,

$$h(x_0) = \inf_{\delta < 0} \sup_{|x - x_0| < \delta} f(x) \leq \varphi(x_0).$$

Namely,  $\varphi \geq h$  except at a finite number of points. Hence,  $\int_a^b \varphi \geq \int_a^b h$  and therefore

$$R \int_a^b f = \inf_{\varphi \geq f} \int_a^b \varphi(x) dx \geq \int_a^b h.$$

We can also derive from the previous discussion that there is a sequence of  $\langle \varphi_n \rangle$  of step functions satisfying  $\varphi \downarrow h$ . By Proposition 6,

$$\int_a^b h = \lim \int_a^b \varphi_n \geq R \int_a^b f.$$

Hence,  $R \int_a^b f = \int_a^b h$ .

(b) First suppose that  $f$  is Riemann integrable and let  $h$  and  $g$  be the upper and lower envelope of  $f$  respectively. By part (a),  $f$  is Riemann integrable implies  $\int_a^b (h - g) = 0$ . Together with the fact that  $h \geq g$ , we conclude that  $h = g$  a.e.. Therefore, by Problem 2.50,  $f$  is continuous except on a set of measure zero.

Note that the argument remains true if we reverse the order, verifying the converse part. Hence, the proposition holds.  $\square$

### 4.3 The Integral of a Nonnegative Function

3.

*Proof.* Suppose that  $E_n = \{x : f(x) > 1/n\}$ . Then,

$$0 = \int f \geq \int_{E_n} f \geq \frac{mE_n}{n}$$

implies  $mE_n = 0$ . Hence,  $m\{x : f(x) > 0\} = m(\bigcup E_n) \leq \sum mE_n = 0$ . Namely,  $f = 0$  a.e.  $\square$

5.

*Proof.* For any fixed  $x_0 \in \mathbb{R}$ , let  $f_n(x) = f \cdot \chi_{(-\infty, x_0 - 1/n]}$ , which is a increasing sequence of nonnegative measurable function whose limit is  $f \cdot \chi_{(-\infty, x_0]}$ . Then by Theorem 10,

$$F(x_0) = \int_{-\infty}^{x_0} f = \int f \cdot \chi_{(-\infty, x_0]} = \lim \int f \cdot \chi_{(-\infty, x_0 - 1/n]} = \lim F(x_0 - 1/n).$$

Meanwhile, since

$$|F(x_0) - F(x_0 + 1/n)| = \left| \int_{x_0}^{x_0 + 1/n} f(x) dx \right| = \left| \int_{-1/n}^0 g(x) dx \right|,$$

where  $g(x) = f(x_0 - x)$ , arguing on  $g$  in a similar manner yields  $F(x_0) = \lim F(x_0 + 1/n)$ . Thus,  $F$  is continuous.  $\square$

6.

*Proof.* By Theorem 9,  $\int f \leq \underline{\lim} \int f_n$ . Meanwhile,  $f_n \leq f$  implies  $\int f_n \leq \int f$  and therefore  $\overline{\lim} \int f_n \leq \int f$ . Hence,  $\int f = \lim \int f_n$ .  $\square$

7.

*Solution.*

(a) Let  $f_n(x) = n \cdot \chi_{[0, 1/n]}$ .  $f_n$  converges to  $f = 0$  except on  $x = 0$ . For each  $n$ ,  $\int f_n = 1$  but  $\int f = 0$ . Hence, the inequality could be strict.

(b) Let  $f_n(x) = \chi_{[n, \infty)}$ . Then  $\langle f_n \rangle$  is a decreasing sequence which converges to  $f = 0$ , the integral of which is 0. However, for every  $n$ ,  $\int f_n = \infty$ .  $\square$

8.

*Proof.* Let  $g_n = \inf\{f_n, f_{n+1}, \dots\}$ . Clear that

$$\int g_n \leq \int f_n. \tag{1}$$

Meanwhile  $\langle g_n \rangle$  is a increasing sequence converging to  $\underline{\lim} f_n$ . Hence, by the Monotone Convergence Theorem and (1)

$$\int \underline{\lim} f_n = \int \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} \int g_n \leq \underline{\lim} \int f_n.$$

$\square$

9.

*Proof.* By Fatou's Lemma,

$$\int_E f \leq \underline{\lim} \int_E f_n. \tag{2}$$

Similarly,  $\int_{\bar{E}} f \leq \underline{\lim} \int_{\bar{E}} f_n$  and therefore

$$\int_E f_n = \int f_n - \int_{\bar{E}} f_n \Rightarrow \overline{\lim} \int_E f_n \leq \int f - \int_{\bar{E}} f = \int_{\bar{E}} f.$$

(2) and the inequality above together implies  $\int_E f_n \rightarrow \int f$ .  $\square$



## 4.4 The General Lebesgue Integral

12.

*Proof.* Note that  $\langle g + f_n \rangle$  is a sequence of nonnegative measurable functions. Hence by Problem 8,

$$\int_E \underline{\lim}(g + f_n) \leq \underline{\lim} \int_E (g + f_n) \Rightarrow \int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n.$$

The second inequality follows immediately from the definition of lower and upper limit. Replacing  $g + f_n$  with  $g - f_n$  and arguing in a similar manner gives the last inequality.  $\square$

13.

*Proof.*  $f_n \geq -h$  implies  $f_n + h \geq 0$ . Hence,  $\int(f_n + h)$  always has a meaning. And since  $g$  is integrable,  $\int f_n = \int(f_n + h) - \int h$  also has a meaning. Similarly,  $\int f$  has a meaning. Meanwhile,

$$\int f = \int(f + h) - \int h \leq \underline{\lim} \int(f_n + h) - \int h = \underline{\lim} \int f_n.$$

$\square$

15.

*Proof.*

(a) By Problem 4, for every  $\varepsilon > 0$ , there exists some simple functions  $\varphi_1 \leq f^+$  and  $\varphi_2 \leq f^-$  such that

$$\int_E f^+ - \int_E \varphi_1 < \varepsilon \quad \text{and} \quad \int_E f^- - \int_E \varphi_2 < \varepsilon.$$

Let  $\varphi = \varphi_1 - \varphi_2$ , which is also a simple function. Meanwhile,

$$\int_E |f - \varphi| \leq \int_E (f^+ - \varphi_1) + \int_E (f^- - \varphi_2) < 2\varepsilon.$$

$\square$

16.

*Proof.* For every integrable  $f$ , by Problem 15, there exists some step function  $\psi = \sum_{k=1}^N c_k \chi_{E_k}$  such that  $\int |f - \psi| < \varepsilon$ . Note that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \psi(x) \cos nx dx = \lim_{n \rightarrow \infty} \sum_{k=1}^N c_k \int_{E_k} \cos nx dx = 0. \quad (3)$$

Hence,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} f(x) \cos nx dx \right| &= \left| \int_{-\infty}^{\infty} (f(x) - \psi(x) + \psi(x)) \cos nx dx \right| \\ &\leq \int_{-\infty}^{\infty} |f(x) - \psi(x)| |\cos nx| dx + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx dx \right| \\ &\leq \varepsilon + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx dx \right| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

□

**18.**

*Proof.* Let  $\langle t_n \rangle$  be any sequence with  $t_n \neq 0$  and tending to 0. Then  $\langle f(x, t_n) \rangle$  is sequence of functions satisfying the hypotheses of Lebesgue Convergence Theorem. Meanwhile,  $f(x, t_n) \rightarrow f$  as  $n \rightarrow \infty$ . Hence,

$$\lim_{n \rightarrow \infty} \int f(x, t_n) dx = \int f(x) dx.$$

Since the choice of  $\langle t_n \rangle$  is arbitrary, by Problem 2.49f,

$$\lim_{t \rightarrow 0} \int f(x, t) dx = \int f(x) dx.$$

If  $f$  is continuous in  $t$  for each  $x$ , then  $\lim_{\Delta t \rightarrow 0} f(x, t + \Delta t) = f(x, t)$  holds for every  $t$ . Therefore, replacing  $t$  with  $\Delta t$  in the previous result yields

$$\lim_{\Delta t \rightarrow 0} \int f(x, t + \Delta t) dx = \int f(x, t) dx.$$

Namely,  $h(t)$  is continuous.

□

## 5 Differentiation and Integration

### 5.1 Differentiation of Monotone Functions

3. "maximum" needs to be changed to "minimum" in both (a) and (b).

*Proof.*

(a) We may assume without loss of generality that  $c = 0$ . Since  $f$  attains a local minimum at  $x = 0$ ,  $f(h) \geq f(0)$  for every  $h$  sufficiently small. Hence, for every small  $h > 0$ ,  $(f(c+h) - f(c))/h > 0$  and therefore  $D_+f(c) \geq 0$ . Meanwhile, by Problem 2.b,

$$-D_-f(0) = D^+f(0) \geq 0 \quad \Rightarrow \quad D_-f(0) \leq 0.$$

The other two inequalities follow immediately from the definitions of upper and lower limits.

(b) If  $f$  has a local minimum at  $a$  or  $b$ , then we only have the right or left half of the inequalities.  $\square$

4.

*Proof.* We first show this for  $g$  with  $D^+g \geq \varepsilon > 0$ . For every  $a \leq x < y \leq b$ , as  $g$  is continuous on  $[a, b]$ ,  $g$  has a maximum in  $[a, b]$  and by Problem 2 and 3,  $g$  can not attain the maximum in  $[a, b)$ . Namely, the restrict of  $f$  to  $[x, y]$  attains the maximum at  $y$ . Hence,  $g(x) \leq g(y)$ .

For every  $f$  with nonnegative  $D^+$ , let  $g(x) = f(x) + \varepsilon x$  where  $\varepsilon > 0$ . Then  $D^+g \geq \varepsilon > 0$ . Hence  $g$  is nondecreasing. Therefore, for every  $a \leq x < y \leq b$ ,

$$g(x) \leq g(y) \quad \Rightarrow \quad f(x) + \varepsilon x \leq f(y) + \varepsilon y.$$

Since the choice of  $\varepsilon$  is arbitrary, this implies  $f(x) \leq f(y)$ .  $\square$

5.a

*Proof.*

$$\begin{aligned} \sup_{t \in (0, h)} \frac{(f+g)(x+t) - (f+g)(x)}{t} &= \sup_{t \in (0, h)} \left( \frac{f(x+t) - f(x)}{t} + \frac{g(x+t) - g(x)}{t} \right) \\ &\leq \sup_{t \in (0, h)} \frac{f(x+t) - f(x)}{t} + \sup_{t \in (0, h)} \frac{g(x+t) - g(x)}{t}. \end{aligned}$$

Letting  $h \rightarrow 0$  yields  $D^+(f+g) \leq D^+f + D^+g$ .  $\square$

### 5.2 Functions of Bounded Variation

7.

*Proof.*

(a) It suffices to show this for monotone functions as each function of bounded variation is the difference of two monotone functions. Suppose that  $f$  is nondecreasing. Then the set  $E = \{f(x) : x > c\}$  is bounded below and hence  $A = \inf E$  is finite. For every  $\varepsilon > 0$ , there exists some  $y > c$  such that  $A \leq f(c) < A + \varepsilon$ . Hence, as  $f$  is nondecreasing,

for every  $x \in (c, y)$ ,  $|f(x) - A| < \varepsilon$ . Namely,  $\lim_{x \rightarrow c+} f(x) = A$ . Similarly,  $\lim_{x \rightarrow c-} f(x)$  exists.

Let  $D_n = \{x : |f(x+) - f(x-)| > 1/n\}$ . Since  $f$  is nondecreasing,  $|f(x) - f(y)| \leq f(b) - f(a) < \infty$  for every  $x, y \in [a, b]$ . Hence,  $D_n$  is finite, otherwise we can choose a sequence  $x_1 < \dots < x_N$  with  $N > (f(b) - f(a))/n$  such that  $f(x_N) - f(x_1) > f(b) - f(a)$ . Therefore,  $\bigcup_{n=1}^{\infty} D_n$ , the set of discontinuities, is countable.

(b) Suppose  $\{x_1, \dots, x_n, \dots\} = \mathbb{Q} \cap [0, 1]$  and define  $f(x) = \sum_{x_n < x} 2^{-n}$ . Clear that  $f$  is monotone and continuous at every irrational point. For each rational  $x = x_k$ ,  $f(x+) - f(x-) = 2^{-k}$ . Hence,  $f$  is discontinuous at each rational point.  $\square$

## 8.

*Proof.*

(a) For every  $\varepsilon > 0$ , there exists some subdivision  $a = x_0 < \dots < x_p = c$  and  $c = x_p < \dots < x_q = b$  of  $[a, c]$  and  $[c, b]$  such that  $T_a^c < t_a^c + \varepsilon$  and  $T_c^b < t_c^b + \varepsilon$ . Hence,  $T_a^c + T_c^b - 2\varepsilon < t_a^c + t_c^b$ . Meanwhile, as  $a = x_0 < \dots < x_q = b$  forms a subdivision of  $[a, b]$ ,  $T_a^b \geq t_a^b = t_a^c + t_c^b$ . Therefore,  $T_a^c + T_c^b - 2\varepsilon < T_a^b$ . Since the choice of  $\varepsilon$  is arbitrary,  $T_a^b + T_c^b \leq T_a^b$ .

To show that  $T_a^b + T_c^b \geq T_a^b$ , let  $a = x_0 < \dots < x_q = b$  be any subdivision of  $[a, b]$  and by adding  $c$  into it, we get subdivisions of  $[a, c]$  and  $[c, b]$ . Suppose that  $c \in (x_k, x_{k+1}]$ , then

$$|f(x_k) - f(c)| + |f(c) - f(x_{k+1})| + t_a^b = t_a^c + t_c^b + |f(x_k) - f(x_{k+1})|,$$

which implies  $t_a^b \leq t_a^c + t_c^b$ . Hence,

$$T_a^b = \sup t_a^b \leq \sup(t_a^c + t_c^b) \leq T_a^c + T_c^b.$$

Thus,  $T_a^b = T_a^c + T_c^b$  and therefore  $T_a^c \leq T_a^b$ .

(b)

$$\begin{aligned} T_a^b(f+g) &= \sup \sum_{i=1}^k |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\leq \sup \sum_{i=1}^k |f(x_i) - f(x_{i-1})| + \sup \sum_{i=1}^k |g(x_i) - g(x_{i-1})| \\ &\leq T_a^b(f) + T_a^b(g). \end{aligned}$$

$$T_a^b(cf) = \sup \sum_{i=1}^k |cf(x_i) - cf(x_{i-1})| = |c| \sup \sum_{i=1}^k |f(x_i) - f(x_{i-1})| = |c| T_a^b(f).$$

$\square$

## 9.

*Proof.* For every  $\varepsilon > 0$ , there exists a subdivision  $a = x_0 < \cdots < x_k = b$  such that  $t_a^b(f) \geq T_a^b(f) - \varepsilon$ . Meanwhile, as  $f_n$  converges to  $f$  pointwisely

$$\begin{aligned} t_a^b(f) &= t_a^b(\lim f_n) \\ &= \sum_{i=1}^k |(\lim f_n)(x_i) - (\lim f_n)(x_{i-1})| \\ &= \lim \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| \\ &\leq \underline{\lim} \sup \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| = \underline{\lim} T_a^b(f_n). \end{aligned}$$

Hence,  $T_a^b(f) - \varepsilon \leq \underline{\lim} T_a^b(f_n)$ . Since the choice of  $\varepsilon$  is arbitrary,  $T_a^b(f) \leq \underline{\lim} T_a^b(f_n)$ .  $\square$

### 10.a

*Solution.* No. Let  $x_k = (k\pi + \pi/2)^{-1/2}$ ,  $k = 0, 1, \dots$  and consider the subdivision  $-1 < 0 < x_n < \cdots < x_0 < 1$ . Then

$$t_n \geq \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \geq \sum_{k=1}^n \frac{2}{(k + 1/2)\pi}.$$

$t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and therefore  $f$  is not of bounded variation on  $[-1, 1]$ .  $\square$

### 11.

*Proof.* By Lemma 4,  $f(x) = f(a) + P_a^x - N_a^x$ . Since  $P_a^x$  and  $N_a^x$  are monotone, by Theorem 3, they are differentiable almost everywhere as  $f$ , a function of bounded variation, does. Hence, for almost every  $x \in [a, b]$ ,

$$\frac{d}{dx}f(x) = \frac{d}{dx}P_a^x - \frac{d}{dx}N_a^x \quad \Rightarrow \quad |f'(x)| \leq \frac{d}{dx}P_a^x + \frac{d}{dx}N_a^x = \frac{d}{dx}T_a^x.$$

Integrating on the both sides yields  $\int_a^b |f'| \leq T_a^b(f)$ .  $\square$

## 5.4 Absolute Continuity

### 12.

*Solution.* The continuous extension of  $x^2 \sin(1/x^2)$  to  $[0, 1]$  is absolutely continuous for all  $[\varepsilon, 1]$  but is not of bounded variation on  $[0, 1]$  and therefore is not absolutely continuous on  $[0, 1]$ .

Suppose that  $f$  is also of bounded variation on  $[0, 1]$ . Then  $f$  is differentiable almost everywhere. Hence  $g(x) = \int_0^x f'(t)dt + f(a)$  is well-defined. For every  $\varepsilon > 0$ , we have

$$g(x) = \int_0^\varepsilon f'(t)dt + \int_\varepsilon^x f'(t)dt + f(0) = \int_0^\varepsilon f'(t)dt + f(x) - f(\varepsilon) + f(0),$$

where the second equality comes from the absolute continuity on  $[\varepsilon, 1]$ . By the continuity of  $f$  at  $x = 0$ ,  $f(\varepsilon) \rightarrow f(0)$ . Hence, letting  $\varepsilon \rightarrow 0$  yields  $g(x) = f(x)$ . Namely,  $f$  is an indefinite integral. Thus, by Theorem 14, it is absolutely continuous.  $\square$

13.

*Proof.* Since absolute continuity implies bounded variation,  $\int_a^b |f'| \leq T_a^b(f)$  by Problem 11. By the definition of  $T$ , for every  $\varepsilon > 0$ , there exists some subdivision  $a = x_0 < \cdots < x_n = b$  such that  $T_a^b(f) > T_a^b(f) - \varepsilon$ . Meanwhile, for every  $i = 1, \dots, n$ ,

$$\int_{x_{i-1}}^{x_i} |f'| \geq \left| \int_{x_{i-1}}^{x_i} f' \right| = |f(x_i) - f(x_{i-1})|,$$

where the second equality is guaranteed by the absolute continuity. Hence,  $\int_a^b |f'| > T_a^b(f) - \varepsilon$  for every  $\varepsilon > 0$ . Thus,  $T_a^b(f) = \int_a^b |f'|$ .

By Lemma 4,  $2P_a^b(f) = T_a^b(f) + f(b) - f(a)$ . Hence,

$$P_a^b(f) = \frac{1}{2} \left( \int_a^b |f'| + f(b) - f(a) \right) = \frac{1}{2} \int_a^b (|f'| + f') = \int_a^b [f']^+.$$

□

14.

*Proof.*

(a) Suppose that  $f$  and  $g$  are absolutely continuous. Then for every  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that for all finite nonoverlapping  $\langle (x_n, y_n) \rangle$  with  $|x_n - y_n| < \varepsilon$ ,

$$\sum |f(x_n) + g(x_n) - f(y_n) - g(y_n)| \leq \sum |f(x_n) - f(y_n)| + |g(x_n) - g(y_n)| \leq 2\varepsilon.$$

Hence,  $f + g$  is also absolutely continuous. Since  $-g$  is absolutely continuous as long as  $g$  is, so is  $f - g$ .

(b) Suppose that  $f$  and  $g$  are absolutely continuous. Then they are bounded, by  $M$  for example. Hence for every  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that for all finite nonoverlapping  $\langle (x_n, y_n) \rangle$  with  $|x_n - y_n| < \varepsilon$ ,

$$\begin{aligned} & \sum |f(x_n)g(x_n) - f(y_n)g(y_n)| \\ &= \sum |f(x_n)g(x_n) - f(x_n)g(y_n) + f(x_n)g(y_n) - f(y_n)g(y_n)| \\ &\leq \sum \{|f(x_n)||g(x_n) - g(y_n)| + |f(x_n) - f(y_n)||g(y_n)|\} \\ &\leq M\varepsilon. \end{aligned}$$

Thus,  $fg$  is also absolutely continuous.

(c) Since  $f$  is continuous on  $[a, b]$ ,  $f$  can achieve its minimum in  $[a, b]$ . Hence,  $|f(x)| \geq m > 0$  as  $f$  is never zero. Therefore for every  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that for all finite nonoverlapping  $\langle (x_n, y_n) \rangle$  with  $|x_n - y_n| < \varepsilon$ ,

$$\sum \left| \frac{1}{f(x_n)} - \frac{1}{f(y_n)} \right| = \sum \left| \frac{f(x_n) - f(y_n)}{f(x_n)f(y_n)} \right| \leq \frac{1}{m^2} \sum |f(x_n) - f(y_n)| \leq \frac{\varepsilon}{m^2}.$$

□

17. Part (a) is wrong. It can be fixed if we further require  $g$  to be monotone increasing.

*Proof.*

(a) For every  $\varepsilon > 0$ , let  $\delta_1$  be the number in the definition of  $F$  corresponding to  $\varepsilon$  and  $\delta_2$  the number in the definition of  $g$  corresponding to  $\delta_1$ . Then for every finite nonoverlapping  $\langle (x_n, y_n) \rangle$  with  $|x_n - y_n| < \delta_2$ ,  $\sum |g(x_n) - g(y_n)| < \delta_1$ . Since  $g$  is monotone increasing,  $(g(x_n), g(y_n))$  are nonoverlapping. Therefore,  $\sum |F(g(x_n)) - F(g(y_n))| < \varepsilon$ . Hence,  $F \circ g$  is absolutely continuous.  $\square$

18.

*Proof.* Without loss of generality, we assume that  $g$  is nondecreasing. Since  $mE = 0$ , for every  $\varepsilon > 0$ , by Proposition 3.15, there exists an open set  $O \supset E$  with  $mO < \varepsilon$ . Meanwhile, there exists a sequence of disjoint open intervals  $\langle I_n = (a_n, b_n) \rangle$  such that  $\bigcup_{n=1}^{\infty} I_n = O$  and  $l(I_n) < \delta$  where  $\delta$  is the number in the definition of absolute continuity. Then  $g[E] \subset \bigcup_{n=1}^{\infty} g[I_n \cap [0, 1]]$ . Since  $g$  is continuous, the image of an interval is still an interval and since  $g$  is also nondecreasing,  $g[I_n \cap [0, 1]] = (g(a'_n), g(b'_n))$ , where  $a'_n = \max\{a_n, 0\}$  and  $b'_n = \min\{b_n, 1\}$ . Finally,

$$m(g[E]) \leq \sum_{n=1}^{\infty} m(g[I_n]) = \sum_{n=1}^{\infty} |g(b'_n) - g(a'_n)| \leq \varepsilon,$$

where the last inequality comes from the absolute continuity of  $g$ . Since the choice of  $\varepsilon$  is arbitrary,  $m(g[E]) = 0$ .  $\square$

20.

*Proof.*

(a) For every  $\varepsilon > 0$ , let  $\delta = \varepsilon/M$ . Then for every  $\langle x_n \rangle_{i=1}^n$  and  $\langle y_n \rangle_{i=1}^n$  with  $|x_n - y_n| \leq \delta$ ,

$$\sum_{i=1}^n |f(x_n) - f(y_n)| \leq M \sum_{i=1}^n |x_n - y_n| \leq \varepsilon,$$

as  $f$  satisfies the Lipschitz condition.

(b) Suppose that  $f$  is absolute continuous and  $|f'|$  is bounded by  $M$ . Then for every  $x$  and  $y$  in the interval,

$$|f(x) - f(y)| = \left| \int_x^y f'(t) dt \right| \leq M|x - y|.$$

Hence,  $f$  satisfies the Lipschitz condition. The converse part has been proved in (a).

(c) It is wrong. A counterexample is  $f(x) = \chi_{[0,1]}$ ,  $x \in (-1, 1)$   $\square$

21.

*Proof.*

(a) Suppose that  $O = \bigcup_{n=1}^{\infty} (c_n, d_n)$  where  $(c_n, d_n)$  are disjoint. Since  $g$  is continuous and increasing,  $g^{-1}(c_n, d_n)$  is still an open interval, denoting it by  $(a_n, b_n)$ , and  $(a_n, b_n)$  are also disjoint. Meanwhile,  $d_n - c_n = f(a_n) - f(b_n) = \int_{a_n}^{b_n} g'$ . Hence,

$$mO = m\left(\bigcup_{n=1}^{\infty} (c_n, d_n)\right) = \sum_{n=1}^{\infty} (d_n - c_n) = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} g' = \int_{g^{-1}[O]} g'.$$

(b) Without loss of generality, we assume that  $d \notin E$ . For every  $\varepsilon > 0$ , there exists an open set  $O \supset E$  with  $mO < \varepsilon$ . By Part (a),

$$\int_{g^{-1}[O] \cap H} g' = \int_{g^{-1}[O]} g' = mO < \varepsilon.$$

Since the choice of  $\varepsilon$  is arbitrary,  $\int_{g^{-1}[O] \cap H} g' = 0$ . Since  $g' > 0$  on  $H$ ,  $g^{-1}[O] \cap H$  has measure zero.

(c) Since  $E$  is measurable, so is  $g^{-1}[E]$ . Meanwhile, by Theorem 3,  $g'$  is measurable, hence  $H$  is also measurable. Therefore,  $F$  is measurable.

We may assume without loss of generality that  $c, d \notin E$ . By Proposition 3.15, there exists some  $G \in G_\delta$  such that  $E \subset G \subset (c, d)$  and  $m(G \setminus E) = 0$ . Since  $g$  is increasing,  $g^{-1}[G] \cap H = F \cup (g^{-1}[G \setminus E] \cap H)$  and by (b),  $g[G \setminus E] \cap H$  is of measure zero. Therefore,  $\int_F g' = \int_{g^{-1}[G] \cap H} g'$ . Namely, it suffices to show the result for  $G \in G_\delta$ .

Suppose that  $G = \bigcap_{n=1}^{\infty} O_n$  where each  $O_n \subset (c, d)$  is open and  $mO_1 < \infty$ . Without loss of generality, we may assume that  $\langle O_n \rangle$  is decreasing. Then  $mG = \lim_{n \rightarrow \infty} mO_n$ . By (a),

$$mO_n = \int_{g^{-1}[O_n]} g' = \int_a^b \chi_{O_n}(g(x)) g'(x) dx.$$

As  $\chi_{O_n}(g(x)) g'(x)$  is bounded by  $|g'|$ ,

$$\lim_{n \rightarrow \infty} \int_a^b \chi_{O_n}(g(x)) g'(x) dx = \int_a^b \chi_G(g(x)) g'(x) dx.$$

Hence,  $mG = \int_{g^{-1}[G] \cap H} g'$ , completing the proof.

(d) By Problem 3.25,  $f \circ g$  is measurable. And since  $g'$  is measurable by Theorem 3,  $(f \circ g)g'$  is also measurable.

Let  $\langle \varphi_n \rangle$  be an increasing sequence of nonnegative simple functions which converges to  $f$ , the existence of which is guaranteed by Problem 4.4. By the monotone convergence theorem,  $\int_c^d f = \lim \int_c^d \varphi_n$ .

For each  $n$ , suppose that  $\varphi_n(y) = \sum_{k=1}^m a_k^{(n)}(y) \chi_{E_k^{(n)}}(y)$ . Then

$$\int_c^d \varphi_n = \sum_{k=1}^m a_k^{(n)} mE_k^{(n)} = \sum_{k=1}^m a_k^{(n)} \int_a^b \chi_{E_k^{(n)}}(g(x)) g'(x) dx = \int_a^b \varphi_n(g(x)) g'(x) dx,$$

where the second equality comes from (c). Since  $g$  is increasing,  $\langle \varphi_n(g(x)) g'(x) \rangle$  is an increasing sequence. Hence,

$$\int_a^b f(g(x)) g'(x) dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_n(g(x)) g'(x) dx.$$

Thus,

$$\int_c^d f(y) dy = \lim_{n \rightarrow \infty} \int_c^d \varphi_n(y) dy = \lim_{n \rightarrow \infty} \int_a^b \varphi_n(g(x)) g'(x) dx = \int_a^b f(g(x)) g'(x) dx.$$

□



## 5.5 Convex Functions

23.

*Proof.*

(a) Suppose that  $x_0 \in (a, b)$  and  $y(x) = m(x - x_0) + \varphi(x_0)$  is a supporting line. As  $[a, b]$  is finite,  $\varphi \geq \min\{\varphi(a), y(a), y(b)\}$ .

(b) If  $\varphi$  is monotone, then the limits exists. If  $\varphi$  is not monotone, then since  $D^+\varphi$  is nondecreasing, there exists some  $[c, d] \subset (a, b)$  such that  $D^+\varphi \leq 0$  on  $(a, c)$  and  $D^+\varphi \geq 0$  on  $(d, b)$ . Namely,  $\varphi$  is monotone on the  $(a, c)$  and  $(d, b)$ . Therefore, the limits also exist.

Consider a finite interval near the finite endpoint. By (a), the limit can not be  $-\infty$  as  $\varphi$  is bounded from below.

(c) If  $x$  and  $y$  are in the interior of  $I$ , the inequality holds by definition. By the continuity of  $\varphi$ , the statement holds for all  $x, y \in I$ .  $\square$

24.

*Proof.* Note that the existence of  $\varphi''$  implies  $\varphi$  is continuously differentiable. Suppose that  $\varphi$  is convex on  $(a, b)$ . Then  $D^+\varphi$  is nondecreasing by Proposition 17, hence  $\varphi''(x) \geq 0$  for each  $x \in (a, b)$ . And the converse of the statement follows from Proposition 18 immediately.  $\square$

25.

*Proof.*

(a)  $\varphi''(t) = b^2 p(p-1)(a+bt)^{p-2}$  which  $\geq 0$  on  $[0, \infty)$  if  $p \geq 1$  and  $\leq 0$  if  $0 < p \leq 1$ .  $\square$

26. TODO

27.

*Proof.* Note that  $\log x$  is concave. Denote  $A_N = \sum_{n=1}^N \alpha_n$  and  $R_N = 1 - A_N$ . The situation where  $\langle \alpha_n \rangle$  is finite is simple. Hence we assume that  $R_N \geq 0$  for all  $N$ . Then for every  $N$ ,

$$\begin{aligned} \log \left( \sum_{n=1}^{\infty} \alpha_n \xi_n \right) &= \log \left( A_N \sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n + R_N \sum_{n=N+1}^{\infty} \frac{\alpha_n}{R_N} \xi_n \right) \\ &\geq A_N \log \left( \sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n \right) + R_N \log \left( \sum_{n=N+1}^{\infty} \frac{\alpha_n}{R_N} \xi_n \right) \\ &\geq A_N \log \left( \sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n \right) \\ &\geq A_N \log \left( \prod_{n=1}^N \xi_n^{\alpha_n/A_N} \right) \end{aligned}$$

Taking exp on the both sides yields

$$\sum_{n=1}^{\infty} \alpha_n \xi_n \geq \left( \prod_{n=1}^N \xi_n^{\alpha_n/A_N} \right)^{A_N} = \prod_{n=1}^N \xi_n^{\alpha_n} \rightarrow \prod_{n=1}^{\infty} \xi_n^{\alpha_n}.$$

$\square$

**28.**

*Proof.* It follows immediately from the Jensen inequality and the fact that  $\log$  is concave. □

## 6 The Classical Banach Spaces

### 6.1 The $L^p$ Spaces

1.

*Proof.* Put  $S = \|f\|_\infty$  and  $T = \|g\|_\infty$ . Then  $|f(t)| \leq S$  and  $|g(t)| \leq T$  a.e. Hence,  $S + T \geq |f(t)| + |g(t)| \geq |f(t) + g(t)|$  a.e. Namely,  $m\{t : |f(t) + g(t)| > S + T\} = 0$ . Thus,  $S + T \geq \|f + g\|_\infty$  by the definition of  $\text{ess sup}$ .  $\square$

2.

*Proof.* Put  $S = \|f\|_\infty$ . Since  $S \geq |f|$  a.e.,

$$\|f\|_p = \left\{ \int_0^1 |f|^p \right\}^{1/p} \leq \left\{ \int_0^1 S^p \right\}^{1/p} = S.$$

Therefore,  $\overline{\lim}_{p \rightarrow \infty} \|f\|_p \leq S$ . For the converse part, let  $\varepsilon$  be any positive number. Then the measure  $\delta$  of  $E = \{t : |f(t)| > S - \varepsilon\}$  is positive. Hence,

$$\left\{ \int_0^1 |f|^p \right\}^{1/p} \geq \left\{ \int_E |f|^p \right\}^{1/p} \geq \delta^{1/p} (S - \varepsilon) \rightarrow S - \varepsilon \quad \text{as } p \rightarrow \infty.$$

Hence,  $\underline{\lim}_{p \rightarrow \infty} \|f\|_p \geq S$ , completing the proof.  $\square$

3.

*Proof.*

$$\|f + g\|_1 = \int |f + g| \leq \int |f| + \int |g| = \|f\|_1 + \|g\|_1.$$

$\square$

4.

*Proof.* For every  $M > \|g\|_\infty$ ,  $|g| \leq M$  a.e. Hence,

$$\int |fg| \leq M \int |f| = \|f\|_1 M.$$

Since the choice of  $M$  is arbitrary,  $\int |fg| \leq \|f\|_1 \|g\|_\infty$ .  $\square$

### 6.2 The Minkowski and Hölder Inequalities

8

*Proof.*

(a) The logarithm function is concave, so

$$\log(a^p/p + b^q/q) \geq \frac{1}{p} \log a^p + \frac{1}{q} \log b^q = \log ab.$$

Taking  $\exp$  on the both sides yields the inequality. The equality holds iff  $a^p = b^q$ .

(b) The case where  $p = \infty$  has been proved in Problem 4 and the case where  $\|f\|_p = 0$  or  $\|g\|_q = 0$  is straightforward. Hence, we assume that  $1 < p, q < \infty$  and  $\|f\|_p \|g\|_q \neq 0$ .

Suppose  $\alpha = \|f\|_p$  and  $\beta = \|g\|_q$ . By Young's inequality,

$$\left| \frac{fg}{\alpha\beta} \right| \leq \frac{1}{p} \left( \frac{|f|}{\alpha} \right)^p + \frac{1}{q} \left( \frac{|g|}{\beta} \right)^q$$

for every  $x$ . Therefore,

$$\int |fg| = \alpha\beta \int \left| \frac{fg}{\alpha\beta} \right| \leq \alpha\beta \int \left\{ \frac{1}{p} \left( \frac{|f|}{\alpha} \right)^p + \frac{1}{q} \left( \frac{|g|}{\beta} \right)^q \right\} = \alpha\beta. \quad (4)$$

The equality holds iff the equality in Young's inequality holds a.e. iff  $\beta|f|^p = \alpha|g|^q$  a.e.

(c) Let  $p' = 1/p$  and  $q' = 1 - p' = -q/p$ . Then for any nonnegative  $c$  and  $d$ , by Young's inequality,

$$cd \leq \frac{c^{p'}}{p'} + \frac{d^{q'}}{q'} = pc^{1/p} - \frac{p}{q} d^{-q/p} \Rightarrow c^{1/p} \geq \frac{cd}{p} + \frac{d^{-q/p}}{q}.$$

Putting  $c = (ab)^p$  and  $d = b^{-p}$  yields the desired inequality.

(d) Just reverse the inequality in (4). □

### 6.3 Convergence and Completeness

9.

*Proof.* Suppose  $\langle f_n \rangle \subset X$  converges to  $f \in X$ . Namely, for every  $\varepsilon > 0$ , there exists some  $N$  such that for all  $n > N$ ,  $\|f_n - f\| < \varepsilon$ . Hence, for every  $n, m > N$ , by Minkowski inequality,

$$\|f_n - f_m\| \leq \|f_n - f\| + \|f - f_m\| < 2\varepsilon.$$

Hence,  $\langle f_n \rangle$  is a Cauchy sequence. □

10.

*Proof.* Suppose  $f_n \rightarrow f$ . Then  $M_n = \|f_n - f\|_\infty = \text{ess sup } |f_n - f| \rightarrow 0$ . Let  $E_n = \{x : |f_n(x) - f(x)| > M_n\}$ , each of which is with measure zero. And therefore  $E = \bigcup_{n=1}^\infty E_n$  is with measure zero. Note that  $\tilde{E} = \{x : |f_n(x) - f(x)| < M_n, \forall n\}$ , which implies the uniform convergence of  $f_n$  since  $M_n \rightarrow 0$ .

For the converse part, the uniform convergence on  $\tilde{E}$  implies that for every  $\varepsilon > 0$ , there exists some  $N$  such that for every  $n > N$  and  $x \in \tilde{E}$ ,  $|f_n(x) - f(x)| < \varepsilon$ . Since  $mE = 0$ , this implies  $\|f_n - f\|_\infty = \text{ess sup } |f_n(x) - f(x)| < \varepsilon$ . Hence,  $f_n \rightarrow f$  in  $L^\infty$ . □

11.

*Proof.* Let  $\langle f_n \rangle \subset L^\infty$  be absolutely summable. Put  $M_n = \|f_n\|_\infty$  and  $A_n = \{t : |f_n(t)| > M_n\}$ . By the definition of  $\|\cdot\|_\infty$ ,  $mA_n = 0$ . Hence,  $A = \bigcup_{n=1}^\infty A_n$  is of measure zero.

Note that  $|f_n(x)| \leq M_n$  for every  $n$  and  $x \in E \setminus A$ . Thus, by the Weierstrass M-test,  $\sum_{n=1}^\infty f_n$  converges uniformly. Hence, on  $E \setminus A$ ,  $\sup |\sum_{n=1}^\infty f_n - \sum_{n=1}^N f_n| \rightarrow 0$  as  $N \rightarrow \infty$ . Since  $mA = 0$ , this implies the summability of  $\langle f_n \rangle$ . □

**13.**

*Proof.* Suppose  $\langle f_n \rangle \subset C$  be absolutely summable. Since for every  $x$ ,  $0 \leq |f_n(x)| \leq \|f_n\|$ ,  $\langle f_n \rangle$  is uniformly convergent on  $[0, 1]$ . Put  $s = \sum_{n=1}^{\infty} f_n$ . Since each  $f_n$  is continuous, so is  $s$ . Therefore,  $s \in C$ .

For every  $\varepsilon > 0$ , there exists some  $N$  such that for every  $n > N$  and  $x \in [0, 1]$ ,  $|s(x) - \sum_{k=1}^n f_k(x)| < \varepsilon$ . Hence,  $\|s - \sum_{k=1}^n f_k\| < \varepsilon$ . Thus,  $\langle f_n \rangle$  is summable and therefore  $C$  is a Banach space.  $\square$

**16.**

*Proof.* Since  $\|f_n - f\| \geq |||f_n| - |f|||$ ,  $f_n \rightarrow f$  in  $L^p$  implies  $\|f_n\| \rightarrow \|f\|$ . For the converse part, note that  $2^p(|f_n|^p + |f|^p) - |f_n - f|^p \geq 0$  and for almost every  $x$ ,

$$2^p(|f_n|^p + |f|^p) - |f_n - f|^p \rightarrow 2^{p+1}|f|^p.$$

By Fatou's Lemma,

$$\begin{aligned} 2^{p+1}\|f\|^p &= 2^{p+1} \int |f|^p \leq \underline{\lim} \int \{2^p(|f_n|^p + |f|^p) - |f_n - f|^p\} \\ &= 2^{p+1}\|f\|^p - \overline{\lim} \|f_n - f\|^p. \end{aligned}$$

Hence,  $\overline{\lim} \|f_n - f\|^p \leq 0$ . Since clear that  $\underline{\lim} \|f_n - f\|^p \geq 0$ ,  $\lim \|f_n - f\| = 0$ , i.e.,  $f_n \rightarrow f$  in  $L^p$ .  $\square$

**17.** I assume that  $1/p + 1/q = 1$ .

*Proof.* Since  $g \in L^p$ ,  $|g|^q$  is integrable on  $E = [0, 1]$  and therefore for every  $\varepsilon > 0$ , there exists some  $\delta$  such that for every  $A \subset E$  with  $mA < \delta$ ,  $\int_A |g|^q < \varepsilon$ . Meanwhile, since  $f_n(x) \rightarrow f(x)$  for almost every  $x$ , by Egoroff's Theorem, there exists some  $A \subset E$  with  $mA < \delta$  such that  $f_n g$  converges to  $f g$  uniformly on  $E \setminus A$ .

From the uniform convergence we conclude

$$\int_{E \setminus A} f g = \lim_{n \rightarrow \infty} \int_{E \setminus A} f_n g. \quad (5)$$

Meanwhile, by Hölder inequality,

$$\left| \int_A (f - f_n) g \right| \leq \int_A |(f - f_n) g| \leq \left\{ \int_A |f_n - f|^p \right\}^{1/p} \left\{ \int_A |g|^q \right\}^{1/q} \leq M \varepsilon^{1/q}.$$

Hence, (5) can be extended to  $E$ .

For  $p = 1$ , this is not true.  $f_n = n\chi_{[0, 1/n]}$  and  $g = \chi_{[0, 1]}$  gives a counterexample.  $\square$

**18.**

*Proof.* By Minkowski inequality,

$$\|g_n f_n - g f\| = \|g_n(f_n - f) + (g_n - g)f\| \leq \|g_n(f_n - f)\| + \|(g_n - g)f\|.$$

Fix  $\varepsilon > 0$ . Since  $f, g_n, g \in L^p$ ,  $|g_n - g|^p |f|^p$  is integrable and therefore there exists some  $\delta > 0$  such that for all subsets with measure  $< \delta$ , the integral of over it  $< \varepsilon$ . Meanwhile,

since  $g_n \rightarrow g$  a.e., by Egoroff's Theorem, there exists some  $A \subset E = [0, 1]$  with  $mA < \delta$  such that  $g_n \rightarrow g$  uniformly on  $E \setminus A$  and therefore there exists some  $N_1 > 0$  such that for all  $n > N_1$ ,  $|g_n(x) - g(x)|^p < \varepsilon$  for  $x \in E \setminus A$ . Thus, for every  $n > N_1$ ,

$$\begin{aligned} \|(g_n - g)f\| &= \left\{ \int_{E \setminus A} |g_n - g|^p |f|^p \right\}^{1/p} + \left\{ \int_A |g_n - g|^p |f|^p \right\}^{1/p} \\ &\leq \sqrt[p]{\varepsilon} \|f\| + \sqrt[p]{\varepsilon} \leq (\|f\| + 1)\varepsilon. \end{aligned}$$

Since  $|g_n| \leq M$ ,  $\|g_n(f_n - f)\| \leq M\|f_n - f\|$ . And since  $f_n \rightarrow f$  in  $L^p$ , there exists some  $N_2 > 0$  such that for all  $n > N_2$ ,  $\|f_n - f\| < \varepsilon$ . Put  $N = \max(N_1, N_2)$ , then for every  $n > N$ ,

$$\|g_n f_n - g f\| \leq (\|f\| + 1 + M)\varepsilon.$$

Hence,  $g_n f_n \rightarrow g f$  in  $L^p$ . □

## 6.4 Approximation in $L^p$

19.

*Proof.* Since  $\|T_\Delta f\| \leq \|T_\Delta |f|\|$  and  $\|f\| = \||f|\|$ , we may assume without loss of generality that  $f \geq 0$ . For  $p > 1$ , by Jensen's inequality,

$$\begin{aligned} \|T_\Delta f\|_p^p &= \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} \left( \frac{1}{\xi_k - \xi_{k-1}} \int_{\xi_{k-1}}^{\xi_k} f \right)^p \\ &\leq \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} \frac{1}{\xi_k - \xi_{k-1}} \int_{\xi_{k-1}}^{\xi_k} f^p \\ &= \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} f^p \\ &= \int_0^1 f^p = \|f\|_p^p. \end{aligned}$$

□

# 11 Measure and Integration

## 11.1 Measure Spaces

1.

*Proof.* Put  $B_1 = A_1$  and  $B_n = A_n \setminus A_{n-1}$  for  $n \geq 2$ .  $(B_n)$  is a sequence of disjoint measurable sets. By the countable additivity of  $\mu$ ,

$$\mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(B_k) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=1}^n B_k\right).$$

Since  $\bigcup_{k=1}^n B_k = \bigcup_{k=1}^n A_k$  for  $k = 1, \dots, n, \dots, \infty$ , this implies  $\mu(\bigcup A_k) = \lim \mu(\bigcup_{k=1}^n A_k)$ .  $\square$

3.

*Proof.*

(a) First,

$$0 = \mu(E_1 \triangle E_2) = \mu(E_1 \setminus E_2 \cup E_2 \setminus E_1) = \mu(E_1 \setminus E_2) + \mu(E_2 \setminus E_1).$$

Together with the nonnegativity of  $\mu$ , we conclude that  $\mu(E_1 \setminus E_2) = \mu(E_2 \setminus E_1) = 0$ . Note that

$$\mu(E_1 \cup E_2) = \mu(E_1 \setminus E_2 \cup E_2) = \mu(E_1 \setminus E_2) + \mu(E_2).$$

Hence,  $\mu(E_1 \cup E_2) = \mu(E_2)$ . Similarly,  $\mu(E_1 \cup E_2) = \mu(E_1)$ . Thus,  $\mu(E_1) = \mu(E_2)$ .

(b) Since  $\mu(E_1 \triangle E_2) = 0$  and  $E_2 \setminus E_1 \subset E_1 \triangle E_2$ , by the completeness of  $\mu$ ,  $E_2 \setminus E_1 \in \mathcal{B}$ . Similarly,  $E_1 \setminus E_2 \in \mathcal{B}$ . In consequence,  $E_1 \cap E_2 = E_1 \setminus (E_1 \setminus E_2) \in \mathcal{B}$  and, therefore,  $E_2 = (E_1 \cap E_2) \cup (E_2 \setminus E_1) \in \mathcal{B}$ .  $\square$

9.

*Proof.*

(a) First, we argue by contradiction to show that  $\mathcal{R}$  and  $\mathcal{R}'$ . Assume that there exists some  $E \in \mathcal{R} \cap \mathcal{R}'$ , that is,  $E \in \mathcal{R}$  and  $E^c \in \mathcal{R}$ . Then  $X = E \cup E^c \in \mathcal{R}$ , which contradict the assumption that  $\mathcal{R}$  is not a  $\sigma$ -algebra. Thus,  $\mathcal{R} \cap \mathcal{R}' = \emptyset$ .

Clear that  $\mathcal{R} \cup \mathcal{R}'$  is a  $\sigma$ -algebra containing  $\mathcal{R}$ . Hence,  $\mathcal{R} \cup \mathcal{R}' \supset \mathcal{B}$ . Meanwhile, since  $\mathcal{B} = \sigma(\mathcal{R})$ ,  $\mathcal{R} \cup \mathcal{R}' \subset \mathcal{B}$ . Thus,  $\mathcal{R} \cup \mathcal{R}' = \mathcal{B}$ .

(b) Since  $\emptyset \in \mathcal{R}$ ,  $\bar{\mu}\emptyset = \mu\emptyset = 0$ . Meanwhile, clear that  $\bar{\mu}$  is nonnegative. Let  $(E_n) \subset \mathcal{B}$  be a sequence of disjoint sets. By part (a), each  $E_n$  is either a element of  $\mathcal{R}$  or  $\mathcal{R}'$ . If all  $E_n \in \mathcal{R}$ , then by the countable additivity of  $\mu$ ,  $\mu(\bigcup E_n) = \sum \mu E_n$ . Suppose there exists some  $E_n$  in  $\mathcal{R}$  and some  $E_m$  in  $\mathcal{R}'$ . Let  $F_1$  and  $F_2$  be the union of theses sets respectively. Since  $\sigma$ -ring is closed under union,  $F_1 \in \mathcal{R}$ , and since  $(\bigcup E_m)^c = \bigcap E_m^c$ ,  $F_2 \in \mathcal{R}'$ . Hence,  $F_1 \cup F_2 \in \mathcal{R}'$ , otherwise,  $F_2 = (F_1 \cup F_2) \setminus F_1$  would be an element of  $\mathcal{R}$ . Therefore,  $\mu(\bigcup E_n) = \infty = \sum \mu E_n$ . Thus,  $\bar{\mu}$  is a measure on  $\mathcal{B}$ .

(c) Clear that  $\bar{\mu}$  is nonnegative and  $\bar{\mu}\emptyset = 0$ . Let  $(E_n) \subset \mathcal{B}$  be disjoint. Note that for  $E \in \mathcal{R}$ ,  $\mu E = \sup\{\mu A : A \subset E, A \in \mathcal{R}\}$ . Hence, it suffices to show that

$$M = \sup\left\{\mu A : A \subset \bigcup_n E_n, A \in \mathcal{R}\right\} = \sum_n \sup\{\mu A : A \subset E_n, A \in \mathcal{R}\} = \sum_n M_n.$$

By definition, for all  $\varepsilon > 0$ , there exists a sequence  $(A_n) \subset \mathcal{R}$  such that  $A_n \subset E_n$  and  $M_n < \mu A_n + \varepsilon/2^n$ . Put  $A = \bigcup A_n$ . Since  $(A_n)$  are disjoint as  $(E_n)$  are,

$$\sum M_n < \varepsilon + \sum \mu A_n = \varepsilon + \mu A.$$

Meanwhile, since  $A \subset \bigcup E_n$  and  $A \in \mathcal{R}$ ,  $\mu A \leq M$ . Therefore,  $\sum M_n < \varepsilon + M$ . Thus,  $\sum M_n \leq M$ .

For the converse, similarly, for every  $\varepsilon > 0$ , there exists an  $A \in \mathcal{R}$  such that  $A \subset \bigcup E_n$  and  $M - \varepsilon > \mu A$ . Put  $A_n = E_n \cap A$ . If  $E_n \in \mathcal{R}$ ,  $A_n \in \mathcal{R}$  by definition. If  $E_n \in \mathcal{R}'$ ,  $A_n = A \setminus E_n^c \in \mathcal{R}$ . Hence,  $A_n \in \mathcal{R}$  for each  $n$ . Thus,

$$M - \varepsilon < \mu A = \sum_n \mu A_n \leq \sum_n M_n,$$

implying that  $M \leq \sum M_n$ . Therefore,  $M = \sum M_n$ , i.e.,  $\underline{\mu}$  is countably additive. Thus, we conclude that  $\underline{\mu}$  is a measure on  $\mathcal{B}$ .

(d) Clear that  $\mu_\beta$  is nonnegative and  $\mu_\beta \emptyset = 0$ . The preceding discussion, *mutatis mutandis*, yields the countable additivity.  $\square$