# Solutions to
# *Introductory Functional Analysis with Applications*

Yunwei Ren

## Contents

# 2 Normed Spaces. Banach Spaces

## 2.3 Further Properties of Normed Spaces

**4.** cf. Prob. 13, Sec 1.2

*Proof.* The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \le \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \le |\alpha_1|\|x_1 - x_2\| + |\alpha_1 - \alpha_2|\|x_2\|.$$

□

**7.**

*Proof.* Let $Y$ and $y_n$ be defined as in the hint. Then $\|y_n\| = 1/n^2$, constituting a convergent number series. However,

$$\sum_{n=1}^{N} y_n = (1, 1/4, \dots, 1/N^2, 0, \dots),$$

which is divergent as $N \to \infty$. □

**8.**

*Proof.* Let $(x_n)$ be a Cauchy sequence in $X$. Hence, for every $n > 0$, there exists some $K_n > 0$ such that for all $p, q > K_n$, $\|x_p - x_q\| < 1/n^2$. Without loss of generality, we may assume that $(K_n)$ is increasing. Since the series $\|x_{K_{n+1}} - x_{K_n}\|$ is bounded by $1/n^2$, it converges. By the hypothesis, the series $(x_{K_{n+1}} - x_{K_n})$ also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1} (x_{K_{i+1}} - x_{K_i}) \to x \quad \text{as } n \to \infty.$$

Now we show that $(x_n)$ converges to $x$. For every $\varepsilon > 0$, since $(x_n)$ is a Cauchy sequence, there exists some $N_1$ such that for all $p, q > N_1$, $\|x_p - x_q\| < \varepsilon$. Meanwhile, since $x_{K_n} \to x$, once $K_n$ is large enough, $\|x - x_{K_n}\| < \varepsilon$. Let $K_n > N_1$. Then for every $n > K_n$

$$\|x_n - x\| \le \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \le 2\varepsilon.$$

Thus, $X$ is complete. □

**9.**

*Proof.* Let $(x_n)$ be an absolutely convergent series in Banach space $X$. Let $s_n = \sum_{i=1}^{n} x_n$. Now we show that $s_n$ is a Cauchy sequence and therefore convergent. Since $\sum_{i=1}^{\infty} \|x_i\| < \infty$, for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n > N$, $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$. Hence, for every $N < p \le q$,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^{q} x_i \right\| \le \sum_{i=p+1}^{q} \|x_i\| < \varepsilon,$$

completing the proof. □

**10.**

*Proof.* Let $(e_n)$ be Schauder basis of $X$. Denote the underlying field of $X$ by $\mathbb{K}$ and let $\mathbb{W} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$ if $\mathbb{K} = \mathbb{C}$. Now we show that

$$S = \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \ldots \right\},$$

a countable subset of $X$, is dense in $X$ to derive the separability.

For every $x \in X$ and $\varepsilon > 0$, by the definition of Schauder basis, there exists $\beta_1, \ldots, \beta_n \in \mathbb{K}$ such that $\|x - (\beta_1 e_1 + \cdots + \beta_n e_n)\| < \varepsilon$. Let $M = \max_i \|e_i\|$. If $M = 0$, then there is nothing to prove. Otherwise, since $\mathbb{W}$ is dense in $\mathbb{K}$, for $i = 1, \ldots, n$, there exists $\alpha_i \in \mathbb{W}$ with $|\alpha_i - \beta_i| < \varepsilon/2^i M$. Hence,

$$\begin{aligned} \left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| &\leq \left\| x - \sum_{i=1}^{n} \beta_i e_i \right\| + \left\| \sum_{i=1}^{n} (\beta_i - \alpha_i) e_i \right\| \\ &\leq \varepsilon + \sum_{i=1}^{n} |\alpha_i - \beta_i| \|e_i\| \\ &\leq 2\varepsilon. \end{aligned}$$

Thus, $S$ is dense in $X$ and therefore $X$ is separable. □

**14.**

*Proof.* Clear that $\|\cdot\|_0$ is nonnegative. And $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$. Meanwhile, $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$. Finally, we show that $\|\hat{x}\|_0 = 0$ implies $\hat{x} = Y$ and invoke Prob. 4, Sec 2.2 to complete the proof. Since $\|\hat{x}\|_0 = 0$, there exists $(x_n) \subset \hat{x}$ which converges to 0. Since $Y$ is closed, $Y$ is complete and so is its cosets. Therefore, $0 \in \hat{x}$, enforcing $\hat{x}$ to be $Y$. □

## 2.4 Finite Dimensional Normed Spaces

**3.**

*Proof.* The reflexive property clearly holds. If there are positive $a$ and $b$ such that $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$ for all $x \in X$, then $\|x\|_1/b \leq \|x\|_0 \leq \|x\|/a$. Hence the relation is symmetric. Next we further suppose there exists positive $c$ and $d$ such that that $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$. Then $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$, giving the transitive property. Thus, the axioms of an equivalence relation hold. □

**4.**

*Proof.* Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Let $E \subset X$ be any open set with respect to $\|\cdot\|$, i.e., for every $x_0 \in E$, there exists some $\delta > 0$ such that $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$. Since $\|\cdot\| \sim \|\cdot\|_0$, there exists some positive $c$ such that $\|x - x_0\| \leq c\|x - x_0\|_0$. Hence, $B = \{x \in X : \|x - x_0\| < \delta/c\} \subset A \subset E$. Namely, $E$ is also open with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. □

**5.**

*Proof.* Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Then for every $x \in X$, there exists some $c > 0$ such that $\|x\|_0 \leq c\|x\|$. Let $(x_n)$ be a Cauchy sequence with respect to $\|\cdot\|$, i.e., for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n, m > N$, $\|x_n - x_m\| < \varepsilon/c$. Hence, $\|x_n - x_m\|_0 < c\|x_n - x_m\| \leq \varepsilon$. Thus, $(x_n)$ is also a Cauchy with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. □

## 2.5 Compactness and Finite Dimension

**5.**

*Proof.* Clear that every point in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a closed bounded, and therefore compact, neighborhood. Hence, $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact. □

**6.**

*Proof.* Let $X$ be a compact metric space and $x$ any point in $X$. Let $E$ be a closed neighborhood of $x$. By Prob 10, $E$ is compact. Thus, $X$ is locally compact. □

**7.**

*Proof.* It suffices to show that $a = \inf_{y \in Y} \|v - y\|$ can actually be obtained. Let $\{b_1, \dots, b_n\}$ be a basis of $Y$ and $y_k = y_{k,1}b_1 + \cdots + y_{k,n}b_n$ a sequence in $Y$ with $\|v - y_k\| \to a$. We may assume without loss of generality that $\|v - y_k\|$ is bounded.

Since $Y$ is a proper subset of $Z$, $v, b_1, \dots, b_n$ are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar $c > 0$ such that for every $k$,

$$\|v - y_{k,1}b_1 - \cdots - y_{k,n}b_n\| \geq c(1 + |y_{k,1}| + \cdots + |y_{k,n}|).$$

Hence, the sequence $(y_{k,1}, \dots, y_{k,n})$ of $n$-tuples is bounded and therefore has a convergent subsequence. Consequently, $(y_k)$ also has a convergent subsequence. Suppose that it converges to $z \in Z$. Note that $\|v - z\| = a$ and as $Y$ is closed, $z \in Y$. Thus, $a$ can be attained in $Y$. □

**8.**

*Proof.* Since the unit ball $B$ with respect to $\|\cdot\|_2$ in $\mathbb{R}^n$ and $\mathbb{C}^n$ is compact and $\|\cdot\|$ is continuous, by 2.5-7, $x \mapsto \|x\|$ can attains its minimum, denoted by $a$, on $B$. Due to the positive definite property of a norm, $a$ is positive. Hence, $0 < a \leq \|x/\|x\|_2\|$. Namely, $a\|x\|_2 \leq \|x\|$. □

**9.**

*Proof.* For every $(x_n) \subset M \subset X$, since $X$ is compact, there exists a subsequence $(x_{n_k})$ of $(x_n)$ which converges to some $y \in X$. Since $M$ is closed, $y \in M$. Hence, $M$ is compact. □

**10.**

*Proof.* From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of $T$ is also continuous, it suffices to show that the image of a closed set $A \subset X$ under $T$ is again a closed set. Since $X$ is compact and $A$ is closed, $A$ is compact. Since $T$ is continuous, by 2.5-6, $T(A)$ is compact and therefore closed. Hence, $T$ is a homeomorphism. □

## 2.7 Bounded and Continuous Linear Operators

**2.**

*Proof.* First suppose $T$ to be bounded and let $A$ be any bounded set in $X$. Then there exists $K < \infty$ such that for all $x \in A$, $\|x\| < K$. Due to the boundedness of $T$, $\|Tx\| \leq \|T\|\|x\| < K\|T\|$. Namely, $T(A)$ is also bounded.

Now suppose that $T$ maps bounded sets in $X$ into bounded sets in $Y$. Clear that the unit ball $B$ of $X$ is bounded and therefore so is $T(B)$. Namely, $\|Tx/\|x\|\|$ is bounded for $x \neq 0$.[1] Hence, $T$ is bounded. □

**3.**

*Proof.* For every $x$ with $\|x\| < 1$, $\|Tx\| \leq \|T\|\|x\| < \|T\|$. □

**4.**

*Proof.* Suppose that the linear operator $T$ is continuous at $x_0 \in \mathcal{D}(T)$. For every $(x_n) \subset \mathcal{D}(T)$ with $\|x_n - x\| \to 0$, by the continuity of $T$ at $x_0$

$$\|Tx_n - Tx\| = \|T(x_n - x + x_0) - Tx_0\| \to 0.$$

Hence, $T$ is continuous. □

**7.**

*Proof.* The inequality implies $\mathcal{N}(T) = 0$. Hence, by Theorem 2.6-10, $T^{-1}$ exists. For every $y \in Y$, suppose that $y = Tx$. Then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

Thus, $T^{-1}$ is bounded. □

**12.**

*Proof.* The compatibility follows immediately from the definition of the supremum. Suppose $\|x\|_1 = \max_j |\xi_j|$ and $\|y\|_2 = \max_j \|\eta_j\|$, then

$$Ax = \begin{bmatrix} x_1\alpha_{11} + \cdots + x_n\alpha_{1n} \\ \vdots \\ x_1\alpha_{r1} + \cdots + x_n\alpha_{rn}. \end{bmatrix}$$

[1] Note that the two $\|\cdot\|$ here are different norms.

Since for all $j$, $x_j \leq \|x_j\|_1$,

$$\frac{\max_j |x_1\alpha_{j1} + \cdots + x_n\alpha_{jn}|}{\|x\|_1} = \max_j \left| \frac{x_1}{\|x\|_1}\alpha_{j1} + \cdots + \frac{x_n}{\|x\|_1}\alpha_{jn} \right| \leq \max_j \sum_{k=1}^{n} |\alpha_{jk}|.$$

Hence,

$$\|A\| \geq \frac{\|Ax\|_2}{\|x\|_1} \quad \text{for all } x. \tag{1}$$

Suppose that maximum of $\sum_{k=1}^n |\alpha_{jk}|$ is obtained at $j = p$. Then choosing $x_k$ to be sgn $\alpha_{pk}$ shows that the equality in (1) can actually be attained. Hence, $\|A\| = \max_j \sum_{k=1}^n |\alpha_{jk}|$. □

## 2.8 Linear Functionals

**8.**

*Proof.* For every $x_1, x_2 \in N(M^*)$, $a, b \in \mathbb{K}$ and $f \in M^*$,

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) = 0.$$

Hence, $ax_1 + bx_2 \in N(M^*)$. Namely, $N(M^*)$ is a vector space. □

**9.**

*Proof.* First we show the uniqueness. Suppose that $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$. Then $0 = (\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)$. Hence,

$$0 = f((\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)) = (\alpha_1 - \alpha_2)f(x_0) + f(y_1) - f(y_2).$$

Since $y_1, y_2 \in \mathcal{N}(f)$, $f(y_1) - f(y_2) = 0$ while $f(x_0) \neq 0$ as $x_0 \notin \mathcal{N}(f)$. Hence, $\alpha_1 = \alpha_2$, which forces $y_1$ and $y_2$ to coincide.

For the existence, it suffices to show that for any fixed $x$, the function $g(\alpha) = f(x - \alpha x_0)$ has a zero. It is easy to verify that $\alpha = f(x)/f(x_0)$ is a zero of $g$. Note that $x_0 \notin \mathcal{N}(f)$ and therefore $f(x_0) \neq 0$. □

**10.**

*Proof.* First we suppose that $x_1, x_2 \in x_0 + \mathcal{N}(f) \in X/\{$. Then together with Prob. 9, $x_i = x_0 + y_i$ where $y_i \in \mathcal{N}(f)$. Hence, for $i = 1, 2$, $f(x_i) = f(x_0) + f(y_i) = f(x_0)$.

For the converse, note that $f(x_1) = f(x_2)$ implies $f(x_1 - x_2) = 0$. Namely, $x_1 - x_2 \in \mathcal{N}(f)$. Hence, $x_1, x_2$ belongs to the same element in $X/\mathcal{N}(f)$.

To show $\operatorname{codim} \mathcal{N}(f) = 1$, we show that $X/\mathcal{N}(f)$ and $\mathbb{K}$ are isomorphic. For every $\hat{x} \in X/\mathcal{N}(f)$, define $I(\hat{x}) = f(x)$. By the previous discussion, this definition is well-defined. Clear that $I$ is linear and therefore is injective. And by the linearity of $f$, $I$ is surjective. Thus, $I$ is an isomorphism between $X/\mathcal{N}(f)$ and $\mathbb{K}$. Hence, $\operatorname{codim} \mathcal{N}(f) = 1$. □

**11.**

*Proof.* Put $N = \mathcal{N}(f_1) = \mathcal{N}(f_2)$ and choose $x_0 \in X \setminus N$. By Prob. 9, for every $x \notin N$, $x = \alpha x_0 + y$ where $y \in N$ and $\alpha \neq 0$. Hence,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha f_1(x_0) + f_1(y)}{\alpha f_2(x_0) + f_2(y)} = \frac{f_1(x_0)}{f_2(x_0)}.$$

□

**12.**

*Proof.* Prob. 10, justifies the discussion on hyperplanes parallel to the $\mathcal{N}(f)$. It suffices to show that $H_1 = b + \mathcal{N}(f)$ for some $b \in X$. Choose $x_1 \in H_1$. Then

$$x \in \mathcal{N}(f) \;\Leftrightarrow\; x + x_1 \in x_1 + \mathcal{N}(f) \;\Leftrightarrow\; f(x + x_1) = f(x) + f(x_1) = 1 \;\Leftrightarrow\; x + x_1 \in H_1.$$

Hence, $H_1 = x_1 + \mathcal{N}(f)$. Namely, $H_1$ is a hyperplane parallel to $\mathcal{N}(f)$. □

**13.**

*Proof.* We argue by contradiction. Assume that there exists a $y_1 \in Y$ such that $f(y_1) \neq c \neq 0$. Then for every $d \in \mathbb{K}$, by the linearity of $f$, $f(dy_1/c) = d$. Contradiction. Hence, $f = 0$ on $Y$. □

**14.**

*Proof.* For every $\varepsilon > 0$, there exists $x_1 \in X$ with $f(x_1) = 1$ such that $\tilde{d} + \varepsilon \geq \|x_1\|$. Hence,

$$\|f\|(\tilde{d} + \varepsilon) \geq \|f\|\|x_1\| \geq |f(x_1)| = 1.$$

Since the choice of $\varepsilon > 0$ is arbitrary, $\|f\|\tilde{d} \geq 1$. Meanwhile, there exists $x_2 \in X$ with $\|x_2\| = 1$ such that $|f(x_2)| \geq \|f\| - \varepsilon$. Put $x_3 = x_2/f(x_2)$. Then $f(x_3) = 1$. Hence,

$$(\|f\| - \varepsilon)\tilde{d} \leq |f(x_2)|\|x_3\| = \|x_2\| = 1,$$

which implies $\|f\|\tilde{d} \leq 1$. Thus, $\|f\|\tilde{d} = 1$. □

**15.**

*Proof.* For every $x$ with $\|x\| \leq 1$, $f(x) \leq \|f\|\|x\| \leq c$. Hence, $x \in X_{c_1}$. Meanwhile, for every $\varepsilon > 0$, by the definition of the supremum, there exists a $x$ with $\|x\| = 1$ such that $|f(x)| > \|f\| - \varepsilon$. By the linearity of $f$, we may remove the $|\cdot|$ on the right side. Hence, $f(x) \notin X_{c_1}$ where $c = \|f\| - \varepsilon$. □