Linear Algebra Done Right

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Contents

3 Linear Map

3.A The Vector Space of Linear Maps

1.

Proof. If T is linear, then $T(0,0,0) = 0$ and therefore $b = 0$. Meanwhile, $T(2,2,2) = 2T(1,1,1)$ implies $12 + 8c = 12 + 2c$. Hence, $c = 0$. The proof of the converse part is trivial. $\square$

3.

Proof. Let e_i be the i-th vector in the standard base of $\mathbb{F}^n$ and suppose that $Te_i = \sum_{j=1}^{n} A_{1,j}e_j$. Then for $x = (x_1, \ldots, x_n)^T \in \mathbb{F}^n$,

$$Tx = T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i T e_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{j,i} e_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} A_{j,i} x_i\right) e_j.$$

$\square$

5.

Proof. Too lengthy to write it down... $\square$

7.

Proof. Let $\{x_0\}$ be a basis of V and λ be a scalar such that $Tx_0 = \lambda x_0$. By the linearity of T, for every $x = kx_0$ in V, $Tx = kTx_0 = k\lambda x_0 = \lambda(kx_0) = \lambda x$. $\square$

9.

Solution. From the additivity condition we can derive that $\varphi(kz) = k\varphi(z)$ for any $k \in \mathbb{Q}$. Hence we can try some functions where $\varphi(iz) = i\varphi(z)$ fails. It turns out that $\varphi(z) = \mathrm{Im}(z)$ is one of the maps required. $\square$

11.

Proof. Let $\{\alpha_1, \ldots, \alpha_p\}$ and $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\}$ be bases of U and V respectively. Then the linear map which maps α_i to $T\alpha_i$ and maps β to 0. Clear that it is the desired linear map. $\square$

13.

Proof. Suppose that v_k is in the span of the other vectors and let $w_i = 0$ for each $i \neq k$ and $w_k \neq 0$. No $T \in \mathcal{L}(V, W)$ can maps v_i to w_i since the linearity of T would force w_k to be 0, leading to a contradiction. $\square$

3.B Null Spaces and Ranges

2.

Proof. Since S maps every vector of V into the null space of T, the map TS is the zero map. Hence $(ST)^2 = S(TS)T = 0$. $\square$

4.

Proof. Suppose $S, T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$ maps and only maps e_1, e_2, e_3 and e_3, e_4, e_5 to the zero vector respectively. Then $e_1, e_2, e_4, e_5 \notin \operatorname{null}(S+T)$, implying that $\dim \operatorname{null}(S+T) < 2$. Hence $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4 : \dim \operatorname{null} T > 2)\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. □

6.

Proof. It follows immediately from the rank-nullity theorem and the fact that $\dim \operatorname{null} T$ and $\dim \operatorname{range} T$ are integers. □

8.

Proof. Let $\{w_1, \ldots, w_m\}$ be a basis of W and $S, T \in \mathcal{L}(V, W)$ be two linear maps such that $\operatorname{range} S = \operatorname{span}(w_1)$ and $\operatorname{range} T = \operatorname{span}(w_2, \ldots, w_n)$. Clear that $\operatorname{range}(S+T) = W$. Hence, the set described is not a subspace of $\mathcal{L}(V, W)$. □

10.

Proof. For every $y \in \operatorname{range} T$ there exists some $x = \sum x_i v_i \in V$ such that

$$y = Ty = T\left(\sum_{i=1}^{n} x_i v_i\right) = \sum_{i=1}^{n} x_i T v_i.$$

Hence, $\operatorname{range} T = \operatorname{span}(Tv_1, \ldots, Tv_n)$. □

12. For readers who familiar with the orbit-stabilizer theorem or just the (group) homomorphism, the proof should be straightforward.

Proof. For every nonzero y in $\operatorname{range} T$, there exists some $x \in V$ such that $Tx = y$. For each $y \neq 0$, we choose one such x, put them all together and put 0 into them to get U. By the construction, clear that $T(U) = \operatorname{range} T$ and , $U \cap \operatorname{null} T = \{0\}$. □

14.

Proof. By the rank-nullity theorem,

$$\dim \operatorname{null} T + \dim \operatorname{range} T = 8 \quad \Rightarrow \quad \dim \operatorname{range} T = 5 = \dim \mathbb{R}^5.$$

Hence, $\operatorname{range} T = \mathbb{R}^5$ and therefore T is surjective. □

16. Actually, the cosets of the kernel partition the whole space.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of $\operatorname{range} T$ and $Tu_i = v_i$ for $i = 1, 2, \ldots, n$. Denote $\operatorname{span}(u_1, \ldots, u_n)$ by U. We now prove that $V = U + \operatorname{null} T$. For every $x \in V$, suppose that $Tx = y = \sum y_i v_i$ and $\tilde{x} = \sum y_i u_i$. Note that $\tilde{x} \in U$ and $T(x - \tilde{x}) = Tx - T\tilde{x} = 0$, i.e., $x - \tilde{x} \in \operatorname{null} T$. Hence, $V = U + \operatorname{null} T$. As both of U and $\operatorname{null} T$ are finite-dimensional, so is V. □

18.

Proof. By the rank-nullity theorem, clear that $\dim V \geq \dim \operatorname{range} T = \dim W$ if there exists some surjective $T \in \mathcal{L}(V, W)$.

Assume that $\dim V \geq \dim W$ and let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be bases of V and W respectively. Then the linear map which maps v_i to w_i for each $1 \leq i \leq m$ is surjective. □

20.

Proof. If T is injective, then for every $y \in \operatorname{range} T$, there exists exactly one $x \in V$ such that $y = Tx$. Let S be the map which maps y to such x. It is linear since for every $y_1, y_2 \in \operatorname{range} T$ and scalar a, b, supposing $Sy_i = x_i$,

$$T(ax_1 + bx_2) = aTx_1 + bTx_2 = ay_1 + by_2.$$

implying $S(ay_1 + by_2) = ax_1 + bx_2 = aSy_1 + bSy_2$. For every $x \in V$, $(ST)x = S(Tx) = x$.

Suppose there exists some $S \in \mathcal{L}(W, V)$ such that $ST = I$. Then

$$Tx_1 = Tx_2 \quad \Rightarrow \quad STx_1 = STx_2 \quad \Rightarrow \quad x_1 = x_2.$$

Hence, T is injective. □

22.

Proof. Let $\tilde{T}$ be the restriction of T to $\operatorname{null} ST$. It is still a linear map since $\operatorname{null} ST$ is a subspace of U. Note that $x \in \operatorname{null} ST$ iff $(ST)x = 0$ iff $Tx \in \operatorname{null} S$. Hence, $\operatorname{range} \tilde{T} \subset \operatorname{null} S$. Thus, by the rank-nullity theorem,

$$\dim \operatorname{range} \tilde{T} \leq \dim \operatorname{null} S \quad \Rightarrow \quad \dim \operatorname{null} ST - \dim \operatorname{null} \tilde{T} \leq \dim \operatorname{null} S.$$

Since $\operatorname{null} \tilde{T} \leq \operatorname{null} T$, this implies $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$. □

24.

Proof. If there exists $S \in \mathcal{L}(W, W)$ such that $T_2 = ST_1$, then $\operatorname{null} T_2 = \operatorname{null} ST_1$. Hence for every $x \in \operatorname{null} T_1$, as $S(T_1 x) = S0 = 0$, $x \in \operatorname{null} T_2$. Therefore, $\operatorname{null} T_1 \subset \operatorname{null} T_2$.

Now we suppose $\operatorname{null} T_1 \subset \operatorname{null} T_2$ and construct S. Note that all we concerns is its behavior on some basis of $\operatorname{range} T_1$. Let $\{w_1, \ldots, w_n\}$ be a basis of $\operatorname{range} T_1$ and $T_1 v_i = w_i$ for $i = 1, \ldots, n$. For each $x \in V$, let $U_x = \{x + y \ : \ y \in \operatorname{null} T_2\}$ and $Sw_k = T_2 x$ if $v_k \in U_x$. It can be verified that S is well-defined and does satisfy the requirement as long as $\operatorname{null} T_1 \subset \operatorname{null} T_2$. □

26.

Proof. Let $\mathcal{P}_n(\mathbb{R}) = \{p \in \mathcal{P}(\mathbb{R}) \ : \ \deg p \leq n\}$, which are some subspaces of $\mathcal{P}(\mathbb{R})$. We now prove that D is a surjective linear map onto $\mathcal{P}_n(\mathbb{R})$ for every nonnegative integer n by induction.

Suppose $Dx = c_0 \neq 0$, then for any $0 \neq c \in \mathcal{P}_0(\mathbb{R})$, $D(cx/c_0) = c$. Hence, D is a surjective map onto $\mathcal{P}_0(\mathbb{R})$. Assume that D is a surjective map onto $\mathcal{P}_{k-1}(\mathbb{R})$ and suppose $Dx^{k+1} = p = a_0 + a_1 x + \cdots + a_k x^k$ where $a_k \neq 0$. For every nonzero b_k and $q = b_0 + b_1 x + \cdots + b_k x^k \in \mathcal{P}_k(\mathbb{R})$, let r be a polynomial with degree $\leq k-1$ such that

$q = b_k/a_k p + r$. By our induction hypothesis, there exists some polynomial $\tilde{r}$ such that $D\tilde{r} = r$. Then

$$D(b_k/a_k x^{k+1} + \tilde{r}) = \frac{b_k}{a_k} Dx^{k+1} + D\tilde{r} = \frac{b_k}{a_k} p + r = q.$$

Hence, D is also a surjective map onto $\mathcal{P}_k(\mathbb{R})$. Thus, D is surjective. □

28. TODO

30. TODO

3.D Invertibility and Isomorphic Vector Spaces

1.

Proof. Clear that the linear map $T^{-1}S^{-1}$ is right and left inverse of ST and therefore ST is invertible. And by the uniqueness of the inverse, $(ST)^{-1} = T^{-1}S^{-1}$. □

3.

Proof. First we suppose the existence of such an operator, then T^{-1} is also the inverse of S. Hence S is invertible and therefore injective.

Now we suppose S is injective. Let $\{u_1, \ldots, u_m\}$ and $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ be bases of U and V respectively. $\{Su_1, \ldots, Su_m\}$ is linearly independent as S is injective and therefore we can expand it to a basis, $\{Su_1, \ldots, Su_m, v_{m+1}, \ldots, v_n\}$, of V. Let $T \in \mathcal{L}(V)$ maps u_i to Su_i for $i = 1, \ldots, m$ and u_j to v_j for $j = m+1, \ldots, n$. T is obviously injective and therefore invertible as V is finite-dimensional. □

5.

Proof. Suppose that such an S exists. Since S is invertible, $\operatorname{range} S = V$. Hence, $\operatorname{range} T_2 = \operatorname{range} T_2 S = \operatorname{range} T_1$.

Now we suppose that $\operatorname{range} T_1 = \operatorname{range} T_2$ and construct S by defining its behavior on a basis of V. Let $\{v_1, \ldots, v_m\}$ be a basis of $\operatorname{null} T_1$. As $\operatorname{range} T_1 = \operatorname{range} T_2$ implies $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$, we can set $Sv_i = u_i$ for $i = 1, \ldots, m$ where $\{u_1, \ldots, u_m\}$ is a basis of $\operatorname{null} T_2$.

Let $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ be a basis of V. Clear that $\{T_1 v_{m+1}, \ldots, T_1 v_n\}$ spans $\operatorname{range} T_1$. It is linearly independent since

$$\begin{aligned}
& x_{m+1} T_1 v_{m+1} + \cdots + x_n T_1 v_n = 0 \\
\Rightarrow\quad & T_1(x_{m+1} v_{m+1} + \cdots + x_n v_n) = 0 \\
\Rightarrow\quad & x_{m+1} v_{m+1} + \cdots + x_n v_n \in \operatorname{null} T_1 \\
\Rightarrow\quad & x_{m+1} = \cdots = x_n = 0.
\end{aligned}$$

Hence, it is a basis of $\operatorname{range} T_1$. Since $\operatorname{range} T_1 = \operatorname{range} T_2$, there exists $u_{m+1}, \ldots, u_n$ such that $T_2 u_i = T_1 v_i$ for $i = m+1, \ldots, n$. It is easy to verify that $u_1, \ldots, u_m, u_{m+1}, \ldots, u_n$ are linearly independent. Finally, for $i = m+1, \ldots, n$, we also set $Sv_i = u_i$. Clear that S is invertible and satisfies the requirement. □

7.

Proof.
(a) For any $A, B \in E$ and scalar a, b,

$$(aA + bB)v = a(Av) + b(Bv) = 0.$$

Hence, E is a subspace of $\mathcal{L}(V, W)$.
(b) Since $v \neq 0$, putting $v_1 = v$, there exists some vectors in V such that $\{v_1, \ldots, v_n\}$ is a basis of V. Let $U = \operatorname{span}(v_2, \ldots, v_n)$. It can be shown that E is isomorphic to $\mathcal{U}, \mathcal{W}$. Hence, $\dim E = (\dim V - 1) \dim W$. □

9.

Proof. If S and T are invertible, then clear that $T^{-1}S^{-1}$ is the inverse of ST. Meanwhile, if S or T is not invertible, therefore not surjective, then

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\} < \dim V.$$

Hence, ST is not surjective and hence not invertible as V is finite-dimensional. Thus, ST is invertible iff S and T are invertible. □

11.

Proof. Since V is finite-dimensional and $S(TU) = (ST)U = I$, both S and U are invertible and the inverses of which are TU and ST respectively. Hence,

$$STU = I \quad \Rightarrow \quad T = S^{-1}U^{-1},$$

implying that T is also invertible and $T^{-1} = US$. □

13.

Proof. It follows almost immediately from Exercise 9 that all of R, S and T are invertible and therefore S is injective. □

15.

Proof. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{F}^{n,1}$ and suppose $Te_i = u_i$. It is easy to verify that $A = (u_1, \ldots, u_n)$ is a m-by-n matrix such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$. □

3.E Products and Quotients of Vector Spaces

2.

Proof. We only prove the result for $m = 2$. It is easy to prove it for arbitrary m in a similar manner. Suppose that $V = V_1 \times V_2$ is finite-dimensional. Then $V_1 \times \{0\}$, a subspace of V, is finite-dimensional. Clear that V_1 is isomorphic to $V_1 \times \{0\}$ and hence it is also of finite dimension. Similarly, V_2 is finite-dimensional. □

4.

Proof. We construct the isomorphism $S : \mathcal{L}(V_1 \times \cdots \times V_n, W) \to \mathcal{L}(V_1, w) \times \cdots \times \mathcal{L}(V_n, W)$ explicitly. For every $T \in \mathcal{L}(V_1 \times \cdots \times V_n, W)$, suppose $T(v_1, \ldots, v_n) = w$. Let $T_i(v_i) = w$ for $i = 1, \ldots, n$ and $ST = (T_1, \ldots, T_n)$. Clear that $T_i \in \mathcal{L}(V_i, W)$ and S is invertible. □

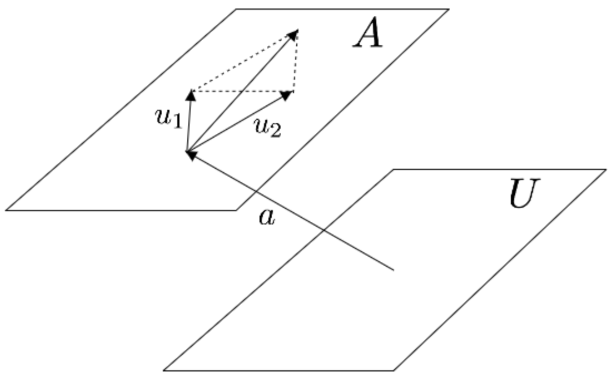
6.

Proof. We may interpret the elements in $\mathcal{L}(\mathbb{F}^n, V)$ as mappings from the "coordinates" to "abstract vectors". With this in mind, we construct the isomorphism S. For every $(v_1, \ldots, v_n) \in V^n$ and $(x_1, \ldots, x_n)^T \in \mathbb{F}^n$, let

$$(S(v_1, \ldots, v_n)) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i v_i.$$

It is easy to verify that S does satisfy the requirement. □

8. We may interpret the set of all possible $\lambda v + (1 - \lambda w)$ as the "line" through v and w. And the idea behind the proof is illustrated in the picture below.



Proof. If A is an affine subset, i.e., there exists some subspace U and $a \in V$ such that $A = a + U$, then for all $\lambda \in \mathbb{F}$ and $v, w \in A$,

$$\lambda v + (1 - \lambda)w = \lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + (u_1 + (1 - \lambda)u_2) \in A$$

where u_1 and u_2 are some elements in U.

Now we suppose $\lambda v + (1 - \lambda)w \in A$ holds, fix $a \in A$ and let $U = \{a_1 - a : a_1 \in A\}$. By the hypothesis, for every scalar λ and $a_1 \in A$, $a + \lambda(a_1 - a) \in A$. Therefore, for every $u_1 = a_1 - a \in U$, $\lambda u_1 \in U$. Meanwhile, let $u_2 = a_2 - a \in U$, $(u_1 + u_2)/2 \in U$ as

$$a + \frac{1}{2}(u_1 + u_2) = a + \frac{1}{2}(a_1 + a_2 - 2a) = \frac{1}{2}a_1 + \frac{1}{2}a_2.$$

Hence, by the previous result, $u_1 + u_2 \in U$. Thus, U is a subspace and $A = a + U$ is an affine subset. □

10.

Proof. Let A be the intersection of every collection of affine subsets of V and suppose A is nonempty. Let $v, w \in A$ and $\lambda \in \mathbb{F}$. Then, by Exercise 8, for every affine subset A_α of V, $\lambda v + (1-\lambda)w \in A_\alpha$. Hence it also belongs to A. Thus, A is also an affine subset of A (as long as nonempty). $\square$

12.

Proof. Let $\{a_1 + U, \ldots, a_m + U\}$ be a basis of V/U and we first prove a small result: for every $v \in V$, there exists an unique list of $v_1, \ldots, v_m \in \mathbb{F}$ such that $v - (v_1a_1 + \cdots v_m a_m) \in U$. Suppose that $v_1', \ldots, v_m'$ is such a list as well. Then

$$(v - (v_1a_1 + \cdots + v_m a_m)) - (v - (v_1'a_1 + \cdots + v_m' a_m)) \in U.$$

Therefore,

$$(v_1 - v_1')a_1 + \cdots + (v_m - v_m')a_m \in U = 0 + U,$$

Hence $v_i' = v_i$ for each $i = 1, \ldots, m$, completing the proof.

Therefore, for every $v \in V$, denoting $v_1 a_i + \cdots + v_m a_m$ as a_v, we may define S to be map which maps v to $(v - a_v, a_v + U)$. Now we show that S is linear and bijective. For every $u, v \in V$ and scalar a, b,

$$\begin{aligned} aSu + bSv &= a(u - a_u, a_u + U) + b(v - a_v, a_v + U) \\ &= ((au + bv) - (aa_u + ba_v), (aa_u + ba_v) + U) \\ &= S(au + bv). \end{aligned}$$

$Su = 0$ iff $(u - a_u, a_u + U) = 0$ iff $u = a_u = 0$ and therefore S is injective. Clear that S is surjective. Thus, S is an isomorphism and V is isomorphic to $U \times (V/U)$. $\square$

16.

Proof. Clear that every vector space with dimension 1 over field $\mathbb{F}$ is isomorphic to $\mathbb{F}$. Hence, it suffices to prove there exists $\varphi \in \mathcal{L}(V, V/U)$ such that $\operatorname{null}\varphi = U$ and the quotient map is just the map we want. $\square$

3.F Duality

1.

Proof. Suppose that $\varphi \in V'$ and is not the zero map. Then, $\varphi(v) = c \neq 0$ for some $v \in V$. By the linearity of φ, for every $0 \neq a \in \mathbb{F}$, $\varphi(av/c) = a$ and $\varphi(0) = 0$, completing the proof. $\square$

3.

Proof. It suffices to prove that there exists $\varphi \in V'$ which maps v to a nonzero element of $\mathbb{F}$. We argue by contradiction. Assume that for all $\varphi \in V'$, $\varphi(v) = 0$. Then $\{v\}^0 = V'$. Hence, $\dim\{v\} = \dim V - \dim\{v\}^0 = 0$, implying that $v = 0$. Contradiction. $\square$

9.

Proof. For every $v = \sum x_i v_i \in V$ and $\psi \in V'$,

$$\begin{aligned}\psi(v) &= \psi(x_1 v_1 + \cdots + x_n v_n) \\ &= \psi(v_1)x_1 + \cdots + \psi(v_n)x_n \\ &= \psi(v_1)\varphi_1(v) + \cdots + \psi(v_n)\varphi_n(v) \\ &= (\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v),\end{aligned}$$

where the third equality comes from the definition of the dual space and the last one comes from the linearity of $\varphi_1, \ldots, \varphi_n$. □

11.

Proof. $\operatorname{rank} A = 1$ iff there exists some nonzero $\alpha \in \mathbb{F}^m$ such that $A = [d_1\alpha \ \ldots \ d_n\alpha]$ iff $A = \alpha[d_1 \ \ldots \ d_n]$. □

15.

Proof. $T' = 0$ iff $\dim W' = \dim \operatorname{null} T'$ iff $\dim W = \dim(\operatorname{range} T)^0$ iff $\operatorname{range} T = 0$ iff $T = 0$. □

19.

Proof. As $U \subset V$ and V is finite-dimensional, $U = V$ iff $\dim U = \dim V$ iff $\dim U^0 = 0$ by 3.106 iff $U^0 = \{0\}$. □

25.

Proof. Note that the RHS of the equality equals to

$$\tilde{U} = \bigcap_{\varphi \in U^0} \operatorname{null} \varphi.$$

For every $u \in U$, since $u \in \operatorname{null} \varphi$ for every $\varphi \in U^0$ by definition. Hence, $U \subset \tilde{U}$. And let $\psi \in U^0$ be a linear functional such that $\operatorname{null} \psi = U$. Then $\dim \tilde{U} \leq \dim \operatorname{null} \psi = \dim U$. Hence, $U = \tilde{U}$. □

29.

Proof. By the hypothesis, for every $\psi \in W'$, $T'(\psi) = \psi \circ T = k\varphi$ for some scalar k. By 3.109, $\dim \operatorname{range} T = \dim \operatorname{range} T' = 1$. Hence, there exists $\psi \in W'$ whose restriction to $\operatorname{range} T$ is an one-to-one map to $\mathbb{F}$. Thus,

$$\operatorname{null} \varphi = \operatorname{null} k\varphi = \operatorname{null}(\psi \circ T) = \operatorname{null} T.$$

□

31. In brief, we choose an arbitrary basis of V and try to express the required basis with it by solving a system of linear equations.

Proof. Let $u_1, \ldots, u_n$ be a basis of V and $A = [\varphi_i(u_j)]$. Now we prove that A is invertible. Suppose

$$x_1 \begin{bmatrix} \varphi_1(u_1) \\ \vdots \\ \varphi_n(u_1) \end{bmatrix} + \cdots + x_n \begin{bmatrix} \varphi_1(u_n) \\ \vdots \\ \varphi_n(u_n) \end{bmatrix} = 0.$$

and $u = x_1 u_1 + \cdots + x_n u_n$. Then, $\varphi_i(u) = 0$ for $i = 1, \ldots, n$. As $\varphi_1, \ldots, \varphi_n$ is a basis of V', this implies $(\operatorname{span}(u))^0 = V'$. Hence, by 3.106, $\dim \operatorname{span}(u) = 0$ and therefore $u = 0$. Thus, the columns of A are linearly independent and therefore A is invertible.

Let

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} A^{-1} \tag{1}$$

and now we prove that $v_1, \ldots, v_n$ is a basis of V and the dual basis of it is exactly $\varphi_1, \ldots, \varphi_n$. Since $u_1, \ldots, u_n$ are linearly independent and A^{-1} is nonsingular, so do $v_1, \ldots, v_n$. Hence, $v_1, \ldots, v_n$ is a basis of V. (1) also implies

$$u_k = \varphi_1(u_k) v_1 + \cdots + \varphi_n(u_1) v_n.$$

Applying φ_i on the both sides for each $k = 1, \ldots, n$ yields

$$\begin{bmatrix} \varphi_i(u_1) \\ \vdots \\ \varphi_i(u_n) \end{bmatrix} = \begin{bmatrix} \varphi_1(u_1) & \cdots & \varphi_n(u_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(u_n) & \cdots & \varphi_n(u_n) \end{bmatrix} \begin{bmatrix} \varphi_i(v_1) \\ \vdots \\ \varphi_i(v_n) \end{bmatrix}.$$

Again, since A is nonsingular, the system of linear equations has exactly one solution $\varphi_i(v_j) = 0$ for $i \neq j$ and $\varphi_i(v_i) = 1$. Namely, $\varphi_1, \ldots, \varphi_n$ is the dual basis of $v_1, \ldots, v_n$. $\square$

37.

Proof.
(a) Since π is surjective, π' is injective by 3.108.
(b) $\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$.
(c) It follows immediately from (a) and (b). $\square$

5 Eigenvalues, Eigenvectors and Invariant Subspaces

5.B Eigenvectors and Upper-Triangular Matrices

Lemma 1. *If λ is an eigenvalue of $T \in \mathcal{L}(V)$ and p is a polynomial, then $p(\lambda)$ is an eigenvalue of $p(T)$. Note that unlike the statement in exercise 11, $\mathbb{F}$ does not required to be $\mathbb{C}$.*

Proof. Suppose that $Tv = \lambda v$ for some $0 \neq v \in V$, then

$$p(T)v = \left(\sum_{k=0}^{n} a_k T^k\right) v = \sum_{k=0}^{n} a_k T^k v = \sum_{k=0}^{n} a_k \lambda^k v = p(\lambda)v.$$

Hence, $p(\lambda)$ is an eigenvalue of $p(T)$. □

1.

Proof.
(a) Since $T^n = 0$ and

$$(I - T)(I + T + \cdots + T^{n-1}) = I + T + \cdots + T^{n-1} - T - \cdots - T^n = I - T^n = I,$$

$I - T$ is invertible and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.
(b) The power series expansion of the function $(1-x)^{-1}$ at $x = 0$ is $1+x+\cdots+x^n+\cdots$. □

3.

Proof. Since 1 is the only eigenvalue of $T^2 = I$ and -1 is not an eigenvalue of T, by Lemma 1, 1 is the only eigenvalue of T and therefore $T = I$. □

5.

Proof. Since $(STS^{-1})^k = S(T(S^{-1}S)TS^{-1} \cdots ST)S^{-1} = ST^kS^{-1}$,

$$p(STS^{-1}) = \sum_{k=0}^{n} a_k (STS^{-1})^k = \sum_{k=0}^{n} a_k ST^kS^{-1} = Sp(T)S^{-1}.$$

□

7.

Proof. It follows immediately from Lemma 1. □

9.

Proof. Since $p(T)v = 0 = 0v$, 0 is an eigenvalue of T. Then by Lemma 1, some of the zeros of p are the eigenvalues of T. Assume that there exists some zero x_0 of p that is not an eigenvalue of p. Then $q = p/(x - x_0)$ is a polynomial of degree less than p and such that $q(T)v = 0$. Contradiction. Hence, every zero of p is an eigenvalue of p. □

11. Note that the proof does not rely on 5.21.

Proof. Suppose that α is an eigenvalue of $p(T)$. If p is a constant polynomial, then there is nothing to be proved. If p is non-constant, then $p(x)-\alpha = c(x-\lambda_1)\cdots(x-\lambda_m)$ where $m \geq 1$. Since α is an eigenvalue of $p(T)$,

$$(p-\alpha)(T) = c(T-\lambda_1 I)\cdots(T-\lambda_m I)$$

is singular. Hence, at least one of $T-\lambda_1 I, \dots, T-\lambda_m T$, denoted by $T-\lambda_k I$, is singular and therefore λ_k is an eigenvalue of T and $p(\lambda_k)=\alpha$.
The converse part is just Lemma 1. $\square$

13.

Proof. Suppose that U is a finite-dimensional T-invariant subspace of W. Then $T|_U$ is an operator on U, a complex vector space. Hence it has an eigenvalue as long as $U = \{0\}$. However, it does not and therefore $U = \{0\}$. $\square$

15.

Proof. $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. $\square$

17.

Proof. Let φ be the map which takes $p \in \mathcal{P}_{n^2}(\mathbb{C})$ to $p(T) \in \mathcal{L}(V)$. It is linear since

$$\varphi(a_1p_1 + a_2p_2) = (a_1p_1 + a_2p_2)(T) = a_1p_1(T) + a_2p_2(T) = a_1\varphi(p_1) + a_2\varphi(p_2).$$

Since $\dim \mathcal{P}_{n^2}(\mathbb{C}) = n^2 + 1$ ans $\dim \mathcal{L}(V) = n^2$, φ is not injective by 3.23. Namely, there exsits nonequal $p_1, p_2 \in \mathcal{P}_{n^2}(\mathbb{C})$ such that $\varphi(p_1) = \varphi(p_2)$. Hence, $\varphi(p_1 - p_2) = (p_1 - p_2)(T) = 0$ where $p_1 - p_2$ is a nonzero polynomial, having zeros in $\mathbb{C}$. Since 0 is the eigenvalue of $(p_1 - p_2)(T)$, one of its zeros is an eigenvalue of T by exercise 11. $\square$