Solutions to
Introductory Functional Analysis with Applications

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2 Normed Spaces. Banach Spaces

2.3 Further Properties of Normed Spaces

4. cf. Prob. 13, Sec 1.2

Proof. The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1|\|x_1 - x_2\| + |\alpha_1 - \alpha_2|\|x_2\|.$$

$\square$

7.

Proof. Let Y and y_n be defined as in the hint. Then $\|y_n\| = 1/n^2$, constituting a convergent number series. However,

$$\sum_{n=1}^{N} y_n = (1, 1/4, \ldots, 1/N^2, 0, \ldots),$$

which is divergent as $N \to \infty$. $\square$

8.

Proof. Let (x_n) be a Cauchy sequence in X. Hence, for every $n > 0$, there exists some $K_n > 0$ such that for all $p, q > K_n$, $\|x_p - x_q\| < 1/n^2$. Without loss of generality, we may assume that (K_n) is increasing. Since the series $\|x_{K_{n+1}} - x_{K_n}\|$ is bounded by $1/n^2$, it converges. By the hypothesis, the series $(x_{K_{n+1}} - x_{K_n})$ also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1}(x_{K_{i+1}} - x_{K_i}) \to x \quad \text{as } n \to \infty.$$

Now we show that (x_n) converges to x. For every $\varepsilon > 0$, since (x_n) is a Cauchy sequence, there exists some N_1 such that for all $p, q > N_1$, $\|x_p - x_q\| < \varepsilon$. Meanwhile, since $x_{K_n} \to x$, once K_n is large enough, $\|x - x_{K_n}\| < \varepsilon$. Let $K_n > N_1$. Then for every $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus, X is complete. $\square$

9.

Proof. Let (x_n) be an absolutely convergent series in Banach space X. Let $s_n = \sum_{i=1}^{n} x_n$. Now we show that s_n is a Cauchy sequence and therefore convergent. Since $\sum_{i=1}^{\infty}\|x_i\| < \infty$, for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n > N$, $\sum_{i=n}^{\infty}\|x_i\| < \varepsilon$. Hence, for every $N < p \leq q$,

$$\|s_q - s_p\| = \left\|\sum_{i=p+1}^{q} x_i\right\| \leq \sum_{i=p+1}^{q}\|x_i\| < \varepsilon,$$

completing the proof. $\square$

10.

Proof. Let (e_n) be Schauder basis of X. Denote the underlying field of X by $\mathbb{K}$ and let $\mathbb{W} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$ if $\mathbb{K} = \mathbb{C}$. Now we show that

$$S = \left\{ \sum_{i=1}^n \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \dots \right\},$$

a countable subset of X, is dense in X to derive the separability.

For every $x \in X$ and $\varepsilon > 0$, by the definition of Schauder basis, there exists $\beta_1, \dots, \beta_n \in \mathbb{K}$ such that $\|x - (\beta_1 e_1 + \cdots + \beta_n e_n)\| < \varepsilon$. Let $M = \max_i \|e_i\|$. If $M = 0$, then there is nothing to prove. Otherwise, since $\mathbb{W}$ is dense in $\mathbb{K}$, for $i = 1, \dots, n$, there exists $\alpha_i \in \mathbb{W}$ with $|\alpha_i - \beta_i| < \varepsilon/2^i M$. Hence,

$$\left\| x - \sum_{i=1}^n \alpha_i e_i \right\| \leq \left\| x - \sum_{i=1}^n \beta_i e_i \right\| + \left\| \sum_{i=1}^n (\beta_i - \alpha_i) e_i \right\|$$

$$\leq \varepsilon + \sum_{i=1}^n |\alpha_i - \beta_i| \|e_i\|$$

$$\leq 2\varepsilon.$$

Thus, S is dense in X and therefore X is separable. $\qquad \square$

14.

Proof. Clear that $\| \cdot \|_0$ is nonnegative. And $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$. Meanwhile, $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$. Finally, we show that $\|\hat{x}\|_0 = 0$ implies $\hat{x} = Y$ and invoke Prob. 4, Sec 2.2 to complete the proof. Since $\|\hat{x}\|_0 = 0$, there exists $(x_n) \subset \hat{x}$ which converges to 0. Since Y is closed, Y is complete and so is its cosets. Therefore, $0 \in \hat{x}$, enforcing $\hat{x}$ to be Y. $\qquad \square$