Matrix Analysis

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1 Eigenvalues, eigenvectors, and similarity

1.1 Introduction

1.

Proof. Let $S = \{x \in \mathbb{R}^n : x^T x = 1\}$, which is clearly a compact subset of $\mathbb{R}^n$. Consider the function $f : x \mapsto x^T Ax$. Since,

$$\|f(x+\delta) - f(x)\| = \|(x^T A)\delta + \delta^T (Ax) + \delta^T A\delta\| \leq K\|\delta\|$$

for every $x \in \mathbb{R}$ and some fixed K, f is continuous. Hence, by Weierstrass's theorem, f attains its maximum value at some point $x \in S$. Namely, (1.0.3) has a solution x. Therefore, there exists some $\lambda \in \mathbb{R}$ such that $2(Ax - \lambda x) = 0$, implying that every real symmetric matrix has at least one real eigenvalue. $\square$

2.

Proof. Let $S = \{x \in \mathbb{R}^n : x^T x = 1\}$ and m be the maximum value of $x \mapsto x^T Ax$ in S. Suppose λ is an eigenvalue of A and $u \neq 0$ is its associated eigenvector, then

$$Au = \lambda u \quad \Rightarrow \quad u^T Au = \lambda\|u\|^2 \quad \Rightarrow \quad (u/\|u\|)^T A(u/\|u\|) = \lambda \quad \Rightarrow \quad m \geq \lambda.$$

Meanwhile, by the previous discussion, m it self is a eigenvalue of A. Hence, it is the largest real eigenvalue of A. $\square$