# Solutions to
# *Introduction to the Theory of Distirbutions*

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## Contents



## 1 Test Functions and Distributions

### 1.2

*Proof.* It suffices to show that $f \equiv 0$ on an open set $O$ iff the restriction of the distribution $\langle f, \cdot \rangle$ onto $O$ is the zero distribution. Suppose that $\langle f, \cdot \rangle|_O \equiv 0$ since the other direction is obvious. Assume, to obtain a contradiction, that $f(x) > 0$ for some $x \in O$. Since $f$ is continuous, there is an open neighborhood $U \subset O$ s.t. $f > \varepsilon$ on $U$ for some $\varepsilon > 0$. Choose a small closed ball $B \subset U$ centered at $x$ and let $\psi$ be the cutoff function with $\operatorname{supp} \psi \subset U$, $0 \le \psi \le 1$ and $\psi \equiv 1$ on $B$. Then

$$0 = \langle f, \psi \rangle = \int_U f\psi > \varepsilon \mu(B) > 0,$$

where $\mu(B)$ is the measure of $B$. Contradiction. Thus, $f \le 0$. Similarly, we can show $f \ge 0$. Therefore, $f \equiv 0$ on $O$.

The result is not true for $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ in general since add a function which is zero a.e. to $f$ does not change the distribution but will change the support of $f$. □

### 1.5

*Proof.* For every compact $K \subset (0, \infty)$, there is an integer $N$ s.t. $1/k \notin K$ for all $k > N$. Hence, for every $\phi \in C_c^\infty(0, \infty)$ with $\operatorname{supp} \phi \subset K$,

$$|\langle u, \phi \rangle| = \left| \sum_{k=0}^{N} \partial^k \phi(1/k) \right| \le \sum_{k=0}^{N} \sup |\partial^k \phi|.$$

Thus, $u$ is a distribution on $(0, \infty)$.

Assume, to obtain a contradiction, that $u = v|_{(0,\infty)}$ for some $v \in \mathscr{D}'(\mathbb{R})$. Let $f \in C_c^\infty(\mathbb{R})$ be a cutoff function with $f \equiv 1$ on $[-1, 1]$. Then, the distribution $fu$ (cf. Sec. 2.5) is of infinite order since its restriction to $1/m$ is $\delta^{(m)}$ for every positive integer $m$. However, since $fu$ is compactly supported, it must have a finite order (cf. Sec. 3.1). Contradiction. □

**1.6**

*Proof.* It follows immediately from the Riesz-Markov theorem. $\square$

**1.7** I am not sure whether the second part can be proved since if we put $f_\varepsilon \equiv 0$ for some $\varepsilon \in (0,1)$, the asymptotic behavior will not change.

*Proof.* It suffices to show $\int f_\varepsilon \phi \to \phi(0)$ as $\varepsilon \to 0$. Let $B_\varepsilon = \{|x| \le \varepsilon\}$. We have

$$\begin{aligned}\left|\int f_\varepsilon \phi - \phi(0)\right| &= \left|\int_{B_\varepsilon} f_\varepsilon(\phi - \phi(0))\right| \\ &\le \sup_{x\in B_\varepsilon} |\phi(x) - \phi(0)| \int |f_\varepsilon| \\ &\le \mu \sup_{x\in B_\varepsilon} |\phi(x) - \phi(0)|.\end{aligned}$$

Since $\phi \in C^\infty(\mathbb{R}^n)$, $\sup_{x\in B_\varepsilon} |\phi(x) - \phi(0)| \to 0$ as $\varepsilon \to 0$. Thus, $f_\varepsilon \to \delta$ in $\mathscr{D}'(\mathbb{R}^n)$. $\square$

**1.9**

*Proof.* Let $u_n(x) := \sum_{k=-n}^{n} c_k e^{ikx}$. For every $\phi \in C_c^\infty(\mathbb{R})$, by repeatedly using integration by parts, we have

$$\begin{aligned}\langle u_n, \phi\rangle &= \int \sum_{k=-n}^{n} c_k e^{ikx} \phi(x)\,\mathrm{d}x \\ &= \sum_{k=-n}^{n} c_k \int e^{ikx}\phi(x)\,\mathrm{d}x \\ &= \sum_{k=-n}^{n} c_k \left(\frac{-1}{ik}\right)^{m+2} \int e^{ikx}\partial^{m+2}\phi(x)\,\mathrm{d}x.\end{aligned}$$

Note that $c_k\left(\frac{-1}{ik}\right)^{m+2} \le O(1/k^2)$ and the $\int e^{ikx}\partial^{m+2}\phi(x)\,\mathrm{d}x$ is bounded. Thus, $\lim\langle u_n, \phi\rangle$ converges for every $\phi$, whence $u$ converges in $\mathscr{D}'(\mathbb{R})$. $\square$