

# Solutions to *Introductory Functional Analysis with Applications*

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## 2 Normed Spaces. Banach Spaces

### 2.3 Further Properties of Normed Spaces

4. cf. Prob. 13, Sec 1.2

*Proof.* The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1| \|x_1 - x_2\| + |\alpha_1 - \alpha_2| \|x_2\|.$$

□

7.

*Proof.* Let  $Y$  and  $y_n$  be defined as in the hint. Then  $\|y_n\| = 1/n^2$ , constituting a convergent number series. However,

$$\sum_{n=1}^N y_n = (1, 1/4, \dots, 1/N^2, 0, \dots),$$

which is divergent as  $N \rightarrow \infty$ .

□

8.

*Proof.* Let  $(x_n)$  be a Cauchy sequence in  $X$ . Hence, for every  $n > 0$ , there exists some  $K_n > 0$  such that for all  $p, q > K_n$ ,  $\|x_p - x_q\| < 1/n^2$ . Without loss of generality, we may assume that  $(K_n)$  is increasing. Since the series  $\|x_{K_{n+1}} - x_{K_n}\|$  is bounded by  $1/n^2$ , it converges. By the hypothesis, the series  $(x_{K_{n+1}} - x_{K_n})$  also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1} (x_{K_{i+1}} - x_{K_i}) \rightarrow x \quad \text{as } n \rightarrow \infty.$$

Now we show that  $(x_n)$  converges to  $x$ . For every  $\varepsilon > 0$ , since  $(x_n)$  is a Cauchy sequence, there exists some  $N_1$  such that for all  $p, q > N_1$ ,  $\|x_p - x_q\| < \varepsilon$ . Meanwhile, since  $x_{K_n} \rightarrow x$ , once  $K_n$  is large enough,  $\|x - x_{K_n}\| < \varepsilon$ . Let  $K_n > N_1$ . Then for every  $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus,  $X$  is complete.

□

9.

*Proof.* Let  $(x_n)$  be an absolutely convergent series in Banach space  $X$ . Let  $s_n = \sum_{i=1}^n x_i$ . Now we show that  $s_n$  is a Cauchy sequence and therefore convergent. Since  $\sum_{i=1}^{\infty} \|x_i\| < \infty$ , for every  $\varepsilon > 0$ , there exists some  $N > 0$  such that for all  $n > N$ ,  $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$ . Hence, for every  $N < p \leq q$ ,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^q x_i \right\| \leq \sum_{i=p+1}^q \|x_i\| < \varepsilon,$$

completing the proof.

□

10.

*Proof.* Let  $(e_n)$  be Schauder basis of  $X$ . Denote the underlying field of  $X$  by  $\mathbb{K}$  and let  $\mathbb{W} = \mathbb{Q}$  if  $\mathbb{K} = \mathbb{R}$  and  $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$  if  $\mathbb{K} = \mathbb{C}$ . Now we show that

$$S = \left\{ \sum_{i=1}^n \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \dots \right\},$$

a countable subset of  $X$ , is dense in  $X$  to derive the separability.

For every  $x \in X$  and  $\varepsilon > 0$ , by the definition of Schauder basis, there exists  $\beta_1, \dots, \beta_n \in \mathbb{K}$  such that  $\|x - (\beta_1 e_1 + \dots + \beta_n e_n)\| < \varepsilon$ . Let  $M = \max_i \|e_i\|$ . If  $M = 0$ , then there is nothing to prove. Otherwise, since  $\mathbb{W}$  is dense in  $\mathbb{K}$ , for  $i = 1, \dots, n$ , there exists  $\alpha_i \in \mathbb{W}$  with  $|\alpha_i - \beta_i| < \varepsilon/2^i M$ . Hence,

$$\begin{aligned} \left\| x - \sum_{i=1}^n \alpha_i e_i \right\| &\leq \left\| x - \sum_{i=1}^n \beta_i e_i \right\| + \left\| \sum_{i=1}^n (\beta_i - \alpha_i) e_i \right\| \\ &\leq \varepsilon + \sum_{i=1}^n |\alpha_i - \beta_i| \|e_i\| \\ &\leq 2\varepsilon. \end{aligned}$$

Thus,  $S$  is dense in  $X$  and therefore  $X$  is separable.  $\square$

14.

*Proof.* Clear that  $\|\cdot\|_0$  is nonnegative. And  $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$ . Meanwhile,  $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$ . Finally, we show that  $\|\hat{x}\|_0 = 0$  implies  $\hat{x} = Y$  and invoke Prob. 4, Sec 2.2 to complete the proof. Since  $\|\hat{x}\|_0 = 0$ , there exists  $(x_n) \subset \hat{x}$  which converges to 0. Since  $Y$  is closed,  $Y$  is complete and so is its cosets. Therefore,  $0 \in \hat{x}$ , enforcing  $\hat{x}$  to be  $Y$ .  $\square$

## 2.4 Finite Dimensional Normed Spaces

3.

*Proof.* The reflexive property clearly holds. If there are positive  $a$  and  $b$  such that  $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$  for all  $x \in X$ , then  $\|x\|_1/b \leq \|x\|_0 \leq \|x\|/a$ . Hence the relation is symmetric. Next we further suppose there exists positive  $c$  and  $d$  such that that  $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$ . Then  $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$ , giving the transitive property. Thus, the axioms of an equivalence relation hold.  $\square$

4.

*Proof.* Suppose the norms  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent. Let  $E \subset X$  be any open set with respect to  $\|\cdot\|$ , i.e., for every  $x_0 \in E$ , there exists some  $\delta > 0$  such that  $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$ . Since  $\|\cdot\| \sim \|\cdot\|_0$ , there exists some positive  $c$  such that  $\|x - x_0\| \leq c\|x - x_0\|_0$ . Hence,  $B = \{x \in X : \|x - x_0\| < \delta/c\} \subset A \subset E$ . Namely,  $E$  is also open with respect to  $\|\cdot\|_0$ . Interchanging the roles of  $\|\cdot\|$  and  $\|\cdot\|_0$  completes the proof.  $\square$

5.

*Proof.* Suppose the norms  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent. Then for every  $x \in X$ , there exists some  $c > 0$  such that  $\|x\|_0 \leq c\|x\|$ . Let  $(x_n)$  be a Cauchy sequence with respect to  $\|\cdot\|$ , i.e., for every  $\varepsilon > 0$ , there exists some  $N > 0$  such that for all  $n, m > N$ ,  $\|x_n - x_m\| < \varepsilon/c$ . Hence,  $\|x_n - x_m\|_0 < c\|x_n - x_m\| \leq \varepsilon$ . Thus,  $(x_n)$  is also a Cauchy with respect to  $\|\cdot\|_0$ . Interchanging the roles of  $\|\cdot\|$  and  $\|\cdot\|_0$  completes the proof.  $\square$

## 2.5 Compactness and Finite Dimension

5.

*Proof.* Clear that every point in  $\mathbb{R}^n$  or  $\mathbb{C}^n$  has a closed bounded, and therefore compact, neighborhood. Hence,  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are locally compact.  $\square$

6.

*Proof.* Let  $X$  be a compact metric space and  $x$  any point in  $X$ . Let  $E$  be a closed neighborhood of  $x$ . By Prob 10,  $E$  is compact. Thus,  $X$  is locally compact.  $\square$

7.

*Proof.* It suffices to show that  $a = \inf_{y \in Y} \|v - y\|$  can actually be obtained. Let  $\{b_1, \dots, b_n\}$  be a basis of  $Y$  and  $y_k = y_{k,1}b_1 + \dots + y_{k,n}b_n$  a sequence in  $Y$  with  $\|v - y_k\| \rightarrow a$ . We may assume without loss of generality that  $\|v - y_k\|$  is bounded.

Since  $Y$  is a proper subset of  $Z$ ,  $v, b_1, \dots, b_n$  are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar  $c > 0$  such that for every  $k$ ,

$$\|v - y_{k,1}b_1 - \dots - y_{k,n}b_n\| \geq c(1 + |y_{k,1}| + \dots + |y_{k,n}|).$$

Hence, the sequence  $(y_{k,1}, \dots, y_{k,n})$  of  $n$ -tuples is bounded and therefore has a convergent subsequence. Consequently,  $(y_k)$  also has a convergent subsequence. Suppose that it converges to  $z \in Z$ . Note that  $\|v - z\| = a$  and as  $Y$  is closed,  $z \in Y$ . Thus,  $a$  can be attained in  $Y$ .  $\square$

8.

*Proof.* Since the unit ball  $B$  with respect to  $\|\cdot\|_2$  in  $\mathbb{R}^n$  and  $\mathbb{C}^n$  is compact and  $\|\cdot\|$  is continuous, by 2.5-7,  $x \mapsto \|x\|$  can attain its minimum, denoted by  $a$ , on  $B$ . Due to the positive definite property of a norm,  $a$  is positive. Hence,  $0 < a \leq \|x\|_2$ . Namely,  $a\|x\|_2 \leq \|x\|$ .  $\square$

9.

*Proof.* For every  $(x_n) \subset M \subset X$ , since  $X$  is compact, there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  which converges to some  $y \in X$ . Since  $M$  is closed,  $y \in M$ . Hence,  $M$  is compact.  $\square$

10.

*Proof.* From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of  $T$  is also continuous, it suffices to show that the image of a closed set  $A \subset X$  under  $T$  is again a closed set. Since  $X$  is compact and  $A$  is closed,  $A$  is compact. Since  $T$  is continuous, by 2.5-6,  $T(A)$  is compact and therefore closed. Hence,  $T$  is a homeomorphism.  $\square$

## 2.7 Bounded and Continuous Linear Operators

2.

*Proof.* First suppose  $T$  to be bounded and let  $A$  be any bounded set in  $X$ . Then there exists  $K < \infty$  such that for all  $x \in A$ ,  $\|x\| < K$ . Due to the boundedness of  $T$ ,  $\|Tx\| \leq \|T\|\|x\| < K\|T\|$ . Namely,  $T(A)$  is also bounded.

Now suppose that  $T$  maps bounded sets in  $X$  into bounded sets in  $Y$ . Clear that the unit ball  $B$  of  $X$  is bounded and therefore so is  $T(B)$ . Namely,  $\|Tx/\|x\|\|$  is bounded for  $x \neq 0$ .<sup>1</sup> Hence,  $T$  is bounded.  $\square$

3.

*Proof.* For every  $x$  with  $\|x\| < 1$ ,  $\|Tx\| \leq \|T\|\|x\| < \|T\|$ .  $\square$

4.

*Proof.* Suppose that the linear operator  $T$  is continuous at  $x_0 \in \mathcal{D}(T)$ . For every  $(x_n) \subset \mathcal{D}(T)$  with  $\|x_n - x_0\| \rightarrow 0$ , by the continuity of  $T$  at  $x_0$

$$\|Tx_n - Tx_0\| = \|T(x_n - x_0 + x_0) - Tx_0\| \rightarrow 0.$$

Hence,  $T$  is continuous.  $\square$

7.

*Proof.* The inequality implies  $\mathcal{N}(T) = 0$ . Hence, by Theorem 2.6-10,  $T^{-1}$  exists. For every  $y \in Y$ , suppose that  $y = Tx$ . Then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

Thus,  $T^{-1}$  is bounded.  $\square$

12.

*Proof.* The compatibility follows immediately from the definition of the supremum. Suppose  $\|x\|_1 = \max_j |\xi_j|$  and  $\|y\|_2 = \max_j \|\eta_j\|$ , then

$$Ax = \begin{bmatrix} x_1\alpha_{11} + \cdots + x_n\alpha_{1n} \\ \vdots \\ x_1\alpha_{r1} + \cdots + x_n\alpha_{rn} \end{bmatrix}$$

---

<sup>1</sup>Note that the two  $\|\cdot\|$  here are different norms.

Since for all  $j$ ,  $x_j \leq \|x_j\|_1$ ,

$$\frac{\max_j |x_1\alpha_{j1} + \cdots + x_n\alpha_{jn}|}{\|x\|_1} = \max_j \left| \frac{x_1}{\|x\|_1}\alpha_{j1} + \cdots + \frac{x_n}{\|x\|_1}\alpha_{jn} \right| \leq \max_j \sum_{k=1}^n |\alpha_{jk}|.$$

Hence,

$$\|A\| \geq \frac{\|Ax\|_2}{\|x\|_1} \quad \text{for all } x. \quad (1)$$

Suppose that maximum of  $\sum_{k=1}^n |\alpha_{jk}|$  is obtained at  $j = p$ . Then choosing  $x_k$  to be  $\text{sgn } \alpha_{pk}$  shows that the equality in (1) can actually be attained. Hence,  $\|A\| = \max_j \sum_{k=1}^n |\alpha_{jk}|$ .  $\square$

## 2.8 Linear Functionals

8.

*Proof.* For every  $x_1, x_2 \in N(M^*)$ ,  $a, b \in \mathbb{K}$  and  $f \in M^*$ ,

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) = 0.$$

Hence,  $ax_1 + bx_2 \in N(M^*)$ . Namely,  $N(M^*)$  is a vector space.  $\square$

9.

*Proof.* First we show the uniqueness. Suppose that  $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$ . Then  $0 = (\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)$ . Hence,

$$0 = f((\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)) = (\alpha_1 - \alpha_2)f(x_0) + f(y_1) - f(y_2).$$

Since  $y_1, y_2 \in \mathcal{N}(f)$ ,  $f(y_1) - f(y_2) = 0$  while  $f(x_0) \neq 0$  as  $x_0 \notin \mathcal{N}(f)$ . Hence,  $\alpha_1 = \alpha_2$ , which forces  $y_1$  and  $y_2$  to coincide.

For the existence, it suffices to show that for any fixed  $x$ , the function  $g(\alpha) = f(x - \alpha x_0)$  has a zero. It is easy to verify that  $\alpha = f(x)/f(x_0)$  is a zero of  $g$ . Note that  $x_0 \notin \mathcal{N}(f)$  and therefore  $f(x_0) \neq 0$ .  $\square$

10.

*Proof.* First we suppose that  $x_1, x_2 \in x_0 + \mathcal{N}(f) \in X/\mathcal{N}(f)$ . Then together with Prob. 9,  $x_i = x_0 + y_i$  where  $y_i \in \mathcal{N}(f)$ . Hence, for  $i = 1, 2$ ,  $f(x_i) = f(x_0) + f(y_i) = f(x_0)$ .

For the converse, note that  $f(x_1) = f(x_2)$  implies  $f(x_1 - x_2) = 0$ . Namely,  $x_1 - x_2 \in \mathcal{N}(f)$ . Hence,  $x_1, x_2$  belongs to the same element in  $X/\mathcal{N}(f)$ .

To show  $\text{codim } \mathcal{N}(f) = 1$ , we show that  $X/\mathcal{N}(f)$  and  $\mathbb{K}$  are isomorphic. For every  $\hat{x} \in X/\mathcal{N}(f)$ , define  $I(\hat{x}) = f(x)$ . By the previous discussion, this definition is well-defined. Clear that  $I$  is linear and therefore is injective. And by the linearity of  $f$ ,  $I$  is surjective. Thus,  $I$  is an isomorphism between  $X/\mathcal{N}(f)$  and  $\mathbb{K}$ . Hence,  $\text{codim } \mathcal{N}(f) = 1$ .  $\square$

11.

*Proof.* Put  $N = \mathcal{N}(f_1) = \mathcal{N}(f_2)$  and choose  $x_0 \in X \setminus N$ . By Prob. 9, for every  $x \notin N$ ,  $x = \alpha x_0 + y$  where  $y \in N$  and  $\alpha \neq 0$ . Hence,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha f_1(x_0) + f_1(y)}{\alpha f_2(x_0) + f_2(y)} = \frac{f_1(x_0)}{f_2(x_0)}.$$

$\square$

**12.**

*Proof.* Prob. 10, justifies the discussion on hyperplanes parallel to the  $\mathcal{N}(f)$ . It suffices to show that  $H_1 = b + \mathcal{N}(f)$  for some  $b \in X$ . Choose  $x_1 \in H_1$ . Then

$$x \in \mathcal{N}(f) \Leftrightarrow x + x_1 \in x_1 + \mathcal{N}(f) \Leftrightarrow f(x + x_1) = f(x) + f(x_1) = 1 \Leftrightarrow x + x_1 \in H_1.$$

Hence,  $H_1 = x_1 + \mathcal{N}(f)$ . Namely,  $H_1$  is a hyperplane parallel to  $\mathcal{N}(f)$ .  $\square$

**13.**

*Proof.* We argue by contradiction. Assume that there exists a  $y_1 \in Y$  such that  $f(y_1) \neq c \neq 0$ . Then for every  $d \in \mathbb{K}$ , by the linearity of  $f$ ,  $f(dy_1/c) = d$ . Contradiction. Hence,  $f = 0$  on  $Y$ .  $\square$

**14.**

*Proof.* For every  $\varepsilon > 0$ , there exists  $x_1 \in X$  with  $f(x_1) = 1$  such that  $\tilde{d} + \varepsilon \geq \|x_1\|$ . Hence,

$$\|f\|(\tilde{d} + \varepsilon) \geq \|f\|\|x_1\| \geq |f(x_1)| = 1.$$

Since the choice of  $\varepsilon > 0$  is arbitrary,  $\|f\|\tilde{d} \geq 1$ . Meanwhile, there exists  $x_2 \in X$  with  $\|x_2\| = 1$  such that  $|f(x_2)| \geq \|f\| - \varepsilon$ . Put  $x_3 = x_2/f(x_2)$ . Then  $f(x_3) = 1$ . Hence,

$$(\|f\| - \varepsilon)\tilde{d} \leq |f(x_2)|\|x_3\| = \|x_2\| = 1,$$

which implies  $\|f\|\tilde{d} \leq 1$ . Thus,  $\|f\|\tilde{d} = 1$ .  $\square$

**15.**

*Proof.* For every  $x$  with  $\|x\| \leq 1$ ,  $f(x) \leq \|f\|\|x\| \leq c$ . Hence,  $x \in X_{c_1}$ . Meanwhile, for every  $\varepsilon > 0$ , by the definition of the supremum, there exists a  $x$  with  $\|x\| = 1$  such that  $|f(x)| > \|f\| - \varepsilon$ . By the linearity of  $f$ , we may remove the  $|\cdot|$  on the right side. Hence,  $f(x) \notin X_{c_1}$  where  $c = \|f\| - \varepsilon$ .  $\square$

## 2.9 Operators on Finite Dimensional Spaces

**8.**

*Proof.* Let  $\{b_2, \dots, b_n\}$  be a basis of  $Z$  and  $\{b_1, \dots, b_n\}$  a basis of  $X$ . Define  $f \in X^*$  to be  $f(b_i) = \delta_{1i}$ . Clear that  $\mathcal{N}(f) = Z$ . By Prob. 11, Sec 2.8,  $f$  is uniquely determined up to a scalar multiple.  $\square$

**12.**

*Proof.* Let  $\varphi : X \rightarrow \mathbb{K}^p$  be defined by  $x \mapsto [f_1(x), \dots, f_p(x)]^T$ . It can be verified that  $\varphi$  is a linear operator. Since  $\dim X = n > p$ ,  $\varphi$  can not be injective. Hence, there exists  $0 \neq x \in X$  such that  $\varphi(x) = 0$ .  $\square$

**13.**

*Proof.* Let  $\{b_1, \dots, b_m\}$  be a basis of  $Z$  and  $\{b_1, \dots, b_n\}$  a basis of  $X$ . Define  $\tilde{f} \in X^*$  to be identical with  $f$  on  $b_1, \dots, b_m$  and 0 on  $b_{m+1}, \dots, b_n$ . Clear that  $\tilde{f}|_Z = f$ .  $\square$

## 2.10 Normed Spaces of Operators. Dual Space

8.

*Proof.* First we construct a linear bijection  $T$  between  $c'_0$  and  $l^1$ . A Schauder basis for  $c_0$  is  $(e_k)$ , where  $e_k = (\delta_{kj})$ . Then for every  $f \in c'_0$ , define  $Tf = (\gamma_k) = (f(e_k))$ . Clear that  $T$  is linear. Now we show that  $Tf = (\gamma_k) \in l^1$ , that is,  $\sum_{k=1}^n |\gamma_k|$  is bounded and therefore convergent. Define  $x_n = (\xi_k^{(n)})$  with

$$\xi_k^{(n)} = \begin{cases} \text{sgn } \gamma_k, & k \leq n, \\ 0, & k > n. \end{cases}$$

Clear that  $x_n \in c_0$ . By the linearity and boundedness of  $f$ ,

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^n |\gamma_k|. \quad (2)$$

Since  $f$  is bounded,  $|f(x_n)| \leq \|f\| \|x_n\| \leq \|f\|$ . Hence,  $\sum \|\gamma_k\|$  is bounded. Thus,  $Tf \in l^1$ .

Meanwhile, for every  $y = (\beta_k) \in l^1$ , define  $Sy = g$  to be the functional  $g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$  for  $x = (\xi_k)$ . On  $c_0$ , the summation does converge and clear that  $g$  is linear and bounded. Hence,  $g \in c'_0$ . It can be verify that  $ST = TS = I$  and  $T$  is linear. Thus,  $c'_0$  and  $l^1$  is isomorphic.

Now we show that  $T$  constructed preserve the norm to complete the proof. For  $x \in c_0$  with  $\|x\| = 1$ ,

$$|f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \gamma_k \right| \leq \sum_{k=1}^{\infty} |\gamma_k| = \|Tf\|.$$

Hence,  $\|f\| \leq \|Tf\|$ . And (2) implies  $\sum_{k=1}^n \|\gamma_k\| \leq \|f\|$ . Letting  $n \rightarrow \infty$  yields  $\|Tf\| \leq \|f\|$ . Thus,  $\|Tf\| = \|f\|$ .  $\square$

9.

*Proof.* Let  $(b_k)$  be a Hamel basis of  $X$  and suppose that  $f, g \in X^*$  coincide on every  $b_k$ . Then for every  $x = \sum_{k=1}^{\infty} \xi_k b_k \in X$ ,

$$f(x) - g(x) = \sum_{k=1}^n \xi_k (f(b_k) - g(b_k)) = 0.$$

Thus,  $f = g$ . Namely,  $f$  is uniquely determined.  $\square$

10.

*Proof.* Let  $(b_k)$  be a Hamel basis of  $X$  and without loss of generality we may assume  $\|b_k\| = 1$ . Justified by Prob. 9, we can define  $T \in X^*$  with  $Tb_k = k$ , which is clearly unbounded.  $\square$

11.

*Proof.* It follows immediately from Prob. 10.  $\square$



**13.**

*Proof.* For any  $f, g \in M^a$  and scalar  $a, b$ ,  $(af + bg)(x) = af(x) + bg(x) = 0$  for every  $x \in M$ . Hence,  $M^a$  is a vector space. For  $(f_n) \subset M^a \subset X'$ , suppose that  $f_n \rightarrow f \in M^*$ . Since  $M'$  is complete, it is closed and therefore  $f \in M'$ . For every  $0 \neq x \in M$ , since  $f_n \rightarrow f$ ,

$$\frac{|f_n(x) - f(x)|}{\|x\|} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Hence,  $f(x) = 0$ . Thus,  $M^a$  is closed.

$$X^a = \{0\} \text{ and } \{0\}^a = X'.$$

□

**14.**

*Proof.* Let  $\{b_1, \dots, b_m\}$  be a basis of  $M$  and  $\{b_1, \dots, b_n\}$  a basis of  $X$ . And let  $\{\beta_1, \dots, \beta_n\}$  be the dual basis. Clear that  $b_1, \dots, b_m \notin M^a$  whereas  $b_{m+1}, \dots, b_n$  does. Together with Prob. 13, this implies  $M^a = \text{span}(b_{m+1}, \dots, b_n)$ . Thus,  $\dim M^a = n - m$ . □

### 3 Inner Product Spaces. Hilbert Spaces

#### 3.1 Inner Product Spaces. Hilbert Spaces

2.

*Proof.*

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle = \|x\|^2 + \|y\|^2,$$

where the last equality comes from the hypothesis of orthogonality. Now we show that for mutually orthogonal  $x_1, \dots, x_m$

$$\left\| \sum_{i=1}^m x_i \right\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

by induction on  $m$ . The case where  $m = 2$  has already been showed and we assume that the equation holds for  $m - 1$ . Since  $x_m$  is orthogonal with each  $i = 1, \dots, m - 1$ ,  $x_m$  is orthogonal to  $x_1 + \dots + x_{m-1}$ . Hence,

$$\left\| \sum_{i=1}^m x_i \right\|^2 = \left\| \sum_{i=1}^{m-1} x_i \right\|^2 + \|x_m\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

completing the proof. □

3.

*Proof.* The equation implies  $\langle x, y \rangle + \langle y, x \rangle = 0$ . The symmetric property of real inner products implies  $\langle x, y \rangle = 0$ . Let  $X = \mathbb{C}$  and  $x = 1, y = i$ . It is easy to verify that  $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = 2$  but  $x$  and  $y$  are not orthogonal. □

7.

*Proof.* It suffices to show that the zero vector is the only vector orthogonal to all vectors. Suppose that  $\langle x_0, x \rangle = 0$  for all  $x \in X$ , then  $\|x_0\|^2 = \langle x_0, x_0 \rangle = 0$ . By the definiteness of the inner product,  $x_0 = 0$ . □

8. We show that any norm satisfying the parallelogram equality can be derived from an inner product.

*Proof.* The proof of (IP3) is trivial and (IP4) follows immediately from the positive-definiteness of the norm. Hence we only show the linearity in the first factor here. For every  $u, v, y \in X$ , from the parallelogram equality we can derive, after some computation, that

$$\begin{aligned} 4\langle u + v, y \rangle &= \|u + v + y\|^2 - \|u + v - y\|^2 \\ &= \|u + y\|^2 - \|u - y\|^2 + \|v + y\|^2 - \|v - y\|^2 \\ &= 4\langle u, y \rangle + 4\langle v, y \rangle. \end{aligned}$$

Namely, (IP1) holds. By induction we can show that  $\langle nu, y \rangle = n\langle u, y \rangle$  for  $n = 1, 2, \dots$ . And since  $\langle -u, y \rangle = \langle 0 - u, y \rangle = \langle 0, y \rangle - \langle u, y \rangle = \langle u, y \rangle$ ,

$$\langle nu, y \rangle = n\langle u, y \rangle, \quad \text{for } n \in \mathbb{Z}.$$

Furthermore, for any positive integer  $m$ ,

$$m \left\langle \frac{n}{m}u, y \right\rangle = mn \left\langle \frac{1}{m}u, y \right\rangle = n\langle u, y \rangle.$$

Dividing the both sides by  $m$  yields

$$\langle qu, y \rangle = q\langle u, y \rangle, \quad \text{for } q \in \mathbb{Q}.$$

For every  $\alpha \in \mathbb{R}$ , let  $(q_n) \subset \mathbb{Q}$  converges to  $\alpha$ . Now we show that  $f(t) = \langle tu, y \rangle$  is continuous at  $t = 0$  and by the additivity we may conclude that  $f$  is continuous on  $\mathbb{R}$ . Since

$$\begin{aligned} 4|f(t)| &= ||tu + y||^2 - ||tu - y||^2 \\ &= (||tu + y|| + ||tu - y||)||tu + y|| - ||tu - y|| \\ &\leq 4t||u||(|t||u|| + ||y||) \rightarrow 0 \end{aligned}$$

as  $t \rightarrow 0$ ,  $f(t)$  is continuous. For every  $\alpha \in \mathbb{R}$ , let  $(q_n) \subset \mathbb{Q}$  be a convergent sequence with limit  $\alpha$ . Then

$$\langle \alpha u, y \rangle = \lim \langle q_n u, y \rangle = \lim q_n \langle u, y \rangle = \alpha \langle u, y \rangle.$$

Hence,  $\langle \cdot, \cdot \rangle$  is linear in the first factor. Thus, it is an inner product. Meanwhile, it is easy to verify that the norm it introduces is exactly the original norm.  $\square$

## 3.2 Further Properties of Inner Product Spaces

7.

*Proof.* First we note that

$$f(\alpha) = ||x + \alpha y||^2 - ||x - \alpha y||^2 = 2\bar{\alpha}\langle x, y \rangle + 2\alpha\langle y, x \rangle.$$

Clear that  $x \perp y$  implies  $f(\alpha) = 0$  for all scalar  $\alpha$ . For the converse, we suppose  $f(\alpha) = 0$  and put  $\alpha = \langle x, y \rangle$ . Then  $0 = f(\alpha) = 2|\langle x, y \rangle|^2$ . Thus,  $x \perp y$ .  $\square$

8.

*Proof.* Clear that  $x \perp y$  implies  $||x + \alpha y|| \geq ||x||$ . Therefore we only show the converse here. Without loss of generality, we assume  $||y|| = 1$ . Then  $||x + \alpha y|| \geq ||x||$  for all scalar  $\alpha$  implies

$$|\alpha|^2 + \bar{\alpha}\langle x, y \rangle + \alpha\overline{\langle x, y \rangle} \geq 0.$$

Put  $\alpha = -\langle x, y \rangle$  and we get

$$0 \leq |\langle x, y \rangle|^2 - 2|\langle x, y \rangle|^2 = -|\langle x, y \rangle|^2,$$

which implies  $\langle x, y \rangle = 0$ . Namely,  $x \perp y$ .  $\square$

9.

*Proof.* For every  $\varepsilon > 0$ , put  $\delta = \varepsilon/\sqrt{b-a}$ . Then for every  $x_1, x_2 \in V$  with  $\|x_1 - x_2\|_\infty < \delta$ ,

$$\|x_1 - x_2\|_2^2 = \int_a^b |x_1(t) - x_2(t)|^2 dt \leq (b-a)\delta^2 = \varepsilon^2.$$

Hence,  $x \mapsto x$  is continuous. □

10.

*Proof.* For every  $u, w \in X$ ,

$$\begin{aligned} \langle Tu, w \rangle &= \frac{1}{4} (\langle T(u+w), u+w \rangle - \langle T(u-w), u-w \rangle) \\ &\quad + \frac{i}{4} (\langle T(u+iw), u+iw \rangle - \langle T(u-iw), u-iw \rangle). \end{aligned}$$

Note that each component of the right hand side is of form  $\langle Tx, x \rangle$  and hence equals to 0. Putting  $w = Tu$  yields  $\|Tu\|^2 = 0$  for all  $u \in X$ . Thus,  $T = 0$ . □

### 3.3 Orthogonal Complements and Direct Sums

7.

*Proof.*

(a)  $x \in A^{\perp\perp}$  iff for all  $y \in A^\perp$ ,  $\langle x, y \rangle = 0$ . By the definition of  $A^\perp$ , the identity holds if  $x \in A$ . Hence,  $A \subset A^{\perp\perp}$ .

(b) For all  $x \in B^\perp$  and  $y \in A \subset B$ ,  $\langle x, y \rangle = 0$  by definition. Hence,  $x \in A^\perp$ . Namely,  $B^\perp \subset A^\perp$ .

(c) We show that  $A^\perp$  is closed (no matter whether  $A$  is or not) and invoke Lemma 3.3-6 to complete the proof. Suppose that  $(x_n) \subset A^\perp$  converges to  $x$ . For all  $y \in A$ ,  $\langle x_n, y \rangle = 0$ . By the continuity of the inner product,  $\langle x, y \rangle = 0$  and therefore  $x \in A^\perp$ . Hence,  $A^\perp$  is closed. Thus,  $A^\perp = A^{\perp\perp\perp}$ . □

8.

*Proof.* We have show this in Prob. 7. □

9.

*Proof.* It has been shown in Lemma 3.3-6 that the closedness of  $Y$  implies  $Y = Y^{\perp\perp}$ . Hence we only show the converse here. For every convergent  $(x_n) \subset Y$ ,  $(x_n) \subset Y^{\perp\perp}$ . Since  $Y^{\perp\perp}$  is closed by Prob. 8, the limit  $x$  of  $(x_n)$  belongs to  $Y^{\perp\perp}$  and hence belongs to  $Y$ . Thus,  $Y$  is closed. □

10. TODO

### 3.4 Orthonormal Sets and Sequences

3.

*Proof.* The situation where  $x$  and  $y$  are linearly dependent is obvious and hence we assume they are linearly independent here. By the homogeneity of the Schwarz inequality, we may assume without loss of generality that  $\|x\| = \|y\| = 1$ . Put  $z = (y - x\langle y, x \rangle) / \|y - x\langle y, x \rangle\|$ . Then  $\{x, z\}$  is orthonormal and therefore by (12\*)

$$|\langle y, x \rangle|^2 + |\langle y, z \rangle|^2 \leq \|y\|^2 = 1.$$

Since  $|\langle y, z \rangle|^2$  is nonnegative, this implies  $|\langle x, y \rangle|^2 \leq 1$ , the Schwarz inequality.  $\square$

7.

*Proof.* For each positive integer  $n$ , by the Schwarz inequality and (12\*),

$$\sum_{k=1}^n |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \sqrt{\sum_{i=1}^n |\langle x, e_i \rangle|^2} \sqrt{\sum_{i=1}^n |\langle y, e_i \rangle|^2} \leq \|x\| \|y\|.$$

Since all terms in the summation is nonnegative, this implies  $\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \|x\| \|y\|$ .  $\square$

8.

*Proof.* It follows immediately from Bessel inequality.  $\square$

### 3.5 Series Related to Orthonormal Sequences

1.

*Proof.* By Theorem 3.5-2,  $\alpha_k = \langle x, e_k \rangle$ . Meanwhile by the definition of the norm,

$$\|x\|^2 = \left\langle \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, x \right\rangle = \sum_{k=1}^{\infty} \langle x, e_k \rangle \langle e_k, x \rangle = \sum_{k=1}^{\infty} |\alpha_k|^2,$$

where the second equality follows from the continuity of the inner product.  $\square$

3.

*Solution.* Put  $x \equiv 1$  on  $[-\pi, \pi]$  and  $e_k = \sin kt$ . Since  $x$  is even but  $e_k$  is odd for every  $k$ , the series does not converges to  $x$ .  $\square$

4.

*Proof.* By the triangle inequality,  $\|x_m + \cdots + x_n\| \leq \|x_m\| + \cdots + \|x_n\|$  for every  $n \geq m > 0$ . Hence the convergence of  $\sum \|x_k\|$  implies that  $s_n$  is a Cauchy sequence.  $\square$

5.

*Proof.* By Prob. 4,  $\sum_{k=1}^n x_k$  is a Cauchy sequence. And since  $H$  is complete,  $\sum_{k=1}^{\infty} x_k$  converges.  $\square$

7.

*Proof.* The existence of  $y$  follows from Theorem 3.5-2(c). And for each  $k$ ,

$$\langle x - y, e_k \rangle = \langle x, e_k \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle e_k, e_j \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0,$$

where the second equality comes from the fact that  $(e_k)$  is orthonormal.  $\square$

8. TODO: Show the validation of the change of the order of summation. Or maybe we can show the equality directly.

*Proof.* We suppose that  $x \in \bar{M}$  here since the proof of the other direction is obvious. Then there exists  $(p_n) \subset M$  such that  $x = \sum_{n=1}^{\infty} p_n$ . For each  $n$ , suppose  $p_n = \sum_{k=1}^{\infty} \langle p_n, e_k \rangle e_k$ . This is valid because  $p_n \in M$  and therefore is a finite linear combination of  $(e_k)$ . In fact, there are only finitely many nonzero term in the summation. Then

$$x = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle p_n, e_k \rangle e_k = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \langle p_n, e_k \rangle \right) e_k.$$

$\square$

9.

*Proof.* First we suppose  $\bar{M}_1 = \bar{M}_2$ . Then by Prob. 8, each  $e_n$  and  $\tilde{e}_n$  can be represented by (a) and (b) respectively.

For the converse, (a) implies, again by Prob. 8,  $e_n \in \bar{M}_2$  and therefore  $M_1 \subset \bar{M}_2$ . Since  $\bar{M}_2$  is closed,  $\bar{M}_1 \subset \bar{M}_2$ . *Mutatis mutandis*, this also shows  $\bar{M}_2 \subset \bar{M}_1$ . Thus,  $\bar{M}_1 = \bar{M}_2$ .  $\square$

10.

*Proof.* Note that for every  $m > 0$ , there are only finite  $e_\kappa$  such that  $\langle x, e_\kappa \rangle \geq 1/m$ . Otherwise we may choose a countable subset of them, which will violate the result in Prob. 8, Sec 3.4. Hence, the collection of all nonzero Fourier coefficient

$$\bigcup_{m=1}^{\infty} \{e_\kappa : \langle x, e_\kappa \rangle \geq 1/m\}$$

is at most countable.  $\square$

### 3.6 Total Orthonormal Sets and Sequences

4.

*Proof.* Suppose that  $x$  and  $y$  satisfy (3). We only show the relation for real cases here. The complex cases can be proved in a similar way. Using (9), Sec 3.1 and (3),

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) = \frac{1}{4} \sum_k (|\langle x + y, e_k \rangle|^2 - |\langle x - y, e_k \rangle|^2).$$

Meanwhile,

$$|\langle x \pm y, e_k \rangle|^2 = \langle x \pm y, e_k \rangle \overline{\langle x \pm y, e_k \rangle} = |\langle x, e_k \rangle|^2 + |\langle y, e_k \rangle|^2 \pm 2 \langle x, e_k \rangle \overline{\langle y, e_k \rangle}.$$

Hence,  $\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$ .  $\square$

6.

*Proof.* Suppose  $M = (e_k)$ . We collect the  $e_k$  which does not belong to  $\text{span}(e_1, \dots, e_{k=1})$  and denote the new sequence by  $(\tilde{e}_k)$ . Clear that  $\text{span}(e_k) = \text{span}(\tilde{e}_k)$  and  $(\tilde{e}_k)$  is linearly independent. Let  $(f_k)$  be the sequence generated from  $(\tilde{e}_k)$  by the Gram-Schmidt process. Then clear that  $(f_k)$  is orthonormal. And since for every  $n$ ,  $\text{span}(\tilde{e}_1, \dots, \tilde{e}_n) = \text{span}(f_1, \dots, f_n)$ ,  $M \subset \text{span}(\tilde{e}_k) = \text{span}(f_k)$ . Finally, since  $M$  is dense in  $H$ ,  $\text{span}(f_k) = H$ . Thus,  $(f_k)$  is a total orthonormal sequence of  $H$ .  $\square$

7.

*Proof.* It follows from the definition of the separable Hilbert space and Prob. 6 immediately.  $\square$

9.

*Proof.*  $\langle v, x \rangle = \langle w, x \rangle$  implies  $\langle v - w, x \rangle = 0$  for all  $x \in M$ , that is,  $x \perp M$ . Since  $M$  is total, by Theorem 3.6-2,  $v - w = 0$ .  $\square$

10.

*Proof.* It follows immediately from Theorem 3.6-2(b).  $\square$

### 3.8 Functionals on Hilbert Spaces

3.

*Proof.* The linearity follows from the sesquilinearity of the inner product and the boundedness from the Schwarz inequality. Furthermore, the Schwarz inequality also implies  $\|f\| \leq \|z\|$ . Meanwhile,  $\|f\| \geq \|f(z/\|z\|)\| = \|z\|$ . Thus,  $\|f\| = \|z\|$ .  $\square$

4.

*Proof.* Clear that the mapping  $z \mapsto f$  is an isomorphism since it is surjective. And by Theorem 2.10-4,  $X'$  is a Hilbert space. Hence,  $X$  is also a Hilbert space.  $\square$

5.

*Proof.* Since  $l^2$  is complete. By Theorem 3.8-1, we may define  $I : (l^2)' \rightarrow l^2$  to be  $f \mapsto z$ . Clear that  $I$  is linear and injective. Meanwhile, it preserves the norm. Furthermore, by Prob. 3, it is surjective. Hence,  $I$  is an isomorphism. Thus,  $l^2$  is isomorphic to its dual.  $\square$

12.

*Proof.* For every  $x \in X$  and  $y \in Y$ ,

$$\begin{aligned} |h(x + \Delta x, y + \Delta y) - h(x, y)| &= |h(\Delta x, y) + h(x, \Delta y) + h(\Delta x, \Delta y)| \\ &\leq |h(\Delta x, y)| + |h(x, \Delta y)| + |h(\Delta x, \Delta y)|. \end{aligned}$$

Since  $h$  is bounded,

$$|h(x + \Delta x, y + \Delta y) - h(x, y)| \leq \|h\|(\|\Delta x\| \|y\| + \|x\| \|\Delta y\| + \|\Delta x\| \|\Delta y\|).$$

Thus,  $h$  is continuous.  $\square$

14.

*Proof.* If  $h(x, x) = 0$ , then for any  $t \in \mathbb{R}$ ,

$$0 \leq h(th(y, x)x + y, th(y, x)x + y) = 2t|h(x, y)|^2 + h(y, y).$$

Hence,  $h(x, y) = 0$ , otherwise we may choose some  $t < 0$  such that the right hand side is negative. Thus, the inequality holds if  $h(x, x) = 0$ .

Now suppose  $h(x, x) \neq 0$ . Put

$$z = y - x \frac{h(y, x)}{h(x, x)} \quad (3)$$

It is easy to verify that  $h(z, x) = 0$ . Multiplying  $z$  on the both sides of (3) yields

$$0 \leq h(z, z) = h\left(z, y - x \frac{h(y, x)}{h(x, x)}\right) = h(z, y) = h(y, y) - \frac{h(x, y)h(y, x)}{h(x, x)}.$$

Thus,  $|h(x, y)|^2 \leq h(x, x)h(y, y)$ . □

### 3.9 Hilbert-Adjoint Operator

1.

*Proof.* By Theorem 3.9-4,  $0^* = (0 + 0)^* = 0^* + 0^*$ . Hence,  $0^* = 0$ . For every  $x, y \in X$ ,

$$\langle (I^* - I)x, y \rangle = \langle I^*x, y \rangle - \langle Ix, y \rangle = \langle x, Iy \rangle - \langle Ix, y \rangle = 0.$$

Hence, by Lemma 3.9-3,  $I = I^*$ . □

2.

*Proof.* By Theorem 3.9-4,  $T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$ . Hence,  $(T^*)^{-1} = (T^{-1})^*$ . □

3.

*Proof.* Since  $\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|$ ,  $T_n^* \rightarrow T^*$  as long as  $T_n \rightarrow T$ . □

4.

*Proof.* It suffices to show that for all  $x_2 \in T^*(M_2^\perp)$  and  $x_1 \in M_1$ ,  $\langle x_1, x_2 \rangle = 0$ .  $x_2 \in T^*(M_2^\perp)$  implies the existence of some  $y_2 \in M_2^\perp$  with  $T^*y_2 = x_2$ . Then

$$\langle x_1, x_2 \rangle = \langle x_1, T^*y_2 \rangle = \langle Tx_1, y_2 \rangle = 0,$$

where the last equality comes from the fact that  $T(M_1) \subset M_2$  and  $y_2 \in M_2^\perp$ . Thus,  $M_1^\perp \supset T^*(M_2^\perp)$ . □

5.

*Proof.* By Prob. 4,  $T^*(M_2^\perp) \subset M_1^\perp$  implies  $M_2^{\perp\perp} \supset T(M_1^{\perp\perp})$ . Since  $M_1$  and  $M_2$  are closed, by Prob. 9, Sec 3.3,  $M_i^{\perp\perp} = M_i$  for  $i = 1, 2$ . Thus,  $T(M_1) \subset M_2$ . The converse part has already been proved in Prob. 4. □



6.

*Proof.*

(a) Since  $T(M_1) = \{0\} \subset H_2$ , by Prob. 4,  $T^*(H_2) \subset M_1^\perp$ .

(b) For every  $y \in [T(H_1)]^\perp$ ,  $\langle y, Tx \rangle = 0$  for all  $x \in H_1$ . Hence,  $\langle T^*y, x \rangle = 0$ . By Lemma 3.8-2,  $T^*y = 0$  and therefore  $y \in \mathcal{N}(T^*)$ . Thus,  $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$ .

(c) Since  $T^{**} = T$ , it follows from (b) that  $[T^*(H_2)]^\perp \subset M_1$ . And since  $M_1$  is closed,  $M_1^{\perp\perp} = M_1$ . Therefore, (a) implies  $[T^*(H_2)]^\perp \supset M_1$ . Thus,  $M_1 = [T^*(H_2)]^\perp$ .  $\square$

7.

*Proof.* It follows immediately from Lemma 3.9-3.  $\square$

8.

*Proof.* For every  $x \in H$  with  $\|x\| = 1$ ,

$$\begin{aligned} \|(I + T^*T)x\| &= \|x + T^*Tx\| = \langle x + T^*Tx, x + T^*Tx \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + \langle x, T^*Tx \rangle + \langle T^*Tx, x \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + \|Tx\|^2 \\ &\geq 1. \end{aligned}$$

Then, by Prob 7, Sec 2.7,  $I + T^*T$  is invertible.  $\square$

9.

*Proof.* If  $T$  can be represent by that form, then  $\mathcal{R}(T)$  can be spanned by  $w_1, \dots, w_n$ . Hence, it is finite dimensional.

Now we suppose that  $T$  has a finite dimensional range. Let  $\{w_1, \dots, w_n\}$  be a orthonormal basis of  $\mathcal{R}(T)$ . Then for every  $x \in H$ ,

$$Tx = \sum_{j=1}^n \varphi_j(x) w_j.$$

Now we show that for each  $j$ ,  $\varphi_j$  is a bounded linear functional and invoke Riesz's Theorem to complete the proof. It is easy to verify the linearity of  $\varphi_j$ . For every  $x$  with norm 1, since  $T$  is bounded and  $(w_j)$  is orthonormal,

$$\|T\| \geq \left\| \sum_{j=1}^n \varphi_j(x) w_j \right\| \geq |\varphi_j(x)|$$

for each  $j = 1, \dots, n$ . Hence, every  $\varphi_j$  is a bounded linear functional and therefore can be represented by  $\varphi_j(x) = \langle x, v_j \rangle$ .  $\square$

## Self-Adjoint, Unitary and Normal Operators

4. We only show the uniqueness here.

*Proof.*  $T_1 + iT_2 = S_1 + iS_2$  implies  $T_1 - iT_2 = S_1 - iS_2$ . Sum these two equations and we get  $T_1 = S_1$ . Meanwhile, it also implies  $i(T_1 + iT_2) = i(S_1 + iS_2)$ . Summing these two gives  $T_2 = S_2$ .  $\square$

**6.**

*Proof.*

(a) We argue by contradiction. Let  $k$  be the smallest positive integer such that  $T^{2k} = 0$ . Then for every  $x \in H$

$$0 = \langle T^{2k}x, x \rangle = \langle T^kx, (T^k)^*x \rangle = \langle T^kx, T^kx \rangle = \|T^kx\|^2.$$

Hence  $T^k = 0$ , which contradicts with the smallest assumption of  $k$ . Hence,  $T^n \neq 0$  for all even positive integer  $n$ .

(b) If  $T^n = 0$  for some positive, not necessarily even, integer  $n$ , then so is  $T^{2n} = 0$ . Hence, by (a),  $T^n \neq 0$  for all  $n \in \mathbb{N}$ .  $\square$

**9.**

*Proof.* Since  $T$  is isometric, it preserves the topology. Hence  $T(H)$  is closed as  $H$  is closed.  $\square$

**10.**

*Proof.* It suffices to show that  $T$  is surjective. Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $X$ . Then  $\{Te_1, \dots, Te_n\}$  is also an orthonormal basis since  $T$  is isometric. Hence,  $T$  is surjective and therefore is unitary.  $\square$

**13.**

*Proof.* It can be verified that  $T_n^*T_n \rightarrow T^*T$  and  $T_nT_n^* \rightarrow TT^*$ . Since  $T_n$  are normal,  $T_n^*T_n = T_nT_n^*$ . Hence,

$$\|T^*T - TT^*\| \leq \|T^*T - T_n^*T_n\| + \|T_nT_n^* - TT^*\| \rightarrow 0$$

as  $n \rightarrow \infty$ . Thus,  $T$  is normal.  $\square$

**15.**

*Proof.* If  $T$  is normal, clear that  $\|T^*x\| = \|Tx\|$ . Now we suppose  $\|Tx\| = \|T^*x\|$  for all  $x \in H$ . Then  $\langle TT^*x, x \rangle = \langle T^*Tx, x \rangle$ . Since  $X$  is complex, by Lemma 3.9-3,  $TT^* = T^*T$ . Namely,  $T$  is normal.

By (a), for every  $x \in H$ ,  $\|T^2x\| = \|T^*Tx\|$ . Hence,

$$\|T^2\| = \sup_{\|x\|=1} \|T^2x\| = \sup_{\|x\|=1} \|T^*Tx\| = \|T^*T\| = \|T\|^2,$$

where the last equality comes from Theorem 3.9-4(e).  $\square$

## 4 Fundamental Theorems for Normed and Banach Spaces

### 4.2 Hahn-Banach Theorem

4.

*Proof.* By the positive homogeneity,  $p(2 \times 0) = 2p(0)$ . Hence,  $p(0) = 0$ . Consequently,  $0 = p(x + (-x)) \leq p(x) + p(-x)$ . Thus,  $-p(x) \leq p(-x)$ .  $\square$

5.

*Proof.* For every  $x, y \in M$  and  $\lambda \in [0, 1]$ ,

$$p(\lambda x + (1 - \lambda)y) \leq \lambda p(x) + (1 - \lambda)p(y) \leq \lambda\gamma + (1 - \lambda)\gamma = \gamma.$$

Hence,  $\lambda x + (1 - \lambda)y \in M$ . Thus,  $M$  is convex.  $\square$

6.

*Proof.* For every  $x, t \in X$ ,

$$p(x - t) \leq p(x) + p(-t) \quad \Rightarrow \quad p(x - t) - p(x) \leq p(-t),$$

and

$$p(x) = p(x - t + t) \leq p(x - t) + p(t) \quad \Rightarrow \quad -p(t) \leq p(x - t) - p(x).$$

Since  $p(0) = 0$  and  $p$  is continuous at 0,  $p(t) \rightarrow 0$  and  $p(-t) \rightarrow 0$  as  $t \rightarrow 0$ . Hence,  $p(x - t) - p(x) \rightarrow 0$  as  $t \rightarrow 0$ , that is,  $p$  is continuous on  $X$ .  $\square$

8.

*Proof.* First,  $p(0) \geq p(0 + 0) - p(0) = 0$ . For nonzero  $x$ , we argue by contradiction. Assume that there exists some  $x$  with  $0 < \|x\| \leq r$  such that  $p(x) < 0$ . Then  $np(x) < 0$  for  $n = 1, 2, \dots$ . For  $n$  sufficiently large,  $n\|x\| > r$  and therefore  $p(nx) \geq 0$ . However, by the subadditivity,  $p(nx) \leq np(x) < 0$ . Contradiction. Thus,  $p(x) \geq 0$  on  $X$ .  $\square$

9.

*Proof.* For all  $x_1 = \alpha_1 x_0, x_2 = \alpha_2 x_0 \in Z$  and scalars  $a_1$  and  $a_2$ ,

$$\begin{aligned} f(a_1 x_1 + a_2 x_2) &= f((a_1 \alpha_1 + a_2 \alpha_2)x_0) = (a_1 \alpha_1 + a_2 \alpha_2)p(x_0) \\ &= a_1 \alpha_1 p(x_0) + a_2 \alpha_2 p(x_0) = a_1 f(x_1) + a_2 f(x_2). \end{aligned}$$

Thus,  $f$  is linear. Now we show that for  $\alpha \in \mathbb{R}$ ,  $\alpha p(x_0) \leq p(\alpha x_0)$  to complete the proof. If  $\alpha \geq 0$ , then it follows from the positive homogeneity. For negative  $\alpha$ ,  $\alpha p(x_0) = -p(-\alpha x_0)$  and by Prob. 4,  $-p(-\alpha x_0) \leq p(\alpha x_0)$ . Thus,  $f(x) \leq p(x)$  for all  $x \in Z$ .  $\square$

10.

*Proof.* Let  $Z$  and  $f$  have the same meaning as in Prob. 9. By Hahn-Banach theorem, there exists a linear extension  $\tilde{f}$  of  $f$  to  $X$  with  $\tilde{f}(x) \leq p(x)$  for all  $x \in X$ . Replacing  $x$  with  $-x$  gives  $\tilde{f}(-x) \leq p(x)$ . Finally, the linearity of  $\tilde{f}$  yields  $-p(-x) \leq \tilde{f}(x)$ .  $\square$

### 4.3 Hahn-Banach Theorem for Normed Spaces

1.

*Proof.* By (2),  $p(2 \times 0) = 2p(0)$ . Hence,  $p(0) = 0$ . And for every  $x \in X$ , by (1),

$$0 = p(0) \leq p(x) + p(-x) = 2p(x),$$

that is,  $p(x) \geq 0$ . □

2.

*Proof.* By (1),  $p(x) = p(x - y + y) \leq p(x - y) + p(y)$ . Therefore,  $p(x) - p(y) \leq p(x - y)$ . Interchange the roles of  $x$  and  $y$  and we obtain  $p(y) - p(x) \leq p(y - x) = p(x - y)$ , where the equality comes from (2). Thus,  $|p(x) - p(y)| \leq p(x - y)$ . □

7.

*Proof.* Define  $\tilde{f}$  to be  $x \mapsto \langle x, x_0 / \|x_0\| \rangle$ . Clear that it is a bounded linear functional on  $X$  and  $\tilde{f}(x_0) = \|x_0\|$ . And by Riesz's Theorem,  $\|\tilde{f}\| = \|x_0\| / \|x_0\| = 1$ . □

8.

*Proof.* It follows immediately from Theorem 4.3-3. □

13.

*Proof.* Just put  $\hat{f} = \tilde{f} / \|x_0\|$ . □

14.

*Proof.* By Prob 13, there exists a  $\hat{f} \in X'$  such that  $\|\hat{f}\| = 1/r$  and  $\hat{f}(x_0) = 1$ . Let hyperplane  $H_0 = \{x \in X : \hat{f}(x) = 1\}$  and half space  $S_0 = \{x \in X : \hat{f}(x) \leq 1\}$ . Then clear that  $x_0 \in H_0$  and for all  $x \in S(0; r)$ ,  $\hat{f}(x) \leq \|\hat{f}\| \|x\| = r/r = 1$ . Hence,  $x \in S_0$ . □

15.

*Proof.* If  $\|x\| = c + 2\varepsilon > c$ , then by Corollary 4.3-4, there exists some  $0 \neq f \in X'$  such that  $|f(x)| / \|f\| \geq c + \varepsilon$ . Consequently, the functional  $g = f / \|f\|$ , which is of norm 1, is such that  $|g(x_0)| > c$ . Contradiction. □

### 4.5 Adjoint Operator

9.

*Proof.* Note that every bounded linear functional is continuous by Theorem 2.7-9. Hence,  $M^a = (\mathcal{R}(T))^a$ . Thus,  $g \in M^a$  iff  $g \in (\mathcal{R}(T))^a$  iff  $g(Tx) = (T^\times g)(x) = 0$  for all  $x \in X$  iff  $T^\times g = 0$  iff  $g \in \mathcal{N}(T^\times)$ . Namely,  $M^a = \mathcal{N}(T^\times)$ . □

10.

*Proof.* For every  $y = Tx \in \mathcal{R}(T)$ ,  $g(Tx) = (T^\times g)(x) = 0$  for all  $g \in \mathcal{N}(T^\times)$ . Hence,  $y \in {}^a\mathcal{N}(T^\times)$ . □

## 4.6 Reflexive Spaces

2.

*Proof.* Since  $Y$  is a closed subspace of a Hilbert space, it is complete. By Lemma 3.3-2, there is some  $y \in Y$  such that  $\|x_0 - y\| = \delta$  and  $z = x_0 - y$  is orthogonal to  $Y$ . Define  $\tilde{f}$  by  $x \mapsto \langle x, z \rangle / \delta$ . Then clear that  $\tilde{f} \in X'$  and  $\tilde{f}(y) = 0$  for all  $y \in Y$ . Meanwhile, by Riesz's Theorem,  $\|\tilde{f}\| = \|z\| / \delta = 1$ . Finally,

$$\tilde{f}(x_0) = \frac{\langle x_0, x_0 - y \rangle}{\delta} = \frac{\langle x_0 - y + y, x_0 - y \rangle}{\delta} = \delta.$$

The proof is then completed.  $\square$

3.

*Proof.* We denote the canonical mapping from  $X$  to  $X''$  by  $C$  and the one from  $X'$  to  $X'''$  by  $D$ . Our goal is to find a  $f \in X'$  for every given  $h \in X'''$  such that  $D(f) = h$ , that is, for every  $g \in X''$ ,  $D(f)(g) = h(g)$ . Since  $X$  is reflexive, there is some  $x \in X$  such that  $g = Cx$ . Put  $f = hC$ , which is clearly an element of  $X'$ . Since

$$h(g) = h(Cx) = (hC)(x) = f(x) \quad \text{and} \quad D(f)(g) = g(f) = (Cx)(f) = f(x),$$

$h = D(hC)$ . Thus,  $X'$  is reflexive.  $\square$

4.

*Proof.* By Prob. 3, the reflexivity of  $X$  implies the reflexivity of  $X'$ . Now we suppose  $X'$  is reflexive. Hence, again by Prob. 3,  $X''$  is reflexive and therefore, by Theorem 4.6-4, is complete. Since  $X$  is isomorphic to  $\mathcal{R}(C) \subset X''$  and  $\mathcal{R}(C)$ , a closed subspace of a reflexive Banach space, is reflexive, so is  $X$ . Thus, a Banach space  $X$  is reflexive iff  $X'$  is reflexive.  $\square$

5.

*Proof.* It suffices to show that  $\delta > 0$  and then putting  $h = \tilde{f}/\delta$  will complete the proof. If  $\delta = 0$ , then by the definition of the infimum, there exists  $(y_n) \subset Y$  which converges to  $x_0$ . Then  $x_0 \in Y$  since  $Y$  is closed, which contradicts our choice of  $x_0$ . Thus,  $\delta > 0$ .  $\square$

6.

*Proof.* We may assume without loss of generality that  $Y_2 \setminus Y_1$  is nonempty. Choose arbitrary  $x_0 \in Y_2 \setminus Y_1 \subset X \setminus Y_1$ . By Prob. 6, there exists some  $h \in X'$  such that  $h(x_0) = 1$  and  $h \in Y^a$ . Thus, the annihilators of  $Y_1$  and  $Y_2$  are different.  $\square$

7.

*Proof.* If  $Y$  is proper, then by Prob. 7, the annihilators of  $Y$  and  $X$  do not coincide, which contradicts our hypothesis. Hence,  $X = Y$ .  $\square$

8.

*Proof.* If  $x \in A$ , then for every  $f \in X'$  whose restriction to  $M$  is 0,  $f(x_0) = 0$  since  $f$ , being bounded, is continuous. For the converse, note that  $f|_M = 0$  implies  $f|_A = 0$ . If  $x_0 \notin A$ , then Prob. 5 guarantees the existence of some  $f \in X'$  which vanishes on  $A$  and is nonzero at  $x_0$ . Contradiction. Thus,  $x_0 \in A$ .  $\square$

9.

*Proof.* If  $M$  is total, then clear that every  $f \in X'$  vanishing on  $M$  is zero everywhere on  $X$ . And the converse part follows immediately from Prob. 8.  $\square$

10.

*Proof.* Let  $\{b_1, \dots, b_n\}$  be a linearly independent subset of  $X$  and define

$$\beta_i : \text{span}\{b_1, \dots, b_n\} \rightarrow \mathbb{F} \quad \text{by} \quad b_j \mapsto \delta_{ij}$$

for  $i = 1, \dots, n$ . By Hahn-Banach Theorem, we can extend them to linear functionals  $\tilde{\beta}_i$  on  $X$ . Suppose that  $f = x_1\tilde{\beta}_1 + \dots + x_n\tilde{\beta}_n = 0$ . Then  $0 = f(b_i) = x_i$  for all  $i$ . Thus,  $\{\tilde{\beta}_1, \dots, \tilde{\beta}_n\}$  is linearly independent.  $\square$

## 4.7 Uniform Boundedness Theorem

1.

*Solution.* Meager, since  $\mathbb{Q}$  is the union of all singleton of rational numbers.  $\square$

5.

*Proof.* First we suppose  $M$  is rare and argue by contradiction. If  $(\bar{M})^c$  is not dense in  $X$ , i.e., there exists some  $x \in X$  and  $r > 0$  such that  $B(x; r) \cap (\bar{M})^c = \emptyset$ . Hence,  $B(x; r) \subset \bar{M}$ , which contradicts the definition of rare subsets. Thus,  $(\bar{M})^c$  is dense in  $X$ .

Now we suppose  $(\bar{M})^c$  is dense in  $X$ . Then for all  $x \in \bar{M}$  and  $r > 0$ , there exists some  $y_r \notin \bar{M}$  but  $y_r \in B(x; r)$ . Hence,  $x$  is not an interior point. Thus,  $M$  is rare.  $\square$

6.

*Proof.* If both  $M$  and  $M^c$  are meager, then so is their union  $X$ , but Baire's theorem says that a complete metric space is nonmeager in itself. Hence,  $M^c$  is nonmeager if  $M$  is.  $\square$

7.

*Proof.* We argue by contradiction. Assume that for all  $x \in X$ ,  $\sup_n \|T_n x\| < \infty$ . Then by the uniform boundedness theorem, there exists some  $c$  such that  $\|T_n\| \leq c$  for all  $n$ . Hence,  $\sup_n \|T_n\| \leq c$ . Contradiction.  $\square$

**10.**

*Proof.* We may assume without loss of generality that  $\eta_1 \neq 0$ . Define  $T_n : c_0 \rightarrow \mathbb{C}$  by  $(\xi_j) \mapsto \sum_{j=1}^n \xi_j \eta_j$ . Clear that  $T_n$  are linear functionals. And since

$$|T_n x| = \left| \sum_{j=1}^n \eta_j \xi_j \right| \leq \max_{j=1, \dots, n} |\xi_j| \sum_{j=1}^n |\eta_j| \leq \|x\| \sum_{j=1}^n |\eta_j|, \quad (4)$$

$T_n$  are bounded and  $\|T_n\| \leq \sum_{j=1}^n |\eta_j|$ . Meanwhile, define  $y = (\gamma_j)$  by

$$\gamma_j = \begin{cases} \operatorname{sgn} \eta_j, & j \leq n, \\ 0, & j > n. \end{cases}$$

Clear that  $y \in c_0$  and  $\|y\| = 1$ . Since  $|T_n y| = \sum_{j=1}^n |\eta_j|$ , together with (4), we conclude  $\|T_n\| = \sum_{j=1}^n |\eta_j|$ .

By Prob. 2, Sec 2.3,  $c_0$  is a Banach space. And for each  $x = (\xi_j) \in c_0$ , since  $\sum \xi_j \eta_j$  converges,  $\|T_n x\|$  is bounded for  $n$  large enough and therefore bounded for all  $n$ . Hence, by the uniform boundedness theorem,  $\sum_{j=1}^n |\eta_j| = \|T_n\| \leq c$  for some fixed  $c$ . Thus,  $\sum |\eta_j| < \infty$ .  $\square$

**11.**

*Proof.* By Prob. 4, Sec 1.4, the Cauchy sequence  $(T_n x)$  is bounded. Thus, by the uniform boundedness theorem,  $(\|T_n\|)$  is bounded.  $\square$

**13.**

*Proof.* Let  $C : X \rightarrow X''$  be the canonical embedding and  $(\varphi_n) = (C x_n)$ . By Lemma 4.6-1,  $\|x_n\| = \|\varphi_n\|$ . Note that  $X''$ , the dual space of  $X'$ , is complete and  $f(x_n) = \varphi_n(f)$ . Thus, by the uniform boundedness theorem,  $(\|x_n\|) = (\|\varphi_n\|)$  is bounded.  $\square$

**14.**

*Proof.*

(a) $\Rightarrow$ (c): It follows immediately from  $|g(T_n x)| \leq \|g\| \|x\| \|T_n\|$ .

(c) $\Rightarrow$ (b): For fixed  $x \in X$ , let  $\varphi_n = C(T_n x)$ , where  $C : Y \rightarrow Y''$  is the canonical embedding. For every  $g \in Y'$ , by (c),  $|\varphi_n(g)| = |g(T_n x)| \leq c_g$ . Since  $Y'$  is complete, by the uniform boundedness theorem,  $(\|\varphi_n\|) = (\|T_n x\|)$  is bounded.

(b) $\Rightarrow$ (a): It is just what the uniform boundedness theorem states.  $\square$

## 4.8 Strong and Weak Convergence

**1.**

*Proof.* The mapping  $\varphi_t : C[a, b] \rightarrow \mathbb{F}$ ,  $x \mapsto x(t)$  is a bounded linear functional on  $C[a, b]$ . Hence, by the definition of weak convergence,  $x_n(t) \rightarrow x(t)$ .  $\square$

**2.**

*Proof.* For every  $f \in Y'$ ,  $fT \in X'$ . Since  $x_n \xrightarrow{w} x_0$ ,  $(fT)(x_n) \rightarrow (fT)(x_0)$ , that is,  $Tx_n \xrightarrow{w} Tx_0$ .  $\square$

4.

*Proof.* If  $x_0 = 0$ , then it is trivial. Otherwise, by Theorem 4.3-3, there exists some  $f \in X'$  such that  $f(x_0) = \|x_0\|$  and  $\|f\| = 1$ . Since  $x_n \xrightarrow{w} x$ ,  $|f(x_n)| \rightarrow |f(x_0)| = \|x_0\|$ . Meanwhile,  $|f(x_n)| \leq \|f\|\|x_n\| = \|x_n\|$ . Thus,  $\underline{\lim} \|x_n\| \geq \|x_0\|$ .  $\square$

5.

*Proof.* If  $\bar{Y} = X$ , then there is nothing to be proved. Otherwise we argue by contradiction. Assume that  $x_0 \in X \setminus \bar{Y} \neq \emptyset$ . Then by Lemma 4.6-7, there exists some  $f \in X'$  such that  $f(Y) = \{0\}$  and  $f(x_0) = \delta > 0$ . However, since  $x_n \in Y$  and  $x_n \xrightarrow{w} x_0$ ,  $f(x_0)$  must be 0. Contradiction. Thus,  $x_0 \in \bar{Y}$ .  $\square$

6.

*Proof.* It follows immediately from Prob. 5.  $\square$

7.

*Proof.* It follows immediately from Prob. 5.  $\square$

8.

*Proof.* For every  $f \in X'$ , by the definition,  $|f(x_n)| < c_f$  for some constant  $c_f$  which does not depend on  $n$ . Let  $g_n = Cx_n$ , where  $C : X \rightarrow X''$  is the canonical embedding. Then for all  $n$ ,  $|g_n(f)| = |f(x_n)| \leq c_f$ . Since  $X'$  is complete, by the uniform boundedness theorem,  $(g_n)$  is bounded and therefore so is  $(x_n)$ .  $\square$

9. Note that we do not regard a sequence with only repeated elements when  $n$  is large as a sequence here.

*Proof.* We argue by contradiction. Assume that  $A$  is unbounded, that is, there exists a  $(a_n) \subset A$  with  $\|a_n\| \geq n$ . Clear that every sequence in  $(a_n)$  is unbounded and therefore, by Prob. 8, is not a weak Cauchy sequence, contradicting our hypothesis. Thus,  $A$  is bounded.  $\square$

10.

*Proof.* Let  $(x_n)$  be a weak Cauchy sequence in  $X$ . Let  $\varphi_n = Cx_n$ , where  $C : X \rightarrow X''$  is the canonical embedding. For every  $f \in X'$ , since  $(x_n)$  is a weak Cauchy sequence,  $(\varphi_n(f)) = (f(x_n))$  is a Cauchy sequence in  $\mathbb{F}$  and therefore  $\lim \varphi_n(f)$  exists. Let  $\varphi : X' \rightarrow \mathbb{F}$  be defined by  $f \mapsto \lim \varphi_n(f)$ . Clear that it is linear. Meanwhile, since  $(x_n)$ , a weak Cauchy sequence, is bounded by Prob. 8,  $(\varphi_n)$  is bounded by, say,  $c$ . Therefore,  $\varphi$  is bounded since  $|\varphi_n(f)| \leq c\|f\|$  for all  $n$ . Thus,  $\varphi \in X''$ .

Since  $X$  is reflexive, there exists some  $x_0 \in X$  such that  $\varphi = Cx_0$ . For all  $f \in X'$ ,

$$f(x_n) = \varphi_n(f) \rightarrow \varphi(f) = f(x_0) \quad \text{as } n \rightarrow \infty.$$

Thus,  $x_n \xrightarrow{w} x_0$ . Thus,  $X$  is weakly complete.  $\square$



## 4.9 Convergence of Sequences of Operators

4.

*Proof.* First suppose that  $f_n \xrightarrow{w} f$ , namely,  $\varphi(f_n) \rightarrow \varphi(f)$  for all  $\varphi \in X''$ . For every  $x \in X$ , denoting the canonical embedding by  $C$ ,

$$|f_n(x) - f(x)| = |(Cx)(f_n) - (Cx)(f)| \rightarrow 0,$$

since  $f_n \xrightarrow{w} f$ . Thus,  $f_n \xrightarrow{w^*} f$ .

Now we suppose  $f_n \xrightarrow{w^*} f$  and  $X$  is reflexive. Then for each  $\varphi \in X''$ , there is an  $x \in X$  such that  $\varphi = Cx$ . Hence,

$$|\varphi(f_n) - \varphi(f)| = |f_n(x) - f(x)| \rightarrow 0,$$

since  $f_n \xrightarrow{w^*} f$ . Thus,  $f_n \xrightarrow{w} f$ . □

6.

*Proof.* If  $T_n \rightarrow T$ , then for every  $\varepsilon > 0$ , there is an  $N$  such that for all  $n > N$ ,  $\|T_n - T\| < \varepsilon$ . Hence, for all  $x$  with  $\|x\| = 1$ ,

$$\|T_n x - T x\| \leq \|T_n - T\| \|x\| < \varepsilon.$$

Now we suppose the converse. Since  $\|T_n - T\| = \sup_{\|x\|=1} \|T_n x - T x\|$ , there is some  $x$  of norm 1 such that

$$\|T_n - T\| - \varepsilon < \|T_n x - T x\| < \varepsilon.$$

Thus,  $\|T_n - T\| \leq 2\varepsilon$ , which implies  $T_n \rightarrow T$ . □

7.

*Proof.* For every  $x \in X$ , since  $T_n x$  converges,  $(\|T_n x\|)$  is bounded. As  $X$  is complete, this implies  $(\|T_n\|)$  is bounded by the uniform boundedness theorem. □

9.

*Proof.* For every  $x \in X$  with  $\|x\| = 1$ ,

$$\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$$

where the first equality comes from the continuity of the norm. Thus,  $\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$ . □

10.

*Proof.* Since  $X$  is separable, we can find a total sequence  $(b_n)$  of  $X$ . Now we choose a subsequence of  $(f_n) \subset X'$  as follows. First we choose a subsequence of  $(f_k(x_n))$  for each fixed  $n$  inductively. Since  $M$  is bounded,  $(f_k(x_1))$  is a bounded sequence in  $\mathbb{R}$  and therefore has a convergent subsequence  $f_{k_i}(x_1)$ . For  $x_2$  we choose the subsequence of  $(f_{k_i}(x_2))$  which is convergent. We proceed in this way and choose a sequence of  $(f_k(x_n))$  for each  $n$ . Now, consider the subsequence  $(g_k)$  of  $(f_n)$  whose  $n$ -th element is the  $n$ -th functional in the  $n$ -th choice. It can be verified that it is a subsequence. And by our construction,  $(f_k(x_n))$  converges for every  $x_n$ . Hence, by Corollary 4.9-7, it is weak\* convergent. □

## 4.12 Open Mapping Theorem

2.

*Solution.* Consider  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto e^x$ . Clear that  $f$  is open. However, it maps  $\mathbb{R}$ , a closed set, onto  $(0, \infty)$ , an open set.  $\square$

4.

*Proof.* It follows from the fact that  $\|x_k\| < 1/2^k$ .  $\square$

5.

*Proof.* The linearity and boundedness is clear. Meanwhile, it is easy to verify that  $T^{-1}((\xi_k)) = (k\xi_k)$ . Let  $x_n = (\xi_k^{(n)})$  be defined by  $\xi_k^{(n)} = \delta_{nk}$ . Then  $\|T^{-1}x_n\| = n$ . Hence,  $T^{-1}$  is unbounded.

This does not contradict with 4.12-2 since  $X$  is not complete. For example, The sequence  $(x_n) \subset X$  where  $x_n = (\xi_k^{(n)})$  is defined by

$$\xi_k^{(n)} = \begin{cases} 1/k, & k < n, \\ 0, & k \geq n \end{cases}$$

is a Cauchy sequence which does not converge in  $X$ .  $\square$

6.

*Proof.* If  $\mathcal{R}(T)$  is closed in  $Y$ , then it is complete and therefore  $T$  is a bijective bounded linear operator from  $X$  onto  $\mathcal{R}(T)$ . Hence, by the open mapping theorem,  $T^{-1}$  is bounded.

If  $T^{-1}$  is bounded, then it is continuous. Therefore,  $\mathcal{R}(T)$ , the preimage of the closed set  $X$ , is closed.  $\square$

7.

*Proof.* The existence of such a  $b$  comes from the definition of boundedness. Since  $T$  is bijective and both  $X$  and  $Y$  are complete, by the bounded inverse theorem,  $T^{-1}$  exists and is bounded. Hence, for all  $y = Tx \in Y$ , there is a  $c > 0$  such that

$$\|x\| = \|T^{-1}y\| \leq c\|y\| = c\|Tx\|.$$

Thus, putting  $a = 1/c > 0$ , we obtain  $a\|x\| \leq \|Tx\|$  for all  $x \in X$ .  $\square$

9.

*Proof.* Let  $J : X_2 \rightarrow X_1$  be the identity map. Clear that  $J$  a bijective linear operator. Then  $\|x\|_1 \leq c\|x\|_2$  implies

$$\frac{\|Jx\|_1}{\|x\|_2} \leq c \quad \text{for all nonzero } x \in X_2.$$

Namely,  $J$  is bounded. Since  $X_1$  and  $X_2$  are complete, by the bounded inverse theorem, the inverse  $I$  of  $J$  is also a bounded linear operator. Thus, for all  $x \in X_2$

$$\|x\|_2 = \|Ix\|_2 \leq \|I\|\|x\|_1.$$

$\square$

### 4.13 Closed Linear Operators. Closed Graph Theorem

5.

*Proof.* The closedness of  $T$  implies that  $\mathcal{G}(T) = \{(x, Tx) : x \in \mathcal{D}(T)\}$  is closed in  $X \times Y$ . Then  $\mathcal{G}(T^{-1}) = \{(Tx, x) : x \in \mathcal{D}(T)\}$  is closed in  $Y \times X$ . Thus,  $T^{-1}$  is closed.  $\square$

6.

*Proof.* Suppose  $Tx_n \rightarrow y$  and  $T\tilde{x}_n \rightarrow \tilde{y}$ . Since  $x_n \rightarrow x \in \mathcal{D}(T)$ , by Theorem 4.13-3,  $Tx = y$ . Similarly,  $Tx = \tilde{y}$ . Thus,  $y = \tilde{y}$ .  $\square$

7.

*Proof.* Since  $X$  is a closed subset of itself, by Lemma 4.13-5,  $T$  is closed and therefore, by Prob. 5, so is  $T^{-1}$ . Since  $Y = \mathcal{R}(T)$  is closed in  $Y$  itself, by the closed graph theorem,  $T^{-1}$  is bounded and therefore continuous.  $\square$

8.

*Proof.*

(a) Let  $(y_n) = (Tx_n) \subset A$  converge to  $y \in Y$ . Since  $C$  is compact,  $(x_n)$  has a convergent sequence, say,  $(x_{n_k})$ . Suppose  $x_{n_k} \rightarrow x \in C$  as  $k \rightarrow \infty$ . By Theorem 4.13-3,  $y = \lim Tx_{n_k} = Tx$ . Therefore,  $y \in A$ . Thus,  $A$  is closed.

(b) Let  $(x_n) \subset B$  converge to  $x \in X$ . Since  $K$  is compact, the sequence  $(y_n) = (Tx_n) \subset K$  has a convergent sequence  $(y_{n_k})$ . Suppose  $y_{n_k} \rightarrow y \in K$ . Since  $T$  is closed,  $y = Tx$ . Hence,  $x \in B$ . Thus,  $B$  is closed.  $\square$