

# Real Analysis

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## Contents

|          |   |           |
|----------|---|-----------|
| <b>3</b> | <b>Lebesgue Measure</b>                               | <b>2</b>  |
| 3.1      | Introduction . . . . .                                | 2         |
| 3.2      | Outer Measure . . . . .                               | 2         |
| 3.3      | Measurable Sets and Lebesgue Measure . . . . .        | 3         |
| 3.5      | Measurable Functions . . . . .                        | 4         |
| 3.6      | Littlewood's Three Principles . . . . .               | 6         |
| <b>4</b> | <b>The Lebesgue Integral</b>                          | <b>7</b>  |
| 4.2      | The Lebesgue Integral of a Bounded Function . . . . . | 7         |
| 4.3      | The Integral of a Nonnegative Function . . . . .      | 7         |
| 4.4      | The General Lebesgue Integral . . . . .               | 9         |
| <b>5</b> | <b>Differentiation and Integration</b>                | <b>11</b> |
| 5.1      | Differentiation of Monotone Functions . . . . .       | 11        |
| 5.2      | Functions of Bounded Variation . . . . .              | 11        |

## 3 Lebesgue Measure

### 3.1 Introduction

1.

*Proof.* Since  $\mathfrak{M}$  is an  $\sigma$ -algebra,  $B \setminus A \in \mathfrak{M}$  as long as  $A, B \in \mathfrak{M}$ . Since  $B \setminus A$  and  $A$  are disjoint,  $mB = mA + m(B \setminus A) \geq mA$  since  $m$  is nonnegative.  $\square$

2.

*Proof.* Let  $A_0 = E_0$  and  $E_k = A_k \setminus A_{k-1}$  for  $k \geq 1$ . Clear that  $E_i$  and  $E_j$  are disjoint for distinct  $i$  and  $j$ ,  $\bigcup A_n = \bigcup E_n$  and  $A_i \subset E_i$  for every  $i$ . Hence,

$$m\left(\bigcup E_n\right) = m\left(\bigcup A_n\right) = \sum mA_n \leq \sum mE_n,$$

where the last inequality comes from Exercise 1.  $\square$

3.

*Proof.* Suppose that  $mA < \infty$ . Then  $mA = m(A \cup \emptyset) = mA + m\emptyset$ , implying that  $m\emptyset = 0$ .  $\square$

### 3.2 Outer Measure

5.

*Proof.* We show that  $\{I_n\}$  must cover the entire  $[0, 1]$  by contradiction. Assume that  $x \notin I_k$  for  $k = 1, 2, \dots, n$ . Then, as  $I_k$  are open and  $n$  is finite, there exists some  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon)$  and  $I_k$  are disjoint for every  $k$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exists some rational number in  $(x - \varepsilon, x + \varepsilon)$ , contradicting with the hypothesis that  $\{I_k\}$  covers all rational numbers between 0 and 1.  $\square$

6.

*Proof.* By the definition of the outer measure, for every  $\varepsilon > 0$ , there exists some collection  $\{I_n\}$  of open intervals that covers  $A$  and  $\sum l(I_n) \leq m^*A + \varepsilon$ . Let  $O = \bigcup I_n$ .  $O$  is a countable union of open sets and therefore is also open. And by Proposition 2,  $m^*O \leq \sum l(I_n)$ . Thus,  $m^*O \leq m^*A + \varepsilon$ .

Let  $\varepsilon_n = 1/n$  and for each  $n$ , by the previous discussion, we can always get an open set  $O_k$  such that  $A \subset O_k$  and  $m^*O \leq m^*A + \varepsilon_m$ . Let  $G$  be the countable intersection of these open sets. Clear that  $G$  is a  $G_\delta$  set covering  $A$  and  $m^*A = m^*G$ .  $\square$

7.

*Proof.* If  $m^*E = \infty$ , it is trivial. Suppose that  $m^*E \leq \infty$ . For any  $x \in \mathbb{R}$ , collection  $\{I_n\}$  of open intervals covers  $E + x$  iff  $\{I_n - x\}$  covers  $E$ . Since the length of intervals is translation invariant, this implies  $m^*(E + x) = m^*E$ .  $\square$

8.

*Proof.* Clear that  $m^*A \leq m^*(A \cup B)$ . Meanwhile,  $m^*(A \cup B) = m^*A + m^*B = m^*B$ . Hence,  $m^*(A \cup B) = m^*B$ .  $\square$

### 3.3 Measurable Sets and Lebesgue Measure

10.

*Proof.*

$$\begin{aligned} mE_1 + mE_2 &= mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2). \end{aligned}$$

□

11.

*Proof.*  $E_n = (n, \infty)$ .

□

12. This is the countable version of Lemma 9.

*Proof.* It suffices to prove  $m^*(A \cap \bigcup E_i) \geq \sum m^*(A \cap E_i)$ . Since  $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^n E_i$  for every  $n$ ,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of  $n$ , we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i),$$

completing the proof.

□

13.

*Proof.* First we suppose that  $m^*E < \infty$ . By Proposition 5, there exists some open set  $O \supset E$  such that  $m^*O \leq m^*E + \varepsilon$ . If  $E$  is measurable, then by the definition,

$$m^*(O \setminus E) = m^*O - m^*E \leq \varepsilon.$$

Namely, (ii) holds. Meanwhile,  $O \subset \mathbb{R}$  is a countable union of disjoint open intervals  $\{I_n\}$ . Since  $mO = m^*O$  is bounded and  $mO = \sum l(I_n)$ , there exists some integer  $N > 0$  such that  $mO - \sum_{n=1}^N l(I_n) < \varepsilon$ . Let  $U = \bigcup_{n=1}^N I_n$ .

$$\begin{aligned} m^*(U \triangle E) &= m^*((U \cup E) \setminus (U \cap E)) \\ &\leq m^*(O \setminus (U \cap E)) \\ &= m^*((O \setminus U) \cup (O \setminus E)) \\ &\leq m^*(O \setminus U) + m^*(O \setminus E) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, (ii) implies (vi). Now we show that (vi) implies (ii). If  $m^*(U \triangle E) < \varepsilon$ , then there exists some countable collection  $\{J_n\}$  of open interval such that

$$\sum l(J_n) \leq m^*(U \triangle E) + \varepsilon < 2\varepsilon.$$

Let  $J = \bigcup J_n$  and  $O = U \cup J$ .  $m^*J < 2\varepsilon$ . And  $O$  is open and covers  $E$ . Meanwhile,

$$m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(J \setminus E) < 3\varepsilon.$$

Hence, (ii) holds.

Now, let  $E$  be an arbitrary set and  $E_n = E \cap (-n, n)$ , which is a set with finite measure. Then by the previous discussion, there exists some open set  $O_n \supset E_n$  with  $m^*(O_n \setminus E_n) < \varepsilon/2^n$ . Let  $O = \bigcup O_n$ , an open set covering  $E$  and

$$m^*(O \setminus E) \leq \sum m^*(O_n \setminus E_n) < 2\varepsilon.$$

Hence, (i) implies (ii). Now we suppose (ii) holds and let  $\varepsilon_n = 1/n$ , then there exists a sequence of open sets  $\langle O_n \rangle$  such that  $m^*(O_n \setminus E) < 1/n$ . Let  $G = \bigcap O_n \in G_\delta$ .  $m^*(G \setminus E) \leq m^*(O_n \setminus E) \leq 1/n$ . Since the left hand side is independent of  $n$ ,  $m^*(G \setminus E) = 0$ . If (iv) holds, then by Lemma 6,  $G \setminus E$  is measurable. Since  $G \in G_\delta$  is also measurable,  $E$  is measurable. Hence, (iv) implies (i).

By the previous result, for any measurable  $E$ , there exists some closed set  $F \subset E$  such that  $\bar{F}$ , which is open, contains  $\text{bar}E$  and  $m^*(\bar{F} \setminus \bar{E}) < \varepsilon$ . Hence,  $m^*(E \setminus F) < \varepsilon$ . We can proceed in a similar manner as we did in the last paragraph to prove that (iii)  $\Rightarrow$  (v)  $\Rightarrow$  (i), leading to the final conclusion.  $\square$

### 3.5 Measurable Functions

19.

*Proof.* For every  $\beta \in \mathbb{R}$ , since  $D$  is measurable, there exists a sequence of  $\alpha_n \in D \cap (\beta - 1/n, \beta)$ . As

$$\{x : f(x) > r\} \Leftrightarrow \bigcup_{n=1}^{\infty} \{x : f(x) > r - 1/n\} \Leftrightarrow \bigcup_{n=1}^{\infty} \{x : f(x) > \alpha_n\}$$

and  $\{x : f(x) > \alpha_n\}$  are measurable, so is  $\{x : f(x) > r\}$ . Hence,  $f$  is measurable.  $\square$

21.

*Proof.*

(a) It follows immediately from  $\{x : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$ .

(b) For  $\alpha \geq 0$ , the sets  $\{x : f(x) > \alpha\}$  and  $\{x : g(x) > \alpha\}$  are the same. And for  $\alpha < 0$ ,

$$\{x : f(x) > \alpha\} = \{x : g(x) > \alpha\} \setminus \bar{D} \quad \text{and} \quad \{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \bar{D}.$$

Hence,  $f$  is measurable iff  $g$  is measurable.  $\square$

22.(d)

*Proof.* Since  $f$  and  $g$  are finite almost everywhere, the set  $A$  consisting of points where  $f + g$  is of the form  $\infty - \infty$  or  $-\infty + \infty$  is of measure zero (and hence measurable). Therefore no matter how it is defined,  $\{x \in A : f + g > \alpha\}$  is measurable for every  $\alpha$ . Namely, the restriction of  $f + g$  to  $A$  is measurable. Meanwhile, clear that the restriction to  $D \setminus A$  is measurable where  $D$  is the domain of  $f$ . Hence, by Exercise 21,  $f$  is measurable.  $\square$

**23.**

*Proof.*

(a) Let  $A_n = \{x : |f(x)| > n\}$ , a sequence of measurable sets. As  $A_{n+1} \subset A_n$ ,  $mA_{n+1} \leq mA_n$ . Since  $A = \bigcap A_n = \{x : |f(x)| = \infty\}$ ,  $mA_1 \leq m[a, b]$  is finite and  $mA = 0$ , by Proposition 14, there exists some  $N$  such that for all  $n \geq N$ ,  $mA_n < \varepsilon/3$ . Set  $M = N$  to complete the proof.

(b) We consider the restriction of  $f$  on to the set  $E = [a, b] \setminus \{x : |f(x)| \geq M\}$ , which is also a measurable real-valued function. To keep our notation simple, we denote the restriction by  $f$  still. For every  $\varepsilon > 0$ , there exists some integer  $N$  with  $0 < 2M/N < \varepsilon$ . Let  $E_n = \{x : x \in [-M + (n-1)\varepsilon, -M + n\varepsilon]\}$  ( $n = 1, 2, \dots, N$ ) and define

$$\varphi(x) = \sum_{i=1}^N f(x_i) \chi_{E_i},$$

where  $x_n \in E_n$  is arbitrary. Clear that  $\varphi$  is a simple function and satisfy all the requirements.

(c) Suppose that  $\varphi(x) = \sum_{i=1}^n \alpha_i \chi_{E_i}$ . For each  $i = 1, \dots, N$ ,  $E_i$  is measurable and therefore by Proposition 15, there exists a finite union  $U_i$  of open intervals such that  $m(U_i \triangle E_i) < \varepsilon$ . Let

$$g(x) = \sum_{i=1}^N \alpha_i \chi_{U_i}.$$

Clear that  $g$  and  $\varphi$  only may differ on a set with measure  $N\varepsilon$ . (d) Suppose that  $g(x) = \sum_{i=1}^N \alpha_i \chi_{U_i}$  is a step function. We may assume without loss of generality that  $U_i$  are disjoint and  $\bigcup U_i = [a, b]$ . And suppose that  $\{x_0 = a < x_1 < \dots < x_N = b\}$  are the endpoints of the intervals. For each  $i = 1, \dots, N-1$ , define

$$f(x) = (x - x_i + \varepsilon)g(x_i - \varepsilon) + (x_i + \varepsilon - x)g(x_i + \varepsilon), \quad x \in (x_i - \varepsilon, x_i + \varepsilon),$$

and  $f(x) = g(x)$  for the other points. (We assume that  $\varepsilon$  is small enough so that  $f$  is well-defined.) Clear that  $f$  is continuous and equals  $g$  except on a set of measure less than  $2N\varepsilon$ .  $\square$

**24.**

*Proof.* For measurable  $f$ , we show that  $\mathcal{A} = \{E : f^{-1}[E] \text{ is measurable}\}$  is a  $\sigma$ -algebra first. As the domain, denoted by  $D$ , of a measurable function is measurable,  $\mathbb{R} \in \mathcal{A}$ . If  $E \in \mathcal{A}$ , then since  $f^{-1}[\bar{E}] = D \cap \overline{f^{-1}[E]}$ ,  $f^{-1}[\bar{E}]$  is also measurable and therefore  $\bar{E} \in \mathcal{A}$ . Suppose that  $\langle E_n \rangle$  is a sequence of sets of  $\mathcal{A}$ . Then, as

$$f^{-1}\left[\bigcup_{n=1}^{\infty} E_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[E_n],$$

$\bigcup E_n \in \mathcal{A}$ . Hence,  $\mathcal{A}$  is a  $\sigma$ -algebra.

By the definition of a measurable function, every open interval belongs to  $\mathcal{A}$ . Since the collection of all Borel sets  $\mathcal{B}$  is the  $\sigma$ -algebra generated by all open intervals,  $\mathcal{B} \subset \mathcal{A}$ . Namely,  $f^{-1}[B]$  is measurable as long as  $B \in \mathcal{B}$ .  $\square$

### 3.6 Littlewood's Three Principles

30.

*Proof.* Let  $\varepsilon_n = 1/n$  and  $\delta_n = \eta/2^n$  ( $n = 0, 1, \dots$ ). By Proposition 24, for each  $n$ , there exists some  $A_n$  with measure less than  $\delta_n$  such that for all  $x \in E_n \setminus A_n$ ,  $|f_m(x) - f(x)| < \varepsilon_n$  for  $m$  large enough. Let  $A = \bigcup_{n=1}^{\infty} A_n$ , the measure of which is less than  $\sum \eta/2^n = \delta$ . Meanwhile, for any  $\varepsilon > 0$ , by construction, for all  $x \in E \setminus A$ ,  $|f_m(x) - f(x)| < \varepsilon$  for  $m$  large enough. Namely,  $f_n$  converges to  $f$  uniformly on  $E \setminus A$ .  $\square$

31.

*Proof.* Let  $\varepsilon_n = \delta/2^n$  ( $n \geq 0$ ), then by Proposition 22, there exists continuous  $g_n$  such that  $E_n = \{x : |f(x) - g_n(x)| \geq \varepsilon_n\}$  is of measure less than  $\varepsilon_n$ . Let  $E = \bigcup E_n$ , the measure of which is less than  $\delta$  and  $g_n$  converges to  $f$  on  $[a, b] \setminus E$ .

By Egoroff's Theorem, there exists some  $A \subset [a, b] \setminus E$  with  $mA < \delta$  such that  $g_n$  converges to  $f$  uniformly on  $[a, b] \setminus (E \cup A)$ . Since  $E \cup A$  is measurable, by Proposition 15, there exists some open set  $O \supset E \cup A$  such that  $m(O \setminus (E \cup A)) < \delta$ . Let  $F = [a, b] \setminus O$ . We know that

1.  $F$  is a closed set.
2.  $mF < 3\delta$ .
3.  $g_n$  converges to  $f$  uniformly on  $F$ .

Hence,  $f$  is continuous on  $F$ . And by Problem 2.40, there exists some continuous function on  $\mathbb{R}$  such that  $\varphi(x) = f(x)$  for  $x \in F$ .

If  $f$  is defined on  $(-\infty, \infty)$ , we can apply the previous result on each  $[n, n+1]$  and "stick" the functions together as we did in Problem 23(c) to get the function required.  $\square$

## 4 The Lebesgue Integral

### 4.2 The Lebesgue Integral of a Bounded Function

2.

*Proof.*

(a) By Problem 2.51,  $h$  is upper semicontinuous as  $f$  is bounded and by Problem 2.50,  $x : h(x) < \lambda$  is open and hence measurable for every  $\lambda \in \mathbb{R}$ . Thus,  $h$  is measurable.

Let  $\varphi(x) \geq f(x)$  be a step function and  $x_0$  any point other than the endpoints of the intervals occurring in  $\varphi$ . Then there exists some  $\delta > 0$  such that for all  $x \in (x_0 - \delta, x_0 + \delta)$ ,  $\varphi(x_0) = \varphi(x) \geq f(x)$ . Hence,

$$h(x_0) = \inf_{\delta < 0} \sup_{|x - x_0| < \delta} f(x) \leq \varphi(x_0).$$

Namely,  $\varphi \geq h$  except at a finite number of points. Hence,  $\int_a^b \varphi \geq \int_a^b h$  and therefore

$$R \int_a^{\bar{b}} f = \inf_{\varphi \geq f} \int_a^b \varphi(x) dx \geq \int_a^b h.$$

We can also derive from the previous discussion that there is a sequence of  $\langle \varphi_n \rangle$  of step functions satisfying  $\varphi \downarrow h$ . By Proposition 6,

$$\int_a^b h = \lim \int_a^b \varphi_n \geq R \int_a^{\bar{b}} f.$$

Hence,  $R \int_a^{\bar{b}} f = \int_a^b h$ .

(b) First suppose that  $f$  is Riemann integrable and let  $h$  and  $g$  be the upper and lower envelope of  $f$  respectively. By part (a),  $f$  is Riemann integrable implies  $\int_a^b (h - g) = 0$ . Together with the fact that  $h \geq g$ , we conclude that  $h = g$  a.e.. Therefore, by Problem 2.50,  $f$  is continuous except on a set of measure zero.

Note that the argument remains true if we reverse the order, verifying the converse part. Hence, the proposition holds.  $\square$

### 4.3 The Integral of a Nonnegative Function

3.

*Proof.* Suppose that  $E_n = \{x : f(x) > 1/n\}$ . Then,

$$0 = \int f \geq \int_{E_n} f \geq \frac{mE_n}{n}$$

implies  $mE_n = 0$ . Hence,  $m\{x : f(x) > 0\} = m(\bigcup E_n) \leq \sum mE_n = 0$ . Namely,  $f = 0$  a.e.  $\square$

5.

*Proof.* For any fixed  $x_0 \in \mathbb{R}$ , let  $f_n(x) = f \cdot \chi_{(-\infty, x_0 - 1/n]}$ , which is a increasing sequence of nonnegative measurable function whose limit is  $f \cdot \chi_{(-\infty, x_0]}$ . Then by Theorem 10,

$$F(x_0) = \int_{-\infty}^{x_0} f = \int f \cdot \chi_{(-\infty, x_0]} = \lim \int f \cdot \chi_{(-\infty, x_0 - 1/n]} = \lim F(x_0 - 1/n).$$

Meanwhile, since

$$|F(x_0) - F(x_0 + 1/n)| = \left| \int_{x_0}^{x_0 + 1/n} f(x) dx \right| = \left| \int_{-1/n}^0 g(x) dx \right|,$$

where  $g(x) = f(x_0 - x)$ , arguing on  $g$  in a similar manner yields  $F(x_0) = \lim F(x_0 + 1/n)$ . Thus,  $F$  is continuous.  $\square$

6.

*Proof.* By Theorem 9,  $\int f \leq \underline{\lim} \int f_n$ . Meanwhile,  $f_n \leq f$  implies  $\int f_n \leq \int f$  and therefore  $\overline{\lim} \int f_n \leq \int f$ . Hence,  $\int f = \lim \int f_n$ .  $\square$

7.

*Solution.*

(a) Let  $f_n(x) = n \cdot \chi_{[0, 1/n]}$ .  $f_n$  converges to  $f = 0$  except on  $x = 0$ . For each  $n$ ,  $\int f_n = 1$  but  $\int f = 0$ . Hence, the inequality could be strict.

(b) Let  $f_n(x) = \chi_{[n, \infty)}$ . Then  $\langle f_n \rangle$  is a decreasing sequence which converges to  $f = 0$ , the integral of which is 0. However, for every  $n$ ,  $\int f_n = \infty$ .  $\square$

8.

*Proof.* Let  $g_n = \inf\{f_n, f_{n+1}, \dots\}$ . Clear that

$$\int g_n \leq \int f_n. \tag{1}$$

Meanwhile  $\langle g_n \rangle$  is a increasing sequence converging to  $\underline{\lim} f_n$ . Hence, by the Monotone Convergence Theorem and (1)

$$\int \underline{\lim} f_n = \int \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} \int g_n \leq \underline{\lim} \int f_n.$$

$\square$

9.

*Proof.* By Fatou's Lemma,

$$\int_E f \leq \underline{\lim} \int_E f_n. \tag{2}$$

Similarly,  $\int_{\bar{E}} f \leq \underline{\lim} \int_{\bar{E}} f_n$  and therefore

$$\int_E f_n = \int f_n - \int_{\bar{E}} f_n \Rightarrow \overline{\lim} \int_E f_n \leq \int f - \int_{\bar{E}} f = \int_{\bar{E}} f.$$

(2) and the inequality above together implies  $\int_E f_n \rightarrow \int f$ .  $\square$



## 4.4 The General Lebesgue Integral

12.

*Proof.* Note that  $\langle g + f_n \rangle$  is a sequence of nonnegative measurable functions. Hence by Problem 8,

$$\int_E \underline{\lim}(g + f_n) \leq \underline{\lim} \int_E (g + f_n) \Rightarrow \int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n.$$

The second inequality follows immediately from the definition of lower and upper limit. Replacing  $g + f_n$  with  $g - f_n$  and arguing in a similar manner gives the last inequality.  $\square$

13.

*Proof.*  $f_n \geq -h$  implies  $f_n + h \geq 0$ . Hence,  $\int (f_n + h)$  always has a meaning. And since  $g$  is integrable,  $\int f_n = \int (f_n + h) - \int h$  also has a meaning. Similarly,  $\int f$  has a meaning. Meanwhile,

$$\int f = \int (f + h) - \int h \leq \underline{\lim} \int (f_n + h) - \int h = \underline{\lim} \int f_n.$$

$\square$

15.

*Proof.*

(a) By Problem 4, for every  $\varepsilon > 0$ , there exists some simple functions  $\varphi_1 \leq f^+$  and  $\varphi_2 \leq f^-$  such that

$$\int_E f^+ - \int_E \varphi_1 < \varepsilon \quad \text{and} \quad \int_E f^- - \int_E \varphi_2 < \varepsilon.$$

Let  $\varphi = \varphi_1 - \varphi_2$ , which is also a simple function. Meanwhile,

$$\int_E |f - \varphi| \leq \int_E (f^+ - \varphi_1) + \int_E (f^- - \varphi_2) < 2\varepsilon.$$

$\square$

16.

*Proof.* For every integrable  $f$ , by Problem 15, there exists some step function  $\psi = \sum_{k=1}^N c_k \chi_{E_k}$  such that  $\int |f - \psi| < \varepsilon$ . Note that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \psi(x) \cos nx dx = \lim_{n \rightarrow \infty} \sum_{k=1}^N c_k \int_{E_k} \cos nx dx = 0. \quad (3)$$

Hence,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} f(x) \cos nx dx \right| &= \left| \int_{-\infty}^{\infty} (f(x) - \psi(x) + \psi(x)) \cos nx dx \right| \\ &\leq \int_{-\infty}^{\infty} |f(x) - \psi(x)| |\cos nx| dx + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx dx \right| \\ &\leq \varepsilon + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx dx \right| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

□

**18.**

*Proof.* Let  $\langle t_n \rangle$  be any sequence with  $t_n \neq 0$  and tending to 0. Then  $\langle f(x, t_n) \rangle$  is sequence of functions satisfying the hypotheses of Lebesgue Convergence Theorem. Meanwhile,  $f(x, t_n) \rightarrow f$  as  $n \rightarrow \infty$ . Hence,

$$\lim_{n \rightarrow \infty} \int f(x, t_n) dx = \int f(x) dx.$$

Since the choice of  $\langle t_n \rangle$  is arbitrary, by Problem 2.49f,

$$\lim_{t \rightarrow 0} \int f(x, t) dx = \int f(x) dx.$$

If  $f$  is continuous in  $t$  for each  $x$ , then  $\lim_{\Delta t \rightarrow 0} f(x, t + \Delta t) = f(x, t)$  holds for every  $t$ . Therefore, replacing  $t$  with  $\Delta t$  in the previous result yields

$$\lim_{\Delta t \rightarrow 0} \int f(x, t + \Delta t) dx = \int f(x, t) dx.$$

Namely,  $h(t)$  is continuous.

□

## 5 Differentiation and Integration

### 5.1 Differentiation of Monotone Functions

3. "maximum" needs to be changed to "minimum" in both (a) and (b).

*Proof.*

(a) We may assume without loss of generality that  $c = 0$ . Since  $f$  attains a local minimum at  $x = 0$ ,  $f(h) \geq f(0)$  for every  $h$  sufficiently small. Hence, for every small  $h > 0$ ,  $(f(c+h) - f(c))/h > 0$  and therefore  $D_+f(c) \geq 0$ . Meanwhile, by Problem 2.b,

$$-D_-f(0) = D^+f(0) \geq 0 \quad \Rightarrow \quad D_-f(0) \leq 0.$$

The other two inequalities follow immediately from the definitions of upper and lower limits.

(b) If  $f$  has a local minimum at  $a$  or  $b$ , then we only have the right or left half of the inequalities.  $\square$

4.

*Proof.* We first show this for  $g$  with  $D^+g \geq \varepsilon > 0$ . For every  $a \leq x < y \leq b$ , as  $g$  is continuous on  $[a, b]$ ,  $g$  has a maximum in  $[a, b]$  and by Problem 2 and 3,  $g$  can not attain the maximum in  $[a, b)$ . Namely, the restrict of  $f$  to  $[x, y]$  attains the maximum at  $y$ . Hence,  $g(x) \leq g(y)$ .

For every  $f$  with nonnegative  $D^+$ , let  $g(x) = f(x) + \varepsilon x$  where  $\varepsilon > 0$ . Then  $D^+g \geq \varepsilon > 0$ . Hence  $g$  is nondecreasing. Therefore, for every  $a \leq x < y \leq b$ ,

$$g(x) \leq g(y) \quad \Rightarrow \quad f(x) + \varepsilon x \leq f(y) + \varepsilon y.$$

Since the choice of  $\varepsilon$  is arbitrary, this implies  $f(x) \leq f(y)$ .  $\square$

5.a

*Proof.*

$$\begin{aligned} \sup_{t \in (0, h)} \frac{(f+g)(x+t) - (f+g)(x)}{t} &= \sup_{t \in (0, h)} \left( \frac{f(x+t) - f(x)}{t} + \frac{g(x+t) - g(x)}{t} \right) \\ &\leq \sup_{t \in (0, h)} \frac{f(x+t) - f(x)}{t} + \sup_{t \in (0, h)} \frac{g(x+t) - g(x)}{t}. \end{aligned}$$

Letting  $h \rightarrow 0$  yields  $D^+(f+g) \leq D^+f + D^+g$ .  $\square$

### 5.2 Functions of Bounded Variation

7.

*Proof.*

(a) It suffices to show this for monotone functions as each function of bounded variation is the difference of two monotone functions. Suppose that  $f$  is nondecreasing. Then the set  $E = \{f(x) : x > c\}$  is bounded below and hence  $A = \inf E$  is finite. For every  $\varepsilon > 0$ , there exists some  $y > c$  such that  $A \leq f(c) < A + \varepsilon$ . Hence, as  $f$  is nondecreasing,

for every  $x \in (c, y)$ ,  $|f(x) - A| < \varepsilon$ . Namely,  $\lim_{x \rightarrow c+} f(x) = A$ . Similarly,  $\lim_{x \rightarrow c-} f(x)$  exists.

Let  $D_n = \{x : |f(x+) - f(x-)| > 1/n\}$ . Since  $f$  is nondecreasing,  $|f(x) - f(y)| \leq f(b) - f(a) < \infty$  for every  $x, y \in [a, b]$ . Hence,  $D_n$  is finite, otherwise we can choose a sequence  $x_1 < \dots < x_N$  with  $N > (f(b) - f(a))/n$  such that  $f(x_N) - f(x_1) > f(b) - f(a)$ . Therefore,  $\bigcup_{n=1}^{\infty} D_n$ , the set of discontinuities, is countable.

(b) Suppose  $\{x_1, \dots, x_n, \dots\} = \mathbb{Q} \cap [0, 1]$  and define  $f(x) = \sum_{x_n < x} 2^{-n}$ . Clear that  $f$  is monotone and continuous at every irrational point. For each rational  $x = x_k$ ,  $f(x+) - f(x-) = 2^{-k}$ . Hence,  $f$  is discontinuous at each rational point.  $\square$

8.

*Proof.*

(a) For every  $\varepsilon > 0$ , there exists some subdivision  $a = x_0 < \dots < x_p = c$  and  $c = x_p < \dots < x_q = b$  of  $[a, c]$  and  $[c, b]$  such that  $T_a^c < t_a^c + \varepsilon$  and  $T_c^b < t_c^b + \varepsilon$ . Hence,  $T_a^c + T_c^b - 2\varepsilon < t_a^c + t_c^b$ . Meanwhile, as  $a = x_0 < \dots < x_q = b$  forms a subdivision of  $[a, b]$ ,  $T_a^b \geq t_a^b = t_a^c + t_c^b$ . Therefore,  $T_a^c + T_c^b - 2\varepsilon < T_a^b$ . Since the choice of  $\varepsilon$  is arbitrary,  $T_a^b + T_c^b \leq T_a^b$ .

To show that  $T_a^b + T_c^b \geq T_a^b$ , let  $a = x_0 < \dots < x_q = b$  be any subdivision of  $[a, b]$  and by adding  $c$  into it, we get subdivisions of  $[a, c]$  and  $[c, b]$ . Suppose that  $c \in (x_k, x_{k+1}]$ , then

$$|f(x_k) - f(c)| + |f(c) - f(x_{k+1})| + t_a^b = t_a^c + t_c^b + |f(x_k) - f(x_{k+1})|,$$

which implies  $t_a^b \leq t_a^c + t_c^b$ . Hence,

$$T_a^b = \sup t_a^b \leq \sup(t_a^c + t_c^b) \leq T_a^c + T_c^b.$$

Thus,  $T_a^b = T_a^c + T_c^b$  and therefore  $T_a^c \leq T_a^b$ .

(b)

$$\begin{aligned} T_a^b(f+g) &= \sup \sum_{i=1}^k |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\leq \sup \sum_{i=1}^k |f(x_i) - f(x_{i-1})| + \sup \sum_{i=1}^k |g(x_i) - g(x_{i-1})| \\ &\leq T_a^b(f) + T_a^b(g). \end{aligned}$$

$$T_a^b(cf) = \sup \sum_{i=1}^k |cf(x_i) - cf(x_{i-1})| = |c| \sup \sum_{i=1}^k |f(x_i) - f(x_{i-1})| = |c| T_a^b(f).$$

$\square$

9.

*Proof.* For every  $\varepsilon > 0$ , there exists a subdivision  $a = x_0 < \cdots < x_k = b$  such that  $t_a^b(f) \geq T_a^b(f) - \varepsilon$ . Meanwhile, as  $f_n$  converges to  $f$  pointwisely

$$\begin{aligned} t_a^b(f) &= t_a^b(\lim f_n) \\ &= \sum_{i=1}^k |(\lim f_n)(x_i) - (\lim f_n)(x_{i-1})| \\ &= \lim \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| \\ &\leq \underline{\lim} \sup \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| = \underline{\lim} T_a^b(f_n). \end{aligned}$$

Hence,  $T_a^b(f) - \varepsilon \leq \underline{\lim} T_a^b(f_n)$ . Since the choice of  $\varepsilon$  is arbitrary,  $T_a^b(f) \leq \underline{\lim} T_a^b(f_n)$ .  $\square$

### 10.a

*Solution.* No. Let  $x_k = (k\pi + \pi/2)^{-1/2}$ ,  $k = 0, 1, \dots$  and consider the subdivision  $-1 < 0 < x_n < \cdots < x_0 < 1$ . Then

$$t_n \geq \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \geq \sum_{k=1}^n \frac{2}{(k + 1/2)\pi}.$$

$t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and therefore  $f$  is not of bounded variation on  $[-1, 1]$ .  $\square$

### 11.

*Proof.* By Lemma 4,  $f(x) = f(a) + P_a^x - N_a^x$ . Since  $P_a^x$  and  $N_a^x$  are monotone, by Theorem 3, they are differentiable almost everywhere as  $f$ , a function of bounded variation, does. Hence, for almost every  $x \in [a, b]$ ,

$$\frac{d}{dx}f(x) = \frac{d}{dx}P_a^x - \frac{d}{dx}N_a^x \quad \Rightarrow \quad |f'(x)| \leq \frac{d}{dx}P_a^x + \frac{d}{dx}N_a^x = \frac{d}{dx}T_a^x.$$

Integrating on the both sides yields  $\int_a^b |f'| \leq T_a^b(f)$ .  $\square$