

Solutions to  
*Introductory Functional Analysis with Applications*

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## 2 Normed Spaces. Banach Spaces

### 2.3 Further Properties of Normed Spaces

4. cf. Prob. 13, Sec 1.2

*Proof.* The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1| \|x_1 - x_2\| + |\alpha_1 - \alpha_2| \|x_2\|.$$

□

7.

*Proof.* Let  $Y$  and  $y_n$  be defined as in the hint. Then  $\|y_n\| = 1/n^2$ , constituting a convergent number series. However,

$$\sum_{n=1}^N y_n = (1, 1/4, \dots, 1/N^2, 0, \dots),$$

which is divergent as  $N \rightarrow \infty$ .

□

8.

*Proof.* Let  $(x_n)$  be a Cauchy sequence in  $X$ . Hence, for every  $n > 0$ , there exists some  $K_n > 0$  such that for all  $p, q > K_n$ ,  $\|x_p - x_q\| < 1/n^2$ . Without loss of generality, we may assume that  $(K_n)$  is increasing. Since the series  $\|x_{K_{n+1}} - x_{K_n}\|$  is bounded by  $1/n^2$ , it converges. By the hypothesis, the series  $(x_{K_{n+1}} - x_{K_n})$  also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1} (x_{K_{i+1}} - x_{K_i}) \rightarrow x \quad \text{as } n \rightarrow \infty.$$

Now we show that  $(x_n)$  converges to  $x$ . For every  $\varepsilon > 0$ , since  $(x_n)$  is a Cauchy sequence, there exists some  $N_1$  such that for all  $p, q > N_1$ ,  $\|x_p - x_q\| < \varepsilon$ . Meanwhile, since  $x_{K_n} \rightarrow x$ , once  $K_n$  is large enough,  $\|x - x_{K_n}\| < \varepsilon$ . Let  $K_n > N_1$ . Then for every  $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus,  $X$  is complete.

□

9.

*Proof.* Let  $(x_n)$  be an absolutely convergent series in Banach space  $X$ . Let  $s_n = \sum_{i=1}^n x_i$ . Now we show that  $s_n$  is a Cauchy sequence and therefore convergent. Since  $\sum_{i=1}^{\infty} \|x_i\| < \infty$ , for every  $\varepsilon > 0$ , there exists some  $N > 0$  such that for all  $n > N$ ,  $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$ . Hence, for every  $N < p \leq q$ ,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^q x_i \right\| \leq \sum_{i=p+1}^q \|x_i\| < \varepsilon,$$

completing the proof.

□

10.

*Proof.* Let  $(e_n)$  be Schauder basis of  $X$ . Denote the underlying field of  $X$  by  $\mathbb{K}$  and let  $\mathbb{W} = \mathbb{Q}$  if  $\mathbb{K} = \mathbb{R}$  and  $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$  if  $\mathbb{K} = \mathbb{C}$ . Now we show that

$$S = \left\{ \sum_{i=1}^n \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \dots \right\},$$

a countable subset of  $X$ , is dense in  $X$  to derive the separability.

For every  $x \in X$  and  $\varepsilon > 0$ , by the definition of Schauder basis, there exists  $\beta_1, \dots, \beta_n \in \mathbb{K}$  such that  $\|x - (\beta_1 e_1 + \dots + \beta_n e_n)\| < \varepsilon$ . Let  $M = \max_i \|e_i\|$ . If  $M = 0$ , then there is nothing to prove. Otherwise, since  $\mathbb{W}$  is dense in  $\mathbb{K}$ , for  $i = 1, \dots, n$ , there exists  $\alpha_i \in \mathbb{W}$  with  $|\alpha_i - \beta_i| < \varepsilon/2^i M$ . Hence,

$$\begin{aligned} \left\| x - \sum_{i=1}^n \alpha_i e_i \right\| &\leq \left\| x - \sum_{i=1}^n \beta_i e_i \right\| + \left\| \sum_{i=1}^n (\beta_i - \alpha_i) e_i \right\| \\ &\leq \varepsilon + \sum_{i=1}^n |\alpha_i - \beta_i| \|e_i\| \\ &\leq 2\varepsilon. \end{aligned}$$

Thus,  $S$  is dense in  $X$  and therefore  $X$  is separable.  $\square$

14.

*Proof.* Clear that  $\|\cdot\|_0$  is nonnegative. And  $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$ . Meanwhile,  $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$ . Finally, we show that  $\|\hat{x}\|_0 = 0$  implies  $\hat{x} = Y$  and invoke Prob. 4, Sec 2.2 to complete the proof. Since  $\|\hat{x}\|_0 = 0$ , there exists  $(x_n) \subset \hat{x}$  which converges to 0. Since  $Y$  is closed,  $Y$  is complete and so is its cosets. Therefore,  $0 \in \hat{x}$ , enforcing  $\hat{x}$  to be  $Y$ .  $\square$

## 2.4 Finite Dimensional Normed Spaces

3.

*Proof.* The reflexive property clearly holds. If there are positive  $a$  and  $b$  such that  $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$  for all  $x \in X$ , then  $\|x\|_1/b \leq \|x\|_0 \leq \|x\|/a$ . Hence the relation is symmetric. Next we further suppose there exists positive  $c$  and  $d$  such that that  $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$ . Then  $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$ , giving the transitive property. Thus, the axioms of an equivalence relation hold.  $\square$

4.

*Proof.* Suppose the norms  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent. Let  $E \subset X$  be any open set with respect to  $\|\cdot\|$ , i.e., for every  $x_0 \in E$ , there exists some  $\delta > 0$  such that  $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$ . Since  $\|\cdot\| \sim \|\cdot\|_0$ , there exists some positive  $c$  such that  $\|x - x_0\| \leq c\|x - x_0\|_0$ . Hence,  $B = \{x \in X : \|x - x_0\| < \delta/c\} \subset A \subset E$ . Namely,  $E$  is also open with respect to  $\|\cdot\|_0$ . Interchanging the roles of  $\|\cdot\|$  and  $\|\cdot\|_0$  completes the proof.  $\square$

5.

*Proof.* Suppose the norms  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent. Then for every  $x \in X$ , there exists some  $c > 0$  such that  $\|x\|_0 \leq c\|x\|$ . Let  $(x_n)$  be a Cauchy sequence with respect to  $\|\cdot\|$ , i.e., for every  $\varepsilon > 0$ , there exists some  $N > 0$  such that for all  $n, m > N$ ,  $\|x_n - x_m\| < \varepsilon/c$ . Hence,  $\|x_n - x_m\|_0 < c\|x_n - x_m\| \leq \varepsilon$ . Thus,  $(x_n)$  is also a Cauchy with respect to  $\|\cdot\|_0$ . Interchanging the roles of  $\|\cdot\|$  and  $\|\cdot\|_0$  completes the proof.  $\square$

## 2.5 Compactness and Finite Dimension

5.

*Proof.* Clear that every point in  $\mathbb{R}^n$  or  $\mathbb{C}^n$  has a closed bounded, and therefore compact, neighborhood. Hence,  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are locally compact.  $\square$

6.

*Proof.* Let  $X$  be a compact metric space and  $x$  any point in  $X$ . Let  $E$  be a closed neighborhood of  $x$ . By Prob 10,  $E$  is compact. Thus,  $X$  is locally compact.  $\square$

7.

*Proof.* It suffices to show that  $a = \inf_{y \in Y} \|v - y\|$  can actually be obtained. Let  $\{b_1, \dots, b_n\}$  be a basis of  $Y$  and  $y_k = y_{k,1}b_1 + \dots + y_{k,n}b_n$  a sequence in  $Y$  with  $\|v - y_k\| \rightarrow a$ . We may assume without loss of generality that  $\|v - y_k\|$  is bounded.

Since  $Y$  is a proper subset of  $Z$ ,  $v, b_1, \dots, b_n$  are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar  $c > 0$  such that for every  $k$ ,

$$\|v - y_{k,1}b_1 - \dots - y_{k,n}b_n\| \geq c(1 + |y_{k,1}| + \dots + |y_{k,n}|).$$

Hence, the sequence  $(y_{k,1}, \dots, y_{k,n})$  of  $n$ -tuples is bounded and therefore has a convergent subsequence. Consequently,  $(y_k)$  also has a convergent subsequence. Suppose that it converges to  $z \in Z$ . Note that  $\|v - z\| = a$  and as  $Y$  is closed,  $z \in Y$ . Thus,  $a$  can be attained in  $Y$ .  $\square$

8.

*Proof.* Since the unit ball  $B$  with respect to  $\|\cdot\|_2$  in  $\mathbb{R}^n$  and  $\mathbb{C}^n$  is compact and  $\|\cdot\|$  is continuous, by 2.5-7,  $x \mapsto \|x\|$  can attain its minimum, denoted by  $a$ , on  $B$ . Due to the positive definite property of a norm,  $a$  is positive. Hence,  $0 < a \leq \|x\|/\|x\|_2$ . Namely,  $a\|x\|_2 \leq \|x\|$ .  $\square$

9.

*Proof.* For every  $(x_n) \subset M \subset X$ , since  $X$  is compact, there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  which converges to some  $y \in X$ . Since  $M$  is closed,  $y \in M$ . Hence,  $M$  is compact.  $\square$

10.

*Proof.* From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of  $T$  is also continuous, it suffices to show that the image of a closed set  $A \subset X$  under  $T$  is again a closed set. Since  $X$  is compact and  $A$  is closed,  $A$  is compact. Since  $T$  is continuous, by 2.5-6,  $T(A)$  is compact and therefore closed. Hence,  $T$  is a homeomorphism.  $\square$

## 2.6 Bounded and Continuous Linear Operators

2.

*Proof.* First suppose  $T$  to be bounded and let  $A$  be any bounded set in  $X$ . Then there exists  $K < \infty$  such that for all  $x \in A$ ,  $\|x\| < K$ . Due to the boundedness of  $T$ ,  $\|Tx\| \leq \|T\|\|x\| < K\|T\|$ . Namely,  $T(A)$  is also bounded.

Now suppose that  $T$  maps bounded sets in  $X$  into bounded sets in  $Y$ . Clear that the unit ball  $B$  of  $X$  is bounded and therefore so is  $T(B)$ . Namely,  $\|Tx/\|x\|\|$  is bounded for  $x \neq 0$ .<sup>1</sup> Hence,  $T$  is bounded.  $\square$

3.

*Proof.* For every  $x$  with  $\|x\| < 1$ ,  $\|Tx\| \leq \|T\|\|x\| < \|T\|$ .  $\square$

4.

*Proof.* Suppose that the linear operator  $T$  is continuous at  $x_0 \in \mathcal{D}(T)$ . For every  $(x_n) \subset \mathcal{D}(T)$  with  $\|x_n - x_0\| \rightarrow 0$ , by the continuity of  $T$  at  $x_0$

$$\|Tx_n - Tx_0\| = \|T(x_n - x_0 + x_0) - Tx_0\| \rightarrow 0.$$

Hence,  $T$  is continuous.  $\square$

7.

*Proof.* The inequality implies  $\mathcal{N}(T) = 0$ . Hence, by Theorem 2.6-10,  $T^{-1}$  exists. For every  $y \in Y$ , suppose that  $y = Tx$ . Then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

Thus,  $T^{-1}$  is bounded.  $\square$

12.

*Proof.* The compatibility follows immediately from the definition of the supremum. Suppose  $\|x\|_1 = \max_j |\xi_j|$  and  $\|y\|_2 = \max_j \|\eta_j\|$ , then

$$Ax = \begin{bmatrix} x_1\alpha_{11} + \cdots + x_n\alpha_{1n} \\ \vdots \\ x_1\alpha_{r1} + \cdots + x_n\alpha_{rn} \end{bmatrix}$$

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<sup>1</sup>Note that the two  $\|\cdot\|$  here are different norms.

Since for all  $j$ ,  $x_j \leq \|x\|_1$ ,

$$\frac{\max_j |x_1\alpha_{j1} + \cdots + x_n\alpha_{jn}|}{\|x\|_1} = \max_j \left| \frac{x_1}{\|x\|_1}\alpha_{j1} + \cdots + \frac{x_n}{\|x\|_1}\alpha_{jn} \right| \leq \max_j \sum_{k=1}^n |\alpha_{jk}|.$$

Hence,

$$\|A\| \geq \frac{\|Ax\|_2}{\|x\|_1} \quad \text{for all } x. \quad (1)$$

Suppose that maximum of  $\sum_{k=1}^n |\alpha_{jk}|$  is obtained at  $j = p$ . Then choosing  $x_k$  to be  $\text{sgn } \alpha_{pk}$  shows that the equality in (1) can actually be attained. Hence,  $\|A\| = \max_j \sum_{k=1}^n |\alpha_{jk}|$ .  $\square$