Real Analysis

Yunwei Ren

Contents

3 Lebesgue Measure

3.1 Introduction

1.

Proof. Since $\mathfrak{M}$ is an σ-algebra, $B \setminus A \in \mathfrak{M}$ as long as $A, B \in \mathfrak{M}$. Since $B \setminus A$ and A are disjoint, $mB = mA + m(B \setminus A) \geq mA$ since m is nonnegative. $\square$

2.

Proof. Let $A_0 = E_0$ and $E_k = A_k \setminus A_{k-1}$ for $k \geq 1$. Clear that E_i and E_j are disjoint for distinct i and j, $\bigcup A_n = \bigcup E_n$ and $A_i \subset E_i$ for every i. Hence,

$$m\left(\bigcup E_n\right) = m\left(\bigcup A_n\right) = \sum mA_n \leq \sum mE_n,$$

where the last inequality comes from Exercise 1. $\square$

3.

Proof. Suppose that $mA < \infty$. Then $mA = m(A \cup \varnothing) = mA + m\varnothing$, implying that $m\varnothing = 0$. $\square$

3.2 Outer Measure

5.

Proof. We show that $\{I_n\}$ must cover the entire $[0,1]$ by contradiction. Assume that $x \notin I_k$ for $k = 1, 2, \ldots, n$. Then, as I_k are open and n is finite, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ and I_k are disjoint for every k. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some rational number in $(x - \varepsilon, x + \varepsilon)$, contradicting with the hypothesis that $\{I_k\}$ covers all rational numbers between 0 and 1. $\square$

6.

Proof. By the definition of the outer measure, for every $\varepsilon > 0$, there exists some collection $\{I_n\}$ of open intervals that covers A and $\sum l(I_n) \leq m^*A + \varepsilon$. Let $O = \bigcup I_n$. O is a countable union of open sets and therefore is also open. And by Proposition 2, $m^*O \leq \sum l(I_n)$. Thus, $m^*O \leq m^*A + \varepsilon$.

Let $\varepsilon_n = 1/n$ and for each n, by the previous discussion, we can always get an open set O_k such that $A \subset O_k$ and $m^*O \leq m^*A + \varepsilon_m$. Let G be the countable intersection of these open sets. Clear that G is a G_δ set covering A and $m^*A = m^*G$. $\square$

7.

Proof. If $m^*E = \infty$, it is trivial. Suppose that $m^*E \leq \infty$. For any $x \in \mathbb{R}$, collection $\{I_n\}$ of open intervals covers $E + x$ iff $\{I_n - x\}$ covers E. Since the length of intervals is translation invariant, this implies $m^*(E + x) = m^*E$. $\square$

8.

Proof. Clear that $m^*A \leq m^*(A \cup B)$. Meanwhile, $m^*(A \cup B) = m^*A + m^*B = m^*B$. Hence, $m^*(A \cup B) = m^*B$. $\square$

3.3 Measurable Sets and Lebesgue Measure

10.

Proof.

$$\begin{aligned} mE_1 + mE_2 &= mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2). \end{aligned}$$

□

11.

Proof. $E_n = (n, \infty)$. □

12. This is the countable version of Lemma 9.

Proof. It suffices to prove $m^*(A \cap \bigcup E_i) \geq \sum m^*(A \cap E_i)$. Since $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i$ for every n,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of n, we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i),$$

completing the proof. □

13.

Proof. First we suppose that $m^*E < \infty$. By Proposition 5, there exists some open set $O \supset E$ such that $m^*O \leq m^*E + \varepsilon$. If E is measurable, then by the definition,

$$m^*(O \setminus E) = m^*O - m^*E \leq \varepsilon.$$

Namely, (ii) holds. Meanwhile, $O \subset \mathbb{R}$ is a countable union of disjoint open intervals $\{I_n\}$. Since $mO = m^*O$ is bounded and $mO = \sum l(I_n)$, there exists some integer $N > 0$ such that $mO - \sum_{n=1}^{N} l(I_n) < \varepsilon$. Let $U = \bigcup_{n=1}^{N} I_n$.

$$\begin{aligned} m^*(U \triangle E) &= m^*((U \cup E) \setminus (U \cap E)) \\ &\leq m^*(O \setminus (U \cap E)) \\ &= m^*((O \setminus U) \cup (O \setminus E)) \\ &\leq m^*(O \setminus U) + m^*(O \setminus E) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, (ii) implies (vi). Now we show that (vi) implies (ii). If $m^*(U \triangle E) < \varepsilon$, then there exists some countable collection $\{J_n\}$ of open interval such that

$$\sum l(J_n) \leq m^*(U \triangle E) + \varepsilon < 2\varepsilon.$$

Let $J = \bigcup J_n$ and $O = U \cup J$. $m^*J < 2\varepsilon$. And O is open and covers E. Meanwhile,

$$m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(J \setminus E) < 3\varepsilon.$$

Hence, (ii) holds.

Now, let E be an arbitrary set and $E_n = E \cap (-n, n)$, which is a set with finite measure. Then by the previous discussion, there exists some open set $O_n \supset E_n$ with $m^*(O_n \setminus E_n) < \varepsilon/2^n$. Let $O = \bigcup O_n$, an open set covering E and

$$m^*(O \setminus E) \leq \sum m^*(O_n \setminus E_n) < 2\varepsilon.$$

Hence, (i) implies (ii). Now we suppose (ii) holds and let $\varepsilon_n = 1/n$, then there exists a sequence of open sets $< O_n >$ such that $m^*(O_n \setminus E) < 1/n$. Let $G = \bigcap O_n \in G_\delta$. $m^*(G \setminus E) \leq m^*(O_n \setminus E) \leq 1/n$. Since the left hand side is independent of n, $m^*(G \setminus E) = 0$. If (iv) holds, then by Lemma 6, $G \setminus E$ is measurable. Since $G \in G_\delta$ is also measurable, E is measurable. Hence, (iv) implies (i).

By the previous result, for any measurable E, there exists some closed set $F \subset E$ such that $\bar{F}$, which is open, contains $barE$ and $m^*(\bar{F} \setminus \bar{E}) < \varepsilon$. Hence, $m^*(E \setminus F) < \varepsilon$. We can proceed in a similar manner as we did in the last paragraph to prove that (iii) $\Rightarrow$ (v) $\Rightarrow$ (i), leading to the final conclusion. □

3.5 Measurable Functions

19.

Proof. For every $\beta \in \mathbb{R}$, since D is measurable, there exists a sequence of $\alpha_n \in D \cap (\beta - 1/n, \beta)$. As

$$\{x : f(x) > r\} \quad \Leftrightarrow \quad \bigcup_{n=1}^{\infty} \{x : f(x) > r - 1/n\} \quad \Leftrightarrow \quad \bigcup_{n=1}^{\infty} \{x : f(x) > \alpha_n\}$$

and $\{x : f(x) > \alpha_n\}$ are measurable, so is $\{x : f(x) > r\}$. Hence, f is measurable. □

21.

Proof.

(a) It follows immediately from $\{x : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$.

(b) For $\alpha \geq 0$, the sets $\{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$ are the same. And for $\alpha < 0$,

$$\{x : f(x) > \alpha\} = \{x : g(x) > \alpha\} \setminus \bar{D} \quad \text{and} \quad \{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \bar{D}.$$

Hence, f is measurable iff g is measurable. □

22.(d)

Proof. Since f and g are finite almost everywhere, the set A consisting of points where $f + g$ is of the form $\infty - \infty$ or $-\infty + \infty$ is of measure zero (and hence measurable). Therefore no matter how it is defined, $\{x \in A : f + g > \alpha\}$ is measurable for every α. Namely, the restriction of $f + g$ to A is measurable. Meanwhile, clear that the restriction to $D \setminus A$ is measurable where D is the domain of f. Hence, by Exercise 21, f is measurable. □

23.

Proof.
(a) Let $A_n = \{x : |f(x)| > n\}$, a sequence of measurable sets. As $A_{n+1} \subset A_n$, $mA_{n+1} \le mA_n$. Since $A = \bigcap A_n = \{x : |f(x)| = \infty\}$, $mA_1 \le m[a,b]$ is finite and $mA = 0$, by Proposition 14, there exists some N such that for all $n \ge N$, $mA_n < \varepsilon/3$. Set $M = N$ to complete the proof.

(b) We consider the restriction of f on to the set $E = [a,b] \setminus \{x : |f(x)| \ge M\}$, which is also a measurable real-valued function. To keep our notation simple, we denote the restriction by f still. For every $\varepsilon > 0$, there exists some integer N with $0 < 2M/N < \varepsilon$. Let $E_n = \{x : x \in [-M + (n-1)\varepsilon, -M + n\varepsilon]\}$ $(n = 1, 2, \ldots, N)$ and define

$$\varphi(x) = \sum_{i=1}^{N} f(x_i)\chi_{E_i},$$

where $x_n \in E_n$ is arbitrary. Clear that φ is a simple function and satisfy all the requirements.

(c) Suppose that $\varphi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$. For each $i = 1, \ldots, N$, E_i is measurable and therefore by Proposition 15, there exists a finite union U_i of open intervals such that $m(U_i \triangle E_i) < \varepsilon$. Let

$$g(x) = \sum_{i=1}^{N} \alpha_i \chi_{U_i}.$$

Clear that g and φ only may differ on a set with measure $N\varepsilon$. (d) Suppose that $g(x) = \sum_{i=1}^{N} \alpha_i \chi_{U_i}$ is a step function. We may assume without loss of generality that U_i are disjoint and $\bigcup U_i = [a,b]$. And suppose that $\{x_0 = a < x_1 < \cdots < x_N = b\}$ are the endpoints of the intervals. For each $i = 1, \ldots, N-1$, define

$$f(x) = (x - x_i + \varepsilon)g(x_i - \varepsilon) + (x_i + \varepsilon - x)g(x_i + \varepsilon), \quad x \in (x_i - \varepsilon, x_i + \varepsilon),$$

and $f(x) = g(x)$ for the other points. (We assume that ε is small enough so that f is well-defined.) Clear that f is continuous and equals g except on a set of measure less then $2N\varepsilon$. □

24.

Proof. For measurable f, we show that $\mathcal{A} = \{E : f^{-1}[E] \text{ is measurable}\}$ is a σ-algebra first. As the domain, denoted by D, of a measurable function is measurable, $\mathbb{R} \in \mathcal{A}$. If $E \in \mathcal{A}$, then since $f^{-1}[\bar{E}] = D \cap \overline{f^{-1}[E]}$, $f^{-1}[\bar{E}]$ is also measurable and therefore $\bar{E} \in \mathcal{A}$. Suppose that $< E_n >$ is a sequence of sets of $\mathcal{A}$. Then, as

$$f^{-1}\left[\bigcup_{n=1}^{\infty} E_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[E_n],$$

$\bigcup E_n \in \mathcal{A}$. Hence, $\mathcal{A}$ is a σ-algebra.

By the definition of a measurable function, every open interval belongs to $\mathcal{A}$. Since the collection of all Borel sets $\mathcal{B}$ is the σ-algebra generated by all open intervals, $\mathcal{B} \subset \mathcal{A}$. Namely, $f^{-1}[B]$ is measurable as long as $B \in \mathcal{B}$. □

3.6 Littlewood's Three Principles

30.

Proof. Let $\varepsilon_n = 1/n$ and $\delta_n = \eta/2^n$ $(n = 0, 1, \dots)$. By Proposition 24, for each n, there exists some A_n with measure less than δ_n such that for all $x \in E_n \setminus A_n$, $|f_m(x) - f(x)| < \varepsilon_n$ for m large enough. Let $A = \bigcup_{n=1}^{\infty} A_n$, the measure of which is less than $\sum \eta/2^n = \delta$. Meanwhile, for any $\varepsilon > 0$, by construction, for all $x \in E \setminus A$, $|f_m(x) - f(x)| < \varepsilon$ for m large enough. Namely, f_n converges to f uniformly on $E \setminus A$. □

31.

Proof. Let $\varepsilon_n = \delta/2^n$ $(n \geq 0)$, then by Proposition 22, there exists continuous g_n such that $E_n = \{x : |f(x) - g_n(x)| \geq \varepsilon_n\}$ is of measure less than ε_n. Let $E = \bigcup E_n$, the measure of which is less than δ and g_n converges to f on $[a, b] \setminus E$.

By Egoroff's Theorem, there exists some $A \subset [a, b] \setminus E$ with $mA < \delta$ such that g_n converges to f uniformly on $[a, b] \setminus (E \cup A)]$. Since $E \cup A$ is measurable, by Proposition 15, there exists some open set $O \supset E \cup A$ such that $m(O \setminus (E \cup A)) < \delta$. Let $F = [a, b] \setminus O$. We know that

1. F is a closed set.
2. $mF < 3\delta$.
3. g_n converges to f uniformly on F.

Hence, f is continuous on F And by Problem 2.40, there exists some continuous function on $\mathbb{R}$ such that $\varphi(x) = f(x)$ for $x \in F$.

If f is defined on $(-\infty, \infty)$, we can apply the previous result on each $[n, n+1]$ and "stick" the functions together as we did in Problem 23(c) to get the function required. □

4 The Lebesgue Integral

4.2 The Lebesgue Integral of a Bounded Function

2.

Proof.
(a) By Problem 2.51, h is upper semicontinuous as f is bounded and by Problem 2.50, $x: h(x) < \lambda$ is open and hence measurable for every $\lambda \in \mathbb{R}$. Thus, h is measurable.

Let $\varphi(x) \geq f(x)$ be a step function and x_0 any point other than the endpoints of the intervals occurring in φ. Then there exists some $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$, $\varphi(x_0) = \varphi(x) \geq f(x)$. Hence,

$$h(x_0) = \inf_{\delta < 0} \sup_{|x - x_0| < \delta} f(x) \leq \varphi(x_0).$$

Namely, $\varphi \geq h$ except at a finite number of points. Hence, $\int_a^b \varphi \geq \int_a^b h$ and therefore

$$R \overline{\int_a^b} f = \inf_{\varphi \geq f} \int_a^b \varphi(x) \mathrm{d}x \geq \int_a^b h.$$

We can also derive from the previous discussion that there is a sequence of $\langle \varphi_n \rangle$ of step functions satisfying $\varphi \downarrow h$. By Proposition 6,

$$\int_a^b h = \lim \int_a^b \varphi_n \geq R \overline{\int_a^b} f.$$

Hence, $R \overline{\int_a^b} f = \int_a^b h$.

(b) First suppose that f is Riemann integrable and let h and g be the upper and lower envelope of f respectively. By part (a), f is Riemann integrable implies $\int_a^b (h - g) = 0$. Together with the fact that $h \geq g$, we conclude that $h = g$ a.e.. Therefore, by Problem 2.50, f is continuous except on a set of measure zero.

Note that the argument remains true if we reverse the order, verifying the converse part. Hence, the proposition holds. $\square$

4.3 The Integral of a Nonnegative Function

3.

Proof. Suppose that $E_n = \{x : f(x) > 1/n\}$. Then,

$$0 = \int f \geq \int_{E_n} f \geq \frac{mE_n}{n}$$

implies $mE_n = 0$. Hence, $m\{x : f(x) > 0\} = m(\bigcup E_n) \leq \sum mE_n = 0$. Namely, $f = 0$ a.e. $\square$

5.

Proof. For any fixed $x_0 \in \mathbb{R}$, let $f_n(x) = f \cdot \chi_{(-\infty, x_0 - 1/n]}$, which is a increasing sequence of nonnegative measurable function whose limit is $f \cdot \chi_{-\infty, x_0}$. Then by Theorem 10,

$$F(x_0) = \int_{-\infty}^{x_0} f = \int f \cdot \chi_{-\infty, x_0} = \lim \int f \cdot \chi_{(-\infty, x_0 - 1/n]} = \lim F(x_0 - 1/n).$$

Meanwhile, since

$$|F(x_0) - F(x_0 + 1/n)| = \left| \int_{x_0}^{x_0 + 1/n} f(x) \mathrm{d}x \right| = \left| \int_{-1/n}^{0} g(x) \mathrm{d}x \right|,$$

where $g(x) = f(x_0 - x)$, arguing on g in a similar manner yields $F(x_0) = \lim F(x_0 + 1/n)$. Thus, F is continuous. □

6.

Proof. By Theorem 9, $\int f \leq \underline{\lim} \int f_n$. Meanwhile, $f_n \leq f$ implies $\int f_n \leq \int f$ and therefore $\overline{\lim} \int f_n \leq \int f$. Hence, $\int f = \lim \int f_n$. □

7.

Solution.

(a) Let $f_n(x) = n \cdot \chi_{[0,1/n]}$. f_n converges to $f = 0$ except on $x = 0$. For each n, $\int f_n = 1$ but $\int f = 0$. Hence, the inequality could be strict.

(b) Let $f_n(x) = \chi_{[n,\infty)}$. Then $\langle f_n \rangle$ is a decreasing sequence which converges to $f = 0$, the integral of which is 0. However, for every n, $\int f_n = \infty$. □

8.

Proof. Let $g_n = \inf\{f_n, f_{n+1}, \ldots\}$. Clear that

$$\int g_n \leq \int f_n. \tag{1}$$

Meanwhile $\langle g_n \rangle$ is a increasing sequence converging to $\underline{\lim} f_n$. Hence, by the Monotone Convergence Theorem and (1)

$$\int \underline{\lim} f_n = \int \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int g_n \leq \underline{\lim} \int f_n.$$

□

9.

Proof. By Fatou's Lemma,

$$\int_E f \leq \underline{\lim} \int_E f_n. \tag{2}$$

Similarly, $\int_{\bar{E}} f \leq \underline{\lim} \int_{\bar{E}} f_n$ and therefore

$$\int_E f_n = \int f_n - \int_{\bar{E}} f_n \quad \Rightarrow \quad \overline{\lim} \int_E f_n \leq \int f - \int_{\bar{E}} f = \int_{\bar{E}} f.$$

(2) and the inequality above together implies $\int_E f_n \to \int f$. □

4.4 The General Lebesgue Integral

12.

Proof. Note that $\langle g + f_n \rangle$ is a sequence of nonnegative measurable functions. Hence by Problem 8,

$$\int_E \underline{\lim}(g + f_n) \le \underline{\lim} \int_E (g + f_n) \quad \Rightarrow \quad \int_E \underline{\lim}\, f_n \le \underline{\lim} \int_E f_n.$$

The second inequality follows immediately from the definition of lower and upper limit. Replacing $g + f_n$ with $g - f_n$ and arguing in a similar manner gives the last inequality. □

13.

Proof. $f_n \ge -h$ implies $f_n + h \ge 0$. Hence, $\int (f_n + h)$ always has a meaning. And since g is integrable, $\int f_n = \int (f_n + h) - \int h$ also has a meaning. Similarly, $\int f$ has a meaning. Meanwhile,

$$\int f = \int (f + h) - \int h \le \underline{\lim} \int (f_n + h) - \int h = \underline{\lim} \int f_n.$$

□

15.

Proof.

(a) By Problem 4, for every $\varepsilon > 0$, there exists some simple functions $\varphi_1 \le f^+$ and $\varphi_2 \le f^-$ such that

$$\int_E f^+ - \int_E \varphi_1 < \varepsilon \quad \text{and} \quad \int_E f^- - \int_E \varphi_2 < \varepsilon.$$

Let $\varphi = \varphi_1 - \varphi_2$, which is also a simple function. Meanwhile,

$$\int_E |f - \varphi| \le \int_E (f^+ - \varphi_1) + \int_E (f^- - \varphi_2) < 2\varepsilon.$$

□

16.

Proof. For every integrable f, by Problem 15, there exists some step function $\psi = \sum_{k=1}^N c_k \chi_{E_k}$ such that $\int |f - \psi| < \varepsilon$. Note that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \psi(x) \cos nx \mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{N} c_k \int_{E_k} \cos nx \mathrm{d}x = 0. \tag{3}$$

Hence,

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} f(x) \cos nx \mathrm{d}x \right| &= \left| \int_{-\infty}^{\infty} (f(x) - \psi(x) + \psi(x)) \cos nx \mathrm{d}x \right| \\
&\le \int_{-\infty}^{\infty} |f(x) - \psi(x)| |\cos nx| \mathrm{d}x + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx \mathrm{d}x \right| \\
&\le \varepsilon + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx \mathrm{d}x \right| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

□

18.

Proof. Let $\langle t_n \rangle$ be any sequence with $t_n \neq 0$ and tending to 0. Then $\langle f(x, t_n) \rangle$ is sequence of functions satisfying the hypotheses of Lebesgue Convergence Theorem. Meanwhile, $f(x, t_n) \to f$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \int f(x, t_n)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

Since the choice of $\langle t_n \rangle$ is arbitrary, by Problem 2.49f,

$$\lim_{t\to 0} \int f(x, t)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

If f is continuous in t for each x, then $\lim_{\Delta t \to 0} f(x, t + \Delta t) = f(x, t)$ holds for every t. Therefore, replacing t with Δt in the previous result yields

$$\lim_{\Delta t\to 0} \int f(x, t + \Delta t)\mathrm{d}x = \int f(x, t)\mathrm{d}x.$$

Namely, $h(t)$ is continuous. □

5 Differentiation and Integration

5.1 Differentiation of Monotone Functions

3. ”maximum” needs to be changed to ”minimum” in both (a) and (b).

Proof.
(a) We may assume without loss of generality that $c = 0$. Since f attains a local minimum at $x = 0$, $f(h) \geq f(0)$ for every h sufficiently small. Hence, for every small $h > 0$, $(f(c+h) - f(c))/h > 0$ and therefore $D_+f(c) \geq 0$. Meanwhile, by Problem 2.b,

$$-D_-f(0) = D^+f(0) \geq 0 \quad \Rightarrow \quad D_-f(0) \leq 0.$$

The other two inequalities follow immediately from the definitions of upper and lower limits.

(b) If f has a local minimum at a or b, then we only have the right or left half of the inequalities. □

4.

Proof. We first show this for g with $D^+g \geq \varepsilon > 0$. For every $a \leq x < y \leq b$, as g is continuous on $[a, b]$, g has a maximum in $[a, b]$ and by Problem 2 and 3, g can not attain the maximum in $[a, b)$. Namely, the restrict of f to $[x, y]$ attains the maximum at y. Hence, $g(x) \leq g(y)$.

For every f with nonnegative D^+, let $g(x) = f(x) + \varepsilon x$ where $\varepsilon > 0$. Then $D^+g \geq \varepsilon > 0$. Hence g is nondecreasing. Therefore, for every $a \leq x < y \leq b$,

$$g(x) \leq g(y) \quad \Rightarrow \quad f(x) + \varepsilon x \leq f(y) + \varepsilon y.$$

Since the choice of ε is arbitrary, this implies $f(x) \leq f(y)$. □

5.a

Proof.

$$\begin{aligned}
\sup_{t\in(0,h)} \frac{(f+g)(x+t) - (f+g)(x)}{t} &= \sup_{t\in(0,h)} \left(\frac{f(x+t) - f(x)}{t} + \frac{g(x+t) - g(x)}{t} \right) \\
&\leq \sup_{t\in(0,h)} \frac{f(x+t) - f(x)}{t} + \sup_{t\in(0,h)} \frac{g(x+t) - g(x)}{t}.
\end{aligned}$$

Letting $h \to 0$ yields $D^+(f + g) \leq D^+f + D^+g$. □