Solutions to *Analysis on Manifolds*

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Contents

2 Differentiation

2.6 Continuously Differentiable Functions

3.

Proof. Since we may choose ε sufficiently closed so that the cube C is contained in a neighbourhood of a, we only need the existence of the partials near a. Furthermore, in the last equation at page 51, we only use the continuity of the partials at a, so we may require only the continuity at a. $\square$

4.

Proof. Suppose that the partials are bounded by M in the open of radius r at a. For every $h \in \mathbb{R}^m$ with $0 < |h| < \varepsilon$, let p_i $(i = 0, \ldots, m)$ have the same meaning as in the proof of Theorem 6.2. Then,

$$|f(a+h) - f(a)| \leq \sum_{j=1}^{m} |f(p_j) - f(p_{j-1})| \leq \sum_{j=1}^{m} Mh_i \leq mM\varepsilon.$$

Thus, f is continuous at a. $\square$

5.(a)

Solution.

$$Df = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}, \quad \det Df = r.$$

$\square$

2.7 The Chain Rule

3.

Solution.

(a) Define $h : \mathbb{R}^2 \to \mathbb{R}^3$ by $(x, y) \mapsto (x, y, g(x, y))$. Then $F = f \circ h$. First we calculate Dh:

$$Dh = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ D_1 g & D_2 g \end{bmatrix}.$$

Then, by the chain rule,

$$DF(x, y) = Df(h(x, y))Dh(x, y) = Df(x, y, g(x, y)) \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ D_1 g & D_2 g \end{bmatrix}. \tag{1}$$

(b) Since F is a constant function, $DF(x, y) = 0$ for all x and y. Hence, from (1) we conclude

$$D_1 g(x, y) = -\frac{y}{D_3 f(g(x, y))}, \quad D_2 g(x, y) = -\frac{x}{D_3 f(g(x, y))}.$$

$\square$