Solutions to
Introductory Functional Analysis with Applications

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4. cf. Prob. 13, Sec 1.2

Proof. The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1|\|x_1 - x_2\| + |\alpha_1 - \alpha_2|\|x_2\|.$$

□

7.

Proof. Let Y and y_n be defined as in the hint. Then $\|y_n\| = 1/n^2$, constituting a convergent number series. However,

$$\sum_{n=1}^{N} y_n = (1, 1/4, \dots, 1/N^2, 0, \dots),$$

which is divergent as $N \to \infty$. □

8.

Proof. Let (x_n) be a Cauchy sequence in X. Hence, for every $n > 0$, there exists some $K_n > 0$ such that for all $p, q > K_n$, $\|x_p - x_q\| < 1/n^2$. Without loss of generality, we may assume that (K_n) is increasing. Since the series $\|x_{K_{n+1}} - x_{K_n}\|$ is bounded by $1/n^2$, it converges. By the hypothesis, the series $(x_{K_{n+1}} - x_{K_n})$ also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1} (x_{K_{i+1}} - x_{K_i}) \to x \quad \text{as } n \to \infty.$$

Now we show that (x_n) converges to x. For every $\varepsilon > 0$, since (x_n) is a Cauchy sequence, there exists some N_1 such that for all $p, q > N_1$, $\|x_p - x_q\| < \varepsilon$. Meanwhile, since $x_{K_n} \to x$, once K_n is large enough, $\|x - x_{K_n}\| < \varepsilon$. Let $K_n > N_1$. Then for every $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus, X is complete. □

9.

Proof. Let (x_n) be an absolutely convergent series in Banach space X. Let $s_n = \sum_{i=1}^{n} x_n$. Now we show that s_n is a Cauchy sequence and therefore convergent. Since $\sum_{i=1}^{\infty} \|x_i\| < \infty$, for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n > N$, $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$. Hence, for every $N < p \leq q$,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^{q} x_i \right\| \leq \sum_{i=p+1}^{q} \|x_i\| < \varepsilon,$$

completing the proof. □

10.

Proof. Let (e_n) be Schauder basis of X. Denote the underlying field of X by $\mathbb{K}$ and let $\mathbb{W} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$ if $\mathbb{K} = \mathbb{C}$. Now we show that

$$S = \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \dots \right\},$$

a countable subset of X, is dense in X to derive the separability.

For every $x \in X$ and $\varepsilon > 0$, by the definition of Schauder basis, there exists $\beta_1, \dots, \beta_n \in \mathbb{K}$ such that $\|x - (\beta_1 e_1 + \cdots + \beta_n e_n)\| < \varepsilon$. Let $M = \max_i \|e_i\|$. If $M = 0$, then there is nothing to prove. Otherwise, since $\mathbb{W}$ is dense in $\mathbb{K}$, for $i = 1, \dots, n$, there exists $\alpha_i \in \mathbb{W}$ with $|\alpha_i - \beta_i| < \varepsilon/2^i M$. Hence,

$$\begin{aligned} \left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| \leq& \left\| x - \sum_{i=1}^{n} \beta_i e_i \right\| + \left\| \sum_{i=1}^{n} (\beta_i - \alpha_i) e_i \right\| \\ \leq& \varepsilon + \sum_{i=1}^{n} |\alpha_i - \beta_i| \|e_i\| \\ \leq& 2\varepsilon. \end{aligned}$$

Thus, S is dense in X and therefore X is separable. □

14.

Proof. Clear that $\|\cdot\|_0$ is nonnegative. And $\|\alpha\hat{x}\|_0 = \inf_{x\in\hat{x}} \|\alpha x\| = |\alpha|\|\hat{x}\|_0$. Meanwhile, $\|\hat{x} + \hat{y}\|_0 = \inf_{z\in\hat{x}+\hat{y}} \|z\| \leq \inf_{z\in\hat{x}} \|z\| + \inf_{z\in\hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$. Finally, we show that $\|\hat{x}\|_0 = 0$ implies $\hat{x} = Y$ and invoke Prob. 4, Sec 2.2 to complete the proof. Since $\|\hat{x}\|_0 = 0$, there exists $(x_n) \subset \hat{x}$ which converges to 0. Since Y is closed, Y is complete and so is its cosets. Therefore, $0 \in \hat{x}$, enforcing $\hat{x}$ to be Y. □

2.4 Finite Dimensional Normed Spaces

3.

Proof. The reflexive property clearly holds. If there are positive a and b such that $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$ for all $x \in X$, then $\|x\|_1/b \leq \|x\|_0 \leq \|x\|/a$. Hence the relation is symmetric. Next we further suppose there exists positive c and d such that that $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$. Then $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$, giving the transitive property. Thus, the axioms of an equivalence relation hold. □

4.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Let $E \subset X$ be any open set with respect to $\|\cdot\|$, i.e., for every $x_0 \in E$, there exists some $\delta > 0$ such that $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$. Since $\|\cdot\| \sim \|\cdot\|_0$, there exists some positive c such that $\|x - x_0\| \leq c\|x - x_0\|_0$. Hence, $B = \{x \in X : \|x - x_0\| < \delta/c\} \subset A \subset E$. Namely, E is also open with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. □

5.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Then for every $x \in X$, there exists some $c > 0$ such that $\|x\|_0 \leq c\|x\|$. Let (x_n) be a Cauchy sequence with respect to $\|\cdot\|$, i.e., for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n, m > N$, $\|x_n - x_m\| < \varepsilon/c$. Hence, $\|x_n - x_m\|_0 < c\|x_n - x_m\| \leq \varepsilon$. Thus, (x_n) is also a Cauchy with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. □

2.5 Compactness and Finite Dimension

5.

Proof. Clear that every point in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a closed bounded, and therefore compact, neighborhood. Hence, $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact. □

6.

Proof. Let X be a compact metric space and x any point in X. Let E be a closed neighborhood of x. By Prob 10, E is compact. Thus, X is locally compact. □

7.

Proof. It suffices to show that $a = \inf_{y \in Y} \|v - y\|$ can actually be obtained. Let $\{b_1, \ldots, b_n\}$ be a basis of Y and $y_k = y_{k,1}b_1 + \cdots + y_{k,n}b_n$ a sequence in Y with $\|v - y_k\| \to a$. We may assume without loss of generality that $\|v - y_k\|$ is bounded.

Since Y is a proper subset of Z, $v, b_1, \ldots, b_n$ are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar $c > 0$ such that for every k,

$$\|v - y_{k,1}b_1 - \cdots - y_{k,n}b_n\| \geq c(1 + |y_{k,1}| + \cdots + |y_{k,n}|).$$

Hence, the sequence $(y_{k,1}, \ldots, y_{k,n})$ of n-tuples is bounded and therefore has a convergent subsequence. Consequently, (y_k) also has a convergent subsequence. Suppose that it converges to $z \in Z$. Note that $\|v - z\| = a$ and as Y is closed, $z \in Y$. Thus, a can be attained in Y. □

8.

Proof. Since the unit ball B with respect to $\|\cdot\|_2$ in $\mathbb{R}^n$ and $\mathbb{C}^n$ is compact and $\|\cdot\|$ is continuous, by 2.5-7, $x \mapsto \|x\|$ can attains its minimum, denoted by a, on B. Due to the positive definite property of a norm, a is positive. Hence, $0 < a \leq \|x/\|x\|_2\|$. Namely, $a\|x\|_2 \leq \|x\|$. □

9.

Proof. For every $(x_n) \subset M \subset X$, since X is compact, there exists a subsequence (x_{n_k}) of (x_n) which converges to some $y \in X$. Since M is closed, $y \in M$. Hence, M is compact. □

10.

Proof. From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of T is also continuous, it suffices to show that the image of a closed set $A \subset X$ under T is again a closed set. Since X is compact and A is closed, A is compact. Since T is continuous, by 2.5-6, $T(A)$ is compact and therefore closed. Hence, T is a homeomorphism. □

2.7 Bounded and Continuous Linear Operators

2.

Proof. First suppose T to be bounded and let A be any bounded set in X. Then there exists $K < \infty$ such that for all $x \in A$, $\|x\| < K$. Due to the boundedness of T, $\|Tx\| \leq \|T\|\|x\| < K\|T\|$. Namely, $T(A)$ is also bounded.

Now suppose that T maps bounded sets in X into bounded sets in Y. Clear that the unit ball B of X is bounded and therefore so is $T(B)$. Namely, $\|Tx/\|x\|\|$ is bounded for $x \neq 0$.[1] Hence, T is bounded. □

3.

Proof. For every x with $\|x\| < 1$, $\|Tx\| \leq \|T\|\|x\| < \|T\|$. □

4.

Proof. Suppose that the linear operator T is continuous at $x_0 \in \mathcal{D}(T)$. For every $(x_n) \subset \mathcal{D}(T)$ with $\|x_n - x\| \to 0$, by the continuity of T at x_0

$$\|Tx_n - Tx\| = \|T(x_n - x + x_0) - Tx_0\| \to 0.$$

Hence, T is continuous. □

7.

Proof. The inequality implies $\mathcal{N}(T) = 0$. Hence, by Theorem 2.6-10, T^{-1} exists. For every $y \in Y$, suppose that $y = Tx$. Then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

Thus, T^{-1} is bounded. □

12.

Proof. The compatibility follows immediately from the definition of the supremum. Suppose $\|x\|_1 = \max_j |\xi_j|$ and $\|y\|_2 = \max_j \|\eta_j\|$, then

$$Ax = \begin{bmatrix} x_1\alpha_{11} + \cdots + x_n\alpha_{1n} \\ \vdots \\ x_1\alpha_{r1} + \cdots + x_n\alpha_{rn}. \end{bmatrix}$$

[1]Note that the two $\|\cdot\|$ here are different norms.

Since for all j, $x_j \le \|x_j\|_1$,

$$\frac{\max_j |x_1\alpha_{j1} + \cdots + x_n\alpha_{jn}|}{\|x\|_1} = \max_j \left| \frac{x_1}{\|x\|_1}\alpha_{j1} + \cdots + \frac{x_n}{\|x\|_1}\alpha_{jn} \right| \le \max_j \sum_{k=1}^{n} |\alpha_{jk}|.$$

Hence,

$$\|A\| \ge \frac{\|Ax\|_2}{\|x\|_1} \quad \text{for all } x. \tag{1}$$

Suppose that maximum of $\sum_{k=1}^n |\alpha_{jk}|$ is obtained at $j = p$. Then choosing x_k to be $\operatorname{sgn} \alpha_{pk}$ shows that the equality in (1) can actually be attained. Hence, $\|A\| = \max_j \sum_{k=1}^n |\alpha_{jk}|$. □

2.8 Linear Functionals

8.

Proof. For every $x_1, x_2 \in N(M^*)$, $a, b \in \mathbb{K}$ and $f \in M^*$,

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) = 0.$$

Hence, $ax_1 + bx_2 \in N(M^*)$. Namely, $N(M^*)$ is a vector space. □

9.

Proof. First we show the uniqueness. Suppose that $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$. Then $0 = (\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)$. Hence,

$$0 = f((\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)) = (\alpha_1 - \alpha_2)f(x_0) + f(y_1) - f(y_2).$$

Since $y_1, y_2 \in \mathcal{N}(f)$, $f(y_1) - f(y_2) = 0$ while $f(x_0) \ne 0$ as $x_0 \notin \mathcal{N}(f)$. Hence, $\alpha_1 = \alpha_2$, which forces y_1 and y_2 to coincide.

For the existence, it suffices to show that for any fixed x, the function $g(\alpha) = f(x - \alpha x_0)$ has a zero. It is easy to verify that $\alpha = f(x)/f(x_0)$ is a zero of g. Note that $x_0 \notin \mathcal{N}(f)$ and therefore $f(x_0) \ne 0$. □

10.

Proof. First we suppose that $x_1, x_2 \in x_0 + \mathcal{N}(f) \in X/\{$. Then together with Prob. 9, $x_i = x_0 + y_i$ where $y_i \in \mathcal{N}(f)$. Hence, for $i = 1, 2$, $f(x_i) = f(x_0) + f(y_i) = f(x_0)$.

For the converse, note that $f(x_1) = f(x_2)$ implies $f(x_1 - x_2) = 0$. Namely, $x_1 - x_2 \in \mathcal{N}(f)$. Hence, x_1, x_2 belongs to the same element in $X/\mathcal{N}(f)$.

To show $\operatorname{codim} \mathcal{N}(f) = 1$, we show that $X/\mathcal{N}(f)$ and $\mathbb{K}$ are isomorphic. For every $\hat{x} \in X/\mathcal{N}(f)$, define $I(\hat{x}) = f(x)$. By the previous discussion, this definition is well-defined. Clear that I is linear and therefore is injective. And by the linearity of f, I is surjective. Thus, I is an isomorphism between $X/\mathcal{N}(f)$ and $\mathbb{K}$. Hence, $\operatorname{codim} \mathcal{N}(f) = 1$. □

11.

Proof. Put $N = \mathcal{N}(f_1) = \mathcal{N}(f_2)$ and choose $x_0 \in X \setminus N$. By Prob. 9, for every $x \notin N$, $x = \alpha x_0 + y$ where $y \in N$ and $\alpha \ne 0$. Hence,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha f_1(x_0) + f_1(y)}{\alpha f_2(x_0) + f_2(y)} = \frac{f_1(x_0)}{f_2(x_0)}.$$

□

12.

Proof. Prob. 10, justifies the discussion on hyperplanes parallel to the $\mathcal{N}(f)$. It suffices to show that $H_1 = b + \mathcal{N}(f)$ for some $b \in X$. Choose $x_1 \in H_1$. Then

$$x \in \mathcal{N}(f) \;\Leftrightarrow\; x + x_1 \in x_1 + \mathcal{N}(f) \;\Leftrightarrow\; f(x + x_1) = f(x) + f(x_1) = 1 \;\Leftrightarrow\; x + x_1 \in H_1.$$

Hence, $H_1 = x_1 + \mathcal{N}(f)$. Namely, H_1 is a hyperplane parallel to $\mathcal{N}(f)$. □

13.

Proof. We argue by contradiction. Assume that there exists a $y_1 \in Y$ such that $f(y_1) \neq c \neq 0$. Then for every $d \in \mathbb{K}$, by the linearity of f, $f(dy_1/c) = d$. Contradiction. Hence, $f = 0$ on Y. □

14.

Proof. For every $\varepsilon > 0$, there exists $x_1 \in X$ with $f(x_1) = 1$ such that $\tilde{d} + \varepsilon \geq \|x_1\|$. Hence,

$$\|f\|(\tilde{d} + \varepsilon) \geq \|f\|\|x_1\| \geq |f(x_1)| = 1.$$

Since the choice of $\varepsilon > 0$ is arbitrary, $\|f\|\tilde{d} \geq 1$. Meanwhile, there exists $x_2 \in X$ with $\|x_2\| = 1$ such that $|f(x_2)| \geq \|f\| - \varepsilon$. Put $x_3 = x_2/f(x_2)$. Then $f(x_3) = 1$. Hence,

$$(\|f\| - \varepsilon)\tilde{d} \leq |f(x_2)|\|x_3\| = \|x_2\| = 1,$$

which implies $\|f\|\tilde{d} \leq 1$. Thus, $\|f\|\tilde{d} = 1$. □

15.

Proof. For every x with $\|x\| \leq 1$, $f(x) \leq \|f\|\|x\| \leq c$. Hence, $x \in X_{c_1}$. Meanwhile, for every $\varepsilon > 0$, by the definition of the supremum, there exists a x with $\|x\| = 1$ such that $|f(x)| > \|f\| - \varepsilon$. By the linearity of f, we may remove the $|\cdot|$ on the right side. Hence, $f(x) \notin X_{c_1}$ where $c = \|f\| - \varepsilon$. □

2.9 Operators on Finite Dimensional Spaces

8.

Proof. Let $\{b_2, \ldots, b_n\}$ be a basis of Z and $\{b_1, \ldots, b_n\}$ a basis of X. Define $f \in X^*$ to be $f(b_i) = \delta_{1i}$. Clear that $\mathcal{N}(f) = Z$. By Prob. 11, Sec 2.8, f is uniquely determined up to a scalar multiple. □

12.

Proof. Let $\varphi : X \to \mathbb{K}^p$ be defined by $x \mapsto [f_1(x), \ldots, f_p(x)]^T$. It can be verified that φ is a linear operator. Since $\dim X = n > p$, φ can not be injective. Hence, there exists $0 \neq x \in X$ such that $\varphi(x) = 0$. □

13.

Proof. Let $\{b_1, \ldots, b_m\}$ be a basis of Z and $\{b_1, \ldots, b_n\}$ a basis of X. Define $\tilde{f} \in X^*$ to be identical with f on $b_1, \ldots, b_m$ and 0 on $b_{m+1}, \ldots, b_n$. Clear that $\tilde{f}|_Z = f$. □

2.10 Normed Spaces of Operators. Dual Space

8.

Proof. First we construct a linear bijection T between c_0' and l^1. A Schauder basis for c_0 is (e_k), where $e_k = (\delta_{kj})$. Then for every $f \in c_0'$, define $Tf = (\gamma_k) = (f(e_k))$. Clear that T is linear. Now we show that $Tf = (\gamma_k) \in l^1$, that is, $\sum_{k=1}^{n} |\gamma_k|$ is bounded and therefore convergent. Define $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \begin{cases} \operatorname{sgn} \gamma_k, & k \leq n, \\ 0, & k > n. \end{cases}$$

Clear that $x_n \in c_0$. By the linearity and boundedness of f,

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|. \tag{2}$$

Since f is bounded, $|f(x_n)| \leq \|f\| \|x_n\| \leq \|f\|$. Hence, $\sum \|\gamma_k\|$ is bounded. Thus, $Tf \in l^1$.

Meanwhile, for every $y = (\beta_k) \in l^1$, define $Sy = g$ to be the functional $g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$ for $x = (\xi_k)$. On c_0, the summation does converge and clear that g is linear and bounded. Hence, $g \in c_0'$. It can be verify that $ST = TS = I$ and T is linear. Thus, c_0' and l^1 is isomorphic.

Now we show that T constructed preserve the norm to complete the proof. For $x \in c_0$ with $\|x\| = 1$,

$$|f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \gamma_k \right| \leq \sum_{k=1}^{\infty} |\gamma_k| = \|Tf\|.$$

Hence, $\|f\| \leq \|Tf\|$. And (2) implies $\sum_{k=1}^{n} \|\gamma| \leq \|f\|$. Letting $n \to \infty$ yields $\|Tf\| \leq \|f\|$. Thus, $\|Tf\| = \|f\|$. $\square$

9.

Proof. Let (b_k) be a Hamel basis of X and suppose that $f, g \in X^*$ coincide on every b_k. Then for every $x = \sum_{k=1}^{\infty} \xi_k b_k \in X$,

$$f(x) - g(x) = \sum_{k=1}^{n} \xi_k (f(b_k) - g(b_k)) = 0.$$

Thus, $f = g$. Namely, f is uniquely determined. $\square$

10.

Proof. Let (b_k) be a Hamel basis of X and without loss of generality we may assume $\|b_k\| = 1$. Justified by Prob. 9, we can define $T \in X^*$ with $Tb_k = k$, which is clearly unbounded. $\square$

11.

Proof. It follows immediately from Prob. 10. $\square$

13.

Proof. For any $f, g \in M^a$ and scalar a, b, $(af + bg)(x) = af(x) + bg(x) = 0$ for every $x \in M$. Hence, M^a is a vector space. For $(f_n) \subset M^a \subset X'$, suppose that $f_n \to f \in M^*$. Since M' is complete, it is closed and therefore $f \in M'$. For every $0 \neq x \in M$, since $f_n \to f$,

$$\frac{|f_n(x) - f(x)|}{\|x\|} \to 0, \text{ as } n \to \infty.$$

Hence, $f(x) = 0$. Thus, M^a is closed.

$X^a = \{0\}$ and $\{0\}^a = X'$. □

14.

Proof. Let $\{b_1, \ldots, b_m\}$ be a basis of M and $\{b_1, \ldots, b_n\}$ a basis of X. And let $\{\beta_1, \ldots, \beta_n\}$ be the dual basis. Clear that $b_1, \ldots, b_m \notin M^a$ whereas $b_{m+1}, \ldots, b_n$ does. Together with Prob. 13, this implies $M^a = \operatorname{span}(b_{m+1}, \ldots, b_n)$. Thus, $\dim M^a = n - m$. □

3 Inner Product Spaces. Hilbert Spaces

3.1 Inner Product Spaces. Hilbert Spaces

2.

Proof.

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle = \|x\|^2 + \|y\|^2,$$

where the last equality comes from the hypothesis of orthogonality. Now we show that for mutually orthogonal $x_1, \ldots, x_m$

$$\left\|\sum_{i=1}^m x_i\right\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

by induction on m. The case where $m = 2$ has already been showed and we assume that the equation holds for $m-1$. Since x_m is orthogonal with each $i = 1, \ldots, m-1$, x_m is orthogonal to $x_1 + \cdots + x_{m-1}$. Hence,

$$\left\|\sum_{i=1}^m x_i\right\|^2 = \left\|\sum_{i=1}^{m-1} x_i\right\|^2 + \|x_m\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

completing the proof. □

3.

Proof. The equation implies $\langle x,y\rangle + \langle y,x\rangle = 0$. The symmetric property of real inner products implies $\langle x,y\rangle = 0$. Let $X = \mathbb{C}$ and $x = 1, y = i$. It is easy to verify that $\|x+y\|^2 = \|x\|^2 + \|y\|^2 = 2$ but x and y are not orthogonal. □

7.

Proof. It suffices to show that the zero vector is the only vector orthogonal to all vectors. Suppose that $\langle x_0, x\rangle = 0$ for all $x \in X$, then $\|x_0\|^2 = \langle x_0, x_0\rangle = 0$. By the definiteness of the inner product, $x_0 = 0$. □

8. We show that any norm satisfying the parallelogram equality can be derived form an inner product.

Proof. The proof of (IP3) is trivial and (IP4) follows immediately from the positive-definiteness of the norm. Hence we only show the linearity in the first factor here. For every $u, v, y \in X$, from the parallelogram equality we can derive, after some computation, that

$$\begin{aligned} 4\langle u+v, y\rangle &= \|u+v+y\|^2 - \|u+v-y\|^2 \\ &= \|u+y\|^2 - \|u-y\|^2 + \|v+y\|^2 - \|v-y\|^2 \\ &= 4\langle u,y\rangle + 4\langle v,y\rangle. \end{aligned}$$

Namely, (IP1) holds. By induction we can show that $\langle nu, y\rangle = n\langle u, y\rangle$ for $n = 1, 2, \ldots$. And since $\langle -u, y\rangle = \langle 0 - u, y\rangle = \langle 0, y\rangle - \langle u, y\rangle = \langle u, y\rangle$,

$$\langle nu, y\rangle = n\langle u, y\rangle, \quad \text{for } n \in \mathbb{Z}.$$

Furthermore, for any positive integer m,

$$m\left\langle \frac{n}{m}u, y\right\rangle = mn\left\langle \frac{1}{m}u, y\right\rangle = n\langle u, y\rangle.$$

Dividing the both sides by m yields

$$\langle qu, y\rangle = q\langle u, y\rangle, \quad \text{for } q \in \mathbb{Q}.$$

For every $\alpha \in \mathbb{R}$, let $(q_n) \subset \mathbb{Q}$ converges to α. Now we show that $f(t) = \langle tu, y\rangle$ is continuous at $t = 0$ and by the additivity we may conclude that f is continuous on $\mathbb{R}$. Since

$$\begin{aligned} 4|f(t)| &= |\|tu + y\|^2 - \|tu - y\|^2| \\ &= (\|tu + y\| + \|tu - y\|)|\|tu + y\| - \|tu - y\|| \\ &\leq 4t\|u\|(t\|u\| + \|y\|) \to 0 \end{aligned}$$

as $t \to 0$, $f(t)$ is continuous. For every $\alpha \in \mathbb{R}$, let $(q_n) \subset \mathbb{Q}$ be a convergent sequence with limit α. Then

$$\langle \alpha u, y\rangle = \lim\langle q_n u, y\rangle = \lim q_n\langle u, y\rangle = \alpha\langle u, y\rangle.$$

Hence, $\langle\cdot,\cdot\rangle$ is linear in the first factor. Thus, it is an inner product. Meanwhile, it is easy to verify that the norm it introduces is exactly the original norm. $\square$

3.2 Further Properties of Inner Product Spaces

7.

Proof. First we note that

$$f(\alpha) = \|x + \alpha y\|^2 - \|x - \alpha y\|^2 = 2\bar{\alpha}\langle x, y\rangle + 2\alpha\langle y, x\rangle.$$

Clear that $x \perp y$ implies $f(\alpha) = 0$ for all scalar α. For the converse, we suppose $f(\alpha) = 0$ and put $\alpha = \langle x, y\rangle$. Then $0 = f(\alpha) = 2|\langle x, y\rangle|$. Thus, $x \perp y$. $\square$

8.

Proof. Clear that $x \perp y$ implies $\|x + \alpha y\| \geq \|x\|$. Therefore we only show the converse here. Without loss of generality, we assume $\|y\| = 1$. Then $\|x + \alpha y\| \geq \|x\|$ for all scalar α implies

$$|\alpha|^2 + \bar{\alpha}\langle x, y\rangle + \alpha\overline{\langle x, y\rangle} \geq 0.$$

Put $\alpha = -\langle x, y\rangle$ and we get

$$0 \leq |\langle x, y\rangle|^2 - 2|\langle x, y\rangle|^2 = -|\langle x, y\rangle|^2,$$

which implies $\langle x, y\rangle = 0$. Namely, $x \perp y$. $\square$

9.

Proof. For every $\varepsilon > 0$, put $\delta = \varepsilon/\sqrt{b-a}$. Then for every $x_1, x_2 \in V$ with $\|x_1 - x_2\|_\infty < \delta$,

$$\|x_1 - x_2\|_2^2 = \int_a^b |x_1(t) - x_2(t)|^2 \mathrm{d}t \le (b-a)\delta^2 = \varepsilon^2.$$

Hence, $x \mapsto x$ is continuous. $\square$

10.

Proof. For every $u, w \in X$,

$$\begin{aligned}\langle Tu, w\rangle =& \frac{1}{4}(\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle) \\ &+ \frac{i}{4}(\langle T(u+iw), u+iw\rangle - \langle T(u-iw), u-iw\rangle).\end{aligned}$$

Note that each component of the right hand side is of form $\langle Tx, x\rangle$ and hence equals to 0. Putting $w = Tu$ yields $\|Tu\|^2 = 0$ for all $u \in X$. Thus, $T = 0$. $\square$

3.3 Orthogonal Complements and Direct Sums

7.

Proof.

(a) $x \in A^{\perp\perp}$ iff for all $y \in A^\perp$, $\langle x, y\rangle = 0$. By the definition of $A^\perp$, the identity holds if $x \in A$. Hence, $A \subset A^{\perp\perp}$.

(b) For all $x \in B^\perp$ and $y \in A \subset B$, $\langle x, y\rangle = 0$ by definition. Hence, $x \in A^\perp$. Namely, $B^\perp \subset A^\perp$.

(c) We show that $A^\perp$ is closed (no matter whether A is or not) and invoke Lemma 3.3-6 to complete the proof. Suppose that $(x_n) \subset A^\perp$ converges to x. For all $y \in A$, $\langle x_n, y\rangle = 0$. By the continuity of the inner product, $\langle x, y\rangle = 0$ and therefore $x \in A^\perp$. Hence, $A^\perp$ is closed. Thus, $A^\perp = A^{\perp\perp\perp}$. $\square$

8.

Proof. We have show this in Prob. 7. $\square$

9.

Proof. It has been shown in Lemma 3.3-6 that the closedness of Y implies $Y = Y^{\perp\perp}$. Hence we only show the converse here. For every convergent $(x_n) \subset Y$, $(x_n) \subset Y^{\perp\perp}$. Since $Y^{\perp\perp}$ is closed by Prob. 8, the limit x of (x_n) belongs to $Y^{\perp\perp}$ and hence belongs to Y. Thus, Y is closed. $\square$

10. TODO

3.4 Orthonormal Sets and Sequences

3.

Proof. The situation where x and y are linearly dependent is obvious and hence we assume they are linearly independent here. By the homogeneity of the Schwarz inequality, we may assume without loss of generality that $\|x\| = \|y\| = 1$. Put $z = (y - x\langle y, x\rangle)/\|y - x\langle y, x\rangle\|$. Then $\{x, z\}$ is orthonormal and therefore by (12*)

$$|\langle y, x\rangle|^2 + |\langle y, z\rangle|^2 \leq \|y\|^2 = 1.$$

Since $|\langle y, z\rangle|^2$ is nonnegative, this implies $|\langle x, y\rangle|^2 \leq 1$, the Schwarz inequality. $\square$

7.

Proof. For each positive integer n, by the Schwarz inequality and (12*),

$$\sum_{k=1}^{n} |\langle x, e_k\rangle\langle y, e_k\rangle| \leq \sqrt{\sum_{i=1}^{n} |\langle x, e_k\rangle|^2} \sqrt{\sum_{i=1}^{n} |\langle y, e_k\rangle|^2} \leq \|x\|\|y\|.$$

Since all terms in the summation is nonnegative, this implies $\sum_{k=1}^{\infty} |\langle x, e_k\rangle\langle y, e_k\rangle| \leq \|x\|\|y\|$. $\square$

8.

Proof. It follows immediately from Bessel inequality. $\square$

3.5 Series Related to Orthonormal Sequences

1.

Proof. By Theorem 3.5-2, $\alpha_k = \langle x, e_k\rangle$. Meanwhile by the definition of the norm,

$$\|x\|^2 = \left\langle \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k, x \right\rangle = \sum_{k=1}^{\infty} \langle x, e_k\rangle\langle e_k, x\rangle = \sum_{k=1}^{\infty} |\alpha_k|^2,$$

where the second equality follows from the continuity of the inner product. $\square$

3.

Solution. Put $x \equiv 1$ on $[-\pi, \pi]$ and $e_k = \sin kt$. Since x is even but e_k is odd for every k, the series does not converges to x. $\square$

4.

Proof. By the triangle inequality, $\|x_m + \cdots + x_n\| \leq \|x_m\| + \cdots + \|x_n\|$ for every $n \geq m > 0$. Hence the convergence of $\sum \|x_k\|$ implies that s_n is a Cauchy sequence. $\square$

5.

Proof. By Prob. 4, $\sum_{k=1}^{n} x_k$ is a Cauchy sequence. And since H is complete, $\sum_{k=1}^{\infty} x_k$ converges. $\square$

7.

Proof. The existence of y follows from Theorem 3.5-2(c). And for each k,

$$\langle x-y, e_k\rangle = \langle x, e_k\rangle - \sum_{j=1}^{\infty}\langle x, e_j\rangle\langle e_k, e_j\rangle = \langle x, e_k\rangle - \langle x, e_k\rangle = 0,$$

where the second equality comes from the fact that (e_k) is orthonormal. □

8. TODO: Show the validation of the change of the order of summation. Or maybe we can show the equality directly.

Proof. We suppose that $x \in \bar{M}$ here since the proof of the other direction is obvious. Then there exists $(p_n) \subset M$ such that $x = \sum_{n=1}^{\infty} p_n$. For each n, suppose $p_n = \sum_{k=1}^{\infty}\langle p_n, e_k\rangle e_k$. This is valid because $p_n \in M$ and therefore is a finite linear combination of (e_k). In fact, there are only finitely many nonzero term in the summation. Then

$$x = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\langle p_n, e_k\rangle e_k = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\langle p_n, e_k\rangle\right) e_k.$$

□

9.

Proof. First we suppose $\bar{M}_1 = \bar{M}_2$. Then by Prob. 8, each e_n and $\tilde{e}_n$ can be represented by (a) and (b) respectively.

For the converse, (a) implies, again by Prob. 8, $e_n \in \bar{M}_2$ and therefore $M_1 \subset \bar{M}_2$. Since $\bar{M}_2$ is closed, $\bar{M}_1 \subset \bar{M}_2$. *Mutatis mutandis*, this also shows $\bar{M}_2 \subset \bar{M}_1$. Thus, $\bar{M}_1 = \bar{M}_2$. □

10.

Proof. Note that for every $m > 0$, there are only finite e_κ such that $\langle x, e_\kappa\rangle \geq 1/m$. Otherwise we may choose a countable subset of them, which will violate the result in Prob. 8, Sec 3.4. Hence, the collection of all nonzero Fourier coefficient

$$\bigcup_{m=1}^{\infty}\{e_\kappa : \langle x, e_\kappa\rangle \geq 1/m\}$$

is at most countable. □

3.6 Total Orthonormal Sets and Sequences

4.

Proof. Suppose that x and y satisfy (3). We only show the relation for real cases here. The complex cases can be proved in a similar way. Using (9), Sec 3.1 and (3),

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}\sum_k \left(|\langle x+y, e_k\rangle|^2 - |\langle x-y, e_k\rangle|^2\right).$$

Meanwhile,

$$|\langle x \pm y, e_k\rangle|^2 = \langle x \pm y, e_k\rangle\overline{\langle x \pm y, e_k\rangle} = |\langle x, e_k\rangle|^2 + |\langle y, e_k\rangle|^2 \pm 2\langle x, e_k\rangle\overline{\langle y, e_k\rangle}.$$

Hence, $\langle x, y\rangle = \sum_k \langle x, e_k\rangle\overline{\langle y, e_k\rangle}$. □

6.

Proof. Suppose $M = (e_k)$. We collect the e_k which does not belong to $\operatorname{span}(e_1, \ldots, e_{k=1})$ and denote the new sequence by $(\tilde{e}_k)$. Clear that $\operatorname{span}(e_k) = \operatorname{span}(\tilde{e}_k)$ and $(\tilde{e}_k)$ is linearly independent. Let (f_k) be the sequence generated from $(\tilde{e}_k)$ by the Gram-Schmidt process. Then clear that (f_k) is orthonormal. And since for every n, $\operatorname{span}(\tilde{e}_1, \ldots, \tilde{e}_n) = \operatorname{span}(f_1, \ldots, f_n)$, $M \subset \operatorname{span}(\tilde{e}_k) = \operatorname{span}(f_k)$. Finally, since M is dense in H, $\overline{\operatorname{span}(f_k)} = H$. Thus, (f_k) is a total orthonormal sequence of H. □

7.

Proof. It follows from the definition of the separable Hilbert space and Prob. 6 immediately. □

9.

Proof. $\langle v, x\rangle = \langle w, x\rangle$ implies $\langle v - w, x\rangle = 0$ for all $x \in M$, that is, $x \perp M$. Since M is total, by Theorem 3.6-2, $v - w = 0$. □

10.

Proof. It follows immediately from Theorem 3.6-2(b). □

3.8 Functionals on Hilbert Spaces

3.

Proof. The linearity follows from the sesquilinearity of the inner product and the boundedness from the Schwarz inequality. Furthermore, the Schwarz inequality also implies $\|f\| \leq \|z\|$. Meanwhile, $\|f\| \geq \|f(z/\|z\|)\| = \|z\|$. Thus, $\|f\| = \|z\|$. □

4.

Proof. Clear that the mapping $z \mapsto f$ is an isomorphism since it is surjective. And by Theorem 2.10-4, X' is a Hilbert space. Hence, X is also a Hilbert space. □

5.

Proof. Since l^2 is complete. By Theorem 3.8-1, we may define $I : (l^2)' \to l^2$ to be $f \mapsto z$. Clear that I is linear and injective. Meanwhile, it preserves the norm. Furthermore, by Prob. 3, it is surjective. Hence, I is an isomorphism. Thus, l^2 is isomorphic to its dual. □

12.

Proof. For every $x \in X$ and $y \in Y$,

$$\begin{aligned}|h(x + \Delta x, y + \Delta y) - h(x, y)| &= |h(\Delta x, y) + h(x, \Delta y) + h(\Delta x, \Delta y)| \\ &\leq |h(\Delta x, y)| + |h(x, \Delta y)| + |h(\Delta x, \Delta y)|.\end{aligned}$$

Since h is bounded,

$$|h(x + \Delta x, y + \Delta y) - h(x, y)| \leq \|h\|(\|\Delta x\|\|y\| + \|\Delta y\|\|x\| + \|\Delta x\|\|\Delta y\|).$$

Thus, h is continuous. □

14.

Proof. If $h(x,x)=0$, then for any $t\in\mathbb{R}$,

$$0\le h(th(y,x)x+y,th(y,x)x+y)=2t|h(x,y)|^2+h(y,y).$$

Hence, $h(x,y)=0$, otherwise we may choose some $t<0$ such that is right hand side is negative. Thus, the inequality holds if $h(x,x)=0$.

Now suppose $h(x,x)\neq 0$. Put

$$z=y-x\frac{h(y,x)}{h(x,x)} \tag{3}$$

It is easy to verify that $h(z,x)=0$. Multiplying z on the both sides of (3) yields

$$0\le h(z,z)=h\left(z,y-x\frac{h(y,x)}{h(x,x)}\right)=h(z,y)=h(y,y)-\frac{h(x,y)h(y,x)}{h(x,x)}.$$

Thus, $|h(x,y)|^2\le h(x,x)h(y,y)$. □

3.9 Hilbert-Adjoint Operator

1.

Proof. By Theorem 3.9-4, $0^*=(0+0)^*=0^*+0^*$. Hence, $0^*=0$. For every $x,y\in X$,

$$\langle (I^*-I)x,y\rangle=\langle I^*x,y\rangle-\langle Ix,y\rangle=\langle x,Iy\rangle-\langle Ix,y\rangle=0.$$

Hence, by Lemma 3.9-3, $I=I^*$. □

2.

Proof. By Theorem 3.9-4, $T^*(T^{-1})^*=(T^{-1}T)^*=I^*=I$. Hence, $(T^*)^{-1}=(T^{-1})^*$. □

3.

Proof. Since $\|T_n^*-T^*\|=\|(T_n-T)^*\|=\|T_n-T\|$, $T_n^*\to T^*$ as long as $T_n\to T$. □

4.

Proof. It suffices to show that for all $x_2\in T^*(M_2^\perp)$ and $x_1\in M_1$, $\langle x_1,x_2\rangle=0$. $x_2\in T^*(M_2^\perp)$ implies the existence of some $y_2\in M_2^\perp$ with $T^*y_2=x_2$. Then

$$\langle x_1,x_2\rangle=\langle x_1,T^*y_2\rangle=\langle Tx_1,y_2\rangle=0,$$

where the last equality comes from the fact that $T(M_1)\subset M_2$ and $y_2\in M_2^\perp$. Thus, $M_1^\perp\supset T^*(M_2^\perp)$. □

5.

Proof. By Prob. 4, $T^*(M_2^\perp)\subset M_1^\perp$ implies $M_2^{\perp\perp}\supset T(M_1^{\perp\perp})$. Since M_1 and M_2 are closed, by Prob. 9, Sec 3.3, $M_i^{\perp\perp}=M_i$ for $i=1,2$. Thus, $T(M_1)\subset M_2$. The converse part has already been proved in Prob. 4. □

6.

Proof.
(a) Since $T(M_1) = \{0\} \subset H_2$, by Prob. 4, $T^*(H_2) \subset M_1^\perp$.
(b) For every $y \in [T(H_1)]^\perp$, $\langle y, Tx \rangle = 0$ for all $x \in H_1$. Hence, $\langle T^*y, x \rangle = 0$. By Lemma 3.8-2, $T^*y = 0$ and therefore $y \in \mathcal{N}(T^*)$. Thus, $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$.
(c) Since $T^{**} = T$, it follows from (b) that $[T^*(H_2)]^\perp \subset M_1$. And since M_1 is closed, $M_1^{\perp\perp} = M_1$. Therefore, (a) implies $[T^*(H_2)]^\perp \supset M_1$. Thus, $M_1 = [T^*(H_2)]^\perp$. □

7.

Proof. It follows immediately from Lemma 3.9-3. □

8.

Proof. For every $x \in H$ with $\|x\| = 1$,

$$\begin{aligned}\|(I + T^*T)x\| = \|x + T^*Tx\| &= \langle x + T^*Tx, x + T^*Tx \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + \langle x, T^*Tx \rangle + \langle T^*Tx, x \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + \|Tx\|^2 \\ &\geq 1.\end{aligned}$$

Then, by Prob 7, Sec 2.7, $I + T^*T$ is invertible. □

9.

Proof. If T can be represent by that form, then $\mathcal{R}(T)$ can be spanned by $w_1, \dots, w_n$. Hence, it is finite dimensional.

Now we suppose that T has a finite dimensional range. Let $\{w_1, \dots, w_n\}$ be a orthonormal basis of $\mathcal{R}(T)$. Then for every $x \in H$,

$$Tx = \sum_{j=1}^n \varphi_j(x) w_j.$$

Now we show that for each j, φ_j is a bounded linear functional and invoke Riesz's Theorem to complete the proof. It is easy to verify the linearity of φ_j. For every x with norm 1, since T is bounded and (w_j) is orthonormal,

$$\|T\| \geq \left\| \sum_{j=1}^n \varphi_j(x) w_j \right\| \geq |\varphi_j(x)|$$

for each $j = 1, \dots, n$. Hence, every φ_j is a bounded linear functional and therefore can be represented by $\varphi_j(x) = \langle x, v_j \rangle$. □

Self-Adjoint, Unitary and Normal Operators

4. We only show the uniqueness here.

Proof. $T_1 + iT_2 = S_1 + iS_2$ implies $T_1 - iT_2 = S_1 - iS_2$. Sum these two equations and we get $T_1 = S_1$. Meanwhile, it also implies $i(T_1 + iT_2) = i(S_1 + iS_2)$. Summing these two gives $T_2 = S_2$. □

6.

Proof.
(a) We argue by contradiction. Let k be the smallest positive integer such that $T^{2k} = 0$. Then for every $x \in H$

$$0 = \langle T^{2k}x, x\rangle = \langle T^k x, (T^k)^* x\rangle = \langle T^k x, T^k x\rangle = \|T^k x\|^2.$$

Hence $T^k = 0$, which contradicts with the smallest assumption of k. Hence, $T^n \neq 0$ for all even positive integer n.

(b) If $T^n = 0$ for some positive, not necessarily even, integer n, then so is $T^{2n} = 0$. Hence, by (a), $T^n \neq 0$ for all $n \in \mathbb{N}$. □

9.

Proof. Since T is isometric, it preserves the topology. Hence $T(H)$ is closed as H is closed. □

10.

Proof. It suffices to show that T is surjective. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of X. Then $\{Te_1, \ldots, Te_n\}$ is also an orthonormal basis since T is isometric. Hence, T is surjective and therefore is unitary. □

13.

Proof. It can be verified that $T_n^* T_n \to T^*T$ and $T_n T_n^* \to TT^*$. Since T_n are normal, $T_n^* T_n = T_n T_n^*$. Hence,

$$\|T^*T - TT^*\| \le \|T^*T - T_n^*T_n\| + \|T_nT_n^* - TT^*\| \to 0$$

as $n \to \infty$. Thus, T is normal. □

15.

Proof. If T is normal, clear that $\|T^*x\| = \|Tx\|$. Now we suppose $\|Tx\| = \|T^*x\|$ for all $x \in H$. Then $\langle TT^*x, x\rangle = \langle T^*Tx, x\rangle$. Since X is complex, by Lemma 3.9-3, $TT^* = T^*T$. Namely, T is normal.

By (a), for every $x \in H$, $\|T^2x\| = \|T^*Tx\|$. Hence,

$$\|T^2\| = \sup_{\|x\|=1} \|T^2x\| = \sup_{\|x\|=1} \|T^*Tx\| = \|T^*T\| = \|T\|^2,$$

where the last equality comes from Theorem 3.9-4(e). □

4 Fundamental Theorems for Normed and Banach Spaces

4.2 Hahn-Banach Theorem

4.

Proof. By the positive homogeneity, $p(2 \times 0) = 2p(0)$. Hence, $p(0) = 0$. Consequently, $0 = p(x + (-x)) \leq p(x) + p(-x)$. Thus, $-p(x) \leq p(-x)$. □

5.

Proof. For every $x, y \in M$ and $\lambda \in [0, 1]$,

$$p(\lambda x + (1 - \lambda)y) \leq \lambda p(x) + (1 - \lambda)p(y) \leq \lambda\gamma + (1 - \lambda)\gamma = \gamma.$$

Hence, $\lambda x + (1 - \lambda)y \in M$. Thus, M is convex. □

6.

Proof. For every $x, t \in X$,

$$p(x - t) \leq p(x) + p(-t) \quad \Rightarrow \quad p(x - t) - p(x) \leq p(-t),$$

and

$$p(x) = p(x - t + t) \leq p(x - t) + p(t) \quad \Rightarrow \quad -p(t) \leq p(x - t) - p(x).$$

Since $p(0) = 0$ and p is continuous at 0, $p(t) \to 0$ and $p(-t) \to 0$ as $t \to 0$. Hence, $p(x - t) - p(x) \to 0$ as $t \to 0$, that is, p is continuous on X. □

8.

Proof. First, $p(0) \geq p(0 + 0) - p(0) = 0$. For nonzero x, we argue by contradiction. Assume that there exists some x with $0 < \|x\| \leq r$ such that $p(x) < 0$. Then $np(x) < 0$ for $n = 1, 2, \ldots$. For n sufficiently large, $n\|x\| > r$ and therefore $p(nx) \geq 0$. However, by the subadditivity, $p(nx) \leq np(x) < 0$. Contradiction. Thus, $p(x) \geq 0$ on X. □

9.

Proof. For all $x_1 = \alpha_1 x_0, x_2 = \alpha_2 x_0 \in Z$ and scalars a_1 and a_2,

$$\begin{aligned} f(a_1 x_1 + a_2 x_2) &= f((a_1\alpha_1 + a_2\alpha_2)x_0) = (a_1\alpha_1 + a_2\alpha_2)p(x_0) \\ &= a_1\alpha_1 p(x_0) + a_2\alpha_2 p(x_0) = a_1 f(x_1) + a_2 f(x_2). \end{aligned}$$

Thus, f is linear. Now we show that for $\alpha \in \mathbb{R}$, $\alpha p(x_0) \leq p(\alpha x_0)$ to complete the proof. If $\alpha \geq 0$, then it follows from the positive homogeneity. For negative α, $\alpha p(x_0) = -p(-\alpha x_0)$ and by Prob. 4, $-p(-\alpha x_0) \leq p(\alpha x_0)$. Thus, $f(x) \leq p(x)$ for all $x \in Z$. □

10.

Proof. Let Z and f have the same meaning as in Prob. 9. By Hahn-Banach theorem, there exists a linear extension $\tilde{f}$ of f to X with $\tilde{f}(x) \leq p(x)$ for all $x \in X$. Replacing x with $-x$ gives $\tilde{f}(-x) \leq p(x)$. Finally, the linearity of $\tilde{f}$ yields $-p(-x) \leq \tilde{f}(x)$. □

4.3 Hahn-Banach Theorem for Normed Spaces

1.

Proof. By (2), $p(2 \times 0) = 2p(0)$. Hence, $p(0) = 0$. And for every $x \in X$, by (1),

$$0 = p(0) \leq p(x) + p(-x) = 2p(x),$$

that is, $p(x) \geq 0$. □

2.

Proof. By (1), $p(x) = p(x - y + y) \leq p(x - y) + p(y)$. Therefore, $p(x) - p(y) \leq p(x - y)$. Interchange the roles of x and y and we obtain $p(y) - p(x) \leq p(y - x) = p(x - y)$, where the equality comes from (2). Thus, $|p(x) - p(y)| \leq p(x - y)$. □

7.

Proof. Define $\tilde{f}$ to be $x \mapsto \langle x, x_0/\|x_0\| \rangle$. Clear that it is a bounded linear functional on X and $\tilde{f}(x_0) = \|x_0\|$. And by Riesz's Theorem, $\|\tilde{f}\| = \|x_0/\|x_0\|\| = 1$. □

8.

Proof. It follows immediately from Theorem 4.3-3. □

13.

Proof. Just put $\hat{f} = \tilde{f}/\|x_0\|$. □

14.

Proof. By Prob 13, there exists a $\hat{f} \in X'$ such that $\|\hat{f}\| = 1/r$ and $\hat{f}(x_0) = 1$. Let hyperplane $H_0 = \{x \in X : \hat{f}(x) = 1\}$ and half space $S_0 = \{x \in X : \hat{f}(x) \leq 1\}$. Then clear that $x_0 \in H_0$ and for all $x \in S(0; r)$, $f(x) \leq \|f\|\|x\| = r/r = 1$. Hence, $x \in S_0$. □

15.

Proof. If $\|x\| = c + 2\varepsilon > c$, then by Corollary 4.3-4, there exists some $0 \neq f \in X'$ such that $|f(x)|/\|f\| \geq c + \varepsilon$. Consequently, the functional $g = f/\|f\|$, which is of norm 1, is such that $|g(x_0)| > c$. Contradiction. □

4.5 Adjoint Operator

9.

Proof. Note that every bounded linear functional is continuous by Theorem 2.7-9. Hence, $M^a = (\mathcal{R}(T))^a$. Thus, $g \in M^a$ iff $g \in (\mathcal{R}(T))^a$ iff $g(Tx) = (T^\times g)(x) = 0$ for all $x \in X$ iff $T^\times g = 0$ iff $g \in \mathcal{N}(T^\times)$. Namely, $M^a = \mathcal{N}(T^\times)$. □

10.

Proof. For every $y = Tx \in \mathcal{R}(T)$, $g(Tx) = (T^\times g)(x) = 0$ for all $g \in \mathcal{N}(T^\times)$. Hence, $y \in^a \mathcal{N}(T^\times)$. □

4.6 Reflexive Spaces

2.

Proof. Since Y is a closed subspace of a Hilbert space, it is complete. By Lemma 3.3-2, there is some $y \in Y$ such that $\|x_0 - y\| = \delta$ and $z = x_0 - y$ is orthogonal to Y. Define $\tilde{f}$ by $x \mapsto \langle x, z \rangle / \delta$. Then clear that $\tilde{f} \in X'$ and $\tilde{f}(y) = 0$ for all $y \in Y$. Meanwhile, by Riesz's Theorem, $\|\tilde{f}\| = \|z\|/\delta = 1$. Finally,

$$\tilde{f}(x_0) = \frac{\langle x_0, x_0 - y \rangle}{\delta} = \frac{\langle x_0 - y + y, x_0 - y \rangle}{\delta} = \delta.$$

The proof is then completed. □

3.

Proof. We denote the canonical mapping from X to X'' by C and the one from X' to X''' by D. Our goal is to find a $f \in X'$ for every given $h \in X'''$ such that $D(f) = h$, that is, for every $g \in X''$, $D(f)(g) = h(g)$. Since X is reflexive, there is some $x \in X$ such that $g = Cx$. Put $f = hC$, which is clearly an element of X'. Since

$$h(g) = h(Cx) = (hC)(x) = f(x) \quad \text{and} \quad D(f)(g) = g(f) = (Cx)(f) = f(x),$$

$h = D(hC)$. Thus, X' is reflexive. □

4.

Proof. By Prob. 3, the reflexivity of X implies the reflexivity of X'. Now we suppose X' is reflexive. Hence, again by Prob. 3, X'' is reflexive and therefore, by Theorem 4.6-4, is complete. Since X is isomorphic to $\mathcal{R}(C) \subset X''$ and $\mathcal{R}(C)$, a closed subspace of a reflexive Banach space, is reflexive, so is X. Thus, a Banach space X is reflexive iff X' is reflexive. □

5.

Proof. It suffices to show that $\delta > 0$ and then putting $h = \tilde{f}/\delta$ will complete the proof. If $\delta = 0$, then by the definition of the infimum, there exists $(y_n) \subset Y$ which converges to x_0. Then $x_0 \in Y$ since Y is closed, which contradicts our choice of x_0. Thus, $\delta > 0$. □

6.

Proof. We may assume without loss of generality that $Y_2 \setminus Y_1$ is nonempty. Choose arbitrary $x_0 \in Y_2 \setminus Y_1 \subset X \setminus Y_1$. By Prob. 6, there exists some $h \in X'$ such that $h(x_0) = 1$ and $h \in Y^a$. Thus, the annihilators of Y_1 and Y_2 are different. □

7.

Proof. If Y is proper, then by Prob. 7, the annihilators of Y and X do not coincide, which contradicts our hypothesis. Hence, $X = Y$. □

8.

Proof. If $x \in A$, then for every $f \in X'$ whose restriction to M is 0, $f(x_0) = 0$ since f, being bounded, is continuous. For the converse, note that $f|_M = 0$ implies $f|_A = 0$. If $x_0 \notin A$, then Prob. 5 guarantees the existence of some $f \in X'$ which vanishes on A and is nonzero at x_0. Contradiction. Thus, $x_0 \in A$. $\square$

9.

Proof. If M is total, then clear that every $f \in X'$ vanishing on M is zero everywhere on X. And the converse part follows immediately from Prob. 8. $\square$

10.

Proof. Let $\{b_1, \ldots, b_n\}$ be a linearly independent subset of X and define

$$\beta_i : \operatorname{span}\{b_1, \ldots, b_n\} \to \mathbb{F} \quad \text{by} \quad b_j \mapsto \delta_{ij}$$

for $i = 1, \ldots, n$. By Hahn-Banach Theorem, we can extend them to linear functionls $\tilde{\beta}_i$ on X. Suppose that $f = x_1\tilde{\beta}_1 + \cdots + x_n\tilde{\beta}_n = 0$. Then $0 = f(b_i) = x_i$ for all i. Thus, $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ is linearly independent. $\square$

4.7 Uniform Boundedness Theorem

1.

Solution. Meager, since $\mathbb{Q}$ is the union of all singleton of rational numbers. $\square$

5.

Proof. First we suppose M is rare and argue by contradiction. If $(\bar{M})^c$ is not dense in X, i.e., there exists some $x \in X$ and $r > 0$ such that $B(x; r) \cap (\bar{M})^c = \varnothing$. Hence, $B(x; r) \subset \bar{M}$, which contradicts the definition of rare subsets. Thus, $(\bar{M})^c$ is dense in X.

Now we suppose $(\bar{M})^c$ is dense in X. Then for all $x \in \bar{M}$ and $r > 0$, there exists some $y_r \notin \bar{M}$ but $y_r \in B(x; r)$. Hence, x is not an interior point. Thus, M is rare. $\square$

6.

Proof. If both M and M^c are meager, then so is their union X, but Baire's theorem says that a complete metric space is nonmeager in itself. Hence, M^c is nonmeager if M is. $\square$

7.

Proof. We argue by contradiction. Assume that for all $x \in X$, $\sup_n \|T_n x\| < \infty$. Then by the uniform boundedness theorem, there exists some c such that $\|T_n\| \le c$ for all n. Hence, $\sup_n \|T_n\| \le c$. Contradiction. $\square$

10.

Proof. We may assume without loss of generality that $\eta_1 \neq 0$. Define $T_n : c_0 \to \mathbb{C}$ by $(\xi_j) \mapsto \sum_{j=1}^n \xi_j \eta_j$. Clear that T_n are linear functionals. And since

$$|T_n x| = \left| \sum_{j=1}^n \eta_j \xi_j \right| \leq \max_{j=1,\ldots,n} |\xi_j| \sum_{j=1}^n |\eta_j| \leq \|x\| \sum_{j=1}^n |\eta_j|, \tag{4}$$

T_n are bounded and $\|T_n\| \leq \sum_{j=1}^n |\eta_j|$. Meanwhile, define $y = (\gamma_j)$ by

$$\gamma_j = \begin{cases} \operatorname{sgn} \eta_j, & j \leq n, \\ 0, & j > n. \end{cases}$$

Clear that $y \in c_0$ and $\|y\| = 1$. Since $|T_n y| = \sum_{j=1}^n |\eta_j|$, together with (4), we conclude $\|T_n\| = \sum_{j=1}^n |\eta_j|$.

By Prob. 2, Sec 2.3, c_0 is a Banach space. And for each $x = (\xi_j) \in c_0$, since $\sum \xi_j \eta_j$ converges, $\|T_n x\|$ is bounded for n large enough and therefore bounded for all n. Hence, by the uniform boundedness theorem, $\sum_{j=1}^n |\eta_j| = \|T_n\| \leq c$ for some fixed c. Thus, $\sum |\eta_j| < \infty$. □

11.

Proof. By Prob. 4, Sec 1.4, the Cauchy sequence $(T_n x)$ is bounded. Thus, by the uniform boundedness theorem, $(\|T_n\|)$ is bounded. □

13.

Proof. Let $C : X \to X''$ be the canonical embedding and $(\varphi_n) = (Cx_n)$. By Lemma 4.6-1, $\|x_n\| = \|\varphi_n\|$. Note that X'', the dual space of X', is complete and $f(x_n) = \varphi_n(f)$. Thus, by the uniform boundedness theorem, $(\|x_n\|) = (\|\varphi_n\|)$ is bounded. □

14.

Proof.

(a)⇒(c): It follows immediately from $|g(T_n x)| \leq \|g\|\|x\|\|T_n\|$.

(c)⇒(b): For fixed $x \in X$, let $\varphi_n = C(T_n x)$, where $C : Y \to Y''$ is the canonical embedding. For every $g \in Y'$, by (c), $|\varphi_n(g)| = |g(T_n x)| \leq c_g$. Since Y' is complete, by the uniform boundedness theorem, $(\|\varphi_n\|) = (\|T_n x\|)$ is bounded.

(b)⇒(a): It is just what the uniform boundedness theorem states. □

4.8 Strong and Weak Convergence

1.

Proof. The mapping $\varphi_t : C[a, b] \to \mathbb{F}$, $x \mapsto x(t)$ is a bounded linear functional on $C[a, b]$. Hence, by the definition of weak convergence, $x_n(t) \to x(t)$. □

2.

Proof. For every $f \in Y'$, $fT \in X'$. Since $x_n \xrightarrow{w} x_0$, $(fT)(x_n) \to (fT)(x_0)$, that is, $Tx_n \xrightarrow{w} Tx_0$. □

4.

Proof. If $x_0 = 0$, then it is trivial. Otherwise, by Theorem 4.3-3, there exists some $f \in X'$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$. Since $x_n \xrightarrow{w} x$, $|f(x_n)| \to |f(x_0)| = \|x_0\|$. Meanwhile, $|f(x_n)| \le \|f\|\|x_n\| = \|x_n\|$. Thus, $\underline{\lim}\, \|x_n\| \ge \|x_0\|$. $\square$

5.

Proof. If $\bar{Y} = X$, then there is nothing to be proved. Otherwise we argue by contradiction. Assume that $x_0 \in X \setminus \bar{Y} \neq \varnothing$. Then by Lemma 4.6-7, there exists some $f \in X'$ such that $f(Y) = \{0\}$ and $f(x_0) = \delta > 0$. However, since $x_n \in Y$ and $x_n \xrightarrow{w} x_0$, $f(x_0)$ must be 0. Contradiction. Thus, $x_0 \in \bar{Y}$. $\square$

6.

Proof. It follows immediately from Prob. 5. $\square$

7.

Proof. It follows immediately from Prob. 5. $\square$

8.

Proof. For every $f \in X'$, by the definition, $|f(x_n)| < c_f$ for some constant c_f which does not depend on n. Let $g_n = Cx_n$, where $C : X \to X''$ is the canonical embedding. Then for all n, $|g_n(f)| = |f(x_n)| \le c_f$. Since X' is complete, by the uniform boundedness theorem, (g_n) is bounded and therefore so is (x_n). $\square$

9. Note that we do not regard a sequence with only repeated elements when n is large as a sequence here.

Proof. We argue by contradiction. Assume that A is unbounded, that is, there exists a $(a_n) \subset A$ with $\|a_n\| \ge n$. Clear that every sequence in (a_n) is unbounded and therefore, by Prob. 8, is not a weak Cauchy sequence, contradicting our hypothesis. Thus, A is bounded. $\square$

10.

Proof. Let (x_n) be a weak Cauchy sequence in X Let $\varphi_n = Cx_n$, where $C : X \to X''$ is the canonical embedding. For every $f \in X'$, since (x_n) is a weak Cauchy sequence, $(\varphi_n(f)) = (f(x_n))$ is a Cauchy sequence in $\mathbb{F}$ and therefore $\lim \varphi_n(f)$ exists. Let $\varphi : X' \to \mathbb{F}$ be defined by $f \mapsto \lim \varphi_n(f)$. Clear that it is linear. Meanwhile, since (x_n), a weak Cauchy sequence, is bounded by Prob. 8, (φ_n) is bounded by, say, c. Therefore, φ is bounded since $|\varphi_n(f)| \le c\|f\|$ for all n. Thus, $\varphi \in X''$.

Since X is reflexive, there exists some $x_0 \in X$ such that $\varphi = Cx_0$. For all $f \in X'$,

$$f(x_n) = \varphi_n(f) \to \varphi(f) = f(x_0) \quad \text{as } n \to \infty.$$

Thus, $x_n \xrightarrow{w} x_0$. Thus, X is weakly complete. $\square$

4.9 Convergence of Sequences of Operators

4.

Proof. First suppose that $f_n \xrightarrow{w} f$, namely, $\varphi(f_n) \to \varphi(f)$ for all $\varphi \in X''$. For every $x \in X$, denoting the canonical embedding by C,

$$|f_n(x) - f(x)| = |(Cx)(f_n) - (Cx)(f)| \to 0,$$

since $f_n \xrightarrow{w} f$. Thus, $f_n \xrightarrow{w^*} f$.

Now we suppose $f_n \xrightarrow{w^*} f$ and X is reflexive. Then for each $\varphi \in X''$, there is an $x \in X$ such that $\varphi = Cx$. Hence,

$$|\varphi(f_n) - \varphi(f)| = |f_n(x) - f(x)| \to 0,$$

since $f_n \xrightarrow{w^*} f$. Thus, $f_n \xrightarrow{w} f$. □

6.

Proof. If $T_n \to T$, then for every $\varepsilon > 0$, there is an N such that for all $n > N$, $\|T_n - T\| < \varepsilon$. Hence, for all x with $\|x\| = 1$,

$$\|T_n x - Tx\| \leq \|T_n - T\|\|x\| < \varepsilon.$$

Now we suppose the converse. Since $\|T_n - T\| = \sup_{\|x\|=1} \|T_n x - Tx\|$, there is some x of norm 1 such that

$$\|T_n - T\| - \varepsilon < \|T_n x - Tx\| < \varepsilon.$$

Thus, $\|T_n - T\| \leq 2\varepsilon$, which implies $T_n \to T$. □

7.

Proof. For every $x \in X$, since $T_n x$ converges, $(\|T_n x\|)$ is bounded. As X is complete, this implies $(\|T_n\|)$ is bounded by the uniform boundedness theorem. □

9.

Proof. For every $x \in X$ with $\|x\| = 1$,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \varliminf_{n\to\infty} \|T_n\|$$

where the first equality comes from the continuity of the norm. Thus, $\|T\| \leq \varliminf_{n\to\infty} \|T_n\|$. □

10.

Proof. Since X is separable, we can find a total sequence (b_n) of X. Now we choose a subsequence of $(f_n) \subset X'$ as follows. First we choose a subsequence of $(f_k(x_n))$ for each fixed n inductively. Since M is bounded, $(f_k(x_1))$ is a bounded sequence in $\mathbb{R}$ and therefore has a convergent subsequence $f_{k_i}(x_1)$. For x_2 we choose the subsequence of $(f_{k_i}(x_2))$ which is convergent. We proceed in this way and choose a sequence of $(f_k(x_n))$ for each n. Now, consider the subsequence (g_k) of (f_n) whose n-th element is the n-th functional in the n-th choice. It can be verified that it is a subsequence. And by our construction, $(f_k(x_n))$ converges for every x_n. Hence, by Corollary 4.9-7, it is weak* convergent. □