Solutions to
A Course in Enumeration

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Contents

1 Fundamental Coefficients

1.1 Elementary Counting Principles

2.

Solution. We compute $N = \#\{(i,j,k) \in \mathbb{N}^3 \,|\, i+j+k = 151, \max\{i,j,k\} \le 75\}$. For fixed $1 \le i \le 75$, j can be chosen between $76 - i$ and 75. Thus,

$$N = \sum_{i=1}^{75} \sum_{j=76-i}^{75} 1 = \sum_{i=1}^{75} i = 2850.$$

□

3.

Proof. The number of subsets of $\{1, \ldots, n+1\}$ is 2^{n+1}. Classify these subsets according the biggest elements in them. The number of subsets whose biggest elements are k equals to the number of subsets of $\{1, \ldots, k\}$ containing k, that is, 2^{k-1}. Thus,

$$2^{n+1} = 1 + \sum_{k=1}^{n+1} 2^{k-1} \quad \Rightarrow \quad 2^{n+1} - 1 = \sum_{k=0}^{n} 2^k.$$

Similarly, we may classify these subsets according to the biggest two elements. Then

$$2^{n+1} - 1 - (n+1) = \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} 2^{i-1} = \sum_{i=1}^{n} 2^{i-1}(n-i+1) = \sum_{i=1}^{n} 2^{i-1}(n-i) + 2^n - 1.$$

Thus, $\sum_{k=1}^{n}(n-k)2^{k-1} = 2^n - n - 1$. □

5.

Proof. We count the number N of triples in $\{1, \ldots, n+1\}$. By definition, $N = \binom{n+1}{3}$. Let S_k be the collection of triples the last elements of which are k. Then $|S_k| = \binom{k-1}{2}$. Thus

$$\binom{n+1}{3} = \sum_{k=3}^{n+1} \binom{k-1}{2} = \sum_{k=2}^{n} \binom{k}{2} = \sum_{k=1}^{n} \binom{k}{2}.$$

□

8.

Proof. Let E be valid k-subset. If $1 \in E$, then we need to choose $k-1$ numbers from $3, \ldots, n$ so that no pair of consecutive integers exists. The number of possible choices is $f(n-2, k-1)$. If $1 \notin E$, then we need to choose k numbers from $2, \ldots, n$ such that no pair of consecutive integers exists. The number of possible choices is $f(n-1, k)$. Hence, we obtain the recurrence relation

$$f(n,k) = f(n-2, k-1) + f(n-1, k).$$

Now we argue by induction on n. Clear that $f(n,0) = 1$ for all n, $f(2,1) = 2$ and $f(n,k) = 0$ for all $n < 2k-1$, all satisfying $f(n,k) = \binom{n-k+1}{k}$. Assume that for all $m < n$, $f(m,k) = \binom{m-k+1}{k}$ holds. Thus,

$$f(n,k) = f(n-2,k-1) + f(n-1,k) = \binom{n-k}{k-1} + \binom{n-k}{k} = \binom{n-k+1}{k}.$$

Let $s(n)$ denote $\sum_{k=0}^{n} f(n,k)$. Then

$$\begin{aligned} s(n-1) + s(n-2) &= f(n-1,0) + \sum_{k=1}^{n-1} f(n-1,k) + \sum_{k=1}^{n-1} f(n-2,k-1) \\ &= 1 + \sum_{k=1}^{n-1} \{f(n-1,k) + f(n-2,k-1)\} \\ &= f(n,0) + \sum_{k=1}^{n-1} f(n,k) = \sum_{k=0}^{n-1} f(n,k) = s(n). \end{aligned}$$

This recurrence relation, together with the fact $s(1) = 2$ and $s(2) = 3$, imply that $s(n) = F_{n+2}$. □

11.

Proof. We argue by contradiction. Let $S_p = \{p + 9k \mid p + 9k \leq 100, p = 0, 1, \dots\}$, $p = 1, \dots, 9$. Clear that $\{S_p\}$ partitions $\{1, \dots, 100\}$; $|S_1| = 12$ and $|S_p| = 11$ for $p = 2, \dots, 9$. If A does not contain two numbers with difference 9. Then for each p, no consecutive elements of S_p can belong to A, which implies that $|S_p \cap A| \leq 6$. Hence, $|A| = \sum_{p=1}^{9} |S_p \cap A| \leq 54$. Contradiction. Thus, A must contain two numbers with difference 9.

For the case $|A| = 54$, this is not true. For a counterexample, put

$$A = \bigcup_{p=1}^{9} \{p + 9k \mid p + 9k \leq 100, k = 1, 3, 5, \dots\}.$$

□