Solutions to *A Second Course in Linear Algebra*

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Contents

1 Linear Algebra

1.3 Linear Transformation

9.

Proof. First we suppose $T^2 = 0$. Let $N = \ker T$ and M the complement of N. Then (a), (b) and (c) hold automatically. Since $T^2 = 0$, $T(M) \subset \ker T$.

Now we suppose the existence of such M and N. By (c), $N \subset \ker T$. For every $x \in V$, if $Tx \neq 0$, then $x \notin N$. Therefore by (a) and (b), x can be uniquely represented by $x = m + n$ where $m \in M$, $n \in N$. Hence,

$$T^2 x = T(Tm) + T(Tn) = 0$$

since $n \in N \subset \ker T$ and $Tm \in T(M) \subset N \subset \ker T$. Thus, $T^2 = 0$. $\square$

10.

Proof. First we suppose $T^2 = I$. Let $M = \{x \in V : Tx = x\}$ and $N = \{x \in V : Tx = -x\}$. Then (b)-(d) hold automatically. For every $x \in V$, $x = (x + Tx)/2 + (x - Tx)/2$. Note that $T(x + Tx) = Tx + x$ and $T(x - Tx) = Tx - x$, that is, $x + Tx \in M$ and $x - Tx \in N$. Hence, $V = M + N$.

Now we suppose the existence of such M and N. Then for every $x \in V$, $x = m + n$ where $m \in M$ and $n \in N$. Hence, $T^2 x = T(Tm) + T(Tn) = Tm - Tn = m + n = x$. Thus, $T^2 = I$. $\square$

1.4 Products and Direct Sums

2.

Proof. Note that for $\oplus_{i \in \Delta} V_i$, the summation $\sum_{i \in \Delta} T_i \pi_i$ is well-defined since for each x, the summation only has finitely many nonzero terms. Hence, we have an analogy of Theorem 4.7 and, consequently, an analogy of Corollary 4.8 for $V = \oplus_{i \in \Delta} V_i$. $\square$

3.

Proof. Define $T : W \to V$ by $\alpha \mapsto (T_i(\alpha))_{i \in \Delta}$. We show that T is linear first. For every $x, y \in W$ and scalars a and b,

$$\begin{aligned} T(ax + by) &= (T_i(ax + by))_{i \in \Delta} = (aT_i(x) + bT_i(y))_{i \in \Delta} \\ &= a(T_i(x))_{i \in \Delta} + b(T_i(y))_{i \in \Delta} = aT(x) + bT(y). \end{aligned}$$

Thus, $T \in \mathrm{Hom}(W, V)$. For the uniqueness, suppose T' is such a linear transformation. Suppose $T'(x) = (T'_i(x))_{i \in \Delta}$. Since the diagram is commutative, for each $p \in \Delta$ and every $x \in W$,

$$T'_p(x) = \pi_p T'(x) = T_p(x).$$

Hence, $T = T'$. Thus, V together with $\{\pi_p : p \in \Delta\}$ satisfies the universal mapping property. $\square$

4.

Proof. Clear that $T_i \in \mathscr{E}(V)$ for all $i = 1, \ldots, n$. And $T_i^2 = \theta_i \pi_i \theta_i \pi_i = \theta_i \pi_i = T_i$ as $\pi_i \theta_i = I_{V_i}$ by Theorem 4.3(a). Similarly, for $i \neq j$, $T_i T_j = 0$ by Theorem 4.3(b). $\square$