# Solutions to
# *A Course in Enumeration*

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## Contents

# 1 Fundamental Coefficients

## 1.1 Elementary Counting Principles

**2.**

*Solution.* We compute $N = \#\{(i,j,k) \in \mathbb{N}^3 \mid i+j+k = 151, \max\{i,j,k\} \leq 75\}$. For fixed $1 \leq i \leq 75$, $j$ can be chosen between $76-i$ and 75. Thus,

$$N = \sum_{i=1}^{75} \sum_{j=76-i}^{75} 1 = \sum_{i=1}^{75} i = 2850.$$

□

**3.**

*Proof.* The number of subsets of $\{1,\ldots,n+1\}$ is $2^{n+1}$. Classify these subsets according the biggest elements in them. The number of subsets whose biggest elements are $k$ equals to the number of subsets of $\{1,\ldots,k\}$ containing $k$, that is, $2^{k-1}$. Thus,

$$2^{n+1} = 1 + \sum_{k=1}^{n+1} 2^{k-1} \quad \Rightarrow \quad 2^{n+1} - 1 = \sum_{k=0}^{n} 2^k.$$

Similarly, we may classify these subsets according to the biggest two elements. Then

$$2^{n+1} - 1 - (n+1) = \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} 2^{i-1} = \sum_{i=1}^{n} 2^{i-1}(n-i+1) = \sum_{i=1}^{n} 2^{i-1}(n-i) + 2^n - 1.$$

Thus, $\sum_{k=1}^{n}(n-k)2^{k-1} = 2^n - n - 1$. □

**5.**

*Proof.* We count the number $N$ of triples in $\{1,\ldots,n+1\}$. By definition, $N = \binom{n+1}{3}$. Let $S_k$ be the collection of triples the last elements of which are $k$. Then $|S_k| = \binom{k-1}{2}$. Thus

$$\binom{n+1}{3} = \sum_{k=3}^{n+1} \binom{k-1}{2} = \sum_{k=2}^{n} \binom{k}{2} = \sum_{k=1}^{n} \binom{k}{2}.$$

□

**8.**

*Proof.* Let $E$ be valid $k$-subset. If $1 \in E$, then we need to choose $k-1$ numbers from $3,\ldots,n$ so that no pair of consecutive integers exists. The number of possible choices is $f(n-2,k-1)$. If $1 \notin E$, then we need to choose $k$ numbers from $2,\ldots,n$ such that no pair of consecutive integers exists. The number of possible choices is $f(n-1,k)$. Hence, we obtain the recurrence relation

$$f(n,k) = f(n-2,k-1) + f(n-1,k).$$

Now we argue by induction on $n$. Clear that $f(n,0) = 1$ for all $n$, $f(2,1) = 2$ and $f(n,k) = 0$ for all $n < 2k-1$, all satisfying $f(n,k) = \binom{n-k+1}{k}$. Assume that for all $m < n$, $f(m,k) = \binom{m-k+1}{k}$ holds. Thus,

$$f(n,k) = f(n-2,k-1) + f(n-1,k) = \binom{n-k}{k-1} + \binom{n-k}{k} = \binom{n-k+1}{k}.$$

Let $s(n)$ denote $\sum_{k=0}^{n} f(n,k)$. Then

$$\begin{aligned}
s(n-1) + s(n-2) &= f(n-1,0) + \sum_{k=1}^{n-1} f(n-1,k) + \sum_{k=1}^{n-1} f(n-2,k-1) \\
&= 1 + \sum_{k=1}^{n-1} \{f(n-1,k) + f(n-2,k-1)\} \\
&= f(n,0) + \sum_{k=1}^{n-1} f(n,k) = \sum_{k=0}^{n-1} f(n,k) = s(n).
\end{aligned}$$

This recurrence relation, together with the fact $s(1) = 2$ and $s(2) = 3$, imply that $s(n) = F_{n+2}$. $\square$

**11.**

*Proof.* We argue by contradiction. Let $S_p = \{p + 9k \mid p + 9k \leq 100, p = 0, 1, \dots\}$, $p = 1, \dots, 9$. Clear that $\{S_p\}$ partitions $\{1, \dots, 100\}$; $|S_1| = 12$ and $|S_p| = 11$ for $p = 2, \dots, 9$. If $A$ does not contain two numbers with difference 9. Then for each $p$, no consecutive elements of $S_p$ can belong to $A$, which implies that $|S_p \cap A| \leq 6$. Hence, $|A| = \sum_{p=1}^{9} |S_p \cap A| \leq 54$. Contradiction. Thus, $A$ must contain two numbers with difference 9.

For the case $|A| = 54$, this is not true. For a counterexample, put

$$A = \bigcup_{p=1}^{9} \{p + 9k \mid p + 9k \leq 100, k = 1, 3, 5, \dots\}.$$

$\square$

## 1.2 Subsets and Binomial Coefficients

**12.**

*Proof.* We count the pairs $(A, B)$ of subsets of $N$ with $|A| = k$ and $|B| = m - k$ and $A \cap B = \varnothing$. We may choose $A \cup B$ first and, then, the elements of $A$ from $A \cup B$. This way yields the left-hand side. Or, we may choose the elements of $A$ from $N$ first and then the elements of $B$ from $N \setminus A$, which yields the right-hand side. Thus, $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

If we let $|A| = k$ range over 0 to $m$, then we count all subsets of $A \cup B$. Thus, $\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$. $\square$

**13.**

*Proof.*

$$\begin{aligned}
\text{L.H.S} &= \left\{\binom{2n}{2n-2k}\binom{2n-2k}{n-k}\right\}\binom{2k}{k} \\
&= \binom{2n}{n-k}\binom{n+k}{n-k}\binom{2k}{k} \\
&= \binom{2n}{n+k}\left\{\binom{n+k}{2k}\binom{2k}{k}\right\} \\
&= \binom{2n}{n+k}\binom{n+k}{k}\binom{n}{k} \\
&= \left\{\binom{2n}{n+k}\binom{n+k}{n}\right\}\binom{n}{k} \\
&= \binom{2n}{n}\binom{n}{k}^2 = \text{R.H.S.}
\end{aligned}$$

□

**17.**

*Proof.* If there exists $i$ and $j$ such that $n_i - n_j \geq 2$, then we may increase the coefficient by replace $n_i$ with $n_i - 1$ and $n_j$ with $n_j + 1$. Since the choices of $n_1, \ldots, n_k$ are finite, this implies that the coefficient must attain its maximum at some $n_1, \ldots, n_k$ with $|n_i - n_j| \leq 1$ for all $i$ and $j$. Meanwhile, since $\max n_i - \min n_i \leq 1$ and $\sum n_i = n$, a fixed number, the choice of $n_1, \ldots, n_k$ is unique up to permutation. Thus, the coefficient attains its maximum at every $n_1, \ldots, n_k$ with $|n_i - n_j| \leq 1$.

We show that $\binom{n}{n_1 n_2 n_3} \leq \frac{3^n}{n+1}$ by induction on $n$. Some computation shows that the inequality holds for $n = 1$. Assume that the inequality holds for all cases less than $n$. And suppose that the coefficient attains its maximum at $m_1 \leq m_2 \leq m_3$. Then

$$\binom{n}{n_1 n_2 n_3} \leq \frac{n!}{m_1! m_2! m_3!} = \frac{(n-1)!}{m_1! m_2! (m_3 - 1)!}\frac{n}{m_3} \leq \frac{3^{n-1}}{n}\frac{n}{(n+1)/3}$$

□

**18.**

*Proof.* Consider the $(k+1) \times (n-k)$-lattice. The number of paths is $\binom{n+1}{k+1}$. Classify the paths according to the height $j$ right before the last $(1,0)$-step. For each $j = 0, \ldots, n-k$, the number equals the number of paths of the $k \times j$-lattice, which is $\binom{k+j}{k}$. Thus,

$$\binom{n+1}{k+1} = \sum_{j=0}^{n-k}\binom{k+j}{k} = \sum_{i=k}^{n}\binom{i}{k} = \sum_{i=0}^{n}\binom{i}{k}.$$

This argument, *mutatis mutandis*, also gives (9). □