# Convex Optimization

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## Contents

# 2 Convex Sets

## 2.1

*Proof.* For $k = 2$, $\theta_1 x_1 + \theta_2 x_2 \in C$ holds by by definition. We argue by induction on $k$ and assume that the inclusion holds for $k < m$. When $k = m$, denoting $\sum_{i=1}^{m-1} \theta_i$ by $s$,

$$\sum_{i=1}^{m} \theta_i x_i = s \sum_{i=1}^{m-1} \frac{\theta_i x_i}{s} + \theta_m x_m.$$

Since $\sum_{i=1}^{m-1} \theta_i / s = 1$, by the induction hypothesis, $\sum_{i=1}^{m-1} \theta_i x_i / s \in C$ . Meanwhile, as $s + \theta_m = 1$, $\sum_{i=1}^{m} \theta_i x_i \in C$, completing the proof. $\square$

## 2.2

*Proof.* Clear that the intersection of two convex sets is still convex. Hence, the intersection of $C \subset \mathbb{R}^n$ and any line is convex as long as $C$ is convex.

Now we suppose that the intersection of $C$ and any line is convex. For any $x_1, x_2 \in C$, $C_l = C \cap \{\theta x_1 + (1-\theta) x_2 : \theta \in \mathbb{R}\}$ is convex and therefore $\theta x_1 + (1-\theta) x_2 \in C_l \subset C$ for every $0 \le \theta \le 1$. Thus, $C$ is convex. $\square$

## 2.8

*Proof.*

(a) It is trivial when $a_1$ and $a_2$ are linearly dependent, so we assume that $a_1$ and $a_2$ are linearly independent. We first tackle the problem for orthonormal $a_1$ and $a_2$ and then reduce the general situation to it.

Suppose that $a_1$ and $a_2$ are orthonormal. Let $S_0 = \operatorname{span}(a_1, a_2)$ and $(b_1, \ldots, b_{n-2})$ a basis of $S_0^{\perp}$. Then

$$x \in S_0 \quad \Leftrightarrow \quad \begin{bmatrix} b_1^T \\ \vdots \\ b_{n-2}^T \end{bmatrix} x = Bx = 0.$$

For $y = (y_1, y_2)^T \in S_0$, $y_1 \le 1$ iff $a_1^T y \le 1$ as $(a_1, a_2)$ is an orthonormal basis of $S_0$. Hence,

$$-1 \le y_1, y_2 \le 1 \quad \Leftrightarrow \quad \begin{bmatrix} a_1^T \\ a_2^T \\ -a_1^T \\ -a_2^T \end{bmatrix} y = Ay \preceq \mathbf{1}.$$

Thus, for orthonormal $a_1$ and $a_2$, $S = \{x : Bx = 0,\ Ax \preceq \mathbf{1}\}$, a polyhedron.

Now we only assume the liner independence of $a_1$ and $a_2$. We know that there exists some invertible $n$ by $n$ matrix[1] $R$ such that $[\tilde{a}_1, \tilde{a_2}] = R[a_1, a_2]$ and $\tilde{a}_1$ and $\tilde{a}_2$ are orthonormal. Denoting the set described in the problem with respect to $u_1$ and $u_2$ by $S(u_1, u_2)$, $x \in S(a_1, a_2)$ iff $Rx \in S(\tilde{a}_1, \tilde{a_2})$ iff $Rx \in \{x : \tilde{B}x = 0,\ \tilde{A}x \preceq 1\}$ where the meaning of $\tilde{A}$ and $\tilde{B}$ are described in the previous passage. Namely,

$$S(a_1, a_2) = \{x : \tilde{B}Rx = 0,\ \tilde{A}Rx \preceq 1\}.$$

[1] We can use $QR$ factorization to construct the matrix explicitly

(b) Yes, and the provided form has already satisfied the requirement.

(c) No. Note that $\langle x, y\rangle_2 \le 1$ for all $y$ with 2-norm 1 implies

$$\|x\|_2 = \langle x, x/\|x\|\rangle_2 \le 1.$$

And by the Cauchy-Schwarz inequality, for every $\|x\| \le 1$, $\langle x, y\rangle_2$ holds for every $\|y\|_2 = 1$. Hence, $S$ is the intersection of the unit ball and $\{x : x \succeq 0\}$, which is not a polyhedron.

(d) Yes. Let $\tilde{S} = \{x \in \mathbb{R}^n : x \succeq 0, \|x\|_\infty \le 1\}$, which is clearly a polyhedron since when $x \succeq 0$, $\|x\|_\infty \le 1$ is equivalent to $[e_1, \dots, e_n]x \preceq \mathbf{1}$ where $e_i$ is the $i$-th vector in the standard basis of $\mathbb{R}^n$.

Now we show that $S = \tilde{S}$. Suppose that $x \succeq 0$. If $\langle x, y\rangle_2 \le 1$ for all $y$ with 1-norm 1, then $x_i = \langle x, e_i\rangle_2 \le 1$. Namely, $\|x\|_\infty \le 1$. Meanwhile, if $\|x\|_\infty \le 1$,

$$\langle x, y\rangle \le \sum_{i=1}^{n} x_i |y_i| \le 1$$

as it is just the weighted average of $x_1, \dots, x_n$. Hence, $S = \tilde{S}$, completing the proof. □

**2.9**

*Proof.*

(a) By the definition,

$$\begin{aligned}
x \in V &\Leftrightarrow \|x - x_0\|_2^2 - \|x - x_i\|_2^2 \le 0 \\
&\Leftrightarrow 2\langle x, x_i - x_0\rangle \le \langle x_i, x_i\rangle - \langle x_0, x_0\rangle \quad \text{for } i = 1, \dots, K \\
&\Leftrightarrow 2\begin{bmatrix} \langle x, x_1 - x_0\rangle \\ \vdots \\ \langle x, x_K - x_0\rangle \end{bmatrix} \preceq \begin{bmatrix} \|x_1\|_2^2 - \|x_0\|_2^2 \\ \vdots \\ \|x_K\|_2^2 - \|x_0\|_2^2 \end{bmatrix} \\
&\Leftrightarrow 2\begin{bmatrix} (x_1 - x_0)^T \\ \vdots \\ (x_K - x_0)^T \end{bmatrix} x \preceq \begin{bmatrix} \|x_1\|_2^2 - \|x_0\|_2^2 \\ \vdots \\ \|x_K\|_2^2 - \|x_0\|_2^2 \end{bmatrix}
\end{aligned}$$

Hence, $V$ is a polyhedron. Intuitively, the border of a Voronoi set are the lines with the same distances to $x_0$ and $x_i$.

(b) Suppose that $P = \{x : \alpha_k^T x \le b_k, k = 1, \dots, K\}$. Let $x_0$ be any point of $P$ and we construct the other points by reflection. For each $k$, let $\tilde{x}_k$ be any point of $\{x : \alpha_k^T x = b_k\}$, $U_k = I - 2\alpha_k\alpha_k^T/\|\alpha_k\|_2^2$, the Householder matrix, and

$$R_k(x) = U_k(x - \tilde{x}_k) + \tilde{x}_k = x + 2\frac{\alpha_k}{\|\alpha_k\|_2^2}(b_k - \alpha_k^T x).$$

It is easy to verified that $P$ is the Voronoi region of $x_0$ with respect to $R_1(x_0), \dots, R_K(x_0)$. □

**2.16**

*Proof.* For every $(a, b_1 + b_2), (c, d_1 + d_2) \in S$ and $0 \le \theta \le 1$, let

$$z_\theta = \theta(a, b_1 + b_2) + (1 - \theta)(c, d_1 + d_2) = (x, y_1 + y_2)$$

where

$$x = \theta a + (1-\theta)c, \quad y_i = \theta b_i + (1-\theta)d_i \quad \text{for } i = 1, 2.$$

Since $S_i$ is convex and $(a, b_i), (c, d_i) \in S_i$,

$$(x, y_i) = \theta(a, b_i) + (1-\theta)(c, d_i) \in S_i.$$

Hence, $S$ is convex. $\square$