Solutions to *Topology*

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2 Topological Spaces and Continuous Functions

13 Basis for a Topology

1.

Proof. Let $\mathcal{T}$ be the topology of X. Since $\mathcal{T}$ is a basis for itself and the hypothesis implies that A is a set in the topology generated by $\mathcal{T}$, $A \in \mathcal{T}$, i.e., A is open. $\square$

4.

Proof.

(a) Put $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$. Since $\varnothing$ and X are contained in all $\mathcal{T}_\alpha$, they are also contained in $\mathcal{T}$. Let $\{U_\beta\}_{\beta \in J}$ be an indexed family of elements of $\mathcal{T}$ and put $U = \bigcup_{\beta \in J} U_\beta$. For every β, since U_β is open with respect to each $\mathcal{T}_\alpha$, by definition, so is $\bigcup_{\beta \in J}$. Similarly, we can show that $\mathcal{T}$ is closed under finite intersection. Thus, $\mathcal{T}$ is a topology.

The union $\bigcup \mathcal{T}_\alpha$, however, may not be a topology. Take $X = \{a, b, c\}$ for example. $\mathcal{T}_a = \{\varnothing, a, X\}$ and $\mathcal{T}_b = \{\varnothing, b, X\}$ are two topologies, but their union is not.

(b) Let $\mathcal{T}$ be the intersection of all topologies containing all $\mathcal{T}_\alpha$. By (a), $\mathcal{T}$ is a topology and clear that it is the unique smallest one. Now, let $\mathcal{T}' = \bigcap T_\alpha$, which is again a topology and is contained in all T_α. It can be verified that $\mathcal{T}'$ is the unique largest one.

(c) $\{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$; $\{\varnothing, X, \{a\}\}$. $\square$

5.

Proof. Let $\mathcal{A}$ be a basis, $\mathcal{T}$ the topology generated by $\mathcal{A}$, $\{\mathcal{T}_\alpha\}$ the collection of all topologies containing $\mathcal{A}$ and $\mathcal{T}' = \bigcap \mathcal{T}_\alpha$. For every union U of elements of $\mathcal{A}$, since, for every α, $\mathcal{A} \subset \mathcal{T}_\alpha$ and $\mathcal{T}_\alpha$ is closed under arbitrary union, $U \in \mathcal{T}_\alpha$. Hence, $\mathcal{T} \subset \mathcal{T}'$. Consequently, $\mathcal{T}'$ is also the intersection of all topologies containing $\mathcal{T}$. Since $\mathcal{T}$ contains itself as a subset, $\mathcal{T}' \subset \mathcal{T}$. Thus, $\mathcal{T} = \mathcal{T}'$.

Consider the collection of all finite intersections of $\mathcal{A}$, which is a basis, and apply the previous result to complete the proof. $\square$

6.

Proof. Let $\mathcal{T}_l$ and $\mathcal{T}_K$ be the topology of $\mathbb{R}_l$ and $\mathbb{R}_k$ respectively. $B = (-1, 1) - K$ is a basis element of $\mathcal{T}_k$ and $0 \in B$. However, no half-open interval containing 0 is in B. Hence, $\mathcal{T}_l$ is no finer than $\mathcal{T}_K$. Conversely, $C = [1, 2)$ is a basis element of $\mathcal{T}_l$ and $1 \in C$, but as $1 \in K$, there is no basis element of $\mathcal{T}_K$ containing 1. Hence, $\mathcal{T}_K$ is no finer than $\mathcal{T}_l$. Thus, they are not comparable. $\square$

8.

Proof.

(a) First clear that $\mathcal{B} \subset \mathcal{T}$. For every $U \in \mathcal{T}$ and $x \in U$, since U is open, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subset U$. Hence, there exists some rational a and b such that $x - \delta < a < x < b < x + \delta$. Thus, by Lemma 13.2, $\mathcal{B}$ generates the standard topology on $\mathbb{R}$.

(b) Since $x \in [\lfloor x \rfloor, \lfloor x \rfloor + 1) \in \mathcal{C}$ for every $x \in \mathbb{R}$, the first condition for a basis is satisfied. Meanwhile, for every $B_1 = [a, b)$ and $B_2 = [c, d)$ in $\mathcal{C}$, if they are not disjoint, $[c, b) = B_1 \cap B_2$ is also in $\mathcal{C}$. Hence, the second condition is satisfied. Thus, $\mathcal{C}$ is a basis.

Since $[\sqrt{2}, 2)$ can not be represented by union of elements in $\mathcal{C}$, $\mathcal{C}$ does not generate the lower limit topology. $\square$

16 The Subspace Topology

1.

Proof. Denote the topologies inherited from X and Y by $\mathcal{T}$ and $\mathcal{T}'$ respectively. For every $E = H \in \mathcal{T}$, supposing that $E = H \cap A$ where H is open in X, then, since $E \subset A \subset Y$, $E = (Y \cap H) \cap A$. Namely, $E \in \mathcal{T}'$. For the converse, suppose that $F = K \cap A$ where K is open in Y, then, for some H open in X, $F = (H \cap Y) \cap A = H \cap A$. Namely, $F \in \mathcal{T}$. Thus, $\mathcal{T} = \mathcal{T}'$. $\square$

2.

Proof. Denote the corresponding subspace topologies by $\mathcal{S}$ and $\mathcal{S}'$ respectively. Clear that $\mathcal{S}'$ is finer than $\mathcal{S}$. The relation, however, may not be strict. As an example, put $Y = \{y\}$. Then both $\mathcal{S}$ and $\mathcal{S}'$ are $\{\varnothing, Y\}$. $\square$

4.

Proof. By Lemma 13.1, (U, V) is open in $X \times Y$ iff $U = \bigcup U_\alpha$ and $V = \bigcup V_\beta$ where all U_α and V_β are open in X and Y respectively. Hence, $\pi_1(U, V) = \bigcup U_\alpha$ and $\pi_2(U, V) = \bigcup V_\beta$ are also open. Thus, π_1 and π_2 are open maps. $\square$

6.

Proof. By Prob. 8(a), Sec. 13, $\{(a, b) : a < b,\ a, b \in \mathbb{Q}\}$ is a basis for $\mathbb{R}$. The result then follows immediately from Theorem 15.1. $\square$

7.

Proof. No. Let $X = \mathbb{Q}$ with the usual order and $Y = \{x : 0 \leq x^2 \leq 2\}$. Y is a proper subset of X and is convex in X but not an interval or a ray. $\square$

9.

Proof. $\mathcal{B}_d = \mathcal{P}(\mathbb{R}) \times \{(b, d) : b < d,\ b, d \in \mathbb{R}\}$ is a basis for $\mathbb{R}_d \times \mathbb{R}$ and by Example 2, Sec. 14, $\mathcal{B}_o = \{\{a\} \times (b, d) : a, b, d \in \mathbb{R},\ b < d\}$ is a basis for the dictionary order topology on $\mathbb{R} \times \mathbb{R}$. Clear that $\mathcal{B}_0 \subset \mathcal{B}_d$. Meanwhile, for every $E \in \mathcal{P}(\mathbb{R})$, $E = \bigcup_{x \in E}\{x\}$. Hence, $\mathcal{B}_d \subset \mathcal{B}_o$. Thus, these two topologies are the same.

The collection $\mathcal{B}$ of all products of open intervals is a basis for the standard topology on $\mathbb{R}^2$. Clear that $\mathcal{B} \subset \mathcal{B}_d$. Meanwhile, $\{0\} \times \mathbb{R}$ is open in $\mathbb{R}_d \times \mathbb{R}$ but not in the standard topological space. Thus, the previous two topologies are strictly finer than the standard topology. $\square$

10.

Proof. Denote these topologies by $\mathcal{T}_i$, $i = 1, 2, 3$, respectively. $[0, 1] \times (1/2, 1] \in \mathcal{T}_1 \setminus \mathcal{T}_2$. Hence, $\mathcal{T}_2$ is no finer than $\mathcal{T}_1$. Meanwhile, since $\{1/2\} \times (1/2, 1) \in \mathcal{T}_2 \setminus \mathcal{T}_1$, $\mathcal{T}_1$ is no finer than $\mathcal{T}_2$. Thus, $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable.

Now we show that $\mathcal{T}_3$ is finer than both $\mathcal{T}_1$ and $\mathcal{T}_2$ and since $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable, this relation is strict. Let $\mathcal{B}_1$ be the collection of all products of open intervals in I and $\mathcal{B}_3$ the collection of all sets of form $\{a\} \times ((b, d) \cap [0, 1])$ where $a \in [0, 1]$. They are bases of $\mathcal{T}_1$ and $\mathcal{T}_3$. respectively. Since every element in $\mathcal{B}_1$ can be represented by an arbitrary union of elements in $\mathcal{B}_3$, $\mathcal{T}_3$ is finer than $\mathcal{T}_1$. Similarly, we assert that $\mathcal{T}_3$ is also finer than $\mathcal{T}_2$. $\square$

17 Closed Sets and Limit Points

2.

Proof. Since A is a subset of $Y \subset X$, $X \setminus A = (Y \setminus A) \cup (X \setminus Y)$. Since A is closed in Y and Y in closed in X, this implies that $X \setminus A$ is open. Thus, A is closed in X. $\square$

4.

Proof. Since A is closed in X, A^c is open in X. Hence, $U \setminus A = U \cap A^c$ is open. Similarly, $A \setminus U$ is closed. $\square$

6.

Proof.

(a) For any $x \in X$, if the neighborhood U of x intersects A, then it intersects B since $A \subset B$. Thus, $\bar{A} \subset \bar{B}$.

(b) Since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$ and $\bar{A} \cup \bar{B}$ is closed set containing $A \cup B$, $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. For the reverse inclusion, suppose that $x \in \bar{A} \cup \bar{B}$. If $x \in \bar{A}$, then all its neighborhood intersects $A \cup B \supset A$. Hence, $x \in \overline{A \cup B}$. Similarly for the case $x \in \bar{B}$. Therefore, $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Thus, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(c) The previous argument, *mutatis mutandis*, yields the inclusion. Let $X = \mathbb{R}$ and $A_n = [0, 1/n]$. Then, $\overline{\bigcup A_n} = [0, 1]$ and $\bigcup \bar{A}_n = [0, 1)$, which do not coincide. $\square$

8.

Proof.

(a) We show that the equality holds. Since $\overline{A \cap B}$ is the smallest closed set containing $A \cap B$ and clear that $\bar{A} \cap \bar{B}$ is closed set containing $A \cap B$, $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. For the reverse inclusion, suppose that $x \in \bar{A} \cap \bar{B}$, then every neighborhood of x intersects both A and B. Hence, $x \in \overline{A \cap B}$. Thus, $\overline{A \cap B} = \bar{A} \cap \bar{B}$.

(b) The previous argument, *mutatis mutandis*, shows that $\overline{\bigcap A_\alpha} \subset \bigcap \bar{A}_\alpha$. The reverse inclusion does not hold in general. For example, let $X = \mathbb{R}$ and $A_n = (0, 1/n)$. Then $\overline{\bigcap A_n} = \varnothing$ but $\bigcap \bar{A}_n = \{0\}$.

(c) We show that $\overline{A \setminus B} \supset \bar{A} \setminus \bar{B}$.

$$\bar{A} \setminus \bar{B} \subset \bar{A} \setminus B = \bar{A} \cap B^c \subset \bar{A} \cap \overline{(B^c)} = \overline{A \cap B^c} = \overline{A \setminus B}$$

where the second equality comes from part (a). The reverse inclusion does not hold in general. For example, let $X = \mathbb{R}$, $A = [0, 1]$ and $B = (0, 1)$. Then $\overline{A \setminus B} = \{0, 1\}$ but $\bar{A} \setminus \bar{B} = \varnothing$. $\square$

10.

Proof. Let X be a simply ordered set and $\mathcal{T}$ the order topology for X. If X contains only one single point, then, vacuously, $\mathcal{T}$ is Hausdorff. Otherwise, let $a < b$ be two distinct point in X. If $(a, b) \neq \varnothing$, that is, there exists some c with $a < c < b$, then $[-\infty, c)$ and $(c, \infty]$ are two disjoint open sets containing a and b respectively. If $(a, b) = \varnothing$, then $[-\infty, b)\setminus(a, b)$ and $(a, \infty]\setminus(a, b)$ are two such sets. Thus, we conclude that $\mathcal{T}$ is Hausdorff. (The case where the $\pm\infty$ can not be attained is similar.) $\square$

12.

Proof. Let Y be a subspace of the Hausdorff space X. Let a, b be two distinct points of Y. Since X is Hausdorff, there are two disjoint sets U and V which contain a and b respectively and are open in X. Hence, $U \cap Y$ and $V \cap Y$ are two disjoint open sets in Y containing a and b respectively. Thus, Y is also Hausdorff. $\square$

14.

Proof. Let $\mathcal{T}$ be the finite complement topology on $\mathbb{R}$. We show that for the sequence (x_n) converges to every point of $\mathbb{R}$. Let x be an arbitrary point of $\mathbb{R}$ and U an neighborhood of x. Since U is open and nonempty, U^c is finite. Hence, U^c contains at most finitely many points in (x_n). Thus, $x_n \in U$ for all sufficiently large n. Namely, $x_n \to x$. $\square$

21.(b)

Solution. $A = (-\infty, 1) \cup (1, 2] \cup \{3\} \cup ([4, 5] \cap \mathbb{Q}) \cup (6, 7) \cup (7, 8]$. $\square$