

Solutions to
Introductory Functional Analysis with Applications

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2 Normed Spaces. Banach Spaces

2.3 Further Properties of Normed Spaces

4. cf. Prob. 13, Sec 1.2

Proof. The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1| \|x_1 - x_2\| + |\alpha_1 - \alpha_2| \|x_2\|.$$

□

7.

Proof. Let Y and y_n be defined as in the hint. Then $\|y_n\| = 1/n^2$, constituting a convergent number series. However,

$$\sum_{n=1}^N y_n = (1, 1/4, \dots, 1/N^2, 0, \dots),$$

which is divergent as $N \rightarrow \infty$.

□

8.

Proof. Let (x_n) be a Cauchy sequence in X . Hence, for every $n > 0$, there exists some $K_n > 0$ such that for all $p, q > K_n$, $\|x_p - x_q\| < 1/n^2$. Without loss of generality, we may assume that (K_n) is increasing. Since the series $\|x_{K_{n+1}} - x_{K_n}\|$ is bounded by $1/n^2$, it converges. By the hypothesis, the series $(x_{K_{n+1}} - x_{K_n})$ also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1} (x_{K_{i+1}} - x_{K_i}) \rightarrow x \quad \text{as } n \rightarrow \infty.$$

Now we show that (x_n) converges to x . For every $\varepsilon > 0$, since (x_n) is a Cauchy sequence, there exists some N_1 such that for all $p, q > N_1$, $\|x_p - x_q\| < \varepsilon$. Meanwhile, since $x_{K_n} \rightarrow x$, once K_n is large enough, $\|x - x_{K_n}\| < \varepsilon$. Let $K_n > N_1$. Then for every $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus, X is complete.

□

9.

Proof. Let (x_n) be an absolutely convergent series in Banach space X . Let $s_n = \sum_{i=1}^n x_i$. Now we show that s_n is a Cauchy sequence and therefore convergent. Since $\sum_{i=1}^{\infty} \|x_i\| < \infty$, for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n > N$, $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$. Hence, for every $N < p \leq q$,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^q x_i \right\| \leq \sum_{i=p+1}^q \|x_i\| < \varepsilon,$$

completing the proof.

□

10.

Proof. Let (e_n) be Schauder basis of X . Denote the underlying field of X by \mathbb{K} and let $\mathbb{W} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$ if $\mathbb{K} = \mathbb{C}$. Now we show that

$$S = \left\{ \sum_{i=1}^n \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \dots \right\},$$

a countable subset of X , is dense in X to derive the separability.

For every $x \in X$ and $\varepsilon > 0$, by the definition of Schauder basis, there exists $\beta_1, \dots, \beta_n \in \mathbb{K}$ such that $\|x - (\beta_1 e_1 + \dots + \beta_n e_n)\| < \varepsilon$. Let $M = \max_i \|e_i\|$. If $M = 0$, then there is nothing to prove. Otherwise, since \mathbb{W} is dense in \mathbb{K} , for $i = 1, \dots, n$, there exists $\alpha_i \in \mathbb{W}$ with $|\alpha_i - \beta_i| < \varepsilon/2^i M$. Hence,

$$\begin{aligned} \left\| x - \sum_{i=1}^n \alpha_i e_i \right\| &\leq \left\| x - \sum_{i=1}^n \beta_i e_i \right\| + \left\| \sum_{i=1}^n (\beta_i - \alpha_i) e_i \right\| \\ &\leq \varepsilon + \sum_{i=1}^n |\alpha_i - \beta_i| \|e_i\| \\ &\leq 2\varepsilon. \end{aligned}$$

Thus, S is dense in X and therefore X is separable. \square

14.

Proof. Clear that $\|\cdot\|_0$ is nonnegative. And $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$. Meanwhile, $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$. Finally, we show that $\|\hat{x}\|_0 = 0$ implies $\hat{x} = Y$ and invoke Prob. 4, Sec 2.2 to complete the proof. Since $\|\hat{x}\|_0 = 0$, there exists $(x_n) \subset \hat{x}$ which converges to 0. Since Y is closed, Y is complete and so is its cosets. Therefore, $0 \in \hat{x}$, enforcing \hat{x} to be Y . \square

2.4 Finite Dimensional Normed Spaces

3.

Proof. The reflexive property clearly holds. If there are positive a and b such that $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$ for all $x \in X$, then $\|x\|_1/b \leq \|x\|_0 \leq \|x\|/a$. Hence the relation is symmetric. Next we further suppose there exists positive c and d such that that $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$. Then $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$, giving the transitive property. Thus, the axioms of an equivalence relation hold. \square

4.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Let $E \subset X$ be any open set with respect to $\|\cdot\|$, i.e., for every $x_0 \in E$, there exists some $\delta > 0$ such that $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$. Since $\|\cdot\| \sim \|\cdot\|_0$, there exists some positive c such that $\|x - x_0\| \leq c\|x - x_0\|_0$. Hence, $B = \{x \in X : \|x - x_0\| < \delta/c\} \subset A \subset E$. Namely, E is also open with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. \square

5.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Then for every $x \in X$, there exists some $c > 0$ such that $\|x\|_0 \leq c\|x\|$. Let (x_n) be a Cauchy sequence with respect to $\|\cdot\|$, i.e., for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n, m > N$, $\|x_n - x_m\| < \varepsilon/c$. Hence, $\|x_n - x_m\|_0 < c\|x_n - x_m\| \leq \varepsilon$. Thus, (x_n) is also a Cauchy with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. \square

2.5 Compactness and Finite Dimension

5.

Proof. Clear that every point in \mathbb{R}^n or \mathbb{C}^n has a closed bounded, and therefore compact, neighborhood. Hence, \mathbb{R}^n and \mathbb{C}^n are locally compact. \square

6.

Proof. Let X be a compact metric space and x any point in X . Let E be a closed neighborhood of x . By Prob 10, E is compact. Thus, X is locally compact. \square

7.

Proof. It suffices to show that $a = \inf_{y \in Y} \|v - y\|$ can actually be obtained. Let $\{b_1, \dots, b_n\}$ be a basis of Y and $y_k = y_{k,1}b_1 + \dots + y_{k,n}b_n$ a sequence in Y with $\|v - y_k\| \rightarrow a$. We may assume without loss of generality that $\|v - y_k\|$ is bounded.

Since Y is a proper subset of Z , v, b_1, \dots, b_n are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar $c > 0$ such that for every k ,

$$\|v - y_{k,1}b_1 - \dots - y_{k,n}b_n\| \geq c(1 + |y_{k,1}| + \dots + |y_{k,n}|).$$

Hence, the sequence $(y_{k,1}, \dots, y_{k,n})$ of n -tuples is bounded and therefore has a convergent subsequence. Consequently, (y_k) also has a convergent subsequence. Suppose that it converges to $z \in Z$. Note that $\|v - z\| = a$ and as Y is closed, $z \in Y$. Thus, a can be attained in Y . \square

8.

Proof. Since the unit ball B with respect to $\|\cdot\|_2$ in \mathbb{R}^n and \mathbb{C}^n is compact and $\|\cdot\|$ is continuous, by 2.5-7, $x \mapsto \|x\|$ can attain its minimum, denoted by a , on B . Due to the positive definite property of a norm, a is positive. Hence, $0 < a \leq \|x\|_2$. Namely, $a\|x\|_2 \leq \|x\|$. \square

9.

Proof. For every $(x_n) \subset M \subset X$, since X is compact, there exists a subsequence (x_{n_k}) of (x_n) which converges to some $y \in X$. Since M is closed, $y \in M$. Hence, M is compact. \square

10.

Proof. From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of T is also continuous, it suffices to show that the image of a closed set $A \subset X$ under T is again a closed set. Since X is compact and A is closed, A is compact. Since T is continuous, by 2.5-6, $T(A)$ is compact and therefore closed. Hence, T is a homeomorphism. \square

2.7 Bounded and Continuous Linear Operators

2.

Proof. First suppose T to be bounded and let A be any bounded set in X . Then there exists $K < \infty$ such that for all $x \in A$, $\|x\| < K$. Due to the boundedness of T , $\|Tx\| \leq \|T\|\|x\| < K\|T\|$. Namely, $T(A)$ is also bounded.

Now suppose that T maps bounded sets in X into bounded sets in Y . Clear that the unit ball B of X is bounded and therefore so is $T(B)$. Namely, $\|Tx/\|x\|\|$ is bounded for $x \neq 0$.¹ Hence, T is bounded. \square

3.

Proof. For every x with $\|x\| < 1$, $\|Tx\| \leq \|T\|\|x\| < \|T\|$. \square

4.

Proof. Suppose that the linear operator T is continuous at $x_0 \in \mathcal{D}(T)$. For every $(x_n) \subset \mathcal{D}(T)$ with $\|x_n - x_0\| \rightarrow 0$, by the continuity of T at x_0

$$\|Tx_n - Tx_0\| = \|T(x_n - x_0 + x_0) - Tx_0\| \rightarrow 0.$$

Hence, T is continuous. \square

7.

Proof. The inequality implies $\mathcal{N}(T) = 0$. Hence, by Theorem 2.6-10, T^{-1} exists. For every $y \in Y$, suppose that $y = Tx$. Then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

Thus, T^{-1} is bounded. \square

12.

Proof. The compatibility follows immediately from the definition of the supremum. Suppose $\|x\|_1 = \max_j |\xi_j|$ and $\|y\|_2 = \max_j \|\eta_j\|$, then

$$Ax = \begin{bmatrix} x_1\alpha_{11} + \cdots + x_n\alpha_{1n} \\ \vdots \\ x_1\alpha_{r1} + \cdots + x_n\alpha_{rn} \end{bmatrix}$$

¹Note that the two $\|\cdot\|$ here are different norms.

Since for all j , $x_j \leq \|x_j\|_1$,

$$\frac{\max_j |x_1\alpha_{j1} + \cdots + x_n\alpha_{jn}|}{\|x\|_1} = \max_j \left| \frac{x_1}{\|x\|_1}\alpha_{j1} + \cdots + \frac{x_n}{\|x\|_1}\alpha_{jn} \right| \leq \max_j \sum_{k=1}^n |\alpha_{jk}|.$$

Hence,

$$\|A\| \geq \frac{\|Ax\|_2}{\|x\|_1} \quad \text{for all } x. \quad (1)$$

Suppose that maximum of $\sum_{k=1}^n |\alpha_{jk}|$ is obtained at $j = p$. Then choosing x_k to be $\text{sgn } \alpha_{pk}$ shows that the equality in (1) can actually be attained. Hence, $\|A\| = \max_j \sum_{k=1}^n |\alpha_{jk}|$. \square

2.8 Linear Functionals

8.

Proof. For every $x_1, x_2 \in N(M^*)$, $a, b \in \mathbb{K}$ and $f \in M^*$,

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) = 0.$$

Hence, $ax_1 + bx_2 \in N(M^*)$. Namely, $N(M^*)$ is a vector space. \square

9.

Proof. First we show the uniqueness. Suppose that $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$. Then $0 = (\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)$. Hence,

$$0 = f((\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)) = (\alpha_1 - \alpha_2)f(x_0) + f(y_1) - f(y_2).$$

Since $y_1, y_2 \in \mathcal{N}(f)$, $f(y_1) - f(y_2) = 0$ while $f(x_0) \neq 0$ as $x_0 \notin \mathcal{N}(f)$. Hence, $\alpha_1 = \alpha_2$, which forces y_1 and y_2 to coincide.

For the existence, it suffices to show that for any fixed x , the function $g(\alpha) = f(x - \alpha x_0)$ has a zero. It is easy to verify that $\alpha = f(x)/f(x_0)$ is a zero of g . Note that $x_0 \notin \mathcal{N}(f)$ and therefore $f(x_0) \neq 0$. \square

10.

Proof. First we suppose that $x_1, x_2 \in x_0 + \mathcal{N}(f) \in X/\mathcal{N}(f)$. Then together with Prob. 9, $x_i = x_0 + y_i$ where $y_i \in \mathcal{N}(f)$. Hence, for $i = 1, 2$, $f(x_i) = f(x_0) + f(y_i) = f(x_0)$.

For the converse, note that $f(x_1) = f(x_2)$ implies $f(x_1 - x_2) = 0$. Namely, $x_1 - x_2 \in \mathcal{N}(f)$. Hence, x_1, x_2 belongs to the same element in $X/\mathcal{N}(f)$.

To show $\text{codim } \mathcal{N}(f) = 1$, we show that $X/\mathcal{N}(f)$ and \mathbb{K} are isomorphic. For every $\hat{x} \in X/\mathcal{N}(f)$, define $I(\hat{x}) = f(x)$. By the previous discussion, this definition is well-defined. Clear that I is linear and therefore is injective. And by the linearity of f , I is surjective. Thus, I is an isomorphism between $X/\mathcal{N}(f)$ and \mathbb{K} . Hence, $\text{codim } \mathcal{N}(f) = 1$. \square

11.

Proof. Put $N = \mathcal{N}(f_1) = \mathcal{N}(f_2)$ and choose $x_0 \in X \setminus N$. By Prob. 9, for every $x \notin N$, $x = \alpha x_0 + y$ where $y \in N$ and $\alpha \neq 0$. Hence,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha f_1(x_0) + f_1(y)}{\alpha f_2(x_0) + f_2(y)} = \frac{f_1(x_0)}{f_2(x_0)}.$$

\square

12.

Proof. Prob. 10, justifies the discussion on hyperplanes parallel to the $\mathcal{N}(f)$. It suffices to show that $H_1 = b + \mathcal{N}(f)$ for some $b \in X$. Choose $x_1 \in H_1$. Then

$$x \in \mathcal{N}(f) \Leftrightarrow x + x_1 \in x_1 + \mathcal{N}(f) \Leftrightarrow f(x + x_1) = f(x) + f(x_1) = 1 \Leftrightarrow x + x_1 \in H_1.$$

Hence, $H_1 = x_1 + \mathcal{N}(f)$. Namely, H_1 is a hyperplane parallel to $\mathcal{N}(f)$. \square

13.

Proof. We argue by contradiction. Assume that there exists a $y_1 \in Y$ such that $f(y_1) \neq c \neq 0$. Then for every $d \in \mathbb{K}$, by the linearity of f , $f(dy_1/c) = d$. Contradiction. Hence, $f = 0$ on Y . \square

14.

Proof. For every $\varepsilon > 0$, there exists $x_1 \in X$ with $f(x_1) = 1$ such that $\tilde{d} + \varepsilon \geq \|x_1\|$. Hence,

$$\|f\|(\tilde{d} + \varepsilon) \geq \|f\|\|x_1\| \geq |f(x_1)| = 1.$$

Since the choice of $\varepsilon > 0$ is arbitrary, $\|f\|\tilde{d} \geq 1$. Meanwhile, there exists $x_2 \in X$ with $\|x_2\| = 1$ such that $|f(x_2)| \geq \|f\| - \varepsilon$. Put $x_3 = x_2/f(x_2)$. Then $f(x_3) = 1$. Hence,

$$(\|f\| - \varepsilon)\tilde{d} \leq |f(x_2)|\|x_3\| = \|x_2\| = 1,$$

which implies $\|f\|\tilde{d} \leq 1$. Thus, $\|f\|\tilde{d} = 1$. \square

15.

Proof. For every x with $\|x\| \leq 1$, $f(x) \leq \|f\|\|x\| \leq c$. Hence, $x \in X_{c_1}$. Meanwhile, for every $\varepsilon > 0$, by the definition of the supremum, there exists a x with $\|x\| = 1$ such that $|f(x)| > \|f\| - \varepsilon$. By the linearity of f , we may remove the $|\cdot|$ on the right side. Hence, $f(x) \notin X_{c_1}$ where $c = \|f\| - \varepsilon$. \square

2.9 Operators on Finite Dimensional Spaces

8.

Proof. Let $\{b_2, \dots, b_n\}$ be a basis of Z and $\{b_1, \dots, b_n\}$ a basis of X . Define $f \in X^*$ to be $f(b_i) = \delta_{1i}$. Clear that $\mathcal{N}(f) = Z$. By Prob. 11, Sec 2.8, f is uniquely determined up to a scalar multiple. \square

12.

Proof. Let $\varphi : X \rightarrow \mathbb{K}^p$ be defined by $x \mapsto [f_1(x), \dots, f_p(x)]^T$. It can be verified that φ is a linear operator. Since $\dim X = n > p$, φ can not be injective. Hence, there exists $0 \neq x \in X$ such that $\varphi(x) = 0$. \square

13.

Proof. Let $\{b_1, \dots, b_m\}$ be a basis of Z and $\{b_1, \dots, b_n\}$ a basis of X . Define $\tilde{f} \in X^*$ to be identical with f on b_1, \dots, b_m and 0 on b_{m+1}, \dots, b_n . Clear that $\tilde{f}|_Z = f$. \square

2.10 Normed Spaces of Operators. Dual Space

8.

Proof. First we construct a linear bijection T between c'_0 and l^1 . A Schauder basis for c_0 is (e_k) , where $e_k = (\delta_{kj})$. Then for every $f \in c'_0$, define $Tf = (\gamma_k) = (f(e_k))$. Clear that T is linear. Now we show that $Tf = (\gamma_k) \in l^1$, that is, $\sum_{k=1}^n |\gamma_k|$ is bounded and therefore convergent. Define $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \begin{cases} \text{sgn } \gamma_k, & k \leq n, \\ 0, & k > n. \end{cases}$$

Clear that $x_n \in c_0$. By the linearity and boundedness of f ,

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^n |\gamma_k|. \quad (2)$$

Since f is bounded, $|f(x_n)| \leq \|f\| \|x_n\| \leq \|f\|$. Hence, $\sum \|\gamma_k\|$ is bounded. Thus, $Tf \in l^1$.

Meanwhile, for every $y = (\beta_k) \in l^1$, define $Sy = g$ to be the functional $g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$ for $x = (\xi_k)$. On c_0 , the summation does converge and clear that g is linear and bounded. Hence, $g \in c'_0$. It can be verify that $ST = TS = I$ and T is linear. Thus, c'_0 and l^1 is isomorphic.

Now we show that T constructed preserve the norm to complete the proof. For $x \in c_0$ with $\|x\| = 1$,

$$|f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \gamma_k \right| \leq \sum_{k=1}^{\infty} |\gamma_k| = \|Tf\|.$$

Hence, $\|f\| \leq \|Tf\|$. And (2) implies $\sum_{k=1}^n \|\gamma_k\| \leq \|f\|$. Letting $n \rightarrow \infty$ yields $\|Tf\| \leq \|f\|$. Thus, $\|Tf\| = \|f\|$. \square

9.

Proof. Let (b_k) be a Hamel basis of X and suppose that $f, g \in X^*$ coincide on every b_k . Then for every $x = \sum_{k=1}^{\infty} \xi_k b_k \in X$,

$$f(x) - g(x) = \sum_{k=1}^n \xi_k (f(b_k) - g(b_k)) = 0.$$

Thus, $f = g$. Namely, f is uniquely determined. \square

10.

Proof. Let (b_k) be a Hamel basis of X and without loss of generality we may assume $\|b_k\| = 1$. Justified by Prob. 9, we can define $T \in X^*$ with $Tb_k = k$, which is clearly unbounded. \square

11.

Proof. It follows immediately from Prob. 10. \square

13.

Proof. For any $f, g \in M^a$ and scalar a, b , $(af + bg)(x) = af(x) + bg(x) = 0$ for every $x \in M$. Hence, M^a is a vector space. For $(f_n) \subset M^a \subset X'$, suppose that $f_n \rightarrow f \in M^*$. Since M' is complete, it is closed and therefore $f \in M'$. For every $0 \neq x \in M$, since $f_n \rightarrow f$,

$$\frac{|f_n(x) - f(x)|}{\|x\|} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Hence, $f(x) = 0$. Thus, M^a is closed.

$$X^a = \{0\} \text{ and } \{0\}^a = X'.$$

□

14.

Proof. Let $\{b_1, \dots, b_m\}$ be a basis of M and $\{b_1, \dots, b_n\}$ a basis of X . And let $\{\beta_1, \dots, \beta_n\}$ be the dual basis. Clear that $b_1, \dots, b_m \notin M^a$ whereas b_{m+1}, \dots, b_n does. Together with Prob. 13, this implies $M^a = \text{span}(b_{m+1}, \dots, b_n)$. Thus, $\dim M^a = n - m$. □

3 Inner Product Spaces. Hilbert Spaces

3.1 Inner Product Spaces. Hilbert Spaces

2.

Proof.

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle = \|x\|^2 + \|y\|^2,$$

where the last equality comes from the hypothesis of orthogonality. Now we show that for mutually orthogonal x_1, \dots, x_m

$$\left\| \sum_{i=1}^m x_i \right\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

by induction on m . The case where $m = 2$ has already been showed and we assume that the equation holds for $m - 1$. Since x_m is orthogonal with each $i = 1, \dots, m - 1$, x_m is orthogonal to $x_1 + \dots + x_{m-1}$. Hence,

$$\left\| \sum_{i=1}^m x_i \right\|^2 = \left\| \sum_{i=1}^{m-1} x_i \right\|^2 + \|x_m\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

completing the proof. □

3.

Proof. The equation implies $\langle x, y \rangle + \langle y, x \rangle = 0$. The symmetric property of real inner products implies $\langle x, y \rangle = 0$. Let $X = \mathbb{C}$ and $x = 1, y = i$. It is easy to verify that $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = 2$ but x and y are not orthogonal. □

7.

Proof. It suffices to show that the zero vector is the only vector orthogonal to all vectors. Suppose that $\langle x_0, x \rangle = 0$ for all $x \in X$, then $\|x_0\|^2 = \langle x_0, x_0 \rangle = 0$. By the definiteness of the inner product, $x_0 = 0$. □

8. We show that any norm satisfying the parallelogram equality can be derived from an inner product.

Proof. The proof of (IP3) is trivial and (IP4) follows immediately from the positive-definiteness of the norm. Hence we only show the linearity in the first factor here. For every $u, v, y \in X$, from the parallelogram equality we can derive, after some computation, that

$$\begin{aligned} 4\langle u + v, y \rangle &= \|u + v + y\|^2 - \|u + v - y\|^2 \\ &= \|u + y\|^2 - \|u - y\|^2 + \|v + y\|^2 - \|v - y\|^2 \\ &= 4\langle u, y \rangle + 4\langle v, y \rangle. \end{aligned}$$

Namely, (IP1) holds. By induction we can show that $\langle nu, y \rangle = n\langle u, y \rangle$ for $n = 1, 2, \dots$. And since $\langle -u, y \rangle = \langle 0 - u, y \rangle = \langle 0, y \rangle - \langle u, y \rangle = -\langle u, y \rangle$,

$$\langle nu, y \rangle = n\langle u, y \rangle, \quad \text{for } n \in \mathbb{Z}.$$

Furthermore, for any positive integer m ,

$$m \left\langle \frac{n}{m}u, y \right\rangle = mn \left\langle \frac{1}{m}u, y \right\rangle = n\langle u, y \rangle.$$

Dividing the both sides by m yields

$$\langle qu, y \rangle = q\langle u, y \rangle, \quad \text{for } q \in \mathbb{Q}.$$

For every $\alpha \in \mathbb{R}$, let $(q_n) \subset \mathbb{Q}$ converges to α . Now we show that $f(t) = \langle tu, y \rangle$ is continuous at $t = 0$ and by the additivity we may conclude that f is continuous on \mathbb{R} . Since

$$\begin{aligned} 4|f(t)| &= ||tu + y||^2 - ||tu - y||^2 \\ &= (||tu + y|| + ||tu - y||)||tu + y|| - ||tu - y|| \\ &\leq 4t||u||(|t||u|| + ||y||) \rightarrow 0 \end{aligned}$$

as $t \rightarrow 0$, $f(t)$ is continuous. For every $\alpha \in \mathbb{R}$, let $(q_n) \subset \mathbb{Q}$ be a convergent sequence with limit α . Then

$$\langle \alpha u, y \rangle = \lim \langle q_n u, y \rangle = \lim q_n \langle u, y \rangle = \alpha \langle u, y \rangle.$$

Hence, $\langle \cdot, \cdot \rangle$ is linear in the first factor. Thus, it is an inner product. Meanwhile, it is easy to verify that the norm it introduces is exactly the original norm. \square

3.2 Further Properties of Inner Product Spaces

7.

Proof. First we note that

$$f(\alpha) = ||x + \alpha y||^2 - ||x - \alpha y||^2 = 2\bar{\alpha}\langle x, y \rangle + 2\alpha\langle y, x \rangle.$$

Clear that $x \perp y$ implies $f(\alpha) = 0$ for all scalar α . For the converse, we suppose $f(\alpha) = 0$ and put $\alpha = \langle x, y \rangle$. Then $0 = f(\alpha) = 2|\langle x, y \rangle|^2$. Thus, $x \perp y$. \square

8.

Proof. Clear that $x \perp y$ implies $||x + \alpha y|| \geq ||x||$. Therefore we only show the converse here. Without loss of generality, we assume $||y|| = 1$. Then $||x + \alpha y|| \geq ||x||$ for all scalar α implies

$$|\alpha|^2 + \bar{\alpha}\langle x, y \rangle + \alpha\overline{\langle x, y \rangle} \geq 0.$$

Put $\alpha = -\langle x, y \rangle$ and we get

$$0 \leq |\langle x, y \rangle|^2 - 2|\langle x, y \rangle|^2 = -|\langle x, y \rangle|^2,$$

which implies $\langle x, y \rangle = 0$. Namely, $x \perp y$. \square

9.

Proof. For every $\varepsilon > 0$, put $\delta = \varepsilon/\sqrt{b-a}$. Then for every $x_1, x_2 \in V$ with $\|x_1 - x_2\|_\infty < \delta$,

$$\|x_1 - x_2\|_2^2 = \int_a^b |x_1(t) - x_2(t)|^2 dt \leq (b-a)\delta^2 = \varepsilon^2.$$

Hence, $x \mapsto x$ is continuous. □

10.

Proof. For every $u, w \in X$,

$$\begin{aligned} \langle Tu, w \rangle &= \frac{1}{4} (\langle T(u+w), u+w \rangle - \langle T(u-w), u-w \rangle) \\ &\quad + \frac{i}{4} (\langle T(u+iw), u+iw \rangle - \langle T(u-iw), u-iw \rangle). \end{aligned}$$

Note that each component of the right hand side is of form $\langle Tx, x \rangle$ and hence equals to 0. Putting $w = Tu$ yields $\|Tu\|^2 = 0$ for all $u \in X$. Thus, $T = 0$. □

3.3 Orthogonal Complements and Direct Sums

7.

Proof.

(a) $x \in A^{\perp\perp}$ iff for all $y \in A^\perp$, $\langle x, y \rangle = 0$. By the definition of A^\perp , the identity holds if $x \in A$. Hence, $A \subset A^{\perp\perp}$.

(b) For all $x \in B^\perp$ and $y \in A \subset B$, $\langle x, y \rangle = 0$ by definition. Hence, $x \in A^\perp$. Namely, $B^\perp \subset A^\perp$.

(c) We show that A^\perp is closed (no matter whether A is or not) and invoke Lemma 3.3-6 to complete the proof. Suppose that $(x_n) \subset A^\perp$ converges to x . For all $y \in A$, $\langle x_n, y \rangle = 0$. By the continuity of the inner product, $\langle x, y \rangle = 0$ and therefore $x \in A^\perp$. Hence, A^\perp is closed. Thus, $A^\perp = A^{\perp\perp\perp}$. □

8.

Proof. We have show this in Prob. 7. □

9.

Proof. It has been shown in Lemma 3.3-6 that the closedness of Y implies $Y = Y^{\perp\perp}$. Hence we only show the converse here. For every convergent $(x_n) \subset Y$, $(x_n) \subset Y^{\perp\perp}$. Since $Y^{\perp\perp}$ is closed by Prob. 8, the limit x of (x_n) belongs to $Y^{\perp\perp}$ and hence belongs to Y . Thus, Y is closed. □

10. TODO

3.4 Orthonormal Sets and Sequences

3.

Proof. The situation where x and y are linearly dependent is obvious and hence we assume they are linearly independent here. By the homogeneity of the Schwarz inequality, we may assume without loss of generality that $\|x\| = \|y\| = 1$. Put $z = (y - x\langle y, x \rangle) / \|y - x\langle y, x \rangle\|$. Then $\{x, z\}$ is orthonormal and therefore by (12*)

$$|\langle y, x \rangle|^2 + |\langle y, z \rangle|^2 \leq \|y\|^2 = 1.$$

Since $|\langle y, z \rangle|^2$ is nonnegative, this implies $|\langle x, y \rangle|^2 \leq 1$, the Schwarz inequality. \square

7.

Proof. For each positive integer n , by the Schwarz inequality and (12*),

$$\sum_{k=1}^n |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \sqrt{\sum_{i=1}^n |\langle x, e_i \rangle|^2} \sqrt{\sum_{i=1}^n |\langle y, e_i \rangle|^2} \leq \|x\| \|y\|.$$

Since all terms in the summation is nonnegative, this implies $\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \|x\| \|y\|$. \square

8.

Proof. It follows immediately from Bessel inequality. \square

3.5 Series Related to Orthonormal Sequences

1.

Proof. By Theorem 3.5-2, $\alpha_k = \langle x, e_k \rangle$. Meanwhile by the definition of the norm,

$$\|x\|^2 = \left\langle \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, x \right\rangle = \sum_{k=1}^{\infty} \langle x, e_k \rangle \langle e_k, x \rangle = \sum_{k=1}^{\infty} |\alpha_k|^2,$$

where the second equality follows from the continuity of the inner product. \square

3.

Solution. Put $x \equiv 1$ on $[-\pi, \pi]$ and $e_k = \sin kt$. Since x is even but e_k is odd for every k , the series does not converges to x . \square

4.

Proof. By the triangle inequality, $\|x_m + \cdots + x_n\| \leq \|x_m\| + \cdots + \|x_n\|$ for every $n \geq m > 0$. Hence the convergence of $\sum \|x_k\|$ implies that s_n is a Cauchy sequence. \square

5.

Proof. By Prob. 4, $\sum_{k=1}^n x_k$ is a Cauchy sequence. And since H is complete, $\sum_{k=1}^{\infty} x_k$ converges. \square

7.

Proof. The existence of y follows from Theorem 3.5-2(c). And for each k ,

$$\langle x - y, e_k \rangle = \langle x, e_k \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle e_k, e_j \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0,$$

where the second equality comes from the fact that (e_k) is orthonormal. \square

8. TODO: Show the validation of the change of the order of summation. Or maybe we can show the equality directly.

Proof. We suppose that $x \in \bar{M}$ here since the proof of the other direction is obvious. Then there exists $(p_n) \subset M$ such that $x = \sum_{n=1}^{\infty} p_n$. For each n , suppose $p_n = \sum_{k=1}^{\infty} \langle p_n, e_k \rangle e_k$. This is valid because $p_n \in M$ and therefore is a finite linear combination of (e_k) . In fact, there are only finitely many nonzero term in the summation. Then

$$x = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle p_n, e_k \rangle e_k = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \langle p_n, e_k \rangle \right) e_k.$$

\square

9.

Proof. First we suppose $\bar{M}_1 = \bar{M}_2$. Then by Prob. 8, each e_n and \tilde{e}_n can be represented by (a) and (b) respectively.

For the converse, (a) implies, again by Prob. 8, $e_n \in \bar{M}_2$ and therefore $M_1 \subset \bar{M}_2$. Since \bar{M}_2 is closed, $\bar{M}_1 \subset \bar{M}_2$. *Mutatis mutandis*, this also shows $\bar{M}_2 \subset \bar{M}_1$. Thus, $\bar{M}_1 = \bar{M}_2$. \square

10.

Proof. Note that for every $m > 0$, there are only finite e_κ such that $\langle x, e_\kappa \rangle \geq 1/m$. Otherwise we may choose a countable subset of them, which will violate the result in Prob. 8, Sec 3.4. Hence, the collection of all nonzero Fourier coefficient

$$\bigcup_{m=1}^{\infty} \{e_\kappa : \langle x, e_\kappa \rangle \geq 1/m\}$$

is at most countable. \square

3.6 Total Orthonormal Sets and Sequences

4.

Proof. Suppose that x and y satisfy (3). We only show the relation for real cases here. The complex cases can be proved in a similar way. Using (9), Sec 3.1 and (3),

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) = \frac{1}{4} \sum_k (|\langle x + y, e_k \rangle|^2 - |\langle x - y, e_k \rangle|^2).$$

Meanwhile,

$$|\langle x \pm y, e_k \rangle|^2 = \langle x \pm y, e_k \rangle \overline{\langle x \pm y, e_k \rangle} = |\langle x, e_k \rangle|^2 + |\langle y, e_k \rangle|^2 \pm 2 \langle x, e_k \rangle \overline{\langle y, e_k \rangle}.$$

Hence, $\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$. \square

6.

Proof. Suppose $M = (e_k)$. We collect the e_k which does not belong to $\text{span}(e_1, \dots, e_{k-1})$ and denote the new sequence by (\tilde{e}_k) . Clear that $\text{span}(e_k) = \text{span}(\tilde{e}_k)$ and (\tilde{e}_k) is linearly independent. Let (f_k) be the sequence generated from (\tilde{e}_k) by the Gram-Schmidt process. Then clear that (f_k) is orthonormal. And since for every n , $\text{span}(\tilde{e}_1, \dots, \tilde{e}_n) = \text{span}(f_1, \dots, f_n)$, $M \subset \text{span}(\tilde{e}_k) = \text{span}(f_k)$. Finally, since M is dense in H , $\text{span}(f_k) = H$. Thus, (f_k) is a total orthonormal sequence of H . \square

7.

Proof. It follows from the definition of the separable Hilbert space and Prob. 6 immediately. \square

9.

Proof. $\langle v, x \rangle = \langle w, x \rangle$ implies $\langle v - w, x \rangle = 0$ for all $x \in M$, that is, $x \perp M$. Since M is total, by Theorem 3.6-2, $v - w = 0$. \square

10.

Proof. It follows immediately from Theorem 3.6-2(b). \square