

Solutions to  
*Introductory Functional Analysis with Applications*

Yunwei Ren

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## 2 Normed Spaces. Banach Spaces

### 2.3 Further Properties of Normed Spaces

4. cf. Prob. 13, Sec 1.2

*Proof.* The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1| \|x_1 - x_2\| + |\alpha_1 - \alpha_2| \|x_2\|.$$

□

7.

*Proof.* Let  $Y$  and  $y_n$  be defined as in the hint. Then  $\|y_n\| = 1/n^2$ , constituting a convergent number series. However,

$$\sum_{n=1}^N y_n = (1, 1/4, \dots, 1/N^2, 0, \dots),$$

which is divergent as  $N \rightarrow \infty$ .

□

8.

*Proof.* Let  $(x_n)$  be a Cauchy sequence in  $X$ . Hence, for every  $n > 0$ , there exists some  $K_n > 0$  such that for all  $p, q > K_n$ ,  $\|x_p - x_q\| < 1/n^2$ . Without loss of generality, we may assume that  $(K_n)$  is increasing. Since the series  $\|x_{K_{n+1}} - x_{K_n}\|$  is bounded by  $1/n^2$ , it converges. By the hypothesis, the series  $(x_{K_{n+1}} - x_{K_n})$  also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1} (x_{K_{i+1}} - x_{K_i}) \rightarrow x \quad \text{as } n \rightarrow \infty.$$

Now we show that  $(x_n)$  converges to  $x$ . For every  $\varepsilon > 0$ , since  $(x_n)$  is a Cauchy sequence, there exists some  $N_1$  such that for all  $p, q > N_1$ ,  $\|x_p - x_q\| < \varepsilon$ . Meanwhile, since  $x_{K_n} \rightarrow x$ , once  $K_n$  is large enough,  $\|x - x_{K_n}\| < \varepsilon$ . Let  $K_n > N_1$ . Then for every  $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus,  $X$  is complete.

□

9.

*Proof.* Let  $(x_n)$  be an absolutely convergent series in Banach space  $X$ . Let  $s_n = \sum_{i=1}^n x_i$ . Now we show that  $s_n$  is a Cauchy sequence and therefore convergent. Since  $\sum_{i=1}^{\infty} \|x_i\| < \infty$ , for every  $\varepsilon > 0$ , there exists some  $N > 0$  such that for all  $n > N$ ,  $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$ . Hence, for every  $N < p \leq q$ ,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^q x_i \right\| \leq \sum_{i=p+1}^q \|x_i\| < \varepsilon,$$

completing the proof.

□

10.

*Proof.* Let  $(e_n)$  be Schauder basis of  $X$ . Denote the underlying field of  $X$  by  $\mathbb{K}$  and let  $\mathbb{W} = \mathbb{Q}$  if  $\mathbb{K} = \mathbb{R}$  and  $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$  if  $\mathbb{K} = \mathbb{C}$ . Now we show that

$$S = \left\{ \sum_{i=1}^n \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \dots \right\},$$

a countable subset of  $X$ , is dense in  $X$  to derive the separability.

For every  $x \in X$  and  $\varepsilon > 0$ , by the definition of Schauder basis, there exists  $\beta_1, \dots, \beta_n \in \mathbb{K}$  such that  $\|x - (\beta_1 e_1 + \dots + \beta_n e_n)\| < \varepsilon$ . Let  $M = \max_i \|e_i\|$ . If  $M = 0$ , then there is nothing to prove. Otherwise, since  $\mathbb{W}$  is dense in  $\mathbb{K}$ , for  $i = 1, \dots, n$ , there exists  $\alpha_i \in \mathbb{W}$  with  $|\alpha_i - \beta_i| < \varepsilon/2^i M$ . Hence,

$$\begin{aligned} \left\| x - \sum_{i=1}^n \alpha_i e_i \right\| &\leq \left\| x - \sum_{i=1}^n \beta_i e_i \right\| + \left\| \sum_{i=1}^n (\beta_i - \alpha_i) e_i \right\| \\ &\leq \varepsilon + \sum_{i=1}^n |\alpha_i - \beta_i| \|e_i\| \\ &\leq 2\varepsilon. \end{aligned}$$

Thus,  $S$  is dense in  $X$  and therefore  $X$  is separable.  $\square$

14.

*Proof.* Clear that  $\|\cdot\|_0$  is nonnegative. And  $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$ . Meanwhile,  $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$ . Finally, we show that  $\|\hat{x}\|_0 = 0$  implies  $\hat{x} = Y$  and invoke Prob. 4, Sec 2.2 to complete the proof. Since  $\|\hat{x}\|_0 = 0$ , there exists  $(x_n) \subset \hat{x}$  which converges to 0. Since  $Y$  is closed,  $Y$  is complete and so is its cosets. Therefore,  $0 \in \hat{x}$ , enforcing  $\hat{x}$  to be  $Y$ .  $\square$

## 2.4 Finite Dimensional Normed Spaces

3.

*Proof.* The reflexive property clearly holds. If there are positive  $a$  and  $b$  such that  $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$  for all  $x \in X$ , then  $\|x\|_1/b \leq \|x\|_0 \leq \|x\|/a$ . Hence the relation is symmetric. Next we further suppose there exists positive  $c$  and  $d$  such that that  $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$ . Then  $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$ , giving the transitive property. Thus, the axioms of an equivalence relation hold.  $\square$

4.

*Proof.* Suppose the norms  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent. Let  $E \subset X$  be any open set with respect to  $\|\cdot\|$ , i.e., for every  $x_0 \in E$ , there exists some  $\delta > 0$  such that  $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$ . Since  $\|\cdot\| \sim \|\cdot\|_0$ , there exists some positive  $c$  such that  $\|x - x_0\| \leq c\|x - x_0\|_0$ . Hence,  $B = \{x \in X : \|x - x_0\| < \delta/c\} \subset A \subset E$ . Namely,  $E$  is also open with respect to  $\|\cdot\|_0$ . Interchanging the roles of  $\|\cdot\|$  and  $\|\cdot\|_0$  completes the proof.  $\square$

5.

*Proof.* Suppose the norms  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent. Then for every  $x \in X$ , there exists some  $c > 0$  such that  $\|x\|_0 \leq c\|x\|$ . Let  $(x_n)$  be a Cauchy sequence with respect to  $\|\cdot\|$ , i.e., for every  $\varepsilon > 0$ , there exists some  $N > 0$  such that for all  $n, m > N$ ,  $\|x_n - x_m\| < \varepsilon/c$ . Hence,  $\|x_n - x_m\|_0 < c\|x_n - x_m\| \leq \varepsilon$ . Thus,  $(x_n)$  is also a Cauchy with respect to  $\|\cdot\|_0$ . Interchanging the roles of  $\|\cdot\|$  and  $\|\cdot\|_0$  completes the proof.  $\square$

## 2.5 Compactness and Finite Dimension

5.

*Proof.* Clear that every point in  $\mathbb{R}^n$  or  $\mathbb{C}^n$  has a closed bounded, and therefore compact, neighborhood. Hence,  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are locally compact.  $\square$

6.

*Proof.* Let  $X$  be a compact metric space and  $x$  any point in  $X$ . Let  $E$  be a closed neighborhood of  $x$ . By Prob 10,  $E$  is compact. Thus,  $X$  is locally compact.  $\square$

7.

*Proof.* It suffices to show that  $a = \inf_{y \in Y} \|v - y\|$  can actually be obtained. Let  $\{b_1, \dots, b_n\}$  be a basis of  $Y$  and  $y_k = y_{k,1}b_1 + \dots + y_{k,n}b_n$  a sequence in  $Y$  with  $\|v - y_k\| \rightarrow a$ . We may assume without loss of generality that  $\|v - y_k\|$  is bounded.

Since  $Y$  is a proper subset of  $Z$ ,  $v, b_1, \dots, b_n$  are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar  $c > 0$  such that for every  $k$ ,

$$\|v - y_{k,1}b_1 - \dots - y_{k,n}b_n\| \geq c(1 + |y_{k,1}| + \dots + |y_{k,n}|).$$

Hence, the sequence  $(y_{k,1}, \dots, y_{k,n})$  of  $n$ -tuples is bounded and therefore has a convergent subsequence. Consequently,  $(y_k)$  also has a convergent subsequence. Suppose that it converges to  $z \in Z$ . Note that  $\|v - z\| = a$  and as  $Y$  is closed,  $z \in Y$ . Thus,  $a$  can be attained in  $Y$ .  $\square$

8.

*Proof.* Since the unit ball  $B$  with respect to  $\|\cdot\|_2$  in  $\mathbb{R}^n$  and  $\mathbb{C}^n$  is compact and  $\|\cdot\|$  is continuous, by 2.5-7,  $x \mapsto \|x\|$  can attain its minimum, denoted by  $a$ , on  $B$ . Due to the positive definite property of a norm,  $a$  is positive. Hence,  $0 < a \leq \|x\|/\|x\|_2$ . Namely,  $a\|x\|_2 \leq \|x\|$ .  $\square$

9.

*Proof.* For every  $(x_n) \subset M \subset X$ , since  $X$  is compact, there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  which converges to some  $y \in X$ . Since  $M$  is closed,  $y \in M$ . Hence,  $M$  is compact.  $\square$

10.

*Proof.* From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of  $T$  is also continuous, it suffices to show that the image of a closed set  $A \subset X$  under  $T$  is again a closed set. Since  $X$  is compact and  $A$  is closed,  $A$  is compact. Since  $T$  is continuous, by 2.5-6,  $T(A)$  is compact and therefore closed. Hence,  $T$  is a homeomorphism.  $\square$

## 2.7 Bounded and Continuous Linear Operators

2.

*Proof.* First suppose  $T$  to be bounded and let  $A$  be any bounded set in  $X$ . Then there exists  $K < \infty$  such that for all  $x \in A$ ,  $\|x\| < K$ . Due to the boundedness of  $T$ ,  $\|Tx\| \leq \|T\|\|x\| < K\|T\|$ . Namely,  $T(A)$  is also bounded.

Now suppose that  $T$  maps bounded sets in  $X$  into bounded sets in  $Y$ . Clear that the unit ball  $B$  of  $X$  is bounded and therefore so is  $T(B)$ . Namely,  $\|Tx/\|x\|\|$  is bounded for  $x \neq 0$ .<sup>1</sup> Hence,  $T$  is bounded.  $\square$

3.

*Proof.* For every  $x$  with  $\|x\| < 1$ ,  $\|Tx\| \leq \|T\|\|x\| < \|T\|$ .  $\square$

4.

*Proof.* Suppose that the linear operator  $T$  is continuous at  $x_0 \in \mathcal{D}(T)$ . For every  $(x_n) \subset \mathcal{D}(T)$  with  $\|x_n - x_0\| \rightarrow 0$ , by the continuity of  $T$  at  $x_0$

$$\|Tx_n - Tx_0\| = \|T(x_n - x_0 + x_0) - Tx_0\| \rightarrow 0.$$

Hence,  $T$  is continuous.  $\square$

7.

*Proof.* The inequality implies  $\mathcal{N}(T) = 0$ . Hence, by Theorem 2.6-10,  $T^{-1}$  exists. For every  $y \in Y$ , suppose that  $y = Tx$ . Then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

Thus,  $T^{-1}$  is bounded.  $\square$

12.

*Proof.* The compatibility follows immediately from the definition of the supremum. Suppose  $\|x\|_1 = \max_j |\xi_j|$  and  $\|y\|_2 = \max_j \|\eta_j\|$ , then

$$Ax = \begin{bmatrix} x_1\alpha_{11} + \cdots + x_n\alpha_{1n} \\ \vdots \\ x_1\alpha_{r1} + \cdots + x_n\alpha_{rn} \end{bmatrix}$$

---

<sup>1</sup>Note that the two  $\|\cdot\|$  here are different norms.

Since for all  $j$ ,  $x_j \leq \|x_j\|_1$ ,

$$\frac{\max_j |x_1\alpha_{j1} + \cdots + x_n\alpha_{jn}|}{\|x\|_1} = \max_j \left| \frac{x_1}{\|x\|_1}\alpha_{j1} + \cdots + \frac{x_n}{\|x\|_1}\alpha_{jn} \right| \leq \max_j \sum_{k=1}^n |\alpha_{jk}|.$$

Hence,

$$\|A\| \geq \frac{\|Ax\|_2}{\|x\|_1} \quad \text{for all } x. \quad (1)$$

Suppose that maximum of  $\sum_{k=1}^n |\alpha_{jk}|$  is obtained at  $j = p$ . Then choosing  $x_k$  to be  $\text{sgn } \alpha_{pk}$  shows that the equality in (1) can actually be attained. Hence,  $\|A\| = \max_j \sum_{k=1}^n |\alpha_{jk}|$ .  $\square$

## 2.8 Linear Functionals

8.

*Proof.* For every  $x_1, x_2 \in N(M^*)$ ,  $a, b \in \mathbb{K}$  and  $f \in M^*$ ,

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) = 0.$$

Hence,  $ax_1 + bx_2 \in N(M^*)$ . Namely,  $N(M^*)$  is a vector space.  $\square$

9.

*Proof.* First we show the uniqueness. Suppose that  $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$ . Then  $0 = (\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)$ . Hence,

$$0 = f((\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)) = (\alpha_1 - \alpha_2)f(x_0) + f(y_1) - f(y_2).$$

Since  $y_1, y_2 \in \mathcal{N}(f)$ ,  $f(y_1) - f(y_2) = 0$  while  $f(x_0) \neq 0$  as  $x_0 \notin \mathcal{N}(f)$ . Hence,  $\alpha_1 = \alpha_2$ , which forces  $y_1$  and  $y_2$  to coincide.

For the existence, it suffices to show that for any fixed  $x$ , the function  $g(\alpha) = f(x - \alpha x_0)$  has a zero. It is easy to verify that  $\alpha = f(x)/f(x_0)$  is a zero of  $g$ . Note that  $x_0 \notin \mathcal{N}(f)$  and therefore  $f(x_0) \neq 0$ .  $\square$

10.

*Proof.* First we suppose that  $x_1, x_2 \in x_0 + \mathcal{N}(f) \in X/\mathcal{N}(f)$ . Then together with Prob. 9,  $x_i = x_0 + y_i$  where  $y_i \in \mathcal{N}(f)$ . Hence, for  $i = 1, 2$ ,  $f(x_i) = f(x_0) + f(y_i) = f(x_0)$ .

For the converse, note that  $f(x_1) = f(x_2)$  implies  $f(x_1 - x_2) = 0$ . Namely,  $x_1 - x_2 \in \mathcal{N}(f)$ . Hence,  $x_1, x_2$  belongs to the same element in  $X/\mathcal{N}(f)$ .

To show  $\text{codim } \mathcal{N}(f) = 1$ , we show that  $X/\mathcal{N}(f)$  and  $\mathbb{K}$  are isomorphic. For every  $\hat{x} \in X/\mathcal{N}(f)$ , define  $I(\hat{x}) = f(x)$ . By the previous discussion, this definition is well-defined. Clear that  $I$  is linear and therefore is injective. And by the linearity of  $f$ ,  $I$  is surjective. Thus,  $I$  is an isomorphism between  $X/\mathcal{N}(f)$  and  $\mathbb{K}$ . Hence,  $\text{codim } \mathcal{N}(f) = 1$ .  $\square$

11.

*Proof.* Put  $N = \mathcal{N}(f_1) = \mathcal{N}(f_2)$  and choose  $x_0 \in X \setminus N$ . By Prob. 9, for every  $x \notin N$ ,  $x = \alpha x_0 + y$  where  $y \in N$  and  $\alpha \neq 0$ . Hence,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha f_1(x_0) + f_1(y)}{\alpha f_2(x_0) + f_2(y)} = \frac{f_1(x_0)}{f_2(x_0)}.$$

$\square$

**12.**

*Proof.* Prob. 10, justifies the discussion on hyperplanes parallel to the  $\mathcal{N}(f)$ . It suffices to show that  $H_1 = b + \mathcal{N}(f)$  for some  $b \in X$ . Choose  $x_1 \in H_1$ . Then

$$x \in \mathcal{N}(f) \Leftrightarrow x + x_1 \in x_1 + \mathcal{N}(f) \Leftrightarrow f(x + x_1) = f(x) + f(x_1) = 1 \Leftrightarrow x + x_1 \in H_1.$$

Hence,  $H_1 = x_1 + \mathcal{N}(f)$ . Namely,  $H_1$  is a hyperplane parallel to  $\mathcal{N}(f)$ .  $\square$

**13.**

*Proof.* We argue by contradiction. Assume that there exists a  $y_1 \in Y$  such that  $f(y_1) \neq c \neq 0$ . Then for every  $d \in \mathbb{K}$ , by the linearity of  $f$ ,  $f(dy_1/c) = d$ . Contradiction. Hence,  $f = 0$  on  $Y$ .  $\square$

**14.**

*Proof.* For every  $\varepsilon > 0$ , there exists  $x_1 \in X$  with  $f(x_1) = 1$  such that  $\tilde{d} + \varepsilon \geq \|x_1\|$ . Hence,

$$\|f\|(\tilde{d} + \varepsilon) \geq \|f\|\|x_1\| \geq |f(x_1)| = 1.$$

Since the choice of  $\varepsilon > 0$  is arbitrary,  $\|f\|\tilde{d} \geq 1$ . Meanwhile, there exists  $x_2 \in X$  with  $\|x_2\| = 1$  such that  $|f(x_2)| \geq \|f\| - \varepsilon$ . Put  $x_3 = x_2/f(x_2)$ . Then  $f(x_3) = 1$ . Hence,

$$(\|f\| - \varepsilon)\tilde{d} \leq |f(x_2)|\|x_3\| = \|x_2\| = 1,$$

which implies  $\|f\|\tilde{d} \leq 1$ . Thus,  $\|f\|\tilde{d} = 1$ .  $\square$

**15.**

*Proof.* For every  $x$  with  $\|x\| \leq 1$ ,  $f(x) \leq \|f\|\|x\| \leq c$ . Hence,  $x \in X_{c_1}$ . Meanwhile, for every  $\varepsilon > 0$ , by the definition of the supremum, there exists a  $x$  with  $\|x\| = 1$  such that  $|f(x)| > \|f\| - \varepsilon$ . By the linearity of  $f$ , we may remove the  $|\cdot|$  on the right side. Hence,  $f(x) \notin X_{c_1}$  where  $c = \|f\| - \varepsilon$ .  $\square$

## 2.9 Operators on Finite Dimensional Spaces

**8.**

*Proof.* Let  $\{b_2, \dots, b_n\}$  be a basis of  $Z$  and  $\{b_1, \dots, b_n\}$  a basis of  $X$ . Define  $f \in X^*$  to be  $f(b_i) = \delta_{1i}$ . Clear that  $\mathcal{N}(f) = Z$ . By Prob. 11, Sec 2.8,  $f$  is uniquely determined up to a scalar multiple.  $\square$

**12.**

*Proof.* Let  $\varphi : X \rightarrow \mathbb{K}^p$  be defined by  $x \mapsto [f_1(x), \dots, f_p(x)]^T$ . It can be verified that  $\varphi$  is a linear operator. Since  $\dim X = n > p$ ,  $\varphi$  can not be injective. Hence, there exists  $0 \neq x \in X$  such that  $\varphi(x) = 0$ .  $\square$

**13.**

*Proof.* Let  $\{b_1, \dots, b_m\}$  be a basis of  $Z$  and  $\{b_1, \dots, b_n\}$  a basis of  $X$ . Define  $\tilde{f} \in X^*$  to be identical with  $f$  on  $b_1, \dots, b_m$  and 0 on  $b_{m+1}, \dots, b_n$ . Clear that  $\tilde{f}|_Z = f$ .  $\square$

## 2.10 Normed Spaces of Operators. Dual Space

8.

*Proof.* First we construct a linear bijection  $T$  between  $c'_0$  and  $l^1$ . A Schauder basis for  $c_0$  is  $(e_k)$ , where  $e_k = (\delta_{kj})$ . Then for every  $f \in c'_0$ , define  $Tf = (\gamma_k) = (f(e_k))$ . Clear that  $T$  is linear. Now we show that  $Tf = (\gamma_k) \in l^1$ , that is,  $\sum_{k=1}^n |\gamma_k|$  is bounded and therefore convergent. Define  $x_n = (\xi_k^{(n)})$  with

$$\xi_k^{(n)} = \begin{cases} \operatorname{sgn} \gamma_k, & k \leq n, \\ 0, & k > n. \end{cases}$$

Clear that  $x_n \in c_0$ . By the linearity and boundedness of  $f$ ,

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^n |\gamma_k|. \quad (2)$$

Since  $f$  is bounded,  $|f(x_n)| \leq \|f\| \|x_n\| \leq \|f\|$ . Hence,  $\sum \|\gamma_k\|$  is bounded. Thus,  $Tf \in l^1$ .

Meanwhile, for every  $y = (\beta_k) \in l^1$ , define  $Sy = g$  to be the functional  $g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$  for  $x = (\xi_k)$ . On  $c_0$ , the summation does converge and clear that  $g$  is linear and bounded. Hence,  $g \in c'_0$ . It can be verify that  $ST = TS = I$  and  $T$  is linear. Thus,  $c'_0$  and  $l^1$  is isomorphic.

Now we show that  $T$  constructed preserve the norm to complete the proof. For  $x \in c_0$  with  $\|x\| = 1$ ,

$$|f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \gamma_k \right| \leq \sum_{k=1}^{\infty} |\gamma_k| = \|Tf\|.$$

Hence,  $\|f\| \leq \|Tf\|$ . And (2) implies  $\sum_{k=1}^n \|\gamma_k\| \leq \|f\|$ . Letting  $n \rightarrow \infty$  yields  $\|Tf\| \leq \|f\|$ . Thus,  $\|Tf\| = \|f\|$ .  $\square$

9.

*Proof.* Let  $(b_k)$  be a Hamel basis of  $X$  and suppose that  $f, g \in X^*$  coincide on every  $b_k$ . Then for every  $x = \sum_{k=1}^{\infty} \xi_k b_k \in X$ ,

$$f(x) - g(x) = \sum_{k=1}^n \xi_k (f(b_k) - g(b_k)) = 0.$$

Thus,  $f = g$ . Namely,  $f$  is uniquely determined.  $\square$

10.

*Proof.* Let  $(b_k)$  be a Hamel basis of  $X$  and without loss of generality we may assume  $\|b_k\| = 1$ . Justified by Prob. 9, we can define  $T \in X^*$  with  $Tb_k = k$ , which is clearly unbounded.  $\square$

11.

*Proof.* It follows immediately from Prob. 10.  $\square$



**13.**

*Proof.* For any  $f, g \in M^a$  and scalar  $a, b$ ,  $(af + bg)(x) = af(x) + bg(x) = 0$  for every  $x \in M$ . Hence,  $M^a$  is a vector space. For  $(f_n) \subset M^a \subset X'$ , suppose that  $f_n \rightarrow f \in M^*$ . Since  $M'$  is complete, it is closed and therefore  $f \in M'$ . For every  $0 \neq x \in M$ , since  $f_n \rightarrow f$ ,

$$\frac{|f_n(x) - f(x)|}{\|x\|} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Hence,  $f(x) = 0$ . Thus,  $M^a$  is closed.

$$X^a = \{0\} \text{ and } \{0\}^a = X'.$$

□

**14.**

*Proof.* Let  $\{b_1, \dots, b_m\}$  be a basis of  $M$  and  $\{b_1, \dots, b_n\}$  a basis of  $X$ . And let  $\{\beta_1, \dots, \beta_n\}$  be the dual basis. Clear that  $b_1, \dots, b_m \notin M^a$  whereas  $b_{m+1}, \dots, b_n$  does. Together with Prob. 13, this implies  $M^a = \text{span}(b_{m+1}, \dots, b_n)$ . Thus,  $\dim M^a = n - m$ . □

### 3 Inner Product Spaces. Hilbert Spaces

#### 3.1 Inner Product Spaces. Hilbert Spaces

2.

*Proof.*

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle = \|x\|^2 + \|y\|^2,$$

where the last equality comes from the hypothesis of orthogonality. Now we show that for mutually orthogonal  $x_1, \dots, x_m$

$$\left\| \sum_{i=1}^m x_i \right\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

by induction on  $m$ . The case where  $m = 2$  has already been showed and we assume that the equation holds for  $m - 1$ . Since  $x_m$  is orthogonal with each  $i = 1, \dots, m - 1$ ,  $x_m$  is orthogonal to  $x_1 + \dots + x_{m-1}$ . Hence,

$$\left\| \sum_{i=1}^m x_i \right\|^2 = \left\| \sum_{i=1}^{m-1} x_i \right\|^2 + \|x_m\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

completing the proof. □

3.

*Proof.* The equation implies  $\langle x, y \rangle + \langle y, x \rangle = 0$ . The symmetric property of real inner products implies  $\langle x, y \rangle = 0$ . Let  $X = \mathbb{C}$  and  $x = 1, y = i$ . It is easy to verify that  $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = 2$  but  $x$  and  $y$  are not orthogonal. □

7.

*Proof.* It suffices to show that the zero vector is the only vector orthogonal to all vectors. Suppose that  $\langle x_0, x \rangle = 0$  for all  $x \in X$ , then  $\|x_0\|^2 = \langle x_0, x_0 \rangle = 0$ . By the definiteness of the inner product,  $x_0 = 0$ . □

8. We show that any norm satisfying the parallelogram equality can be derived from an inner product.

*Proof.* The proof of (IP3) is trivial and (IP4) follows immediately from the positive-definiteness of the norm. Hence we only show the linearity in the first factor here. For every  $u, v, y \in X$ , from the parallelogram equality we can derive, after some computation, that

$$\begin{aligned} 4\langle u + v, y \rangle &= \|u + v + y\|^2 - \|u + v - y\|^2 \\ &= \|u + y\|^2 - \|u - y\|^2 + \|v + y\|^2 - \|v - y\|^2 \\ &= 4\langle u, y \rangle + 4\langle v, y \rangle. \end{aligned}$$

Namely, (IP1) holds. By induction we can show that  $\langle nu, y \rangle = n\langle u, y \rangle$  for  $n = 1, 2, \dots$ . And since  $\langle -u, y \rangle = \langle 0 - u, y \rangle = \langle 0, y \rangle - \langle u, y \rangle = -\langle u, y \rangle$ ,

$$\langle nu, y \rangle = n\langle u, y \rangle, \quad \text{for } n \in \mathbb{Z}.$$

Furthermore, for any positive integer  $m$ ,

$$m \left\langle \frac{n}{m}u, y \right\rangle = mn \left\langle \frac{1}{m}u, y \right\rangle = n\langle u, y \rangle.$$

Dividing the both sides by  $m$  yields

$$\langle qu, y \rangle = q\langle u, y \rangle, \quad \text{for } q \in \mathbb{Q}.$$

For every  $\alpha \in \mathbb{R}$ , let  $(q_n) \subset \mathbb{Q}$  converges to  $\alpha$ . Now we show that  $f(t) = \langle tu, y \rangle$  is continuous at  $t = 0$  and by the additivity we may conclude that  $f$  is continuous on  $\mathbb{R}$ . Since

$$\begin{aligned} 4|f(t)| &= ||tu + y||^2 - ||tu - y||^2 \\ &= (||tu + y|| + ||tu - y||)||tu + y|| - ||tu - y|| \\ &\leq 4t||u||(|t||u|| + ||y||) \rightarrow 0 \end{aligned}$$

as  $t \rightarrow 0$ ,  $f(t)$  is continuous. For every  $\alpha \in \mathbb{R}$ , let  $(q_n) \subset \mathbb{Q}$  be a convergent sequence with limit  $\alpha$ . Then

$$\langle \alpha u, y \rangle = \lim \langle q_n u, y \rangle = \lim q_n \langle u, y \rangle = \alpha \langle u, y \rangle.$$

Hence,  $\langle \cdot, \cdot \rangle$  is linear in the first factor. Thus, it is an inner product. Meanwhile, it is easy to verify that the norm it introduces is exactly the original norm.  $\square$

## 3.2 Further Properties of Inner Product Spaces

7.

*Proof.* First we note that

$$f(\alpha) = ||x + \alpha y||^2 - ||x - \alpha y||^2 = 2\bar{\alpha}\langle x, y \rangle + 2\alpha\langle y, x \rangle.$$

Clear that  $x \perp y$  implies  $f(\alpha) = 0$  for all scalar  $\alpha$ . For the converse, we suppose  $f(\alpha) = 0$  and put  $\alpha = \langle x, y \rangle$ . Then  $0 = f(\alpha) = 2|\langle x, y \rangle|^2$ . Thus,  $x \perp y$ .  $\square$

8.

*Proof.* Clear that  $x \perp y$  implies  $||x + \alpha y|| \geq ||x||$ . Therefore we only show the converse here. Without loss of generality, we assume  $||y|| = 1$ . Then  $||x + \alpha y|| \geq ||x||$  for all scalar  $\alpha$  implies

$$|\alpha|^2 + \bar{\alpha}\langle x, y \rangle + \alpha\overline{\langle x, y \rangle} \geq 0.$$

Put  $\alpha = -\langle x, y \rangle$  and we get

$$0 \leq |\langle x, y \rangle|^2 - 2|\langle x, y \rangle|^2 = -|\langle x, y \rangle|^2,$$

which implies  $\langle x, y \rangle = 0$ . Namely,  $x \perp y$ .  $\square$

9.

*Proof.* For every  $\varepsilon > 0$ , put  $\delta = \varepsilon/\sqrt{b-a}$ . Then for every  $x_1, x_2 \in V$  with  $\|x_1 - x_2\|_\infty < \delta$ ,

$$\|x_1 - x_2\|_2^2 = \int_a^b |x_1(t) - x_2(t)|^2 dt \leq (b-a)\delta^2 = \varepsilon^2.$$

Hence,  $x \mapsto x$  is continuous. □

10.

*Proof.* For every  $u, w \in X$ ,

$$\begin{aligned} \langle Tu, w \rangle &= \frac{1}{4} (\langle T(u+w), u+w \rangle - \langle T(u-w), u-w \rangle) \\ &\quad + \frac{i}{4} (\langle T(u+iw), u+iw \rangle - \langle T(u-iw), u-iw \rangle). \end{aligned}$$

Note that each component of the right hand side is of form  $\langle Tx, x \rangle$  and hence equals to 0. Putting  $w = Tu$  yields  $\|Tu\|^2 = 0$  for all  $u \in X$ . Thus,  $T = 0$ . □

### 3.3 Orthogonal Complements and Direct Sums

7.

*Proof.*

(a)  $x \in A^{\perp\perp}$  iff for all  $y \in A^\perp$ ,  $\langle x, y \rangle = 0$ . By the definition of  $A^\perp$ , the identity holds if  $x \in A$ . Hence,  $A \subset A^{\perp\perp}$ .

(b) For all  $x \in B^\perp$  and  $y \in A \subset B$ ,  $\langle x, y \rangle = 0$  by definition. Hence,  $x \in A^\perp$ . Namely,  $B^\perp \subset A^\perp$ .

(c) We show that  $A^\perp$  is closed (no matter whether  $A$  is or not) and invoke Lemma 3.3-6 to complete the proof. Suppose that  $(x_n) \subset A^\perp$  converges to  $x$ . For all  $y \in A$ ,  $\langle x_n, y \rangle = 0$ . By the continuity of the inner product,  $\langle x, y \rangle = 0$  and therefore  $x \in A^\perp$ . Hence,  $A^\perp$  is closed. Thus,  $A^\perp = A^{\perp\perp\perp}$ . □

8.

*Proof.* We have show this in Prob. 7. □

9.

*Proof.* It has been shown in Lemma 3.3-6 that the closedness of  $Y$  implies  $Y = Y^{\perp\perp}$ . Hence we only show the converse here. For every convergent  $(x_n) \subset Y$ ,  $(x_n) \subset Y^{\perp\perp}$ . Since  $Y^{\perp\perp}$  is closed by Prob. 8, the limit  $x$  of  $(x_n)$  belongs to  $Y^{\perp\perp}$  and hence belongs to  $Y$ . Thus,  $Y$  is closed. □

10. TODO

### 3.4 Orthonormal Sets and Sequences

3.

*Proof.* The situation where  $x$  and  $y$  are linearly dependent is obvious and hence we assume they are linearly independent here. By the homogeneity of the Schwarz inequality, we may assume without loss of generality that  $\|x\| = \|y\| = 1$ . Put  $z = (y - x\langle y, x \rangle) / \|y - x\langle y, x \rangle\|$ . Then  $\{x, z\}$  is orthonormal and therefore by (12\*)

$$|\langle y, x \rangle|^2 + |\langle y, z \rangle|^2 \leq \|y\|^2 = 1.$$

Since  $|\langle y, z \rangle|^2$  is nonnegative, this implies  $|\langle x, y \rangle|^2 \leq 1$ , the Schwarz inequality.  $\square$

7.

*Proof.* For each positive integer  $n$ , by the Schwarz inequality and (12\*),

$$\sum_{k=1}^n |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \sqrt{\sum_{i=1}^n |\langle x, e_i \rangle|^2} \sqrt{\sum_{i=1}^n |\langle y, e_i \rangle|^2} \leq \|x\| \|y\|.$$

Since all terms in the summation is nonnegative, this implies  $\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \|x\| \|y\|$ .  $\square$

8.

*Proof.* It follows immediately from Bessel inequality.  $\square$

### 3.5 Series Related to Orthonormal Sequences

1.

*Proof.* By Theorem 3.5-2,  $\alpha_k = \langle x, e_k \rangle$ . Meanwhile by the definition of the norm,

$$\|x\|^2 = \left\langle \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, x \right\rangle = \sum_{k=1}^{\infty} \langle x, e_k \rangle \langle e_k, x \rangle = \sum_{k=1}^{\infty} |\alpha_k|^2,$$

where the second equality follows from the continuity of the inner product.  $\square$

3.

*Solution.* Put  $x \equiv 1$  on  $[-\pi, \pi]$  and  $e_k = \sin kt$ . Since  $x$  is even but  $e_k$  is odd for every  $k$ , the series does not converges to  $x$ .  $\square$

4.

*Proof.* By the triangle inequality,  $\|x_m + \cdots + x_n\| \leq \|x_m\| + \cdots + \|x_n\|$  for every  $n \geq m > 0$ . Hence the convergence of  $\sum \|x_k\|$  implies that  $s_n$  is a Cauchy sequence.  $\square$

5.

*Proof.* By Prob. 4,  $\sum_{k=1}^n x_k$  is a Cauchy sequence. And since  $H$  is complete,  $\sum_{k=1}^{\infty} x_k$  converges.  $\square$

7.

*Proof.* The existence of  $y$  follows from Theorem 3.5-2(c). And for each  $k$ ,

$$\langle x - y, e_k \rangle = \langle x, e_k \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle e_k, e_j \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0,$$

where the second equality comes from the fact that  $(e_k)$  is orthonormal.  $\square$

8. TODO: Show the validation of the change of the order of summation. Or maybe we can show the equality directly.

*Proof.* We suppose that  $x \in \bar{M}$  here since the proof of the other direction is obvious. Then there exists  $(p_n) \subset M$  such that  $x = \sum_{n=1}^{\infty} p_n$ . For each  $n$ , suppose  $p_n = \sum_{k=1}^{\infty} \langle p_n, e_k \rangle e_k$ . This is valid because  $p_n \in M$  and therefore is a finite linear combination of  $(e_k)$ . In fact, there are only finitely many nonzero term in the summation. Then

$$x = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle p_n, e_k \rangle e_k = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \langle p_n, e_k \rangle \right) e_k.$$

$\square$

9.

*Proof.* First we suppose  $\bar{M}_1 = \bar{M}_2$ . Then by Prob. 8, each  $e_n$  and  $\tilde{e}_n$  can be represented by (a) and (b) respectively.

For the converse, (a) implies, again by Prob. 8,  $e_n \in \bar{M}_2$  and therefore  $M_1 \subset \bar{M}_2$ . Since  $\bar{M}_2$  is closed,  $\bar{M}_1 \subset \bar{M}_2$ . *Mutatis mutandis*, this also shows  $\bar{M}_2 \subset \bar{M}_1$ . Thus,  $\bar{M}_1 = \bar{M}_2$ .  $\square$

10.

*Proof.* Note that for every  $m > 0$ , there are only finite  $e_\kappa$  such that  $\langle x, e_\kappa \rangle \geq 1/m$ . Otherwise we may choose a countable subset of them, which will violate the result in Prob. 8, Sec 3.4. Hence, the collection of all nonzero Fourier coefficient

$$\bigcup_{m=1}^{\infty} \{e_\kappa : \langle x, e_\kappa \rangle \geq 1/m\}$$

is at most countable.  $\square$

### 3.6 Total Orthonormal Sets and Sequences

4.

*Proof.* Suppose that  $x$  and  $y$  satisfy (3). We only show the relation for real cases here. The complex cases can be proved in a similar way. Using (9), Sec 3.1 and (3),

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) = \frac{1}{4} \sum_k (|\langle x + y, e_k \rangle|^2 - |\langle x - y, e_k \rangle|^2).$$

Meanwhile,

$$|\langle x \pm y, e_k \rangle|^2 = \langle x \pm y, e_k \rangle \overline{\langle x \pm y, e_k \rangle} = |\langle x, e_k \rangle|^2 + |\langle y, e_k \rangle|^2 \pm 2 \langle x, e_k \rangle \overline{\langle y, e_k \rangle}.$$

Hence,  $\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$ .  $\square$

6.

*Proof.* Suppose  $M = (e_k)$ . We collect the  $e_k$  which does not belong to  $\text{span}(e_1, \dots, e_{k=1})$  and denote the new sequence by  $(\tilde{e}_k)$ . Clear that  $\text{span}(e_k) = \text{span}(\tilde{e}_k)$  and  $(\tilde{e}_k)$  is linearly independent. Let  $(f_k)$  be the sequence generated from  $(\tilde{e}_k)$  by the Gram-Schmidt process. Then clear that  $(f_k)$  is orthonormal. And since for every  $n$ ,  $\text{span}(\tilde{e}_1, \dots, \tilde{e}_n) = \text{span}(f_1, \dots, f_n)$ ,  $M \subset \text{span}(\tilde{e}_k) = \text{span}(f_k)$ . Finally, since  $M$  is dense in  $H$ ,  $\text{span}(f_k) = H$ . Thus,  $(f_k)$  is a total orthonormal sequence of  $H$ .  $\square$

7.

*Proof.* It follows from the definition of the separable Hilbert space and Prob. 6 immediately.  $\square$

9.

*Proof.*  $\langle v, x \rangle = \langle w, x \rangle$  implies  $\langle v - w, x \rangle = 0$  for all  $x \in M$ , that is,  $x \perp M$ . Since  $M$  is total, by Theorem 3.6-2,  $v - w = 0$ .  $\square$

10.

*Proof.* It follows immediately from Theorem 3.6-2(b).  $\square$

### 3.8 Functionals on Hilbert Spaces

3.

*Proof.* The linearity follows from the sesquilinearity of the inner product and the boundedness from the Schwarz inequality. Furthermore, the Schwarz inequality also implies  $\|f\| \leq \|z\|$ . Meanwhile,  $\|f\| \geq \|f(z/\|z\|)\| = \|z\|$ . Thus,  $\|f\| = \|z\|$ .  $\square$

4.

*Proof.* Clear that the mapping  $z \mapsto f$  is an isomorphism since it is surjective. And by Theorem 2.10-4,  $X'$  is a Hilbert space. Hence,  $X$  is also a Hilbert space.  $\square$

5.

*Proof.* Since  $l^2$  is complete. By Theorem 3.8-1, we may define  $I : (l^2)' \rightarrow l^2$  to be  $f \mapsto z$ . Clear that  $I$  is linear and injective. Meanwhile, it preserves the norm. Furthermore, by Prob. 3, it is surjective. Hence,  $I$  is an isomorphism. Thus,  $l^2$  is isomorphic to its dual.  $\square$

12.

*Proof.* For every  $x \in X$  and  $y \in Y$ ,

$$\begin{aligned} |h(x + \Delta x, y + \Delta y) - h(x, y)| &= |h(\Delta x, y) + h(x, \Delta y) + h(\Delta x, \Delta y)| \\ &\leq |h(\Delta x, y)| + |h(x, \Delta y)| + |h(\Delta x, \Delta y)|. \end{aligned}$$

Since  $h$  is bounded,

$$|h(x + \Delta x, y + \Delta y) - h(x, y)| \leq \|h\|(\|\Delta x\| \|y\| + \|\Delta y\| \|x\| + \|\Delta x\| \|\Delta y\|).$$

Thus,  $h$  is continuous.  $\square$

**14.**

*Proof.* If  $h(x, x) = 0$ , then for any  $t \in \mathbb{R}$ ,

$$0 \leq h(th(y, x)x + y, th(y, x)x + y) = 2t|h(x, y)|^2 + h(y, y).$$

Hence,  $h(x, y) = 0$ , otherwise we may choose some  $t < 0$  such that the right hand side is negative. Thus, the inequality holds if  $h(x, x) = 0$ .

Now suppose  $h(x, x) \neq 0$ . Put

$$z = y - x \frac{h(y, x)}{h(x, x)} \tag{3}$$

It is easy to verify that  $h(z, x) = 0$ . Multiplying  $z$  on the both sides of (3) yields

$$0 \leq h(z, z) = h\left(z, y - x \frac{h(y, x)}{h(x, x)}\right) = h(z, y) = h(y, y) - \frac{h(x, y)h(y, x)}{h(x, x)}.$$

Thus,  $|h(x, y)|^2 \leq h(x, x)h(y, y)$ . □