

Solutions to *Introductory Functional Analysis with Applications*

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2 Normed Spaces. Banach Spaces

2.3 Further Properties of Normed Spaces

4. cf. Prob. 13, Sec 1.2

Proof. The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1| \|x_1 - x_2\| + |\alpha_1 - \alpha_2| \|x_2\|.$$

□

7.

Proof. Let Y and y_n be defined as in the hint. Then $\|y_n\| = 1/n^2$, constituting a convergent number series. However,

$$\sum_{n=1}^N y_n = (1, 1/4, \dots, 1/N^2, 0, \dots),$$

which is divergent as $N \rightarrow \infty$.

□

8.

Proof. Let (x_n) be a Cauchy sequence in X . Hence, for every $n > 0$, there exists some $K_n > 0$ such that for all $p, q > K_n$, $\|x_p - x_q\| < 1/n^2$. Without loss of generality, we may assume that (K_n) is increasing. Since the series $\|x_{K_{n+1}} - x_{K_n}\|$ is bounded by $1/n^2$, it converges. By the hypothesis, the series $(x_{K_{n+1}} - x_{K_n})$ also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1} (x_{K_{i+1}} - x_{K_i}) \rightarrow x \quad \text{as } n \rightarrow \infty.$$

Now we show that (x_n) converges to x . For every $\varepsilon > 0$, since (x_n) is a Cauchy sequence, there exists some N_1 such that for all $p, q > N_1$, $\|x_p - x_q\| < \varepsilon$. Meanwhile, since $x_{K_n} \rightarrow x$, once K_n is large enough, $\|x - x_{K_n}\| < \varepsilon$. Let $K_n > N_1$. Then for every $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus, X is complete.

□

9.

Proof. Let (x_n) be an absolutely convergent series in Banach space X . Let $s_n = \sum_{i=1}^n x_i$. Now we show that s_n is a Cauchy sequence and therefore convergent. Since $\sum_{i=1}^{\infty} \|x_i\| < \infty$, for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n > N$, $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$. Hence, for every $N < p \leq q$,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^q x_i \right\| \leq \sum_{i=p+1}^q \|x_i\| < \varepsilon,$$

completing the proof.

□

10.

Proof. Let (e_n) be Schauder basis of X . Denote the underlying field of X by \mathbb{K} and let $\mathbb{W} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$ if $\mathbb{K} = \mathbb{C}$. Now we show that

$$S = \left\{ \sum_{i=1}^n \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \dots \right\},$$

a countable subset of X , is dense in X to derive the separability.

For every $x \in X$ and $\varepsilon > 0$, by the definition of Schauder basis, there exists $\beta_1, \dots, \beta_n \in \mathbb{K}$ such that $\|x - (\beta_1 e_1 + \dots + \beta_n e_n)\| < \varepsilon$. Let $M = \max_i \|e_i\|$. If $M = 0$, then there is nothing to prove. Otherwise, since \mathbb{W} is dense in \mathbb{K} , for $i = 1, \dots, n$, there exists $\alpha_i \in \mathbb{W}$ with $|\alpha_i - \beta_i| < \varepsilon/2^i M$. Hence,

$$\begin{aligned} \left\| x - \sum_{i=1}^n \alpha_i e_i \right\| &\leq \left\| x - \sum_{i=1}^n \beta_i e_i \right\| + \left\| \sum_{i=1}^n (\beta_i - \alpha_i) e_i \right\| \\ &\leq \varepsilon + \sum_{i=1}^n |\alpha_i - \beta_i| \|e_i\| \\ &\leq 2\varepsilon. \end{aligned}$$

Thus, S is dense in X and therefore X is separable. \square

14.

Proof. Clear that $\|\cdot\|_0$ is nonnegative. And $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$. Meanwhile, $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$. Finally, we show that $\|\hat{x}\|_0 = 0$ implies $\hat{x} = Y$ and invoke Prob. 4, Sec 2.2 to complete the proof. Since $\|\hat{x}\|_0 = 0$, there exists $(x_n) \subset \hat{x}$ which converges to 0. Since Y is closed, Y is complete and so is its cosets. Therefore, $0 \in \hat{x}$, enforcing \hat{x} to be Y . \square

2.4 Finite Dimensional Normed Spaces

3.

Proof. The reflexive property clearly holds. If there are positive a and b such that $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$ for all $x \in X$, then $\|x\|_1/b \leq \|x\|_0 \leq \|x\|/a$. Hence the relation is symmetric. Next we further suppose there exists positive c and d such that that $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$. Then $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$, giving the transitive property. Thus, the axioms of an equivalence relation hold. \square

4.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Let $E \subset X$ be any open set with respect to $\|\cdot\|$, i.e., for every $x_0 \in E$, there exists some $\delta > 0$ such that $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$. Since $\|\cdot\| \sim \|\cdot\|_0$, there exists some positive c such that $\|x - x_0\| \leq c\|x - x_0\|_0$. Hence, $B = \{x \in X : \|x - x_0\| < \delta/c\} \subset A \subset E$. Namely, E is also open with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. \square

5.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Then for every $x \in X$, there exists some $c > 0$ such that $\|x\|_0 \leq c\|x\|$. Let (x_n) be a Cauchy sequence with respect to $\|\cdot\|$, i.e., for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n, m > N$, $\|x_n - x_m\| < \varepsilon/c$. Hence, $\|x_n - x_m\|_0 < c\|x_n - x_m\| \leq \varepsilon$. Thus, (x_n) is also a Cauchy with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. \square

2.5 Compactness and Finite Dimension

5.

Proof. Clear that every point in \mathbb{R}^n or \mathbb{C}^n has a closed bounded, and therefore compact, neighborhood. Hence, \mathbb{R}^n and \mathbb{C}^n are locally compact. \square

6.

Proof. Let X be a compact metric space and x any point in X . Let E be a closed neighborhood of x . By Prob 10, E is compact. Thus, X is locally compact. \square

7.

Proof. It suffices to show that $a = \inf_{y \in Y} \|v - y\|$ can actually be obtained. Let $\{b_1, \dots, b_n\}$ be a basis of Y and $y_k = y_{k,1}b_1 + \dots + y_{k,n}b_n$ a sequence in Y with $\|v - y_k\| \rightarrow a$. We may assume without loss of generality that $\|v - y_k\|$ is bounded.

Since Y is a proper subset of Z , v, b_1, \dots, b_n are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar $c > 0$ such that for every k ,

$$\|v - y_{k,1}b_1 - \dots - y_{k,n}b_n\| \geq c(1 + |y_{k,1}| + \dots + |y_{k,n}|).$$

Hence, the sequence $(y_{k,1}, \dots, y_{k,n})$ of n -tuples is bounded and therefore has a convergent subsequence. Consequently, (y_k) also has a convergent subsequence. Suppose that it converges to $z \in Z$. Note that $\|v - z\| = a$ and as Y is closed, $z \in Y$. Thus, a can be attained in Y . \square

8.

Proof. Since the unit ball B with respect to $\|\cdot\|_2$ in \mathbb{R}^n and \mathbb{C}^n is compact and $\|\cdot\|$ is continuous, by 2.5-7, $x \mapsto \|x\|$ can attain its minimum, denoted by a , on B . Due to the positive definite property of a norm, a is positive. Hence, $0 < a \leq \|x\|_2$. Namely, $a\|x\|_2 \leq \|x\|$. \square

9.

Proof. For every $(x_n) \subset M \subset X$, since X is compact, there exists a subsequence (x_{n_k}) of (x_n) which converges to some $y \in X$. Since M is closed, $y \in M$. Hence, M is compact. \square

10.

Proof. From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of T is also continuous, it suffices to show that the image of a closed set $A \subset X$ under T is again a closed set. Since X is compact and A is closed, A is compact. Since T is continuous, by 2.5-6, $T(A)$ is compact and therefore closed. Hence, T is a homeomorphism. \square

2.7 Bounded and Continuous Linear Operators

2.

Proof. First suppose T to be bounded and let A be any bounded set in X . Then there exists $K < \infty$ such that for all $x \in A$, $\|x\| < K$. Due to the boundedness of T , $\|Tx\| \leq \|T\|\|x\| < K\|T\|$. Namely, $T(A)$ is also bounded.

Now suppose that T maps bounded sets in X into bounded sets in Y . Clear that the unit ball B of X is bounded and therefore so is $T(B)$. Namely, $\|Tx/\|x\|\|$ is bounded for $x \neq 0$.¹ Hence, T is bounded. \square

3.

Proof. For every x with $\|x\| < 1$, $\|Tx\| \leq \|T\|\|x\| < \|T\|$. \square

4.

Proof. Suppose that the linear operator T is continuous at $x_0 \in \mathcal{D}(T)$. For every $(x_n) \subset \mathcal{D}(T)$ with $\|x_n - x_0\| \rightarrow 0$, by the continuity of T at x_0

$$\|Tx_n - Tx_0\| = \|T(x_n - x_0 + x_0) - Tx_0\| \rightarrow 0.$$

Hence, T is continuous. \square

7.

Proof. The inequality implies $\mathcal{N}(T) = 0$. Hence, by Theorem 2.6-10, T^{-1} exists. For every $y \in Y$, suppose that $y = Tx$. Then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

Thus, T^{-1} is bounded. \square

12.

Proof. The compatibility follows immediately from the definition of the supremum. Suppose $\|x\|_1 = \max_j |\xi_j|$ and $\|y\|_2 = \max_j \|\eta_j\|$, then

$$Ax = \begin{bmatrix} x_1\alpha_{11} + \cdots + x_n\alpha_{1n} \\ \vdots \\ x_1\alpha_{r1} + \cdots + x_n\alpha_{rn} \end{bmatrix}$$

¹Note that the two $\|\cdot\|$ here are different norms.

Since for all j , $x_j \leq \|x_j\|_1$,

$$\frac{\max_j |x_1\alpha_{j1} + \cdots + x_n\alpha_{jn}|}{\|x\|_1} = \max_j \left| \frac{x_1}{\|x\|_1}\alpha_{j1} + \cdots + \frac{x_n}{\|x\|_1}\alpha_{jn} \right| \leq \max_j \sum_{k=1}^n |\alpha_{jk}|.$$

Hence,

$$\|A\| \geq \frac{\|Ax\|_2}{\|x\|_1} \quad \text{for all } x. \quad (1)$$

Suppose that maximum of $\sum_{k=1}^n |\alpha_{jk}|$ is obtained at $j = p$. Then choosing x_k to be $\text{sgn } \alpha_{pk}$ shows that the equality in (1) can actually be attained. Hence, $\|A\| = \max_j \sum_{k=1}^n |\alpha_{jk}|$. \square

2.8 Linear Functionals

8.

Proof. For every $x_1, x_2 \in N(M^*)$, $a, b \in \mathbb{K}$ and $f \in M^*$,

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) = 0.$$

Hence, $ax_1 + bx_2 \in N(M^*)$. Namely, $N(M^*)$ is a vector space. \square

9.

Proof. First we show the uniqueness. Suppose that $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$. Then $0 = (\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)$. Hence,

$$0 = f((\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)) = (\alpha_1 - \alpha_2)f(x_0) + f(y_1) - f(y_2).$$

Since $y_1, y_2 \in \mathcal{N}(f)$, $f(y_1) - f(y_2) = 0$ while $f(x_0) \neq 0$ as $x_0 \notin \mathcal{N}(f)$. Hence, $\alpha_1 = \alpha_2$, which forces y_1 and y_2 to coincide.

For the existence, it suffices to show that for any fixed x , the function $g(\alpha) = f(x - \alpha x_0)$ has a zero. It is easy to verify that $\alpha = f(x)/f(x_0)$ is a zero of g . Note that $x_0 \notin \mathcal{N}(f)$ and therefore $f(x_0) \neq 0$. \square

10.

Proof. First we suppose that $x_1, x_2 \in x_0 + \mathcal{N}(f) \in X/\mathcal{N}(f)$. Then together with Prob. 9, $x_i = x_0 + y_i$ where $y_i \in \mathcal{N}(f)$. Hence, for $i = 1, 2$, $f(x_i) = f(x_0) + f(y_i) = f(x_0)$.

For the converse, note that $f(x_1) = f(x_2)$ implies $f(x_1 - x_2) = 0$. Namely, $x_1 - x_2 \in \mathcal{N}(f)$. Hence, x_1, x_2 belongs to the same element in $X/\mathcal{N}(f)$.

To show $\text{codim } \mathcal{N}(f) = 1$, we show that $X/\mathcal{N}(f)$ and \mathbb{K} are isomorphic. For every $\hat{x} \in X/\mathcal{N}(f)$, define $I(\hat{x}) = f(x)$. By the previous discussion, this definition is well-defined. Clear that I is linear and therefore is injective. And by the linearity of f , I is surjective. Thus, I is an isomorphism between $X/\mathcal{N}(f)$ and \mathbb{K} . Hence, $\text{codim } \mathcal{N}(f) = 1$. \square

11.

Proof. Put $N = \mathcal{N}(f_1) = \mathcal{N}(f_2)$ and choose $x_0 \in X \setminus N$. By Prob. 9, for every $x \notin N$, $x = \alpha x_0 + y$ where $y \in N$ and $\alpha \neq 0$. Hence,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha f_1(x_0) + f_1(y)}{\alpha f_2(x_0) + f_2(y)} = \frac{f_1(x_0)}{f_2(x_0)}.$$

\square

12.

Proof. Prob. 10, justifies the discussion on hyperplanes parallel to the $\mathcal{N}(f)$. It suffices to show that $H_1 = b + \mathcal{N}(f)$ for some $b \in X$. Choose $x_1 \in H_1$. Then

$$x \in \mathcal{N}(f) \Leftrightarrow x + x_1 \in x_1 + \mathcal{N}(f) \Leftrightarrow f(x + x_1) = f(x) + f(x_1) = 1 \Leftrightarrow x + x_1 \in H_1.$$

Hence, $H_1 = x_1 + \mathcal{N}(f)$. Namely, H_1 is a hyperplane parallel to $\mathcal{N}(f)$. \square

13.

Proof. We argue by contradiction. Assume that there exists a $y_1 \in Y$ such that $f(y_1) \neq c \neq 0$. Then for every $d \in \mathbb{K}$, by the linearity of f , $f(dy_1/c) = d$. Contradiction. Hence, $f = 0$ on Y . \square

14.

Proof. For every $\varepsilon > 0$, there exists $x_1 \in X$ with $f(x_1) = 1$ such that $\tilde{d} + \varepsilon \geq \|x_1\|$. Hence,

$$\|f\|(\tilde{d} + \varepsilon) \geq \|f\|\|x_1\| \geq |f(x_1)| = 1.$$

Since the choice of $\varepsilon > 0$ is arbitrary, $\|f\|\tilde{d} \geq 1$. Meanwhile, there exists $x_2 \in X$ with $\|x_2\| = 1$ such that $|f(x_2)| \geq \|f\| - \varepsilon$. Put $x_3 = x_2/f(x_2)$. Then $f(x_3) = 1$. Hence,

$$(\|f\| - \varepsilon)\tilde{d} \leq |f(x_2)|\|x_3\| = \|x_2\| = 1,$$

which implies $\|f\|\tilde{d} \leq 1$. Thus, $\|f\|\tilde{d} = 1$. \square

15.

Proof. For every x with $\|x\| \leq 1$, $f(x) \leq \|f\|\|x\| \leq c$. Hence, $x \in X_{c_1}$. Meanwhile, for every $\varepsilon > 0$, by the definition of the supremum, there exists a x with $\|x\| = 1$ such that $|f(x)| > \|f\| - \varepsilon$. By the linearity of f , we may remove the $|\cdot|$ on the right side. Hence, $f(x) \notin X_{c_1}$ where $c = \|f\| - \varepsilon$. \square

2.9 Operators on Finite Dimensional Spaces

8.

Proof. Let $\{b_2, \dots, b_n\}$ be a basis of Z and $\{b_1, \dots, b_n\}$ a basis of X . Define $f \in X^*$ to be $f(b_i) = \delta_{1i}$. Clear that $\mathcal{N}(f) = Z$. By Prob. 11, Sec 2.8, f is uniquely determined up to a scalar multiple. \square

12.

Proof. Let $\varphi : X \rightarrow \mathbb{K}^p$ be defined by $x \mapsto [f_1(x), \dots, f_p(x)]^T$. It can be verified that φ is a linear operator. Since $\dim X = n > p$, φ can not be injective. Hence, there exists $0 \neq x \in X$ such that $\varphi(x) = 0$. \square

13.

Proof. Let $\{b_1, \dots, b_m\}$ be a basis of Z and $\{b_1, \dots, b_n\}$ a basis of X . Define $\tilde{f} \in X^*$ to be identical with f on b_1, \dots, b_m and 0 on b_{m+1}, \dots, b_n . Clear that $\tilde{f}|_Z = f$. \square

2.10 Normed Spaces of Operators. Dual Space

8.

Proof. First we construct a linear bijection T between c'_0 and l^1 . A Schauder basis for c_0 is (e_k) , where $e_k = (\delta_{kj})$. Then for every $f \in c'_0$, define $Tf = (\gamma_k) = (f(e_k))$. Clear that T is linear. Now we show that $Tf = (\gamma_k) \in l^1$, that is, $\sum_{k=1}^n |\gamma_k|$ is bounded and therefore convergent. Define $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \begin{cases} \text{sgn } \gamma_k, & k \leq n, \\ 0, & k > n. \end{cases}$$

Clear that $x_n \in c_0$. By the linearity and boundedness of f ,

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^n |\gamma_k|. \quad (2)$$

Since f is bounded, $|f(x_n)| \leq \|f\| \|x_n\| \leq \|f\|$. Hence, $\sum \|\gamma_k\|$ is bounded. Thus, $Tf \in l^1$.

Meanwhile, for every $y = (\beta_k) \in l^1$, define $Sy = g$ to be the functional $g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$ for $x = (\xi_k)$. On c_0 , the summation does converge and clear that g is linear and bounded. Hence, $g \in c'_0$. It can be verify that $ST = TS = I$ and T is linear. Thus, c'_0 and l^1 is isomorphic.

Now we show that T constructed preserve the norm to complete the proof. For $x \in c_0$ with $\|x\| = 1$,

$$|f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \gamma_k \right| \leq \sum_{k=1}^{\infty} |\gamma_k| = \|Tf\|.$$

Hence, $\|f\| \leq \|Tf\|$. And (2) implies $\sum_{k=1}^n \|\gamma_k\| \leq \|f\|$. Letting $n \rightarrow \infty$ yields $\|Tf\| \leq \|f\|$. Thus, $\|Tf\| = \|f\|$. \square

9.

Proof. Let (b_k) be a Hamel basis of X and suppose that $f, g \in X^*$ coincide on every b_k . Then for every $x = \sum_{k=1}^{\infty} \xi_k b_k \in X$,

$$f(x) - g(x) = \sum_{k=1}^n \xi_k (f(b_k) - g(b_k)) = 0.$$

Thus, $f = g$. Namely, f is uniquely determined. \square

10.

Proof. Let (b_k) be a Hamel basis of X and without loss of generality we may assume $\|b_k\| = 1$. Justified by Prob. 9, we can define $T \in X^*$ with $Tb_k = k$, which is clearly unbounded. \square

11.

Proof. It follows immediately from Prob. 10. \square

13.

Proof. For any $f, g \in M^a$ and scalar a, b , $(af + bg)(x) = af(x) + bg(x) = 0$ for every $x \in M$. Hence, M^a is a vector space. For $(f_n) \subset M^a \subset X'$, suppose that $f_n \rightarrow f \in M^*$. Since M' is complete, it is closed and therefore $f \in M'$. For every $0 \neq x \in M$, since $f_n \rightarrow f$,

$$\frac{|f_n(x) - f(x)|}{\|x\|} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Hence, $f(x) = 0$. Thus, M^a is closed.

$$X^a = \{0\} \text{ and } \{0\}^a = X'.$$

□

14.

Proof. Let $\{b_1, \dots, b_m\}$ be a basis of M and $\{b_1, \dots, b_n\}$ a basis of X . And let $\{\beta_1, \dots, \beta_n\}$ be the dual basis. Clear that $b_1, \dots, b_m \notin M^a$ whereas b_{m+1}, \dots, b_n does. Together with Prob. 13, this implies $M^a = \text{span}(b_{m+1}, \dots, b_n)$. Thus, $\dim M^a = n - m$. □

3 Inner Product Spaces. Hilbert Spaces

3.1 Inner Product Spaces. Hilbert Spaces

2.

Proof.

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle = \|x\|^2 + \|y\|^2,$$

where the last equality comes from the hypothesis of orthogonality. Now we show that for mutually orthogonal x_1, \dots, x_m

$$\left\| \sum_{i=1}^m x_i \right\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

by induction on m . The case where $m = 2$ has already been showed and we assume that the equation holds for $m - 1$. Since x_m is orthogonal with each $i = 1, \dots, m - 1$, x_m is orthogonal to $x_1 + \dots + x_{m-1}$. Hence,

$$\left\| \sum_{i=1}^m x_i \right\|^2 = \left\| \sum_{i=1}^{m-1} x_i \right\|^2 + \|x_m\|^2 = \sum_{i=1}^m \|x_i\|^2,$$

completing the proof. □

3.

Proof. The equation implies $\langle x, y \rangle + \langle y, x \rangle = 0$. The symmetric property of real inner products implies $\langle x, y \rangle = 0$. Let $X = \mathbb{C}$ and $x = 1, y = i$. It is easy to verify that $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = 2$ but x and y are not orthogonal. □

7.

Proof. It suffices to show that the zero vector is the only vector orthogonal to all vectors. Suppose that $\langle x_0, x \rangle = 0$ for all $x \in X$, then $\|x_0\|^2 = \langle x_0, x_0 \rangle = 0$. By the definiteness of the inner product, $x_0 = 0$. □

8. We show that any norm satisfying the parallelogram equality can be derived from an inner product.

Proof. The proof of (IP3) is trivial and (IP4) follows immediately from the positive-definiteness of the norm. Hence we only show the linearity in the first factor here. For every $u, v, y \in X$, from the parallelogram equality we can derive, after some computation, that

$$\begin{aligned} 4\langle u + v, y \rangle &= \|u + v + y\|^2 - \|u + v - y\|^2 \\ &= \|u + y\|^2 - \|u - y\|^2 + \|v + y\|^2 - \|v - y\|^2 \\ &= 4\langle u, y \rangle + 4\langle v, y \rangle. \end{aligned}$$

Namely, (IP1) holds. By induction we can show that $\langle nu, y \rangle = n\langle u, y \rangle$ for $n = 1, 2, \dots$. And since $\langle -u, y \rangle = \langle 0 - u, y \rangle = \langle 0, y \rangle - \langle u, y \rangle = -\langle u, y \rangle$,

$$\langle nu, y \rangle = n\langle u, y \rangle, \quad \text{for } n \in \mathbb{Z}.$$

Furthermore, for any positive integer m ,

$$m \left\langle \frac{n}{m}u, y \right\rangle = mn \left\langle \frac{1}{m}u, y \right\rangle = n\langle u, y \rangle.$$

Dividing the both sides by m yields

$$\langle qu, y \rangle = q\langle u, y \rangle, \quad \text{for } q \in \mathbb{Q}.$$

For every $\alpha \in \mathbb{R}$, let $(q_n) \subset \mathbb{Q}$ converges to α . Now we show that $f(t) = \langle tu, y \rangle$ is continuous at $t = 0$ and by the additivity we may conclude that f is continuous on \mathbb{R} . Since

$$\begin{aligned} 4|f(t)| &= ||tu + y||^2 - ||tu - y||^2 \\ &= (||tu + y|| + ||tu - y||)||tu + y|| - ||tu - y|| \\ &\leq 4t||u||(|t||u|| + ||y||) \rightarrow 0 \end{aligned}$$

as $t \rightarrow 0$, $f(t)$ is continuous. For every $\alpha \in \mathbb{R}$, let $(q_n) \subset \mathbb{Q}$ be a convergent sequence with limit α . Then

$$\langle \alpha u, y \rangle = \lim \langle q_n u, y \rangle = \lim q_n \langle u, y \rangle = \alpha \langle u, y \rangle.$$

Hence, $\langle \cdot, \cdot \rangle$ is linear in the first factor. Thus, it is an inner product. Meanwhile, it is easy to verify that the norm it introduces is exactly the original norm. \square

3.2 Further Properties of Inner Product Spaces

7.

Proof. First we note that

$$f(\alpha) = ||x + \alpha y||^2 - ||x - \alpha y||^2 = 2\bar{\alpha}\langle x, y \rangle + 2\alpha\langle y, x \rangle.$$

Clear that $x \perp y$ implies $f(\alpha) = 0$ for all scalar α . For the converse, we suppose $f(\alpha) = 0$ and put $\alpha = \langle x, y \rangle$. Then $0 = f(\alpha) = 2|\langle x, y \rangle|^2$. Thus, $x \perp y$. \square

8.

Proof. Clear that $x \perp y$ implies $||x + \alpha y|| \geq ||x||$. Therefore we only show the converse here. Without loss of generality, we assume $||y|| = 1$. Then $||x + \alpha y|| \geq ||x||$ for all scalar α implies

$$|\alpha|^2 + \bar{\alpha}\langle x, y \rangle + \alpha\overline{\langle x, y \rangle} \geq 0.$$

Put $\alpha = -\langle x, y \rangle$ and we get

$$0 \leq |\langle x, y \rangle|^2 - 2|\langle x, y \rangle|^2 = -|\langle x, y \rangle|^2,$$

which implies $\langle x, y \rangle = 0$. Namely, $x \perp y$. \square

9.

Proof. For every $\varepsilon > 0$, put $\delta = \varepsilon/\sqrt{b-a}$. Then for every $x_1, x_2 \in V$ with $\|x_1 - x_2\|_\infty < \delta$,

$$\|x_1 - x_2\|_2^2 = \int_a^b |x_1(t) - x_2(t)|^2 dt \leq (b-a)\delta^2 = \varepsilon^2.$$

Hence, $x \mapsto x$ is continuous. □

10.

Proof. For every $u, w \in X$,

$$\begin{aligned} \langle Tu, w \rangle &= \frac{1}{4} (\langle T(u+w), u+w \rangle - \langle T(u-w), u-w \rangle) \\ &\quad + \frac{i}{4} (\langle T(u+iw), u+iw \rangle - \langle T(u-iw), u-iw \rangle). \end{aligned}$$

Note that each component of the right hand side is of form $\langle Tx, x \rangle$ and hence equals to 0. Putting $w = Tu$ yields $\|Tu\|^2 = 0$ for all $u \in X$. Thus, $T = 0$. □

3.3 Orthogonal Complements and Direct Sums

7.

Proof.

(a) $x \in A^{\perp\perp}$ iff for all $y \in A^\perp$, $\langle x, y \rangle = 0$. By the definition of A^\perp , the identity holds if $x \in A$. Hence, $A \subset A^{\perp\perp}$.

(b) For all $x \in B^\perp$ and $y \in A \subset B$, $\langle x, y \rangle = 0$ by definition. Hence, $x \in A^\perp$. Namely, $B^\perp \subset A^\perp$.

(c) We show that A^\perp is closed (no matter whether A is or not) and invoke Lemma 3.3-6 to complete the proof. Suppose that $(x_n) \subset A^\perp$ converges to x . For all $y \in A$, $\langle x_n, y \rangle = 0$. By the continuity of the inner product, $\langle x, y \rangle = 0$ and therefore $x \in A^\perp$. Hence, A^\perp is closed. Thus, $A^\perp = A^{\perp\perp\perp}$. □

8.

Proof. We have show this in Prob. 7. □

9.

Proof. It has been shown in Lemma 3.3-6 that the closedness of Y implies $Y = Y^{\perp\perp}$. Hence we only show the converse here. For every convergent $(x_n) \subset Y$, $(x_n) \subset Y^{\perp\perp}$. Since $Y^{\perp\perp}$ is closed by Prob. 8, the limit x of (x_n) belongs to $Y^{\perp\perp}$ and hence belongs to Y . Thus, Y is closed. □

10. TODO

3.4 Orthonormal Sets and Sequences

3.

Proof. The situation where x and y are linearly dependent is obvious and hence we assume they are linearly independent here. By the homogeneity of the Schwarz inequality, we may assume without loss of generality that $\|x\| = \|y\| = 1$. Put $z = (y - x\langle y, x \rangle) / \|y - x\langle y, x \rangle\|$. Then $\{x, z\}$ is orthonormal and therefore by (12*)

$$|\langle y, x \rangle|^2 + |\langle y, z \rangle|^2 \leq \|y\|^2 = 1.$$

Since $|\langle y, z \rangle|^2$ is nonnegative, this implies $|\langle x, y \rangle|^2 \leq 1$, the Schwarz inequality. \square

7.

Proof. For each positive integer n , by the Schwarz inequality and (12*),

$$\sum_{k=1}^n |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \sqrt{\sum_{i=1}^n |\langle x, e_i \rangle|^2} \sqrt{\sum_{i=1}^n |\langle y, e_i \rangle|^2} \leq \|x\| \|y\|.$$

Since all terms in the summation is nonnegative, this implies $\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \|x\| \|y\|$. \square

8.

Proof. It follows immediately from Bessel inequality. \square

3.5 Series Related to Orthonormal Sequences

1.

Proof. By Theorem 3.5-2, $\alpha_k = \langle x, e_k \rangle$. Meanwhile by the definition of the norm,

$$\|x\|^2 = \left\langle \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, x \right\rangle = \sum_{k=1}^{\infty} \langle x, e_k \rangle \langle e_k, x \rangle = \sum_{k=1}^{\infty} |\alpha_k|^2,$$

where the second equality follows from the continuity of the inner product. \square

3.

Solution. Put $x \equiv 1$ on $[-\pi, \pi]$ and $e_k = \sin kt$. Since x is even but e_k is odd for every k , the series does not converges to x . \square

4.

Proof. By the triangle inequality, $\|x_m + \cdots + x_n\| \leq \|x_m\| + \cdots + \|x_n\|$ for every $n \geq m > 0$. Hence the convergence of $\sum \|x_k\|$ implies that s_n is a Cauchy sequence. \square

5.

Proof. By Prob. 4, $\sum_{k=1}^n x_k$ is a Cauchy sequence. And since H is complete, $\sum_{k=1}^{\infty} x_k$ converges. \square

7.

Proof. The existence of y follows from Theorem 3.5-2(c). And for each k ,

$$\langle x - y, e_k \rangle = \langle x, e_k \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle e_k, e_j \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0,$$

where the second equality comes from the fact that (e_k) is orthonormal. \square

8. TODO: Show the validation of the change of the order of summation. Or maybe we can show the equality directly.

Proof. We suppose that $x \in \bar{M}$ here since the proof of the other direction is obvious. Then there exists $(p_n) \subset M$ such that $x = \sum_{n=1}^{\infty} p_n$. For each n , suppose $p_n = \sum_{k=1}^{\infty} \langle p_n, e_k \rangle e_k$. This is valid because $p_n \in M$ and therefore is a finite linear combination of (e_k) . In fact, there are only finitely many nonzero term in the summation. Then

$$x = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle p_n, e_k \rangle e_k = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \langle p_n, e_k \rangle \right) e_k.$$

\square

9.

Proof. First we suppose $\bar{M}_1 = \bar{M}_2$. Then by Prob. 8, each e_n and \tilde{e}_n can be represented by (a) and (b) respectively.

For the converse, (a) implies, again by Prob. 8, $e_n \in \bar{M}_2$ and therefore $M_1 \subset \bar{M}_2$. Since \bar{M}_2 is closed, $\bar{M}_1 \subset \bar{M}_2$. *Mutatis mutandis*, this also shows $\bar{M}_2 \subset \bar{M}_1$. Thus, $\bar{M}_1 = \bar{M}_2$. \square

10.

Proof. Note that for every $m > 0$, there are only finite e_κ such that $\langle x, e_\kappa \rangle \geq 1/m$. Otherwise we may choose a countable subset of them, which will violate the result in Prob. 8, Sec 3.4. Hence, the collection of all nonzero Fourier coefficient

$$\bigcup_{m=1}^{\infty} \{e_\kappa : \langle x, e_\kappa \rangle \geq 1/m\}$$

is at most countable. \square

3.6 Total Orthonormal Sets and Sequences

4.

Proof. Suppose that x and y satisfy (3). We only show the relation for real cases here. The complex cases can be proved in a similar way. Using (9), Sec 3.1 and (3),

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) = \frac{1}{4} \sum_k (|\langle x + y, e_k \rangle|^2 - |\langle x - y, e_k \rangle|^2).$$

Meanwhile,

$$|\langle x \pm y, e_k \rangle|^2 = \langle x \pm y, e_k \rangle \overline{\langle x \pm y, e_k \rangle} = |\langle x, e_k \rangle|^2 + |\langle y, e_k \rangle|^2 \pm 2\langle x, e_k \rangle \overline{\langle y, e_k \rangle}.$$

Hence, $\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$. \square

6.

Proof. Suppose $M = (e_k)$. We collect the e_k which does not belong to $\text{span}(e_1, \dots, e_{k=1})$ and denote the new sequence by (\tilde{e}_k) . Clear that $\text{span}(e_k) = \text{span}(\tilde{e}_k)$ and (\tilde{e}_k) is linearly independent. Let (f_k) be the sequence generated from (\tilde{e}_k) by the Gram-Schmidt process. Then clear that (f_k) is orthonormal. And since for every n , $\text{span}(\tilde{e}_1, \dots, \tilde{e}_n) = \text{span}(f_1, \dots, f_n)$, $M \subset \text{span}(\tilde{e}_k) = \text{span}(f_k)$. Finally, since M is dense in H , $\text{span}(f_k) = H$. Thus, (f_k) is a total orthonormal sequence of H . \square

7.

Proof. It follows from the definition of the separable Hilbert space and Prob. 6 immediately. \square

9.

Proof. $\langle v, x \rangle = \langle w, x \rangle$ implies $\langle v - w, x \rangle = 0$ for all $x \in M$, that is, $x \perp M$. Since M is total, by Theorem 3.6-2, $v - w = 0$. \square

10.

Proof. It follows immediately from Theorem 3.6-2(b). \square

3.8 Functionals on Hilbert Spaces

3.

Proof. The linearity follows from the sesquilinearity of the inner product and the boundedness from the Schwarz inequality. Furthermore, the Schwarz inequality also implies $\|f\| \leq \|z\|$. Meanwhile, $\|f\| \geq \|f(z/\|z\|)\| = \|z\|$. Thus, $\|f\| = \|z\|$. \square

4.

Proof. Clear that the mapping $z \mapsto f$ is an isomorphism since it is surjective. And by Theorem 2.10-4, X' is a Hilbert space. Hence, X is also a Hilbert space. \square

5.

Proof. Since l^2 is complete. By Theorem 3.8-1, we may define $I : (l^2)' \rightarrow l^2$ to be $f \mapsto z$. Clear that I is linear and injective. Meanwhile, it preserves the norm. Furthermore, by Prob. 3, it is surjective. Hence, I is an isomorphism. Thus, l^2 is isomorphic to its dual. \square

12.

Proof. For every $x \in X$ and $y \in Y$,

$$\begin{aligned} |h(x + \Delta x, y + \Delta y) - h(x, y)| &= |h(\Delta x, y) + h(x, \Delta y) + h(\Delta x, \Delta y)| \\ &\leq |h(\Delta x, y)| + |h(x, \Delta y)| + |h(\Delta x, \Delta y)|. \end{aligned}$$

Since h is bounded,

$$|h(x + \Delta x, y + \Delta y) - h(x, y)| \leq \|h\|(\|\Delta x\| \|y\| + \|\Delta y\| \|x\| + \|\Delta x\| \|\Delta y\|).$$

Thus, h is continuous. \square

14.

Proof. If $h(x, x) = 0$, then for any $t \in \mathbb{R}$,

$$0 \leq h(th(y, x)x + y, th(y, x)x + y) = 2t|h(x, y)|^2 + h(y, y).$$

Hence, $h(x, y) = 0$, otherwise we may choose some $t < 0$ such that the right hand side is negative. Thus, the inequality holds if $h(x, x) = 0$.

Now suppose $h(x, x) \neq 0$. Put

$$z = y - x \frac{h(y, x)}{h(x, x)} \quad (3)$$

It is easy to verify that $h(z, x) = 0$. Multiplying z on the both sides of (3) yields

$$0 \leq h(z, z) = h\left(z, y - x \frac{h(y, x)}{h(x, x)}\right) = h(z, y) = h(y, y) - \frac{h(x, y)h(y, x)}{h(x, x)}.$$

Thus, $|h(x, y)|^2 \leq h(x, x)h(y, y)$. □

3.9 Hilbert-Adjoint Operator

1.

Proof. By Theorem 3.9-4, $0^* = (0 + 0)^* = 0^* + 0^*$. Hence, $0^* = 0$. For every $x, y \in X$,

$$\langle (I^* - I)x, y \rangle = \langle I^*x, y \rangle - \langle Ix, y \rangle = \langle x, Iy \rangle - \langle Ix, y \rangle = 0.$$

Hence, by Lemma 3.9-3, $I = I^*$. □

2.

Proof. By Theorem 3.9-4, $T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$. Hence, $(T^*)^{-1} = (T^{-1})^*$. □

3.

Proof. Since $\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|$, $T_n^* \rightarrow T^*$ as long as $T_n \rightarrow T$. □

4.

Proof. It suffices to show that for all $x_2 \in T^*(M_2^\perp)$ and $x_1 \in M_1$, $\langle x_1, x_2 \rangle = 0$. $x_2 \in T^*(M_2^\perp)$ implies the existence of some $y_2 \in M_2^\perp$ with $T^*y_2 = x_2$. Then

$$\langle x_1, x_2 \rangle = \langle x_1, T^*y_2 \rangle = \langle Tx_1, y_2 \rangle = 0,$$

where the last equality comes from the fact that $T(M_1) \subset M_2$ and $y_2 \in M_2^\perp$. Thus, $M_1^\perp \supset T^*(M_2^\perp)$. □

5.

Proof. By Prob. 4, $T^*(M_2^\perp) \subset M_1^\perp$ implies $M_2^{\perp\perp} \supset T(M_1^{\perp\perp})$. Since M_1 and M_2 are closed, by Prob. 9, Sec 3.3, $M_i^{\perp\perp} = M_i$ for $i = 1, 2$. Thus, $T(M_1) \subset M_2$. The converse part has already been proved in Prob. 4. □

6.

Proof.

(a) Since $T(M_1) = \{0\} \subset H_2$, by Prob. 4, $T^*(H_2) \subset M_1^\perp$.

(b) For every $y \in [T(H_1)]^\perp$, $\langle y, Tx \rangle = 0$ for all $x \in H_1$. Hence, $\langle T^*y, x \rangle = 0$. By Lemma 3.8-2, $T^*y = 0$ and therefore $y \in \mathcal{N}(T^*)$. Thus, $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$.

(c) Since $T^{**} = T$, it follows from (b) that $[T^*(H_2)]^\perp \subset M_1$. And since M_1 is closed, $M_1^{\perp\perp} = M_1$. Therefore, (a) implies $[T^*(H_2)]^\perp \supset M_1$. Thus, $M_1 = [T^*(H_2)]^\perp$. \square

7.

Proof. It follows immediately from Lemma 3.9-3. \square

8.

Proof. For every $x \in H$ with $\|x\| = 1$,

$$\begin{aligned} \|(I + T^*T)x\| &= \|x + T^*Tx\| = \langle x + T^*Tx, x + T^*Tx \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + \langle x, T^*Tx \rangle + \langle T^*Tx, x \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + \|Tx\|^2 \\ &\geq 1. \end{aligned}$$

Then, by Prob 7, Sec 2.7, $I + T^*T$ is invertible. \square

9.

Proof. If T can be represent by that form, then $\mathcal{R}(T)$ can be spanned by w_1, \dots, w_n . Hence, it is finite dimensional.

Now we suppose that T has a finite dimensional range. Let $\{w_1, \dots, w_n\}$ be a orthonormal basis of $\mathcal{R}(T)$. Then for every $x \in H$,

$$Tx = \sum_{j=1}^n \varphi_j(x) w_j.$$

Now we show that for each j , φ_j is a bounded linear functional and invoke Riesz's Theorem to complete the proof. It is easy to verify the linearity of φ_j . For every x with norm 1, since T is bounded and (w_j) is orthonormal,

$$\|T\| \geq \left\| \sum_{j=1}^n \varphi_j(x) w_j \right\| \geq |\varphi_j(x)|$$

for each $j = 1, \dots, n$. Hence, every φ_j is a bounded linear functional and therefore can be represented by $\varphi_j(x) = \langle x, v_j \rangle$. \square

Self-Adjoint, Unitary and Normal Operators

4. We only show the uniqueness here.

Proof. $T_1 + iT_2 = S_1 + iS_2$ implies $T_1 - iT_2 = S_1 - iS_2$. Sum these two equations and we get $T_1 = S_1$. Meanwhile, it also implies $i(T_1 + iT_2) = i(S_1 + iS_2)$. Summing these two gives $T_2 = S_2$. \square

6.

Proof.

(a) We argue by contradiction. Let k be the smallest positive integer such that $T^{2k} = 0$. Then for every $x \in H$

$$0 = \langle T^{2k}x, x \rangle = \langle T^kx, (T^k)^*x \rangle = \langle T^kx, T^kx \rangle = \|T^kx\|^2.$$

Hence $T^k = 0$, which contradicts with the smallest assumption of k . Hence, $T^n \neq 0$ for all even positive integer n .

(b) If $T^n = 0$ for some positive, not necessarily even, integer n , then so is $T^{2n} = 0$. Hence, by (a), $T^n \neq 0$ for all $n \in \mathbb{N}$. \square

9.

Proof. Since T is isometric, it preserves the topology. Hence $T(H)$ is closed as H is closed. \square

10.

Proof. It suffices to show that T is surjective. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of X . Then $\{Te_1, \dots, Te_n\}$ is also an orthonormal basis since T is isometric. Hence, T is surjective and therefore is unitary. \square

13.

Proof. It can be verified that $T_n^*T_n \rightarrow T^*T$ and $T_nT_n^* \rightarrow TT^*$. Since T_n are normal, $T_n^*T_n = T_nT_n^*$. Hence,

$$\|T^*T - TT^*\| \leq \|T^*T - T_n^*T_n\| + \|T_nT_n^* - TT^*\| \rightarrow 0$$

as $n \rightarrow \infty$. Thus, T is normal. \square

15.

Proof. If T is normal, clear that $\|T^*x\| = \|Tx\|$. Now we suppose $\|Tx\| = \|T^*x\|$ for all $x \in H$. Then $\langle TT^*x, x \rangle = \langle T^*Tx, x \rangle$. Since X is complex, by Lemma 3.9-3, $TT^* = T^*T$. Namely, T is normal.

By (a), for every $x \in H$, $\|T^2x\| = \|T^*Tx\|$. Hence,

$$\|T^2\| = \sup_{\|x\|=1} \|T^2x\| = \sup_{\|x\|=1} \|T^*Tx\| = \|T^*T\| = \|T\|^2,$$

where the last equality comes from Theorem 3.9-4(e). \square

4 Fundamental Theorems for Normed and Banach Spaces

4.2 Hahn-Banach Theorem

4.

Proof. By the positive homogeneity, $p(2 \times 0) = 2p(0)$. Hence, $p(0) = 0$. Consequently, $0 = p(x + (-x)) \leq p(x) + p(-x)$. Thus, $-p(x) \leq p(-x)$. \square

5.

Proof. For every $x, y \in M$ and $\lambda \in [0, 1]$,

$$p(\lambda x + (1 - \lambda)y) \leq \lambda p(x) + (1 - \lambda)p(y) \leq \lambda\gamma + (1 - \lambda)\gamma = \gamma.$$

Hence, $\lambda x + (1 - \lambda)y \in M$. Thus, M is convex. \square

6.

Proof. For every $x, t \in X$,

$$p(x - t) \leq p(x) + p(-t) \quad \Rightarrow \quad p(x - t) - p(x) \leq p(-t),$$

and

$$p(x) = p(x - t + t) \leq p(x - t) + p(t) \quad \Rightarrow \quad -p(t) \leq p(x - t) - p(x).$$

Since $p(0) = 0$ and p is continuous at 0, $p(t) \rightarrow 0$ and $p(-t) \rightarrow 0$ as $t \rightarrow 0$. Hence, $p(x - t) - p(x) \rightarrow 0$ as $t \rightarrow 0$, that is, p is continuous on X . \square

8.

Proof. First, $p(0) \geq p(0 + 0) - p(0) = 0$. For nonzero x , we argue by contradiction. Assume that there exists some x with $0 < \|x\| \leq r$ such that $p(x) < 0$. Then $np(x) < 0$ for $n = 1, 2, \dots$. For n sufficiently large, $n\|x\| > r$ and therefore $p(nx) \geq 0$. However, by the subadditivity, $p(nx) \leq np(x) < 0$. Contradiction. Thus, $p(x) \geq 0$ on X . \square

9.

Proof. For all $x_1 = \alpha_1 x_0, x_2 = \alpha_2 x_0 \in Z$ and scalars a_1 and a_2 ,

$$\begin{aligned} f(a_1 x_1 + a_2 x_2) &= f((a_1 \alpha_1 + a_2 \alpha_2) x_0) = (a_1 \alpha_1 + a_2 \alpha_2) p(x_0) \\ &= a_1 \alpha_1 p(x_0) + a_2 \alpha_2 p(x_0) = a_1 f(x_1) + a_2 f(x_2). \end{aligned}$$

Thus, f is linear. Now we show that for $\alpha \in \mathbb{R}$, $\alpha p(x_0) \leq p(\alpha x_0)$ to complete the proof. If $\alpha \geq 0$, then it follows from the positive homogeneity. For negative α , $\alpha p(x_0) = -p(-\alpha x_0)$ and by Prob. 4, $-p(-\alpha x_0) \leq p(\alpha x_0)$. Thus, $f(x) \leq p(x)$ for all $x \in Z$. \square

10.

Proof. Let Z and f have the same meaning as in Prob. 9. By Hahn-Banach theorem, there exists a linear extension \tilde{f} of f to X with $\tilde{f}(x) \leq p(x)$ for all $x \in X$. Replacing x with $-x$ gives $\tilde{f}(-x) \leq p(x)$. Finally, the linearity of \tilde{f} yields $-p(-x) \leq \tilde{f}(x)$. \square

4.3 Hahn-Banach Theorem for Normed Spaces

1.

Proof. By (2), $p(2 \times 0) = 2p(0)$. Hence, $p(0) = 0$. And for every $x \in X$, by (1),

$$0 = p(0) \leq p(x) + p(-x) = 2p(x),$$

that is, $p(x) \geq 0$. □

2.

Proof. By (1), $p(x) = p(x - y + y) \leq p(x - y) + p(y)$. Therefore, $p(x) - p(y) \leq p(x - y)$. Interchange the roles of x and y and we obtain $p(y) - p(x) \leq p(y - x) = p(x - y)$, where the equality comes from (2). Thus, $|p(x) - p(y)| \leq p(x - y)$. □

7.

Proof. Define \tilde{f} to be $x \mapsto \langle x, x_0 / \|x_0\| \rangle$. Clear that it is a bounded linear functional on X and $\tilde{f}(x_0) = \|x_0\|$. And by Riesz's Theorem, $\|\tilde{f}\| = \|x_0\| / \|x_0\| = 1$. □

8.

Proof. It follows immediately from Theorem 4.3-3. □

13.

Proof. Just put $\hat{f} = \tilde{f} / \|x_0\|$. □

14.

Proof. By Prob 13, there exists a $\hat{f} \in X'$ such that $\|\hat{f}\| = 1/r$ and $\hat{f}(x_0) = 1$. Let hyperplane $H_0 = \{x \in X : \hat{f}(x) = 1\}$ and half space $S_0 = \{x \in X : \hat{f}(x) \leq 1\}$. Then clear that $x_0 \in H_0$ and for all $x \in S(0; r)$, $\hat{f}(x) \leq \|\hat{f}\| \|x\| = r/r = 1$. Hence, $x \in S_0$. □

15.

Proof. If $\|x\| = c + 2\varepsilon > c$, then by Corollary 4.3-4, there exists some $0 \neq f \in X'$ such that $|f(x)| / \|f\| \geq c + \varepsilon$. Consequently, the functional $g = f / \|f\|$, which is of norm 1, is such that $|g(x_0)| > c$. Contradiction. □

4.5 Adjoint Operator

9.

Proof. Note that every bounded linear functional is continuous by Theorem 2.7-9. Hence, $M^a = (\mathcal{R}(T))^a$. Thus, $g \in M^a$ iff $g \in (\mathcal{R}(T))^a$ iff $g(Tx) = (T^\times g)(x) = 0$ for all $x \in X$ iff $T^\times g = 0$ iff $g \in \mathcal{N}(T^\times)$. Namely, $M^a = \mathcal{N}(T^\times)$. □

10.

Proof. For every $y = Tx \in \mathcal{R}(T)$, $g(Tx) = (T^\times g)(x) = 0$ for all $g \in \mathcal{N}(T^\times)$. Hence, $y \in {}^a\mathcal{N}(T^\times)$. □

4.6 Reflexive Spaces

2.

Proof. Since Y is a closed subspace of a Hilbert space, it is complete. By Lemma 3.3-2, there is some $y \in Y$ such that $\|x_0 - y\| = \delta$ and $z = x_0 - y$ is orthogonal to Y . Define \tilde{f} by $x \mapsto \langle x, z \rangle / \delta$. Then clear that $\tilde{f} \in X'$ and $\tilde{f}(y) = 0$ for all $y \in Y$. Meanwhile, by Riesz's Theorem, $\|\tilde{f}\| = \|z\| / \delta = 1$. Finally,

$$\tilde{f}(x_0) = \frac{\langle x_0, x_0 - y \rangle}{\delta} = \frac{\langle x_0 - y + y, x_0 - y \rangle}{\delta} = \delta.$$

The proof is then completed. \square

3.

Proof. We denote the canonical mapping from X to X'' by C and the one from X' to X''' by D . Our goal is to find a $f \in X'$ for every given $h \in X'''$ such that $D(f) = h$, that is, for every $g \in X''$, $D(f)(g) = h(g)$. Since X is reflexive, there is some $x \in X$ such that $g = Cx$. Put $f = hC$, which is clearly an element of X' . Since

$$h(g) = h(Cx) = (hC)(x) = f(x) \quad \text{and} \quad D(f)(g) = g(f) = (Cx)(f) = f(x),$$

$h = D(hC)$. Thus, X' is reflexive. \square

4.

Proof. By Prob. 3, the reflexivity of X implies the reflexivity of X' . Now we suppose X' is reflexive. Hence, again by Prob. 3, X'' is reflexive and therefore, by Theorem 4.6-4, is complete. Since X is isomorphic to $\mathcal{R}(C) \subset X''$ and $\mathcal{R}(C)$, a closed subspace of a reflexive Banach space, is reflexive, so is X . Thus, a Banach space X is reflexive iff X' is reflexive. \square

5.

Proof. It suffices to show that $\delta > 0$ and then putting $h = \tilde{f}/\delta$ will complete the proof. If $\delta = 0$, then by the definition of the infimum, there exists $(y_n) \subset Y$ which converges to x_0 . Then $x_0 \in Y$ since Y is closed, which contradicts our choice of x_0 . Thus, $\delta > 0$. \square

6.

Proof. We may assume without loss of generality that $Y_2 \setminus Y_1$ is nonempty. Choose arbitrary $x_0 \in Y_2 \setminus Y_1 \subset X \setminus Y_1$. By Prob. 6, there exists some $h \in X'$ such that $h(x_0) = 1$ and $h \in Y^a$. Thus, the annihilators of Y_1 and Y_2 are different. \square

7.

Proof. If Y is proper, then by Prob. 7, the annihilators of Y and X do not coincide, which contradicts our hypothesis. Hence, $X = Y$. \square

8.

Proof. If $x \in A$, then for every $f \in X'$ whose restriction to M is 0, $f(x_0) = 0$ since f , being bounded, is continuous. For the converse, note that $f|_M = 0$ implies $f|_A = 0$. If $x_0 \notin A$, then Prob. 5 guarantees the existence of some $f \in X'$ which vanishes on A and is nonzero at x_0 . Contradiction. Thus, $x_0 \in A$. \square

9.

Proof. If M is total, then clear that every $f \in X'$ vanishing on M is zero everywhere on X . And the converse part follows immediately from Prob. 8. \square

10.

Proof. Let $\{b_1, \dots, b_n\}$ be a linearly independent subset of X and define

$$\beta_i : \text{span}\{b_1, \dots, b_n\} \rightarrow \mathbb{F} \quad \text{by} \quad b_j \mapsto \delta_{ij}$$

for $i = 1, \dots, n$. By Hahn-Banach Theorem, we can extend them to linear functionals $\tilde{\beta}_i$ on X . Suppose that $f = x_1\tilde{\beta}_1 + \dots + x_n\tilde{\beta}_n = 0$. Then $0 = f(b_i) = x_i$ for all i . Thus, $\{\tilde{\beta}_1, \dots, \tilde{\beta}_n\}$ is linearly independent. \square

4.7 Uniform Boundedness Theorem

1.

Solution. Meager, since \mathbb{Q} is the union of all singleton of rational numbers. \square

5.

Proof. First we suppose M is rare and argue by contradiction. If $(\bar{M})^c$ is not dense in X , i.e., there exists some $x \in X$ and $r > 0$ such that $B(x; r) \cap (\bar{M})^c = \emptyset$. Hence, $B(x; r) \subset \bar{M}$, which contradicts the definition of rare subsets. Thus, $(\bar{M})^c$ is dense in X .

Now we suppose $(\bar{M})^c$ is dense in X . Then for all $x \in \bar{M}$ and $r > 0$, there exists some $y_r \notin \bar{M}$ but $y_r \in B(x; r)$. Hence, x is not an interior point. Thus, M is rare. \square

6.

Proof. If both M and M^c are meager, then so is their union X , but Baire's theorem says that a complete metric space is nonmeager in itself. Hence, M^c is nonmeager if M is. \square

7.

Proof. We argue by contradiction. Assume that for all $x \in X$, $\sup_n \|T_n x\| < \infty$. Then by the uniform boundedness theorem, there exists some c such that $\|T_n\| \leq c$ for all n . Hence, $\sup_n \|T_n\| \leq c$. Contradiction. \square

10.

Proof. We may assume without loss of generality that $\eta_1 \neq 0$. Define $T_n : c_0 \rightarrow \mathbb{C}$ by $(\xi_j) \mapsto \sum_{j=1}^n \xi_j \eta_j$. Clear that T_n are linear functionals. And since

$$|T_n x| = \left| \sum_{j=1}^n \eta_j \xi_j \right| \leq \max_{j=1, \dots, n} |\xi_j| \sum_{j=1}^n |\eta_j| \leq \|x\| \sum_{j=1}^n |\eta_j|, \quad (4)$$

T_n are bounded and $\|T_n\| \leq \sum_{j=1}^n |\eta_j|$. Meanwhile, define $y = (\gamma_j)$ by

$$\gamma_j = \begin{cases} \operatorname{sgn} \eta_j, & j \leq n, \\ 0, & j > n. \end{cases}$$

Clear that $y \in c_0$ and $\|y\| = 1$. Since $|T_n y| = \sum_{j=1}^n |\eta_j|$, together with (4), we conclude $\|T_n\| = \sum_{j=1}^n |\eta_j|$.

By Prob. 2, Sec 2.3, c_0 is a Banach space. And for each $x = (\xi_j) \in c_0$, since $\sum \xi_j \eta_j$ converges, $\|T_n x\|$ is bounded for n large enough and therefore bounded for all n . Hence, by the uniform boundedness theorem, $\sum_{j=1}^n |\eta_j| = \|T_n\| \leq c$ for some fixed c . Thus, $\sum |\eta_j| < \infty$. \square

11.

Proof. By Prob. 4, Sec 1.4, the Cauchy sequence $(T_n x)$ is bounded. Thus, by the uniform boundedness theorem, $(\|T_n\|)$ is bounded. \square

13.

Proof. Let $C : X \rightarrow X''$ be the canonical embedding and $(\varphi_n) = (C x_n)$. By Lemma 4.6-1, $\|x_n\| = \|\varphi_n\|$. Note that X'' , the dual space of X' , is complete and $f(x_n) = \varphi_n(f)$. Thus, by the uniform boundedness theorem, $(\|x_n\|) = (\|\varphi_n\|)$ is bounded. \square

14.

Proof.

(a) \Rightarrow (c): It follows immediately from $|g(T_n x)| \leq \|g\| \|x\| \|T_n\|$.

(c) \Rightarrow (b): For fixed $x \in X$, let $\varphi_n = C(T_n x)$, where $C : Y \rightarrow Y''$ is the canonical embedding. For every $g \in Y'$, by (c), $|\varphi_n(g)| = |g(T_n x)| \leq c_g$. Since Y' is complete, by the uniform boundedness theorem, $(\|\varphi_n\|) = (\|T_n x\|)$ is bounded.

(b) \Rightarrow (a): It is just what the uniform boundedness theorem states. \square

4.8 Strong and Weak Convergence

1.

Proof. The mapping $\varphi_t : C[a, b] \rightarrow \mathbb{F}$, $x \mapsto x(t)$ is a bounded linear functional on $C[a, b]$. Hence, by the definition of weak convergence, $x_n(t) \rightarrow x(t)$. \square

2.

Proof. For every $f \in Y'$, $fT \in X'$. Since $x_n \xrightarrow{w} x_0$, $(fT)(x_n) \rightarrow (fT)(x_0)$, that is, $Tx_n \xrightarrow{w} Tx_0$. \square

4.

Proof. If $x_0 = 0$, then it is trivial. Otherwise, by Theorem 4.3-3, there exists some $f \in X'$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$. Since $x_n \xrightarrow{w} x$, $|f(x_n)| \rightarrow |f(x_0)| = \|x_0\|$. Meanwhile, $|f(x_n)| \leq \|f\|\|x_n\| = \|x_n\|$. Thus, $\underline{\lim} \|x_n\| \geq \|x_0\|$. \square

5.

Proof. If $\bar{Y} = X$, then there is nothing to be proved. Otherwise we argue by contradiction. Assume that $x_0 \in X \setminus \bar{Y} \neq \emptyset$. Then by Lemma 4.6-7, there exists some $f \in X'$ such that $f(Y) = \{0\}$ and $f(x_0) = \delta > 0$. However, since $x_n \in Y$ and $x_n \xrightarrow{w} x_0$, $f(x_0)$ must be 0. Contradiction. Thus, $x_0 \in \bar{Y}$. \square

6.

Proof. It follows immediately from Prob. 5. \square

7.

Proof. It follows immediately from Prob. 5. \square

8.

Proof. For every $f \in X'$, by the definition, $|f(x_n)| < c_f$ for some constant c_f which does not depend on n . Let $g_n = Cx_n$, where $C : X \rightarrow X''$ is the canonical embedding. Then for all n , $|g_n(f)| = |f(x_n)| \leq c_f$. Since X' is complete, by the uniform boundedness theorem, (g_n) is bounded and therefore so is (x_n) . \square

9. Note that we do not regard a sequence with only repeated elements when n is large as a sequence here.

Proof. We argue by contradiction. Assume that A is unbounded, that is, there exists a $(a_n) \subset A$ with $\|a_n\| \geq n$. Clear that every sequence in (a_n) is unbounded and therefore, by Prob. 8, is not a weak Cauchy sequence, contradicting our hypothesis. Thus, A is bounded. \square

10.

Proof. Let (x_n) be a weak Cauchy sequence in X . Let $\varphi_n = Cx_n$, where $C : X \rightarrow X''$ is the canonical embedding. For every $f \in X'$, since (x_n) is a weak Cauchy sequence, $(\varphi_n(f)) = (f(x_n))$ is a Cauchy sequence in \mathbb{F} and therefore $\lim \varphi_n(f)$ exists. Let $\varphi : X' \rightarrow \mathbb{F}$ be defined by $f \mapsto \lim \varphi_n(f)$. Clear that it is linear. Meanwhile, since (x_n) , a weak Cauchy sequence, is bounded by Prob. 8, (φ_n) is bounded by, say, c . Therefore, φ is bounded since $|\varphi_n(f)| \leq c\|f\|$ for all n . Thus, $\varphi \in X''$.

Since X is reflexive, there exists some $x_0 \in X$ such that $\varphi = Cx_0$. For all $f \in X'$,

$$f(x_n) = \varphi_n(f) \rightarrow \varphi(f) = f(x_0) \quad \text{as } n \rightarrow \infty.$$

Thus, $x_n \xrightarrow{w} x_0$. Thus, X is weakly complete. \square

4.9 Convergence of Sequences of Operators

4.

Proof. First suppose that $f_n \xrightarrow{w} f$, namely, $\varphi(f_n) \rightarrow \varphi(f)$ for all $\varphi \in X''$. For every $x \in X$, denoting the canonical embedding by C ,

$$|f_n(x) - f(x)| = |(Cx)(f_n) - (Cx)(f)| \rightarrow 0,$$

since $f_n \xrightarrow{w} f$. Thus, $f_n \xrightarrow{w^*} f$.

Now we suppose $f_n \xrightarrow{w^*} f$ and X is reflexive. Then for each $\varphi \in X''$, there is an $x \in X$ such that $\varphi = Cx$. Hence,

$$|\varphi(f_n) - \varphi(f)| = |f_n(x) - f(x)| \rightarrow 0,$$

since $f_n \xrightarrow{w^*} f$. Thus, $f_n \xrightarrow{w} f$. □

6.

Proof. If $T_n \rightarrow T$, then for every $\varepsilon > 0$, there is an N such that for all $n > N$, $\|T_n - T\| < \varepsilon$. Hence, for all x with $\|x\| = 1$,

$$\|T_n x - T x\| \leq \|T_n - T\| \|x\| < \varepsilon.$$

Now we suppose the converse. Since $\|T_n - T\| = \sup_{\|x\|=1} \|T_n x - T x\|$, there is some x of norm 1 such that

$$\|T_n - T\| - \varepsilon < \|T_n x - T x\| < \varepsilon.$$

Thus, $\|T_n - T\| \leq 2\varepsilon$, which implies $T_n \rightarrow T$. □

7.

Proof. For every $x \in X$, since $T_n x$ converges, $(\|T_n x\|)$ is bounded. As X is complete, this implies $(\|T_n\|)$ is bounded by the uniform boundedness theorem. □

9.

Proof. For every $x \in X$ with $\|x\| = 1$,

$$\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$$

where the first equality comes from the continuity of the norm. Thus, $\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$. □

10.

Proof. Since X is separable, we can find a total sequence (b_n) of X . Now we choose a subsequence of $(f_n) \subset X'$ as follows. First we choose a subsequence of $(f_k(x_n))$ for each fixed n inductively. Since M is bounded, $(f_k(x_1))$ is a bounded sequence in \mathbb{R} and therefore has a convergent subsequence $f_{k_i}(x_1)$. For x_2 we choose the subsequence of $(f_{k_i}(x_2))$ which is convergent. We proceed in this way and choose a sequence of $(f_k(x_n))$ for each n . Now, consider the subsequence (g_k) of (f_n) whose n -th element is the n -th functional in the n -th choice. It can be verified that it is a subsequence. And by our construction, $(f_k(x_n))$ converges for every x_n . Hence, by Corollary 4.9-7, it is weak* convergent. □

4.12 Open Mapping Theorem

2.

Solution. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto e^x$. Clear that f is open. However, it maps \mathbb{R} , a closed set, onto $(0, \infty)$, an open set. \square

4.

Proof. It follows from the fact that $\|x_k\| < 1/2^k$. \square

5.

Proof. The linearity and boundedness is clear. Meanwhile, it is easy to verify that $T^{-1}((\xi_k)) = (k\xi_k)$. Let $x_n = (\xi_k^{(n)})$ be defined by $\xi_k^{(n)} = \delta_{nk}$. Then $\|T^{-1}x_n\| = n$. Hence, T^{-1} is unbounded.

This does not contradict with 4.12-2 since X is not complete. For example, The sequence $(x_n) \subset X$ where $x_n = (\xi_k^{(n)})$ is defined by

$$\xi_k^{(n)} = \begin{cases} 1/k, & k < n, \\ 0, & k \geq n \end{cases}$$

is a Cauchy sequence which does not converge in X . \square

6.

Proof. If $\mathcal{R}(T)$ is closed in Y , then it is complete and therefore T is a bijective bounded linear operator from X onto $\mathcal{R}(T)$. Hence, by the open mapping theorem, T^{-1} is bounded.

If T^{-1} is bounded, then it is continuous. Therefore, $\mathcal{R}(T)$, the preimage of the closed set X , is closed. \square

7.

Proof. The existence of such a b comes from the definition of boundedness. Since T is bijective and both X and Y are complete, by the bounded inverse theorem, T^{-1} exists and is bounded. Hence, for all $y = Tx \in Y$, there is a $c > 0$ such that

$$\|x\| = \|T^{-1}y\| \leq c\|y\| = c\|Tx\|.$$

Thus, putting $a = 1/c > 0$, we obtain $a\|x\| \leq \|Tx\|$ for all $x \in X$. \square

9.

Proof. Let $J : X_2 \rightarrow X_1$ be the identity map. Clear that J a bijective linear operator. Then $\|x\|_1 \leq c\|x\|_2$ implies

$$\frac{\|Jx\|_1}{\|x\|_2} \leq c \quad \text{for all nonzero } x \in X_2.$$

Namely, J is bounded. Since X_1 and X_2 are complete, by the bounded inverse theorem, the inverse I of J is also a bounded linear operator. Thus, for all $x \in X_2$

$$\|x\|_2 = \|Ix\|_2 \leq \|I\|\|x\|_1.$$

\square