Linear Algebra Done Right

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Contents

3 Linear Map

3.A The Vector Space of Linear Maps

1.

Proof. If T is linear, then $T(0,0,0) = 0$ and therefore $b = 0$. Meanwhile, $T(2,2,2) = 2T(1,1,1)$ implies $12 + 8c = 12 + 2c$. Hence, $c = 0$. The proof of the converse part is trivial. $\square$

3.

Proof. Let e_i be the i-th vector in the standard base of $\mathbb{F}^n$ and suppose that $Te_i = \sum_{j=1}^{n} A_{1,j}e_j$. Then for $x = (x_1, \ldots, x_n)^T \in \mathbb{F}^n$,

$$Tx = T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i T e_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{j,i} e_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} A_{j,i} x_i\right) e_j.$$

$\square$

5.

Proof. Too lengthy to write it down... $\square$

7.

Proof. Let $\{x_0\}$ be a basis of V and λ be a scalar such that $Tx_0 = \lambda x_0$. By the linearity of T, for every $x = kx_0$ in V, $Tx = kTx_0 = k\lambda x_0 = \lambda(kx_0) = \lambda x$. $\square$

9.

Solution. From the additivity condition we can derive that $\varphi(kz) = k\varphi(z)$ for any $k \in \mathbb{Q}$. Hence we can try some functions where $\varphi(iz) = i\varphi(z)$ fails. It turns out that $\varphi(z) = \mathrm{Im}(z)$ is one of the maps required. $\square$

11.

Proof. Let $\{\alpha_1, \ldots, \alpha_p\}$ and $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\}$ be bases of U and V respectively. Then the linear map which maps α_i to $T\alpha_i$ and maps β to 0. Clear that it is the desired linear map. $\square$

13.

Proof. Suppose that v_k is in the span of the other vectors and let $w_i = 0$ for each $i \neq k$ and $w_k \neq 0$. No $T \in \mathcal{L}(V, W)$ can maps v_i to w_i since the linearity of T would force w_k to be 0, leading to a contradiction. $\square$

3.B Null Spaces and Ranges

2.

Proof. Since S maps every vector of V into the null space of T, the map TS is the zero map. Hence $(ST)^2 = S(TS)T = 0$. $\square$

4.

Proof. Suppose $S, T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$ maps and only maps e_1, e_2, e_3 and e_3, e_4, e_5 to the zero vector respectively. Then $e_1, e_2, e_4, e_5 \notin \operatorname{null}(S+T)$, implying that $\dim \operatorname{null}(S+T) < 2$. Hence $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4 : \dim \operatorname{null} T > 2)\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. □

6.

Proof. It follows immediately from the rank-nullity theorem and the fact that $\dim \operatorname{null} T$ and $\dim \operatorname{range} T$ are integers. □

8.

Proof. Let $\{w_1, \dots, w_m\}$ be a basis of W and $S, T \in \mathcal{L}(V, W)$ be two linear maps such that $\operatorname{range} S = \operatorname{span}(w_1)$ and $\operatorname{range} T = \operatorname{span}(w_2, \dots, w_n)$. Clear that $\operatorname{range}(S+T) = W$. Hence, the set described is not a subspace of $\mathcal{L}(V, W)$. □

10.

Proof. For every $y \in \operatorname{range} T$ there exists some $x = \sum x_i v_i \in V$ such that

$$y = Ty = T\left(\sum_{i=1}^{n} x_i v_i\right) = \sum_{i=1}^{n} x_i T v_i.$$

Hence, $\operatorname{range} T = \operatorname{span}(Tv_1, \dots, Tv_n)$. □

12. For readers who familiar with the orbit-stabilizer theorem or just the (group) homomorphism, the proof should be straightforward.

Proof. For every nonzero y in $\operatorname{range} T$, there exists some $x \in V$ such that $Tx = y$. For each $y \neq 0$, we choose one such x, put them all together and put 0 into them to get U. By the construction, clear that $T(U) = \operatorname{range} T$ and , $U \cap \operatorname{null} T = \{0\}$. □

14.

Proof. By the rank-nullity theorem,

$$\dim \operatorname{null} T + \dim \operatorname{range} T = 8 \quad \Rightarrow \quad \dim \operatorname{range} T = 5 = \dim \mathbb{R}^5.$$

Hence, $\operatorname{range} T = \mathbb{R}^5$ and therefore T is surjective. □

16. Actually, the cosets of the kernel partition the whole space.

Proof. Let $\{v_1, \dots, v_n\}$ be a basis of $\operatorname{range} T$ and $Tu_i = v_i$ for $i = 1, 2, \dots, n$. Denote $\operatorname{span}(u_1, \dots, u_n)$ by U. We now prove that $V = U + \operatorname{null} T$. For every $x \in V$, suppose that $Tx = y = \sum y_i v_i$ and $\tilde{x} = \sum y_i u_i$. Note that $\tilde{x} \in U$ and $T(x - \tilde{x}) = Tx - T\tilde{x} = 0$, i.e., $x - \tilde{x} \in \operatorname{null} T$. Hence, $V = U + \operatorname{null} T$. As both of U and $\operatorname{null} T$ are finite-dimensional, so is V. □

18.

Proof. By the rank-nullity theorem, clear that $\dim V \geq \dim \operatorname{range} T = \dim W$ if there exists some surjective $T \in \mathcal{L}(V, W)$.

Assume that $\dim V \geq \dim W$ and let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be bases of V and W respectively. Then the linear map which maps v_i to w_i for each $1 \leq i \leq m$ is surjective. $\square$

20.

Proof. If T is injective, then for every $y \in \operatorname{range} T$, there exists exactly one $x \in V$ such that $y = Tx$. Let S be the map which maps y to such x. It is linear since for every $y_1, y_2 \in \operatorname{range} T$ and scalar a, b, supposing $Sy_i = x_i$,

$$T(ax_1 + bx_2) = aTx_1 + bTx_2 = ay_1 + by_2.$$

implying $S(ay_1 + by_2) = ax_1 + bx_2 = aSy_1 + bSy_2$. For every $x \in V$, $(ST)x = S(Tx) = x$.

Suppose there exists some $S \in \mathcal{L}(W, V)$ such that $ST = I$. Then

$$Tx_1 = Tx_2 \quad \Rightarrow \quad STx_1 = STx_2 \quad \Rightarrow \quad x_1 = x_2.$$

Hence, T is injective. $\square$

22.

Proof. Let $\tilde{T}$ be the restriction of T to $\operatorname{null} ST$. It is still a linear map since $\operatorname{null} ST$ is a subspace of U. Note that $x \in \operatorname{null} ST$ iff $(ST)x = 0$ iff $Tx \in \operatorname{null} S$. Hence, $\operatorname{range} \tilde{T} \subset \operatorname{null} S$. Thus, by the rank-nullity theorem,

$$\dim \operatorname{range} \tilde{T} \leq \dim \operatorname{null} S \quad \Rightarrow \quad \dim \operatorname{null} ST - \dim \operatorname{null} \tilde{T} \leq \dim \operatorname{null} S.$$

Since $\operatorname{null} \tilde{T} \leq \operatorname{null} T$, this implies $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$. $\square$

24.

Proof. If there exists $S \in \mathcal{L}(W, W)$ such that $T_2 = ST_1$, then $\operatorname{null} T_2 = \operatorname{null} ST_1$. Hence for every $x \in \operatorname{null} T_1$, as $S(T_1 x) = S0 = 0$, $x \in \operatorname{null} T_2$. Therefore, $\operatorname{null} T_1 \subset \operatorname{null} T_2$.

Now we suppose $\operatorname{null} T_1 \subset \operatorname{null} T_2$ and construct S. Note that all we concerns is its behavior on some basis of $\operatorname{range} T_1$. Let $\{w_1, \ldots, w_n\}$ be a basis of $\operatorname{range} T_1$ and $T_1 v_i = w_i$ for $i = 1, \ldots, n$. For each $x \in V$, let $U_x = \{x + y : y \in \operatorname{null} T_2\}$ and $Sw_k = T_2 x$ if $v_k \in U_x$. It can be verified that S is well-defined and does satisfy the requirement as long as $\operatorname{null} T_1 \subset \operatorname{null} T_2$. $\square$

26.

Proof. Let $\mathcal{P}_n(\mathbb{R}) = \{p \in \mathcal{P}(\mathbb{R}) : \deg p \leq n\}$, which are some subspaces of $\mathcal{P}(\mathbb{R})$. We now prove that D is a surjective linear map onto $\mathcal{P}_n(\mathbb{R})$ for every nonnegative integer n by induction.

Suppose $Dx = c_0 \neq 0$, then for any $0 \neq c \in \mathcal{P}_0(\mathbb{R})$, $D(cx/c_0) = c$. Hence, D is a surjective map onto $\mathcal{P}_0(\mathbb{R})$. Assume that D is a surjective map onto $\mathcal{P}_{k-1}(\mathbb{R})$ and suppose $Dx^{k+1} = p = a_0 + a_1 x + \cdots + a_k x^k$ where $a_k \neq 0$. For every nonzero b_k and $q = b_0 + b_1 x + \cdots + b_k x^k \in \mathcal{P}_k(\mathbb{R})$, let r be a polynomial with degree $\leq k - 1$ such that

$q = b_k/a_k p + r$. By our induction hypothesis, there exists some polynomial $\tilde{r}$ such that $D\tilde{r} = r$. Then

$$D(b_k/a_k x^{k+1} + \tilde{r}) = \frac{b_k}{a_k} D x^{k+1} + D\tilde{r} = \frac{b_k}{a_k} p + r = q.$$

Hence, D is also a surjective map onto $\mathcal{P}_k(\mathbb{R})$. Thus, D is surjective. □

28. TODO

30. TODO

3.D Invertibility and Isomorphic Vector Spaces

1.

Proof. Clear that the linear map $T^{-1}S^{-1}$ is right and left inverse of ST and therefore ST is invertible. And by the uniqueness of the inverse, $(ST)^{-1} = T^{-1}S^{-1}$. □

3.

Proof. First we suppose the existence of such an operator, then T^{-1} is also the inverse of S. Hence S is invertible and therefore injective.

Now we suppose S is injective. Let $\{u_1, \ldots, u_m\}$ and $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ be bases of U and V respectively. $\{Su_1, \ldots, Su_m\}$ is linearly independent as S is injective and therefore we can expand it to a basis, $\{Su_1, \ldots, Su_m, v_{m+1}, \ldots, v_n\}$, of V. Let $T \in \mathcal{L}(V)$ maps u_i to Su_i for $i = 1, \ldots, m$ and u_j to v_j for $j = m+1, \ldots, n$. T is obviously injective and therefore invertible as V is finite-dimensional. □

5.

Proof. Suppose that such an S exists. Since S is invertible, $\operatorname{range} S = V$. Hence, $\operatorname{range} T_2 = \operatorname{range} T_2 S = \operatorname{range} T_1$.

Now we suppose that $\operatorname{range} T_1 = \operatorname{range} T_2$ and construct S by defining its behavior on a basis of V. Let $\{v_1, \ldots, v_m\}$ be a basis of $\operatorname{null} T_1$. As $\operatorname{range} T_1 = \operatorname{range} T_2$ implies $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$, we can set $Sv_i = u_i$ for $i = 1, \ldots, m$ where $\{u_1, \ldots, u_m\}$ is a basis of $\operatorname{null} T_2$.

Let $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ be a basis of V. Clear that $\{T_1 v_{m+1}, \ldots, T_1 v_n\}$ spans $\operatorname{range} T_1$. It is linearly independent since

$$\begin{aligned}
& x_{m+1} T_1 v_{m+1} + \cdots + x_n T_1 v_n = 0 \\
\Rightarrow \quad & T_1(x_{m+1} v_{m+1} + \cdots + x_n v_n) = 0 \\
\Rightarrow \quad & x_{m+1} v_{m+1} + \cdots + x_n v_n \in \operatorname{null} T_1 \\
\Rightarrow \quad & x_{m+1} = \cdots = x_n = 0.
\end{aligned}$$

Hence, it is a basis of $\operatorname{range} T_1$. Since $\operatorname{range} T_1 = \operatorname{range} T_2$, there exists $u_{m+1}, \ldots, u_n$ such that $T_2 u_i = T_1 v_i$ for $i = m+1, \ldots, n$. It is easy to verify that $u_1, \ldots, u_m, u_{m+1}, \ldots, u_n$ are linearly independent. Finally, for $i = m+1, \ldots, n$, we also set $Sv_i = u_i$. Clear that S is invertible and satisfies the requirement. □

7.

Proof.
(a) For any $A, B \in E$ and scalar a, b,

$$(aA + bB)v = a(Av) + b(Bv) = 0.$$

Hence, E is a subspace of $\mathcal{L}(V, W)$.
(b) Since $v \neq 0$, putting $v_1 = v$, there exists some vectors in V such that $\{v_1, \ldots, v_n\}$ is a basis of V. Let $U = \operatorname{span}(v_2, \ldots, v_n)$. It can be shown that E is isomorphic to $\mathcal{U}, \mathcal{W}$. Hence, $\dim E = (\dim V - 1) \dim W$. □

9.

Proof. If S and T are invertible, then clear that $T^{-1}S^{-1}$ is the inverse of ST. Meanwhile, if S or T is not invertible, therefore not surjective, then

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\} < \dim V.$$

Hence, ST is not surjective and hence not invertible as V is finite-dimensional. Thus, ST is invertible iff S and T are invertible. □

11.

Proof. Since V is finite-dimensional and $S(TU) = (ST)U = I$, both S and U are invertible and the inverses of which are TU and ST respectively. Hence,

$$STU = I \quad \Rightarrow \quad T = S^{-1}U^{-1},$$

implying that T is also invertible and $T^{-1} = US$. □

13.

Proof. It follows almost immediately from Exercise 9 that all of R, S and T are invertible and therefore S is injective. □

15.

Proof. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{F}^{n,1}$ and suppose $Te_i = u_i$. It is easy to verify that $A = \left[\begin{smallmatrix} u_1 & \cdots & u_n \end{smallmatrix}\right]$ is a m-by-n matrix such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$. □