Solutions to *Topology*

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13 Basis for a Topology

1.

Proof. Let $\mathcal{T}$ be the topology of X. Since $\mathcal{T}$ is a basis for itself and the hypothesis implies that A is a set in the topology generated by $\mathcal{T}$, $A \in \mathcal{T}$, i.e., A is open. □

4.

Proof.

(a) Put $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$. Since $\varnothing$ and X are contained in all $\mathcal{T}_\alpha$, they are also contained in $\mathcal{T}$. Let $\{U_\beta\}_{\beta \in J}$ be an indexed family of elements of $\mathcal{T}$ and put $U = \bigcup_{\beta \in J} U_\beta$. For every β, since U_β is open with respect to each $\mathcal{T}_\alpha$, by definition, so is $\bigcup_{\beta \in J}$. Similarly, we can show that $\mathcal{T}$ is closed under finite intersection. Thus, $\mathcal{T}$ is a topology.

The union $\bigcup \mathcal{T}_\alpha$, however, may not be a topology. Take $X = \{a, b, c\}$ for example. $\mathcal{T}_a = \{\varnothing, a, X\}$ and $\mathcal{T}_b = \{\varnothing, b, X\}$ are two topologies, but their union is not.

(b) Let $\mathcal{T}$ be the intersection of all topologies containing all $\mathcal{T}_\alpha$. By (a), $\mathcal{T}$ is a topology and clear that it is the unique smallest one. Now, let $\mathcal{T}' = \bigcap T_\alpha$, which is again a topology and is contained in all T_α. It can be verified that $\mathcal{T}'$ is the unique largest one.

(c) $\{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$; $\{\varnothing, X, \{a\}\}$. □

5.

Proof. Let $\mathcal{A}$ be a basis, $\mathcal{T}$ the topology generated by $\mathcal{A}$, $\{\mathcal{T}_\alpha\}$ the collection of all topologies containing $\mathcal{A}$ and $\mathcal{T}' = \bigcap \mathcal{T}_\alpha$. For every union U of elements of $\mathcal{A}$, since, for every α, $\mathcal{A} \subset \mathcal{T}_\alpha$ and $\mathcal{T}_\alpha$ is closed under arbitrary union, $U \in \mathcal{T}_\alpha$. Hence, $\mathcal{T} \subset \mathcal{T}'$. Consequently, $\mathcal{T}'$ is also the intersection of all topologies containing $\mathcal{T}$. Since $\mathcal{T}$ contains itself as a subset, $\mathcal{T}' \subset \mathcal{T}$. Thus, $\mathcal{T} = \mathcal{T}'$.

Consider the collection of all finite intersections of $\mathcal{A}$, which is a basis, and apply the previous result to complete the proof. □

6.

Proof. Let $\mathcal{T}_l$ and $\mathcal{T}_K$ be the topology of $\mathbb{R}_l$ and $\mathbb{R}_k$ respectively. $B = (-1, 1) - K$ is a basis element of $\mathcal{T}_k$ and $0 \in B$. However, no half-open interval containing 0 is in B. Hence, $\mathcal{T}_l$ is no finer than $\mathcal{T}_K$. Conversely, $C = [1, 2)$ is a basis element of $\mathcal{T}_l$ and $1 \in C$, but as $1 \in K$, there is no basis element of $\mathcal{T}_K$ containing 1. Hence, $\mathcal{T}_K$ is no finer than $\mathcal{T}_l$. Thus, they are not comparable. □

8.

Proof.

(a) First clear that $\mathcal{B} \subset \mathcal{T}$. For every $U \in \mathcal{T}$ and $x \in U$, since U is open, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subset U$. Hence, there exists some rational a and b such that $x - \delta < a < x < b < x + \delta$. Thus, by Lemma 13.2, $\mathcal{B}$ generates the standard topology on $\mathbb{R}$.

(b) Since $x \in [\lfloor x \rfloor, \lfloor x \rfloor + 1) \in \mathcal{C}$ for every $x \in \mathbb{R}$, the first condition for a basis is satisfied. Meanwhile, for every $B_1 = [a, b)$ and $B_2 = [c, d)$ in $\mathcal{C}$, if they are not disjoint, $[c, b) = B_1 \cap B_2$ is also in $\mathcal{C}$. Hence, the second condition is satisfied. Thus, $\mathcal{C}$ is a basis.

Since $[\sqrt{2}, 2)$ can not be represented by union of elements in $\mathcal{C}$, $\mathcal{C}$ does not generate the lower limit topology. $\square$