Convex Optimization

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Contents

2 Convex Sets

2.1 Definition of convexity

1.

Proof. For $k = 2$, $\theta_1 x_1 + \theta_2 x_2 \in C$ holds by by definition. We argue by induction on k and assume that the inclusion holds for $k < m$. When $k = m$, denoting $\sum_{i=1}^{m-1} \theta_i$ by s,

$$\sum_{i=1}^{m} \theta_i x_i = s \sum_{i=1}^{m-1} \frac{\theta_i x_i}{s} + \theta_m x_m.$$

Since $\sum_{i=1}^{m-1} \theta_i / s = 1$, by the induction hypothesis, $\sum_{i=1}^{m-1} \theta_i x_i / s \in C$. Meanwhile, as $s + \theta_m = 1$, $\sum_{i=1}^{m} \theta_i x_i \in C$, completing the proof. $\square$

2.

Proof. Clear that the intersection of two convex sets is still convex. Hence, the intersection of $C \subset \mathbb{R}^n$ and any line is convex as long as C is convex.

Now we suppose that the intersection of C and any line is convex. For any $x_1, x_2 \in C$, $C_l = C \cap \{\theta x_1 + (1-\theta) x_2 : \theta \in \mathbb{R}\}$ is convex and therefore $\theta x_1 + (1-\theta) x_2 \in C_l \subset C$ for every $0 \leq \theta \leq 1$. Thus, C is convex.

The above argument, *mutatis mutandis*, gives the second result. $\square$

3.

Proof. For every $\theta \in [0, 1]$, the process of bisecting the interval implies there exists a series $\langle \delta_n \rangle$ whose sum is θ. Hence, for every $a, b \in C$, $x_n = a + (b - a) \sum_{n=1}^{\infty} \delta_n$ converges to $a + \theta(b - a)$. Meanwhile, the midpoint convexity implies $x_n \in C$ for every n. And since C is closed, $a + \theta(b - a) \in C$. Thus, C is convex. $\square$

4.

Proof. Let D be the intersection of all convex sets containing C. If $x \in C$, then its is a convex combination of some points in C. Hence, for every convex set containing C, it contains x. Therefore, $\mathbf{conv}\, C \subset D$. For the converse, since $\mathbf{conv}\, C$ itself is a convex set containing C, $D \subset \mathbf{conv}\, C$. Thus, $\mathbf{conv}\, C = D$. $\square$