Linear Algebra Done Right

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Contents

3 Linear Map

3.A The Vector Space of Linear Maps

1.

Proof. If T is linear, then $T(0,0,0) = 0$ and therefore $b = 0$. Meanwhile, $T(2,2,2) = 2T(1,1,1)$ implies $12 + 8c = 12 + 2c$. Hence, $c = 0$. The proof of the converse part is trivial. $\square$

3.

Proof. Let e_i be the i-th vector in the standard base of $\mathbb{F}^n$ and suppose that $Te_i = \sum_{j=1}^{n} A_{1,j}e_j$. Then for $x = (x_1, \ldots, x_n)^T \in \mathbb{F}^n$,

$$Tx = T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i Te_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{j,i}e_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} A_{j,i}x_i\right)e_j.$$

$\square$

5.

Proof. Too lengthy to write it down... $\square$

7.

Proof. Let $\{x_0\}$ be a basis of V and λ be a scalar such that $Tx_0 = \lambda x_0$. By the linearity of T, for every $x = kx_0$ in V, $Tx = kTx_0 = k\lambda x_0 = \lambda(kx_0) = \lambda x$. $\square$

9.

Solution. From the additivity condition we can derive that $\varphi(kz) = k\varphi(z)$ for any $k \in \mathbb{Q}$. Hence we can try some functions where $\varphi(iz) = i\varphi(z)$ fails. It turns out that $\varphi(z) = \mathrm{Im}(z)$ is one of the maps required. $\square$

11.

Proof. Let $\{\alpha_1, \ldots, \alpha_p\}$ and $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\}$ be bases of U and V respectively. Then the linear map which maps α_i to $T\alpha_i$ and maps β to 0. Clear that it is the desired linear map. $\square$

13.

Proof. Suppose that v_k is in the span of the other vectors and let $w_i = 0$ for each $i \neq k$ and $w_k \neq 0$. No $T \in \mathcal{L}(V, W)$ can maps v_i to w_i since the linearity of T would force w_k to be 0, leading to a contradiction. $\square$

3.B Null Spaces and Ranges

2.

Proof. Since S maps every vector of V into the null space of T, the map TS is the zero map. Hence $(ST)^2 = S(TS)T = 0$. $\square$

4.

Proof. Suppose $S, T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$ maps and only maps e_1, e_2, e_3 and e_3, e_4, e_5 to the zero vector respectively. Then $e_1, e_2, e_4, e_5 \notin \operatorname{null}(S+T)$, implying that $\dim \operatorname{null}(S+T) < 2$. Hence $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4 : \dim \operatorname{null} T > 2)\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. $\square$

6.

Proof. It follows immediately from the rank-nullity theorem and the fact that $\dim \operatorname{null} T$ and $\dim \operatorname{range} T$ are integers. $\square$

8.

Proof. Let $\{w_1, \ldots, w_m\}$ be a basis of W and $S, T \in \mathcal{L}(V, W)$ be two linear maps such that $\operatorname{range} S = \operatorname{span}(w_1)$ and $\operatorname{range} T = \operatorname{span}(w_2, \ldots, w_n)$. Clear that $\operatorname{range}(S+T) = W$. Hence, the set described is not a subspace of $\mathcal{L}(V, W)$. $\square$

10.

Proof. For every $y \in \operatorname{range} T$ there exists some $x = \sum x_i v_i \in V$ such that

$$y = Ty = T\left(\sum_{i=1}^{n} x_i v_i\right) = \sum_{i=1}^{n} x_i T v_i.$$

Hence, $\operatorname{range} T = \operatorname{span}(Tv_1, \ldots, Tv_n)$. $\square$

12. For readers who familiar with the orbit-stabilizer theorem or just the (group) homomorphism, the proof should be straightforward.

Proof. For every nonzero y in $\operatorname{range} T$, there exists some $x \in V$ such that $Tx = y$. For each $y \neq 0$, we choose one such x, put them all together and put 0 into them to get U. By the construction, clear that $T(U) = \operatorname{range} T$ and , $U \cap \operatorname{null} T = \{0\}$. $\square$

14.

Proof. By the rank-nullity theorem,

$$\dim \operatorname{null} T + \dim \operatorname{range} T = 8 \quad \Rightarrow \quad \dim \operatorname{range} T = 5 = \dim \mathbb{R}^5.$$

Hence, $\operatorname{range} T = \mathbb{R}^5$ and therefore T is surjective. $\square$

16. Actually, the cosets of the kernel partition the whole space.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of $\operatorname{range} T$ and $Tu_i = v_i$ for $i = 1, 2, \ldots, n$. Denote $\operatorname{span}(u_1, \ldots, u_n)$ by U. We now prove that $V = U + \operatorname{null} T$. For every $x \in V$, suppose that $Tx = y = \sum y_i v_i$ and $\tilde{x} = \sum y_i u_i$. Note that $\tilde{x} \in U$ and $T(x - \tilde{x}) = Tx - T\tilde{x} = 0$, i.e., $x - \tilde{x} \in \operatorname{null} T$. Hence, $V = U + \operatorname{null} T$. As both of U and $\operatorname{null} T$ are finite-dimensional, so is V. $\square$

18.

Proof. By the rank-nullity theorem, clear that $\dim V \geq \dim \operatorname{range} T = \dim W$ if there exists some surjective $T \in \mathcal{L}(V, W)$.

Assume that $\dim V \geq \dim W$ and let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be bases of V and W respectively. Then the linear map which maps v_i to w_i for each $1 \leq i \leq m$ is surjective. $\square$

20.

Proof. If T is injective, then for every $y \in \operatorname{range} T$, there exists exactly one $x \in V$ such that $y = Tx$. Let S be the map which maps y to such x. It is linear since for every $y_1, y_2 \in \operatorname{range} T$ and scalar a, b, supposing $Sy_i = x_i$,

$$T(ax_1 + bx_2) = aTx_1 + bTx_2 = ay_1 + by_2.$$

implying $S(ay_1 + by_2) = ax_1 + bx_2 = aSy_1 + bSy_2$. For every $x \in V$, $(ST)x = S(Tx) = x$.

Suppose there exists some $S \in \mathcal{L}(W, V)$ such that $ST = I$. Then

$$Tx_1 = Tx_2 \quad \Rightarrow \quad STx_1 = STx_2 \quad \Rightarrow \quad x_1 = x_2.$$

Hence, T is injective. $\square$

22.

Proof. Let $\tilde{T}$ be the restriction of T to $\operatorname{null} ST$. It is still a linear map since $\operatorname{null} ST$ is a subspace of U. Note that $x \in \operatorname{null} ST$ iff $(ST)x = 0$ iff $Tx \in \operatorname{null} S$. Hence, $\operatorname{range} \tilde{T} \subset \operatorname{null} S$. Thus, by the rank-nullity theorem,

$$\dim \operatorname{range} \tilde{T} \leq \dim \operatorname{null} S \quad \Rightarrow \quad \dim \operatorname{null} ST - \dim \operatorname{null} \tilde{T} \leq \dim \operatorname{null} S.$$

Since $\operatorname{null} \tilde{T} \leq \operatorname{null} T$, this implies $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$. $\square$

24.

Proof. If there exists $S \in \mathcal{L}(W, W)$ such that $T_2 = ST_1$, then $\operatorname{null} T_2 = \operatorname{null} ST_1$. Hence for every $x \in \operatorname{null} T_1$, as $S(T_1 x) = S0 = 0$, $x \in \operatorname{null} T_2$. Therefore, $\operatorname{null} T_1 \subset \operatorname{null} T_2$.

Now we suppose $\operatorname{null} T_1 \subset \operatorname{null} T_2$ and construct S. Note that all we concerns is its behavior on some basis of $\operatorname{range} T_1$. Let $\{w_1, \ldots, w_n\}$ be a basis of $\operatorname{range} T_1$ and $T_1 v_i = w_i$ for $i = 1, \ldots, n$. For each $x \in V$, let $U_x = \{x + y \ : \ y \in \operatorname{null} T_2\}$ and $Sw_k = T_2 x$ if $v_k \in U_x$. It can be verified that S is well-defined and does satisfy the requirement as long as $\operatorname{null} T_1 \subset \operatorname{null} T_2$. $\square$

26.

Proof. Let $\mathcal{P}_n(\mathbb{R}) = \{p \in \mathcal{P}(\mathbb{R}) \ : \ \deg p \leq n\}$, which are some subspaces of $\mathcal{P}(\mathbb{R})$. We now prove that D is a surjective linear map onto $\mathcal{P}_n(\mathbb{R})$ for every nonnegative integer n by induction.

Suppose $Dx = c_0 \neq 0$, then for any $0 \neq c \in \mathcal{P}_0(\mathbb{R})$, $D(cx/c_0) = c$. Hence, D is a surjective map onto $\mathcal{P}_0(\mathbb{R})$. Assume that D is a surjective map onto $\mathcal{P}_{k-1}(\mathbb{R})$ and suppose $Dx^{k+1} = p = a_0 + a_1 x + \cdots + a_k x^k$ where $a_k \neq 0$. For every nonzero b_k and $q = b_0 + b_1 x + \cdots + b_k x^k \in \mathcal{P}_k(\mathbb{R})$, let r be a polynomial with degree $\leq k - 1$ such that

$q = b_k/a_k p + r$. By our induction hypothesis, there exists some polynomial $\tilde{r}$ such that $D\tilde{r} = r$. Then

$$D(b_k/a_k x^{k+1} + \tilde{r}) = \frac{b_k}{a_k} Dx^{k+1} + D\tilde{r} = \frac{b_k}{a_k} p + r = q.$$

Hence, D is also a surjective map onto $\mathcal{P}_k(\mathbb{R})$. Thus, D is surjective. □

28. TODO

30. TODO

3.D Invertibility and Isomorphic Vector Spaces

1.

Proof. Clear that the linear map $T^{-1}S^{-1}$ is right and left inverse of ST and therefore ST is invertible. And by the uniqueness of the inverse, $(ST)^{-1} = T^{-1}S^{-1}$. □

3.

Proof. First we suppose the existence of such an operator, then T^{-1} is also the inverse of S. Hence S is invertible and therefore injective.

Now we suppose S is injective. Let $\{u_1, \ldots, u_m\}$ and $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ be bases of U and V respectively. $\{Su_1, \ldots, Su_m\}$ is linearly independent as S is injective and therefore we can expand it to a basis, $\{Su_1, \ldots, Su_m, v_{m+1}, \ldots, v_n\}$, of V. Let $T \in \mathcal{L}(V)$ maps u_i to Su_i for $i = 1, \ldots, m$ and u_j to v_j for $j = m+1, \ldots, n$. T is obviously injective and therefore invertible as V is finite-dimensional. □

5.

Proof. Suppose that such an S exists. Since S is invertible, $\operatorname{range} S = V$. Hence, $\operatorname{range} T_2 = \operatorname{range} T_2 S = \operatorname{range} T_1$.

Now we suppose that $\operatorname{range} T_1 = \operatorname{range} T_2$ and construct S by defining its behavior on a basis of V. Let $\{v_1, \ldots, v_m\}$ be a basis of $\operatorname{null} T_1$. As $\operatorname{range} T_1 = \operatorname{range} T_2$ implies $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$, we can set $Sv_i = u_i$ for $i = 1, \ldots, m$ where $\{u_1, \ldots, u_m\}$ is a basis of $\operatorname{null} T_2$.

Let $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ be a basis of V. Clear that $\{T_1 v_{m+1}, \ldots, T_1 v_n\}$ spans $\operatorname{range} T_1$. It is linearly independent since

$$\begin{aligned}
& x_{m+1} T_1 v_{m+1} + \cdots + x_n T_1 v_n = 0 \\
\Rightarrow\quad & T_1(x_{m+1} v_{m+1} + \cdots + x_n v_n) = 0 \\
\Rightarrow\quad & x_{m+1} v_{m+1} + \cdots + x_n v_n \in \operatorname{null} T_1 \\
\Rightarrow\quad & x_{m+1} = \cdots = x_n = 0.
\end{aligned}$$

Hence, it is a basis of $\operatorname{range} T_1$. Since $\operatorname{range} T_1 = \operatorname{range} T_2$, there exists $u_{m+1}, \ldots, u_n$ such that $T_2 u_i = T_1 v_i$ for $i = m+1, \ldots, n$. It is easy to verify that $u_1, \ldots, u_m, u_{m+1}, \ldots, u_n$ are linearly independent. Finally, for $i = m+1, \ldots, n$, we also set $Sv_i = u_i$. Clear that S is invertible and satisfies the requirement. □

7.

Proof.
(a) For any $A, B \in E$ and scalar a, b,

$$(aA + bB)v = a(Av) + b(Bv) = 0.$$

Hence, E is a subspace of $\mathcal{L}(V, W)$.
(b) Since $v \neq 0$, putting $v_1 = v$, there exists some vectors in V such that $\{v_1, \ldots, v_n\}$ is a basis of V. Let $U = \operatorname{span}(v_2, \ldots, v_n)$. It can be shown that E is isomorphic to $\mathcal{U}, \mathcal{W}$. Hence, $\dim E = (\dim V - 1) \dim W$. □

9.

Proof. If S and T are invertible, then clear that $T^{-1}S^{-1}$ is the inverse of ST. Meanwhile, if S or T is not invertible, therefore not surjective, then

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\} < \dim V.$$

Hence, ST is not surjective and hence not invertible as V is finite-dimensional. Thus, ST is invertible iff S and T are invertible. □

11.

Proof. Since V is finite-dimensional and $S(TU) = (ST)U = I$, both S and U are invertible and the inverses of which are TU and ST respectively. Hence,

$$STU = I \quad \Rightarrow \quad T = S^{-1}U^{-1},$$

implying that T is also invertible and $T^{-1} = US$. □

13.

Proof. It follows almost immediately from Exercise 9 that all of R, S and T are invertible and therefore S is injective. □

15.

Proof. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{F}^{n,1}$ and suppose $Te_i = u_i$. It is easy to verify that $A = (u_1, \ldots, u_n)$ is a m-by-n matrix such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$. □

3.E Products and Quotients of Vector Spaces

2.

Proof. We only prove the result for $m = 2$. It is easy to prove it for arbitrary m in a similar manner. Suppose that $V = V_1 \times V_2$ is finite-dimensional. Then $V_1 \times \{0\}$, a subspace of V, is finite-dimensional. Clear that V_1 is isomorphic to $V_1 \times \{0\}$ and hence it is also of finite dimension. Similarly, V_2 is finite-dimensional. □

4.

Proof. We construct the isomorphism $S : \mathcal{L}(V_1 \times \cdots \times V_n, W) \to \mathcal{L}(V_1, w) \times \cdots \times \mathcal{L}(V_n, W)$ explicitly. For every $T \in \mathcal{L}(V_1 \times \cdots \times V_n, W)$, suppose $T(v_1, \ldots, v_n) = w$. Let $T_i(v_i) = w$ for $i = 1, \ldots, n$ and $ST = (T_1, \ldots, T_n)$. Clear that $T_i \in \mathcal{L}(V_i, W)$ and S is invertible. □

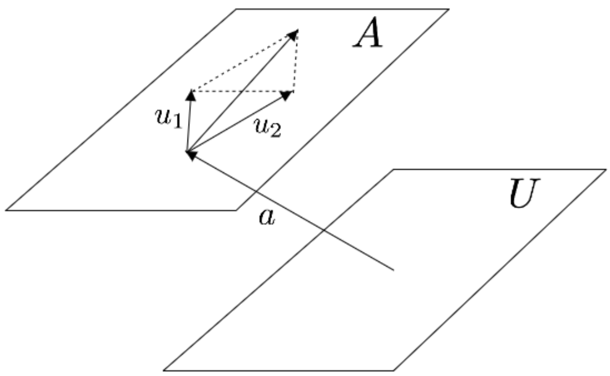
6.

Proof. We may interpret the elements in $\mathcal{L}(\mathbb{F}^n, V)$ as mappings from the "coordinates" to "abstract vectors". With this in mind, we construct the isomorphism S. For every $(v_1, \ldots, v_n) \in V^n$ and $(x_1, \ldots, x_n)^T \in \mathbb{F}^n$, let

$$(S(v_1, \ldots, v_n)) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i v_i.$$

It is easy to verify that S does satisfy the requirement. □

8. We may interpret the set of all possible $\lambda v + (1 - \lambda w)$ as the "line" through v and w. And the idea behind the proof is illustrated in the picture below.



Proof. If A is an affine subset, i.e., there exists some subspace U and $a \in V$ such that $A = a + U$, then for all $\lambda \in \mathbb{F}$ and $v, w \in A$,

$$\lambda v + (1 - \lambda)w = \lambda(a + u_1) + (1 - \lambda)(a + u_2) = a + (u_1 + (1 - \lambda)u_2) \in A$$

where u_1 and u_2 are some elements in U.

Now we suppose $\lambda v + (1 - \lambda)w \in A$ holds, fix $a \in A$ and let $U = \{a_1 - a : a_1 \in A\}$. By the hypothesis, for every scalar λ and $a_1 \in A$, $a + \lambda(a_1 - a) \in A$. Therefore, for every $u_1 = a_1 - a \in U$, $\lambda u_1 \in U$. Meanwhile, let $u_2 = a_2 - a \in U$, $(u_1 + u_2)/2 \in U$ as

$$a + \frac{1}{2}(u_1 + u_2) = a + \frac{1}{2}(a_1 + a_2 - 2a) = \frac{1}{2}a_1 + \frac{1}{2}a_2.$$

Hence, by the previous result, $u_1 + u_2 \in U$. Thus, U is a subspace and $A = a + U$ is an affine subset. □

10.

Proof. Let A be the intersection of every collection of affine subsets of V and suppose A is nonempty. Let $v, w \in A$ and $\lambda \in \mathbb{F}$. Then, by Exercise 8, for every affine subset A_α of V, $\lambda v + (1-\lambda)w \in A_\alpha$. Hence it also belongs to A. Thus, A is also an affine subset of A (as long as nonempty). $\square$

12.

Proof. Let $\{a_1 + U, \ldots, a_m + U\}$ be a basis of V/U and we first prove a small result: for every $v \in V$, there exists an unique list of $v_1, \ldots, v_m \in \mathbb{F}$ such that $v - (v_1 a_1 + \cdots v_m a_m) \in U$. Suppose that $v_1', \ldots, v_m'$ is such a list as well. Then

$$(v - (v_1 a_1 + \cdots + v_m a_m)) - (v - (v_1' a_1 + \cdots + v_m' a_m)) \in U.$$

Therefore,

$$(v_1 - v_1')a_1 + \cdots + (v_m - v_m')a_m \in U = 0 + U,$$

Hence $v_i' = v_i$ for each $i = 1, \ldots, m$, completing the proof.

Therefore, for every $v \in V$, denoting $v_1 a_i + \cdots + v_m a_m$ as a_v, we may define S to be map which maps v to $(v - a_v, a_v + U)$. Now we show that S is linear and bijective. For every $u, v \in V$ and scalar a, b,

$$\begin{aligned} aSu + bSv &= a(u - a_u, a_u + U) + b(v - a_v, a_v + U) \\ &= ((au + bv) - (aa_u + ba_v), (aa_u + ba_v) + U) \\ &= S(au + bv). \end{aligned}$$

$Su = 0$ iff $(u - a_u, a_u + U) = 0$ iff $u = a_u = 0$ and therefore S is injective. Clear that S is surjective. Thus, S is an isomorphism and V is isomorphic to $U \times (V/U)$. $\square$

16.

Proof. Clear that every vector space with dimension 1 over field $\mathbb{F}$ is isomorphic to $\mathbb{F}$. Hence, it suffices to prove there exists $\varphi \in \mathcal{L}(V, V/U)$ such that $\operatorname{null}\varphi = U$ and the quotient map is just the map we want. $\square$

3.F Duality

1.

Proof. Suppose that $\varphi \in V'$ and is not the zero map. Then, $\varphi(v) = c \neq 0$ for some $v \in V$. By the linearity of φ, for every $0 \neq a \in \mathbb{F}$, $\varphi(av/c) = a$ and $\varphi(0) = 0$, completing the proof. $\square$

3.

Proof. It suffices to prove that there exists $\varphi \in V'$ which maps v to a nonzero element of $\mathbb{F}$. We argue by contradiction. Assume that for all $\varphi \in V'$, $\varphi(v) = 0$. Then $\{v\}^0 = V'$. Hence, $\dim\{v\} = \dim V - \dim\{v\}^0 = 0$, implying that $v = 0$. Contradiction. $\square$

9.

Proof. For every $v = \sum x_i v_i \in V$ and $\psi \in V'$,

$$\begin{aligned}\psi(v) &= \psi(x_1 v_1 + \cdots + x_n v_n)\\ &= \psi(v_1)x_1 + \cdots + \psi(v_n)x_n\\ &= \psi(v_1)\varphi_1(v) + \cdots + \psi(v_n)\varphi_n(v)\\ &= (\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v),\end{aligned}$$

where the third equality comes from the definition of the dual space and the last one comes from the linearity of $\varphi_1, \ldots, \varphi_n$. □

11.

Proof. $\operatorname{rank} A = 1$ iff there exists some nonzero $\alpha \in \mathbb{F}^m$ such that $A = [d_1\alpha \; \ldots \; d_n\alpha]$ iff $A = \alpha[d_1 \; \ldots \; d_n]$. □

15.

Proof. $T' = 0$ iff $\dim W' = \dim \operatorname{null} T'$ iff $\dim W = \dim(\operatorname{range} T)^0$ iff $\operatorname{range} T = 0$ iff $T = 0$. □

19.

Proof. As $U \subset V$ and V is finite-dimensional, $U = V$ iff $\dim U = \dim V$ iff $\dim U^0 = 0$ by 3.106 iff $U^0 = \{0\}$. □

25.

Proof. Note that the RHS of the equality equals to

$$\tilde{U} = \bigcap_{\varphi \in U^0} \operatorname{null} \varphi.$$

For every $u \in U$, since $u \in \operatorname{null} \varphi$ for every $\varphi \in U^0$ by definition. Hence, $U \subset \tilde{U}$. And let $\psi \in U^0$ be a linear functional such that $\operatorname{null} \psi = U$. Then $\dim \tilde{U} \leq \dim \operatorname{null} \psi = \dim U$. Hence, $U = \tilde{U}$. □

29.

Proof. By the hypothesis, for every $\psi \in W'$, $T'(\psi) = \psi \circ T = k\varphi$ for some scalar k. By 3.109, $\dim \operatorname{range} T = \dim \operatorname{range} T' = 1$. Hence, there exists $\psi \in W'$ whose restriction to $\operatorname{range} T$ is an one-to-one map to $\mathbb{F}$. Thus,

$$\operatorname{null} \varphi = \operatorname{null} k\varphi = \operatorname{null}(\psi \circ T) = \operatorname{null} T.$$

□

31. In brief, we choose an arbitrary basis of V and try to express the required basis with it by solving a system of linear equations.

Proof. Let $u_1, \dots, u_n$ be a basis of V and $A = [\varphi_i(u_j)]$. Now we prove that A is invertible. Suppose

$$x_1 \begin{bmatrix} \varphi_1(u_1) \\ \vdots \\ \varphi_n(u_1) \end{bmatrix} + \cdots + x_n \begin{bmatrix} \varphi_1(u_n) \\ \vdots \\ \varphi_n(u_n) \end{bmatrix} = 0.$$

and $u = x_1u_1 + \cdots + x_nu_n$. Then, $\varphi_i(u) = 0$ for $i = 1, \dots, n$. As $\varphi_1, \dots, \varphi_n$ is a basis of V', this implies $(\operatorname{span}(u))^0 = V'$. Hence, by 3.106, $\dim \operatorname{span}(u) = 0$ and therefore $u = 0$. Thus, the columns of A are linearly independent and therefore A is invertible.

Let

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} A^{-1} \tag{1}$$

and now we prove that $v_1, \dots, v_n$ is a basis of V and the dual basis of it is exactly $\varphi_1, \dots, \varphi_n$. Since $u_1, \dots, u_n$ are linearly independent and A^{-1} is nonsingular, so do $v_1, \dots, v_n$. Hence, $v_1, \dots, v_n$ is a basis of V. (1) also implies

$$u_k = \varphi_1(u_k)v_1 + \cdots + \varphi_n(u_1)v_n.$$

Applying φ_i on the both sides for each $k = 1, \dots, n$ yields

$$\begin{bmatrix} \varphi_i(u_1) \\ \vdots \\ \varphi_i(u_n) \end{bmatrix} = \begin{bmatrix} \varphi_1(u_1) & \cdots & \varphi_n(u_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(u_n) & \cdots & \varphi_n(u_n) \end{bmatrix} \begin{bmatrix} \varphi_i(v_1) \\ \vdots \\ \varphi_i(v_n) \end{bmatrix}.$$

Again, since A is nonsingular, the system of linear equations has exactly one solution $\varphi_i(v_j) = 0$ for $i \neq j$ and $\varphi_i(v_i) = 1$. Namely, $\varphi_1, \dots, \varphi_n$ is the dual basis of $v_1, \dots, v_n$. □

37.

Proof.
(a) Since π is surjective, π' is injective by 3.108.
(b) $\operatorname{range} \pi' = (\operatorname{null} \pi)^0 = U^0$.
(c) It follows immediately from (a) and (b). □

5 Eigenvalues, Eigenvectors and Invariant Subspaces

5.B Eigenvectors and Upper-Triangular Matrices

Lemma 1. *If λ is an eigenvalue of $T \in \mathcal{L}(V)$ and p is a polynomial, then $p(\lambda)$ is an eigenvalue of $p(T)$. Note that unlike the statement in exercise 11, $\mathbb{F}$ does not required to be $\mathbb{C}$.*

Proof. Suppose that $Tv = \lambda v$ for some $0 \neq v \in V$, then

$$p(T)v = \left(\sum_{k=0}^{n} a_k T^k\right) v = \sum_{k=0}^{n} a_k T^k v = \sum_{k=0}^{n} a_k \lambda^k v = p(\lambda)v.$$

Hence, $p(\lambda)$ is an eigenvalue of $p(T)$. □

1.

Proof.
(a) Since $T^n = 0$ and

$$(I - T)(I + T + \cdots + T^{n-1}) = I + T + \cdots + T^{n-1} - T - \cdots - T^n = I - T^n = I,$$

$I - T$ is invertible and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.
(b) The power series expansion of the function $(1-x)^{-1}$ at $x = 0$ is $1+x+\cdots+x^n+\cdots$. □

3.

Proof. Since 1 is the only eigenvalue of $T^2 = I$ and -1 is not an eigenvalue of T, by Lemma 1, 1 is the only eigenvalue of T and therefore $T = I$. □

5.

Proof. Since $(STS^{-1})^k = S(T(S^{-1}S)TS^{-1} \cdots ST)S^{-1} = ST^kS^{-1}$,

$$p(STS^{-1}) = \sum_{k=0}^{n} a_k (STS^{-1})^k = \sum_{k=0}^{n} a_k ST^k S^{-1} = Sp(T)S^{-1}.$$

□

7.

Proof. It follows immediately from Lemma 1. □

9.

Proof. Since $p(T)v = 0 = 0v$, 0 is an eigenvalue of T. Then by Lemma 1, some of the zeros of p are the eigenvalues of T. Assume that there exists some zero x_0 of p that is not an eigenvalue of p. Then $q = p/(x - x_0)$ is a polynomial of degree less than p and such that $q(T)v = 0$. Contradiction. Hence, every zero of p is an eigenvalue of p. □

11. Note that the proof does not rely on 5.21.

Proof. Suppose that α is an eigenvalue of $p(T)$. If p is a constant polynomial, then there is nothing to be proved. If p is non-constant, then $p(x) - \alpha = c(x - \lambda_1) \cdots (x - \lambda_m)$ where $m \geq 1$. Since α is an eigenvalue of $p(T)$,

$$(p - \alpha)(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)$$

is singular. Hence, at least one of $T - \lambda_1 I, \dots, T - \lambda_m T$, denoted by $T - \lambda_k I$, is singular and therefore λ_k is an eigenvalue of T and $p(\lambda_k) = \alpha$.
The converse part is just Lemma 1. □

13.

Proof. Suppose that U is a finite-dimensional T-invariant subspace of W. Then $T|_U$ is an operator on U, a complex vector space. Hence it has an eigenvalue as long as $U = \{0\}$. However, it does not and therefore $U = \{0\}$. □

15.

Proof. $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. □

17.

Proof. Let φ be the map which takes $p \in \mathcal{P}_{n^2}(\mathbb{C})$ to $p(T) \in \mathcal{L}(V)$. It is linear since

$$\varphi(a_1 p_1 + a_2 p_2) = (a_1 p_1 + a_2 p_2)(T) = a_1 p_1(T) + a_2 p_2(T) = a_1 \varphi(p_1) + a_2 \varphi(p_2).$$

Since $\dim \mathcal{P}_{n^2}(\mathbb{C}) = n^2 + 1$ ans $\dim \mathcal{L}(V) = n^2$, φ is not injective by 3.23. Namely, there exsits nonequal $p_1, p_2 \in \mathcal{P}_{n^2}(\mathbb{C})$ such that $\varphi(p_1) = \varphi(p_2)$. Hence, $\varphi(p_1 - p_2) = (p_1 - p_2)(T) = 0$ where $p_1 - p_2$ is a nonzero polynomial, having zeros in $\mathbb{C}$. Since 0 is the eigenvalue of $(p_1 - p_2)(T)$, one of its zeros is an eigenvalue of T by exercise 11. □

7 Operators on Inner Product Spaces

Note that on the first page of this chapter, the author says that all the inner product spaces appeared in this chapter should be assumed to be finite-dimensional if not specified.

7.A Self-Adjoint and Normal Operators

2.

Proof. $\operatorname{null}(T-\lambda I) = (\operatorname{range}(T-\lambda I)^*)^\perp = (\operatorname{range}(T^*-\bar{\lambda}I))^\perp$. Hence, $\dim\operatorname{null}(T-\lambda I) > 0$ iff $\dim\operatorname{range}(T^* - \bar{\lambda}I) < \dim V$ iff $\dim\operatorname{null}(T^* - \bar{\lambda}I) > 0$. Thus, λ is an eigenvalue of T iff $\bar{\lambda}$ is an eigenvalue of T^*. □

4.

Proof. If follows immediately from 7.7. □

8.

Proof. Let $S, T \in \mathcal{L}(V)$ be self-adjoint and $a, b \in \mathbb{R}$. Then by 7.5,

$$(aS + bT)^* = \bar{a}S^* + \bar{b}T^* = aS + bT.$$

Hence, the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$. □

10.

Proof. The n-by-n matrices

$$A = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}$$

are both normal, but $C = B - A$, which is nilpotent, is not. □

12.

Proof. Since T is normal and 3 and 4 are two eigenvalues of T, there exists an orthonormal list u_1, u_2 such that $Tu_1 = 3u_1$ and $Tu_2 = 4u_2$. Let $v = u_1 + u_2$. Then $\|v\| = \sqrt{2}$ and

$$\|Tv\| = \|3u_1 + 4u_2)\| = 5.$$

□

16.

Proof. 7.20 implies $\operatorname{null} T = \operatorname{null} T^*$ and therefore, by 7.7, $\operatorname{range} T = \operatorname{range} T^*$. □

7.B The Spectral Theorem

4.

Proof. If T is normal, then by the complex spectral theorem, clear that all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

Now we show the converse. Clear that the union of the orthonormal bases of each $E(\lambda_i, T)$ is an orthonormal basis consisting of eigenvectors of T. $\square$

6.

Proof. Suppose $T \in \mathbb{L}(V)$ is self-adjoint and $Tu = \lambda u$ for some $u \neq 0$. Then by 7.21, $\lambda u = Tu = T^* u = \bar{\lambda} u$. Hence, $\lambda = \bar{\lambda}$ and therefore λ is real.

By the complex spectral theorem, T has a matrix $M = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with respect to some orthonormal basis. If λ_i, the eigenvalues of T, are all real. Then the conjugate transpose of M is still M. Hence, $T = T^*$. Namely, T is self-adjoint. $\square$

8.

Proof. Let

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

be an 8-by-8 matrix. $A^9 = A^8 = 0$ but $A^2 \neq A$. $\square$

12.

Proof. We argue by contradiction. Assume that for every eigenvalue λ' of T, $|\lambda - \lambda'| > \varepsilon$. Since T is self-adjoint, there exists an orthonormal basis $e_1, \dots, e_n$ such that $Te_i = \lambda_i e_i$ for $i = 1, \dots, n$. Then

$$Tv - \lambda v = \sum_{i=1}^{n} \langle v, e_i \rangle (\lambda_i - \lambda) e_i.$$

Hence,

$$\|Tv - \lambda v\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2 |\lambda_i - \lambda|^2 \geq \varepsilon^2 \sum_{i=1}^{n} |\langle v, e_i \rangle|^2 = \varepsilon^2.$$

Contradiction. Hence, T has an eigenvalue λ' such that $|\lambda - \lambda'| < \varepsilon$. $\square$

14.

Proof. If there exists an inner product on U which makes T into a self-adjoint operator, then by the spectral theorem, U has a basis consisting of eigenvectors of T. Now we suppose that such a basis, denoted by $u_1, \dots, u_n$, exists. Define $\langle e_i, e_j \rangle$ to be 0 if $i \neq j$ and 1 if $i = j$. It is easy to verify that it does define an inner product and $u_1, \dots, u_n$ is an orthonormal basis. $\square$

7.C Positive Operators and Isometries

2.

Proof. The hypothesis implies $T^2v = v$ and therefore 1 is an eigenvalue of T^2 and v is its associated eigenvector. Note that T^2 is still a positive operator and T is its positive square root. Hence, v is also an eigenvector of T associated with eigenvalue 1. Namely, $v = Tv = w$. □

4.

Proof. $\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2 \geq 0$ for all $v \in V$. Hence, T^*T is a positive operator on V. Similarly, TT^* is a positive operator on W. □

6.

Proof. If T is positive, T is self-adjoint and so does T^k. Meanwhile, T has a diagonal matrix $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with respect to some orthonormal basis and $\lambda_i \geq 0$ for each $i = 1, \dots, n$. With respect to the same basis, T^k has the matrix $\operatorname{diag}(\lambda_1^k, \dots, \lambda_n^k)$. Since $\lambda_i^k \geq 0$, T^k is positive by 7.35. □

8.

Proof. Suppose T is positive and invertible. Then $\langle v, v\rangle_T = \langle Tv, v\rangle \geq 0$. By 7.35, $T = R^*R$ for some $R \in \mathcal{L}(V)$. Since T is invertible, both R^* and R are invertible. Hence, $0 = \langle v, v\rangle_T = \langle Tv, v\rangle = \|Rv\|^2$ iff $Rv = 0$ iff $v = 0$. Meanwhile, for every $u, v \in V$ and scalar a, b

$$\langle au + bv, w\rangle_T = a\langle Tu, w\rangle + b\langle Tv, w\rangle = a\langle v, w\rangle_T + b\langle v, w\rangle_T.$$

Finally,

$$\langle u, v\rangle_T = \langle Ru, Rv\rangle = \overline{\langle Rv, Ru\rangle} = \overline{\langle v, u\rangle_T}.$$

Thus, $\langle \cdot, \cdot\rangle_T$ is an inner product on V.

Now we suppose $\langle \cdot, \cdot\rangle_T$ is an inner product on V. Hence, by the positivity of inner product, T is positive and by the previous discussion, $Tv = 0$ iff $v = 0$. Hence, T is invertible. □

14.

Proof. As T is self-adjoint by Exercise 21 of Section 7.A, so does $-T$. Suppose that

$$f(x) = x_0 + x_1 \cos x + \cdots + x_n \cos nx + y_1 \sin x + \cdots + y_n \sin nx.$$

Then

$$f''(x) = -\sum_{k=1}^{n} x_k k^2 \cos kx - \sum_{k=1}^{n} y_k k^2 \sin kx.$$

Note that for all $i \neq j$, the integral of $(\cos ix \sin jx)$ is 0. Thus,

$$
\begin{aligned}
\langle f, f'' \rangle &= -\int_{-\pi}^{\pi} \left(x_0 + \sum_{k=1}^{n} (x_k \cos kx + y_k \sin kx) \right) \sum_{k=1}^{n} (k^2 x_k \cos kx + k^2 y_k \sin kx) \mathrm{d}x \\
&= -\int_{-\pi}^{\pi} \left(\sum_{k=1}^{n} x_k^2 k^2 \cos^2 kx + \sum_{k=1}^{n} y_k^2 k^2 \sin^2 kx \right) \mathrm{d}x \\
&= -2\pi \sum_{k=1}^{n} k^2 (x_k^2 + y_k^2) \leq 0.
\end{aligned}
$$

Therefore, $-T$ is a positive operator. □

7.D Polar Decomposition and Singular Value Decomposition

3.

Proof. As $\sqrt{T^*T}$ is positive, $(\sqrt{T^*T})^* = \sqrt{T^*T}$. The polar decomposition asserts that there exists an isometry S_1 such that $T = S_1\sqrt{T^*T}$. Replacing T with T^* yields

$$
T^* = S_2\sqrt{TT^*} \quad \Rightarrow \quad T = \sqrt{TT^*}S_2^*,
$$

where S_2^* is also an isometry. □

8. This result can be used to prove Exercise 1 in a simple way.

Proof. Note that R^2 and T^*T are both positive (and thereby self-adjoint). Hence, $R^2 - T^*T$ is also self-adjoint. For all $v \in V$,

$$
\langle (R^2 - T^*T)v, v \rangle = \langle R^2 v, v \rangle - \langle T^*Tv, v \rangle = \|Rv\|^2 - \|Tv\|^2.
$$

Since S is an isometry, $\|Rv\| = \|SRv\| = \|Tv\|$. Therefore, $\langle (R^2 - T^*T)v, v \rangle = 0$. Thus, $R^2 = T^*T$ and so does their unique positive square roots. □

9.

Proof. As $T = S\sqrt{T^*T}$ for some isometry S and every isometry is invertible, T is invertible iff $\sqrt{T^*T}$ is invertible. If T is invertible and $T = S_1\sqrt{T^*T} = S_2\sqrt{T^*T}$, multiplying $(\sqrt{T^*T})^{-1}$ on the both sides yields $S_1 = S_2$.

Now we suppose that such an isometry S is not unique and show that $\sqrt{T^*T}$ is not invertible to complete the proof. Suppose $T = S_1\sqrt{T^*T} = S_2\sqrt{T^*T}$ where $S_1 \neq S_2$. Then, since $S_1 - S_2 \neq 0$ but $(S_1 - S_2)\sqrt{T^*T} = 0$, $\sqrt{T^*T}$ is not invertible and so is T. □

13.

Proof. By the discussion in Exercise 9, T is invertible iff $\sqrt{T^*T}$ is invertible iff 0 is not an eigenvalue of $\sqrt{T^*T}$. □

8 Operators on Complex Vector Spaces

8.A Generalized Eigenvectors and Nilpotent Operators

3.

Proof. Suppose $\dim V = N$.

$$\begin{aligned}
v \in G(T^{-1}, \lambda^{-1}) &\Leftrightarrow (T^{-1}v - \lambda^{-1}I)^n v = 0 \\
&\Leftrightarrow \left(\sum_{k=0}^{n} \binom{n}{k} T^{-k}(-\lambda^{-1})^{n-k} \right) v = 0 \\
&\Leftrightarrow T^n(-\lambda)^n \left(\sum_{k=0}^{n} \binom{n}{k} T^{-k}(-\lambda)^{k-n} \right) v = 0 \\
&\Leftrightarrow \left(\sum_{k=0}^{n} \binom{n}{k} T^{n-k}(-\lambda)^{k} \right) v = 0 \\
&\Leftrightarrow (T - \lambda I)^n v = 0 \\
&\Leftrightarrow v \in G(T, \lambda).
\end{aligned}$$

□

5. Intuitively, $\operatorname{null} T, \ldots, \operatorname{null} T^n$ is a sequence of subspaces where the preceding ones are contained by succeeding ones. And $T^k v$ lies in the additional part between two successive subspaces.

Proof. Note that $T^{m-1}v \neq 0$ but $T^m v = 0$ implies for $k = 1, \ldots, m$

$$T^{m-k}v \in \operatorname{null} T^k, \quad T^{m-k}v \notin \operatorname{null} T^{k-1}.$$

Therefore, $T^{m-k}v \notin \operatorname{span}(T^{m-1}v, \ldots, T^{m-k+1}v)$; otherwise, $T^{m-k}v = x_1 T^{m-1}v + \cdots + x_{m-k+1}T^{m-k+1}v$, implying that $T^{m-k}v \in \operatorname{null} T^{k-1}$. Hence, $v, \ldots, T^{m-1}v$ are linearly independent. □

7.

Proof. It follows immediately from 8.19. □

9.

Proof. Suppose that v is nonzero and $(TS - \lambda I)v = 0$. Then

$$(STS - \lambda S)v = 0 \quad \Rightarrow \quad (ST - \lambda I)Sv = 0.$$

If $Sv = 0$, then $0 = (TS - \lambda I)v = T(Sv) - \lambda v$ implies $\lambda = 0$. If $Sv \neq 0$, then λ is an eigenvalue of ST and therefore equals 0 by Exercise 7. Hence, in every case, $\lambda = 0$. Thus, TS is also nilpotent. □

13.

Proof. We are going to show that $N^{k-1} = 0$ if $N^k = 0$ for $k > 1$ to conclude that $N = 0$. Suppose that $N^k = 0$ and let $M = N^{k-1}$. Then for every $v \in V$,

$$\|M^*Mv\|^2 = \langle M^*Mv, M^*Mv\rangle = \langle M^*MM^*Mv, v\rangle = \langle M^*M^*MMv, v\rangle.$$

Since $M^2 = N^{2k-2} = 0$, this implies $M^*M = 0$. Hence, with the polar decomposition, $M = 0$. $\square$

15.

Proof. Suppose $\dim V = n$. Since $\operatorname{null} N^{n-1} \neq \operatorname{null} N^n$,

$$0 < \dim N < \cdots < \dim N^n.$$

Hence, $\dim \operatorname{null} N^j = j$ for $0 \leq j \leq n$. Since $\dim \operatorname{null} N^n = n$, $\operatorname{null} N^n = V$ and therefore N is nilpotent. $\square$