

Real Analysis

Yunwei Ren

Contents

3	Lebesgue Measure	3
3.1	Introduction	3
3.2	Outer Measure	3
3.3	Measurable Sets and Lebesgue Measure	4
3.5	Measurable Functions	5
3.6	Littlewood's Three Principles	7
4	The Lebesgue Integral	8
4.2	The Lebesgue Integral of a Bounded Function	8
4.3	The Integral of a Nonnegative Function	8
4.4	The General Lebesgue Integral	10
5	Differentiation and Integration	12
5.1	Differentiation of Monotone Functions	12
5.2	Functions of Bounded Variation	12
5.4	Absolute Continuity	14
5.5	Convex Functions	18
6	The Classical Banach Spaces	20
6.1	The L^p Spaces	20
6.2	The Minkowski and Hölder Inequalities	20
6.3	Convergence and Completeness	21
6.4	Approximation in L^p	23
7	Metric Spaces	24
7.7	Compact Metric Space	24
7.8	Baire Category	25
7.10	The Ascoli-Arzelá Theorem	25
8	Topological Spaces	27
8.1	Fundamental Notions	27
8.2	Bases and Countability	27
8.3	The Separation Axioms and Continuous Real-Valued Functions	28
8.4	Connectedness	30
9	Compact and Locally Compact Spaces	31
9.1	Compact Spaces	31
9.2	Countable Compactness and the Bolzano-Weierstrass Property	31

11 Measure and Integration	33
11.1 Measure Spaces	33
11.2 Measurable Functions	35
11.3 Integration	37
11.4 General Convergence Theorems	38
11.5 Signed Measures	38
11.6 The Radon-Nikodym Theorem	40
11.7 The L^p Spaces	43
12 Measure and Outer Measure	45
12.1 Outer Measure and Measurability	45
12.2 The Extension Theorem	45

3 Lebesgue Measure

3.1 Introduction

1.

Proof. Since \mathfrak{M} is an σ -algebra, $B \setminus A \in \mathfrak{M}$ as long as $A, B \in \mathfrak{M}$. Since $B \setminus A$ and A are disjoint, $mB = mA + m(B \setminus A) \geq mA$ since m is nonnegative. \square

2.

Proof. Let $A_0 = E_0$ and $E_k = A_k \setminus A_{k-1}$ for $k \geq 1$. Clear that E_i and E_j are disjoint for distinct i and j , $\bigcup A_n = \bigcup E_n$ and $A_i \subset E_i$ for every i . Hence,

$$m\left(\bigcup E_n\right) = m\left(\bigcup A_n\right) = \sum mA_n \leq \sum mE_n,$$

where the last inequality comes from Exercise 1. \square

3.

Proof. Suppose that $mA < \infty$. Then $mA = m(A \cup \emptyset) = mA + m\emptyset$, implying that $m\emptyset = 0$. \square

3.2 Outer Measure

5.

Proof. We show that $\{I_n\}$ must cover the entire $[0, 1]$ by contradiction. Assume that $x \notin I_k$ for $k = 1, 2, \dots, n$. Then, as I_k are open and n is finite, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ and I_k are disjoint for every k . Since \mathbb{Q} is dense in \mathbb{R} , there exists some rational number in $(x - \varepsilon, x + \varepsilon)$, contradicting with the hypothesis that $\{I_k\}$ covers all rational numbers between 0 and 1. \square

6.

Proof. By the definition of the outer measure, for every $\varepsilon > 0$, there exists some collection $\{I_n\}$ of open intervals that covers A and $\sum l(I_n) \leq m^*A + \varepsilon$. Let $O = \bigcup I_n$. O is a countable union of open sets and therefore is also open. And by Proposition 2, $m^*O \leq \sum l(I_n)$. Thus, $m^*O \leq m^*A + \varepsilon$.

Let $\varepsilon_n = 1/n$ and for each n , by the previous discussion, we can always get an open set O_k such that $A \subset O_k$ and $m^*O \leq m^*A + \varepsilon_m$. Let G be the countable intersection of these open sets. Clear that G is a G_δ set covering A and $m^*A = m^*G$. \square

7.

Proof. If $m^*E = \infty$, it is trivial. Suppose that $m^*E \leq \infty$. For any $x \in \mathbb{R}$, collection $\{I_n\}$ of open intervals covers $E + x$ iff $\{I_n - x\}$ covers E . Since the length of intervals is translation invariant, this implies $m^*(E + x) = m^*E$. \square

8.

Proof. Clear that $m^*A \leq m^*(A \cup B)$. Meanwhile, $m^*(A \cup B) = m^*A + m^*B = m^*B$. Hence, $m^*(A \cup B) = m^*B$. \square

3.3 Measurable Sets and Lebesgue Measure

10.

Proof.

$$\begin{aligned} mE_1 + mE_2 &= mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2). \end{aligned}$$

□

11.

Proof. $E_n = (n, \infty)$.

□

12. This is the countable version of Lemma 9.

Proof. It suffices to prove $m^*(A \cap \bigcup E_i) \geq \sum m^*(A \cap E_i)$. Since $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^n E_i$ for every n ,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of n , we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i),$$

completing the proof.

□

13.

Proof. First we suppose that $m^*E < \infty$. By Proposition 5, there exists some open set $O \supset E$ such that $m^*O \leq m^*E + \varepsilon$. If E is measurable, then by the definition,

$$m^*(O \setminus E) = m^*O - m^*E \leq \varepsilon.$$

Namely, (ii) holds. Meanwhile, $O \subset \mathbb{R}$ is a countable union of disjoint open intervals $\{I_n\}$. Since $mO = m^*O$ is bounded and $mO = \sum l(I_n)$, there exists some integer $N > 0$ such that $mO - \sum_{n=1}^N l(I_n) < \varepsilon$. Let $U = \bigcup_{n=1}^N I_n$.

$$\begin{aligned} m^*(U \triangle E) &= m^*((U \cup E) \setminus (U \cap E)) \\ &\leq m^*(O \setminus (U \cap E)) \\ &= m^*((O \setminus U) \cup (O \setminus E)) \\ &\leq m^*(O \setminus U) + m^*(O \setminus E) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, (ii) implies (vi). Now we show that (vi) implies (ii). If $m^*(U \triangle E) < \varepsilon$, then there exists some countable collection $\{J_n\}$ of open interval such that

$$\sum l(J_n) \leq m^*(U \triangle E) + \varepsilon < 2\varepsilon.$$

Let $J = \bigcup J_n$ and $O = U \cup J$. $m^*J < 2\varepsilon$. And O is open and covers E . Meanwhile,

$$m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(J \setminus E) < 3\varepsilon.$$

Hence, (ii) holds.

Now, let E be an arbitrary set and $E_n = E \cap (-n, n)$, which is a set with finite measure. Then by the previous discussion, there exists some open set $O_n \supset E_n$ with $m^*(O_n \setminus E_n) < \varepsilon/2^n$. Let $O = \bigcup O_n$, an open set covering E and

$$m^*(O \setminus E) \leq \sum m^*(O_n \setminus E_n) < 2\varepsilon.$$

Hence, (i) implies (ii). Now we suppose (ii) holds and let $\varepsilon_n = 1/n$, then there exists a sequence of open sets $\langle O_n \rangle$ such that $m^*(O_n \setminus E) < 1/n$. Let $G = \bigcap O_n \in G_\delta$. $m^*(G \setminus E) \leq m^*(O_n \setminus E) \leq 1/n$. Since the left hand side is independent of n , $m^*(G \setminus E) = 0$. If (iv) holds, then by Lemma 6, $G \setminus E$ is measurable. Since $G \in G_\delta$ is also measurable, E is measurable. Hence, (iv) implies (i).

By the previous result, for any measurable E , there exists some closed set $F \subset E$ such that \bar{F} , which is open, contains $\text{bar}E$ and $m^*(\bar{F} \setminus \bar{E}) < \varepsilon$. Hence, $m^*(E \setminus F) < \varepsilon$. We can proceed in a similar manner as we did in the last paragraph to prove that (iii) \Rightarrow (v) \Rightarrow (i), leading to the final conclusion. \square

3.5 Measurable Functions

19.

Proof. For every $\beta \in \mathbb{R}$, since D is measurable, there exists a sequence of $\alpha_n \in D \cap (\beta - 1/n, \beta)$. As

$$\{x : f(x) > r\} \Leftrightarrow \bigcup_{n=1}^{\infty} \{x : f(x) > r - 1/n\} \Leftrightarrow \bigcup_{n=1}^{\infty} \{x : f(x) > \alpha_n\}$$

and $\{x : f(x) > \alpha_n\}$ are measurable, so is $\{x : f(x) > r\}$. Hence, f is measurable. \square

21.

Proof.

(a) It follows immediately from $\{x : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$.

(b) For $\alpha \geq 0$, the sets $\{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$ are the same. And for $\alpha < 0$,

$$\{x : f(x) > \alpha\} = \{x : g(x) > \alpha\} \setminus \bar{D} \quad \text{and} \quad \{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \bar{D}.$$

Hence, f is measurable iff g is measurable. \square

22.(d)

Proof. Since f and g are finite almost everywhere, the set A consisting of points where $f + g$ is of the form $\infty - \infty$ or $-\infty + \infty$ is of measure zero (and hence measurable). Therefore no matter how it is defined, $\{x \in A : f + g > \alpha\}$ is measurable for every α . Namely, the restriction of $f + g$ to A is measurable. Meanwhile, clear that the restriction to $D \setminus A$ is measurable where D is the domain of f . Hence, by Exercise 21, f is measurable. \square

23.

Proof.

(a) Let $A_n = \{x : |f(x)| > n\}$, a sequence of measurable sets. As $A_{n+1} \subset A_n$, $mA_{n+1} \leq mA_n$. Since $A = \bigcap A_n = \{x : |f(x)| = \infty\}$, $mA_1 \leq m[a, b]$ is finite and $mA = 0$, by Proposition 14, there exists some N such that for all $n \geq N$, $mA_n < \varepsilon/3$. Set $M = N$ to complete the proof.

(b) We consider the restriction of f on to the set $E = [a, b] \setminus \{x : |f(x)| \geq M\}$, which is also a measurable real-valued function. To keep our notation simple, we denote the restriction by f still. For every $\varepsilon > 0$, there exists some integer N with $0 < 2M/N < \varepsilon$. Let $E_n = \{x : x \in [-M + (n-1)\varepsilon, -M + n\varepsilon]\}$ ($n = 1, 2, \dots, N$) and define

$$\varphi(x) = \sum_{i=1}^N f(x_i) \chi_{E_i},$$

where $x_n \in E_n$ is arbitrary. Clear that φ is a simple function and satisfy all the requirements.

(c) Suppose that $\varphi(x) = \sum_{i=1}^n \alpha_i \chi_{E_i}$. For each $i = 1, \dots, N$, E_i is measurable and therefore by Proposition 15, there exists a finite union U_i of open intervals such that $m(U_i \triangle E_i) < \varepsilon$. Let

$$g(x) = \sum_{i=1}^N \alpha_i \chi_{U_i}.$$

Clear that g and φ only may differ on a set with measure $N\varepsilon$. (d) Suppose that $g(x) = \sum_{i=1}^N \alpha_i \chi_{U_i}$ is a step function. We may assume without loss of generality that U_i are disjoint and $\bigcup U_i = [a, b]$. And suppose that $\{x_0 = a < x_1 < \dots < x_N = b\}$ are the endpoints of the intervals. For each $i = 1, \dots, N-1$, define

$$f(x) = (x - x_i + \varepsilon)g(x_i - \varepsilon) + (x_i + \varepsilon - x)g(x_i + \varepsilon), \quad x \in (x_i - \varepsilon, x_i + \varepsilon),$$

and $f(x) = g(x)$ for the other points. (We assume that ε is small enough so that f is well-defined.) Clear that f is continuous and equals g except on a set of measure less than $2N\varepsilon$. \square

24.

Proof. For measurable f , we show that $\mathcal{A} = \{E : f^{-1}[E] \text{ is measurable}\}$ is a σ -algebra first. As the domain, denoted by D , of a measurable function is measurable, $\mathbb{R} \in \mathcal{A}$. If $E \in \mathcal{A}$, then since $f^{-1}[\bar{E}] = D \cap \overline{f^{-1}[E]}$, $f^{-1}[\bar{E}]$ is also measurable and therefore $\bar{E} \in \mathcal{A}$. Suppose that $\langle E_n \rangle$ is a sequence of sets of \mathcal{A} . Then, as

$$f^{-1}\left[\bigcup_{n=1}^{\infty} E_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[E_n],$$

$\bigcup E_n \in \mathcal{A}$. Hence, \mathcal{A} is a σ -algebra.

By the definition of a measurable function, every open interval belongs to \mathcal{A} . Since the collection of all Borel sets \mathcal{B} is the σ -algebra generated by all open intervals, $\mathcal{B} \subset \mathcal{A}$. Namely, $f^{-1}[B]$ is measurable as long as $B \in \mathcal{B}$. \square

3.6 Littlewood's Three Principles

30.

Proof. Let $\varepsilon_n = 1/n$ and $\delta_n = \eta/2^n$ ($n = 0, 1, \dots$). By Proposition 24, for each n , there exists some A_n with measure less than δ_n such that for all $x \in E_n \setminus A_n$, $|f_m(x) - f(x)| < \varepsilon_n$ for m large enough. Let $A = \bigcup_{n=1}^{\infty} A_n$, the measure of which is less than $\sum \eta/2^n = \delta$. Meanwhile, for any $\varepsilon > 0$, by construction, for all $x \in E \setminus A$, $|f_m(x) - f(x)| < \varepsilon$ for m large enough. Namely, f_n converges to f uniformly on $E \setminus A$. \square

31.

Proof. Let $\varepsilon_n = \delta/2^n$ ($n \geq 0$), then by Proposition 22, there exists continuous g_n such that $E_n = \{x : |f(x) - g_n(x)| \geq \varepsilon_n\}$ is of measure less than ε_n . Let $E = \bigcup E_n$, the measure of which is less than δ and g_n converges to f on $[a, b] \setminus E$.

By Egoroff's Theorem, there exists some $A \subset [a, b] \setminus E$ with $mA < \delta$ such that g_n converges to f uniformly on $[a, b] \setminus (E \cup A)$. Since $E \cup A$ is measurable, by Proposition 15, there exists some open set $O \supset E \cup A$ such that $m(O \setminus (E \cup A)) < \delta$. Let $F = [a, b] \setminus O$. We know that

1. F is a closed set.
2. $mF < 3\delta$.
3. g_n converges to f uniformly on F .

Hence, f is continuous on F . And by Problem 2.40, there exists some continuous function on \mathbb{R} such that $\varphi(x) = f(x)$ for $x \in F$.

If f is defined on $(-\infty, \infty)$, we can apply the previous result on each $[n, n+1]$ and "stick" the functions together as we did in Problem 23(c) to get the function required. \square

4 The Lebesgue Integral

4.2 The Lebesgue Integral of a Bounded Function

2.

Proof.

(a) By Problem 2.51, h is upper semicontinuous as f is bounded and by Problem 2.50, $x : h(x) < \lambda$ is open and hence measurable for every $\lambda \in \mathbb{R}$. Thus, h is measurable.

Let $\varphi(x) \geq f(x)$ be a step function and x_0 any point other than the endpoints of the intervals occurring in φ . Then there exists some $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$, $\varphi(x_0) = \varphi(x) \geq f(x)$. Hence,

$$h(x_0) = \inf_{\delta < 0} \sup_{|x - x_0| < \delta} f(x) \leq \varphi(x_0).$$

Namely, $\varphi \geq h$ except at a finite number of points. Hence, $\int_a^b \varphi \geq \int_a^b h$ and therefore

$$R \int_a^{\bar{b}} f = \inf_{\varphi \geq f} \int_a^b \varphi(x) dx \geq \int_a^b h.$$

We can also derive from the previous discussion that there is a sequence of $\langle \varphi_n \rangle$ of step functions satisfying $\varphi \downarrow h$. By Proposition 6,

$$\int_a^b h = \lim \int_a^b \varphi_n \geq R \int_a^{\bar{b}} f.$$

Hence, $R \int_a^{\bar{b}} f = \int_a^b h$.

(b) First suppose that f is Riemann integrable and let h and g be the upper and lower envelope of f respectively. By part (a), f is Riemann integrable implies $\int_a^b (h - g) = 0$. Together with the fact that $h \geq g$, we conclude that $h = g$ a.e.. Therefore, by Problem 2.50, f is continuous except on a set of measure zero.

Note that the argument remains true if we reverse the order, verifying the converse part. Hence, the proposition holds. \square

4.3 The Integral of a Nonnegative Function

3.

Proof. Suppose that $E_n = \{x : f(x) > 1/n\}$. Then,

$$0 = \int f \geq \int_{E_n} f \geq \frac{mE_n}{n}$$

implies $mE_n = 0$. Hence, $m\{x : f(x) > 0\} = m(\bigcup E_n) \leq \sum mE_n = 0$. Namely, $f = 0$ a.e. \square

5.

Proof. For any fixed $x_0 \in \mathbb{R}$, let $f_n(x) = f \cdot \chi_{(-\infty, x_0 - 1/n]}$, which is a increasing sequence of nonnegative measurable function whose limit is $f \cdot \chi_{(-\infty, x_0]}$. Then by Theorem 10,

$$F(x_0) = \int_{-\infty}^{x_0} f = \int f \cdot \chi_{(-\infty, x_0]} = \lim \int f \cdot \chi_{(-\infty, x_0 - 1/n]} = \lim F(x_0 - 1/n).$$

Meanwhile, since

$$|F(x_0) - F(x_0 + 1/n)| = \left| \int_{x_0}^{x_0 + 1/n} f(x) dx \right| = \left| \int_{-1/n}^0 g(x) dx \right|,$$

where $g(x) = f(x_0 - x)$, arguing on g in a similar manner yields $F(x_0) = \lim F(x_0 + 1/n)$. Thus, F is continuous. \square

6.

Proof. By Theorem 9, $\int f \leq \underline{\lim} \int f_n$. Meanwhile, $f_n \leq f$ implies $\int f_n \leq \int f$ and therefore $\overline{\lim} \int f_n \leq \int f$. Hence, $\int f = \lim \int f_n$. \square

7.

Solution.

(a) Let $f_n(x) = n \cdot \chi_{[0, 1/n]}$. f_n converges to $f = 0$ except on $x = 0$. For each n , $\int f_n = 1$ but $\int f = 0$. Hence, the inequality could be strict.

(b) Let $f_n(x) = \chi_{[n, \infty)}$. Then $\langle f_n \rangle$ is a decreasing sequence which converges to $f = 0$, the integral of which is 0. However, for every n , $\int f_n = \infty$. \square

8.

Proof. Let $g_n = \inf\{f_n, f_{n+1}, \dots\}$. Clear that

$$\int g_n \leq \int f_n. \quad (1)$$

Meanwhile $\langle g_n \rangle$ is a increasing sequence converging to $\underline{\lim} f_n$. Hence, by the Monotone Convergence Theorem and (1)

$$\int \underline{\lim} f_n = \int \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} \int g_n \leq \underline{\lim} \int f_n.$$

\square

9.

Proof. By Fatou's Lemma,

$$\int_E f \leq \underline{\lim} \int_E f_n. \quad (2)$$

Similarly, $\int_{\bar{E}} f \leq \underline{\lim} \int_{\bar{E}} f_n$ and therefore

$$\int_E f_n = \int f_n - \int_{\bar{E}} f_n \Rightarrow \overline{\lim} \int_E f_n \leq \int f - \int_{\bar{E}} f = \int_{\bar{E}} f.$$

(2) and the inequality above together implies $\int_E f_n \rightarrow \int f$. \square

4.4 The General Lebesgue Integral

12.

Proof. Note that $\langle g + f_n \rangle$ is a sequence of nonnegative measurable functions. Hence by Problem 8,

$$\int_E \underline{\lim}(g + f_n) \leq \underline{\lim} \int_E (g + f_n) \Rightarrow \int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n.$$

The second inequality follows immediately from the definition of lower and upper limit. Replacing $g + f_n$ with $g - f_n$ and arguing in a similar manner gives the last inequality. \square

13.

Proof. $f_n \geq -h$ implies $f_n + h \geq 0$. Hence, $\int(f_n + h)$ always has a meaning. And since g is integrable, $\int f_n = \int(f_n + h) - \int h$ also has a meaning. Similarly, $\int f$ has a meaning. Meanwhile,

$$\int f = \int(f + h) - \int h \leq \underline{\lim} \int(f_n + h) - \int h = \underline{\lim} \int f_n.$$

\square

15.

Proof.

(a) By Problem 4, for every $\varepsilon > 0$, there exists some simple functions $\varphi_1 \leq f^+$ and $\varphi_2 \leq f^-$ such that

$$\int_E f^+ - \int_E \varphi_1 < \varepsilon \quad \text{and} \quad \int_E f^- - \int_E \varphi_2 < \varepsilon.$$

Let $\varphi = \varphi_1 - \varphi_2$, which is also a simple function. Meanwhile,

$$\int_E |f - \varphi| \leq \int_E (f^+ - \varphi_1) + \int_E (f^- - \varphi_2) < 2\varepsilon.$$

\square

16.

Proof. For every integrable f , by Problem 15, there exists some step function $\psi = \sum_{k=1}^N c_k \chi_{E_k}$ such that $\int |f - \psi| < \varepsilon$. Note that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \psi(x) \cos nx dx = \lim_{n \rightarrow \infty} \sum_{k=1}^N c_k \int_{E_k} \cos nx dx = 0. \quad (3)$$

Hence,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} f(x) \cos nx dx \right| &= \left| \int_{-\infty}^{\infty} (f(x) - \psi(x) + \psi(x)) \cos nx dx \right| \\ &\leq \int_{-\infty}^{\infty} |f(x) - \psi(x)| |\cos nx| dx + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx dx \right| \\ &\leq \varepsilon + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx dx \right| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

□

18.

Proof. Let $\langle t_n \rangle$ be any sequence with $t_n \neq 0$ and tending to 0. Then $\langle f(x, t_n) \rangle$ is sequence of functions satisfying the hypotheses of Lebesgue Convergence Theorem. Meanwhile, $f(x, t_n) \rightarrow f$ as $n \rightarrow \infty$. Hence,

$$\lim_{n \rightarrow \infty} \int f(x, t_n) dx = \int f(x) dx.$$

Since the choice of $\langle t_n \rangle$ is arbitrary, by Problem 2.49f,

$$\lim_{t \rightarrow 0} \int f(x, t) dx = \int f(x) dx.$$

If f is continuous in t for each x , then $\lim_{\Delta t \rightarrow 0} f(x, t + \Delta t) = f(x, t)$ holds for every t . Therefore, replacing t with Δt in the previous result yields

$$\lim_{\Delta t \rightarrow 0} \int f(x, t + \Delta t) dx = \int f(x, t) dx.$$

Namely, $h(t)$ is continuous.

□

5 Differentiation and Integration

5.1 Differentiation of Monotone Functions

3. "maximum" needs to be changed to "minimum" in both (a) and (b).

Proof.

(a) We may assume without loss of generality that $c = 0$. Since f attains a local minimum at $x = 0$, $f(h) \geq f(0)$ for every h sufficiently small. Hence, for every small $h > 0$, $(f(c+h) - f(c))/h > 0$ and therefore $D_+f(c) \geq 0$. Meanwhile, by Problem 2.b,

$$-D_-f(0) = D^+f(0) \geq 0 \quad \Rightarrow \quad D_-f(0) \leq 0.$$

The other two inequalities follow immediately from the definitions of upper and lower limits.

(b) If f has a local minimum at a or b , then we only have the right or left half of the inequalities. \square

4.

Proof. We first show this for g with $D^+g \geq \varepsilon > 0$. For every $a \leq x < y \leq b$, as g is continuous on $[a, b]$, g has a maximum in $[a, b]$ and by Problem 2 and 3, g can not attain the maximum in $[a, b)$. Namely, the restrict of f to $[x, y]$ attains the maximum at y . Hence, $g(x) \leq g(y)$.

For every f with nonnegative D^+ , let $g(x) = f(x) + \varepsilon x$ where $\varepsilon > 0$. Then $D^+g \geq \varepsilon > 0$. Hence g is nondecreasing. Therefore, for every $a \leq x < y \leq b$,

$$g(x) \leq g(y) \quad \Rightarrow \quad f(x) + \varepsilon x \leq f(y) + \varepsilon y.$$

Since the choice of ε is arbitrary, this implies $f(x) \leq f(y)$. \square

5.a

Proof.

$$\begin{aligned} \sup_{t \in (0, h)} \frac{(f+g)(x+t) - (f+g)(x)}{t} &= \sup_{t \in (0, h)} \left(\frac{f(x+t) - f(x)}{t} + \frac{g(x+t) - g(x)}{t} \right) \\ &\leq \sup_{t \in (0, h)} \frac{f(x+t) - f(x)}{t} + \sup_{t \in (0, h)} \frac{g(x+t) - g(x)}{t}. \end{aligned}$$

Letting $h \rightarrow 0$ yields $D^+(f+g) \leq D^+f + D^+g$. \square

5.2 Functions of Bounded Variation

7.

Proof.

(a) It suffices to show this for monotone functions as each function of bounded variation is the difference of two monotone functions. Suppose that f is nondecreasing. Then the set $E = \{f(x) : x > c\}$ is bounded below and hence $A = \inf E$ is finite. For every $\varepsilon > 0$, there exists some $y > c$ such that $A \leq f(c) < A + \varepsilon$. Hence, as f is nondecreasing,

for every $x \in (c, y)$, $|f(x) - A| < \varepsilon$. Namely, $\lim_{x \rightarrow c+} f(x) = A$. Similarly, $\lim_{x \rightarrow c-} f(x)$ exists.

Let $D_n = \{x : |f(x+) - f(x-)| > 1/n\}$. Since f is nondecreasing, $|f(x) - f(y)| \leq f(b) - f(a) < \infty$ for every $x, y \in [a, b]$. Hence, D_n is finite, otherwise we can choose a sequence $x_1 < \dots < x_N$ with $N > (f(b) - f(a))/n$ such that $f(x_N) - f(x_1) > f(b) - f(a)$. Therefore, $\bigcup_{n=1}^{\infty} D_n$, the set of discontinuities, is countable.

(b) Suppose $\{x_1, \dots, x_n, \dots\} = \mathbb{Q} \cap [0, 1]$ and define $f(x) = \sum_{x_n < x} 2^{-n}$. Clear that f is monotone and continuous at every irrational point. For each rational $x = x_k$, $f(x+) - f(x-) = 2^{-k}$. Hence, f is discontinuous at each rational point. \square

8.

Proof.

(a) For every $\varepsilon > 0$, there exists some subdivision $a = x_0 < \dots < x_p = c$ and $c = x_p < \dots < x_q = b$ of $[a, c]$ and $[c, b]$ such that $T_a^c < t_a^c + \varepsilon$ and $T_c^b < t_c^b + \varepsilon$. Hence, $T_a^c + T_c^b - 2\varepsilon < t_a^c + t_c^b$. Meanwhile, as $a = x_0 < \dots < x_q = b$ forms a subdivision of $[a, b]$, $T_a^b \geq t_a^b = t_a^c + t_c^b$. Therefore, $T_a^c + T_c^b - 2\varepsilon < T_a^b$. Since the choice of ε is arbitrary, $T_a^b + T_c^b \leq T_a^b$.

To show that $T_a^b + T_c^b \geq T_a^b$, let $a = x_0 < \dots < x_q = b$ be any subdivision of $[a, b]$ and by adding c into it, we get subdivisions of $[a, c]$ and $[c, b]$. Suppose that $c \in (x_k, x_{k+1}]$, then

$$|f(x_k) - f(c)| + |f(c) - f(x_{k+1})| + t_a^b = t_a^c + t_c^b + |f(x_k) - f(x_{k+1})|,$$

which implies $t_a^b \leq t_a^c + t_c^b$. Hence,

$$T_a^b = \sup t_a^b \leq \sup(t_a^c + t_c^b) \leq T_a^c + T_c^b.$$

Thus, $T_a^b = T_a^c + T_c^b$ and therefore $T_a^c \leq T_a^b$.

(b)

$$\begin{aligned} T_a^b(f+g) &= \sup \sum_{i=1}^k |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\leq \sup \sum_{i=1}^k |f(x_i) - f(x_{i-1})| + \sup \sum_{i=1}^k |g(x_i) - g(x_{i-1})| \\ &\leq T_a^b(f) + T_a^b(g). \end{aligned}$$

$$T_a^b(cf) = \sup \sum_{i=1}^k |cf(x_i) - cf(x_{i-1})| = |c| \sup \sum_{i=1}^k |f(x_i) - f(x_{i-1})| = |c| T_a^b(f).$$

\square

9.

Proof. For every $\varepsilon > 0$, there exists a subdivision $a = x_0 < \cdots < x_k = b$ such that $t_a^b(f) \geq T_a^b(f) - \varepsilon$. Meanwhile, as f_n converges to f pointwisely

$$\begin{aligned} t_a^b(f) &= t_a^b(\lim f_n) \\ &= \sum_{i=1}^k |(\lim f_n)(x_i) - (\lim f_n)(x_{i-1})| \\ &= \lim \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| \\ &\leq \underline{\lim} \sup \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| = \underline{\lim} T_a^b(f_n). \end{aligned}$$

Hence, $T_a^b(f) - \varepsilon \leq \underline{\lim} T_a^b(f_n)$. Since the choice of ε is arbitrary, $T_a^b(f) \leq \underline{\lim} T_a^b(f_n)$. \square

10.a

Solution. No. Let $x_k = (k\pi + \pi/2)^{-1/2}$, $k = 0, 1, \dots$ and consider the subdivision $-1 < 0 < x_n < \cdots < x_0 < 1$. Then

$$t_n \geq \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \geq \sum_{k=1}^n \frac{2}{(k + 1/2)\pi}.$$

$t_n \rightarrow \infty$ as $n \rightarrow \infty$ and therefore f is not of bounded variation on $[-1, 1]$. \square

11.

Proof. By Lemma 4, $f(x) = f(a) + P_a^x - N_a^x$. Since P_a^x and N_a^x are monotone, by Theorem 3, they are differentiable almost everywhere as f , a function of bounded variation, does. Hence, for almost every $x \in [a, b]$,

$$\frac{d}{dx} f(x) = \frac{d}{dx} P_a^x - \frac{d}{dx} N_a^x \quad \Rightarrow \quad |f'(x)| \leq \frac{d}{dx} P_a^x + \frac{d}{dx} N_a^x = \frac{d}{dx} T_a^x.$$

Integrating on the both sides yields $\int_a^b |f'| \leq T_a^b(f)$. \square

5.4 Absolute Continuity

12.

Solution. The continuous extension of $x^2 \sin(1/x^2)$ to $[0, 1]$ is absolutely continuous for all $[\varepsilon, 1]$ but is not of bounded variation on $[0, 1]$ and therefore is not absolutely continuous on $[0, 1]$.

Suppose that f is also of bounded variation on $[0, 1]$. Then f is differentiable almost everywhere. Hence $g(x) = \int_0^x f'(t)dt + f(a)$ is well-defined. For every $\varepsilon > 0$, we have

$$g(x) = \int_0^\varepsilon f'(t)dt + \int_\varepsilon^x f'(t)dt + f(0) = \int_0^\varepsilon f'(t)dt + f(x) - f(\varepsilon) + f(0),$$

where the second equality comes from the absolute continuity on $[\varepsilon, 1]$. By the continuity of f at $x = 0$, $f(\varepsilon) \rightarrow f(0)$. Hence, letting $\varepsilon \rightarrow 0$ yields $g(x) = f(x)$. Namely, f is an indefinite integral. Thus, by Theorem 14, it is absolutely continuous. \square

13.

Proof. Since absolute continuity implies bounded variation, $\int_a^b |f'| \leq T_a^b(f)$ by Problem 11. By the definition of T , for every $\varepsilon > 0$, there exists some subdivision $a = x_0 < \cdots < x_n = b$ such that $t_a^b(f) > T_a^b(f) - \varepsilon$. Meanwhile, for every $i = 1, \dots, n$,

$$\int_{x_{i-1}}^{x_i} |f'| \geq \left| \int_{x_{i-1}}^{x_i} f' \right| = |f(x_i) - f(x_{i-1})|,$$

where the second equality is guaranteed by the absolute continuity. Hence, $\int_a^b |f'| > T_a^b(f) - \varepsilon$ for every $\varepsilon > 0$. Thus, $T_a^b(f) = \int_a^b |f'|$.

By Lemma 4, $2P_a^b(f) = T_a^b(f) + f(b) - f(a)$. Hence,

$$P_a^b(f) = \frac{1}{2} \left(\int_a^b |f'| + f(b) - f(a) \right) = \frac{1}{2} \int_a^b (|f'| + f') = \int_a^b [f']^+.$$

□

14.

Proof.

(a) Suppose that f and g are absolutely continuous. Then for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for all finite nonoverlapping $\langle (x_n, y_n) \rangle$ with $|x_n - y_n| < \varepsilon$,

$$\sum |f(x_n) + g(x_n) - f(y_n) - g(y_n)| \leq \sum |f(x_n) - f(y_n)| + |g(x_n) - g(y_n)| \leq 2\varepsilon.$$

Hence, $f + g$ is also absolutely continuous. Since $-g$ is absolutely continuous as long as g is, so is $f - g$.

(b) Suppose that f and g are absolutely continuous. Then they are bounded, by M for example. Hence for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for all finite nonoverlapping $\langle (x_n, y_n) \rangle$ with $|x_n - y_n| < \varepsilon$,

$$\begin{aligned} & \sum |f(x_n)g(x_n) - f(y_n)g(y_n)| \\ &= \sum |f(x_n)g(x_n) - f(x_n)g(y_n) + f(x_n)g(y_n) - f(y_n)g(y_n)| \\ &\leq \sum \{|f(x_n)||g(x_n) - g(y_n)| + |f(x_n) - f(y_n)||g(y_n)|\} \\ &\leq M\varepsilon. \end{aligned}$$

Thus, fg is also absolutely continuous.

(c) Since f is continuous on $[a, b]$, f can achieve its minimum in $[a, b]$. Hence, $|f(x)| \geq m > 0$ as f is never zero. Therefore for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for all finite nonoverlapping $\langle (x_n, y_n) \rangle$ with $|x_n - y_n| < \varepsilon$,

$$\sum \left| \frac{1}{f(x_n)} - \frac{1}{f(y_n)} \right| = \sum \left| \frac{f(y_n) - f(x_n)}{f(x_n)f(y_n)} \right| \leq \frac{1}{m^2} \sum |f(x_n) - f(y_n)| \leq \frac{\varepsilon}{m^2}.$$

□

17. Part (a) is wrong. It can be fixed if we further require g to be monotone increasing.

Proof.

(a) For every $\varepsilon > 0$, let δ_1 be the number in the definition of F corresponding to ε and δ_2 the number in the definition of g corresponding to δ_1 . Then for every finite nonoverlapping $\langle (x_n, y_n) \rangle$ with $|x_n - y_n| < \delta_2$, $\sum |g(x_n) - g(y_n)| < \delta_1$. Since g is monotone increasing, $(g(x_n), g(y_n))$ are nonoverlapping. Therefore, $\sum |F(g(x_n)) - F(g(y_n))| < \varepsilon$. Hence, $F \circ g$ is absolutely continuous. \square

18.

Proof. Without loss of generality, we assume that g is nondecreasing. Since $mE = 0$, for every $\varepsilon > 0$, by Proposition 3.15, there exists an open set $O \supset E$ with $mO < \varepsilon$. Meanwhile, there exists a sequence of disjoint open intervals $\langle I_n = (a_n, b_n) \rangle$ such that $\bigcup_{n=1}^{\infty} I_n = O$ and $l(I_n) < \delta$ where δ is the number in the definition of absolute continuity. Then $g[E] \subset \bigcup_{n=1}^{\infty} g[I_n \cap [0, 1]]$. Since g is continuous, the image of an interval is still an interval and since g is also nondecreasing, $g[I_n \cap [0, 1]] = (g(a'_n), g(b'_n))$, where $a'_n = \max\{a_n, 0\}$ and $b'_n = \min\{b_n, 1\}$. Finally,

$$m(g[E]) \leq \sum_{n=1}^{\infty} m(g[I_n]) = \sum_{n=1}^{\infty} |g(b'_n) - g(a'_n)| \leq \varepsilon,$$

where the last inequality comes from the absolute continuity of g . Since the choice of ε is arbitrary, $m(g[E]) = 0$. \square

20.

Proof.

(a) For every $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then for every $\langle x_n \rangle_{i=1}^n$ and $\langle y_n \rangle_{i=1}^n$ with $|x_n - y_n| \leq \delta$,

$$\sum_{i=1}^n |f(x_n) - f(y_n)| \leq M \sum_{i=1}^n |x_n - y_n| \leq \varepsilon,$$

as f satisfies the Lipschitz condition.

(b) Suppose that f is absolute continuous and $|f'|$ is bounded by M . Then for every x and y in the interval,

$$|f(x) - f(y)| = \left| \int_x^y f'(t) dt \right| \leq M|x - y|.$$

Hence, f satisfies the Lipschitz condition. The converse part has been proved in (a).

(c) It is wrong. A counterexample is $f(x) = \chi_{[0,1]}$, $x \in (-1, 1)$ \square

21.

Proof.

(a) Suppose that $O = \bigcup_{n=1}^{\infty} (c_n, d_n)$ where (c_n, d_n) are disjoint. Since g is continuous and increasing, $g^{-1}(c_n, d_n)$ is still an open interval, denoting it by (a_n, b_n) , and (a_n, b_n) are also disjoint. Meanwhile, $d_n - c_n = f(a_n) - f(b_n) = \int_{a_n}^{b_n} g'$. Hence,

$$mO = m\left(\bigcup_{n=1}^{\infty} (c_n, d_n)\right) = \sum_{n=1}^{\infty} (d_n - c_n) = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} g' = \int_{g^{-1}[O]} g'.$$

(b) Without loss of generality, we assume that $d \notin E$. For every $\varepsilon > 0$, there exists an open set $O \supset E$ with $mO < \varepsilon$. By Part (a),

$$\int_{g^{-1}[O] \cap H} g' = \int_{g^{-1}[O]} g' = mO < \varepsilon.$$

Since the choice of ε is arbitrary, $\int_{g^{-1}[O] \cap H} g' = 0$. Since $g' > 0$ on H , $g^{-1}[O] \cap H$ has measure zero.

(c) Since E is measurable, so is $g^{-1}[E]$. Meanwhile, by Theorem 3, g' is measurable, hence H is also measurable. Therefore, F is measurable.

We may assume without loss of generality that $c, d \notin E$. By Proposition 3.15, there exists some $G \in G_\delta$ such that $E \subset G \subset (c, d)$ and $m(G \setminus E) = 0$. Since g is increasing, $g^{-1}[G] \cap H = F \cup (g^{-1}[G \setminus E] \cap H)$ and by (b), $g[G \setminus E] \cap H$ is of measure zero. Therefore, $\int_F g' = \int_{g^{-1}[G] \cap H} g'$. Namely, it suffices to show the result for $G \in G_\delta$.

Suppose that $G = \bigcap_{n=1}^{\infty} O_n$ where each $O_n \subset (c, d)$ is open and $mO_1 < \infty$. Without loss of generality, we may assume that $\langle O_n \rangle$ is decreasing. Then $mG = \lim_{n \rightarrow \infty} mO_n$. By (a),

$$mO_n = \int_{g^{-1}[O_n]} g' = \int_a^b \chi_{O_n}(g(x)) g'(x) dx.$$

As $\chi_{O_n}(g(x)) g'(x)$ is bounded by $|g'|$,

$$\lim_{n \rightarrow \infty} \int_a^b \chi_{O_n}(g(x)) g'(x) dx = \int_a^b \chi_G(g(x)) g'(x) dx.$$

Hence, $mG = \int_{g^{-1}[G] \cap H} g'$, completing the proof.

(d) By Problem 3.25, $f \circ g$ is measurable. And since g' is measurable by Theorem 3, $(f \circ g)g'$ is also measurable.

Let $\langle \varphi_n \rangle$ be an increasing sequence of nonnegative simple functions which converges to f , the existence of which is guaranteed by Problem 4.4. By the monotone convergence theorem, $\int_c^d f = \lim \int_c^d \varphi_n$.

For each n , suppose that $\varphi_n(y) = \sum_{k=1}^m a_k^{(n)}(y) \chi_{E_k^{(n)}}(y)$. Then

$$\int_c^d \varphi_n = \sum_{k=1}^m a_k^{(n)} mE_k^{(n)} = \sum_{k=1}^m a_k^{(n)} \int_a^b \chi_{E_k^{(n)}}(g(x)) g'(x) dx = \int_a^b \varphi_n(g(x)) g'(x) dx,$$

where the second equality comes from (c). Since g is increasing, $\langle \varphi_n(g(x)) g'(x) \rangle$ is an increasing sequence. Hence,

$$\int_a^b f(g(x)) g'(x) dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_n(g(x)) g'(x) dx.$$

Thus,

$$\int_c^d f(y) dy = \lim_{n \rightarrow \infty} \int_c^d \varphi_n(y) dy = \lim_{n \rightarrow \infty} \int_a^b \varphi_n(g(x)) g'(x) dx = \int_a^b f(g(x)) g'(x) dx.$$

□

5.5 Convex Functions

23.

Proof.

(a) Suppose that $x_0 \in (a, b)$ and $y(x) = m(x - x_0) + \varphi(x_0)$ is a supporting line. As $[a, b]$ is finite, $\varphi \geq \min\{\varphi(a), y(a), y(b)\}$.

(b) If φ is monotone, then the limits exists. If φ is not monotone, then since $D^+\varphi$ is nondecreasing, there exists some $[c, d] \subset (a, b)$ such that $D^+\varphi \leq 0$ on (a, c) and $D^+\varphi \geq 0$ on (d, b) . Namely, φ is monotone on the (a, c) and (d, b) . Therefore, the limits also exist.

Consider a finite interval near the finite endpoint. By (a), the limit can not be $-\infty$ as φ is bounded from below.

(c) If x and y are in the interior of I , the inequality holds by definition. By the continuity of φ , the statement holds for all $x, y \in I$. \square

24.

Proof. Note that the existence of φ'' implies φ is continuously differentiable. Suppose that φ is convex on (a, b) . Then $D^+\varphi$ is nondecreasing by Proposition 17, hence $\varphi''(x) \geq 0$ for each $x \in (a, b)$. And the converse of the statement follows from Proposition 18 immediately. \square

25.

Proof.

(a) $\varphi''(t) = b^2 p(p-1)(a+bt)^{p-2}$ which ≥ 0 on $[0, \infty)$ if $p \geq 1$ and ≤ 0 if $0 < p \leq 1$. \square

26. TODO

27.

Proof. Note that $\log x$ is concave. Denote $A_N = \sum_{n=1}^N \alpha_n$ and $R_N = 1 - A_N$. The situation where $\langle \alpha_n \rangle$ is finite is simple. Hence we assume that $R_N \geq 0$ for all N . Then for every N ,

$$\begin{aligned} \log \left(\sum_{n=1}^{\infty} \alpha_n \xi_n \right) &= \log \left(A_N \sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n + R_N \sum_{n=N+1}^{\infty} \frac{\alpha_n}{R_N} \xi_n \right) \\ &\geq A_N \log \left(\sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n \right) + R_N \log \left(\sum_{n=N+1}^{\infty} \frac{\alpha_n}{R_N} \xi_n \right) \\ &\geq A_N \log \left(\sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n \right) \\ &\geq A_N \log \left(\prod_{n=1}^N \xi_n^{\alpha_n/A_N} \right) \end{aligned}$$

Taking exp on the both sides yields

$$\sum_{n=1}^{\infty} \alpha_n \xi_n \geq \left(\prod_{n=1}^N \xi_n^{\alpha_n/A_N} \right)^{A_N} = \prod_{n=1}^N \xi_n^{\alpha_n} \rightarrow \prod_{n=1}^{\infty} \xi_n^{\alpha_n}.$$

\square

28.

Proof. It follows immediately from the Jensen inequality and the fact that \log is concave. □

6 The Classical Banach Spaces

6.1 The L^p Spaces

1.

Proof. Put $S = \|f\|_\infty$ and $T = \|g\|_\infty$. Then $|f(t)| \leq S$ and $|g(t)| \leq T$ a.e. Hence, $S + T \geq |f(t)| + |g(t)| \geq |f(t) + g(t)|$ a.e. Namely, $m\{t : |f(t) + g(t)| > S + T\} = 0$. Thus, $S + T \geq \|f + g\|_\infty$ by the definition of ess sup . \square

2.

Proof. Put $S = \|f\|_\infty$. Since $S \geq |f|$ a.e.,

$$\|f\|_p = \left\{ \int_0^1 |f|^p \right\}^{1/p} \leq \left\{ \int_0^1 S^p \right\}^{1/p} = S.$$

Therefore, $\overline{\lim}_{p \rightarrow \infty} \|f\|_p \leq S$. For the converse part, let ε be any positive number. Then the measure δ of $E = \{t : |f(t)| > S - \varepsilon\}$ is positive. Hence,

$$\left\{ \int_0^1 |f|^p \right\}^{1/p} \geq \left\{ \int_E |f|^p \right\}^{1/p} \geq \delta^{1/p} (S - \varepsilon) \rightarrow S - \varepsilon \quad \text{as } p \rightarrow \infty.$$

Hence, $\underline{\lim}_{p \rightarrow \infty} \|f\|_p \geq S$, completing the proof. \square

3.

Proof.

$$\|f + g\|_1 = \int |f + g| \leq \int |f| + \int |g| = \|f\|_1 + \|g\|_1.$$

\square

4.

Proof. For every $M > \|g\|_\infty$, $|g| \leq M$ a.e. Hence,

$$\int |fg| \leq M \int |f| = \|f\|_1 M.$$

Since the choice of M is arbitrary, $\int |fg| \leq \|f\|_1 \|g\|_\infty$. \square

6.2 The Minkowski and Hölder Inequalities

8

Proof.

(a) The logarithm function is concave, so

$$\log(a^p/p + b^q/q) \geq \frac{1}{p} \log a^p + \frac{1}{q} \log b^q = \log ab.$$

Taking \exp on the both sides yields the inequality. The equality holds iff $a^p = b^q$.

(b) The case where $p = \infty$ has been proved in Problem 4 and the case where $\|f\|_p = 0$ or $\|g\|_q = 0$ is straightforward. Hence, we assume that $1 < p, q < \infty$ and $\|f\|_p \|g\|_q \neq 0$.

Suppose $\alpha = \|f\|_p$ and $\beta = \|g\|_q$. By Young's inequality,

$$\left| \frac{fg}{\alpha\beta} \right| \leq \frac{1}{p} \left(\frac{|f|}{\alpha} \right)^p + \frac{1}{q} \left(\frac{|g|}{\beta} \right)^q$$

for every x . Therefore,

$$\int |fg| = \alpha\beta \int \left| \frac{fg}{\alpha\beta} \right| \leq \alpha\beta \int \left\{ \frac{1}{p} \left(\frac{|f|}{\alpha} \right)^p + \frac{1}{q} \left(\frac{|g|}{\beta} \right)^q \right\} = \alpha\beta. \quad (4)$$

The equality holds iff the equality in Young's inequality holds a.e. iff $\beta|f|^p = \alpha|g|^q$ a.e.

(c) Let $p' = 1/p$ and $q' = 1 - p' = -q/p$. Then for any nonnegative c and d , by Young's inequality,

$$cd \leq \frac{c^{p'}}{p'} + \frac{d^{q'}}{q'} = pc^{1/p} - \frac{p}{q} d^{-q/p} \Rightarrow c^{1/p} \geq \frac{cd}{p} + \frac{d^{-q/p}}{q}.$$

Putting $c = (ab)^p$ and $d = b^{-p}$ yields the desired inequality.

(d) Just reverse the inequality in (4). □

6.3 Convergence and Completeness

9.

Proof. Suppose $\langle f_n \rangle \subset X$ converges to $f \in X$. Namely, for every $\varepsilon > 0$, there exists some N such that for all $n > N$, $\|f_n - f\| < \varepsilon$. Hence, for every $n, m > N$, by Minkowski inequality,

$$\|f_n - f_m\| \leq \|f_n - f\| + \|f - f_m\| < 2\varepsilon.$$

Hence, $\langle f_n \rangle$ is a Cauchy sequence. □

10.

Proof. Suppose $f_n \rightarrow f$. Then $M_n = \|f_n - f\|_\infty = \text{ess sup } |f_n - f| \rightarrow 0$. Let $E_n = \{x : |f_n(x) - f(x)| > M_n\}$, each of which is with measure zero. And therefore $E = \bigcup_{n=1}^\infty E_n$ is with measure zero. Note that $\tilde{E} = \{x : |f_n(x) - f(x)| < M_n, \forall n\}$, which implies the uniform convergence of f_n since $M_n \rightarrow 0$.

For the converse part, the uniform convergence on \tilde{E} implies that for every $\varepsilon > 0$, there exists some N such that for every $n > N$ and $x \in \tilde{E}$, $|f_n(x) - f(x)| < \varepsilon$. Since $mE = 0$, this implies $\|f_n - f\|_\infty = \text{ess sup } |f_n(x) - f(x)| < \varepsilon$. Hence, $f_n \rightarrow f$ in L^∞ . □

11.

Proof. Let $\langle f_n \rangle \subset L^\infty$ be absolutely summable. Put $M_n = \|f_n\|_\infty$ and $A_n = \{t : |f_n(t)| > M_n\}$. By the definition of $\|\cdot\|_\infty$, $mA_n = 0$. Hence, $A = \bigcup_{n=1}^\infty A_n$ is of measure zero.

Note that $|f_n(x)| \leq M_n$ for every n and $x \in E \setminus A$. Thus, by the Weierstrass M-test, $\sum_{n=1}^\infty f_n$ converges uniformly. Hence, on $E \setminus A$, $\sup |\sum_{n=1}^\infty f_n - \sum_{n=1}^N f_n| \rightarrow 0$ as $N \rightarrow \infty$. Since $mA = 0$, this implies the summability of $\langle f_n \rangle$. □

13.

Proof. Suppose $\langle f_n \rangle \subset C$ be absolutely summable. Since for every x , $0 \leq |f_n(x)| \leq \|f_n\|$, $\langle f_n \rangle$ is uniformly convergent on $[0, 1]$. Put $s = \sum_{n=1}^{\infty} f_n$. Since each f_n is continuous, so is s . Therefore, $s \in C$.

For every $\varepsilon > 0$, there exists some N such that for every $n > N$ and $x \in [0, 1]$, $|s(x) - \sum_{k=1}^n f_k(x)| < \varepsilon$. Hence, $\|s - \sum_{k=1}^n f_k\| < \varepsilon$. Thus, $\langle f_n \rangle$ is summable and therefore C is a Banach space. \square

16.

Proof. Since $\|f_n - f\| \geq |||f_n| - |f|||$, $f_n \rightarrow f$ in L^p implies $\|f_n\| \rightarrow \|f\|$. For the converse part, note that $2^p(|f_n|^p + |f|^p) - |f_n - f|^p \geq 0$ and for almost every x ,

$$2^p(|f_n|^p + |f|^p) - |f_n - f|^p \rightarrow 2^{p+1}|f|^p.$$

By Fatou's Lemma,

$$\begin{aligned} 2^{p+1}\|f\|^p &= 2^{p+1} \int |f|^p \leq \underline{\lim} \int \{2^p(|f_n|^p + |f|^p) - |f_n - f|^p\} \\ &= 2^{p+1}\|f\|^p - \overline{\lim} \|f_n - f\|^p. \end{aligned}$$

Hence, $\overline{\lim} \|f_n - f\|^p \leq 0$. Since clear that $\underline{\lim} \|f_n - f\|^p \geq 0$, $\lim \|f_n - f\| = 0$, i.e., $f_n \rightarrow f$ in L^p . \square

17. I assume that $1/p + 1/q = 1$.

Proof. Since $g \in L^p$, $|g|^q$ is integrable on $E = [0, 1]$ and therefore for every $\varepsilon > 0$, there exists some δ such that for every $A \subset E$ with $mA < \delta$, $\int_A |g|^q < \varepsilon$. Meanwhile, since $f_n(x) \rightarrow f(x)$ for almost every x , by Egoroff's Theorem, there exists some $A \subset E$ with $mA < \delta$ such that $f_n g$ converges to $f g$ uniformly on $E \setminus A$.

From the uniform convergence we conclude

$$\int_{E \setminus A} f g = \lim_{n \rightarrow \infty} \int_{E \setminus A} f_n g. \quad (5)$$

Meanwhile, by Hölder inequality,

$$\left| \int_A (f - f_n) g \right| \leq \int_A |(f - f_n) g| \leq \left\{ \int_A |f_n - f|^p \right\}^{1/p} \left\{ \int_A |g|^q \right\}^{1/q} \leq M \varepsilon^{1/q}.$$

Hence, (5) can be extended to E .

For $p = 1$, this is not true. $f_n = n\chi_{[0, 1/n]}$ and $g = \chi_{[0, 1]}$ gives a counterexample. \square

18.

Proof. By Minkowski inequality,

$$\|g_n f_n - g f\| = \|g_n(f_n - f) + (g_n - g)f\| \leq \|g_n(f_n - f)\| + \|(g_n - g)f\|.$$

Fix $\varepsilon > 0$. Since $f, g_n, g \in L^p$, $|g_n - g|^p |f|^p$ is integrable and therefore there exists some $\delta > 0$ such that for all subsets with measure $< \delta$, the integral of over it $< \varepsilon$. Meanwhile,

since $g_n \rightarrow g$ a.e., by Egoroff's Theorem, there exists some $A \subset E = [0, 1]$ with $mA < \delta$ such that $g_n \rightarrow g$ uniformly on $E \setminus A$ and therefore there exists some $N_1 > 0$ such that for all $n > N_1$, $|g_n(x) - g(x)|^p < \varepsilon$ for $x \in E \setminus A$. Thus, for every $n > N_1$,

$$\begin{aligned} \|(g_n - g)f\| &= \left\{ \int_{E \setminus A} |g_n - g|^p |f|^p \right\}^{1/p} + \left\{ \int_A |g_n - g|^p |f|^p \right\}^{1/p} \\ &\leq \sqrt[p]{\varepsilon} \|f\| + \sqrt[p]{\varepsilon} \leq (\|f\| + 1)\varepsilon. \end{aligned}$$

Since $|g_n| \leq M$, $\|g_n(f_n - f)\| \leq M\|f_n - f\|$. And since $f_n \rightarrow f$ in L^p , there exists some $N_2 > 0$ such that for all $n > N_2$, $\|f_n - f\| < \varepsilon$. Put $N = \max(N_1, N_2)$, then for every $n > N$,

$$\|g_n f_n - g f\| \leq (\|f\| + 1 + M)\varepsilon.$$

Hence, $g_n f_n \rightarrow g f$ in L^p . □

6.4 Approximation in L^p

19.

Proof. Since $\|T_\Delta f\| \leq \|T_\Delta |f|\|$ and $\|f\| = \||f|\|$, we may assume without loss of generality that $f \geq 0$. For $p > 1$, by Jensen's inequality,

$$\begin{aligned} \|T_\Delta f\|_p^p &= \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} \left(\frac{1}{\xi_k - \xi_{k-1}} \int_{\xi_{k-1}}^{\xi_k} f \right)^p \\ &\leq \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} \frac{1}{\xi_k - \xi_{k-1}} \int_{\xi_{k-1}}^{\xi_k} f^p \\ &= \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} f^p \\ &= \int_0^1 f^p = \|f\|_p^p. \end{aligned}$$

□

7 Metric Spaces

7.7 Compact Metric Space

27.

Proof. If $\rho(F, K) > 0$, then clear that $F \cap K = \emptyset$. For the reverse direction, consider the function $h(x) = \rho(x, F) = \inf_{y \in F} \rho(x, y)$. Clear that for $x \in K$, $h(x) \leq \rho(K, F)$.

First, we show that h is continuous. Let x be fixed. For every $x' \in X$

$$h(x) \leq \rho(x, y) \leq \rho(x, x') + \rho(x', y), \quad \forall y \in F.$$

Take infimum on the right hand side and we get

$$h(x) \leq \rho(x, x') + h(x') \quad \Rightarrow \quad h(x) - h(x') \leq \rho(x, x').$$

Similarly, we have $h(x') - h(x) \leq \rho(x, x')$. Thus, h is continuous.

Since K is compact and h is continuous, h attains its infimum c at some point $x_0 \in K$. Assume, to obtain a contradiction, that $c = 0$. Then, for every $\varepsilon > 0$, there is a $y \in F$ s.t. $\rho(x_0, y) < \varepsilon$. Namely, x_0 is a cluster point of F . Since F is closed, $x_0 \in F$, which contradicts with $F \cap K = \emptyset$. Thus, $c > 0$ and therefore $\rho(F, K) > 0$. \square

28.

Proof.

(a) Let $\varepsilon > 0$ be fixed. Since f is uniformly continuous, there exists some $\delta > 0$ s.t. for every $x_1, x_2 \in X$ with $\rho(x_1, x_2) < \delta$, $\rho(f(x_1), f(x_2)) < \varepsilon$. Let $B \subset X$ be a ball with radius $\delta/2$. Then $f(B) \subset Y$ is contained by some ball of radius ε . Hence, we can cover Y with finitely many balls of radius ε as long as we can cover X with finitely many balls of radius $\delta/2$ and this can be done since X is totally bounded.

(b) Clear that $X = (0, 1)$ is totally bounded while $Y = (1, \infty)$ is not. The function $1/x$ a continuous function that maps X onto Y . Hence, the result does not hold. \square

29.

Proof.

(a) First, by the definition of open cover and open sets, $\varphi(x) > 0$. Meanwhile, since X is compact, it is bounded and, therefore, $\varphi(x) < \infty$.

(b) Let x be fixed and r be such that there exists some $O \in \mathcal{U}$ with $B_{x,r} \subset O$. Let $s = r - \rho(x, y)$. If $s \leq 0$, then clear that $s \leq \varphi(y)$. If $s > 0$, then we have $B_{y,s} \subset B_{x,r} \subset O$. Hence, $s \leq \varphi(y)$. Thus, $r - \rho(x, y) \leq \varphi(y)$. Take supremum on the left hand side and we get $\varphi(x) - \rho(x, y) \leq \varphi(y)$.

(c) It follows immediately from (b).

(d) Let (x_n) be a sequence s.t. $\varphi(x_n) \rightarrow \varepsilon$. Since X is sequentially compact, (x_n) has a convergent subsequence. For the sake of convenience, we assume the subsequence is (x_n) itself. Suppose $x_n \rightarrow x$. Since φ is continuous, $\varphi(x) = \lim \varphi(x_n) = \varepsilon$. Thus, by (a), $\varepsilon > 0$.

(e) Let δ be any positive number that is less than ε . For every x , since $\delta < \varepsilon = \inf \varphi$, $\delta < \varphi(x)$ and, therefore, $B_{x,\delta} \subset O$ for some $O \in \mathcal{U}$. \square

7.8 Baire Category

31.(a)

Proof. Suppose that F is nowhere dense. Since F is closed, F^c is dense. Therefore, for every point $x \in F$, every neighborhood of x contains a point of F^c . Thus, F contains no open set. For the reverse direction, since F contains no open set, every neighborhood of every $x \in X$ contains a point in F^c , which implies that F^c is dense. \square

34.

Proof. For the first part, it suffices to show that ∂E is nowhere dense for any set $E \subset X$. Since ∂E is closed and contains no open set, by 31.(a), it is nowhere dense.

For the second part, assume, to obtain a contradiction, that F is not nowhere dense. Then by 31.(a), it contains a nonempty open set O . By 32.(a), O is also meager, which contradicts the Baire category theorem. \square

35.

Proof. Let E be a subset set of the complete metric space X . If E is residual, then $E^c = \bigcup_n K_n$ where K_n are nowhere dense sets. Hence,

$$E = \bigcap K_n^c \supset \bigcap (\text{cl } K_n)^c.$$

Note that $(\text{cl } K_n)^c$ is a G_δ . Meanwhile, since each $(\text{cl } K_n)^c$ is a dense open set and X is complete, By the theorem of Baire, $\bigcap (\text{cl } K_n)^c$ is also dense. The proof of the reverse direction is similar. \square

39.

Proof. Let E_m has the same meaning as in the proof of Theorem 32. Let $O := \bigcup_m E_m^\circ$. By Prop. 31, O is a dense residual open set. For every $x \in O$, $x \in E_m^\circ$ for some m . Since E_m° is open, there is a neighborhood U of x which is contained by E_m° . Thus, \mathcal{F} is uniformly bounded by m in U . \square

7.10 The Ascoli-Arzelá Theorem

47.

Proof. Let x be an arbitrary fixed point in X and $(x_n) \subset X$ a sequence that converges to x . Clear that $K = \{x\} \cup (x_n)$ is (sequently) compact. Hence, f_n converges to f uniformly on K . Thus, f is also continuous on K and therefore at x . Namely, f is continuous. \square

49.

Proof. Let x be an arbitrary point in X . For every $\varepsilon > 0$, since \mathcal{F} is equicontinuous, there is an open neighborhood O of x s.t. $\sigma(f(x), f(y)) < \varepsilon$ for every $y \in O$ and $f \in \mathcal{F}$. Now, for every $f^+ \in \mathcal{F}^+$, suppose $f_n \rightarrow f$. We have

$$\sigma(f^+(x), f^+(y)) = \sigma\left(\lim_n f_n(x), \lim_m f_m(y)\right) = \lim_n \lim_m \sigma(f_n(x), f_m(y)) \leq \varepsilon,$$

where the second equality comes from the continuity of σ and the last inequality comes from the equicontinuity of \mathcal{F} . \square

50. I assume that the norm on $C[0, 1]$ is the sup norm.

Proof. Let $\mathcal{F} := \{f \in C[0, 1] \mid \|f\|_\alpha \leq 1\}$. First, we show that \mathcal{F} is equicontinuous. By the definition of $\|\cdot\|_\alpha$, $f \in \mathcal{F}$ only if it is bounded by 1 and $|f(x) - f(y)| \leq |x - y|^\alpha$ for all $x, y \in [0, 1]$. For every $x \in X$ and $\varepsilon > 0$, consider the $\sqrt[\alpha]{\varepsilon}$ -ball B centered at x . For every $y \in B$, $|f(x) - f(y)| \leq |x - y|^\alpha < \varepsilon$. Hence, \mathcal{F} is equicontinuous. Let (f_n) be a sequence in \mathcal{F} . Since f_n is bounded by 1, by Corollary 41, (f_n) contains a convergent subsequence. Thus, \mathcal{F} is a (sequentially) compact subset of $C[0, 1]$. \square

8 Topological Spaces

8.1 Fundamental Notions

3.

Proof. If A is open, then clear that $x \in A \subset A$. For the converse, let $E = \bigcup_{x \in A} O_x$ where $x \in O_x \subset A$. Since $O_x \subset A$, $E \subset A$. Meanwhile, for every $x \in A$, $x \in O_x \subset E$. Hence, $A = E$. Thus, A is open since E is the union of open sets. \square

7.

Proof.

(a) We argue by contradiction. Assume $x \in F^c$. Since F is closed, F^c is open and therefore there is neighborhood O of x s.t. $O \cap F \neq \emptyset$, which contradicts the fact that x is a cluster point. Thus, $x \in F$.

(b) Let $y := f(x)$ and $y_n := f(x_n)$. Let O be an arbitrary neighborhood of y . Since f is continuous, there is a neighborhood U of x s.t. $f(U) \subset O$. Since $x = \lim x_n$, there is an integer N s.t. $x_n \in U$ for every $n > N$. Hence, $y_n = f(x_n) \in f(U) \subset O$ for all $n > N$. Thus, $y_n \rightarrow y$.

(c) The previous argument, *mutatis mutandis*, yields the result. \square

10.

Proof.

(a) Suppose that both A_1 and A_2 are open. Let $f_1 := f|_{A_1}$, $x \in A$ and $y = f(x)$. We may assume without loss of generality that $x \in A_1$. For every neighborhood O of y , since f_1 is continuous, there is a neighborhood $A_1 \cap U$ s.t. $f(A_1 \cap U) \subset O$ where U is an open set of X . Since $A_1 \cap U$ is still open, f is continuous at x . Thus, f is continuous.

(b) Let $f(x) = 0$ on $A_1 = (-1, 0)$ and $f(x) = 1$ on $A_2 = [0, 1]$. \square

8.2 Bases and Contability

11.

Proof.

(a) Suppose for every $B \in \mathcal{B}$ containing x , there is a $y \in B \cap E$. Let O be an arbitrary open set containing x . By the definition of bases, there is a base set B s.t. $x \in B \subset \mathcal{B}$. Thus, $O \cap E \supset B \cap E \neq \emptyset$. Namely, $x \in \text{cl } E$. The reversed direction follows immediately from the definition of cluster points.

(b) If there is sequence in E that converges to x , clear that $x \in \text{cl } E$. Now we show the reverse. Let $\mathcal{B}_x = (B_k)$ be a countable base at x and $S_n = \bigcap_{k=1}^n B_k$. Choose $x_n \in S_n$. For every open set O containing x , there exists a B_m s.t. $x \in B_m \subset O$. Hence, for every $n > m$, $x_n \in S_n \subset S_m \subset B_m \subset O$. Thus, $x_n \rightarrow x$.

(c) It follows immediately from (b). \square

13.

Proof. By the construction of \mathcal{B} and Prop. 5, \mathcal{B} is a base for some topology on X . Let \mathcal{T} be a topology containing \mathcal{C} . Since $X \in \mathcal{T}$ and \mathcal{T} is closed under finite intersection, $\mathcal{T} \supset \mathcal{B}$. Thus, \mathcal{T} contains the topology generated by \mathcal{B} . Since the choice of \mathcal{T} is arbitrary, we conclude that \mathcal{B} is a base for the weakest topology containing \mathcal{C} . \square

16.

Proof. Let \mathcal{B} be a countable base for the topology on X and \mathcal{U} an open cover of X . For each $B \in \mathcal{B}$, if there is some $U \in \mathcal{U}$ with $B \subset U$, then pick that U . This yields a at most countable subset \mathcal{V} of \mathcal{U} . Now we show that \mathcal{V} covers X . For every $x \in X$, since \mathcal{U} covers X , there is a $U \in \mathcal{U}$ s.t. $x \in U$. Meanwhile, since \mathcal{B} is a base, there is a $B \in \mathcal{B}$ s.t. $x \in B \subset U$. Hence, by our construction, there is a $U' \in \mathcal{U}$ containing x is picked. Thus, x is covered by \mathcal{V} . \square

8.3 The Separation Axioms and Continuous Real-Valued Functions

18.

Proof.

(a) If $x \neq y$. then $d(x, y) > 0$. It suffices to choose the open balls of radius $d(x, y)/2$ centered at x and y respectively.

(b) Let $O_1 = \{x \mid \rho(x, F_1) < \rho(x, F_2)\}$ and $O_2 = \{x \mid \rho(x, F_1) > \rho(x, F_2)\}$. Clear that O_1 and O_2 are disjoint. Since F_1 and F_2 are closed and disjoint, $\rho(F_1, F_2) > 0$. Hence, for every $x \in F_1$, $\rho(x, F_1) = 0$ and $\rho(x, F_2) > 0$. Therefore, $F_1 \subset O_1$. Similarly, $F_2 \subset O_2$. Meanwhile, for every $x \in O_1$, the open ball centered at x and of radius $(\rho(x, F_2) - \rho(x, F_1))/3$ is contained in O_1 . Therefore, O_1 is open and similarly for O_2 . Thus, X is normal. \square

20.

Proof. If f is continuous, then clear that $\{x \mid f(x) < a\} = f^{-1}((-\infty, a))$ and $\{x \mid f(x) > a\} = f^{-1}((a, \infty))$ are open. Now we show the reverse. For every open interval (a, b) , $f^{-1}((a, b)) = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b))$ is open. Meanwhile, every open set O in \mathbb{R} is a union $\bigcup I_\alpha$ of open intervals. Thus, $f^{-1}(O) = \bigcup f^{-1}(I_\alpha)$ is open. Namely, f is continuous. The second results follows immediately from $\{f \geq a\}$ is closed iff $\{f < a\}$ is open. \square

23. It seems that in (a), the Hausdorff condition is not necessary.

Proof.

(a) First, suppose that X is normal. Let $G = O^c$. F and G are disjoint closed set and, therefore, there are disjoint open sets U and V s.t. $F \subset U$ and $G \subset V$. To show $\text{cl} U \subset O$, note that $O = G^c \supset V^c \supset U$, since U and V are disjoint. Since V^c is closed, $V^c \supset \text{cl} U$. Thus, $F \subset U$ and $\text{cl} U \subset O$.

For the reverse, let F_1 and F_2 be two disjoint closed sets. Then F_1^c is an open set containing F_2 and therefore there exists an open U_2 s.t. $F_2 \subset U_2$ and $\text{cl} U_2 \subset F_1^c$. Note that $\text{cl} U_2$ and F_1 are again disjoint closed sets. Hence, similarly, we can find an open U_1 s.t. $F_1 \subset U_1$ and $\text{cl} U_1 \subset (\text{cl} U_2)^c$. Thus, X is normal.

(b) We index the sequence by n instead of r . Suppose that N is the smallest integer s.t. $r = p2^{-N} < 1$. By (a), we may find a open set U_N s.t. $F \subset U_N$ and $\text{cl } U_N \subset O$. Now, U_N is again an open set containing F and therefore we can find U_{N+1} s.t. $F \subset U_{N+1}$ and $\text{cl } U_{N+1} \subset U_N$. Proceed iteratively and we get the required sequence.

(c) Clear that $0 \leq f \leq 1$, $f \equiv 0$ on F and $f \equiv 1$ on O^c . Hence, it suffices to show the continuity. For every $x \in X$ and $\varepsilon > 0$, choose $r_1, r_2 \in \mathbb{Q}$ such that

$$f(x) - \varepsilon < r_1 < f(x) < r_2 < f(x) + \varepsilon.$$

Let $U = U_{r_2} \setminus \text{cl } U_{r_1}$. Clear that U is open. Meanwhile,

$$f(x) < r_2 \Rightarrow \inf\{r \mid x \in U_r\} < r_2 \Rightarrow \exists r < r_2 \text{ s.t. } x \in U_r.$$

Hence, $x \in U_{r_2}$. If $x \in \text{cl } U_{r_1}$, then $x \in U_r$ for all $r > r_1$ since $U_r \supset \text{cl } U_{r_1}$ and, therefore, $f(x) \leq r_1$. Contradiction. Thus, $x \notin U_{r_1}$. Hence, U is an open neighborhood of x . For every $y \in U$, clear that $f(x) < r_2$. Also, $y \notin \text{cl } U_{r_1}$ implies that $f(y) \geq r_1$. Hence, $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. Thus, f is continuous.

(d) If X is normal, then clear that the function described in (c) satisfies the requirements. For the reverse, let $O_1 = \{x \mid f(x) < 1/2\}$ and $O_2 = \{x \mid f(x) > 1/2\}$. Clear that O_1 and O_2 are disjoint and since f is continuous, they are open. Meanwhile, as $f \equiv 0$ on A and $f \equiv 1$ on B , O_1 contains A and O_2 contains B . Thus, X is normal. \square

24. The function in (f) should be $g = \varphi k / (1 - |\varphi k|)$.

Proof.

(a) Obvious.

(b) Clear that B and C are disjoint. Since f is continuous, so is h and therefore B and C are closed. By Urysohn's lemma, there exists continuous $0 \leq h_{1,B}, h_{1,C} \leq 1$ s.t. $h_{1,B} \equiv 1$ (resp. $h_{1,C} \equiv 1$) on B (resp. C) and vanishes on C (resp. B). Let $h_1 = (-h_{1,B} + h_{1,C})/3$. Then $h_1 \equiv -1/3$ on B and $h_1 \equiv 1/3$ on C . Meanwhile, h_1 is continuous and $|h_1| \leq 1/3$. Let $x \in A$. If $h(x) \leq -1/3$, then $|h(x) - h_1(x)| = |h(x) + 1/3| < 2/3$. Similarly for x with $h(x) \geq 1/3$. If $-1/3 < h(x) < 1/3$, then $|h(x) - h_1(x)| < |h(x)| + |h_1(x)| \leq 2/3$. Thus, $|h - h_1| < 2/3$.

(c) Suppose that we have constructed h_n . Let $s_n = \sum_{i=1}^n h_i$. Let

$$B_n = \{x \mid h(x) - s(x) < -2^n/3^{n+1}\} \quad \text{and} \quad C_n = \{x \mid h(x) - s(x) > -2^n/3^{n+1}\}.$$

The previous argument, *mutatis mutandis*, yields a continuous function h_{n+1} s.t. $|h_{n+1}| < 2^n/3^{n+1}$ and $|h - s_n - h_{n+1}| = |h - s_{n+1}| < 2^{n+1}/3^{n+1}$ for all $x \in A$.

(d) By the Weierstrass M -test, h_n is uniformly summable. Hence, $k = \sum_{n=1}^{\infty} h_n$ is continuous as each h_n is. Clear that $|k| \leq 1$. Moreover, by the estimation in (c), $h = k$ on A .

(e) By Urysohn's lemma, there is a continuous function φ on X s.t. $\varphi \equiv 1$ on A and $\varphi \equiv 0$ on $\{x \mid k(x) = 1\}$.

(f) Let $g = \varphi k / (1 - |\varphi k|)$. By (e), g is well-defined on entire X . Also, g is continuous and for $x \in A$

$$g(x) = \frac{\varphi(x)k(x)}{1 - |\varphi(x)k(x)|} = \frac{1 \times h(x)}{1 - |1 \times h(x)|} = \frac{\frac{f}{1+|f|}}{1 - \frac{|f|}{1+|f|}} = f.$$

\square

26.

Proof. Let \mathcal{J} be the topology generated by \mathcal{F} . Since every $f \in \mathcal{F}$ is continuous, $\mathcal{J} \subset \mathcal{T}$. Let $O \in \mathcal{T}$. For every $x \in O$, there is a continuous function $f \in \mathcal{F}$ s.t. $f(x) = 1$ and vanishes on O^c . Namely,

$$U_x := f^{-1}(\mathbb{R} \setminus \{0\}) \in \mathcal{J}$$

is an open set with $x \in U_x \subset O$. Clear that $O = \bigcup_x U_x$. Thus, $O \in \mathcal{J}$. Namely, $\mathcal{T} = \mathcal{J}$. \square

8.4 Connectedness

32.

Proof. Assume that G is not connected and let (O_1, O_2) be a separation of G . Since each G_α is connected, G_α is contained by exactly one of O_1 and O_2 since otherwise $(G_\alpha \cap O_1, G_\alpha \cap O_2)$ would be a separation of G_α . Thus, there are two of $\{G_\alpha\}$ s.t. one is contained by O_1 and the other contained by O_2 and hence they have no point in common. Contradiction. \square

33.

Proof. Assume that B is not connected and let (O_1, O_2) be a separation of B . Since A is connected, either $A \cap O_1$ or $A \cap O_2$ is empty. Assume without loss of generality that $A \cap O_2 = \emptyset$. Then, $O_2 \subset \partial A$ and therefore has empty interior. Contradiction. \square

35.

Proof.

(a) If X is not connected, then we choose two points x, y from each set of a separation. Clear that they can not be connected by some arc since the image of $[0, 1]$ under a continuous map is still connected.

(b) Assume that X is not connected and let (O_1, O_2) be a separation. Since each of $X_1 = \{(x, y) \mid x = 0, -1 \leq y \leq 1\}$ and $X_2 = \{(x, y) \mid y = \sin x, 0 < x \leq 1\}$ is connected, they are contained in, say, O_1 and O_2 respectively. Clear that any neighborhood of $(0, 0)$ contains points in X_2 and therefore $O_1 \cap O_2 \neq \emptyset$. Contradiction.

Now we show that X is not arcwise connected. (TODO)

(c) Let $x \in G$ and H be the points of G that can be connected to x by a polygonal arc. For every $y \in H$, since G is open, there is an open ball B centered at y with $B \subset G$. Clear that we can connect every $z \in B$ by the arc connecting x and y , and the segment from y to z . Thus, H is open. Now, we show that $K = G \setminus H$ is open. Let y be a point in K and B a small open ball centered at y that is contained in G . Assume, to obtain a contradiction, that $B \cap H \neq \emptyset$. Then the previous argument, *mutatis mutandis*, show that y can be connected to x . Contradiction. Thus, H is both open and closed in G . Since G is connected, $H = G$. Thus, G is arcwise connected. \square

9 Compact and Locally Compact Spaces

9.1 Compact Spaces

2. I further assume that X is Hausdorff.

Proof. Assume, to obtain a contradiction, that $F_n = K_n \setminus O \neq \emptyset$ for every n . Since $F_n \supset F_{n+1}$, $\{F_n\}$ is collection of closed subsets of compact set K_1 with finite intersection property. Hence, $\bigcap F_n$ is nonempty, contradicting with $\bigcap K_n \subset O$. \square

3.

Proof. Let F be a closed set and $x \notin F$. Since X is Hausdorff, for every $y \in F$, there are two disjoint open sets U_y and O_y s.t. $y \in U_y$ and $x \in O_y$. Since X is compact, so is F . Note that $\{U_y\}$ is an open cover for F . Hence, it has a finite subcover $\{U_{y_i}\}_{i=1}^n$. Let $U = \bigcup U_{y_i}$ and $O = \bigcap O_{y_i}$. Clear that they are disjoint open sets s.t. $F \subset U$ and $x \in O$. \square

6.

Proof. Let $\varepsilon > 0$ be fixed. Since \mathcal{F} is equicontinuous, for every $x \in X$, there is a neighborhood O_x s.t. for every $x' \in X$, $\sigma(f(x), f(x')) < \varepsilon$. Clear that $\{O_x\}$ is an open cover and since X is compact, it has a finite subcover $\{O_{x_i}\}_{i=1}^m$.

For x_i , since $f_n(x_i) \rightarrow f(x_i)$, there is an integer N_i s.t. for every $n > N_i$, $\sigma(f_n(x_i), f(x_i)) < \varepsilon$. Since $\sigma(f_n(x_i), f_n(x)) < \varepsilon$ holds for all n , $\sigma(f(x_i), f(x)) \leq \varepsilon$. Hence, for every $x \in O_{x_i}$ and $n > N_i$,

$$\sigma(f_n(x), f(x)) \leq \sigma(f_n(x), f_n(x_i)) + \sigma(f_n(x_i), f(x_i)) + \sigma(f(x_i), f(x)) < 3\varepsilon.$$

Let $N = \max N_{x_i}$ and we get the desired result. \square

9.2 Countable Compactness and the Bolzano-Weierstrass Property

9.

Proof.

(a) It follows immediately from the definition and Problem 8.20.

(b) For every $\alpha \in \mathbb{R}$,

$$f + g < \alpha \quad \text{iff} \quad f < \alpha - g \quad \text{iff} \quad \exists q \in \mathbb{Q} \text{ s.t. } f < q, q < \alpha - g.$$

Hence,

$$\{f + g < \alpha\} = \bigcup_{q \in \mathbb{Q}} \{f < q\} \cap \{g < \alpha - q\},$$

which is open. Thus, $f + g$ is also upper semicontinuous.

(c) Since (f_n) is a decreasing sequence, we can write $f(x) = \inf_n f_n(x)$. Hence, for every $\alpha \in \mathbb{R}$, $f < \alpha$ iff there exists some n s.t. $f_n(x) < \alpha$. Hence,

$$\{f < \alpha\} = \bigcup_n \{f_n < \alpha\},$$

which is open. Thus, f is also upper semicontinuous.

(d) Note that $(f_n - f)$ is a decreasing sequence of upper semicontinuous functions that converges to 0. Hence, by Dini's theorem, the convergence is uniform.

(e) Suppose that $x \in \{f < \alpha\}$. Let ε be a positive real number. Since $f_n \rightarrow f$ uniformly, there is an integer n s.t. $|f(y) - f_n(y)| < \varepsilon$ for all $y \in X$. Meanwhile, since f_n is upper semicontinuous, there is a $\delta > 0$ s.t. for every y in the δ -ball B centered at x , $f_n(y) < f_n(x) + \varepsilon$. Hence, for every $y \in B$,

$$f(y) = f(y) - f_n(y) + f_n(y) - f_n(x) + f_n(x) - f(x) + f(x) \leq 3\varepsilon + f(x).$$

Thus, for sufficiently small $\varepsilon > 0$, we have $B \subset \{f < \alpha\}$. Namely, $\{f < \alpha\}$ is open whence f is upper semicontinuous. \square

10.

Proof.

(i. \Rightarrow iii.) Let f be a bounded continuous real-valued function and $M := \sup f < \infty$. Let $F_n = \{f \geq M - 1/n\}$. Since f is continuous, F_n is closed. Note that (F_n) is a countable family of closed sets with finite intersection property. Hence, $\bigcap F_n = \{f \geq M\}$ is nonempty as X is countably compact. Namely, the maximum can be attained.

(iii. \Rightarrow ii.) Let f be a continuous function and assume, to obtain a contradiction, that f is unbounded. Then the function $-1/(|f| + 1)$ is a continuous bounded function whose maximum can not be attained. Contradiction.

(ii. \Rightarrow i.) Assume, to obtain a contradiction, that X does not have the Bolzano-Weierstrass property, that is, there is a sequence (x_n) in X that has no cluster point. Then $F := \{x_n\}_{n=1}^{\infty}$ is closed. Define $f : F \rightarrow \mathbb{R}$ by $f(x_n) = n$. Note that f is continuous on F and by Tietze's extension theorem, it can be continuously extended to X . However, f is unbounded, contradicting (ii.). Thus X has the Bolzano-Weierstrass and, therefore, is countably compact. \square

11 Measure and Integration

11.1 Measure Spaces

1.

Proof. Put $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. (B_n) is a sequence of disjoint measurable sets. By the countable additivity of μ ,

$$\mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(B_k) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=1}^n B_k\right).$$

Since $\bigcup_{k=1}^n B_k = \bigcup_{k=1}^n A_k$ for $k = 1, \dots, n, \dots, \infty$, this implies $\mu(\bigcup A_k) = \lim \mu(\bigcup_{k=1}^n A_k)$. \square

3.

Proof.

(a) First,

$$0 = \mu(E_1 \triangle E_2) = \mu(E_1 \setminus E_2 \cup E_2 \setminus E_1) = \mu(E_1 \setminus E_2) + \mu(E_2 \setminus E_1).$$

Together with the nonnegativity of μ , we conclude that $\mu(E_1 \setminus E_2) = \mu(E_2 \setminus E_1) = 0$. Note that

$$\mu(E_1 \cup E_2) = \mu(E_1 \setminus E_2 \cup E_2) = \mu(E_1 \setminus E_2) + \mu(E_2).$$

Hence, $\mu(E_1 \cup E_2) = \mu(E_2)$. Similarly, $\mu(E_1 \cup E_2) = \mu(E_1)$. Thus, $\mu(E_1) = \mu(E_2)$.

(b) Since $\mu(E_1 \triangle E_2) = 0$ and $E_2 \setminus E_1 \subset E_1 \triangle E_2$, by the completeness of μ , $E_2 \setminus E_1 \in \mathcal{B}$. Similarly, $E_1 \setminus E_2 \in \mathcal{B}$. In consequence, $E_1 \cap E_2 = E_1 \setminus (E_1 \setminus E_2) \in \mathcal{B}$ and, therefore, $E_2 = (E_1 \cap E_2) \cup (E_2 \setminus E_1) \in \mathcal{B}$. \square

7.

Proof. Let \mathcal{B}_0 be the collection of all sets $E = A \cup B$ where $B \in \mathcal{B}$ and $A \subset C$, $C \in \mathcal{B}$, $\mu C = 0$. Clear that $\mathcal{B} \subset \mathcal{B}_0$. Now we show that it is a σ -algebra. Since $X \in \mathcal{B}$, $X \in \mathcal{B}_0$. Let $E_n = A_n \cup B_n$ be a sequence of elements of \mathcal{B}_0 . Then, $\bigcup E_n = (\bigcup A_n) \cup (\bigcup B_n)$ also belongs to \mathcal{B}_0 since $\bigcup B_n \in \mathcal{B}$ and $\bigcup A_n \subset \bigcup C_n$, which is a countable union of sets of measure zero. Hence, \mathcal{B}_0 is closed under countable union. Now, let $E = A \cup B \in \mathcal{B}_0$. Note that

$$E^c = A^c \cap B^c = (C \setminus A) \cup (B^c \setminus C),$$

where $C \setminus A \subset C$ and $B^c \setminus C \in \mathcal{B}$. Hence, \mathcal{B}_0 is closed under complement. Thus, it is a σ -algebra.

We define $\mu_0 : \mathcal{B}_0 \rightarrow [0, \infty]$ by $\mu_0 E = \mu_0(A \cup B) = \mu B$. First, we show that it is well-defined, that is, if $E = A' \cup B'$, then $\mu B = \mu B'$. Since $C \in \mathcal{B}$ contains A , $(A \cup B) \setminus C \in \mathcal{B}$. Meanwhile, since $\mu C = 0$,

$$\mu B = \mu((A \cup B) \setminus C) = \mu(E \setminus C). \quad (6)$$

Since $E \setminus C \subset E \cup C'$,

$$\mu(E \setminus C) \leq \mu(E \cup C') = \mu((A' \cup B') \cup C') = \mu B', \quad (7)$$

where the measurability of $E \cup C'$ and the last equality both comes from the fact that $A' \subset C' \in \mathcal{B}$ and $\mu C' = 0$. Combine (6) and (7) and we get $\mu B \leq \mu B'$. Interchanging the role of $A \cup B$ and $A' \cup B'$ yields $\mu B \geq \mu B'$. Hence, $\mu B = \mu B'$ and, in consequence, μ_0 is well-defined. Meanwhile, clear that for $E \in \mathcal{B}$, $\mu E = \mu_0 E$.

Finally, we show that μ_0 is a measure. Clear that μ_0 is nonnegative and $\mu_0 \emptyset = 0$. Let $\langle E_n \rangle \subset \mathcal{B}_0$ be a sequence of disjoint sets. Then

$$\mu_0 \left(\bigcup E_n \right) = \mu_0 \left(\bigcup A_n \cup \bigcup B_n \right) = \mu \left(\bigcup B_n \right) = \sum \mu B_n = \sum \mu_0 E_n.$$

Namely, μ_0 is countably additive. Thus, μ_0 is a measure. \square

9.

Proof.

(a) First, we argue by contradiction to show that \mathcal{R} and \mathcal{R}' . Assume that there exists some $E \in \mathcal{R} \cap \mathcal{R}'$, that is, $E \in \mathcal{R}$ and $E^c \in \mathcal{R}$. Then $X = E \cup E^c \in \mathcal{R}$, which contradicts the assumption that \mathcal{R} is not a σ -algebra. Thus, $\mathcal{R} \cap \mathcal{R}' = \emptyset$.

Clear that $\mathcal{R} \cup \mathcal{R}'$ is a σ -algebra containing \mathcal{R} . Hence, $\mathcal{R} \cup \mathcal{R}' \supset \mathcal{B}$. Meanwhile, since $\mathcal{B} = \sigma(\mathcal{R})$, $\mathcal{R} \cup \mathcal{R}' \subset \mathcal{B}$. Thus, $\mathcal{R} \cup \mathcal{R}' = \mathcal{B}$.

(b) Since $\emptyset \in \mathcal{R}$, $\bar{\mu} \emptyset = \mu \emptyset = 0$. Meanwhile, clear that $\bar{\mu}$ is nonnegative. Let $\langle E_n \rangle \subset \mathcal{B}$ be a sequence of disjoint sets. By part (a), each E_n is either an element of \mathcal{R} or \mathcal{R}' . If all $E_n \in \mathcal{R}$, then by the countable additivity of μ , $\mu(\bigcup E_n) = \sum \mu E_n$. Suppose there exists some E_n in \mathcal{R} and some E_m in \mathcal{R}' . Let F_1 and F_2 be the union of theses sets respectively. Since σ -ring is closed under union, $F_1 \in \mathcal{R}$, and since $(\bigcup E_m)^c = \bigcap E_m^c$, $F_2 \in \mathcal{R}'$. Hence, $F_1 \cup F_2 \in \mathcal{R}'$, otherwise, $F_2 = (F_1 \cup F_2) \setminus F_1$ would be an element of \mathcal{R} . Therefore, $\mu(\bigcup E_n) = \infty = \sum \mu E_n$. Thus, $\bar{\mu}$ is a measure on \mathcal{B} .

(c) Clear that μ is nonnegative and $\mu \emptyset = 0$. Let $\langle E_n \rangle \subset \mathcal{B}$ be disjoint. Note that for $E \in \mathcal{R}$, $\mu E = \sup \{ \mu A : A \subset E, A \in \mathcal{R} \}$. Hence, it suffices to show that

$$M = \sup \left\{ \mu A : A \subset \bigcup_n E_n, A \in \mathcal{R} \right\} = \sum_n \sup \{ \mu A : A \subset E_n, A \in \mathcal{R} \} = \sum_n M_n.$$

By definition, for all $\varepsilon > 0$, there exists a sequence $\langle A_n \rangle \subset \mathcal{R}$ such that $A_n \subset E_n$ and $M_n < \mu A_n + \varepsilon/2^n$. Put $A = \bigcup A_n$. Since $\langle A_n \rangle$ are disjoint as $\langle E_n \rangle$ are,

$$\sum M_n < \varepsilon + \sum \mu A_n = \varepsilon + \mu A.$$

Meanwhile, since $A \subset \bigcup E_n$ and $A \in \mathcal{R}$, $\mu A \leq M$. Therefore, $\sum M_n < \varepsilon + M$. Thus, $\sum M_n \leq M$.

For the converse, similarly, for every $\varepsilon > 0$, there exists an $A \in \mathcal{R}$ such that $A \subset \bigcup E_n$ and $M - \varepsilon > \mu A$. Put $A_n = E_n \cap A$. If $E_n \in \mathcal{R}$, $A_n \in \mathcal{R}$ by definition. If $E_n \in \mathcal{R}'$, $A_n = A \setminus E_n^c \in \mathcal{R}$. Hence, $A_n \in \mathcal{R}$ for each n . Thus,

$$M - \varepsilon < \mu A = \sum_n \mu A_n \leq \sum_n M_n,$$

implying that $M \leq \sum M_n$. Therefore, $M = \sum M_n$, i.e., μ is countably additive. Thus, we conclude that μ is a measure on \mathcal{B} .

(d) Clear that μ_β is nonnegative and $\mu_\beta \emptyset = 0$. The preceding discussion, *mutatis mutandis*, yields the countable additivity. \square

11.2 Measurable Functions

10.

Proof. For every integers n and k , let

$$\begin{aligned} E_{n,k} &= \{x : k2^{-n} \leq f(x) < (k+1)2^{-n}\}, (k \leq 2^{2n}) \\ E_{n,2^{2n}+1} &= \{x : f(x) \geq (2^{2n}+1)2^{-n}\}, \\ \varphi_n &= 2^{-n} \sum_{k=0}^{2^{2n}+1} k \chi_{E_{n,k}} \end{aligned}$$

Since f is measurable, all $E_{n,k}$ are measurable. Thus, $\langle \varphi_n \rangle$ is a sequence of nonnegative simple functions. Clear that for fixed n , $\langle E_{n,k} \rangle_k$ are disjoint. Let $x \in X$ be fixed. If $x \in E_{n,k}$ for some $k \leq 2^{2n}$, then $x \in E_{n+1,2k} \cup E_{n+1,2k+1}$. Hence, $\varphi_{n+1}(x) \geq 2k/2^{-(n+1)} = \varphi_n(x)$. If $x \in E_{n,2^{2n}+1}$, then $x \in E_{n+1,k'}$ for some $k' \geq 2^{2n+2}$. Hence, $\varphi_{n+1}(x) \geq 2k'/2^{-(n+1)} = \varphi_n(x)$. Thus, $\varphi_{n+1} \geq \varphi_n$ for all n .

Now, we show that φ_n converges to f pointwisely. Let $x \in X$ be fixed. If $f(x) = \infty$, then $\varphi_n(x) = 2^{-n}(2^{2n}+1) \rightarrow \infty$ as $n \rightarrow \infty$. If $f(x) < \infty$, then $f(x) < 2^N$ for some integer N . For all $n > N$, $x \in E_{n,k_n}$ where $k_n = \lfloor 2^n f(x) \rfloor$. Thus,

$$f(x) - \varphi_n(x) = f(x) - 2^{-n} \lfloor 2^n f(x) \rfloor \rightarrow 0$$

as $n \rightarrow \infty$. Namely, $\varphi_n(x) \rightarrow f(x)$.

If the measure space is σ -finite, then let $(X_n) \subset X$ be a sequence of measurable sets such that $X_n \subset X_{n+1}$, $\mu X_n < \infty$ and $X = \bigcup X_n$. Replacing $E_{n,k}$ with $E_{n,k} \cap X_n$ yields a sequence $\langle \varphi_n \rangle$ satisfying all previous requirements and vanishing outside X_n for each n . \square

11.

Proof. Put $F_\alpha = \{x : f(x) \leq \alpha\}$, $G_\alpha = \{x : g(x) \leq \alpha\}$, $E = \{x : f(x) \neq g(x)\}$ and $E_\alpha = \{x \in E : g(x) \leq \alpha\}$. Then $G_\alpha = (F_\alpha \setminus E) \cup E_\alpha$. Since F is measurable, all F_α are measurable. Since $f = g$ a.e., E is of measure zero. Meanwhile, since μ is complete, $E_\alpha \subset E$ is measurable. Thus, G_α is measurable. Namely, g is measurable. \square

13.

Proof. Note that f_n converges to f in measure iff for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mu\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} = 0.$$

(a) By definition, for every $\varepsilon_m = 2^{-m}$, there exists some integer N_m such that for all $n \geq N_m$, $\mu\{x : |f_n(x) - f(x)| \geq \varepsilon_m\} < \varepsilon_m$. Consider the subsequence $\langle f_{N_m} \rangle_m$. We show that it converges to f almost everywhere. Put $E_m = \{x : |f_{N_m} - f(x)| \geq \varepsilon_m\}$ and $E = \limsup E_m$. Then, for each k ,

$$\mu E \leq \bigcup_{m=k}^{\infty} E_m \leq \sum_{m=k}^{\infty} 2^{-m+1} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

For every $x \notin E$, $x \notin \bigcup_{m=k}^{\infty} E_m$ for some k . Then for all $m > k$, $|f_{N_m}(x) - f(x)| < \varepsilon_{N_m}$. Hence, $f_{N_m}(x) \rightarrow f(x)$. Namely, $f_{N_m} \rightarrow f$ almost everywhere.

(b) First we prove a lemma: Let $\langle E_n \rangle$ be a sequence of measurable subset of A . Then $\limsup \mu E_n \leq \mu(\limsup E_n)$. Let $F_N = \bigcup_{n=N}^{\infty} E_n$. Clear that $F_{n+1} \subset F_n$ and $\mu F_1 < \infty$. Hence, by Prop. 2,

$$\limsup \mu E_n \leq \lim \mu F_n = \mu \left(\bigcap_{n=1}^{\infty} F_n \right) = \mu(\limsup E_n).$$

Thus, the lemma holds.

For fixed $\varepsilon > 0$, let $E_n = \{x \in A : |f_n(x) - f(x)| \geq \varepsilon\}$. We show that $\lim \mu E_n = 0$. First, clear that $0 \leq \limsup \mu E_n$. Meanwhile, if $x \in \limsup E_n$, then x belongs to infinitely many E_n . As a consequence, f_n does not converges to f at x . Since f_n converges to f a.e., $\mu(\limsup E_n) = 0$. Note that all $E_n \subset A$ are of finite measure. Hence, by the preceding lemma, $\limsup \mu E_n \leq 0$. Thus, $\lim \mu E_n = 0$. Let F_n denote $\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$ and G the collection of points at which f_n does not converge to f . Since all f_n vanishes outside A , for a point outside to belong to F_n , it has to belong to G , a set of measure zero. Therefore, $E_n \subset F_n \subset E_n \cup G$, implying that $\mu F_n = \mu E_n$. Thus, f_n converges to f in measure.

(c) By definition, for each positive integer k , there is an integer N_k such that for all $n, m \geq N_k$, $\mu\{x \in X : |f_n(x) - f_m(x)| \geq 2^{-k}\} < 2^{-k}$. We may assume without loss of generality that N_k is increasing. Put $E_k = \{x : |f_{N_{k+1}}(x) - f_{N_k}(x)| \geq 2^{-k}\}$ and $E = \limsup E_k$. By our construction, $\mu E = 0$. For $x \notin E$, $|f_{N_{k+1}}(x) - f_{N_k}(x)| < 2^{-k}$ for large k and, therefore, the number series $\sum (f_{N_{k+1}}(x) - f_{N_k}(x))$ converges to some point, say, $g(x)$. Hence, f_{N_k} converges to $f = f_{N_1} + g$ almost everywhere. Since all f_{N_k} are measurable, f is measurable.

Now we show that f_n converges to f in measure. Let D be the set of points at which f_{N_k} does not converge to f . For every $\varepsilon > 0$, let $F_n = \{x \in X \setminus D : |f_n(x) - f(x)| \geq \varepsilon\}$. Note that for all sufficiently large N_k ,

$$\begin{aligned} F_n &\subset \{x \in X \setminus D : |f_n(x) - f_{N_k}(x)| + |f_{N_k}(x) - f(x)| \geq \varepsilon\} \\ &\subset \{x \in X \setminus D : |f_n(x) - f_{N_k}(x)| \geq \varepsilon/2\}, \end{aligned}$$

where the measure of the last set can be less than ε for sufficiently large n and N_k as $\langle f \rangle$ is Cauchy in measure. Since D is of measure zero, we conclude that $\langle f_n \rangle$ converges to f in measure. \square

16.

Proof. Egoroff: Let (X, \mathcal{B}, μ) be a measure space and $E \subset X$ is of finite measure. Let $\langle f_n \rangle$ be a sequence of measurable functions which converge to some function f a.e. on E . Then for every $\eta > 0$, there is a subset $A \subset E$ with $\mu A < \eta$ such that f_n converges to f uniformly on $E \setminus A$.

We may assume without loss of generality that all f_n vanish outside E . Then, by Prob. 13(b), f_n converges to f in measure over E . Fix $\eta > 0$. First, we construct A . Put $\delta_m = \delta/2^m$. For every m , there exists some integer N_m and a measurable set A_m with $\mu A_m < \delta_m$ such that for all $n > N_m$ and $x \notin A_m$, $|f_n(x) - f(x)| < \delta_m$. Put $A = \bigcup A_m$. Clear that $\mu A < \delta$.

Now we show that f_n converges to f uniformly on $E \setminus A$. Fix $x \in E \setminus A$. For every $\varepsilon > 0$, suppose there is an m such that $0 < \delta_m < \varepsilon$. For all $n > N_m$, since $x \notin A$, $|f_n(x) - f(x)| < \delta_m < \varepsilon$. Thus, $f_n \rightarrow f$ uniformly on $E \setminus A$. \square

11.3 Integration

19.

Proof. Since $|\int_E f| \leq \int_E |f|$, it suffices to show the result for nonnegative f . Fix $\varepsilon > 0$. By definition, there is a nonnegative simple function $\varphi = \sum_{i=1}^n c_i \chi_{E_i}$ such that $\int f < \int \varphi + \varepsilon/2$. Put $M = \max_i c_i$ and $\delta = \varepsilon/2Mn$. Then, for every measurable E with $\mu E < \delta$, we have

$$\int_E f < \int_E \varphi + \varepsilon/2 = \sum_{i=1}^n c_i \mu(E_i \cap E) + \varepsilon/2 \leq Mn\delta + \varepsilon/2 = \varepsilon.$$

□

20.

Proof. We show here Fatou's Lemma: Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions which converges to a function f in measure on a measurable set E . Then $\int_E f \leq \liminf \int_E f_n$.

Since the collection of limits point of $\int_E f_n$ forms a closed set, there exists a subsequence $\langle f_{n_k} \rangle_k$ such that $\lim \int_E f_{n_k} = \liminf \int_E f_n$. Since f_{n_k} also converges to f in measure, by Prob. 13(a), there is a subsequence $\langle f_{n_{k_j}} \rangle$ which converges to f a.e. on E . Hence, by Theorem 10,

$$\int_E f \leq \liminf_j \int_E f_{n_{k_j}} = \lim_j \int_E f_{n_{k_j}} = \liminf_n \int_E f_n.$$

□

21.

Proof.

(a) We may assume without loss of generality that f is nonnegative since replacing f by $|f|$ does not change the integrability and the set $E = \{x : f(x) \neq 0\}$. For every positive integer n , since $\int f < \infty$, the set $E_n = \{x : f(x) \geq 1/n\}$ is of finite measure. Thus, $E = \bigcup_{n=1}^{\infty} E_n$ is of σ -finite measure.

(b) It follows immediately from part (a) and Prop. 7.

(c) If $f \geq 0$, then the existence of such a φ comes directly from the definition. For general cases, let $f = f^+ - f^-$ and φ^+, φ^- two simple functions such that

$$\int |f^+ - \varphi^+| < \varepsilon/2 \quad \text{and} \quad \int |f^- - \varphi^-| < \varepsilon/2.$$

Note that $\varphi = \varphi^+ - \varphi^-$ is also a simple function and

$$\int |f - \varphi| \leq \int |f^+ - \varphi^+| + \int |f^- - \varphi^-| < \varepsilon.$$

□

22.

Proof.

(a) Clear that ν is nonnegative and $\nu\emptyset = 0$. Let $\langle E_n \rangle$ be a sequence of disjoint measurable sets and $E = \bigcup_n E_n$. By Corollary 14, we have

$$\nu E = \int_E g d\mu = \int_E \sum g \chi_{E_n} d\mu = \sum \int_E g \chi_{E_n} d\mu = \sum \int_{E_n} g d\mu = \sum \nu E_n.$$

Thus, ν is a measure.

(b) First, we show the identity for an arbitrary simple function $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$ where E_k are disjoint.

$$\int \varphi d\nu = \sum_{k=1}^n c_k \nu E_k = \sum_{k=1}^n c_k \int g \chi_{E_k} d\mu = \int \varphi g d\mu.$$

Let f be a nonnegative measurable function and $\langle \varphi_n \rangle$ a increasing sequence of simple functions converging to f , the existence of which is guaranteed by Prop. 7. Then, By the monotone convergence theorem,

$$\int f d\nu = \lim \int \varphi_n d\nu = \lim \int \varphi_n g d\mu.$$

Note that $\langle \varphi_n g \rangle$ is a increasing sequence of functions converging to fg and with $\varphi_n g \leq fg$. Hence, again by the monotone convergence theorem,

$$\lim \int \varphi_n g d\mu = \int fg d\mu.$$

Thus, $\int f d\nu = \int fg d\mu$. □

11.4 General Convergence Theorems

24.

Proof. Since $\mu_n E$ is increasing for every E , such limits do exists. Clear that μ is nonnegative and $\mu\emptyset = 0$. Let $\langle E_k \rangle$ be a sequence of disjoint measurable sets. Then

$$\mu \left(\bigcup_{k=1}^{\infty} E_k \right) = \lim_{n \rightarrow \infty} \mu_n \left(\bigcup_{k=1}^n E_k \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \mu_n(E_k).$$

Since for fixed k , $\mu_n(E_k) \leq \mu_{n+1}(E_k)$, it is valid to change the order of the limit and the summation, which implies that $\mu(\bigcup E_k) = \sum \mu E_k$. Thus, μ is a measure. □

11.5 Signed Measures

27.

Proof.

(a) Consider the usual Lebesgue measure on \mathbb{R} . Let A be any countable subset of \mathbb{R} and $B = \mathbb{R} \setminus A$. Clear that A is negative set while B is a positive set. Namely, A and B form a Hahn decomposition of \mathbb{R} for μ .

(b) Let $\{A_1, B_1\}$ and $\{A_2, B_2\}$ be two Hahn decomposition of X for ν and A_1 and A_2 are two positive sets. We show that $A_1 \triangle A_2$ is a null set. Since the roles of A_1 and A_2 are interchangeable, it suffices to show that $A_1 \setminus A_2$ is a null set. Since A_1 is positive, every subset $E \subset A_1 \setminus A_2 \subset A_1$ is of nonnegative measure. Meanwhile, $A_1 \setminus A_2$ is also contained in B_2 , a negative set. Hence, $\nu E \leq 0$. Thus, $\nu E = 0$, implying that $A_1 \triangle A_2$ is a null set. \square

28.

Proof. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν and A and B be such that $X = A \cup B$ and $\nu^+(A) = \nu^-(B) = 0$. For every $E \subset A$,

$$\nu E = \nu^+ E - \nu^- E = -\nu^- E \leq 0.$$

Hence, A is a negative set. Similarly, B is positive set. Thus, $\{A, B\}$ is a Hahn decomposition of X .

Let $\nu = \nu_1 + \nu_2$ be another Jordan decomposition of ν and $\{C, D\}$ be the corresponding Hahn decomposition. By Prob. 27(b), $\{A, B\}$ and $\{C, D\}$ only differ by two null sets. Thus, $\nu_1 = \nu^+$ and $\nu_2 = \nu^-$. Namely, the decomposition is unique. \square

31.

Proof. Clear that

$$\left| \int_E f d\nu \right| \leq \left| \int_E f d\nu^+ \right| + \left| \int_E f d\nu^- \right| \leq M\nu^+ E + M\nu^- E = M|\nu|(E).$$

Let $\{A, B\}$ be the corresponding Hahn decomposition of X and A is the positive set. Then define f by

$$f(x) = \begin{cases} 1, & x \in A, \\ -1, & x \notin A. \end{cases}$$

Clear that $|f| \leq 1$ and

$$\int_E f d\nu = \int_E f d\nu^+ - \int_E f d\nu^- = \mu^+(A \cap E) + \nu^-(A \cap B) = |\nu|(E).$$

\square

32.

Proof.

(a) Put $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$, which can be verified to be a measure. For every $E \subset X$, suppose $\mu E \leq \nu E$. Then

$$(\mu \wedge \nu)(E) = \frac{1}{2}(\mu E + \nu E - |\mu - \nu|(E)) = \frac{1}{2}(\mu E + \nu E - \nu E + \mu E) = \mu E.$$

Similarly, $(\mu \wedge \nu)(E) = \nu E$ if $\nu E \leq \mu E$. Hence, $\mu \wedge \nu$ is smaller than both μ and ν . Note that $(\mu \wedge \nu)(E) = \min\{\mu E, \nu E\}$. Thus, clear that it is larger than any other signed measure smaller than μ and ν .

(b) Put $\mu \vee \nu = \frac{1}{2}(|\mu - \nu| + \mu + \nu)$. The previous argument, *mutatis mutandis*, shows that $(\mu \vee \nu)(E) = \max\{\mu E, \nu E\}$. Thus, it is the smallest measure larger than μ and ν . Meanwhile, clear that $\mu \wedge \nu + \mu \vee \nu = \mu + \nu$.

(c) Suppose that μ and ν are mutually singular and let $\{A, B\}$ be such that $A \cup B = X$, $\mu A = \nu B = 0$. Then

$$(\mu \wedge \nu)(E) \leq (\mu \wedge \nu)(E \cap A) + (\mu \wedge \nu)(E \cap B) \leq \mu A + \nu B = 0.$$

For the converse, suppose that $\mu \wedge \nu = 0$. If $\mu = 0$ or $\nu = 0$, then $\mu \perp \nu$ holds vacuously. Suppose that both μ and ν are nonzero. Since the roles of μ and ν are interchangeable, we may assume without loss of generality that $\mu E = 0$ and $\nu E > 0$ for some measurable E . Then, $\mu E^c \neq 0$, forcing νE^c to be zero. Therefore, $\mu E = \nu E^c = 0$, implying that $\mu \perp \nu$. \square

11.6 The Radon-Nikodym Theorem

33.

Proof. Suppose $X = \bigcup_{n=1}^{\infty} X_i$ and $\mu X_i < \infty$ for each n and X_i are disjoint. Then both $\mu|_{X_i}$ and $\nu|_{X_i}$, the restrictions to X_i , are finite. In consequence, by the Radon-Nikodym theorem for finite measure, there is a nonnegative measurable function $f_i : X_i \rightarrow \mathbb{R}$ such that $\nu(E \cap X_i) = \int_{(E \cap X_i)} f_i d\mu$. Without loss of generality, we may consider f_i to be a function on X (instead of X_i) that vanishes outside X_i .

Put $f = \sum f_i$. Since X_i are disjoint and f_i vanishes outside X_i , the summation does make sense. Meanwhile, clear that f is nonnegative and measurable. Note that for each measurable E ,

$$\nu E = \sum_{n=1}^{\infty} \nu(E \cap X_i) = \sum_{n=1}^{\infty} \int_E f_i d\mu = \int_E f d\mu,$$

where the last equality comes from Corollary 14. Namely, $\nu E = \int_E f d\mu$.

Finally, we show that f is unique up to almost equality. Let g be a nonnegative measurable function with this property. Then, $g|_{X_i}$, the restriction of g to X_i , equals to f_i a.e. $[\mu]$. Thus, $g = f$ a.e. $[\mu]$. \square

34. Radon-Nikodym derivatives

Proof.

(a) It suffices to show the result for simple functions. Let $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$ be a simple function. Then

$$\int \varphi d\nu = \sum_{k=1}^n c_k \nu E_k.$$

Meanwhile,

$$\int \varphi \left[\frac{d\nu}{d\mu} \right] d\mu = \sum_{k=1}^n c_k \int_{E_k} \left[\frac{d\nu}{d\mu} \right] d\mu = \sum_{k=1}^n c_k \nu E_k.$$

Thus, $\int \varphi d\nu = \int \varphi [d\nu/d\mu] d\mu$. \square

35.

Proof.

(d) Let ρ_0, ρ_1 be two measures with $\rho_0 \perp \mu$, $\rho_1 \ll \mu$ and $\nu = \rho_0 + \rho_1$. We show that $\rho_0 = \nu_0$ and $\rho_1 = \nu_1$. Since $\nu_0 \perp \mu$ and $\rho_0 \perp \mu$, there exists measurable A, B and C, D such that $A \cup B = C \cup D = X$, $A \cap B = C \cap D = \emptyset$ and $\nu_0 A = \mu B = \rho_0 C = \mu D = 0$. Put $U = A \cap C$ and $V = B \cup D$. Note that

$$\begin{aligned} U \cup V &= (A \cap C) \cup (B \cup D) = (A \cup B \cup D) \cap (C \cup B \cup D) = X, \\ U \cap V &= (A \cap C) \cap (B \cup D) = (A \cap C \cap B) \cup (C \cap B \cap D) = \emptyset. \end{aligned}$$

For every measurable E , if $E \subset U$, then $\nu_0 E = \rho_0 E = 0$ and $(\nu_0 + \nu_1)(E) = (\rho_0 + \rho_1)(E)$ implies that $\nu_1 E = \rho_1 E$. If $E \subset V$, then $\mu E = 0$, implying that $\nu_1 E = \rho_1 E = 0$ and, therefore, $\nu_0 E = \rho_0 E$. Since U and V partitions X , this implies that $\nu_0 = \rho_0$ and $\nu_1 = \rho_1$ for all measurable E . \square

36.

Proof. We show that: Let (X, \mathcal{B}, μ) be a σ -finite signed measure space, and let ν be a signed measure on \mathcal{B} with $\nu \ll \mu$. Then there is a measurable function such that for all measurable E we have $\nu E = \int_E f d\mu$. Furthermore, the function f is unique up to almost equality with respect to μ .

Let $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ be the Jordan decompositions. Clear that $\nu^+ \ll \mu^+$ and $\nu^- \ll \mu^-$. Hence, by the Radon-Nikodym theorem for measures, there exists nonnegative g and h such that

$$\nu^+ E = \int_E g d\mu^+ \quad \text{and} \quad \nu^- E = \int_E h d\mu^-$$

for all measurable E . Put $f = g - h$. Clear that it is measurable. Meanwhile,

$$\nu E = \nu^+ E - \nu^- E = \int_E g d\mu^+ - \int_E h d\mu^- = \int_E (g - h) d\mu$$

where the last equality comes from the mutual singularity of μ^+ and μ^- . And the argument in Prob. 33, *mutatis mutandis*, gives the uniqueness. \square

40.

Proof. Let I denote the index set of $\{X_\alpha\}$ and, just for convenience, let \sum_J denote $\sum_{\alpha \in J} \mu(E \cap X_\alpha)$.

(a) First, we suppose that E is of finite measure. Let J be any finite index subset of I . Then, since X_α are disjoint, $\mu E \geq \sum_J$. Hence, $\mu E \geq \sum_I$. For the converse, since, by our previous result, all \sum_J are bounded by μE , \sum_I is finite. For each positive integer n , there is a finite subset J_n of I such that $\sum_I - 1/n < \sum_{J_n}$. Put

$$J = \bigcup_{n=1}^{\infty} J_n \quad \text{and} \quad Y = \bigcup_{\beta \in J} X_\beta.$$

We show that (1) $\mu(Y \cap E) \leq \sum_I$ and (2) $\mu E = \mu(Y \cap E)$ to complete the proof. Since $\{X_\beta\}_{\beta \in J}$ is a countable collection of disjoint sets,

$$\mu(Y \cap E) = \sum_{\beta \in J} \mu(X_\beta \cap E) \leq \sum_{\alpha \in I} \mu(X_\alpha \cap E).$$

Namely, (1) holds. To show (2), we first show that, for each $\alpha \in I$, the set $X_\alpha \cap (E \setminus Y)$ is of measure zero. Assume, to obtain a contradiction, that there is some α such that $\mu(X_\alpha \cap (E \setminus Y)) = \delta > 0$. Since

$$X_\alpha \cap (E \setminus Y) = X_\alpha \cap E \cap \left(\bigcap_{\beta \in J} X_\beta^c \right),$$

this implies that $\alpha \notin J_n$ for all n and $\mu(X_\alpha \cap E) = \delta$. Since $\delta > 1/n$ for some large n , this leads to the contradiction

$$\sum_{J_n \cup \{\alpha\}} = \sum_{J_n} + \mu(X_\alpha \cap E) > \sum_I - \frac{1}{n} + \delta \geq \sum_I.$$

Hence, $\mu(X_\alpha \cap (E \setminus Y)) = 0$ for all n and, therefore, $\mu(E \setminus Y) = 0$. Note that $E \setminus (Y \cap E) = E \setminus Y$, this implies that $\mu(E \setminus (Y \cap E)) = 0$. In consequence, $\mu E = \mu(Y \cap E)$. Thus, $\mu E \leq \sum_I$.

Now suppose $\mu E = \infty$. If all \sum_J are finite □

40.

Proof. Let I denote the index set of $\{X_\alpha\}$. Fix a measurable E . Put

$$J = \{\alpha \in I : \mu(E \cap X_\alpha) > 0\} \quad \text{and} \quad Y = \bigcup_{\beta \in J} X_\beta.$$

First we show that $\mu E = \mu(E \cap Y)$. Clear that $\mu E \geq \mu(E \cap Y)$. For the converse, consider the set $E \setminus (E \cap Y) = E \setminus Y$. For every X_α , if $\alpha \notin J$, by the construction of J , $\mu(X_\alpha \cap (E \setminus Y)) = 0$. If $\alpha \in J$, then

$$X_\alpha \cap (E \setminus Y) = X_\alpha \cap E \cap \left(\bigcap_{\beta \in J} X_\beta \right) = \emptyset.$$

As a result, $\mu(X_\alpha \cap (E \setminus Y)) = 0$ for all $\alpha \in I$. Since $\{X_\alpha\}$ is a decomposition, this implies that $\mu(E \setminus Y) = 0$. Therefore, $\mu E \leq \mu(E \cap Y)$. Thus, $\mu E = \mu(E \cap Y)$.

(a) For each positive integer n , put

$$J_n = \{\alpha \in I : \mu(E \cap X_\alpha) > 1/n\}.$$

Clear that $J = \bigcup_n J_n$. If J is uncountable, then there must exist some uncountable J_n , which implies that $\mu E = \sum \mu(X_\alpha \cap E) = \infty$. If J is countable, then

$$\mu E = \mu(E \cap Y) = \sum_{\beta \in J} \mu(E \cap X_\beta) = \sum_{\alpha \in I} \mu(E \cap X_\alpha).$$

Thus, $\mu E = \sum \mu(E \cap X_\alpha)$. □

11.7 The L^p Spaces

41.

Proof. First, we prove the following lemma: For $a, b \geq 0$, $|a - b|^p \leq 2|a^p - b^p|$. It suffices to show that $(a - b)^p \leq 2(a^p - b^p)$ for all $a \geq b \geq 0$. If $p = 1$, then the inequality holds trivially. Suppose $p > 1$ and put $h(x) = (x - b)^p - 2(x^p - b^p)$. Clear that $h(b) = 0$. Meanwhile, for $x \geq b$,

$$h'(x) = p(x - b)^{p-1} - 2px^{p-1} = px^{p-1} \left(\left(1 - \frac{b}{x}\right)^{p-1} - 2 \right) < 0.$$

Thus, $h(x) \leq 0$ for all $x \geq b$, which implies that $|a - b|^p \leq 2|a^p - b^p|$ for all $a, b \geq 0$.

Since $|f|^p$ is integrable, by Prob. 21(a), the set on which f does not vanish is of σ -finite measure. Hence, $\int |f|^p = \sup \int \varphi$ as φ ranges over all simple functions that each vanishes outside a set of finite measure. Thus, for every $\varepsilon > 0$, there is a nonnegative simple function $\tilde{\varphi} \leq |f|^p$, vanishing outside a set E of finite measure, such that $\int (|f|^p - \tilde{\varphi}) < \varepsilon^p/2$. Put $\varphi = \sqrt[p]{\tilde{\varphi}}$, which is also a nonnegative simple function that vanishes outside E . Meanwhile, by the previous inequality,

$$\|f - \varphi\|_p^p = \int |f - \varphi|^p \leq 2 \int (|f|^p - \tilde{\varphi}) < \varepsilon^p.$$

Namely, Prop. 26 holds. □

42.

Proof. We may assume without loss of generality that g is nonnegative. Assume, to obtain a contradiction, that $\text{ess sup } |g| > M$, that is, the measure of $E = \{t : g(t) > M + \eta\}$ is nonzero for some positive η . Meanwhile, since μ is finite, $\mu E < \infty$. Let $\varphi = \chi_E$, which is clearly a simple function. Then

$$\left| \int g\varphi \right| \geq (M + \eta)\mu E > M\|\varphi\|_1.$$

Contradiction. Hence, $\text{ess sup } |g| \leq M$, implying that $g \in L^\infty$. □

43. The case $p = 1$ is left undone.

Proof. Suppose that $p > 1$. Let $\langle X_n \rangle$ be such that $\mu X_n < \infty$ and $X = \bigcup X_n$. Furthermore, we may assume without loss of generality that X_n are disjoint. Put $g_n = \sum g\chi_{X_n}$. By Lemma 27, for n , $g\chi_{X_n} \in L^q$ and $\|g_n\|_q \leq M$. Since $g_n \rightarrow g$, by Fatou's lemma, $\|g\|_q \leq M$. Thus, $g \in L^q$. □

44.

Proof. Note that

$$\int |f|^p = \sum \int |f|^p \chi_{E_n} = \sum \int |f_n|^p = \sum \|f_n\|_p^p.$$

Thus, $f \in L^p$ iff $\sum \|f_n\|_p^p < \infty$. □

45.

Proof. For every $f \in L^p$ with $\|f\|_p = 1$,

$$\int |fg| \leq \|f\|_p \|g\|_q = \|g\|_q.$$

Hence, $\|F\| \leq \|g\|_q$. For the reverse inequality, put

$$f = (\operatorname{sgn} g)|g|^{q-1} = (\operatorname{sgn} g)|g|^{p/q}.$$

Note that $|f|^p = |g|^q$. Therefore, $g \in L^q$ implies $f \in L^p$. Meanwhile, $\|f\|_p^p = \|g\|_q^q$. Hence,

$$\|F\| \|f\|_p \geq |F(f)| = \int |g|^q = \|g\|_q^q \quad \Rightarrow \quad \|F\| \geq \|g\|_q.$$

Thus, $\|F\| = \|g\|_q$. □

12 Measure and Outer Measure

12.1 Outer Measure and Measurability

1.

Proof. Suppose that $E \subset X$ and there is a measurable B with $\bar{\mu}B = 0$ such that $E \subset B$. We show that E is measurable, that is, for every $A \subset X$ with finite outer measure, $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Since $A \cap E \subset E \subset B$ and μ^* is monotone, $\mu^*(A \cap E) \leq \mu^*(B) = \bar{\mu}B = 0$. Again by the monotonicity, $\mu^*(A) \geq \mu^*(A \cap E^c)$. Thus, E is measurable, implying that $\bar{\mu}$ is complete. \square

2.

Proof. From the countable subadditivity we obtain that $\mu^*(A \cap E) \leq \sum \mu^*(A \cap E_i)$. For the converse, first we consider just E_1 and E_2 . Since E_1 is measurable,

$$\mu^*(A \cap E) = \mu^*(A \cap E \cap E_1) + \mu^*(A \cap E \cap E_1^c) \geq \mu^*(A \cap E_1) + \mu^*(A \cap E_2).$$

By induction on n we get $\mu^*(A \cap E) \geq \sum_{i=1}^n \mu^*(A \cap E_i)$. Let $n \rightarrow \infty$ and the proof is completed. \square

12.2 The Extension Theorem

4.

Proof.

(a) Since $\{D_j\}$ partitions A , $C_i \subset A$ for each i and \mathcal{C} is closed under intersection, by condition (i), $\mu C_i = \sum_j \mu(C_i \cap D_j)$. Similarly for μD_j . Thus,

$$\sum_i \mu C_i = \sum_{i,j} \mu(C_i \cap D_j) = \sum_j \sum_i \mu(C_i \cap D_j) = \sum_j \mu D_j.$$

This result implies that the definition of μ on \mathcal{A} is well-defined.

(b) Since every $A \in \mathcal{A}$ is a finite union of sets of \mathcal{C} , it suffices to show that $\mu C \geq \sum \mu C_i$. Then, from this condition, the countable additivity follows. Since μ is nonnegative and monotone, $\mu C \geq \sum_{i=1}^n \mu C_i$ for all positive integer n . Let $n \rightarrow \infty$ and we get $\mu C \geq \sum \mu C_i$. \square

7.

Proof. To prove the "if" part, let $\varepsilon_n = 1/n$ and $A_n \in \mathcal{A}_\delta$ be such that $\mu^*(E \setminus A_n) < \varepsilon_n$. Put $A = \bigcup A_n$. Since the collection of μ^* -measurable sets is a σ -algebra, A is measurable. Meanwhile, since $\mu^*(E \setminus A) \leq \mu^*(E \setminus A_n) < \varepsilon_n$, $E \setminus A$ is of μ^* -measure zero. Hence, it is measurable. Thus, $E = A \cup (E \setminus A)$ is also measurable.

For the converse, suppose that E is measurable and let $\varepsilon > 0$ be fixed. Note that E^c is also measurable. Hence, by Prop. 6, there is a set $A \in \mathcal{A}_\sigma$ with $E^c \subset A$ and

$$\varepsilon > \mu^*(A \setminus E^c) = \mu^*(A \cap E).$$

Then, $A^c \in \mathcal{A}_\delta$, $A^c \subset E$ and $\mu^*(E \setminus A^c) = \mu^*(E \cap E) < \varepsilon$. \square