Solutions to
Introductory Functional Analysis with Applications

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2 Normed Spaces. Banach Spaces

2.3 Further Properties of Normed Spaces

4. cf. Prob. 13, Sec 1.2

Proof. The continuity of addition and multiplication follows respectively from the inequalities

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\|$$

and

$$\|\alpha_1 x_1 - \alpha_2 x_2\| = \|\alpha_1 x_1 - \alpha_1 x_2 + \alpha_1 x_2 - \alpha_2 x_2\| \leq |\alpha_1|\|x_1 - x_2\| + |\alpha_1 - \alpha_2|\|x_2\|.$$

□

7.

Proof. Let Y and y_n be defined as in the hint. Then $\|y_n\| = 1/n^2$, constituting a convergent number series. However,

$$\sum_{n=1}^{N} y_n = (1, 1/4, \ldots, 1/N^2, 0, \ldots),$$

which is divergent as $N \to \infty$. □

8.

Proof. Let (x_n) be a Cauchy sequence in X. Hence, for every $n > 0$, there exists some $K_n > 0$ such that for all $p, q > K_n$, $\|x_p - x_q\| < 1/n^2$. Without loss of generality, we may assume that (K_n) is increasing. Since the series $\|x_{K_{n+1}} - x_{K_n}\|$ is bounded by $1/n^2$, it converges. By the hypothesis, the series $(x_{K_{n+1}} - x_{K_n})$ also converges. Hence,

$$x_{K_n} = x_{K_1} + \sum_{i=1}^{n-1}(x_{K_{i+1}} - x_{K_i}) \to x \quad \text{as } n \to \infty.$$

Now we show that (x_n) converges to x. For every $\varepsilon > 0$, since (x_n) is a Cauchy sequence, there exists some N_1 such that for all $p, q > N_1$, $\|x_p - x_q\| < \varepsilon$. Meanwhile, since $x_{K_n} \to x$, once K_n is large enough, $\|x - x_{K_n}\| < \varepsilon$. Let $K_n > N_1$. Then for every $n > K_n$

$$\|x_n - x\| \leq \|x_n - x_{K_n}\| + \|x_{K_n} - x\| \leq 2\varepsilon.$$

Thus, X is complete. □

9.

Proof. Let (x_n) be an absolutely convergent series in Banach space X. Let $s_n = \sum_{i=1}^{n} x_n$. Now we show that s_n is a Cauchy sequence and therefore convergent. Since $\sum_{i=1}^{\infty} \|x_i\| < \infty$, for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n > N$, $\sum_{i=n}^{\infty} \|x_i\| < \varepsilon$. Hence, for every $N < p \leq q$,

$$\|s_q - s_p\| = \left\| \sum_{i=p+1}^{q} x_i \right\| \leq \sum_{i=p+1}^{q} \|x_i\| < \varepsilon,$$

completing the proof. □

10.

Proof. Let (e_n) be Schauder basis of X. Denote the underlying field of X by $\mathbb{K}$ and let $\mathbb{W} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{W} = \{p + iq : p, q \in \mathbb{Q}\}$ if $\mathbb{K} = \mathbb{C}$. Now we show that

$$S = \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{W}, n = 1, 2, \ldots \right\},$$

a countable subset of X, is dense in X to derive the separability.

For every $x \in X$ and $\varepsilon > 0$, by the definition of Schauder basis, there exists $\beta_1, \ldots, \beta_n \in \mathbb{K}$ such that $\|x - (\beta_1 e_1 + \cdots + \beta_n e_n)\| < \varepsilon$. Let $M = \max_i \|e_i\|$. If $M = 0$, then there is nothing to prove. Otherwise, since $\mathbb{W}$ is dense in $\mathbb{K}$, for $i = 1, \ldots, n$, there exists $\alpha_i \in \mathbb{W}$ with $|\alpha_i - \beta_i| < \varepsilon/2^i M$. Hence,

$$\begin{aligned}
\left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| \leq & \left\| x - \sum_{i=1}^{n} \beta_i e_i \right\| + \left\| \sum_{i=1}^{n} (\beta_i - \alpha_i) e_i \right\| \\
\leq & \varepsilon + \sum_{i=1}^{n} |\alpha_i - \beta_i| \|e_i\| \\
\leq & 2\varepsilon.
\end{aligned}$$

Thus, S is dense in X and therefore X is separable. □

14.

Proof. Clear that $\| \cdot \|_0$ is nonnegative. And $\|\alpha \hat{x}\|_0 = \inf_{x \in \hat{x}} \|\alpha x\| = |\alpha| \|\hat{x}\|_0$. Meanwhile, $\|\hat{x} + \hat{y}\|_0 = \inf_{z \in \hat{x} + \hat{y}} \|z\| \leq \inf_{z \in \hat{x}} \|z\| + \inf_{z \in \hat{y}} \|z\| = \|\hat{x}\|_0 + \|\hat{y}\|_0$. Finally, we show that $\|\hat{x}\|_0 = 0$ implies $\hat{x} = Y$ and invoke Prob. 4, Sec 2.2 to complete the proof. Since $\|\hat{x}\|_0 = 0$, there exists $(x_n) \subset \hat{x}$ which converges to 0. Since Y is closed, Y is complete and so is its cosets. Therefore, $0 \in \hat{x}$, enforcing $\hat{x}$ to be Y. □

2.4 Finite Dimensional Normed Spaces

3.

Proof. The reflexive property clearly holds. If there are positive a and b such that $a\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0$ for all $x \in X$, then $\|x\|_1/b \leq \|x\|_0 \leq \|x\|/a$. Hence the relation is symmetric. Next we further suppose there exists positive c and d such that that $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$. Then $ac\|x\|_0 \leq \|x\|_2 \leq bd\|x\|_0$, giving the transitive property. Thus, the axioms of an equivalence relation hold. □

4.

Proof. Suppose the norms $\| \cdot \|$ and $\| \cdot \|_0$ are equivalent. Let $E \subset X$ be any open set with respect to $\| \cdot \|$, i.e., for every $x_0 \in E$, there exists some $\delta > 0$ such that $A = \{x \in X : \|x - x_0\| < \delta\} \subset E$. Since $\| \cdot \| \sim \| \cdot \|_0$, there exists some positive c such that that $\|x - x_0\| \leq c\|x - x_0\|_0$. Hence, $B = \{x \in X : \|x - x_0\| < \delta/c\} \subset A \subset E$. Namely, E is also open with respect to $\| \cdot \|_0$. Interchanging the roles of $\| \cdot \|$ and $\| \cdot \|_0$ completes the proof. □

5.

Proof. Suppose the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Then for every $x \in X$, there exists some $c > 0$ such that $\|x\|_0 \le c\|x\|$. Let (x_n) be a Cauchy sequence with respect to $\|\cdot\|$, i.e., for every $\varepsilon > 0$, there exists some $N > 0$ such that for all $n, m > N$, $\|x_n - x_m\| < \varepsilon/c$. Hence, $\|x_n - x_m\|_0 < c\|x_n - x_m\| \le \varepsilon$. Thus, (x_n) is also a Cauchy with respect to $\|\cdot\|_0$. Interchanging the roles of $\|\cdot\|$ and $\|\cdot\|_0$ completes the proof. □

2.5 Compactness and Finite Dimension

5.

Proof. Clear that every point in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a closed bounded, and therefore compact, neighborhood. Hence, $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact. □

6.

Proof. Let X be a compact metric space and x any point in X. Let E be a closed neighborhood of x. By Prob 10, E is compact. Thus, X is locally compact. □

7.

Proof. It suffices to show that $a = \inf_{y \in Y} \|v - y\|$ can actually be obtained. Let $\{b_1, \ldots, b_n\}$ be a basis of Y and $y_k = y_{k,1}b_1 + \cdots + y_{k,n}b_n$ a sequence in Y with $\|v - y_k\| \to a$. We may assume without loss of generality that $\|v - y_k\|$ is bounded.

Since Y is a proper subset of Z, $v, b_1, \ldots, b_n$ are linearly independent. Therefore, by Lemma 2.4-1, there exists a scalar $c > 0$ such that for every k,

$$\|v - y_{k,1}b_1 - \cdots - y_{k,n}b_n\| \ge c(1 + |y_{k,1}| + \cdots + |y_{k,n}|).$$

Hence, the sequence $(y_{k,1}, \ldots, y_{k,n})$ of n-tuples is bounded and therefore has a convergent subsequence. Consequently, (y_k) also has a convergent subsequence. Suppose that it converges to $z \in Z$. Note that $\|v - z\| = a$ and as Y is closed, $z \in Y$. Thus, a can be attained in Y. □

8.

Proof. Since the unit ball B with respect to $\|\cdot\|_2$ in $\mathbb{R}^n$ and $\mathbb{C}^n$ is compact and $\|\cdot\|$ is continuous, by 2.5-7, $x \mapsto \|x\|$ can attains its minimum, denoted by a, on B. Due to the positive definite property of a norm, a is positive. Hence, $0 < a \le \|x/\|x\|_2\|$. Namely, $a\|x\|_2 \le \|x\|$. □

9.

Proof. For every $(x_n) \subset M \subset X$, since X is compact, there exists a subsequence (x_{n_k}) of (x_n) which converges to some $y \in X$. Since M is closed, $y \in M$. Hence, M is compact. □

10.

Proof. From 1.3-4 and the definition of closed sets, we conclude that a mapping is continuous iff the preimage of a closed set under it is also a closed set. Hence, to show that the inverse of T is also continuous, it suffices to show that the image of a closed set $A \subset X$ under T is again a closed set. Since X is compact and A is closed, A is compact. Since T is continuous, by 2.5-6, $T(A)$ is compact and therefore closed. Hence, T is a homeomorphism. $\square$

2.7 Bounded and Continuous Linear Operators

2.

Proof. First suppose T to be bounded and let A be any bounded set in X. Then there exists $K < \infty$ such that for all $x \in A$, $\|x\| < K$. Due to the boundedness of T, $\|Tx\| \leq \|T\|\|x\| < K\|T\|$. Namely, $T(A)$ is also bounded.

Now suppose that T maps bounded sets in X into bounded sets in Y. Clear that the unit ball B of X is bounded and therefore so is $T(B)$. Namely, $\|Tx/\|x\|\|$ is bounded for $x \neq 0$.[1] Hence, T is bounded. $\square$

3.

Proof. For every x with $\|x\| < 1$, $\|Tx\| \leq \|T\|\|x\| < \|T\|$. $\square$

4.

Proof. Suppose that the linear operator T is continuous at $x_0 \in \mathcal{D}(T)$. For every $(x_n) \subset \mathcal{D}(T)$ with $\|x_n - x\| \to 0$, by the continuity of T at x_0

$$\|Tx_n - Tx\| = \|T(x_n - x + x_0) - Tx_0\| \to 0.$$

Hence, T is continuous. $\square$

7.

Proof. The inequality implies $\mathcal{N}(T) = 0$. Hence, by Theorem 2.6-10, T^{-1} exists. For every $y \in Y$, suppose that $y = Tx$. Then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

Thus, T^{-1} is bounded. $\square$

12.

Proof. The compatibility follows immediately from the definition of the supremum. Suppose $\|x\|_1 = \max_j |\xi_j|$ and $\|y\|_2 = \max_j \|\eta_j\|$, then

$$Ax = \begin{bmatrix} x_1\alpha_{11} + \cdots + x_n\alpha_{1n} \\ \vdots \\ x_1\alpha_{r1} + \cdots + x_n\alpha_{rn}. \end{bmatrix}$$

[1] Note that the two $\|\cdot\|$ here are different norms.

Since for all j, $x_j \le \|x_j\|_1$,

$$\frac{\max_j |x_1\alpha_{j1} + \cdots + x_n\alpha_{jn}|}{\|x\|_1} = \max_j \left| \frac{x_1}{\|x\|_1}\alpha_{j1} + \cdots + \frac{x_n}{\|x\|_1}\alpha_{jn} \right| \le \max_j \sum_{k=1}^{n} |\alpha_{jk}|.$$

Hence,

$$\|A\| \ge \frac{\|Ax\|_2}{\|x\|_1} \quad \text{for all } x. \tag{1}$$

Suppose that maximum of $\sum_{k=1}^n |\alpha_{jk}|$ is obtained at $j = p$. Then choosing x_k to be $\operatorname{sgn} \alpha_{pk}$ shows that the equality in (1) can actually be attained. Hence, $\|A\| = \max_j \sum_{k=1}^n |\alpha_{jk}|$. □

2.8 Linear Functionals

8.

Proof. For every $x_1, x_2 \in N(M^*)$, $a, b \in \mathbb{K}$ and $f \in M^*$,

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) = 0.$$

Hence, $ax_1 + bx_2 \in N(M^*)$. Namely, $N(M^*)$ is a vector space. □

9.

Proof. First we show the uniqueness. Suppose that $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$. Then $0 = (\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)$. Hence,

$$0 = f((\alpha_1 - \alpha_2)x_0 + (y_1 - y_2)) = (\alpha_1 - \alpha_2)f(x_0) + f(y_1) - f(y_2).$$

Since $y_1, y_2 \in \mathcal{N}(f)$, $f(y_1) - f(y_2) = 0$ while $f(x_0) \ne 0$ as $x_0 \notin \mathcal{N}(f)$. Hence, $\alpha_1 = \alpha_2$, which forces y_1 and y_2 to coincide.

For the existence, it suffices to show that for any fixed x, the function $g(\alpha) = f(x - \alpha x_0)$ has a zero. It is easy to verify that $\alpha = f(x)/f(x_0)$ is a zero of g. Note that $x_0 \notin \mathcal{N}(f)$ and therefore $f(x_0) \ne 0$. □

10.

Proof. First we suppose that $x_1, x_2 \in x_0 + \mathcal{N}(f) \in X/\{$. Then together with Prob. 9, $x_i = x_0 + y_i$ where $y_i \in \mathcal{N}(f)$. Hence, for $i = 1, 2$, $f(x_i) = f(x_0) + f(y_i) = f(x_0)$.

For the converse, note that $f(x_1) = f(x_2)$ implies $f(x_1 - x_2) = 0$. Namely, $x_1 - x_2 \in \mathcal{N}(f)$. Hence, x_1, x_2 belongs to the same element in $X/\mathcal{N}(f)$.

To show $\operatorname{codim} \mathcal{N}(f) = 1$, we show that $X/\mathcal{N}(f)$ and $\mathbb{K}$ are isomorphic. For every $\hat{x} \in X/\mathcal{N}(f)$, define $I(\hat{x}) = f(x)$. By the previous discussion, this definition is well-defined. Clear that I is linear and therefore is injective. And by the linearity of f, I is surjective. Thus, I is an isomorphism between $X/\mathcal{N}(f)$ and $\mathbb{K}$. Hence, $\operatorname{codim} \mathcal{N}(f) = 1$. □

11.

Proof. Put $N = \mathcal{N}(f_1) = \mathcal{N}(f_2)$ and choose $x_0 \in X \setminus N$. By Prob. 9, for every $x \notin N$, $x = \alpha x_0 + y$ where $y \in N$ and $\alpha \ne 0$. Hence,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha f_1(x_0) + f_1(y)}{\alpha f_2(x_0) + f_2(y)} = \frac{f_1(x_0)}{f_2(x_0)}.$$

□

12.

Proof. Prob. 10, justifies the discussion on hyperplanes parallel to the $\mathcal{N}(f)$. It suffices to show that $H_1 = b + \mathcal{N}(f)$ for some $b \in X$. Choose $x_1 \in H_1$. Then

$$x \in \mathcal{N}(f) \Leftrightarrow x + x_1 \in x_1 + \mathcal{N}(f) \Leftrightarrow f(x + x_1) = f(x) + f(x_1) = 1 \Leftrightarrow x + x_1 \in H_1.$$

Hence, $H_1 = x_1 + \mathcal{N}(f)$. Namely, H_1 is a hyperplane parallel to $\mathcal{N}(f)$. $\square$

13.

Proof. We argue by contradiction. Assume that there exists a $y_1 \in Y$ such that $f(y_1) \neq c \neq 0$. Then for every $d \in \mathbb{K}$, by the linearity of f, $f(dy_1/c) = d$. Contradiction. Hence, $f = 0$ on Y. $\square$

14.

Proof. For every $\varepsilon > 0$, there exists $x_1 \in X$ with $f(x_1) = 1$ such that $\tilde{d} + \varepsilon \geq \|x_1\|$. Hence,

$$\|f\|(\tilde{d} + \varepsilon) \geq \|f\|\|x_1\| \geq |f(x_1)| = 1.$$

Since the choice of $\varepsilon > 0$ is arbitrary, $\|f\|\tilde{d} \geq 1$. Meanwhile, there exists $x_2 \in X$ with $\|x_2\| = 1$ such that $|f(x_2)| \geq \|f\| - \varepsilon$. Put $x_3 = x_2/f(x_2)$. Then $f(x_3) = 1$. Hence,

$$(\|f\| - \varepsilon)\tilde{d} \leq |f(x_2)|\|x_3\| = \|x_2\| = 1,$$

which implies $\|f\|\tilde{d} \leq 1$. Thus, $\|f\|\tilde{d} = 1$. $\square$

15.

Proof. For every x with $\|x\| \leq 1$, $f(x) \leq \|f\|\|x\| \leq c$. Hence, $x \in X_{c_1}$. Meanwhile, for every $\varepsilon > 0$, by the definition of the supremum, there exists a x with $\|x\| = 1$ such that $|f(x)| > \|f\| - \varepsilon$. By the linearity of f, we may remove the $|\cdot|$ on the right side. Hence, $f(x) \notin X_{c_1}$ where $c = \|f\| - \varepsilon$. $\square$

2.9 Operators on Finite Dimensional Spaces

8.

Proof. Let $\{b_2, \ldots, b_n\}$ be a basis of Z and $\{b_1, \ldots, b_n\}$ a basis of X. Define $f \in X^*$ to be $f(b_i) = \delta_{1i}$. Clear that $\mathcal{N}(f) = Z$. By Prob. 11, Sec 2.8, f is uniquely determined up to a scalar multiple. $\square$

12.

Proof. Let $\varphi : X \to \mathbb{K}^p$ be defined by $x \mapsto [f_1(x), \ldots, f_p(x)]^T$. It can be verified that φ is a linear operator. Since $\dim X = n > p$, φ can not be injective. Hence, there exists $0 \neq x \in X$ such that $\varphi(x) = 0$. $\square$

13.

Proof. Let $\{b_1, \ldots, b_m\}$ be a basis of Z and $\{b_1, \ldots, b_n\}$ a basis of X. Define $\tilde{f} \in X^*$ to be identical with f on $b_1, \ldots, b_m$ and 0 on $b_{m+1}, \ldots, b_n$. Clear that $\tilde{f}|_Z = f$. $\square$

2.10 Normed Spaces of Operators. Dual Space

8.

Proof. First we construct a linear bijection T between c_0' and l^1. A Schauder basis for c_0 is (e_k), where $e_k = (\delta_{kj})$. Then for every $f \in c_0'$, define $Tf = (\gamma_k) = (f(e_k))$. Clear that T is linear. Now we show that $Tf = (\gamma_k) \in l^1$, that is, $\sum_{k=1}^n |\gamma_k|$ is bounded and therefore convergent. Define $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \begin{cases} \operatorname{sgn} \gamma_k, & k \le n, \\ 0, & k > n. \end{cases}$$

Clear that $x_n \in c_0$. By the linearity and boundedness of f,

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|. \tag{2}$$

Since f is bounded, $|f(x_n)| \le \|f\| \|x_n\| \le \|f\|$. Hence, $\sum \|\gamma_k\|$ is bounded. Thus, $Tf \in l^1$.

Meanwhile, for every $y = (\beta_k) \in l^1$, define $Sy = g$ to be the functional $g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$ for $x = (\xi_k)$. On c_0, the summation does converge and clear that g is linear and bounded. Hence, $g \in c_0'$. It can be verify that $ST = TS = I$ and T is linear. Thus, c_0' and l^1 is isomorphic.

Now we show that T constructed preserve the norm to complete the proof. For $x \in c_0$ with $\|x\| = 1$,

$$|f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \gamma_k \right| \le \sum_{k=1}^{\infty} |\gamma_k| = \|Tf\|.$$

Hence, $\|f\| \le \|Tf\|$. And (2) implies $\sum_{k=1}^n \|\gamma\| \le \|f\|$. Letting $n \to \infty$ yields $\|Tf\| \le \|f\|$. Thus, $\|Tf\| = \|f\|$. □

9.

Proof. Let (b_k) be a Hamel basis of X and suppose that $f, g \in X^*$ coincide on every b_k. Then for every $x = \sum_{k=1}^{\infty} \xi_k b_k \in X$,

$$f(x) - g(x) = \sum_{k=1}^{n} \xi_k (f(b_k) - g(b_k)) = 0.$$

Thus, $f = g$. Namely, f is uniquely determined. □

10.

Proof. Let (b_k) be a Hamel basis of X and without loss of generality we may assume $\|b_k\| = 1$. Justified by Prob. 9, we can define $T \in X^*$ with $Tb_k = k$, which is clearly unbounded. □

11.

Proof. It follows immediately from Prob. 10. □

13.

Proof. For any $f, g \in M^a$ and scalar a, b, $(af + bg)(x) = af(x) + bg(x) = 0$ for every $x \in M$. Hence, M^a is a vector space. For $(f_n) \subset M^a \subset X'$, suppose that $f_n \to f \in M^*$. Since M' is complete, it is closed and therefore $f \in M'$. For every $0 \neq x \in M$, since $f_n \to f$,

$$\frac{|f_n(x) - f(x)|}{\|x\|} \to 0, \text{ as } n \to \infty.$$

Hence, $f(x) = 0$. Thus, M^a is closed.

$X^a = \{0\}$ and $\{0\}^a = X'$. □

14.

Proof. Let $\{b_1, \ldots, b_m\}$ be a basis of M and $\{b_1, \ldots, b_n\}$ a basis of X. And let $\{\beta_1, \ldots, \beta_n\}$ be the dual basis. Clear that $b_1, \ldots, b_m \notin M^a$ whereas $b_{m+1}, \ldots, b_n$ does. Together with Prob. 13, this implies $M^a = \operatorname{span}(b_{m+1}, \ldots, b_n)$. Thus, $\dim M^a = n - m$. □

3 Inner Product Spaces. Hilbert Spaces

3.1 Inner Product Spaces. Hilbert Spaces

2.

Proof.

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle = \|x\|^2 + \|y\|^2,$$

where the last equality comes from the hypothesis of orthogonality. Now we show that for mutually orthogonal $x_1, \ldots, x_m$

$$\left\|\sum_{i=1}^{m} x_i\right\|^2 = \sum_{i=1}^{m} \|x_i\|^2,$$

by induction on m. The case where $m = 2$ has already been showed and we assume that the equation holds for $m-1$. Since x_m is orthogonal with each $i = 1, \ldots, m-1$, x_m is orthogonal to $x_1 + \cdots + x_{m-1}$. Hence,

$$\left\|\sum_{i=1}^{m} x_i\right\|^2 = \left\|\sum_{i=1}^{m-1} x_i\right\|^2 + \|x_m\|^2 = \sum_{i=1}^{m} \|x_i\|^2,$$

completing the proof. □

3.

Proof. The equation implies $\langle x, y\rangle + \langle y, x\rangle = 0$. The symmetric property of real inner products implies $\langle x, y\rangle = 0$. Let $X = \mathbb{C}$ and $x = 1, y = i$. It is easy to verify that $\|x+y\|^2 = \|x\|^2 + \|y\|^2 = 2$ but x and y are not orthogonal. □

7.

Proof. It suffices to show that the zero vector is the only vector orthogonal to all vectors. Suppose that $\langle x_0, x\rangle = 0$ for all $x \in X$, then $\|x_0\|^2 = \langle x_0, x_0\rangle = 0$. By the definiteness of the inner product, $x_0 = 0$. □

8. We show that any norm satisfying the parallelogram equality can be derived form an inner product.

Proof. The proof of (IP3) is trivial and (IP4) follows immediately from the positive-definiteness of the norm. Hence we only show the linearity in the first factor here. For every $u, v, y \in X$, from the parallelogram equality we can derive, after some computation, that

$$\begin{aligned} 4\langle u+v, y\rangle &= \|u+v+y\|^2 - \|u+v-y\|^2 \\ &= \|u+y\|^2 - \|u-y\|^2 + \|v+y\|^2 - \|v-y\|^2 \\ &= 4\langle u, y\rangle + 4\langle v, y\rangle. \end{aligned}$$

Namely, (IP1) holds. By induction we can show that $\langle nu, y\rangle = n\langle u, y\rangle$ for $n = 1, 2, \ldots$. And since $\langle -u, y\rangle = \langle 0 - u, y\rangle = \langle 0, y\rangle - \langle u, y\rangle = \langle u, y\rangle$,

$$\langle nu, y\rangle = n\langle u, y\rangle, \quad \text{for } n \in \mathbb{Z}.$$

Furthermore, for any positive integer m,

$$m\left\langle \frac{n}{m}u, y\right\rangle = mn\left\langle \frac{1}{m}u, y\right\rangle = n\langle u, y\rangle.$$

Dividing the both sides by m yields

$$\langle qu, y\rangle = q\langle u, y\rangle, \quad \text{for } q \in \mathbb{Q}.$$

For every $\alpha \in \mathbb{R}$, let $(q_n) \subset \mathbb{Q}$ converges to α. Now we show that $f(t) = \langle tu, y\rangle$ is continuous at $t = 0$ and by the additivity we may conclude that f is continuous on $\mathbb{R}$. Since

$$\begin{aligned} 4|f(t)| &= |\|tu + y\|^2 - \|tu - y\|^2| \\ &= (\|tu + y\| + \|tu - y\|)|\|tu + y\| - \|tu - y\|| \\ &\leq 4t\|u\|(t\|u\| + \|y\|) \to 0 \end{aligned}$$

as $t \to 0$, $f(t)$ is continuous. For every $\alpha \in \mathbb{R}$, let $(q_n) \subset \mathbb{Q}$ be a convergent sequence with limit α. Then

$$\langle \alpha u, y\rangle = \lim\langle q_n u, y\rangle = \lim q_n\langle u, y\rangle = \alpha\langle u, y\rangle.$$

Hence, $\langle \cdot, \cdot\rangle$ is linear in the first factor. Thus, it is an inner product. Meanwhile, it is easy to verify that the norm it introduces is exactly the original norm. □