Linear Algebra Done Right

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Contents

3 Linear Map

3.A The Vector Space of Linear Maps

1.

Proof. If T is linear, then $T(0,0,0) = 0$ and therefore $b = 0$. Meanwhile, $T(2,2,2) = 2T(1,1,1)$ implies $12 + 8c = 12 + 2c$. Hence, $c = 0$. The proof of the converse part is trivial. $\square$

3.

Proof. Let e_i be the i-th vector in the standard base of $\mathbb{F}^n$ and suppose that $Te_i = \sum_{j=1}^{n} A_{1,j} e_j$. Then for $x = (x_1, \ldots, x_n)^T \in \mathbb{F}^n$,

$$Tx = T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i T e_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{j,i} e_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} A_{j,i} x_i\right) e_j.$$

$\square$

5.

Proof. Too lengthy to write it down... $\square$

7.

Proof. Let $\{x_0\}$ be a basis of V and λ be a scalar such that $Tx_0 = \lambda x_0$. By the linearity of T, for every $x = kx_0$ in V, $Tx = kTx_0 = k\lambda x_0 = \lambda(kx_0) = \lambda x$. $\square$

9.

Solution. From the additivity condition we can derive that $\varphi(kz) = k\varphi(z)$ for any $k \in \mathbb{Q}$. Hence we can try some functions where $\varphi(iz) = i\varphi(z)$ fails. It turns out that $\varphi(z) = \mathrm{Im}(z)$ is one of the such maps required. $\square$

11.

Proof. Let $\{\alpha_1, \ldots, \alpha_p\}$ and $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\}$ be bases of U and V respectively. Then the linear map which maps α_i to $T\alpha_i$ and maps β to 0. Clear that it is the desired linear map. $\square$

13.

Proof. Suppose that v_k is in the span of the other vectors and let $w_i = 0$ for each $i \neq k$ and $w_k \neq 0$. No $T \in \mathcal{L}(V, W)$ can maps v_i to w_i since the linearity of T would force w_k to be 0, leading to a contradiction. $\square$