Statistical Inference

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Contents

1 Probability Theory

1.5(a)

Solution. A U.S. birth results in female identical twins. □

1.5(b)

Solution.

$$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C) = P(A|B)P(B|C)P(C) = \frac{1}{2}\frac{1}{3}\frac{1}{90} = \frac{1}{540}.$$

□

1.24(b)

Solution. Suppose $E_i = \{\text{head first appears on } i\text{th toss}\}$, then

$$P(\text{A wins}) = P(\bigcup_{i=k}^{\infty} E_{2k+1}) = \sum_{k=0}^{\infty} P(E_{2k+1}) = \sum_{k=0}^{\infty} p(1-p)^{2k+1} = \frac{p}{1-(1-p)^2}.$$

□

1.31(a)

Proof. To get the average $(x_1+\cdots+x_n)/n$, we need the unordered sample to be $\{x_1, x_2, \ldots, x_n\}$. The number of ordered samples which results in it is $n!$ and there are n^n ordered samples in total. Hence, the probability is $n!/n^n$.

For any other resulted average, there will exist some double counting when counting the ordered samples. Therefore, the outcome with average $(x_1 + \cdots + x_n)/n$ is most likely. □

1.33

Solution.

$$\begin{aligned} P(\text{male}|\text{color-blind}) &= P(\text{color-blind}|\text{male})\frac{P(\text{male})}{P(\text{color-blind})} \\ &= 0.05 \times \frac{0.5}{0.5 \times 0.05 + 0.5 \times 0.0025} \\ &= \frac{20}{21} = 0.9524. \end{aligned}$$

□

1.36

Solution. The probabilities of all shots being missed and the target being hit exactly once are $(4/5)^5 = 0.32768$ and $5 \times (1/5)(4/5)^4 = 0.4096$ respectively. Hence,

$$P(\text{being hit at least twice}) = 1 - 0.32768 - 0.4096 = 0.26272.$$

And

$$\begin{aligned}
&P(\text{being hit at least twice}|\text{being hit at least once}) \\
=&\frac{P(\text{being hit at least twice})}{P(\text{being hit at least once})} = \frac{0.26272}{0.4096} = 0.6414.
\end{aligned}$$

□

1.39(a)

Proof. A and B are mutually exclusive means that $A \cap B = \varnothing$. Hence, $P(A \cap B) = 0$. However, $P(A), P(B) > 0$. Therefore, $P(A \cap B) \neq P(A)P(B)$. □

1.39(b)

Proof. As A and B are independent, $P(A \cap B) = P(A)P(B) > 0$, which implies that $A \cap B \neq \varnothing$. □

Notes on 1.39 An intuitive proof: Since A and B are mutually exclusive, if we know that A did not happen, then the possibility that B happened will increase. Hence, they are not independent.

1.52

Proof. Clear that $g(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\mathrm{d}x &= \int_{x_0}^{\infty} \frac{f(x)}{1 - F(x_0)}\mathrm{d}x \\
&= \frac{1}{1 - F(x_0)}\left(\int_{-\infty}^{\infty} f(x)\mathrm{d}x - \int_{-\infty}^{x_0} f(x)\mathrm{d}x\right) \\
&= \frac{1}{1 - F(x_0)}(1 - F(x_0)) = 1.
\end{aligned}$$

Hence, by Theorem 1.6.5, $g(x)$ is a pdf. □