

# Real Analysis

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## 3 Lebesgue Measure

### 3.1 Introduction

1.

*Proof.* Since  $\mathfrak{M}$  is an  $\sigma$ -algebra,  $B \setminus A \in \mathfrak{M}$  as long as  $A, B \in \mathfrak{M}$ . Since  $B \setminus A$  and  $A$  are disjoint,  $mB = mA + m(B \setminus A) \geq mA$  since  $m$  is nonnegative.  $\square$

2.

*Proof.* Let  $A_0 = E_0$  and  $E_k = A_k \setminus A_{k-1}$  for  $k \geq 1$ . Clear that  $E_i$  and  $E_j$  are disjoint for distinct  $i$  and  $j$ ,  $\bigcup A_n = \bigcup E_n$  and  $A_i \subset E_i$  for every  $i$ . Hence,

$$m\left(\bigcup E_n\right) = m\left(\bigcup A_n\right) = \sum mA_n \leq \sum mE_n,$$

where the last inequality comes from Exercise 1.  $\square$

3.

*Proof.* Suppose that  $mA < \infty$ . Then  $mA = m(A \cup \emptyset) = mA + m\emptyset$ , implying that  $m\emptyset = 0$ .  $\square$

### 3.2 Outer Measure

5.

*Proof.* We show that  $\{I_n\}$  must cover the entire  $[0, 1]$  by contradiction. Assume that  $x \notin I_k$  for  $k = 1, 2, \dots, n$ . Then, as  $I_k$  are open and  $n$  is finite, there exists some  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon)$  and  $I_k$  are disjoint for every  $k$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exists some rational number in  $(x - \varepsilon, x + \varepsilon)$ , contradicting with the hypothesis that  $\{I_k\}$  covers all rational numbers between 0 and 1.  $\square$

6.

*Proof.* By the definition of the outer measure, for every  $\varepsilon > 0$ , there exists some collection  $\{I_n\}$  of open intervals that covers  $A$  and  $\sum l(I_n) \leq m^*A + \varepsilon$ . Let  $O = \bigcup I_n$ .  $O$  is a countable union of open sets and therefore is also open. And by Proposition 2,  $m^*O \leq \sum l(I_n)$ . Thus,  $m^*O \leq m^*A + \varepsilon$ .

Let  $\varepsilon_n = 1/n$  and for each  $n$ , by the previous discussion, we can always get an open set  $O_k$  such that  $A \subset O_k$  and  $m^*O \leq m^*A + \varepsilon_m$ . Let  $G$  be the countable intersection of these open sets. Clear that  $G$  is a  $G_\delta$  set covering  $A$  and  $m^*A = m^*G$ .  $\square$

7.

*Proof.* If  $m^*E = \infty$ , it is trivial. Suppose that  $m^*E \leq \infty$ . For any  $x \in \mathbb{R}$ , collection  $\{I_n\}$  of open intervals covers  $E + x$  iff  $\{I_n - x\}$  covers  $E$ . Since the length of intervals is translation invariant, this implies  $m^*(E + x) = m^*E$ .  $\square$

8.

*Proof.* Clear that  $m^*A \leq m^*(A \cup B)$ . Meanwhile,  $m^*(A \cup B) = m^*A + m^*B = m^*B$ . Hence,  $m^*(A \cup B) = m^*B$ .  $\square$

### 3.3 Measurable Sets and Lebesgue Measure

10.

*Proof.*

$$\begin{aligned} mE_1 + mE_2 &= mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2). \end{aligned}$$

□

11.

*Proof.*  $E_n = (n, \infty)$ .

□

12. This is the countable version of Lemma 9.

*Proof.* It suffices to prove  $m^*(A \cap \bigcup E_i) \geq \sum m^*(A \cap E_i)$ . Since  $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^n E_i$  for every  $n$ ,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of  $n$ , we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i),$$

completing the proof.

□

13.

*Proof.* First we suppose that  $m^*E < \infty$ . By Proposition 5, there exists some open set  $O \supset E$  such that  $m^*O \leq m^*E + \varepsilon$ . If  $E$  is measurable, then by the definition,

$$m^*(O \setminus E) = m^*O - m^*E \leq \varepsilon.$$

Namely, (ii) holds. Meanwhile,  $O \subset \mathbb{R}$  is a countable union of disjoint open intervals  $\{I_n\}$ . Since  $mO = m^*O$  is bounded and  $mO = \sum l(I_n)$ , there exists some integer  $N > 0$  such that  $mO - \sum_{n=1}^N l(I_n) < \varepsilon$ . Let  $U = \bigcup_{n=1}^N I_n$ .

$$\begin{aligned} m^*(U \triangle E) &= m^*((U \cup E) \setminus (U \cap E)) \\ &\leq m^*(O \setminus (U \cap E)) \\ &= m^*((O \setminus U) \cup (O \setminus E)) \\ &\leq m^*(O \setminus U) + m^*(O \setminus E) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, (ii) implies (vi). Now we show that (vi) implies (ii). If  $m^*(U \triangle E) < \varepsilon$ , then there exists some countable collection  $\{J_n\}$  of open interval such that

$$\sum l(J_n) \leq m^*(U \triangle E) + \varepsilon < 2\varepsilon.$$

Let  $J = \bigcup J_n$  and  $O = U \cup J$ .  $m^*J < 2\varepsilon$ . And  $O$  is open and covers  $E$ . Meanwhile,

$$m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(J \setminus E) < 3\varepsilon.$$

Hence, (ii) holds.

Now, let  $E$  be an arbitrary set and  $E_n = E \cap (-n, n)$ , which is a set with finite measure. Then by the previous discussion, there exists some open set  $O_n \supset E_n$  with  $m^*(O_n \setminus E_n) < \varepsilon/2^n$ . Let  $O = \bigcup O_n$ , an open set covering  $E$  and

$$m^*(O \setminus E) \leq \sum m^*(O_n \setminus E_n) < 2\varepsilon.$$

Hence, (i) implies (ii). Now we suppose (ii) holds and let  $\varepsilon_n = 1/n$ , then there exists a sequence of open sets  $\langle O_n \rangle$  such that  $m^*(O_n \setminus E) < 1/n$ . Let  $G = \bigcap O_n \in G_\delta$ .  $m^*(G \setminus E) \leq m^*(O_n \setminus E) \leq 1/n$ . Since the left hand side is independent of  $n$ ,  $m^*(G \setminus E) = 0$ . If (iv) holds, then by Lemma 6,  $G \setminus E$  is measurable. Since  $G \in G_\delta$  is also measurable,  $E$  is measurable. Hence, (iv) implies (i).

By the previous result, for any measurable  $E$ , there exists some closed set  $F \subset E$  such that  $\bar{F}$ , which is open, contains  $\text{bar}E$  and  $m^*(\bar{F} \setminus \bar{E}) < \varepsilon$ . Hence,  $m^*(E \setminus F) < \varepsilon$ . We can proceed in a similar manner as we did in the last paragraph to prove that (iii)  $\Rightarrow$  (v)  $\Rightarrow$  (i), leading to the final conclusion.  $\square$

### 3.5 Measurable Functions

19.

*Proof.* For every  $\beta \in \mathbb{R}$ , since  $D$  is measurable, there exists a sequence of  $\alpha_n \in D \cap (\beta - 1/n, \beta)$ . As

$$\{x : f(x) > r\} \Leftrightarrow \bigcup_{n=1}^{\infty} \{x : f(x) > r - 1/n\} \Leftrightarrow \bigcup_{n=1}^{\infty} \{x : f(x) > \alpha_n\}$$

and  $\{x : f(x) > \alpha_n\}$  are measurable, so is  $\{x : f(x) > r\}$ . Hence,  $f$  is measurable.  $\square$

21.

*Proof.*

(a) It follows immediately from  $\{x : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$ .

(b) For  $\alpha \geq 0$ , the sets  $\{x : f(x) > \alpha\}$  and  $\{x : g(x) > \alpha\}$  are the same. And for  $\alpha < 0$ ,

$$\{x : f(x) > \alpha\} = \{x : g(x) > \alpha\} \setminus \bar{D} \quad \text{and} \quad \{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \bar{D}.$$

Hence,  $f$  is measurable iff  $g$  is measurable.  $\square$

22.(d)

*Proof.* Since  $f$  and  $g$  are finite almost everywhere, the set  $A$  consisting of points where  $f + g$  is of the form  $\infty - \infty$  or  $-\infty + \infty$  is of measure zero (and hence measurable). Therefore no matter how it is defined,  $\{x \in A : f + g > \alpha\}$  is measurable for every  $\alpha$ . Namely, the restriction of  $f + g$  to  $A$  is measurable. Meanwhile, clear that the restriction to  $D \setminus A$  is measurable where  $D$  is the domain of  $f$ . Hence, by Exercise 21,  $f$  is measurable.  $\square$

**23.**

*Proof.*

(a) Let  $A_n = \{x : |f(x)| > n\}$ , a sequence of measurable sets. As  $A_{n+1} \subset A_n$ ,  $mA_{n+1} \leq mA_n$ . Since  $A = \bigcap A_n = \{x : |f(x)| = \infty\}$ ,  $mA_1 \leq m[a, b]$  is finite and  $mA = 0$ , by Proposition 14, there exists some  $N$  such that for all  $n \geq N$ ,  $mA_n < \varepsilon/3$ . Set  $M = N$  to complete the proof.

(b) We consider the restriction of  $f$  on to the set  $E = [a, b] \setminus \{x : |f(x)| \geq M\}$ , which is also a measurable real-valued function. To keep our notation simple, we denote the restriction by  $f$  still. For every  $\varepsilon > 0$ , there exists some integer  $N$  with  $0 < 2M/N < \varepsilon$ . Let  $E_n = \{x : x \in [-M + (n-1)\varepsilon, -M + n\varepsilon]\}$  ( $n = 1, 2, \dots, N$ ) and define

$$\varphi(x) = \sum_{i=1}^N f(x_i) \chi_{E_i},$$

where  $x_n \in E_n$  is arbitrary. Clear that  $\varphi$  is a simple function and satisfy all the requirements.

(c) Suppose that  $\varphi(x) = \sum_{i=1}^n \alpha_i \chi_{E_i}$ . For each  $i = 1, \dots, N$ ,  $E_i$  is measurable and therefore by Proposition 15, there exists a finite union  $U_i$  of open intervals such that  $m(U_i \triangle E_i) < \varepsilon$ . Let

$$g(x) = \sum_{i=1}^N \alpha_i \chi_{U_i}.$$

Clear that  $g$  and  $\varphi$  only may differ on a set with measure  $N\varepsilon$ . (d) Suppose that  $g(x) = \sum_{i=1}^N \alpha_i \chi_{U_i}$  is a step function. We may assume without loss of generality that  $U_i$  are disjoint and  $\bigcup U_i = [a, b]$ . And suppose that  $\{x_0 = a < x_1 < \dots < x_N = b\}$  are the endpoints of the intervals. For each  $i = 1, \dots, N-1$ , define

$$f(x) = (x - x_i + \varepsilon)g(x_i - \varepsilon) + (x_i + \varepsilon - x)g(x_i + \varepsilon), \quad x \in (x_i - \varepsilon, x_i + \varepsilon),$$

and  $f(x) = g(x)$  for the other points. (We assume that  $\varepsilon$  is small enough so that  $f$  is well-defined.) Clear that  $f$  is continuous and equals  $g$  except on a set of measure less than  $2N\varepsilon$ .  $\square$

**24.**

*Proof.* For measurable  $f$ , we show that  $\mathcal{A} = \{E : f^{-1}[E] \text{ is measurable}\}$  is a  $\sigma$ -algebra first. As the domain, denoted by  $D$ , of a measurable function is measurable,  $\mathbb{R} \in \mathcal{A}$ . If  $E \in \mathcal{A}$ , then since  $f^{-1}[\bar{E}] = D \cap \overline{f^{-1}[E]}$ ,  $f^{-1}[\bar{E}]$  is also measurable and therefore  $\bar{E} \in \mathcal{A}$ . Suppose that  $\langle E_n \rangle$  is a sequence of sets of  $\mathcal{A}$ . Then, as

$$f^{-1}\left[\bigcup_{n=1}^{\infty} E_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[E_n],$$

$\bigcup E_n \in \mathcal{A}$ . Hence,  $\mathcal{A}$  is a  $\sigma$ -algebra.

By the definition of a measurable function, every open interval belongs to  $\mathcal{A}$ . Since the collection of all Borel sets  $\mathcal{B}$  is the  $\sigma$ -algebra generated by all open intervals,  $\mathcal{B} \subset \mathcal{A}$ . Namely,  $f^{-1}[B]$  is measurable as long as  $B \in \mathcal{B}$ .  $\square$

### 3.6 Littlewood's Three Principles

30.

*Proof.* Let  $\varepsilon_n = 1/n$  and  $\delta_n = \eta/2^n$  ( $n = 0, 1, \dots$ ). By Proposition 24, for each  $n$ , there exists some  $A_n$  with measure less than  $\delta_n$  such that for all  $x \in E_n \setminus A_n$ ,  $|f_m(x) - f(x)| < \varepsilon_n$  for  $m$  large enough. Let  $A = \bigcup_{n=1}^{\infty} A_n$ , the measure of which is less than  $\sum \eta/2^n = \delta$ . Meanwhile, for any  $\varepsilon > 0$ , by construction, for all  $x \in E \setminus A$ ,  $|f_m(x) - f(x)| < \varepsilon$  for  $m$  large enough. Namely,  $f_n$  converges to  $f$  uniformly on  $E \setminus A$ .  $\square$

31.

*Proof.* Let  $\varepsilon_n = \delta/2^n$  ( $n \geq 0$ ), then by Proposition 22, there exists continuous  $g_n$  such that  $E_n = \{x : |f(x) - g_n(x)| \geq \varepsilon_n\}$  is of measure less than  $\varepsilon_n$ . Let  $E = \bigcup E_n$ , the measure of which is less than  $\delta$  and  $g_n$  converges to  $f$  on  $[a, b] \setminus E$ .

By Egoroff's Theorem, there exists some  $A \subset [a, b] \setminus E$  with  $mA < \delta$  such that  $g_n$  converges to  $f$  uniformly on  $[a, b] \setminus (E \cup A)$ . Since  $E \cup A$  is measurable, by Proposition 15, there exists some open set  $O \supset E \cup A$  such that  $m(O \setminus (E \cup A)) < \delta$ . Let  $F = [a, b] \setminus O$ . We know that

1.  $F$  is a closed set.
2.  $mF < 3\delta$ .
3.  $g_n$  converges to  $f$  uniformly on  $F$ .

Hence,  $f$  is continuous on  $F$ . And by Problem 2.40, there exists some continuous function on  $\mathbb{R}$  such that  $\varphi(x) = f(x)$  for  $x \in F$ .

If  $f$  is defined on  $(-\infty, \infty)$ , we can apply the previous result on each  $[n, n+1]$  and "stick" the functions together as we did in Problem 23(c) to get the function required.  $\square$

## 4 The Lebesgue Integral

### 4.2 The Lebesgue Integral of a Bounded Function

2.

*Proof.*

(a) By Problem 2.51,  $h$  is upper semicontinuous as  $f$  is bounded and by Problem 2.50,  $x : h(x) < \lambda$  is open and hence measurable for every  $\lambda \in \mathbb{R}$ . Thus,  $h$  is measurable.

Let  $\varphi(x) \geq f(x)$  be a step function and  $x_0$  any point other than the endpoints of the intervals occurring in  $\varphi$ . Then there exists some  $\delta > 0$  such that for all  $x \in (x_0 - \delta, x_0 + \delta)$ ,  $\varphi(x_0) = \varphi(x) \geq f(x)$ . Hence,

$$h(x_0) = \inf_{\delta < 0} \sup_{|x - x_0| < \delta} f(x) \leq \varphi(x_0).$$

Namely,  $\varphi \geq h$  except at a finite number of points. Hence,  $\int_a^b \varphi \geq \int_a^b h$  and therefore

$$R \int_a^{\bar{b}} f = \inf_{\varphi \geq f} \int_a^b \varphi(x) dx \geq \int_a^b h.$$

We can also derive from the previous discussion that there is a sequence of  $\langle \varphi_n \rangle$  of step functions satisfying  $\varphi \downarrow h$ . By Proposition 6,

$$\int_a^b h = \lim \int_a^b \varphi_n \geq R \int_a^{\bar{b}} f.$$

Hence,  $R \int_a^{\bar{b}} f = \int_a^b h$ .

(b) First suppose that  $f$  is Riemann integrable and let  $h$  and  $g$  be the upper and lower envelope of  $f$  respectively. By part (a),  $f$  is Riemann integrable implies  $\int_a^b (h - g) = 0$ . Together with the fact that  $h \geq g$ , we conclude that  $h = g$  a.e.. Therefore, by Problem 2.50,  $f$  is continuous except on a set of measure zero.

Note that the argument remains true if we reverse the order, verifying the converse part. Hence, the proposition holds.  $\square$

### 4.3 The Integral of a Nonnegative Function

3.

*Proof.* Suppose that  $E_n = \{x : f(x) > 1/n\}$ . Then,

$$0 = \int f \geq \int_{E_n} f \geq \frac{mE_n}{n}$$

implies  $mE_n = 0$ . Hence,  $m\{x : f(x) > 0\} = m(\bigcup E_n) \leq \sum mE_n = 0$ . Namely,  $f = 0$  a.e.  $\square$



5.

*Proof.* For any fixed  $x_0 \in \mathbb{R}$ , let  $f_n(x) = f \cdot \chi_{(-\infty, x_0 - 1/n]}$ , which is a increasing sequence of nonnegative measurable function whose limit is  $f \cdot \chi_{(-\infty, x_0]}$ . Then by Theorem 10,

$$F(x_0) = \int_{-\infty}^{x_0} f = \int f \cdot \chi_{(-\infty, x_0]} = \lim \int f \cdot \chi_{(-\infty, x_0 - 1/n]} = \lim F(x_0 - 1/n).$$

Meanwhile, since

$$|F(x_0) - F(x_0 + 1/n)| = \left| \int_{x_0}^{x_0 + 1/n} f(x) dx \right| = \left| \int_{-1/n}^0 g(x) dx \right|,$$

where  $g(x) = f(x_0 - x)$ , arguing on  $g$  in a similar manner yields  $F(x_0) = \lim F(x_0 + 1/n)$ . Thus,  $F$  is continuous.  $\square$

6.

*Proof.* By Theorem 9,  $\int f \leq \underline{\lim} \int f_n$ . Meanwhile,  $f_n \leq f$  implies  $\int f_n \leq \int f$  and therefore  $\overline{\lim} \int f_n \leq \int f$ . Hence,  $\int f = \lim \int f_n$ .  $\square$

7.

*Solution.*

(a) Let  $f_n(x) = n \cdot \chi_{[0, 1/n]}$ .  $f_n$  converges to  $f = 0$  except on  $x = 0$ . For each  $n$ ,  $\int f_n = 1$  but  $\int f = 0$ . Hence, the inequality could be strict.

(b) Let  $f_n(x) = \chi_{[n, \infty)}$ . Then  $\langle f_n \rangle$  is a decreasing sequence which converges to  $f = 0$ , the integral of which is 0. However, for every  $n$ ,  $\int f_n = \infty$ .  $\square$

8.

*Proof.* Let  $g_n = \inf\{f_n, f_{n+1}, \dots\}$ . Clear that

$$\int g_n \leq \int f_n. \tag{1}$$

Meanwhile  $\langle g_n \rangle$  is a increasing sequence converging to  $\underline{\lim} f_n$ . Hence, by the Monotone Convergence Theorem and (1)

$$\int \underline{\lim} f_n = \int \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} \int g_n \leq \underline{\lim} \int f_n.$$

$\square$

9.

*Proof.* By Fatou's Lemma,

$$\int_E f \leq \underline{\lim} \int_E f_n. \tag{2}$$

Similarly,  $\int_{\bar{E}} f \leq \underline{\lim} \int_{\bar{E}} f_n$  and therefore

$$\int_E f_n = \int f_n - \int_{\bar{E}} f_n \Rightarrow \overline{\lim} \int_E f_n \leq \int f - \int_{\bar{E}} f = \int_{\bar{E}} f.$$

(2) and the inequality above together implies  $\int_E f_n \rightarrow \int f$ .  $\square$

## 4.4 The General Lebesgue Integral

12.

*Proof.* Note that  $\langle g + f_n \rangle$  is a sequence of nonnegative measurable functions. Hence by Problem 8,

$$\int_E \underline{\lim}(g + f_n) \leq \underline{\lim} \int_E (g + f_n) \Rightarrow \int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n.$$

The second inequality follows immediately from the definition of lower and upper limit. Replacing  $g + f_n$  with  $g - f_n$  and arguing in a similar manner gives the last inequality.  $\square$

13.

*Proof.*  $f_n \geq -h$  implies  $f_n + h \geq 0$ . Hence,  $\int(f_n + h)$  always has a meaning. And since  $g$  is integrable,  $\int f_n = \int(f_n + h) - \int h$  also has a meaning. Similarly,  $\int f$  has a meaning. Meanwhile,

$$\int f = \int(f + h) - \int h \leq \underline{\lim} \int(f_n + h) - \int h = \underline{\lim} \int f_n.$$

$\square$

15.

*Proof.*

(a) By Problem 4, for every  $\varepsilon > 0$ , there exists some simple functions  $\varphi_1 \leq f^+$  and  $\varphi_2 \leq f^-$  such that

$$\int_E f^+ - \int_E \varphi_1 < \varepsilon \quad \text{and} \quad \int_E f^- - \int_E \varphi_2 < \varepsilon.$$

Let  $\varphi = \varphi_1 - \varphi_2$ , which is also a simple function. Meanwhile,

$$\int_E |f - \varphi| \leq \int_E (f^+ - \varphi_1) + \int_E (f^- - \varphi_2) < 2\varepsilon.$$

$\square$

16.

*Proof.* For every integrable  $f$ , by Problem 15, there exists some step function  $\psi = \sum_{k=1}^N c_k \chi_{E_k}$  such that  $\int |f - \psi| < \varepsilon$ . Note that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \psi(x) \cos nx dx = \lim_{n \rightarrow \infty} \sum_{k=1}^N c_k \int_{E_k} \cos nx dx = 0. \quad (3)$$

Hence,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} f(x) \cos nx dx \right| &= \left| \int_{-\infty}^{\infty} (f(x) - \psi(x) + \psi(x)) \cos nx dx \right| \\ &\leq \int_{-\infty}^{\infty} |f(x) - \psi(x)| |\cos nx| dx + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx dx \right| \\ &\leq \varepsilon + \left| \int_{-\infty}^{\infty} \psi(x) \cos nx dx \right| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

□

**18.**

*Proof.* Let  $\langle t_n \rangle$  be any sequence with  $t_n \neq 0$  and tending to 0. Then  $\langle f(x, t_n) \rangle$  is sequence of functions satisfying the hypotheses of Lebesgue Convergence Theorem. Meanwhile,  $f(x, t_n) \rightarrow f$  as  $n \rightarrow \infty$ . Hence,

$$\lim_{n \rightarrow \infty} \int f(x, t_n) dx = \int f(x) dx.$$

Since the choice of  $\langle t_n \rangle$  is arbitrary, by Problem 2.49f,

$$\lim_{t \rightarrow 0} \int f(x, t) dx = \int f(x) dx.$$

If  $f$  is continuous in  $t$  for each  $x$ , then  $\lim_{\Delta t \rightarrow 0} f(x, t + \Delta t) = f(x, t)$  holds for every  $t$ . Therefore, replacing  $t$  with  $\Delta t$  in the previous result yields

$$\lim_{\Delta t \rightarrow 0} \int f(x, t + \Delta t) dx = \int f(x, t) dx.$$

Namely,  $h(t)$  is continuous.

□

## 5 Differentiation and Integration

### 5.1 Differentiation of Monotone Functions

3. "maximum" needs to be changed to "minimum" in both (a) and (b).

*Proof.*

(a) We may assume without loss of generality that  $c = 0$ . Since  $f$  attains a local minimum at  $x = 0$ ,  $f(h) \geq f(0)$  for every  $h$  sufficiently small. Hence, for every small  $h > 0$ ,  $(f(c+h) - f(c))/h > 0$  and therefore  $D_+f(c) \geq 0$ . Meanwhile, by Problem 2.b,

$$-D_-f(0) = D^+f(0) \geq 0 \quad \Rightarrow \quad D_-f(0) \leq 0.$$

The other two inequalities follow immediately from the definitions of upper and lower limits.

(b) If  $f$  has a local minimum at  $a$  or  $b$ , then we only have the right or left half of the inequalities.  $\square$

4.

*Proof.* We first show this for  $g$  with  $D^+g \geq \varepsilon > 0$ . For every  $a \leq x < y \leq b$ , as  $g$  is continuous on  $[a, b]$ ,  $g$  has a maximum in  $[a, b]$  and by Problem 2 and 3,  $g$  can not attain the maximum in  $[a, b)$ . Namely, the restrict of  $f$  to  $[x, y]$  attains the maximum at  $y$ . Hence,  $g(x) \leq g(y)$ .

For every  $f$  with nonnegative  $D^+$ , let  $g(x) = f(x) + \varepsilon x$  where  $\varepsilon > 0$ . Then  $D^+g \geq \varepsilon > 0$ . Hence  $g$  is nondecreasing. Therefore, for every  $a \leq x < y \leq b$ ,

$$g(x) \leq g(y) \quad \Rightarrow \quad f(x) + \varepsilon x \leq f(y) + \varepsilon y.$$

Since the choice of  $\varepsilon$  is arbitrary, this implies  $f(x) \leq f(y)$ .  $\square$

5.a

*Proof.*

$$\begin{aligned} \sup_{t \in (0, h)} \frac{(f+g)(x+t) - (f+g)(x)}{t} &= \sup_{t \in (0, h)} \left( \frac{f(x+t) - f(x)}{t} + \frac{g(x+t) - g(x)}{t} \right) \\ &\leq \sup_{t \in (0, h)} \frac{f(x+t) - f(x)}{t} + \sup_{t \in (0, h)} \frac{g(x+t) - g(x)}{t}. \end{aligned}$$

Letting  $h \rightarrow 0$  yields  $D^+(f+g) \leq D^+f + D^+g$ .  $\square$

### 5.2 Functions of Bounded Variation

7.

*Proof.*

(a) It suffices to show this for monotone functions as each function of bounded variation is the difference of two monotone functions. Suppose that  $f$  is nondecreasing. Then the set  $E = \{f(x) : x > c\}$  is bounded below and hence  $A = \inf E$  is finite. For every  $\varepsilon > 0$ , there exists some  $y > c$  such that  $A \leq f(c) < A + \varepsilon$ . Hence, as  $f$  is nondecreasing,

for every  $x \in (c, y)$ ,  $|f(x) - A| < \varepsilon$ . Namely,  $\lim_{x \rightarrow c+} f(x) = A$ . Similarly,  $\lim_{x \rightarrow c-} f(x)$  exists.

Let  $D_n = \{x : |f(x+) - f(x-)| > 1/n\}$ . Since  $f$  is nondecreasing,  $|f(x) - f(y)| \leq f(b) - f(a) < \infty$  for every  $x, y \in [a, b]$ . Hence,  $D_n$  is finite, otherwise we can choose a sequence  $x_1 < \dots < x_N$  with  $N > (f(b) - f(a))/n$  such that  $f(x_N) - f(x_1) > f(b) - f(a)$ . Therefore,  $\bigcup_{n=1}^{\infty} D_n$ , the set of discontinuities, is countable.

(b) Suppose  $\{x_1, \dots, x_n, \dots\} = \mathbb{Q} \cap [0, 1]$  and define  $f(x) = \sum_{x_n < x} 2^{-n}$ . Clear that  $f$  is monotone and continuous at every irrational point. For each rational  $x = x_k$ ,  $f(x+) - f(x-) = 2^{-k}$ . Hence,  $f$  is discontinuous at each rational point.  $\square$

## 8.

*Proof.*

(a) For every  $\varepsilon > 0$ , there exists some subdivision  $a = x_0 < \dots < x_p = c$  and  $c = x_p < \dots < x_q = b$  of  $[a, c]$  and  $[c, b]$  such that  $T_a^c < t_a^c + \varepsilon$  and  $T_c^b < t_c^b + \varepsilon$ . Hence,  $T_a^c + T_c^b - 2\varepsilon < t_a^c + t_c^b$ . Meanwhile, as  $a = x_0 < \dots < x_q = b$  forms a subdivision of  $[a, b]$ ,  $T_a^b \geq t_a^b = t_a^c + t_c^b$ . Therefore,  $T_a^c + T_c^b - 2\varepsilon < T_a^b$ . Since the choice of  $\varepsilon$  is arbitrary,  $T_a^b + T_c^b \leq T_a^b$ .

To show that  $T_a^b + T_c^b \geq T_a^b$ , let  $a = x_0 < \dots < x_q = b$  be any subdivision of  $[a, b]$  and by adding  $c$  into it, we get subdivisions of  $[a, c]$  and  $[c, b]$ . Suppose that  $c \in (x_k, x_{k+1}]$ , then

$$|f(x_k) - f(c)| + |f(c) - f(x_{k+1})| + t_a^b = t_a^c + t_c^b + |f(x_k) - f(x_{k+1})|,$$

which implies  $t_a^b \leq t_a^c + t_c^b$ . Hence,

$$T_a^b = \sup t_a^b \leq \sup(t_a^c + t_c^b) \leq T_a^c + T_c^b.$$

Thus,  $T_a^b = T_a^c + T_c^b$  and therefore  $T_a^c \leq T_a^b$ .

(b)

$$\begin{aligned} T_a^b(f+g) &= \sup \sum_{i=1}^k |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\leq \sup \sum_{i=1}^k |f(x_i) - f(x_{i-1})| + \sup \sum_{i=1}^k |g(x_i) - g(x_{i-1})| \\ &\leq T_a^b(f) + T_a^b(g). \end{aligned}$$

$$T_a^b(cf) = \sup \sum_{i=1}^k |cf(x_i) - cf(x_{i-1})| = |c| \sup \sum_{i=1}^k |f(x_i) - f(x_{i-1})| = |c| T_a^b(f).$$

$\square$

## 9.

*Proof.* For every  $\varepsilon > 0$ , there exists a subdivision  $a = x_0 < \cdots < x_k = b$  such that  $t_a^b(f) \geq T_a^b(f) - \varepsilon$ . Meanwhile, as  $f_n$  converges to  $f$  pointwisely

$$\begin{aligned} t_a^b(f) &= t_a^b(\lim f_n) \\ &= \sum_{i=1}^k |(\lim f_n)(x_i) - (\lim f_n)(x_{i-1})| \\ &= \lim \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| \\ &\leq \underline{\lim} \sup \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| = \underline{\lim} T_a^b(f_n). \end{aligned}$$

Hence,  $T_a^b(f) - \varepsilon \leq \underline{\lim} T_a^b(f_n)$ . Since the choice of  $\varepsilon$  is arbitrary,  $T_a^b(f) \leq \underline{\lim} T_a^b(f_n)$ .  $\square$

### 10.a

*Solution.* No. Let  $x_k = (k\pi + \pi/2)^{-1/2}$ ,  $k = 0, 1, \dots$  and consider the subdivision  $-1 < 0 < x_n < \cdots < x_0 < 1$ . Then

$$t_n \geq \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \geq \sum_{k=1}^n \frac{2}{(k + 1/2)\pi}.$$

$t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and therefore  $f$  is not of bounded variation on  $[-1, 1]$ .  $\square$

### 11.

*Proof.* By Lemma 4,  $f(x) = f(a) + P_a^x - N_a^x$ . Since  $P_a^x$  and  $N_a^x$  are monotone, by Theorem 3, they are differentiable almost everywhere as  $f$ , a function of bounded variation, does. Hence, for almost every  $x \in [a, b]$ ,

$$\frac{d}{dx} f(x) = \frac{d}{dx} P_a^x - \frac{d}{dx} N_a^x \quad \Rightarrow \quad |f'(x)| \leq \frac{d}{dx} P_a^x + \frac{d}{dx} N_a^x = \frac{d}{dx} T_a^x.$$

Integrating on the both sides yields  $\int_a^b |f'| \leq T_a^b(f)$ .  $\square$

## 5.4 Absolute Continuity

### 12.

*Solution.* The continuous extension of  $x^2 \sin(1/x^2)$  to  $[0, 1]$  is absolutely continuous for all  $[\varepsilon, 1]$  but is not of bounded variation on  $[0, 1]$  and therefore is not absolutely continuous on  $[0, 1]$ .

Suppose that  $f$  is also of bounded variation on  $[0, 1]$ . Then  $f$  is differentiable almost everywhere. Hence  $g(x) = \int_0^x f'(t)dt + f(a)$  is well-defined. For every  $\varepsilon > 0$ , we have

$$g(x) = \int_0^\varepsilon f'(t)dt + \int_\varepsilon^x f'(t)dt + f(0) = \int_0^\varepsilon f'(t)dt + f(x) - f(\varepsilon) + f(0),$$

where the second equality comes from the absolute continuity on  $[\varepsilon, 1]$ . By the continuity of  $f$  at  $x = 0$ ,  $f(\varepsilon) \rightarrow f(0)$ . Hence, letting  $\varepsilon \rightarrow 0$  yields  $g(x) = f(x)$ . Namely,  $f$  is an indefinite integral. Thus, by Theorem 14, it is absolutely continuous.  $\square$

13.

*Proof.* Since absolute continuity implies bounded variation,  $\int_a^b |f'| \leq T_a^b(f)$  by Problem 11. By the definition of  $T$ , for every  $\varepsilon > 0$ , there exists some subdivision  $a = x_0 < \cdots < x_n = b$  such that  $t_a^b(f) > T_a^b(f) - \varepsilon$ . Meanwhile, for every  $i = 1, \dots, n$ ,

$$\int_{x_{i-1}}^{x_i} |f'| \geq \left| \int_{x_{i-1}}^{x_i} f' \right| = |f(x_i) - f(x_{i-1})|,$$

where the second equality is guaranteed by the absolute continuity. Hence,  $\int_a^b |f'| > T_a^b(f) - \varepsilon$  for every  $\varepsilon > 0$ . Thus,  $T_a^b(f) = \int_a^b |f'|$ .

By Lemma 4,  $2P_a^b(f) = T_a^b(f) + f(b) - f(a)$ . Hence,

$$P_a^b(f) = \frac{1}{2} \left( \int_a^b |f'| + f(b) - f(a) \right) = \frac{1}{2} \int_a^b (|f'| + f') = \int_a^b [f']^+.$$

□

14.

*Proof.*

(a) Suppose that  $f$  and  $g$  are absolutely continuous. Then for every  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that for all finite nonoverlapping  $\langle (x_n, y_n) \rangle$  with  $|x_n - y_n| < \varepsilon$ ,

$$\sum |f(x_n) + g(x_n) - f(y_n) - g(y_n)| \leq \sum |f(x_n) - f(y_n)| + |g(x_n) - g(y_n)| \leq 2\varepsilon.$$

Hence,  $f + g$  is also absolutely continuous. Since  $-g$  is absolutely continuous as long as  $g$  is, so is  $f - g$ .

(b) Suppose that  $f$  and  $g$  are absolutely continuous. Then they are bounded, by  $M$  for example. Hence for every  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that for all finite nonoverlapping  $\langle (x_n, y_n) \rangle$  with  $|x_n - y_n| < \varepsilon$ ,

$$\begin{aligned} & \sum |f(x_n)g(x_n) - f(y_n)g(y_n)| \\ &= \sum |f(x_n)g(x_n) - f(x_n)g(y_n) + f(x_n)g(y_n) - f(y_n)g(y_n)| \\ &\leq \sum \{|f(x_n)||g(x_n) - g(y_n)| + |f(x_n) - f(y_n)||g(y_n)|\} \\ &\leq M\varepsilon. \end{aligned}$$

Thus,  $fg$  is also absolutely continuous.

(c) Since  $f$  is continuous on  $[a, b]$ ,  $f$  can achieve its minimum in  $[a, b]$ . Hence,  $|f(x)| \geq m > 0$  as  $f$  is never zero. Therefore for every  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that for all finite nonoverlapping  $\langle (x_n, y_n) \rangle$  with  $|x_n - y_n| < \varepsilon$ ,

$$\sum \left| \frac{1}{f(x_n)} - \frac{1}{f(y_n)} \right| = \sum \left| \frac{f(y_n) - f(x_n)}{f(x_n)f(y_n)} \right| \leq \frac{1}{m^2} \sum |f(x_n) - f(y_n)| \leq \frac{\varepsilon}{m^2}.$$

□

17. Part (a) is wrong. It can be fixed if we further require  $g$  to be monotone increasing.

*Proof.*

(a) For every  $\varepsilon > 0$ , let  $\delta_1$  be the number in the definition of  $F$  corresponding to  $\varepsilon$  and  $\delta_2$  the number in the definition of  $g$  corresponding to  $\delta_1$ . Then for every finite nonoverlapping  $\langle (x_n, y_n) \rangle$  with  $|x_n - y_n| < \delta_2$ ,  $\sum |g(x_n) - g(y_n)| < \delta_1$ . Since  $g$  is monotone increasing,  $(g(x_n), g(y_n))$  are nonoverlapping. Therefore,  $\sum |F(g(x_n)) - F(g(y_n))| < \varepsilon$ . Hence,  $F \circ g$  is absolutely continuous.  $\square$

18.

*Proof.* Without loss of generality, we assume that  $g$  is nondecreasing. Since  $mE = 0$ , for every  $\varepsilon > 0$ , by Proposition 3.15, there exists an open set  $O \supset E$  with  $mO < \varepsilon$ . Meanwhile, there exists a sequence of disjoint open intervals  $\langle I_n = (a_n, b_n) \rangle$  such that  $\bigcup_{n=1}^{\infty} I_n = O$  and  $l(I_n) < \delta$  where  $\delta$  is the number in the definition of absolute continuity. Then  $g[E] \subset \bigcup_{n=1}^{\infty} g[I_n \cap [0, 1]]$ . Since  $g$  is continuous, the image of an interval is still an interval and since  $g$  is also nondecreasing,  $g[I_n \cap [0, 1]] = (g(a'_n), g(b'_n))$ , where  $a'_n = \max\{a_n, 0\}$  and  $b'_n = \min\{b_n, 1\}$ . Finally,

$$m(g[E]) \leq \sum_{n=1}^{\infty} m(g[I_n]) = \sum_{n=1}^{\infty} |g(b'_n) - g(a'_n)| \leq \varepsilon,$$

where the last inequality comes from the absolute continuity of  $g$ . Since the choice of  $\varepsilon$  is arbitrary,  $m(g[E]) = 0$ .  $\square$

20.

*Proof.*

(a) For every  $\varepsilon > 0$ , let  $\delta = \varepsilon/M$ . Then for every  $\langle x_n \rangle_{i=1}^n$  and  $\langle y_n \rangle_{i=1}^n$  with  $|x_n - y_n| \leq \delta$ ,

$$\sum_{i=1}^n |f(x_n) - f(y_n)| \leq M \sum_{i=1}^n |x_n - y_n| \leq \varepsilon,$$

as  $f$  satisfies the Lipschitz condition.

(b) Suppose that  $f$  is absolute continuous and  $|f'|$  is bounded by  $M$ . Then for every  $x$  and  $y$  in the interval,

$$|f(x) - f(y)| = \left| \int_x^y f'(t) dt \right| \leq M|x - y|.$$

Hence,  $f$  satisfies the Lipschitz condition. The converse part has been proved in (a).

(c) It is wrong. A counterexample is  $f(x) = \chi_{[0,1]}$ ,  $x \in (-1, 1)$   $\square$

21.

*Proof.*

(a) Suppose that  $O = \bigcup_{n=1}^{\infty} (c_n, d_n)$  where  $(c_n, d_n)$  are disjoint. Since  $g$  is continuous and increasing,  $g^{-1}(c_n, d_n)$  is still an open interval, denoting it by  $(a_n, b_n)$ , and  $(a_n, b_n)$  are also disjoint. Meanwhile,  $d_n - c_n = f(a_n) - f(b_n) = \int_{a_n}^{b_n} g'$ . Hence,

$$mO = m\left(\bigcup_{n=1}^{\infty} (c_n, d_n)\right) = \sum_{n=1}^{\infty} (d_n - c_n) = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} g' = \int_{g^{-1}[O]} g'.$$



(b) Without loss of generality, we assume that  $d \notin E$ . For every  $\varepsilon > 0$ , there exists an open set  $O \supset E$  with  $mO < \varepsilon$ . By Part (a),

$$\int_{g^{-1}[O] \cap H} g' = \int_{g^{-1}[O]} g' = mO < \varepsilon.$$

Since the choice of  $\varepsilon$  is arbitrary,  $\int_{g^{-1}[O] \cap H} g' = 0$ . Since  $g' > 0$  on  $H$ ,  $g^{-1}[O] \cap H$  has measure zero.

(c) Since  $E$  is measurable, so is  $g^{-1}[E]$ . Meanwhile, by Theorem 3,  $g'$  is measurable, hence  $H$  is also measurable. Therefore,  $F$  is measurable.

We may assume without loss of generality that  $c, d \notin E$ . By Proposition 3.15, there exists some  $G \in G_\delta$  such that  $E \subset G \subset (c, d)$  and  $m(G \setminus E) = 0$ . Since  $g$  is increasing,  $g^{-1}[G] \cap H = F \cup (g^{-1}[G \setminus E] \cap H)$  and by (b),  $g[G \setminus E] \cap H$  is of measure zero. Therefore,  $\int_F g' = \int_{g^{-1}[G] \cap H} g'$ . Namely, it suffices to show the result for  $G \in G_\delta$ .

Suppose that  $G = \bigcap_{n=1}^{\infty} O_n$  where each  $O_n \subset (c, d)$  is open and  $mO_1 < \infty$ . Without loss of generality, we may assume that  $\langle O_n \rangle$  is decreasing. Then  $mG = \lim_{n \rightarrow \infty} mO_n$ . By (a),

$$mO_n = \int_{g^{-1}[O_n]} g' = \int_a^b \chi_{O_n}(g(x)) g'(x) dx.$$

As  $\chi_{O_n}(g(x)) g'(x)$  is bounded by  $|g'|$ ,

$$\lim_{n \rightarrow \infty} \int_a^b \chi_{O_n}(g(x)) g'(x) dx = \int_a^b \chi_G(g(x)) g'(x) dx.$$

Hence,  $mG = \int_{g^{-1}[G] \cap H} g'$ , completing the proof.

(d) By Problem 3.25,  $f \circ g$  is measurable. And since  $g'$  is measurable by Theorem 3,  $(f \circ g)g'$  is also measurable.

Let  $\langle \varphi_n \rangle$  be an increasing sequence of nonnegative simple functions which converges to  $f$ , the existence of which is guaranteed by Problem 4.4. By the monotone convergence theorem,  $\int_c^d f = \lim \int_c^d \varphi_n$ .

For each  $n$ , suppose that  $\varphi_n(y) = \sum_{k=1}^m a_k^{(n)}(y) \chi_{E_k^{(n)}}(y)$ . Then

$$\int_c^d \varphi_n = \sum_{k=1}^m a_k^{(n)} mE_k^{(n)} = \sum_{k=1}^m a_k^{(n)} \int_a^b \chi_{E_k^{(n)}}(g(x)) g'(x) dx = \int_a^b \varphi_n(g(x)) g'(x) dx,$$

where the second equality comes from (c). Since  $g$  is increasing,  $\langle \varphi_n(g(x)) g'(x) \rangle$  is an increasing sequence. Hence,

$$\int_a^b f(g(x)) g'(x) dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_n(g(x)) g'(x) dx.$$

Thus,

$$\int_c^d f(y) dy = \lim_{n \rightarrow \infty} \int_c^d \varphi_n(y) dy = \lim_{n \rightarrow \infty} \int_a^b \varphi_n(g(x)) g'(x) dx = \int_a^b f(g(x)) g'(x) dx.$$

□

## 5.5 Convex Functions

23.

*Proof.*

(a) Suppose that  $x_0 \in (a, b)$  and  $y(x) = m(x - x_0) + \varphi(x_0)$  is a supporting line. As  $[a, b]$  is finite,  $\varphi \geq \min\{\varphi(a), y(a), y(b)\}$ .

(b) If  $\varphi$  is monotone, then the limits exists. If  $\varphi$  is not monotone, then since  $D^+\varphi$  is nondecreasing, there exists some  $[c, d] \subset (a, b)$  such that  $D^+\varphi \leq 0$  on  $(a, c)$  and  $D^+\varphi \geq 0$  on  $(d, b)$ . Namely,  $\varphi$  is monotone on the  $(a, c)$  and  $(d, b)$ . Therefore, the limits also exist.

Consider a finite interval near the finite endpoint. By (a), the limit can not be  $-\infty$  as  $\varphi$  is bounded from below.

(c) If  $x$  and  $y$  are in the interior of  $I$ , the inequality holds by definition. By the continuity of  $\varphi$ , the statement holds for all  $x, y \in I$ .  $\square$

24.

*Proof.* Note that the existence of  $\varphi''$  implies  $\varphi$  is continuously differentiable. Suppose that  $\varphi$  is convex on  $(a, b)$ . Then  $D^+\varphi$  is nondecreasing by Proposition 17, hence  $\varphi''(x) \geq 0$  for each  $x \in (a, b)$ . And the converse of the statement follows from Proposition 18 immediately.  $\square$

25.

*Proof.*

(a)  $\varphi''(t) = b^2 p(p-1)(a+bt)^{p-2}$  which  $\geq 0$  on  $[0, \infty)$  if  $p \geq 1$  and  $\leq 0$  if  $0 < p \leq 1$ .  $\square$

26. TODO

27.

*Proof.* Note that  $\log x$  is concave. Denote  $A_N = \sum_{n=1}^N \alpha_n$  and  $R_N = 1 - A_N$ . The situation where  $\langle \alpha_n \rangle$  is finite is simple. Hence we assume that  $R_N \geq 0$  for all  $N$ . Then for every  $N$ ,

$$\begin{aligned} \log \left( \sum_{n=1}^{\infty} \alpha_n \xi_n \right) &= \log \left( A_N \sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n + R_N \sum_{n=N+1}^{\infty} \frac{\alpha_n}{R_N} \xi_n \right) \\ &\geq A_N \log \left( \sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n \right) + R_N \log \left( \sum_{n=N+1}^{\infty} \frac{\alpha_n}{R_N} \xi_n \right) \\ &\geq A_N \log \left( \sum_{n=1}^N \frac{\alpha_n}{A_N} \xi_n \right) \\ &\geq A_N \log \left( \prod_{n=1}^N \xi_n^{\alpha_n/A_N} \right) \end{aligned}$$

Taking exp on the both sides yields

$$\sum_{n=1}^{\infty} \alpha_n \xi_n \geq \left( \prod_{n=1}^N \xi_n^{\alpha_n/A_N} \right)^{A_N} = \prod_{n=1}^N \xi_n^{\alpha_n} \rightarrow \prod_{n=1}^{\infty} \xi_n^{\alpha_n}.$$

$\square$

**28.**

*Proof.* It follows immediately from the Jensen inequality and the fact that  $\log$  is concave. □

## 6 The Classical Banach Spaces

### 6.1 The $L^p$ Spaces

1.

*Proof.* Put  $S = \|f\|_\infty$  and  $T = \|g\|_\infty$ . Then  $|f(t)| \leq S$  and  $|g(t)| \leq T$  a.e. Hence,  $S + T \geq |f(t)| + |g(t)| \geq |f(t) + g(t)|$  a.e. Namely,  $m\{t : |f(t) + g(t)| > S + T\} = 0$ . Thus,  $S + T \geq \|f + g\|_\infty$  by the definition of  $\text{ess sup}$ .  $\square$

2.

*Proof.* Put  $S = \|f\|_\infty$ . Since  $S \geq |f|$  a.e.,

$$\|f\|_p = \left\{ \int_0^1 |f|^p \right\}^{1/p} \leq \left\{ \int_0^1 S^p \right\}^{1/p} = S.$$

Therefore,  $\overline{\lim}_{p \rightarrow \infty} \|f\|_p \leq S$ . For the converse part, let  $\varepsilon$  be any positive number. Then the measure  $\delta$  of  $E = \{t : |f(t)| > S - \varepsilon\}$  is positive. Hence,

$$\left\{ \int_0^1 |f|^p \right\}^{1/p} \geq \left\{ \int_E |f|^p \right\}^{1/p} \geq \delta^{1/p} (S - \varepsilon) \rightarrow S - \varepsilon \quad \text{as } p \rightarrow \infty.$$

Hence,  $\underline{\lim}_{p \rightarrow \infty} \|f\|_p \geq S$ , completing the proof.  $\square$

3.

*Proof.*

$$\|f + g\|_1 = \int |f + g| \leq \int |f| + \int |g| = \|f\|_1 + \|g\|_1.$$

$\square$

4.

*Proof.* For every  $M > \|g\|_\infty$ ,  $|g| \leq M$  a.e. Hence,

$$\int |fg| \leq M \int |f| = \|f\|_1 M.$$

Since the choice of  $M$  is arbitrary,  $\int |fg| \leq \|f\|_1 \|g\|_\infty$ .  $\square$

### 6.2 The Minkowski and Hölder Inequalities

8

*Proof.*

(a) The logarithm function is concave, so

$$\log(a^p/p + b^q/q) \geq \frac{1}{p} \log a^p + \frac{1}{q} \log b^q = \log ab.$$

Taking  $\exp$  on the both sides yields the inequality. The equality holds iff  $a^p = b^q$ .

(b) The case where  $p = \infty$  has been proved in Problem 4 and the case where  $\|f\|_p = 0$  or  $\|g\|_q = 0$  is straightforward. Hence, we assume that  $1 < p, q < \infty$  and  $\|f\|_p \|g\|_q \neq 0$ .

Suppose  $\alpha = \|f\|_p$  and  $\beta = \|g\|_q$ . By Young's inequality,

$$\left| \frac{fg}{\alpha\beta} \right| \leq \frac{1}{p} \left( \frac{|f|}{\alpha} \right)^p + \frac{1}{q} \left( \frac{|g|}{\beta} \right)^q$$

for every  $x$ . Therefore,

$$\int |fg| = \alpha\beta \int \left| \frac{fg}{\alpha\beta} \right| \leq \alpha\beta \int \left\{ \frac{1}{p} \left( \frac{|f|}{\alpha} \right)^p + \frac{1}{q} \left( \frac{|g|}{\beta} \right)^q \right\} = \alpha\beta. \quad (4)$$

The equality holds iff the equality in Young's inequality holds a.e. iff  $\beta|f|^p = \alpha|g|^q$  a.e.

(c) Let  $p' = 1/p$  and  $q' = 1 - p' = -q/p$ . Then for any nonnegative  $c$  and  $d$ , by Young's inequality,

$$cd \leq \frac{c^{p'}}{p'} + \frac{d^{q'}}{q'} = pc^{1/p} - \frac{p}{q} d^{-q/p} \Rightarrow c^{1/p} \geq \frac{cd}{p} + \frac{d^{-q/p}}{q}.$$

Putting  $c = (ab)^p$  and  $d = b^{-p}$  yields the desired inequality.

(d) Just reverse the inequality in (4). □

## 6.3 Convergence and Completeness

9.

*Proof.* Suppose  $\langle f_n \rangle \subset X$  converges to  $f \in X$ . Namely, for every  $\varepsilon > 0$ , there exists some  $N$  such that for all  $n > N$ ,  $\|f_n - f\| < \varepsilon$ . Hence, for every  $n, m > N$ , by Minkowski inequality,

$$\|f_n - f_m\| \leq \|f_n - f\| + \|f - f_m\| < 2\varepsilon.$$

Hence,  $\langle f_n \rangle$  is a Cauchy sequence. □

10.

*Proof.* Suppose  $f_n \rightarrow f$ . Then  $M_n = \|f_n - f\|_\infty = \text{ess sup } |f_n - f| \rightarrow 0$ . Let  $E_n = \{x : |f_n(x) - f(x)| > M_n\}$ , each of which is with measure zero. And therefore  $E = \bigcup_{n=1}^\infty E_n$  is with measure zero. Note that  $\tilde{E} = \{x : |f_n(x) - f(x)| < M_n, \forall n\}$ , which implies the uniform convergence of  $f_n$  since  $M_n \rightarrow 0$ .

For the converse part, the uniform convergence on  $\tilde{E}$  implies that for every  $\varepsilon > 0$ , there exists some  $N$  such that for every  $n > N$  and  $x \in \tilde{E}$ ,  $|f_n(x) - f(x)| < \varepsilon$ . Since  $mE = 0$ , this implies  $\|f_n - f\|_\infty = \text{ess sup } |f_n(x) - f(x)| < \varepsilon$ . Hence,  $f_n \rightarrow f$  in  $L^\infty$ . □

11.

*Proof.* Let  $\langle f_n \rangle \subset L^\infty$  be absolutely summable. Put  $M_n = \|f_n\|_\infty$  and  $A_n = \{t : |f_n(t)| > M_n\}$ . By the definition of  $\|\cdot\|_\infty$ ,  $mA_n = 0$ . Hence,  $A = \bigcup_{n=1}^\infty A_n$  is of measure zero.

Note that  $|f_n(x)| \leq M_n$  for every  $n$  and  $x \in E \setminus A$ . Thus, by the Weierstrass M-test,  $\sum_{n=1}^\infty f_n$  converges uniformly. Hence, on  $E \setminus A$ ,  $\sup |\sum_{n=1}^\infty f_n - \sum_{n=1}^N f_n| \rightarrow 0$  as  $N \rightarrow \infty$ . Since  $mA = 0$ , this implies the summability of  $\langle f_n \rangle$ . □

**13.**

*Proof.* Suppose  $\langle f_n \rangle \subset C$  be absolutely summable. Since for every  $x$ ,  $0 \leq |f_n(x)| \leq \|f_n\|$ ,  $\langle f_n \rangle$  is uniformly convergent on  $[0, 1]$ . Put  $s = \sum_{n=1}^{\infty} f_n$ . Since each  $f_n$  is continuous, so is  $s$ . Therefore,  $s \in C$ .

For every  $\varepsilon > 0$ , there exists some  $N$  such that for every  $n > N$  and  $x \in [0, 1]$ ,  $|s(x) - \sum_{k=1}^n f_k(x)| < \varepsilon$ . Hence,  $\|s - \sum_{k=1}^n f_k\| < \varepsilon$ . Thus,  $\langle f_n \rangle$  is summable and therefore  $C$  is a Banach space.  $\square$

**16.**

*Proof.* Since  $\|f_n - f\| \geq |||f_n| - |f|||$ ,  $f_n \rightarrow f$  in  $L^p$  implies  $\|f_n\| \rightarrow \|f\|$ . For the converse part, note that  $2^p(|f_n|^p + |f|^p) - |f_n - f|^p \geq 0$  and for almost every  $x$ ,

$$2^p(|f_n|^p + |f|^p) - |f_n - f|^p \rightarrow 2^{p+1}|f|^p.$$

By Fatou's Lemma,

$$\begin{aligned} 2^{p+1}\|f\|^p &= 2^{p+1} \int |f|^p \leq \underline{\lim} \int \{2^p(|f_n|^p + |f|^p) - |f_n - f|^p\} \\ &= 2^{p+1}\|f\|^p - \overline{\lim} \|f_n - f\|^p. \end{aligned}$$

Hence,  $\overline{\lim} \|f_n - f\|^p \leq 0$ . Since clear that  $\underline{\lim} \|f_n - f\|^p \geq 0$ ,  $\lim \|f_n - f\| = 0$ , i.e.,  $f_n \rightarrow f$  in  $L^p$ .  $\square$

**17.** I assume that  $1/p + 1/q = 1$ .

*Proof.* Since  $g \in L^p$ ,  $|g|^q$  is integrable on  $E = [0, 1]$  and therefore for every  $\varepsilon > 0$ , there exists some  $\delta$  such that for every  $A \subset E$  with  $mA < \delta$ ,  $\int_A |g|^q < \varepsilon$ . Meanwhile, since  $f_n(x) \rightarrow f(x)$  for almost every  $x$ , by Egoroff's Theorem, there exists some  $A \subset E$  with  $mA < \delta$  such that  $f_n g$  converges to  $f g$  uniformly on  $E \setminus A$ .

From the uniform convergence we conclude

$$\int_{E \setminus A} f g = \lim_{n \rightarrow \infty} \int_{E \setminus A} f_n g. \quad (5)$$

Meanwhile, by Hölder inequality,

$$\left| \int_A (f - f_n) g \right| \leq \int_A |(f - f_n) g| \leq \left\{ \int_A |f_n - f|^p \right\}^{1/p} \left\{ \int_A |g|^q \right\}^{1/q} \leq M \varepsilon^{1/q}.$$

Hence, (5) can be extended to  $E$ .

For  $p = 1$ , this is not true.  $f_n = n\chi_{[0, 1/n]}$  and  $g = \chi_{[0, 1]}$  gives a counterexample.  $\square$

**18.**

*Proof.* By Minkowski inequality,

$$\|g_n f_n - g f\| = \|g_n(f_n - f) + (g_n - g)f\| \leq \|g_n(f_n - f)\| + \|(g_n - g)f\|.$$

Fix  $\varepsilon > 0$ . Since  $f, g_n, g \in L^p$ ,  $|g_n - g|^p |f|^p$  is integrable and therefore there exists some  $\delta > 0$  such that for all subsets with measure  $< \delta$ , the integral of over it  $< \varepsilon$ . Meanwhile,

since  $g_n \rightarrow g$  a.e., by Egoroff's Theorem, there exists some  $A \subset E = [0, 1]$  with  $mA < \delta$  such that  $g_n \rightarrow g$  uniformly on  $E \setminus A$  and therefore there exists some  $N_1 > 0$  such that for all  $n > N_1$ ,  $|g_n(x) - g(x)|^p < \varepsilon$  for  $x \in E \setminus A$ . Thus, for every  $n > N_1$ ,

$$\begin{aligned} \|(g_n - g)f\| &= \left\{ \int_{E \setminus A} |g_n - g|^p |f|^p \right\}^{1/p} + \left\{ \int_A |g_n - g|^p |f|^p \right\}^{1/p} \\ &\leq \sqrt[p]{\varepsilon} \|f\| + \sqrt[p]{\varepsilon} \leq (\|f\| + 1)\varepsilon. \end{aligned}$$

Since  $|g_n| \leq M$ ,  $\|g_n(f_n - f)\| \leq M\|f_n - f\|$ . And since  $f_n \rightarrow f$  in  $L^p$ , there exists some  $N_2 > 0$  such that for all  $n > N_2$ ,  $\|f_n - f\| < \varepsilon$ . Put  $N = \max(N_1, N_2)$ , then for every  $n > N$ ,

$$\|g_n f_n - g f\| \leq (\|f\| + 1 + M)\varepsilon.$$

Hence,  $g_n f_n \rightarrow g f$  in  $L^p$ . □

## 6.4 Approximation in $L^p$

19.

*Proof.* Since  $\|T_\Delta f\| \leq \|T_\Delta |f|\|$  and  $\|f\| = \||f|\|$ , we may assume without loss of generality that  $f \geq 0$ . For  $p > 1$ , by Jensen's inequality,

$$\begin{aligned} \|T_\Delta f\|_p^p &= \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} \left( \frac{1}{\xi_k - \xi_{k-1}} \int_{\xi_{k-1}}^{\xi_k} f \right)^p \\ &\leq \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} \frac{1}{\xi_k - \xi_{k-1}} \int_{\xi_{k-1}}^{\xi_k} f^p \\ &= \sum_{k=1}^m \int_{\xi_{k-1}}^{\xi_k} f^p \\ &= \int_0^1 f^p = \|f\|_p^p. \end{aligned}$$

□

## 7 Metric Spaces

### 7.7 Compact Metric Space

27.

*Proof.* If  $\rho(F, K) > 0$ , then clear that  $F \cap K = \emptyset$ . For the reverse direction, consider the function  $h(x) = \rho(x, F) = \inf_{y \in F} \rho(x, y)$ . Clear that for  $x \in K$ ,  $h(x) \leq \rho(K, F)$ .

First, we show that  $h$  is continuous. Let  $x$  be fixed. For every  $x' \in X$

$$h(x) \leq \rho(x, y) \leq \rho(x, x') + \rho(x', y), \quad \forall y \in F.$$

Take infimum on the right hand side and we get

$$h(x) \leq \rho(x, x') + h(x') \quad \Rightarrow \quad h(x) - h(x') \leq \rho(x, x').$$

Similarly, we have  $h(x') - h(x) \leq \rho(x, x')$ . Thus,  $h$  is continuous.

Since  $K$  is compact and  $h$  is continuous,  $h$  attains its infimum  $c$  at some point  $x_0 \in K$ . Assume, to obtain a contradiction, that  $c = 0$ . Then, for every  $\varepsilon > 0$ , there is a  $y \in F$  s.t.  $\rho(x_0, y) < \varepsilon$ . Namely,  $x_0$  is a cluster point of  $F$ . Since  $F$  is closed,  $x_0 \in F$ , which contradicts with  $F \cap K = \emptyset$ . Thus,  $c > 0$  and therefore  $\rho(F, K) > 0$ .  $\square$

28.

*Proof.*

(a) Let  $\varepsilon > 0$  be fixed. Since  $f$  is uniformly continuous, there exists some  $\delta > 0$  s.t. for every  $x_1, x_2 \in X$  with  $\rho(x_1, x_2) < \delta$ ,  $\rho(f(x_1), f(x_2)) < \varepsilon$ . Let  $B \subset X$  be a ball with radius  $\delta/2$ . Then  $f(B) \subset Y$  is contained by some ball of radius  $\varepsilon$ . Hence, we can cover  $Y$  with finitely many balls of radius  $\varepsilon$  as long as we can cover  $X$  with finitely many balls of radius  $\delta/2$  and this can be done since  $X$  is totally bounded.

(b) Clear that  $X = (0, 1)$  is totally bounded while  $Y = (1, \infty)$  is not. The function  $1/x$  a continuous function that maps  $X$  onto  $Y$ . Hence, the result does not hold.  $\square$

29.

*Proof.*

(a) First, by the definition of open cover and open sets,  $\varphi(x) > 0$ . Meanwhile, since  $X$  is compact, it is bounded and, therefore,  $\varphi(x) < \infty$ .

(b) Let  $x$  be fixed and  $r$  be such that there exists some  $O \in \mathcal{U}$  with  $B_{x,r} \subset O$ . Let  $s = r - \rho(x, y)$ . If  $s \leq 0$ , then clear that  $s \leq \varphi(y)$ . If  $s > 0$ , then we have  $B_{y,s} \subset B_{x,r} \subset O$ . Hence,  $s \leq \varphi(y)$ . Thus,  $r - \rho(x, y) \leq \varphi(y)$ . Take supremum on the left hand side and we get  $\varphi(x) - \rho(x, y) \leq \varphi(y)$ .

(c) It follows immediately from (b).

(d) Let  $(x_n)$  be a sequence s.t.  $\varphi(x_n) \rightarrow \varepsilon$ . Since  $X$  is sequentially compact,  $(x_n)$  has a convergent subsequence. For the sake of convenience, we assume the subsequence is  $(x_n)$  itself. Suppose  $x_n \rightarrow x$ . Since  $\varphi$  is continuous,  $\varphi(x) = \lim \varphi(x_n) = \varepsilon$ . Thus, by (a),  $\varepsilon > 0$ .

(e) Let  $\delta$  be any positive number that is less than  $\varepsilon$ . For every  $x$ , since  $\delta < \varepsilon = \inf \varphi$ ,  $\delta < \varphi(x)$  and, therefore,  $B_{x,\delta} \subset O$  for some  $O \in \mathcal{U}$ .  $\square$



## 7.8 Baire Category

### 31.(a)

*Proof.* Suppose that  $F$  is nowhere dense. Since  $F$  is closed,  $F^c$  is dense. Therefore, for every point  $x \in F$ , every neighborhood of  $x$  contains a point of  $F^c$ . Thus,  $F$  contains no open set. For the reverse direction, since  $F$  contains no open set, every neighborhood of every  $x \in X$  contains a point in  $F^c$ , which implies that  $F^c$  is dense.  $\square$

### 34.

*Proof.* For the first part, it suffices to show that  $\partial E$  is nowhere dense for any set  $E \subset X$ . Since  $\partial E$  is closed and contains no open set, by 31.(a), it is nowhere dense.

For the second part, assume, to obtain a contradiction, that  $F$  is not nowhere dense. Then by 31.(a), it contains a nonempty open set  $O$ . By 32.(a),  $O$  is also meager, which contradicts the Baire category theorem.  $\square$

### 35.

*Proof.* Let  $E$  be a subset set of the complete metric space  $X$ . If  $E$  is residual, then  $E^c = \bigcup_n K_n$  where  $K_n$  are nowhere dense sets. Hence,

$$E = \bigcap K_n^c \supset \bigcap (\text{cl } K_n)^c.$$

Note that  $(\text{cl } K_n)^c$  is a  $G_\delta$ . Meanwhile, since each  $(\text{cl } K_n)^c$  is a dense open set and  $X$  is complete, By the theorem of Baire,  $\bigcap (\text{cl } K_n)^c$  is also dense. The proof of the reverse direction is similar.  $\square$

### 39.

*Proof.* Let  $E_m$  has the same meaning as in the proof of Theorem 32. Let  $O := \bigcup_m E_m^\circ$ . By Prop. 31,  $O$  is a dense residual open set. For every  $x \in O$ ,  $x \in E_m^\circ$  for some  $m$ . Since  $E_m^\circ$  is open, there is a neighborhood  $U$  of  $x$  which is contained by  $E_m^\circ$ . Thus,  $\mathcal{F}$  is uniformly bounded by  $m$  in  $U$ .  $\square$

## 7.10 The Ascoli-Arzelá Theorem

### 47.

*Proof.* Let  $x$  be an arbitrary fixed point in  $X$  and  $(x_n) \subset X$  a sequence that converges to  $x$ . Clear that  $K = \{x\} \cup (x_n)$  is (sequently) compact. Hence,  $f_n$  converges to  $f$  uniformly on  $K$ . Thus,  $f$  is also continuous on  $K$  and therefore at  $x$ . Namely,  $f$  is continuous.  $\square$

### 49.

*Proof.* Let  $x$  be an arbitrary point in  $X$ . For every  $\varepsilon > 0$ , since  $\mathcal{F}$  is equicontinuous, there is an open neighborhood  $O$  of  $x$  s.t.  $\sigma(f(x), f(y)) < \varepsilon$  for every  $y \in O$  and  $f \in \mathcal{F}$ . Now, for every  $f^+ \in \mathcal{F}^+$ , suppose  $f_n \rightarrow f$ . We have

$$\sigma(f^+(x), f^+(y)) = \sigma\left(\lim_n f_n(x), \lim_m f_m(y)\right) = \lim_n \lim_m \sigma(f_n(x), f_m(y)) \leq \varepsilon,$$

where the second equality comes from the continuity of  $\sigma$  and the last inequality comes from the equicontinuity of  $\mathcal{F}$ .  $\square$

**50.** I assume that the norm on  $C[0, 1]$  is the sup norm.

*Proof.* Let  $\mathcal{F} := \{f \in C[0, 1] \mid \|f\|_\alpha \leq 1\}$ . First, we show that  $\mathcal{F}$  is equicontinuous. By the definition of  $\|\cdot\|_\alpha$ ,  $f \in \mathcal{F}$  only if it is bounded by 1 and  $|f(x) - f(y)| \leq |x - y|^\alpha$  for all  $x, y \in [0, 1]$ . For every  $x \in X$  and  $\varepsilon > 0$ , consider the  $\sqrt[\alpha]{\varepsilon}$ -ball  $B$  centered at  $x$ . For every  $y \in B$ ,  $|f(x) - f(y)| \leq |x - y|^\alpha < \varepsilon$ . Hence,  $\mathcal{F}$  is equicontinuous. Let  $(f_n)$  be a sequence in  $\mathcal{F}$ . Since  $f_n$  is bounded by 1, by Corollary 41,  $(f_n)$  contains a convergent subsequence. Thus,  $\mathcal{F}$  is a (sequentially) compact subset of  $C[0, 1]$ .  $\square$

## 8 Topological Spaces

### 8.1 Fundamental Notions

3.

*Proof.* If  $A$  is open, then clear that  $x \in A \subset A$ . For the converse, let  $E = \bigcup_{x \in A} O_x$  where  $x \in O_x \subset A$ . Since  $O_x \subset A$ ,  $E \subset A$ . Meanwhile, for every  $x \in A$ ,  $x \in O_x \subset E$ . Hence,  $A = E$ . Thus,  $A$  is open since  $E$  is the union of open sets.  $\square$

7.

*Proof.*

(a) We argue by contradiction. Assume  $x \in F^c$ . Since  $F$  is closed,  $F^c$  is open and therefore there is neighborhood  $O$  of  $x$  s.t.  $O \cap F \neq \emptyset$ , which contradicts the fact that  $x$  is a cluster point. Thus,  $x \in F$ .

(b) Let  $y := f(x)$  and  $y_n := f(x_n)$ . Let  $O$  be an arbitrary neighborhood of  $y$ . Since  $f$  is continuous, there is a neighborhood  $U$  of  $x$  s.t.  $f(U) \subset O$ . Since  $x = \lim x_n$ , there is an integer  $N$  s.t.  $x_n \in U$  for every  $n > N$ . Hence,  $y_n = f(x_n) \in f(U) \subset O$  for all  $n > N$ . Thus,  $y_n \rightarrow y$ .

(c) The previous argument, *mutatis mutandis*, yields the result.  $\square$

10.

*Proof.*

(a) Suppose that both  $A_1$  and  $A_2$  are open. Let  $f_1 := f|_{A_1}$ ,  $x \in A$  and  $y = f(x)$ . We may assume without loss of generality that  $x \in A_1$ . For every neighborhood  $O$  of  $y$ , since  $f_1$  is continuous, there is a neighborhood  $A_1 \cap U$  s.t.  $f(A_1 \cap U) \subset O$  where  $U$  is an open set of  $X$ . Since  $A_1 \cap U$  is still open,  $f$  is continuous at  $x$ . Thus,  $f$  is continuous.

(b) Let  $f(x) = 0$  on  $A_1 = (-1, 0)$  and  $f(x) = 1$  on  $A_2 = [0, 1]$ .  $\square$

# 11 Measure and Integration

## 11.1 Measure Spaces

1.

*Proof.* Put  $B_1 = A_1$  and  $B_n = A_n \setminus A_{n-1}$  for  $n \geq 2$ .  $(B_n)$  is a sequence of disjoint measurable sets. By the countable additivity of  $\mu$ ,

$$\mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(B_k) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=1}^n B_k\right).$$

Since  $\bigcup_{k=1}^n B_k = \bigcup_{k=1}^n A_k$  for  $k = 1, \dots, n, \dots, \infty$ , this implies  $\mu(\bigcup A_k) = \lim \mu(\bigcup_{k=1}^n A_k)$ .  $\square$

3.

*Proof.*

(a) First,

$$0 = \mu(E_1 \triangle E_2) = \mu(E_1 \setminus E_2 \cup E_2 \setminus E_1) = \mu(E_1 \setminus E_2) + \mu(E_2 \setminus E_1).$$

Together with the nonnegativity of  $\mu$ , we conclude that  $\mu(E_1 \setminus E_2) = \mu(E_2 \setminus E_1) = 0$ . Note that

$$\mu(E_1 \cup E_2) = \mu(E_1 \setminus E_2 \cup E_2) = \mu(E_1 \setminus E_2) + \mu(E_2).$$

Hence,  $\mu(E_1 \cup E_2) = \mu(E_2)$ . Similarly,  $\mu(E_1 \cup E_2) = \mu(E_1)$ . Thus,  $\mu(E_1) = \mu(E_2)$ .

(b) Since  $\mu(E_1 \triangle E_2) = 0$  and  $E_2 \setminus E_1 \subset E_1 \triangle E_2$ , by the completeness of  $\mu$ ,  $E_2 \setminus E_1 \in \mathcal{B}$ . Similarly,  $E_1 \setminus E_2 \in \mathcal{B}$ . In consequence,  $E_1 \cap E_2 = E_1 \setminus (E_1 \setminus E_2) \in \mathcal{B}$  and, therefore,  $E_2 = (E_1 \cap E_2) \cup (E_2 \setminus E_1) \in \mathcal{B}$ .  $\square$

7.

*Proof.* Let  $\mathcal{B}_0$  be the collection of all sets  $E = A \cup B$  where  $B \in \mathcal{B}$  and  $A \subset C$ ,  $C \in \mathcal{B}$ ,  $\mu C = 0$ . Clear that  $\mathcal{B} \subset \mathcal{B}_0$ . Now we show that it is a  $\sigma$ -algebra. Since  $X \in \mathcal{B}$ ,  $X \in \mathcal{B}_0$ . Let  $E_n = A_n \cup B_n$  be a sequence of elements of  $\mathcal{B}_0$ . Then,  $\bigcup E_n = (\bigcup A_n) \cup (\bigcup B_n)$  also belongs to  $\mathcal{B}_0$  since  $\bigcup B_n \in \mathcal{B}$  and  $\bigcup A_n \subset \bigcup C_n$ , which is a countable union of sets of measure zero. Hence,  $\mathcal{B}_0$  is closed under countable union. Now, let  $E = A \cup B \in \mathcal{B}_0$ . Note that

$$E^c = A^c \cap B^c = (C \setminus A) \cup (B^c \setminus C),$$

where  $C \setminus A \subset C$  and  $B^c \setminus C \in \mathcal{B}$ . Hence,  $\mathcal{B}_0$  is closed under complement. Thus, it is a  $\sigma$ -algebra.

We define  $\mu_0 : \mathcal{B}_0 \rightarrow [0, \infty]$  by  $\mu_0 E = \mu_0(A \cup B) = \mu B$ . First, we show that it is well-defined, that is, if  $E = A' \cup B'$ , then  $\mu B = \mu B'$ . Since  $C \in \mathcal{B}$  contains  $A$ ,  $(A \cup B) \setminus C \in \mathcal{B}$ . Meanwhile, since  $\mu C = 0$ ,

$$\mu B = \mu((A \cup B) \setminus C) = \mu(E \setminus C). \quad (6)$$

Since  $E \setminus C \subset E \cup C'$ ,

$$\mu(E \setminus C) \leq \mu(E \cup C') = \mu((A' \cup B') \cup C') = \mu B', \quad (7)$$

where the measurability of  $E \cup C'$  and the last equality both comes from the fact that  $A' \subset C' \in \mathcal{B}$  and  $\mu C' = 0$ . Combine (6) and (7) and we get  $\mu B \leq \mu B'$ . Interchanging the role of  $A \cup B$  and  $A' \cup B'$  yields  $\mu B \geq \mu B'$ . Hence,  $\mu B = \mu B'$  and, in consequence,  $\mu_0$  is well-defined. Meanwhile, clear that for  $E \in \mathcal{B}$ ,  $\mu E = \mu_0 E$ .

Finally, we show that  $\mu_0$  is a measure. Clear that  $\mu_0$  is nonnegative and  $\mu_0 \emptyset = 0$ . Let  $\langle E_n \rangle \subset \mathcal{B}_0$  be a sequence of disjoint sets. Then

$$\mu_0 \left( \bigcup E_n \right) = \mu_0 \left( \bigcup A_n \cup \bigcup B_n \right) = \mu \left( \bigcup B_n \right) = \sum \mu B_n = \sum \mu_0 E_n.$$

Namely,  $\mu_0$  is countably additive. Thus,  $\mu_0$  is a measure.  $\square$

## 9.

*Proof.*

(a) First, we argue by contradiction to show that  $\mathcal{R}$  and  $\mathcal{R}'$  are disjoint. Assume that there exists some  $E \in \mathcal{R} \cap \mathcal{R}'$ , that is,  $E \in \mathcal{R}$  and  $E^c \in \mathcal{R}$ . Then  $X = E \cup E^c \in \mathcal{R}$ , which contradicts the assumption that  $\mathcal{R}$  is not a  $\sigma$ -algebra. Thus,  $\mathcal{R} \cap \mathcal{R}' = \emptyset$ .

Clear that  $\mathcal{R} \cup \mathcal{R}'$  is a  $\sigma$ -algebra containing  $\mathcal{R}$ . Hence,  $\mathcal{R} \cup \mathcal{R}' \supset \mathcal{B}$ . Meanwhile, since  $\mathcal{B} = \sigma(\mathcal{R})$ ,  $\mathcal{R} \cup \mathcal{R}' \subset \mathcal{B}$ . Thus,  $\mathcal{R} \cup \mathcal{R}' = \mathcal{B}$ .

(b) Since  $\emptyset \in \mathcal{R}$ ,  $\bar{\mu} \emptyset = \mu \emptyset = 0$ . Meanwhile, clear that  $\bar{\mu}$  is nonnegative. Let  $\langle E_n \rangle \subset \mathcal{B}$  be a sequence of disjoint sets. By part (a), each  $E_n$  is either an element of  $\mathcal{R}$  or  $\mathcal{R}'$ . If all  $E_n \in \mathcal{R}$ , then by the countable additivity of  $\mu$ ,  $\mu(\bigcup E_n) = \sum \mu E_n$ . Suppose there exists some  $E_n$  in  $\mathcal{R}$  and some  $E_m$  in  $\mathcal{R}'$ . Let  $F_1$  and  $F_2$  be the union of these sets respectively. Since  $\sigma$ -ring is closed under union,  $F_1 \in \mathcal{R}$ , and since  $(\bigcup E_m)^c = \bigcap E_m^c$ ,  $F_2 \in \mathcal{R}'$ . Hence,  $F_1 \cup F_2 \in \mathcal{R}'$ , otherwise,  $F_2 = (F_1 \cup F_2) \setminus F_1$  would be an element of  $\mathcal{R}$ . Therefore,  $\mu(\bigcup E_n) = \infty = \sum \mu E_n$ . Thus,  $\bar{\mu}$  is a measure on  $\mathcal{B}$ .

(c) Clear that  $\mu$  is nonnegative and  $\mu \emptyset = 0$ . Let  $\langle E_n \rangle \subset \mathcal{B}$  be disjoint. Note that for  $E \in \mathcal{R}$ ,  $\mu E = \sup \{ \mu A : A \subset E, A \in \mathcal{R} \}$ . Hence, it suffices to show that

$$M = \sup \left\{ \mu A : A \subset \bigcup_n E_n, A \in \mathcal{R} \right\} = \sum_n \sup \{ \mu A : A \subset E_n, A \in \mathcal{R} \} = \sum_n M_n.$$

By definition, for all  $\varepsilon > 0$ , there exists a sequence  $\langle A_n \rangle \subset \mathcal{R}$  such that  $A_n \subset E_n$  and  $M_n < \mu A_n + \varepsilon/2^n$ . Put  $A = \bigcup A_n$ . Since  $\langle A_n \rangle$  are disjoint as  $\langle E_n \rangle$  are,

$$\sum M_n < \varepsilon + \sum \mu A_n = \varepsilon + \mu A.$$

Meanwhile, since  $A \subset \bigcup E_n$  and  $A \in \mathcal{R}$ ,  $\mu A \leq M$ . Therefore,  $\sum M_n < \varepsilon + M$ . Thus,  $\sum M_n \leq M$ .

For the converse, similarly, for every  $\varepsilon > 0$ , there exists an  $A \in \mathcal{R}$  such that  $A \subset \bigcup E_n$  and  $M - \varepsilon > \mu A$ . Put  $A_n = E_n \cap A$ . If  $E_n \in \mathcal{R}$ ,  $A_n \in \mathcal{R}$  by definition. If  $E_n \in \mathcal{R}'$ ,  $A_n = A \setminus E_n^c \in \mathcal{R}$ . Hence,  $A_n \in \mathcal{R}$  for each  $n$ . Thus,

$$M - \varepsilon < \mu A = \sum_n \mu A_n \leq \sum_n M_n,$$

implying that  $M \leq \sum M_n$ . Therefore,  $M = \sum M_n$ , i.e.,  $\mu$  is countably additive. Thus, we conclude that  $\mu$  is a measure on  $\mathcal{B}$ .

(d) Clear that  $\mu_\beta$  is nonnegative and  $\mu_\beta \emptyset = 0$ . The preceding discussion, *mutatis mutandis*, yields the countable additivity.  $\square$

## 11.2 Measurable Functions

10.

*Proof.* For every integers  $n$  and  $k$ , let

$$\begin{aligned} E_{n,k} &= \{x : k2^{-n} \leq f(x) < (k+1)2^{-n}\}, (k \leq 2^{2n}) \\ E_{n,2^{2n}+1} &= \{x : f(x) \geq (2^{2n}+1)2^{-n}\}, \\ \varphi_n &= 2^{-n} \sum_{k=0}^{2^{2n}+1} k \chi_{E_{n,k}} \end{aligned}$$

Since  $f$  is measurable, all  $E_{n,k}$  are measurable. Thus,  $\langle \varphi_n \rangle$  is a sequence of nonnegative simple functions. Clear that for fixed  $n$ ,  $\langle E_{n,k} \rangle_k$  are disjoint. Let  $x \in X$  be fixed. If  $x \in E_{n,k}$  for some  $k \leq 2^{2n}$ , then  $x \in E_{n+1,2k} \cup E_{n+1,2k+1}$ . Hence,  $\varphi_{n+1}(x) \geq 2k/2^{-(n+1)} = \varphi_n(x)$ . If  $x \in E_{n,2^{2n}+1}$ , then  $x \in E_{n+1,k'}$  for some  $k' \geq 2^{2n+2}$ . Hence,  $\varphi_{n+1}(x) \geq 2k'/2^{-(n+1)} = \varphi_n(x)$ . Thus,  $\varphi_{n+1} \geq \varphi_n$  for all  $n$ .

Now, we show that  $\varphi_n$  converges to  $f$  pointwisely. Let  $x \in X$  be fixed. If  $f(x) = \infty$ , then  $\varphi_n(x) = 2^{-n}(2^{2n}+1) \rightarrow \infty$  as  $n \rightarrow \infty$ . If  $f(x) < \infty$ , then  $f(x) < 2^N$  for some integer  $N$ . For all  $n > N$ ,  $x \in E_{n,k_n}$  where  $k_n = \lfloor 2^n f(x) \rfloor$ . Thus,

$$f(x) - \varphi_n(x) = f(x) - 2^{-n} \lfloor 2^n f(x) \rfloor \rightarrow 0$$

as  $n \rightarrow \infty$ . Namely,  $\varphi_n(x) \rightarrow f(x)$ .

If the measure space is  $\sigma$ -finite, then let  $(X_n) \subset X$  be a sequence of measurable sets such that  $X_n \subset X_{n+1}$ ,  $\mu X_n < \infty$  and  $X = \bigcup X_n$ . Replacing  $E_{n,k}$  with  $E_{n,k} \cap X_n$  yields a sequence  $\langle \varphi_n \rangle$  satisfying all previous requirements and vanishing outside  $X_n$  for each  $n$ .  $\square$

11.

*Proof.* Put  $F_\alpha = \{x : f(x) \leq \alpha\}$ ,  $G_\alpha = \{x : g(x) \leq \alpha\}$ ,  $E = \{x : f(x) \neq g(x)\}$  and  $E_\alpha = \{x \in E : g(x) \leq \alpha\}$ . Then  $G_\alpha = (F_\alpha \setminus E) \cup E_\alpha$ . Since  $F$  is measurable, all  $F_\alpha$  are measurable. Since  $f = g$  a.e.,  $E$  is of measure zero. Meanwhile, since  $\mu$  is complete,  $E_\alpha \subset E$  is measurable. Thus,  $G_\alpha$  is measurable. Namely,  $g$  is measurable.  $\square$

13.

*Proof.* Note that  $f_n$  converges to  $f$  in measure iff for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mu\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} = 0.$$

(a) By definition, for every  $\varepsilon_m = 2^{-m}$ , there exists some integer  $N_m$  such that for all  $n \geq N_m$ ,  $\mu\{x : |f_n(x) - f(x)| \geq \varepsilon_m\} < \varepsilon_m$ . Consider the subsequence  $\langle f_{N_m} \rangle_m$ . We show that it converges to  $f$  almost everywhere. Put  $E_m = \{x : |f_{N_m} - f(x)| \geq \varepsilon_m\}$  and  $E = \limsup E_m$ . Then, for each  $k$ ,

$$\mu E \leq \bigcup_{m=k}^{\infty} E_m \leq \sum_{m=k}^{\infty} 2^{-m+1} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

For every  $x \notin E$ ,  $x \notin \bigcup_{m=k}^{\infty} E_m$  for some  $k$ . Then for all  $m > k$ ,  $|f_{N_m}(x) - f(x)| < \varepsilon_{N_m}$ . Hence,  $f_{N_m}(x) \rightarrow f(x)$ . Namely,  $f_{N_m} \rightarrow f$  almost everywhere.

(b) First we prove a lemma: Let  $\langle E_n \rangle$  be a sequence of measurable subset of  $A$ . Then  $\limsup \mu E_n \leq \mu(\limsup E_n)$ . Let  $F_N = \bigcup_{n=N}^{\infty} E_n$ . Clear that  $F_{n+1} \subset F_n$  and  $\mu F_1 < \infty$ . Hence, by Prop. 2,

$$\limsup \mu E_n \leq \lim \mu F_n = \mu \left( \bigcap_{n=1}^{\infty} F_n \right) = \mu(\limsup E_n).$$

Thus, the lemma holds.

For fixed  $\varepsilon > 0$ , let  $E_n = \{x \in A : |f_n(x) - f(x)| \geq \varepsilon\}$ . We show that  $\lim \mu E_n = 0$ . First, clear that  $0 \leq \limsup \mu E_n$ . Meanwhile, if  $x \in \limsup E_n$ , then  $x$  belongs to infinitely many  $E_n$ . As a consequence,  $f_n$  does not converges to  $f$  at  $x$ . Since  $f_n$  converges to  $f$  a.e.,  $\mu(\limsup E_n) = 0$ . Note that all  $E_n \subset A$  are of finite measure. Hence, by the preceding lemma,  $\limsup \mu E_n \leq 0$ . Thus,  $\lim \mu E_n = 0$ . Let  $F_n$  denote  $\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$  and  $G$  the collection of points at which  $f_n$  does not converge to  $f$ . Since all  $f_n$  vanishes outside  $A$ , for a point outside to belong to  $F_n$ , it has to belong to  $G$ , a set of measure zero. Therefore,  $E_n \subset F_n \subset E_n \cup G$ , implying that  $\mu F_n = \mu E_n$ . Thus,  $f_n$  converges to  $f$  in measure.

(c) By definition, for each positive integer  $k$ , there is an integer  $N_k$  such that for all  $n, m \geq N_k$ ,  $\mu\{x \in X : |f_n(x) - f_m(x)| \geq 2^{-k}\} < 2^{-k}$ . We may assume without loss of generality that  $N_k$  is increasing. Put  $E_k = \{x : |f_{N_{k+1}}(x) - f_{N_k}(x)| \geq 2^{-k}\}$  and  $E = \limsup E_k$ . By our construction,  $\mu E = 0$ . For  $x \notin E$ ,  $|f_{N_{k+1}}(x) - f_{N_k}(x)| < 2^{-k}$  for large  $k$  and, therefore, the number series  $\sum (f_{N_{k+1}}(x) - f_{N_k}(x))$  converges to some point, say,  $g(x)$ . Hence,  $f_{N_k}$  converges to  $f = f_{N_1} + g$  almost everywhere. Since all  $f_{N_k}$  are measurable,  $f$  is measurable.

Now we show that  $f_n$  converges to  $f$  in measure. Let  $D$  be the set of points at which  $f_{N_k}$  does not converge to  $f$ . For every  $\varepsilon > 0$ , let  $F_n = \{x \in X \setminus D : |f_n(x) - f(x)| \geq \varepsilon\}$ . Note that for all sufficiently large  $N_k$ ,

$$\begin{aligned} F_n &\subset \{x \in X \setminus D : |f_n(x) - f_{N_k}(x)| + |f_{N_k}(x) - f(x)| \geq \varepsilon\} \\ &\subset \{x \in X \setminus D : |f_n(x) - f_{N_k}(x)| \geq \varepsilon/2\}, \end{aligned}$$

where the measure of the last set can be less than  $\varepsilon$  for sufficiently large  $n$  and  $N_k$  as  $\langle f \rangle$  is Cauchy in measure. Since  $D$  is of measure zero, we conclude that  $\langle f_n \rangle$  converges to  $f$  in measure.  $\square$

## 16.

*Proof.* Egoroff: Let  $(X, \mathcal{B}, \mu)$  be a measure space and  $E \subset X$  is of finite measure. Let  $\langle f_n \rangle$  be a sequence of measurable functions which converge to some function  $f$  a.e. on  $E$ . Then for every  $\eta > 0$ , there is a subset  $A \subset E$  with  $\mu A < \eta$  such that  $f_n$  converges to  $f$  uniformly on  $E \setminus A$ .

We may assume without loss of generality that all  $f_n$  vanish outside  $E$ . Then, by Prob. 13(b),  $f_n$  converges to  $f$  in measure over  $E$ . Fix  $\eta > 0$ . First, we construct  $A$ . Put  $\delta_m = \delta/2^m$ . For every  $m$ , there exists some integer  $N_m$  and a measurable set  $A_m$  with  $\mu A_m < \delta_m$  such that for all  $n > N_m$  and  $x \notin A_m$ ,  $|f_n(x) - f(x)| < \delta_m$ . Put  $A = \bigcup A_m$ . Clear that  $\mu A < \delta$ .

Now we show that  $f_n$  converges to  $f$  uniformly on  $E \setminus A$ . Fix  $x \in E \setminus A$ . For every  $\varepsilon > 0$ , suppose there is an  $m$  such that  $0 < \delta_m < \varepsilon$ . For all  $n > N_m$ , since  $x \notin A$ ,  $|f_n(x) - f(x)| < \delta_m < \varepsilon$ . Thus,  $f_n \rightarrow f$  uniformly on  $E \setminus A$ .  $\square$

## 11.3 Integration

19.

*Proof.* Since  $|\int_E f| \leq \int_E |f|$ , it suffices to show the result for nonnegative  $f$ . Fix  $\varepsilon > 0$ . By definition, there is a nonnegative simple function  $\varphi = \sum_{i=1}^n c_i \chi_{E_i}$  such that  $\int f < \int \varphi + \varepsilon/2$ . Put  $M = \max_i c_i$  and  $\delta = \varepsilon/2Mn$ . Then, for every measurable  $E$  with  $\mu E < \delta$ , we have

$$\int_E f < \int_E \varphi + \varepsilon/2 = \sum_{i=1}^n c_i \mu(E_i \cap E) + \varepsilon/2 \leq Mn\delta + \varepsilon/2 = \varepsilon.$$

□

20.

*Proof.* We show here Fatou's Lemma: Let  $\langle f_n \rangle$  be a sequence of nonnegative measurable functions which converges to a function  $f$  in measure on a measurable set  $E$ . Then  $\int_E f \leq \liminf \int_E f_n$ .

Since the collection of limits point of  $\int_E f_n$  forms a closed set, there exists a subsequence  $\langle f_{n_k} \rangle_k$  such that  $\lim \int_E f_{n_k} = \liminf \int_E f_n$ . Since  $f_{n_k}$  also converges to  $f$  in measure, by Prob. 13(a), there is a subsequence  $\langle f_{n_{k_j}} \rangle$  which converges to  $f$  a.e. on  $E$ . Hence, by Theorem 10,

$$\int_E f \leq \liminf_j \int_E f_{n_{k_j}} = \lim_j \int_E f_{n_{k_j}} = \liminf_n \int_E f_n.$$

□

21.

*Proof.*

(a) We may assume without loss of generality that  $f$  is nonnegative since replacing  $f$  by  $|f|$  does not change the integrability and the set  $E = \{x : f(x) \neq 0\}$ . For every positive integer  $n$ , since  $\int f < \infty$ , the set  $E_n = \{x : f(x) \geq 1/n\}$  is of finite measure. Thus,  $E = \bigcup_{n=1}^{\infty} E_n$  is of  $\sigma$ -finite measure.

(b) It follows immediately from part (a) and Prop. 7.

(c) If  $f \geq 0$ , then the existence of such a  $\varphi$  comes directly from the definition. For general cases, let  $f = f^+ - f^-$  and  $\varphi^+, \varphi^-$  two simple functions such that

$$\int |f^+ - \varphi^+| < \varepsilon/2 \quad \text{and} \quad \int |f^- - \varphi^-| < \varepsilon/2.$$

Note that  $\varphi = \varphi^+ - \varphi^-$  is also a simple function and

$$\int |f - \varphi| \leq \int |f^+ - \varphi^+| + \int |f^- - \varphi^-| < \varepsilon.$$

□



**22.**

*Proof.*

(a) Clear that  $\nu$  is nonnegative and  $\nu\emptyset = 0$ . Let  $\langle E_n \rangle$  be a sequence of disjoint measurable sets and  $E = \bigcup_n E_n$ . By Corollary 14, we have

$$\nu E = \int_E g d\mu = \int_E \sum g \chi_{E_n} d\mu = \sum \int_E g \chi_{E_n} d\mu = \sum \int_{E_n} g d\mu = \sum \nu E_n.$$

Thus,  $\nu$  is a measure.

(b) First, we show the identity for an arbitrary simple function  $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$  where  $E_k$  are disjoint.

$$\int \varphi d\nu = \sum_{k=1}^n c_k \nu E_k = \sum_{k=1}^n c_k \int g \chi_{E_k} d\mu = \int \varphi g d\mu.$$

Let  $f$  be a nonnegative measurable function and  $\langle \varphi_n \rangle$  a increasing sequence of simple functions converging to  $f$ , the existence of which is guaranteed by Prop. 7. Then, By the monotone convergence theorem,

$$\int f d\nu = \lim \int \varphi_n d\nu = \lim \int \varphi_n g d\mu.$$

Note that  $\langle \varphi_n g \rangle$  is a increasing sequence of functions converging to  $fg$  and with  $\varphi_n g \leq fg$ . Hence, again by the monotone convergence theorem,

$$\lim \int \varphi_n g d\mu = \int fg d\mu.$$

Thus,  $\int f d\nu = \int fg d\mu$ . □

## 11.4 General Convergence Theorems

**24.**

*Proof.* Since  $\mu_n E$  is increasing for every  $E$ , such limits do exists. Clear that  $\mu$  is nonnegative and  $\mu\emptyset = 0$ . Let  $\langle E_k \rangle$  be a sequence of disjoint measurable sets. Then

$$\mu \left( \bigcup_{k=1}^{\infty} E_k \right) = \lim_{n \rightarrow \infty} \mu_n \left( \bigcup_{k=1}^n E_k \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \mu_n(E_k).$$

Since for fixed  $k$ ,  $\mu_n(E_k) \leq \mu_{n+1}(E_k)$ , it is valid to change the order of the limit and the summation, which implies that  $\mu(\bigcup E_k) = \sum \mu E_k$ . Thus,  $\mu$  is a measure. □

## 11.5 Signed Measures

**27.**

*Proof.*

(a) Consider the usual Lebesgue measure on  $\mathbb{R}$ . Let  $A$  be any countable subset of  $\mathbb{R}$  and  $B = \mathbb{R} \setminus A$ . Clear that  $A$  is negative set while  $B$  is a positive set. Namely,  $A$  and  $B$  form a Hahn decomposition of  $\mathbb{R}$  for  $\mu$ .

(b) Let  $\{A_1, B_1\}$  and  $\{A_2, B_2\}$  be two Hahn decomposition of  $X$  for  $\nu$  and  $A_1$  and  $A_2$  are two positive sets. We show that  $A_1 \triangle A_2$  is a null set. Since the roles of  $A_1$  and  $A_2$  are interchangeable, it suffices to show that  $A_1 \setminus A_2$  is a null set. Since  $A_1$  is positive, every subset  $E \subset A_1 \setminus A_2 \subset A_1$  is of nonnegative measure. Meanwhile,  $A_1 \setminus A_2$  is also contained in  $B_2$ , a negative set. Hence,  $\nu E \leq 0$ . Thus,  $\nu E = 0$ , implying that  $A_1 \triangle A_2$  is a null set.  $\square$

**28.**

*Proof.* Let  $\nu = \nu^+ - \nu^-$  be the Jordan decomposition of  $\nu$  and  $A$  and  $B$  be such that  $X = A \cup B$  and  $\nu^+(A) = \nu^-(B) = 0$ . For every  $E \subset A$ ,

$$\nu E = \nu^+ E - \nu^- E = -\nu^- E \leq 0.$$

Hence,  $A$  is a negative set. Similarly,  $B$  is positive set. Thus,  $\{A, B\}$  is a Hahn decomposition of  $X$ .

Let  $\nu = \nu_1 + \nu_2$  be another Jordan decomposition of  $\nu$  and  $\{C, D\}$  be the corresponding Hahn decomposition. By Prob. 27(b),  $\{A, B\}$  and  $\{C, D\}$  only differ by two null sets. Thus,  $\nu_1 = \nu^+$  and  $\nu_2 = \nu^-$ . Namely, the decomposition is unique.  $\square$

**31.**

*Proof.* Clear that

$$\left| \int_E f d\nu \right| \leq \left| \int_E f d\nu^+ \right| + \left| \int_E f d\nu^- \right| \leq M\nu^+ E + M\nu^- E = M|\nu|(E).$$

Let  $\{A, B\}$  be the corresponding Hahn decomposition of  $X$  and  $A$  is the positive set. Then define  $f$  by

$$f(x) = \begin{cases} 1, & x \in A, \\ -1, & x \notin A. \end{cases}$$

Clear that  $|f| \leq 1$  and

$$\int_E f d\nu = \int_E f d\nu^+ - \int_E f d\nu^- = \mu^+(A \cap E) + \nu^-(A \cap B) = |\nu|(E).$$

$\square$

**32.**

*Proof.*

(a) Put  $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$ , which can be verified to be a measure. For every  $E \subset X$ , suppose  $\mu E \leq \nu E$ . Then

$$(\mu \wedge \nu)(E) = \frac{1}{2}(\mu E + \nu E - |\mu - \nu|(E)) = \frac{1}{2}(\mu E + \nu E - \nu E + \mu E) = \mu E.$$

Similarly,  $(\mu \wedge \nu)(E) = \nu E$  if  $\nu E \leq \mu E$ . Hence,  $\mu \wedge \nu$  is smaller than both  $\mu$  and  $\nu$ . Note that  $(\mu \wedge \nu)(E) = \min\{\mu E, \nu E\}$ . Thus, clear that it is larger than any other signed measure smaller than  $\mu$  and  $\nu$ .

(b) Put  $\mu \vee \nu = \frac{1}{2}(|\mu - \nu| + \mu + \nu)$ . The previous argument, *mutatis mutandis*, shows that  $(\mu \vee \nu)(E) = \max\{\mu E, \nu E\}$ . Thus, it is the smallest measure larger than  $\mu$  and  $\nu$ . Meanwhile, clear that  $\mu \wedge \nu + \mu \vee \nu = \mu + \nu$ .

(c) Suppose that  $\mu$  and  $\nu$  are mutually singular and let  $\{A, B\}$  be such that  $A \cup B = X$ ,  $\mu A = \nu B = 0$ . Then

$$(\mu \wedge \nu)(E) \leq (\mu \wedge \nu)(E \cap A) + (\mu \wedge \nu)(E \cap B) \leq \mu A + \nu B = 0.$$

For the converse, suppose that  $\mu \wedge \nu = 0$ . If  $\mu = 0$  or  $\nu = 0$ , then  $\mu \perp \nu$  holds vacuously. Suppose that both  $\mu$  and  $\nu$  are nonzero. Since the roles of  $\mu$  and  $\nu$  are interchangeable, we may assume without loss of generality that  $\mu E = 0$  and  $\nu E > 0$  for some measurable  $E$ . Then,  $\mu E^c \neq 0$ , forcing  $\nu E^c$  to be zero. Therefore,  $\mu E = \nu E^c = 0$ , implying that  $\mu \perp \nu$ .  $\square$

## 11.6 The Radon-Nikodym Theorem

### 33.

*Proof.* Suppose  $X = \bigcup_{n=1}^{\infty} X_i$  and  $\mu X_i < \infty$  for each  $n$  and  $X_i$  are disjoint. Then both  $\mu|_{X_i}$  and  $\nu|_{X_i}$ , the restrictions to  $X_i$ , are finite. In consequence, by the Radon-Nikodym theorem for finite measure, there is a nonnegative measurable function  $f_i : X_i \rightarrow \mathbb{R}$  such that  $\nu(E \cap X_i) = \int_{(E \cap X_i)} f_i d\mu$ . Without loss of generality, we may consider  $f_i$  to be a function on  $X$  (instead of  $X_i$ ) that vanishes outside  $X_i$ .

Put  $f = \sum f_i$ . Since  $X_i$  are disjoint and  $f_i$  vanishes outside  $X_i$ , the summation does make sense. Meanwhile, clear that  $f$  is nonnegative and measurable. Note that for each measurable  $E$ ,

$$\nu E = \sum_{n=1}^{\infty} \nu(E \cap X_i) = \sum_{n=1}^{\infty} \int_E f_i d\mu = \int_E f d\mu,$$

where the last equality comes from Corollary 14. Namely,  $\nu E = \int_E f d\mu$ .

Finally, we show that  $f$  is unique up to almost equality. Let  $g$  be a nonnegative measurable function with this property. Then,  $g|_{X_i}$ , the restriction of  $g$  to  $X_i$ , equals to  $f_i$  a.e.  $[\mu]$ . Thus,  $g = f$  a.e.  $[\mu]$ .  $\square$

### 34. Radon-Nikodym derivatives

*Proof.*

(a) It suffices to show the result for simple functions. Let  $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$  be a simple function. Then

$$\int \varphi d\nu = \sum_{k=1}^n c_k \nu E_k.$$

Meanwhile,

$$\int \varphi \left[ \frac{d\nu}{d\mu} \right] d\mu = \sum_{k=1}^n c_k \int_{E_k} \left[ \frac{d\nu}{d\mu} \right] d\mu = \sum_{k=1}^n c_k \nu E_k.$$

Thus,  $\int \varphi d\nu = \int \varphi [d\nu/d\mu] d\mu$ .  $\square$

35.

*Proof.*

(d) Let  $\rho_0, \rho_1$  be two measures with  $\rho_0 \perp \mu$ ,  $\rho_1 \ll \mu$  and  $\nu = \rho_0 + \rho_1$ . We show that  $\rho_0 = \nu_0$  and  $\rho_1 = \nu_1$ . Since  $\nu_0 \perp \mu$  and  $\rho_0 \perp \mu$ , there exists measurable  $A, B$  and  $C, D$  such that  $A \cup B = C \cup D = X$ ,  $A \cap B = C \cap D = \emptyset$  and  $\nu_0 A = \mu B = \rho_0 C = \mu D = 0$ . Put  $U = A \cap C$  and  $V = B \cup D$ . Note that

$$\begin{aligned} U \cup V &= (A \cap C) \cup (B \cup D) = (A \cup B \cup D) \cap (C \cup B \cup D) = X, \\ U \cap V &= (A \cap C) \cap (B \cup D) = (A \cap C \cap B) \cup (C \cap B \cap D) = \emptyset. \end{aligned}$$

For every measurable  $E$ , if  $E \subset U$ , then  $\nu_0 E = \rho_0 E = 0$  and  $(\nu_0 + \nu_1)(E) = (\rho_0 + \rho_1)(E)$  implies that  $\nu_1 E = \rho_1 E$ . If  $E \subset V$ , then  $\mu E = 0$ , implying that  $\nu_1 E = \rho_1 E = 0$  and, therefore,  $\nu_0 E = \rho_0 E$ . Since  $U$  and  $V$  partitions  $X$ , this implies that  $\nu_0 = \rho_0$  and  $\nu_1 = \rho_1$  for all measurable  $E$ .  $\square$

36.

*Proof.* We show that: Let  $(X, \mathcal{B}, \mu)$  be a  $\sigma$ -finite signed measure space, and let  $\nu$  be a signed measure on  $\mathcal{B}$  with  $\nu \ll \mu$ . Then there is a measurable function such that for all measurable  $E$  we have  $\nu E = \int_E f d\mu$ . Furthermore, the function  $f$  is unique up to almost equality with respect to  $\mu$ .

Let  $\mu = \mu^+ - \mu^-$  and  $\nu = \nu^+ - \nu^-$  be the Jordan decompositions. Clear that  $\nu^+ \ll \mu^+$  and  $\nu^- \ll \mu^-$ . Hence, by the Radon-Nikodym theorem for measures, there exists nonnegative  $g$  and  $h$  such that

$$\nu^+ E = \int_E g d\mu^+ \quad \text{and} \quad \nu^- E = \int_E h d\mu^-$$

for all measurable  $E$ . Put  $f = g - h$ . Clear that it is measurable. Meanwhile,

$$\nu E = \nu^+ E - \nu^- E = \int_E g d\mu^+ - \int_E h d\mu^- = \int_E (g - h) d\mu$$

where the last equality comes from the mutual singularity of  $\mu^+$  and  $\mu^-$ . And the argument in Prob. 33, *mutatis mutandis*, gives the uniqueness.  $\square$

40.

*Proof.* Let  $I$  denote the index set of  $\{X_\alpha\}$  and, just for convenience, let  $\sum_J$  denote  $\sum_{\alpha \in J} \mu(E \cap X_\alpha)$ .

(a) First, we suppose that  $E$  is of finite measure. Let  $J$  be any finite index subset of  $I$ . Then, since  $X_\alpha$  are disjoint,  $\mu E \geq \sum_J$ . Hence,  $\mu E \geq \sum_I$ . For the converse, since, by our previous result, all  $\sum_J$  are bounded by  $\mu E$ ,  $\sum_I$  is finite. For each positive integer  $n$ , there is a finite subset  $J_n$  of  $I$  such that  $\sum_I - 1/n < \sum_{J_n}$ . Put

$$J = \bigcup_{n=1}^{\infty} J_n \quad \text{and} \quad Y = \bigcup_{\beta \in J} X_\beta.$$

We show that (1)  $\mu(Y \cap E) \leq \sum_I$  and (2)  $\mu E = \mu(Y \cap E)$  to complete the proof. Since  $\{X_\beta\}_{\beta \in J}$  is a countable collection of disjoint sets,

$$\mu(Y \cap E) = \sum_{\beta \in J} \mu(X_\beta \cap E) \leq \sum_{\alpha \in I} \mu(X_\alpha \cap E).$$

Namely, (1) holds. To show (2), we first show that, for each  $\alpha \in I$ , the set  $X_\alpha \cap (E \setminus Y)$  is of measure zero. Assume, to obtain a contradiction, that there is some  $\alpha$  such that  $\mu(X_\alpha \cap (E \setminus Y)) = \delta > 0$ . Since

$$X_\alpha \cap (E \setminus Y) = X_\alpha \cap E \cap \left( \bigcap_{\beta \in J} X_\beta^c \right),$$

this implies that  $\alpha \notin J_n$  for all  $n$  and  $\mu(X_\alpha \cap E) = \delta$ . Since  $\delta > 1/n$  for some large  $n$ , this leads to the contradiction

$$\sum_{J_n \cup \{\alpha\}} = \sum_{J_n} + \mu(X_\alpha \cap E) > \sum_I - \frac{1}{n} + \delta \geq \sum_I.$$

Hence,  $\mu(X_\alpha \cap (E \setminus Y)) = 0$  for all  $n$  and, therefore,  $\mu(E \setminus Y) = 0$ . Note that  $E \setminus (Y \cap E) = E \setminus Y$ , this implies that  $\mu(E \setminus (Y \cap E)) = 0$ . In consequence,  $\mu E = \mu(Y \cap E)$ . Thus,  $\mu E \leq \sum_I$ .

Now suppose  $\mu E = \infty$ . If all  $\sum_J$  are finite □

**40.**

*Proof.* Let  $I$  denote the index set of  $\{X_\alpha\}$ . Fix a measurable  $E$ . Put

$$J = \{\alpha \in I : \mu(E \cap X_\alpha) > 0\} \quad \text{and} \quad Y = \bigcup_{\beta \in J} X_\beta.$$

First we show that  $\mu E = \mu(E \cap Y)$ . Clear that  $\mu E \geq \mu(E \cap Y)$ . For the converse, consider the set  $E \setminus (E \cap Y) = E \setminus Y$ . For every  $X_\alpha$ , if  $\alpha \notin J$ , by the construction of  $J$ ,  $\mu(X_\alpha \cap (E \setminus Y)) = 0$ . If  $\alpha \in J$ , then

$$X_\alpha \cap (E \setminus Y) = X_\alpha \cap E \cap \left( \bigcap_{\beta \in J} X_\beta \right) = \emptyset.$$

As a result,  $\mu(X_\alpha \cap (E \setminus Y)) = 0$  for all  $\alpha \in I$ . Since  $\{X_\alpha\}$  is a decomposition, this implies that  $\mu(E \setminus Y) = 0$ . Therefore,  $\mu E \leq \mu(E \cap Y)$ . Thus,  $\mu E = \mu(E \cap Y)$ .

(a) For each positive integer  $n$ , put

$$J_n = \{\alpha \in I : \mu(E \cap X_\alpha) > 1/n\}.$$

Clear that  $J = \bigcup_n J_n$ . If  $J$  is uncountable, then there must exist some uncountable  $J_n$ , which implies that  $\mu E = \sum \mu(X_\alpha \cap E) = \infty$ . If  $J$  is countable, then

$$\mu E = \mu(E \cap Y) = \sum_{\beta \in J} \mu(E \cap X_\beta) = \sum_{\alpha \in I} \mu(E \cap X_\alpha).$$

Thus,  $\mu E = \sum \mu(E \cap X_\alpha)$ . □

## 11.7 The $L^p$ Spaces

41.

*Proof.* First, we prove the following lemma: For  $a, b \geq 0$ ,  $|a - b|^p \leq 2|a^p - b^p|$ . It suffices to show that  $(a - b)^p \leq 2(a^p - b^p)$  for all  $a \geq b \geq 0$ . If  $p = 1$ , then the inequality holds trivially. Suppose  $p > 1$  and put  $h(x) = (x - b)^p - 2(x^p - b^p)$ . Clear that  $h(b) = 0$ . Meanwhile, for  $x \geq b$ ,

$$h'(x) = p(x - b)^{p-1} - 2px^{p-1} = px^{p-1} \left( \left(1 - \frac{b}{x}\right)^{p-1} - 2 \right) < 0.$$

Thus,  $h(x) \leq 0$  for all  $x \geq b$ , which implies that  $|a - b|^p \leq 2|a^p - b^p|$  for all  $a, b \geq 0$ .

Since  $|f|^p$  is integrable, by Prob. 21(a), the set on which  $f$  does not vanish is of  $\sigma$ -finite measure. Hence,  $\int |f|^p = \sup \int \varphi$  as  $\varphi$  ranges over all simple functions that each vanishes outside a set of finite measure. Thus, for every  $\varepsilon > 0$ , there is a nonnegative simple function  $\tilde{\varphi} \leq |f|^p$ , vanishing outside a set  $E$  of finite measure, such that  $\int (|f|^p - \tilde{\varphi}) < \varepsilon^p/2$ . Put  $\varphi = \sqrt[p]{\tilde{\varphi}}$ , which is also a nonnegative simple function that vanishes outside  $E$ . Meanwhile, by the previous inequality,

$$\|f - \varphi\|_p^p = \int |f - \varphi|^p \leq 2 \int (|f|^p - \tilde{\varphi}) < \varepsilon^p.$$

Namely, Prop. 26 holds. □

42.

*Proof.* We may assume without loss of generality that  $g$  is nonnegative. Assume, to obtain a contradiction, that  $\text{ess sup } |g| > M$ , that is, the measure of  $E = \{t : g(t) > M + \eta\}$  is nonzero for some positive  $\eta$ . Meanwhile, since  $\mu$  is finite,  $\mu E < \infty$ . Let  $\varphi = \chi_E$ , which is clearly a simple function. Then

$$\left| \int g\varphi \right| \geq (M + \eta)\mu E > M\|\varphi\|_1.$$

Contradiction. Hence,  $\text{ess sup } |g| \leq M$ , implying that  $g \in L^\infty$ . □

43. The case  $p = 1$  is left undone.

*Proof.* Suppose that  $p > 1$ . Let  $\langle X_n \rangle$  be such that  $\mu X_n < \infty$  and  $X = \bigcup X_n$ . Furthermore, we may assume without loss of generality that  $X_n$  are disjoint. Put  $g_n = \sum g\chi_{X_n}$ . By Lemma 27, for  $n$ ,  $g\chi_{X_n} \in L^q$  and  $\|g_n\|_q \leq M$ . Since  $g_n \rightarrow g$ , by Fatou's lemma,  $\|g\|_q \leq M$ . Thus,  $g \in L^q$ . □

44.

*Proof.* Note that

$$\int |f|^p = \sum \int |f|^p \chi_{E_n} = \sum \int |f_n|^p = \sum \|f_n\|_p^p.$$

Thus,  $f \in L^p$  iff  $\sum \|f_n\|_p^p < \infty$ . □

**45.**

*Proof.* For every  $f \in L^p$  with  $\|f\|_p = 1$ ,

$$\int |fg| \leq \|f\|_p \|g\|_q = \|g\|_q.$$

Hence,  $\|F\| \leq \|g\|_q$ . For the reverse inequality, put

$$f = (\operatorname{sgn} g)|g|^{q-1} = (\operatorname{sgn} g)|g|^{p/q}.$$

Note that  $|f|^p = |g|^q$ . Therefore,  $g \in L^q$  implies  $f \in L^p$ . Meanwhile,  $\|f\|_p^p = \|g\|_q^q$ . Hence,

$$\|F\| \|f\|_p \geq |F(f)| = \int |g|^q = \|g\|_q^q \quad \Rightarrow \quad \|F\| \geq \|g\|_q.$$

Thus,  $\|F\| = \|g\|_q$ . □

## 12 Measure and Outer Measure

### 12.1 Outer Measure and Measurability

1.

*Proof.* Suppose that  $E \subset X$  and there is a measurable  $B$  with  $\bar{\mu}B = 0$  such that  $E \subset B$ . We show that  $E$  is measurable, that is, for every  $A \subset X$  with finite outer measure,  $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ . Since  $A \cap E \subset E \subset B$  and  $\mu^*$  is monotone,  $\mu^*(A \cap E) \leq \mu^*(B) = \bar{\mu}B = 0$ . Again by the monotonicity,  $\mu^*(A) \geq \mu^*(A \cap E^c)$ . Thus,  $E$  is measurable, implying that  $\bar{\mu}$  is complete.  $\square$

2.

*Proof.* From the countable subadditivity we obtain that  $\mu^*(A \cap E) \leq \sum \mu^*(A \cap E_i)$ . For the converse, first we consider just  $E_1$  and  $E_2$ . Since  $E_1$  is measurable,

$$\mu^*(A \cap E) = \mu^*(A \cap E \cap E_1) + \mu^*(A \cap E \cap E_1^c) \geq \mu^*(A \cap E_1) + \mu^*(A \cap E_2).$$

By induction on  $n$  we get  $\mu^*(A \cap E) \geq \sum_{i=1}^n \mu^*(A \cap E_i)$ . Let  $n \rightarrow \infty$  and the proof is completed.  $\square$

### 12.2 The Extension Theorem

4.

*Proof.*

(a) Since  $\{D_j\}$  partitions  $A$ ,  $C_i \subset A$  for each  $i$  and  $\mathcal{C}$  is closed under intersection, by condition (i),  $\mu C_i = \sum_j \mu(C_i \cap D_j)$ . Similarly for  $\mu D_j$ . Thus,

$$\sum_i \mu C_i = \sum_{i,j} \mu(C_i \cap D_j) = \sum_j \sum_i \mu(C_i \cap D_j) = \sum_j \mu D_j.$$

This result implies that the definition of  $\mu$  on  $\mathcal{A}$  is well-defined.

(b) Since every  $A \in \mathcal{A}$  is a finite union of sets of  $\mathcal{C}$ , it suffices to show that  $\mu C \geq \sum \mu C_i$ . Then, from this condition, the countable additivity follows. Since  $\mu$  is nonnegative and monotone,  $\mu C \geq \sum_{i=1}^n \mu C_i$  for all positive integer  $n$ . Let  $n \rightarrow \infty$  and we get  $\mu C \geq \sum \mu C_i$ .  $\square$

7.

*Proof.* To prove the "if" part, let  $\varepsilon_n = 1/n$  and  $A_n \in \mathcal{A}_\delta$  be such that  $\mu^*(E \setminus A_n) < \varepsilon_n$ . Put  $A = \bigcup A_n$ . Since the collection of  $\mu^*$ -measurable sets is a  $\sigma$ -algebra,  $A$  is measurable. Meanwhile, since  $\mu^*(E \setminus A) \leq \mu^*(E \setminus A_n) < \varepsilon_n$ ,  $E \setminus A$  is of  $\mu^*$ -measure zero. Hence, it is measurable. Thus,  $E = A \cup (E \setminus A)$  is also measurable.

For the converse, suppose that  $E$  is measurable and let  $\varepsilon > 0$  be fixed. Note that  $E^c$  is also measurable. Hence, by Prop. 6, there is a set  $A \in \mathcal{A}_\sigma$  with  $E^c \subset A$  and

$$\varepsilon > \mu^*(A \setminus E^c) = \mu^*(A \cap E).$$

Then,  $A^c \in \mathcal{A}_\delta$ ,  $A^c \subset E$  and  $\mu^*(E \setminus A^c) = \mu^*(E \cap E) < \varepsilon$ .  $\square$