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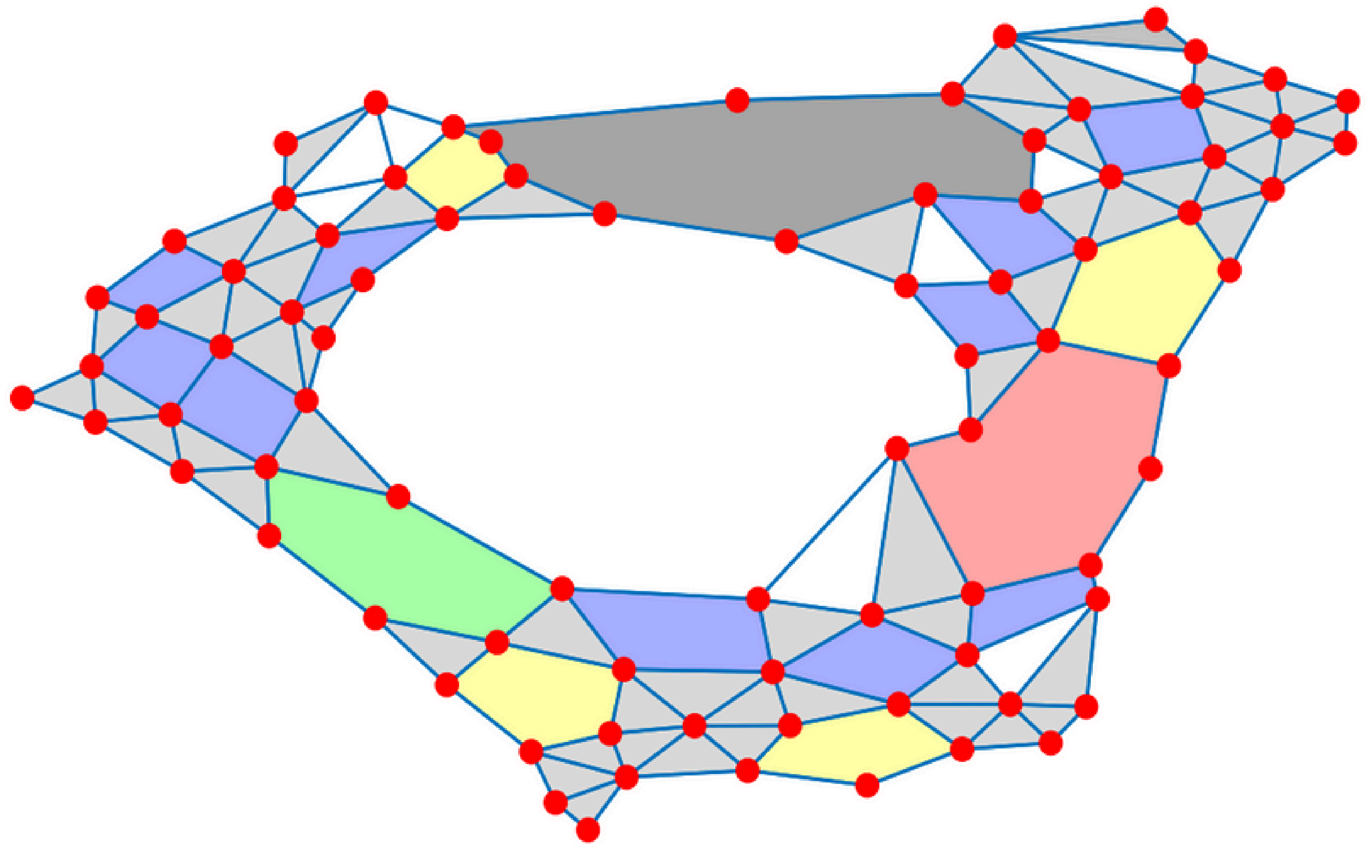
Joint Topology and Dictionary Learning for Sparse Data Representation over Cell Complexes



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Signal Processing

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Signal Processing is the branch of engineering and science aimed to the mainpulation of signals.

↓

A **signal** can be seen as a general function living in a certain Hlbert space, i.e. a mapping from a **domain** to a **co-domain**:

$$\mathbf{x} : \mathcal{D} \to \mathcal{C}$$

We can choose $\mathcal{D}$ and $\mathcal{C}$ in order to describe different phenomena.

Example of signals

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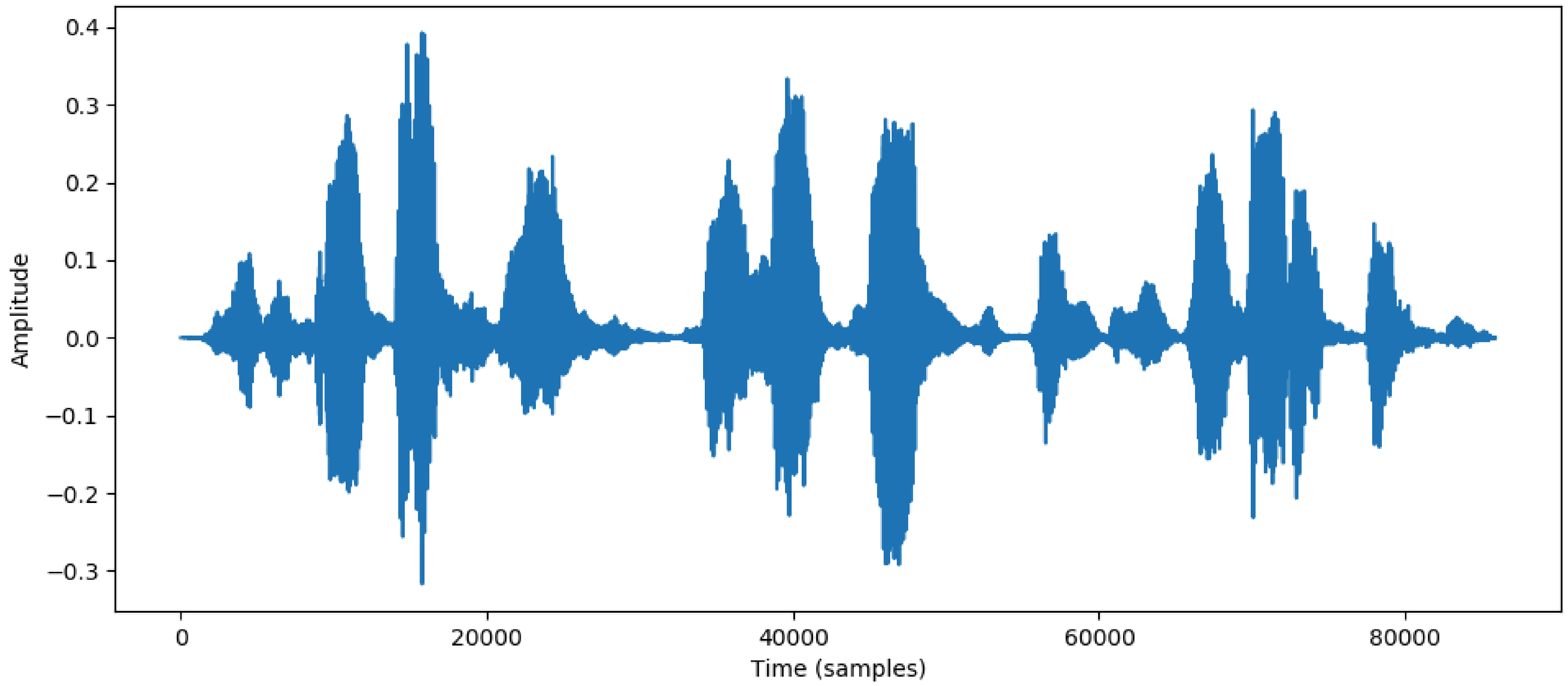
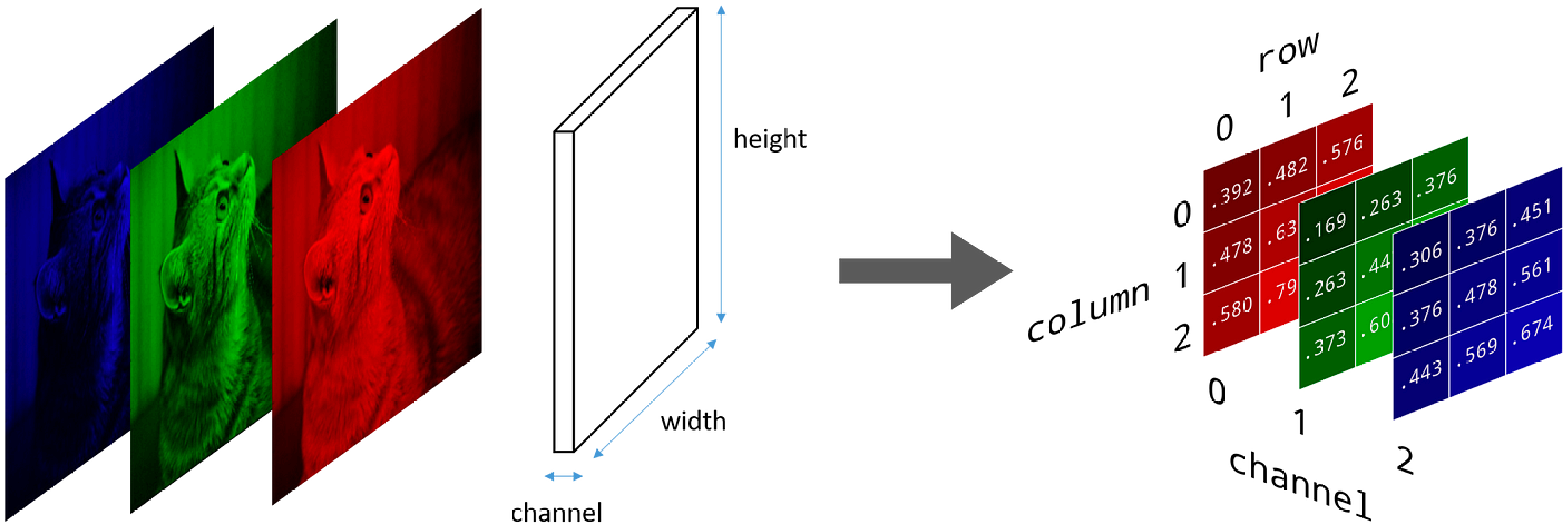
Audio Signals



$$\mathcal{D} = \{0, 1, \ldots, N\} \longrightarrow \mathcal{C} = \mathbb{R}$$

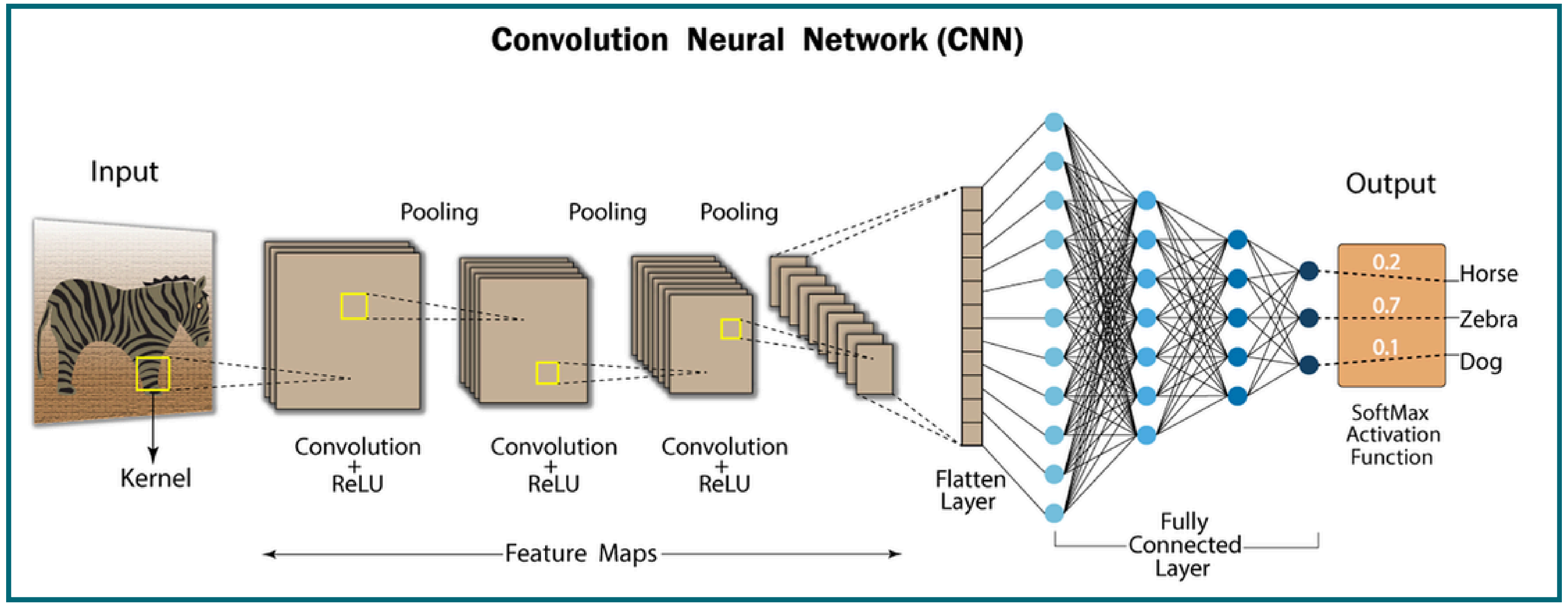
Example of signals

Image Signals



$$\mathcal{D} = \{(0,0),\dots,(0,w),(1,0),\dots,(h,w)\} \longrightarrow \mathcal{C} = [0,255]^3$$

Signal Processing for Data Science



images ⟷ grid domain

translation equivariance ⟷ convolution

Inductive bias:

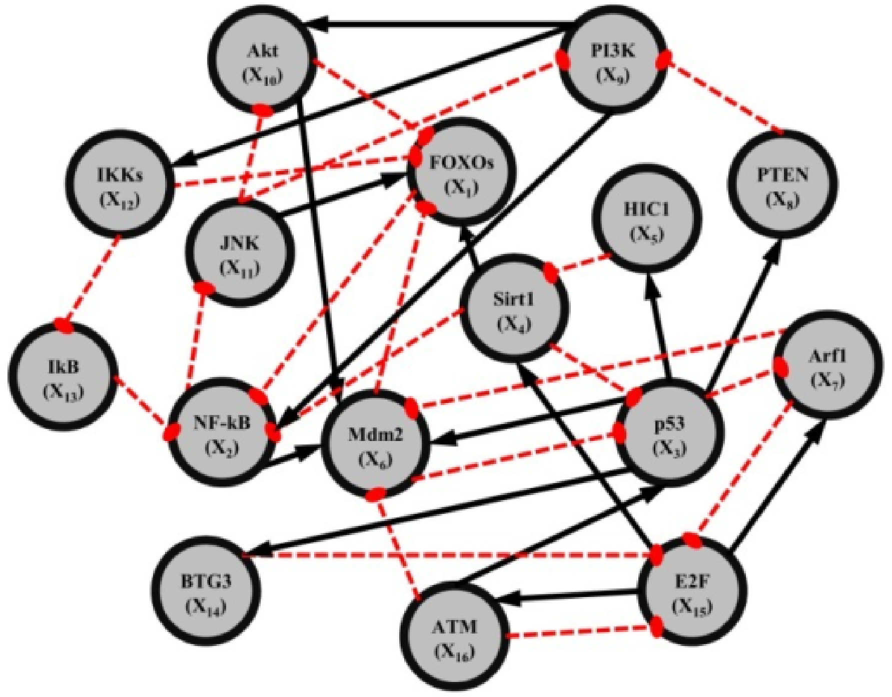
Enrich a ***data-driven*** approach with a **model** about solution space/data generating process

Relational inductive bias:

Use some ***geometrical insights*** on data to bias the model solution. We generally use data **domain symmetries** and **local relations**

Example of signals

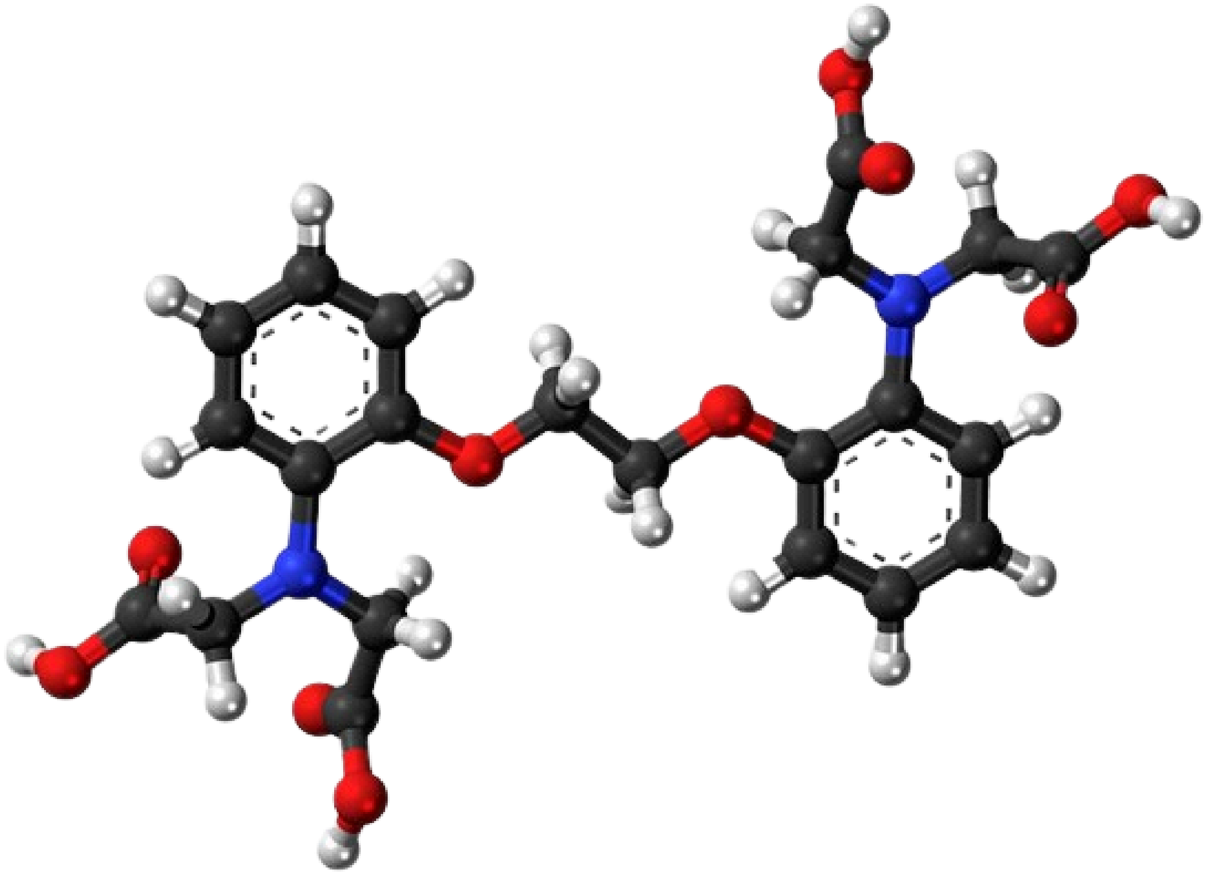
Graph Signals



$$\mathcal{D} = \mathcal{V} \subset \mathbb{N} \longrightarrow \mathcal{C} = \mathbb{R}^N$$

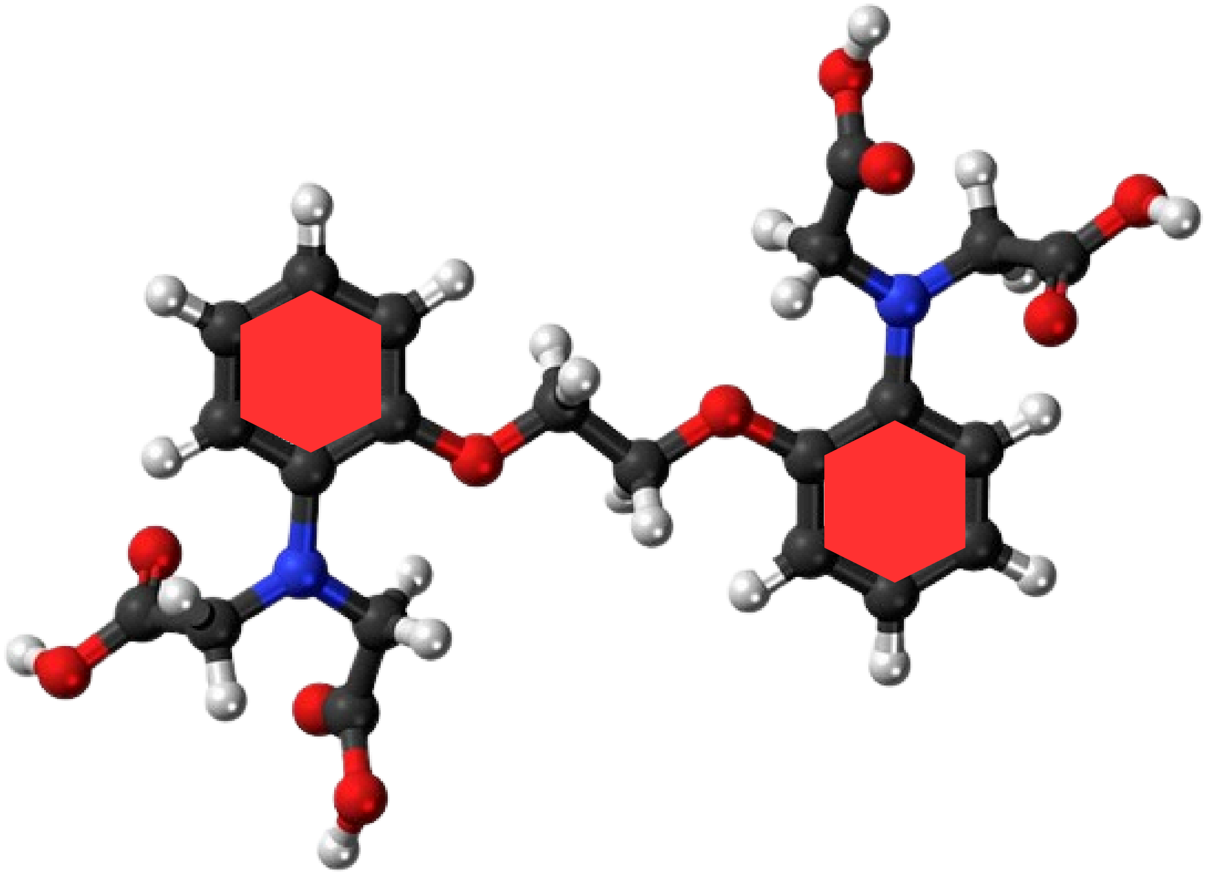
Beyond graphs

Graphs can't describe **multiway relations** between domain entities...



Beyond graphs

Graphs can't describe **multiway relations** between domain entities...



... in TSP we generally focus on signal defined over general higher order domains (simplicial complexes, cell complexes etc...)

Non-Euclidean Spaces

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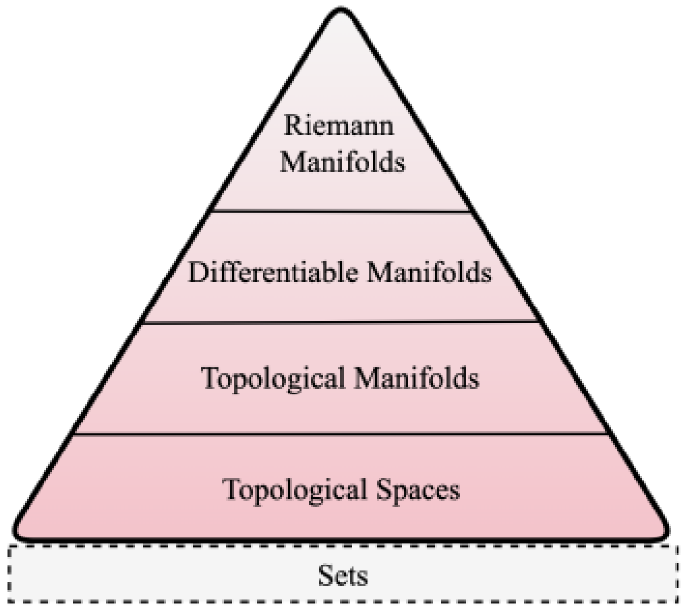
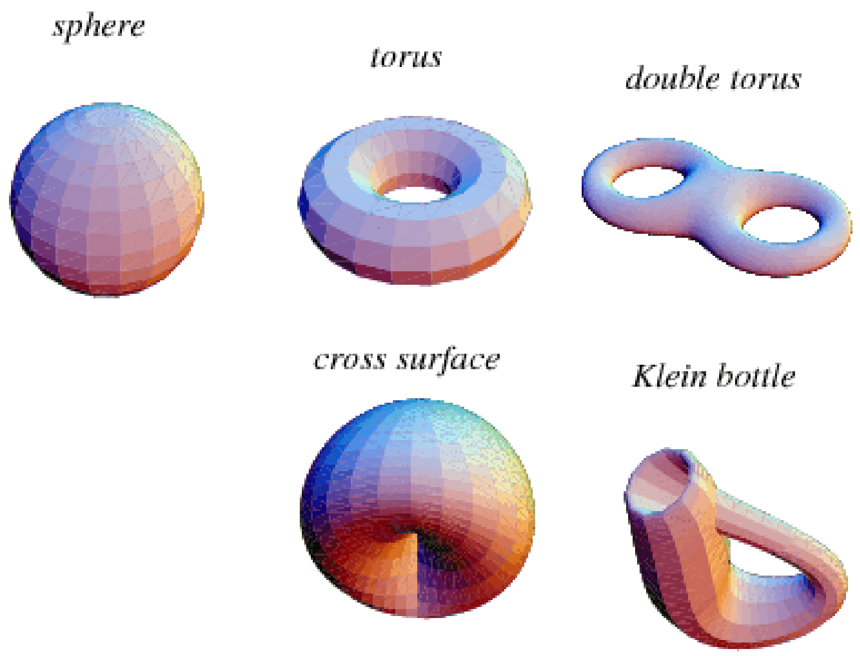
We need to generalize to data defined over non-Euclidean domains → Typically data lives over a manifold → How to leverage an unknown manifold structure? → How to identify invariants of a manifold? → How to introduce this invariants in the learning algorithm? → Topological Signal Processing

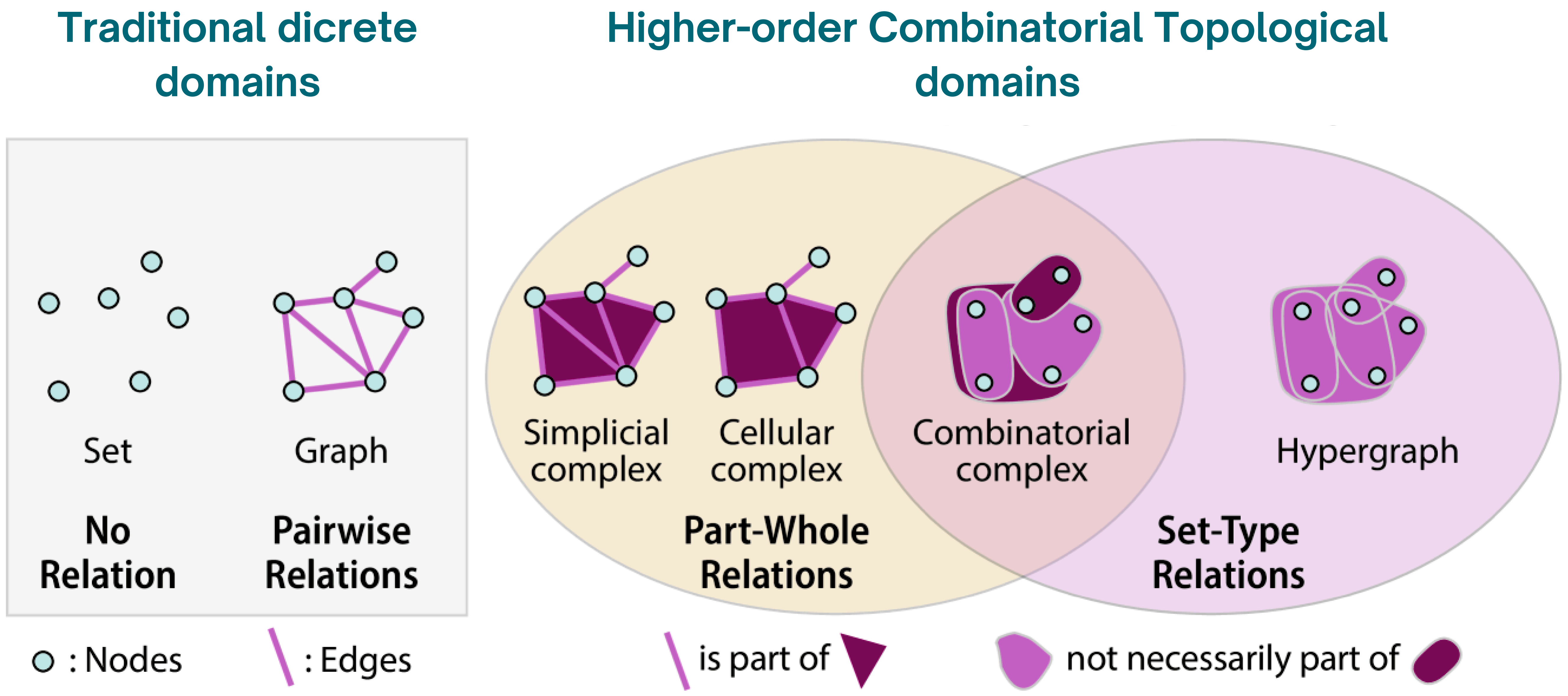
Topological Signal Processing

- express local relations
- discover invariances
- exploit permutation invariance





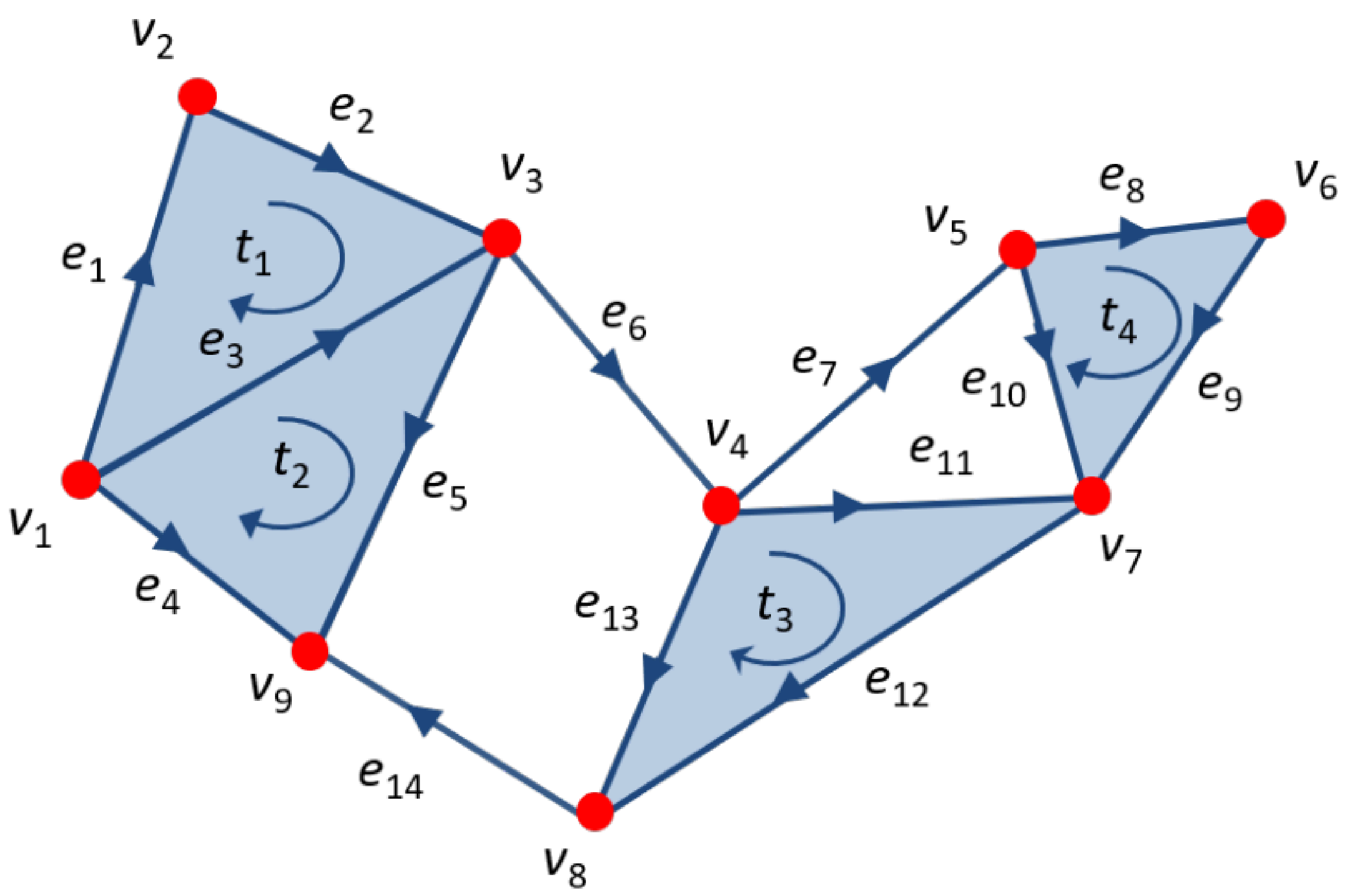
Topological Domains



Simplicial Complex

Given a finite set $V = \{v_0, \ldots, v_{N-1}\}$ of N points (vertices), a **order-k simlpex** σ_i^k is an unordered set $\{v_{i_0}, \ldots, v_{i_k}\}$ of $k+1$ points with $0 \leq i_j \leq N-1$ for $v_{i_j} \neq v_{i_n}$, and for all $i_j \neq i_n$.

A **simplicial complex** $\mathcal{X}$ is a finite collection of simplices that is closed under inclusion of faces, i.e., if $\sigma_i \in \mathcal{X}$ then all faces of σ_i also belong to $\mathcal{X}$



Simplicial vs Cell Complex

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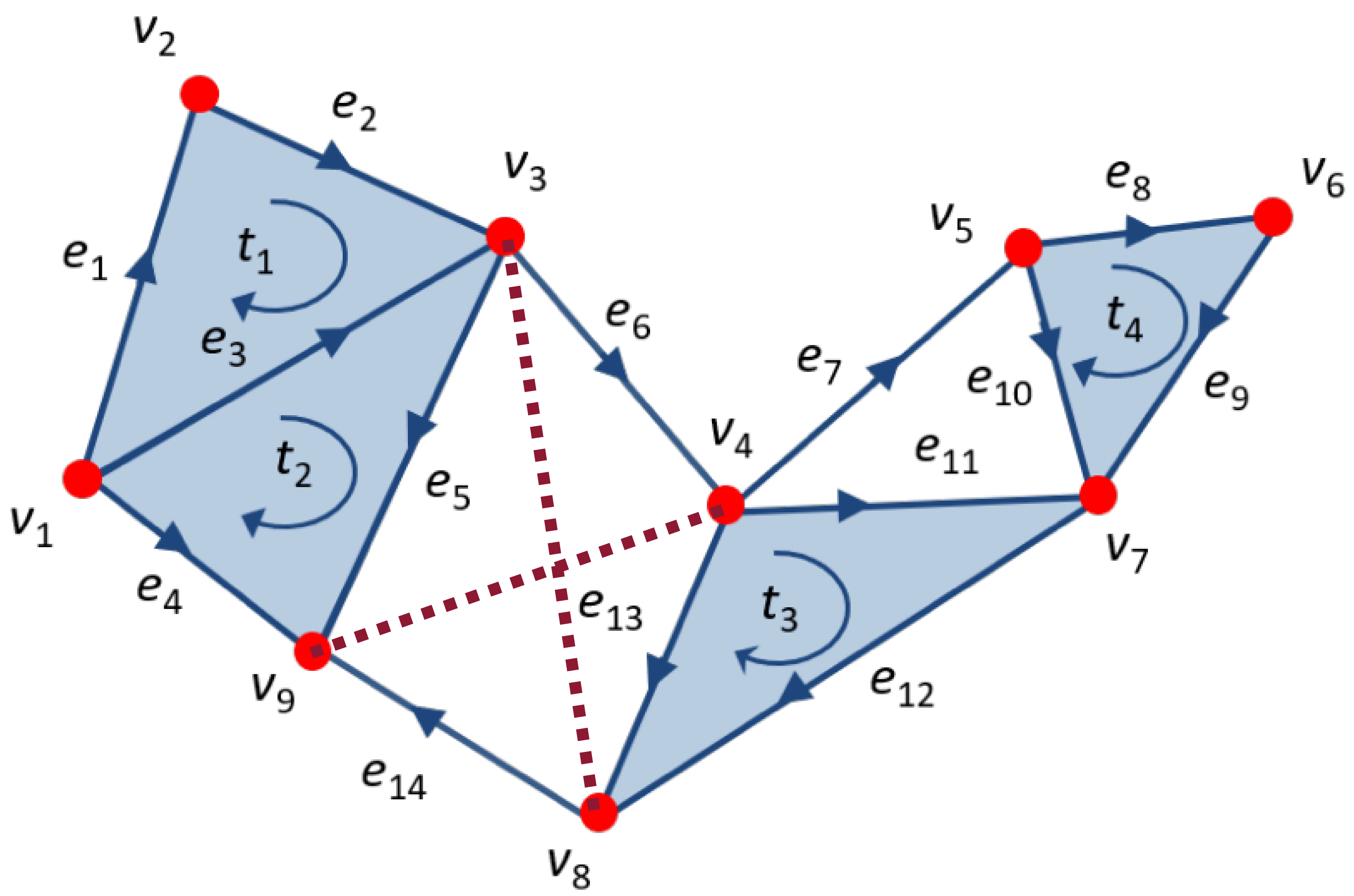
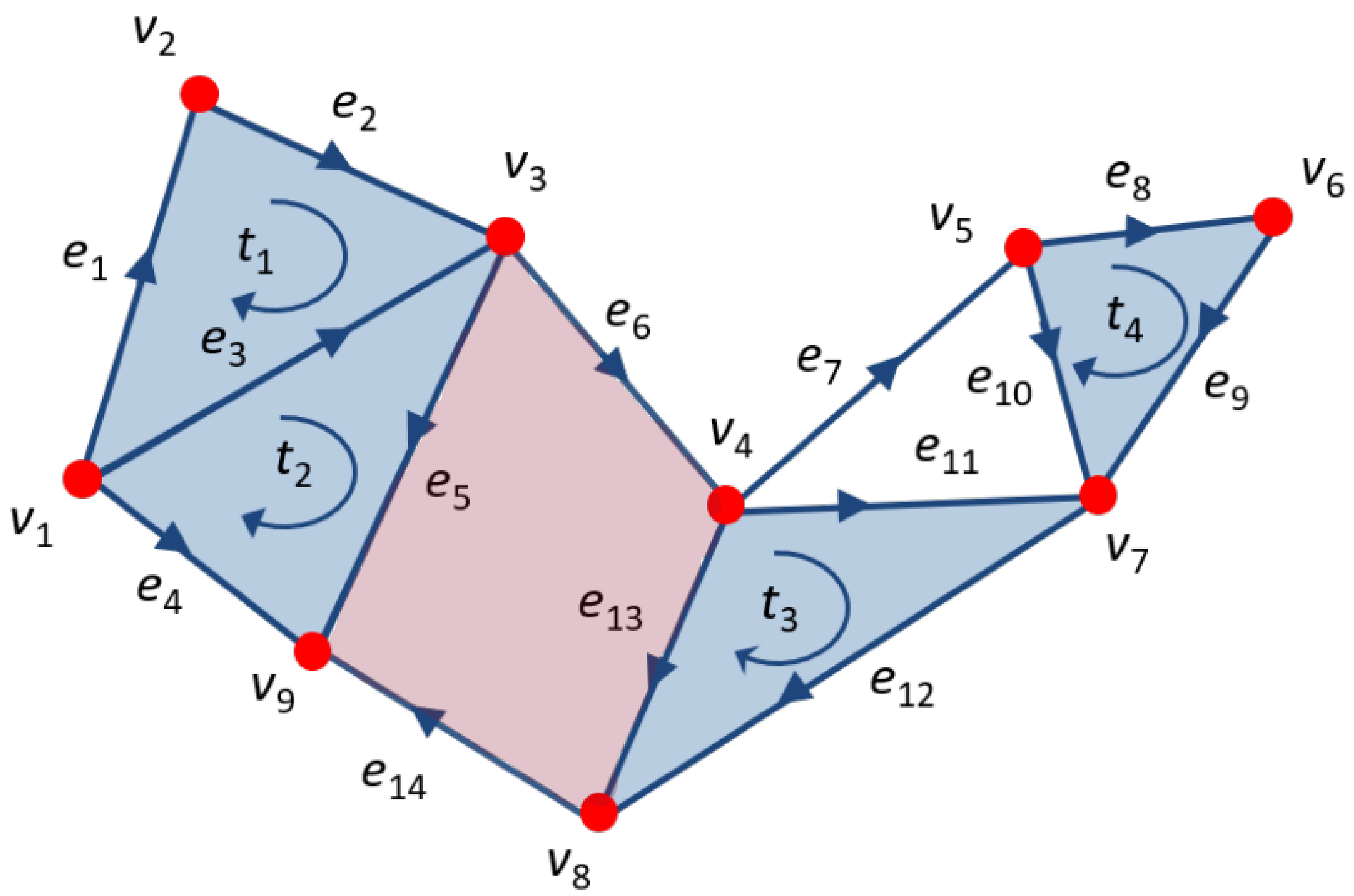


The **inclusion property** of **Simplicial Complexes** could be too restrictive

Simplicial vs Cell Complex

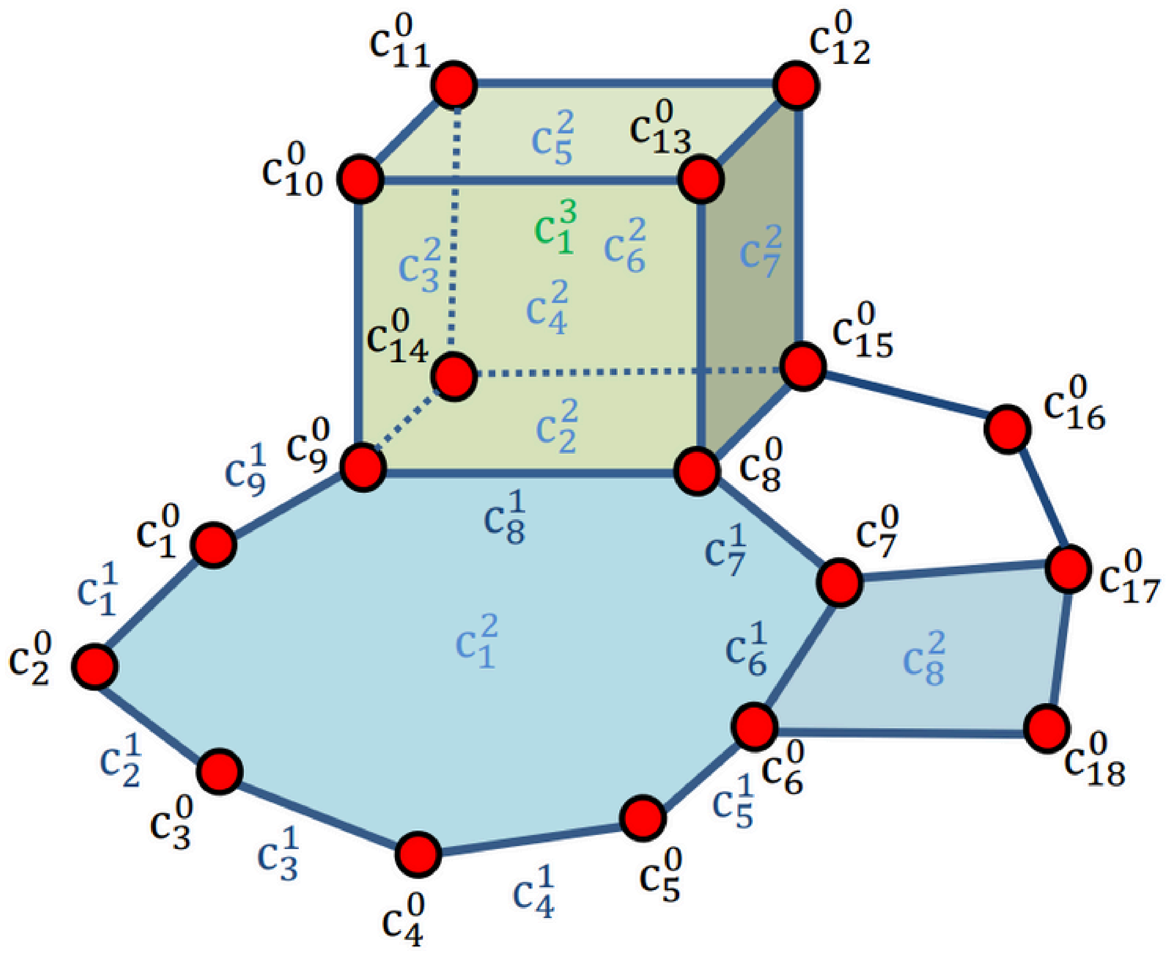


Cell Complexes allow for a more general definition avoiding the inclusion property assumption.

Cell Complex

A **Regular Cell Complex** is a topological space $\mathcal{X}$ together with a partition $\{\mathcal{X}_\sigma\}_{\sigma \in \mathcal{P}_\mathcal{X}}$ of subspaces of $\mathcal{X}_\sigma$ called cells, where $\mathcal{P}_\mathcal{X}$ is the indexing set of $\mathcal{X}_\sigma$, such that:

1. For each $c \in \mathcal{X}$, every sufficiently small neighborhood of c intersects finitely many $\mathcal{X}_\sigma$;
2. For all τ, σ we have that $\mathcal{X}_\tau \cap \mathcal{X}_\sigma \neq \emptyset$ if and only if $\mathcal{X}_\tau \subseteq \overline{\mathcal{X}_\sigma}$, where $\overline{\mathcal{X}_\sigma}$ is the closure of the cell;
3. Every $\mathcal{X}_\sigma$ is homeomorphic to $\mathbb{R}^k$ for some k;
4. For every $\sigma \in \mathcal{P}_\mathcal{X}$ there is a homeomorphism ϕ of a closed ball in $\mathbb{R}^k$ to $\mathcal{X}_\sigma$ such that the restriction of ϕ to the interior of the ball is a homeomorphism onto $\mathcal{X}_\sigma$.



Cell Complex

We can represent all the needed information to describe cell complex with few **algebric operators**.

The **Incidence Matrices** establish which k-cells are incident to which (k - 1)-cells:

$$\mathbf{B}_k[i,j] = \begin{cases} 0, & \text{if } \sigma_i^{k-1} \not\subset \sigma_j^k \\ 1, & \text{if } \sigma_i^{k-1} \subset \sigma_j^k \text{ and } \sigma_i^{k-1} \sim \sigma_j^k \\ -1, & \text{if } \sigma_i^{k-1} \subset \sigma_j^k \text{ and } \sigma_i^{k-1} \nsim \sigma_j^k \end{cases}$$

The **Laplacian Matrices** :

$$\begin{cases} L_0 & = \mathbf{B}_1 \mathbf{B}_1^T \\ L_k & = \underbrace{\mathbf{B}_k^T \mathbf{B}_k}_{L_k^{(d)}} + \underbrace{\mathbf{B}_{k+1} \mathbf{B}_{k+1}^T}_{L_k^{(u)}} \\ L_K & = \mathbf{B}_K^T \mathbf{B}_K \end{cases}$$

where the **lower Laplacian** $L_k^{(d)}$ and the **upper Laplacian** $L_k^{(u)}$ encode respectively the lower and upper connectivity among k-order cells

Topological Signal Processing

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A **k-topological signal** $\mathbf{s}_k$ over a K-th order **regular cell complex** $\mathcal{X}^K$ is defined as a collection of mappings from the set of all k-cells contained in the complex to real numbers, i.e. $\mathbf{s}_k : D_k \rightarrow \mathbb{R}$ such that:

$$\mathbf{s}_k = [\mathbf{s}_k(\sigma_1^k), \ldots, \mathbf{s}_k(\sigma_i^k), \ldots, \mathbf{s}_k(\sigma_{N_k}^k)] \in \mathbb{R}^{N_k}$$

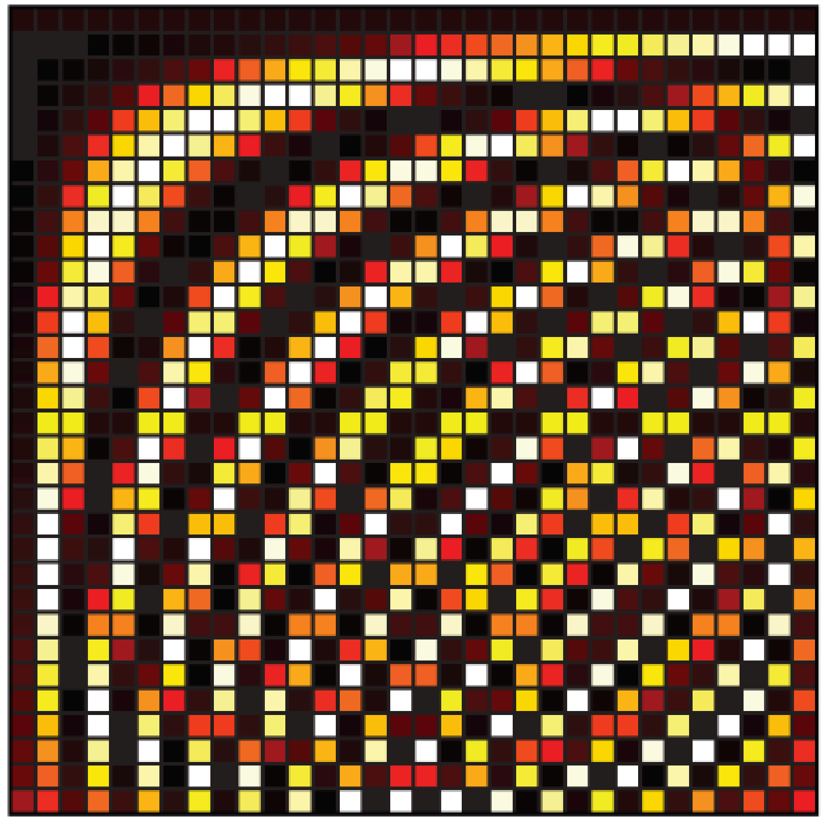
where $D_k := \{\sigma_i^k : \sigma_i^k \in \mathcal{X}^k\}$ is the set of k-cells in $\mathcal{X}^K$ and $|D_k| = N_k$.

How do we find sparse representations of topological signals?

Sparse Representation

Topological Signal	**Dictionary** (change-of-basis)	**Sparse Representation**
$\mathbf{y} \in \mathbb{C}^M$	$D \in \mathbb{C}^{M \times N}$	$\mathbf{x} \in \mathbb{C}^N$

$=$



Sparse Representation

Objective:

Find the sparse representation that minimizies the approximation error:

$$\|y - Dx\|_2^2$$

Topological Signal

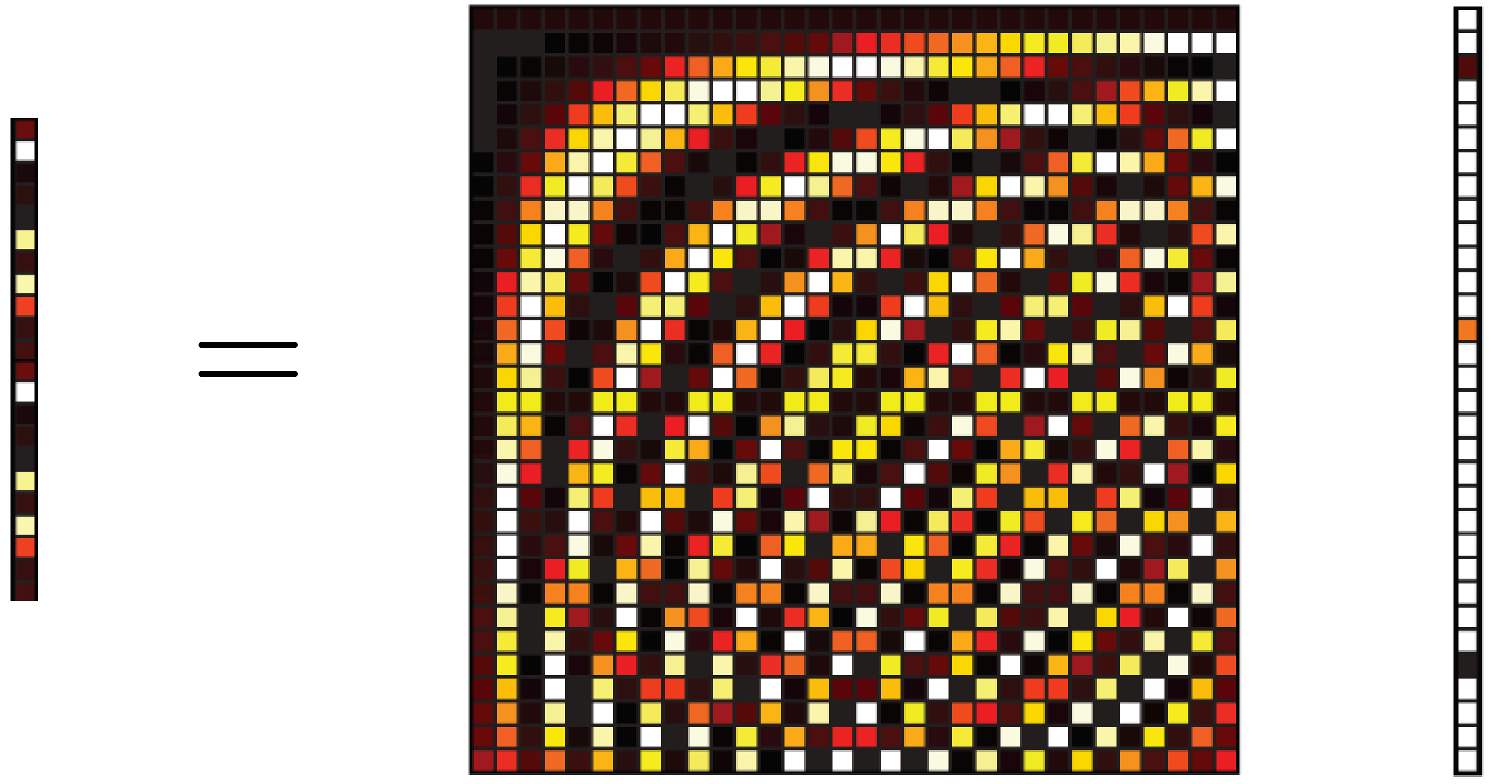
$\mathbf{y} \in \mathbb{C}^M$

Dictionary
(change-of-basis)

$D \in \mathbb{C}^{M \times N}$

Sparse Representation

$\mathbf{x} \in \mathbb{C}^N$



Dictionary Learning

Idea:

1. Build a **learnable** dictionary and **exploit geometrical structure** of the topology to parameterize it.
2. Use **Hodge theory** to impose a "***separated***" parameterization.
3. Make the dictionary **overcomplete** to exploit redundacy
4. Setup an efficient iterative **alternating-direction algorithm** to solve the resulting **non-convex** problem

Define the learnable dictionary as a J-th order **Cell Convolutional FIR Filter**

$$D = h^{(id)} I + \sum_{j=1}^{J} h_j (L_k)^j$$

Dictionary Learning

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Define the learnable dictionary as a J-th order **Cell Convolutional FIR Filter**

$$D = \underline{h^{(id)}} I + \sum_{j=1}^{J} \underline{h_j} (L_k)^j$$

$$\boxed{\mathbf{h} \in \mathbb{R}^{J+1}}$$

Dictionary Learning

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Recall the definition of k-th order Hodge Laplacian matrix

$$L_k = \mathbf{B}_k^T \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^T = L_k^{(d)} + L_k^{(u)}$$

Exploit the **Hodge Decomposition** for topological signals in order to **independently parameterize** lower and upper information

$$D = h^{(id)} I + \sum_{j=1}^{J} h_j^{(u)} (L_k^{(u)})^j + h_j^{(d)} (L_k^{(d)})^j$$

Dictionary Learning

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Independently parameterize lower and upper information

$$D = h^{(id)} I + \sum_{j=1}^{J} h_j^{(u)} (L_k^{(u)})^j + h_j^{(d)} (L_k^{(d)})^j$$

$$\mathbf{h} \in \mathbb{R}^{2J+1}$$

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Create an **overcomplete** dictionary by concatenating P sub-dictionaries, i.e.:

$$\mathbf{D} = \{D_1, D_2, \ldots, D_P\} \in \mathbb{R}^{M \times PM}$$

where each sub-dictionary is a FIR Cell Filter with **Separated Hodge Laplacian** parameterization

$$D_i = h_i^{(id)} I + \sum_{j=1}^{J} h_{ij}^{(u)} (L_k^{(u)})^j + h_{ij}^{(d)} (L_k^{(d)})^j \quad , \; i = 1, \ldots, P$$

Dictionary Learning

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$$\boxed{\dim(\mathbf{x}) = MP > M = \dim(\mathbf{y})}$$

Overcomplete dictionary with **Separated Hodge Laplacian** parameterization

$$D_i = h_i^{(id)} I + \sum_{j=1}^{J} h_{ij}^{(u)} (L_k^{(u)})^j + h_{ij}^{(d)} (L_k^{(d)})^j$$

$$\boxed{\mathbf{h} \in \mathbb{R}^{(2J+1)P}}$$

Dictionary Learning

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Impose the **non-convex optimization problem** for **Topological Dictionary Learning**

$$(\mathbf{h}^*, \mathbf{X}^*) = \begin{cases} \arg\min_{\mathbf{h},\mathbf{X}} \|\mathbf{Y} - \mathbf{D}\mathbf{X}\|_F^2 + \gamma\|\mathbf{h}\|_2^2 \\ s.t. \\ a)\ \|x_i\|_0 \leq K_0, \quad i = 1, \ldots, N \\ b)\ 0 \preceq D_p \preceq dI, \quad p = 1, \ldots, P \\ c)\ (d-\varepsilon)I \preceq \sum_{p=1}^{P} D_p \preceq (d+\varepsilon)I \\ d)\ D_p \quad \text{defined as before}, \quad p = 1, \ldots, P \end{cases}$$

Frequency behaviour control

Dictionary Learning

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Idea:

1. Build a **learnable** dictionary and **exploit geometrical structure** of the topology to parameterize it.
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3. Make the dictionary **overcomplete** to exploit redundacy
4. Setup an efficient iterative **alternating-direction algorithm** to solve the resulting **non-convex** problem

Decompose the problem in two sub-problems and alternate the optimization of the **dictionary** and the **sparse representation** keeping one of the two variables fixed.

$$\mathbf{h}^{[t+1]} = \begin{cases} \arg\min_{\mathbf{h}} \|\mathbf{Y} - \mathbf{D}(\mathbf{h})\mathbf{X}^{[t]}\|_F^2 + \gamma\|\mathbf{h}\|_2^2 \\ s.t. \\ b)\ 0 \preceq \mathbf{D}_p \preceq dI, \quad p = 1, \dots, P \\ c)\ (d-\varepsilon)I \preceq \sum_{p=1}^{P} \mathbf{D}_p \preceq (d+\varepsilon)I \\ d)\ \mathbf{D}_p \quad \text{defined as before}, \quad p = 1, \dots, P \end{cases}$$

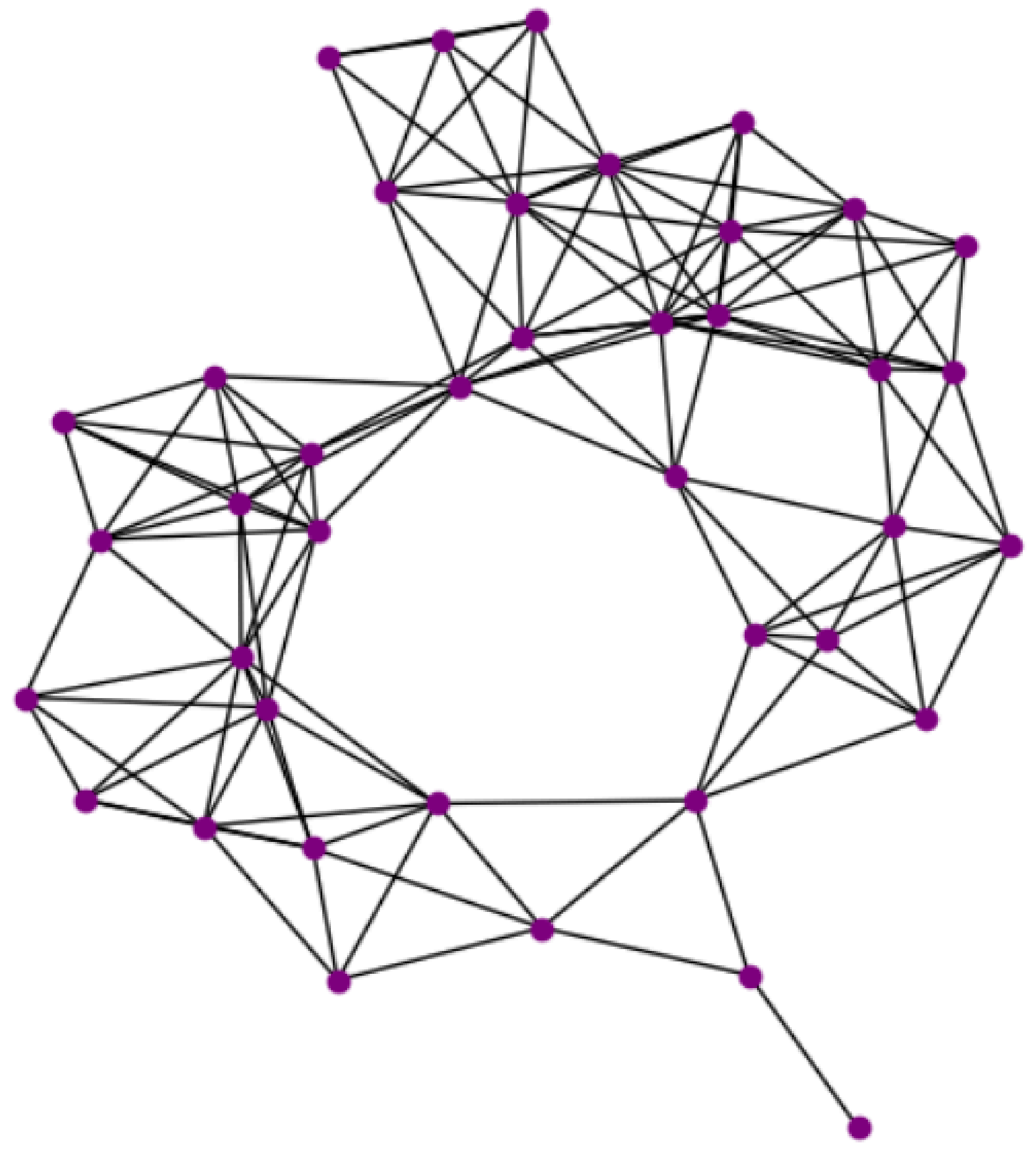
SDP

$$\mathbf{X}^{[t+1]} = \begin{cases} \arg\min_{\mathbf{X}} \|\mathbf{Y} - \mathbf{D}(\mathbf{h}^{[t+1]})\mathbf{X}\|_F^2 \\ s.t. \\ a)\ \|x_i\|_0 \leq K_0, \quad i = 1, \dots, N \end{cases}$$

OMP

Synthetic Data

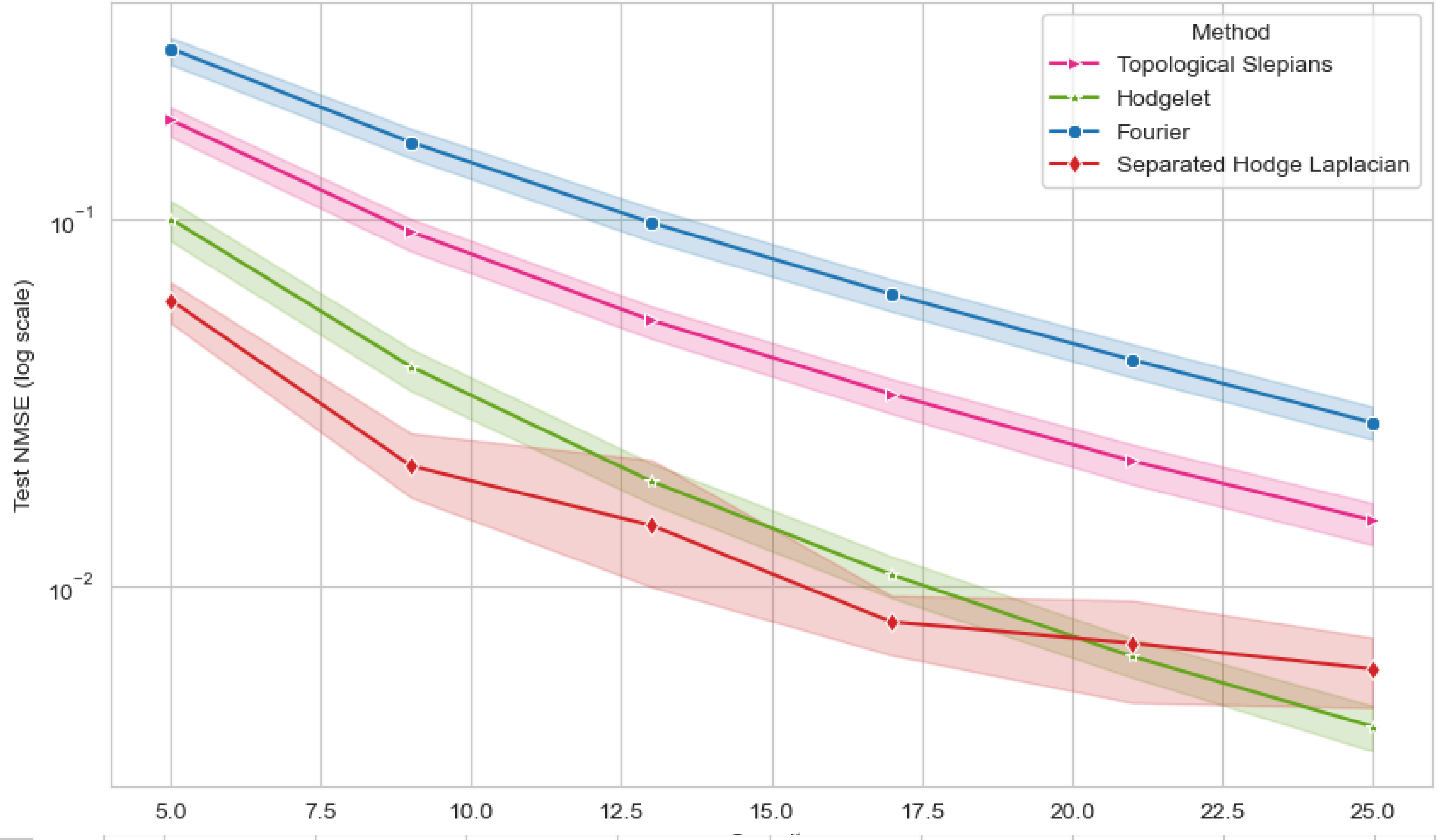
- Simulate 3 different types of dataset using 3 different generating functions for each proposed dictionary parameterization:
  - **Edge Laplacian**
  - **Joint Hodge Laplacian**
  - **Separated Hodge Laplacian**
- Set **P=3** and **J=2**
- **10 datasets** for each parameterization
- **220 edge signals** for each dataset defined over the same simulated **second order cell complex**
- The cell complex has:
  - **40** nodes
  - **100** edges
  - a max of **62** possible triangles

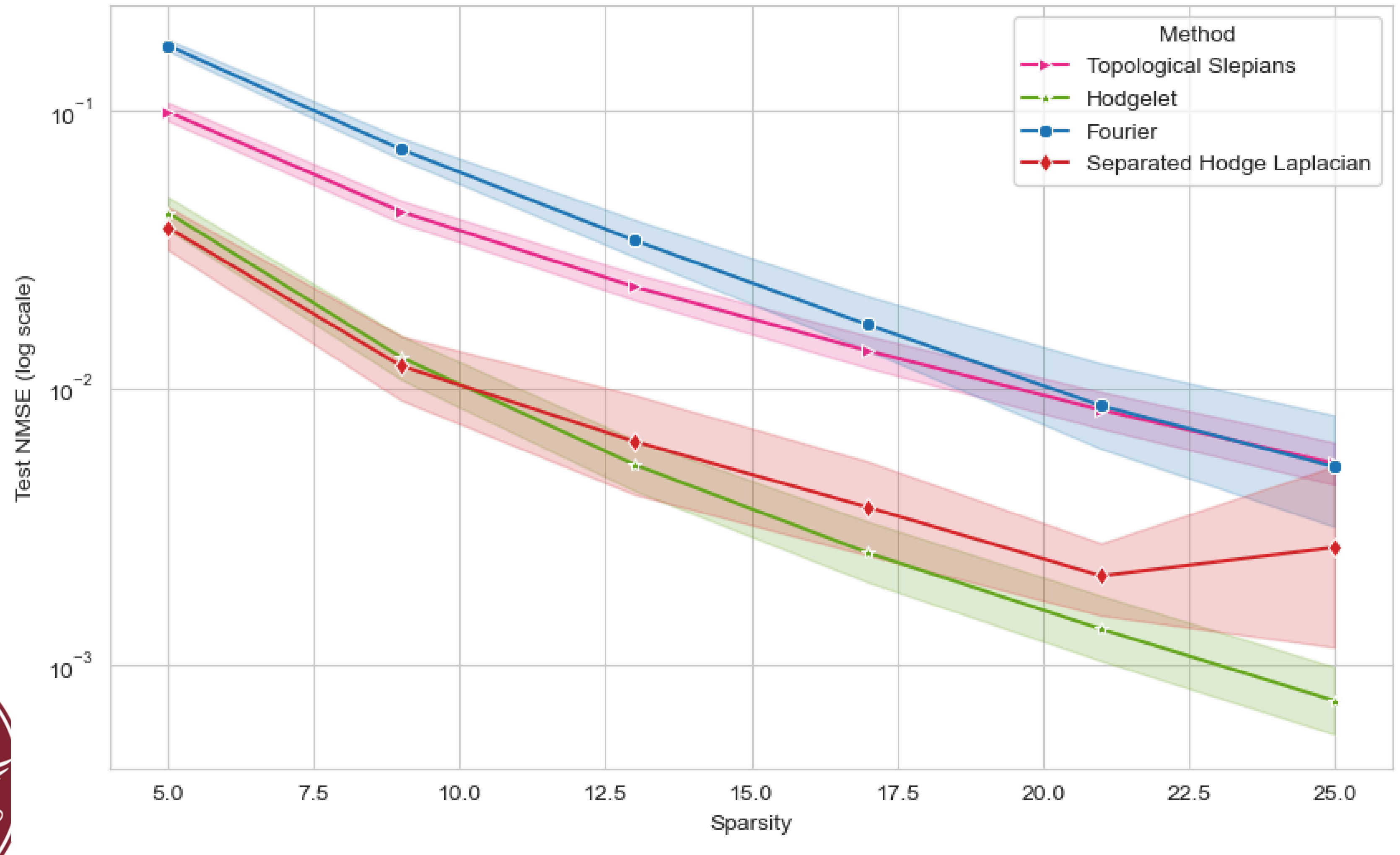


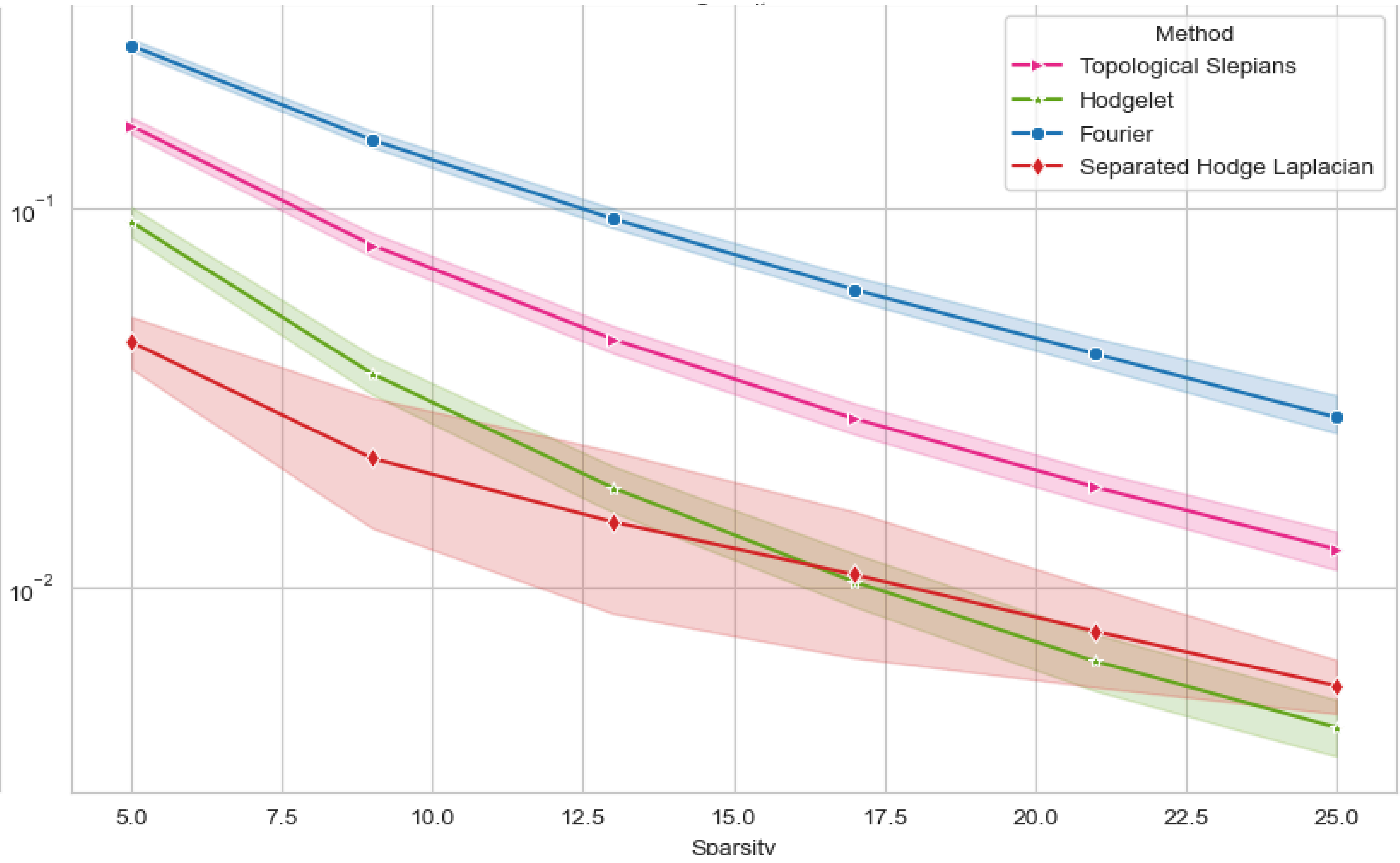
Dictionary Learning

Results on synthetic data

Comparison between dictionary learning algorithm and sparse representation with ***analytical dictionaries***



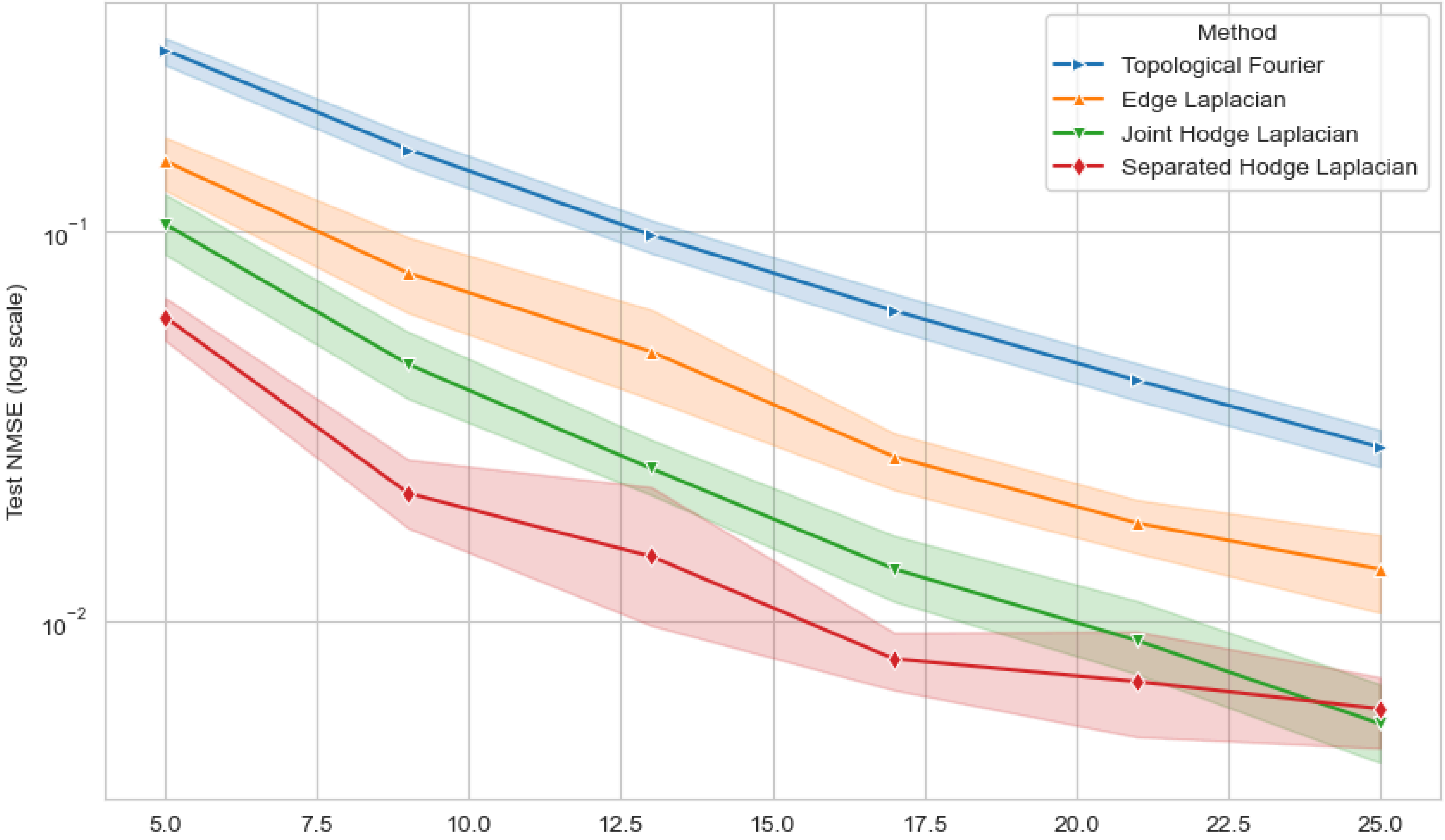


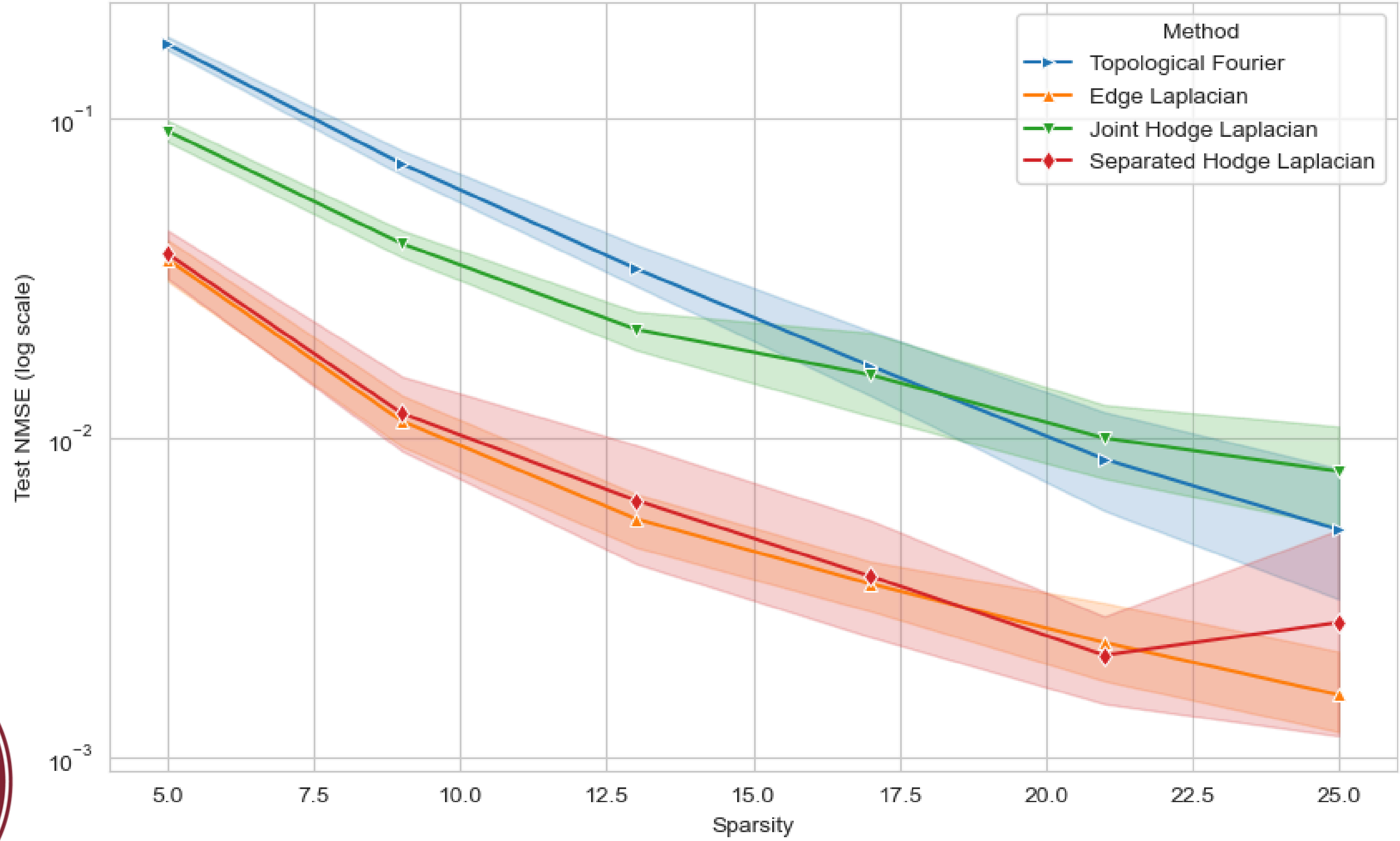


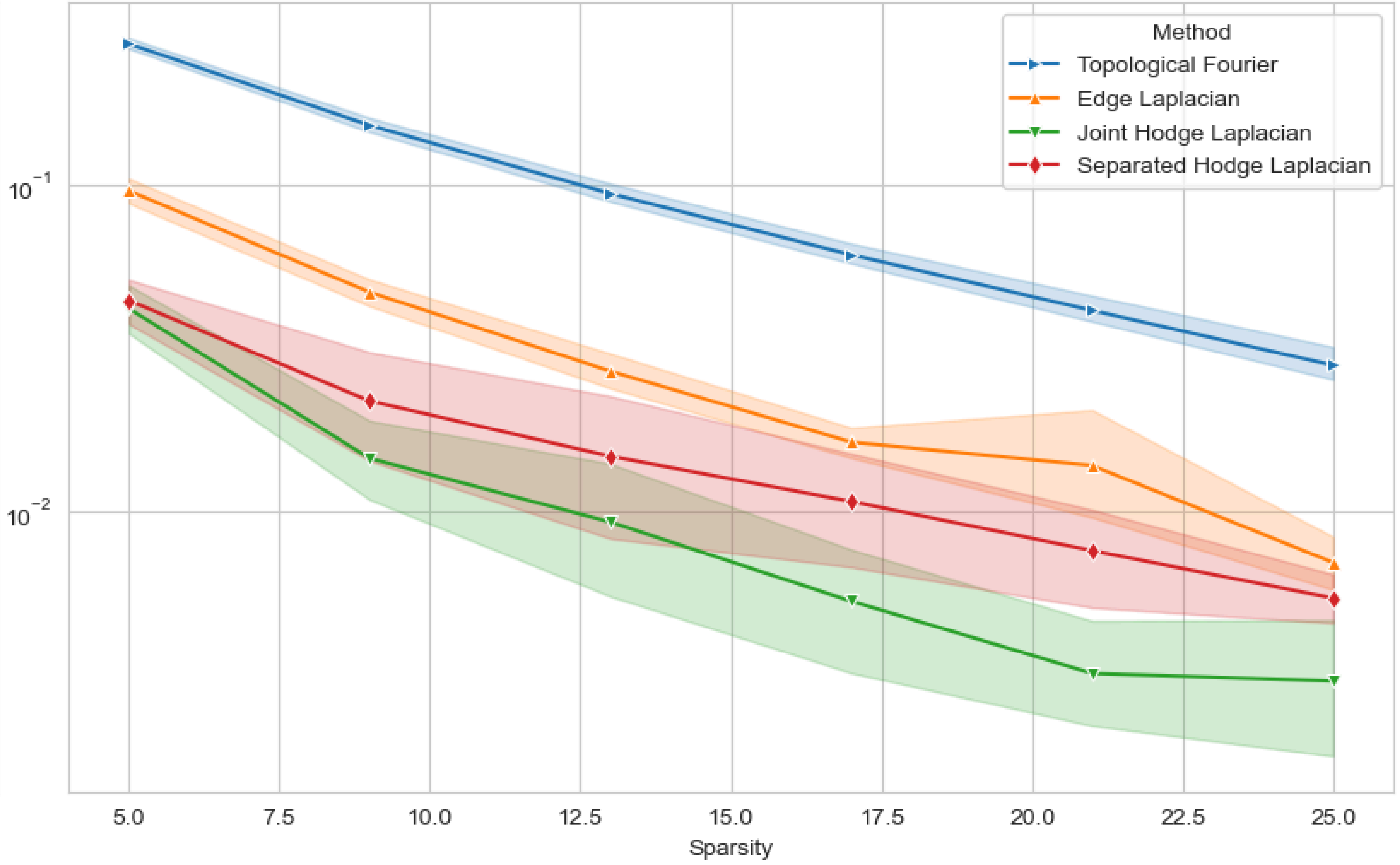
Dictionary Learning

Results on synthetic data

Comparison between dictionary learning algorithm with different ***dictionary parameterizations***







Topology Learning Integration

Introduce **Topology Learning** process at the end of the alternating-direction algorithm:

- Begin with a **‘full’ (‘empty’) topology**
- Iteratively update the **diagonal masking matrix** T by setting a diagonal element to 0 (1)
- **Update the topology** by multiplying the incidence matrix by the masking matrix

$$L_k^{(u)} = B_{k+1} T B_{k+1}^T$$

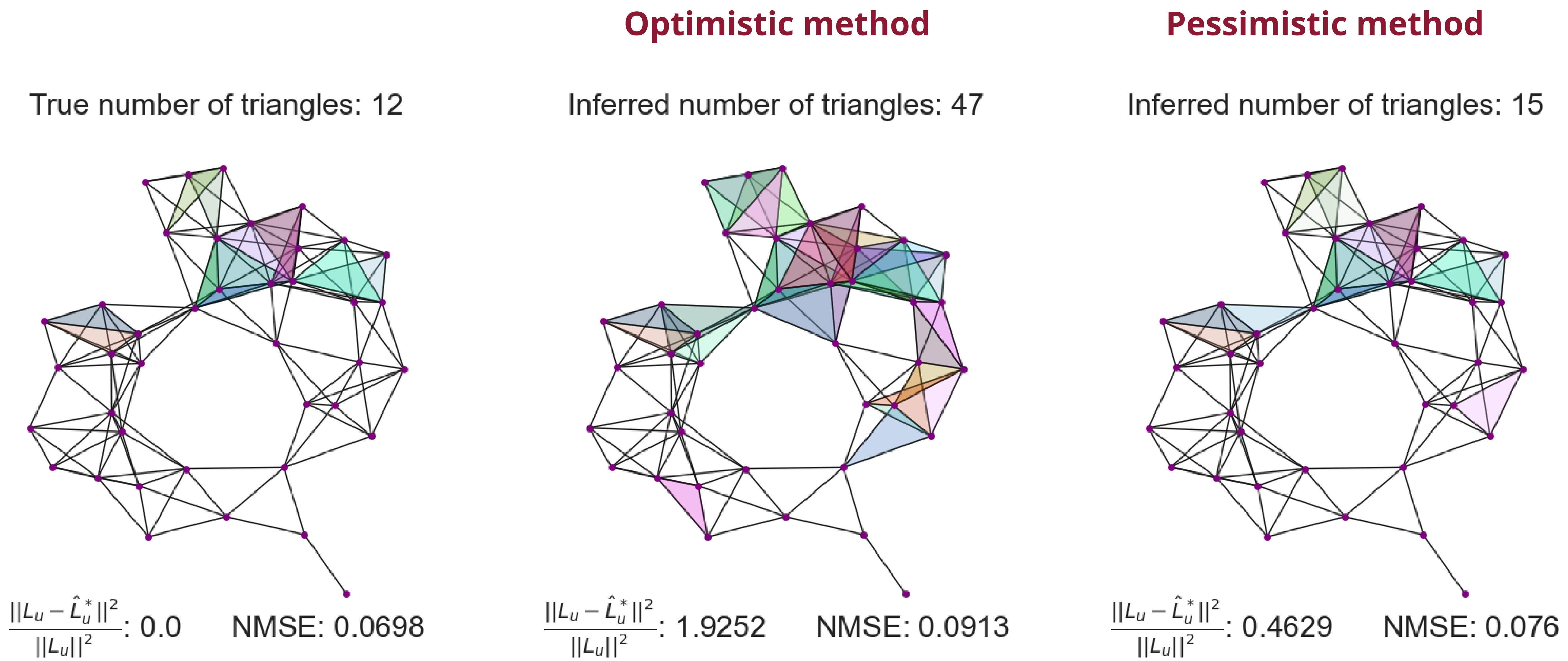
- Greedly select the optimal topology that minimizes approximation error
- Newly apply the Alternating-direction algorithm with the new selected Laplacian

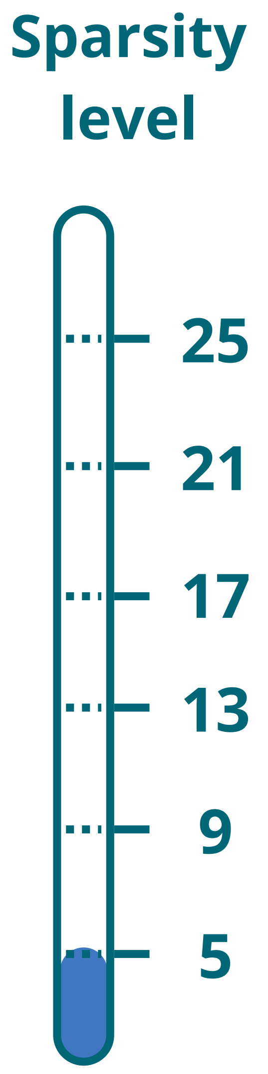
New relaxed assumptions:

- So far we assumed a **full cell complex** as ‘full’ i.e., all the cells belong to the topology
- Now we assume to only know the k-skeleton of our topology $L_k^{(d)}$
- Greedly infer the upper adjacency structure, and jointly learn the dictioanry the upper Laplacian $L_k^{(u)}$

Joint Topology and Dictionary Learning

Results on synthetic data





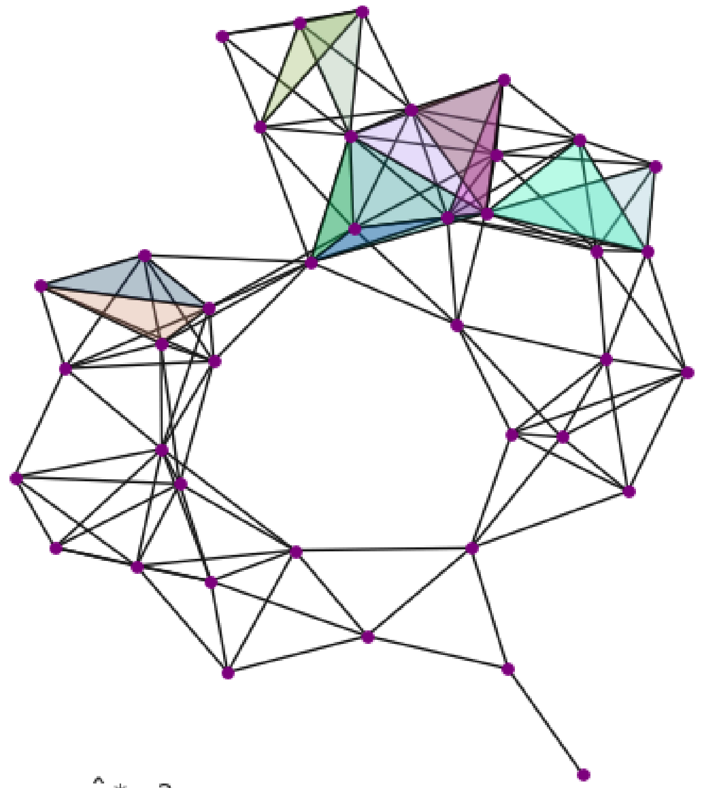
% of included triangles in the true topology

Joint Topology and Dictionary Learning

Results on synthetic data

Optimistic method | **Pessimistic method**

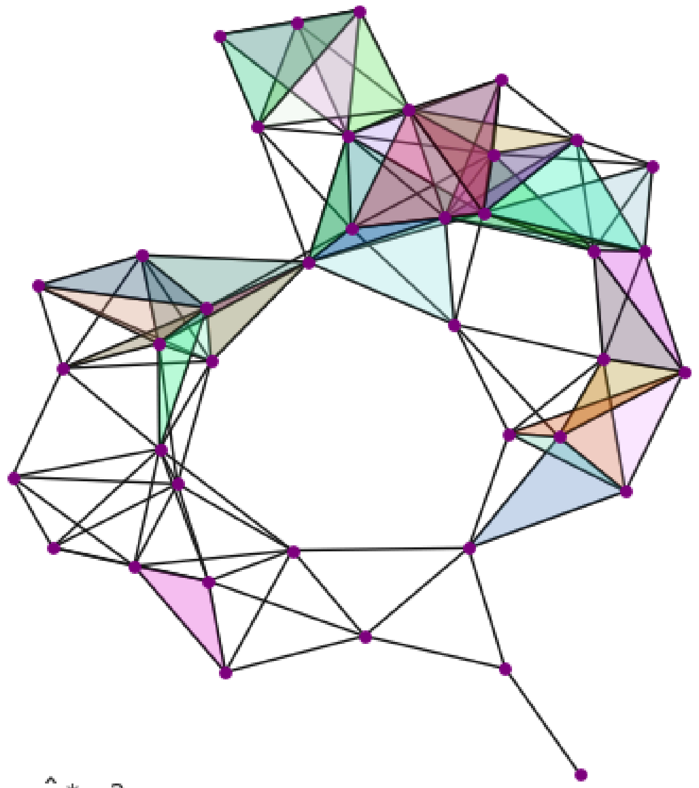
True number of triangles: 12



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0378

Optimistic method

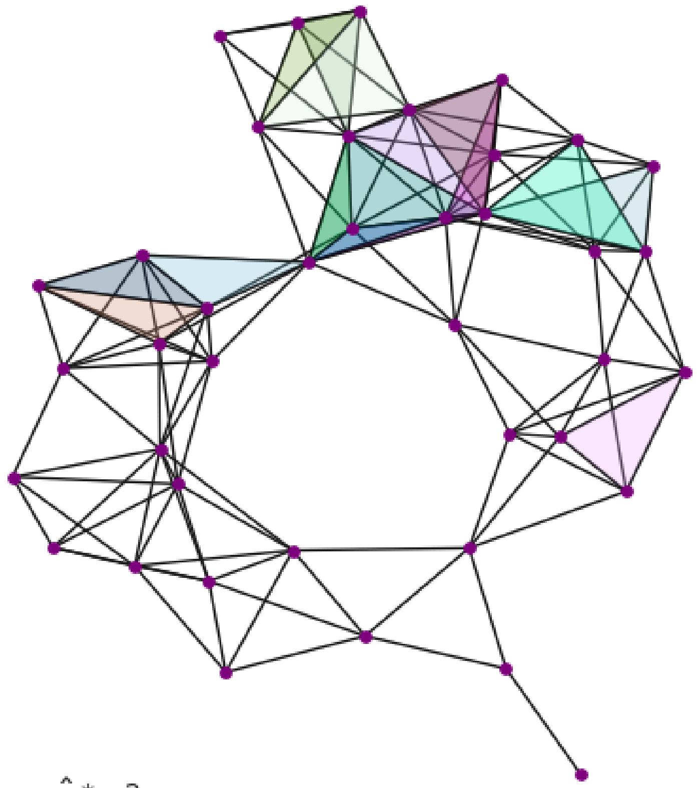
Inferred number of triangles: 42



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 1.7043 NMSE: 0.0345

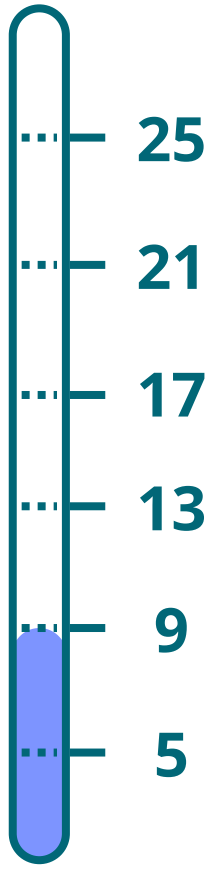
Pessimistic method

Inferred number of triangles: 16



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.5345 NMSE: 0.029

Sparsity level

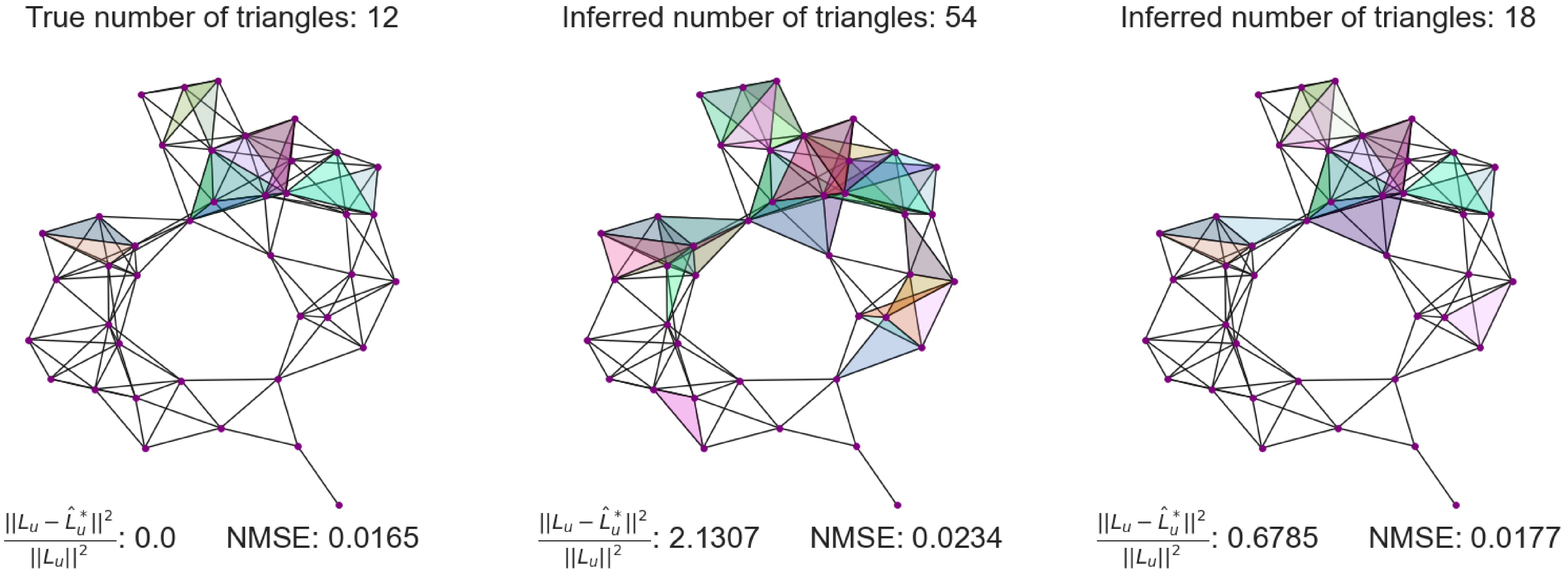


% of included triangles in the true topology

Joint Topology and Dictionary Learning

Results on synthetic data

Optimistic method **Pessimistic method**



Sparsity level

25
21
17
13
9
5

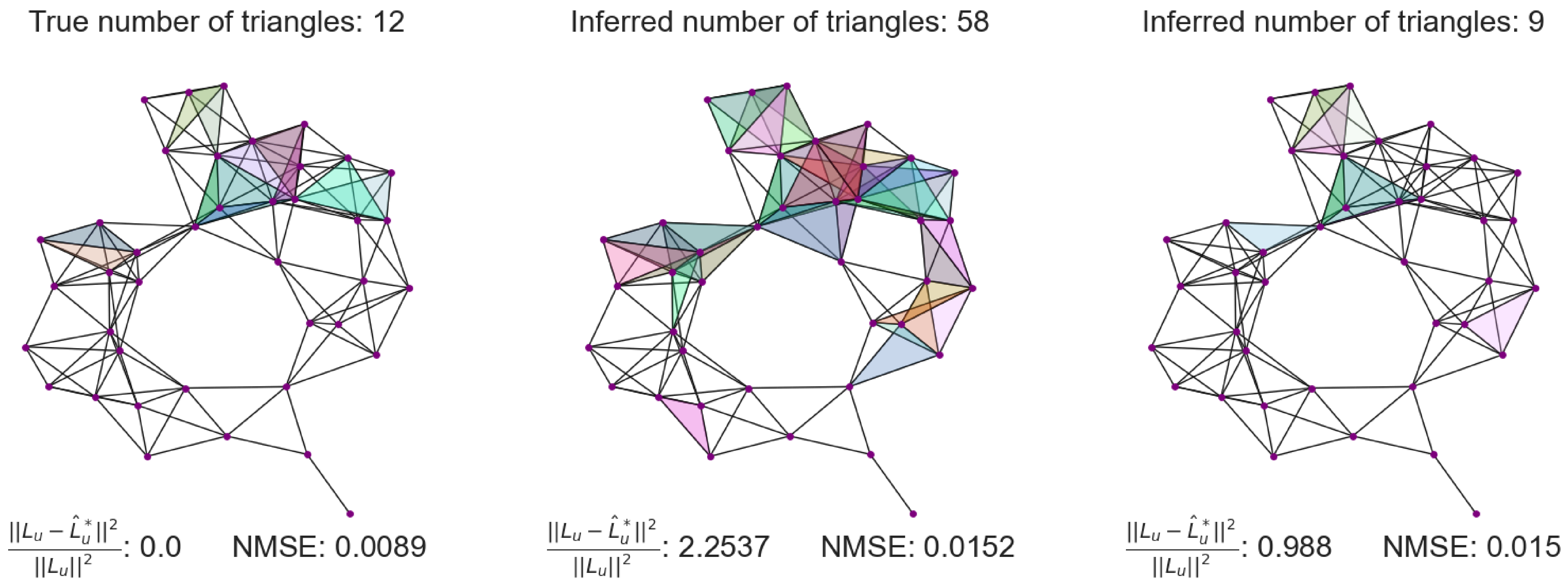
% of included triangles in the true topology

20% 50% 80% 90%

Joint Topology and Dictionary Learning

Results on synthetic data

Optimistic method **Pessimistic method**



Sparsity level

25
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Joint Topology and Dictionary Learning

Results on synthetic data

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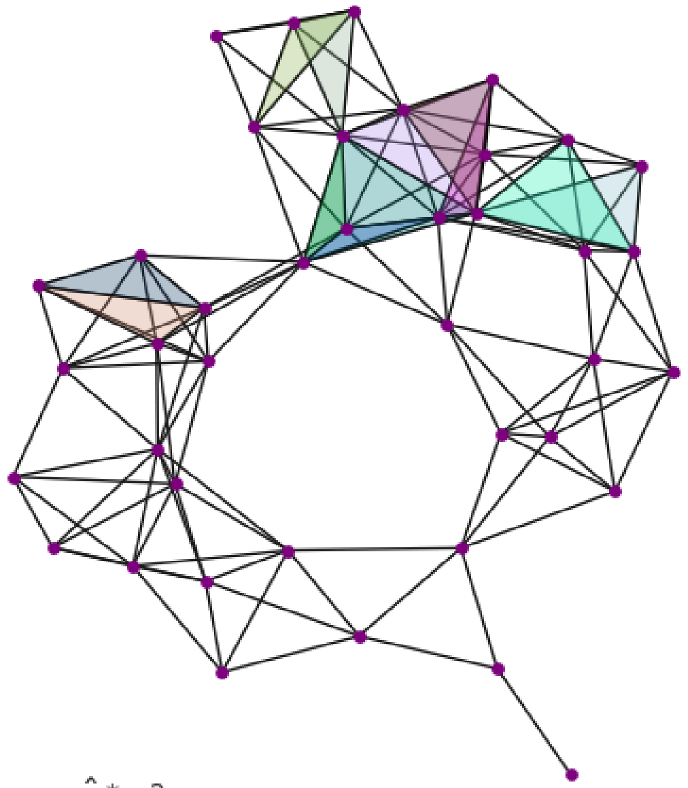
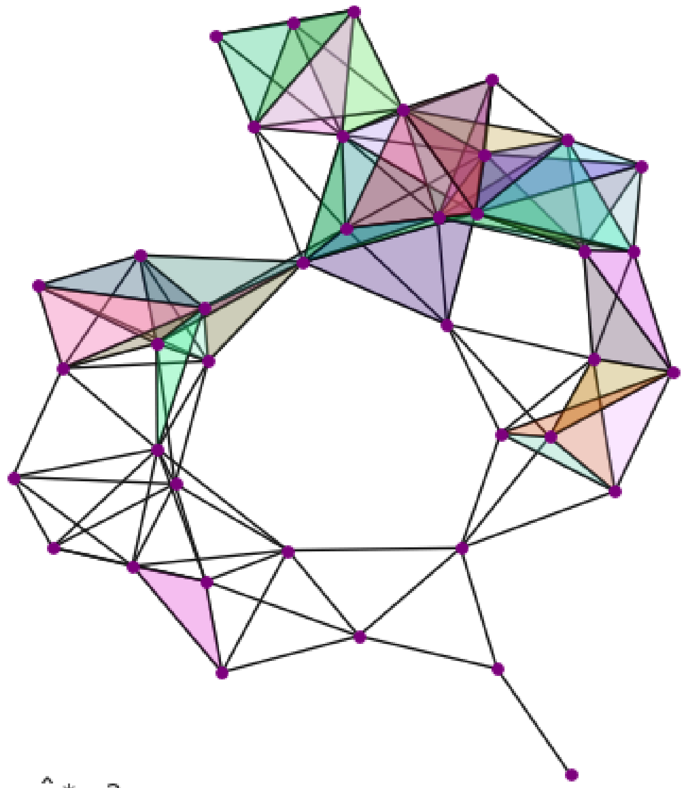


$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0073

Optimistic method

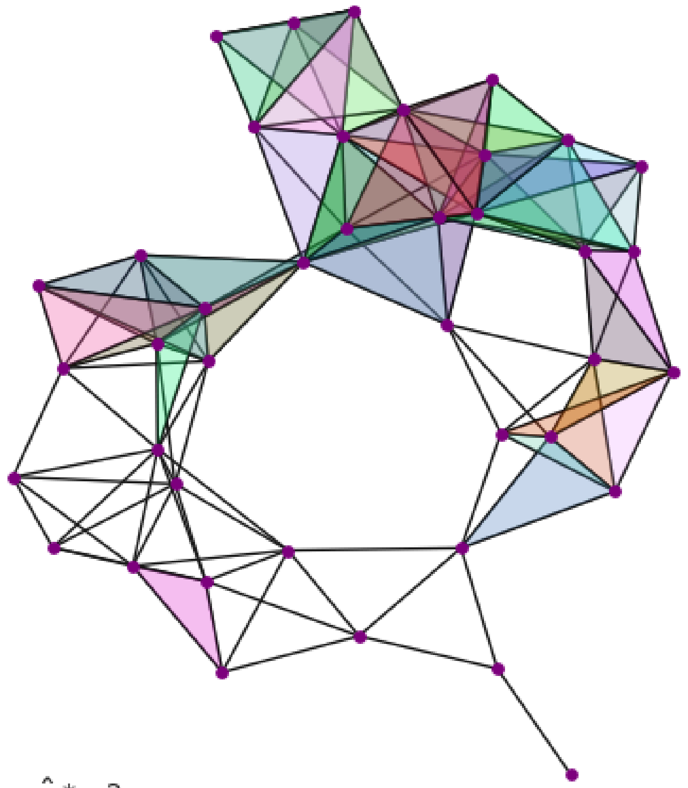
Inferred number of triangles: 49



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 1.9781 NMSE: 0.0083

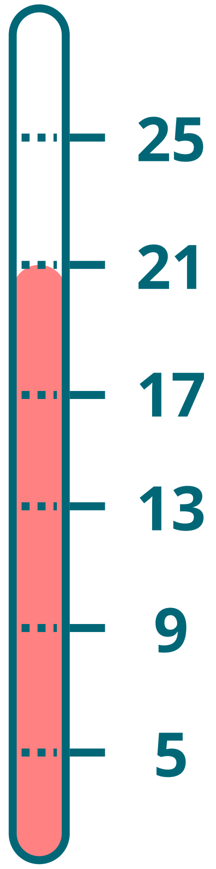
Pessimistic method

Inferred number of triangles: 62



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 1.0764 NMSE: 0.0498

Sparsity level

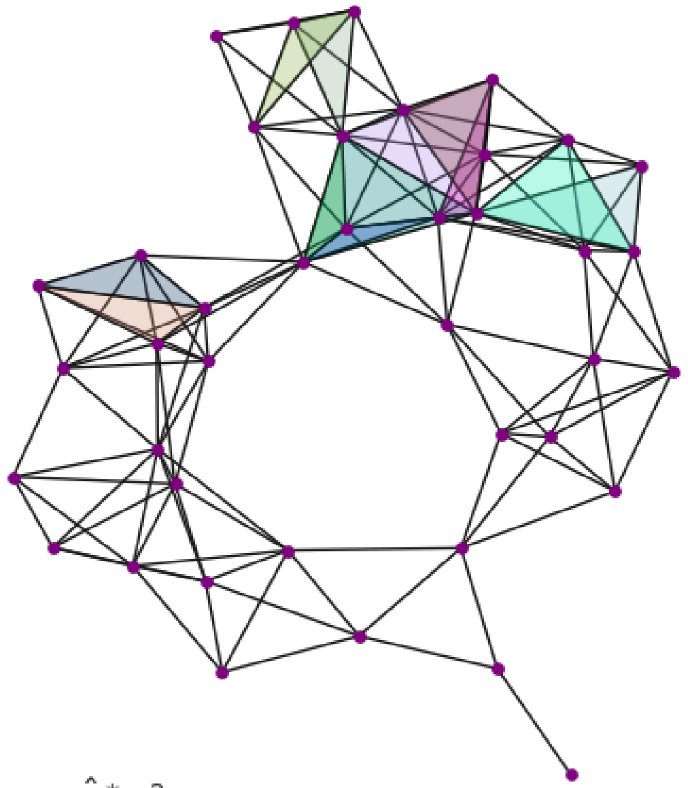


% of included triangles in the true topology

Joint Topology and Dictionary Learning

Results on synthetic data

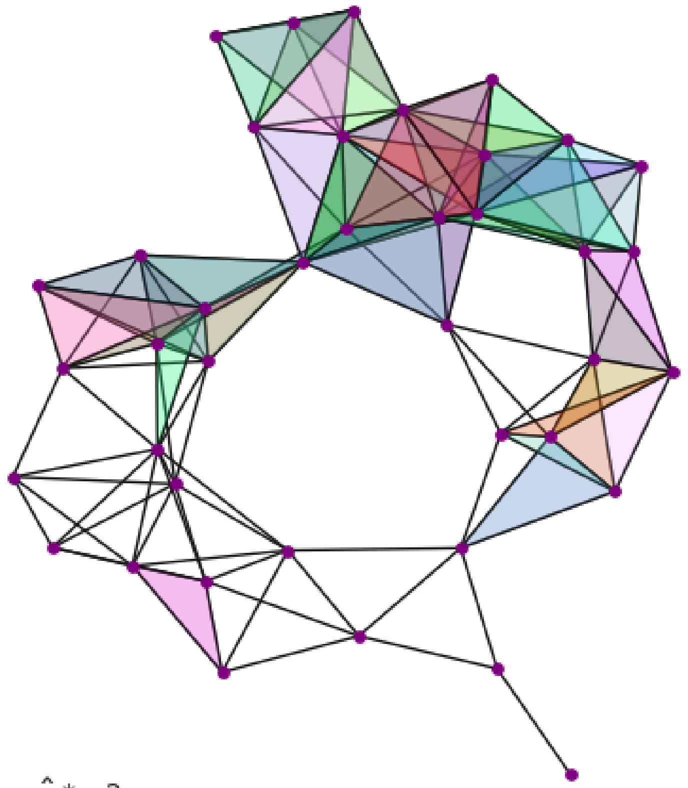
True number of triangles: 12



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0123

Optimistic method

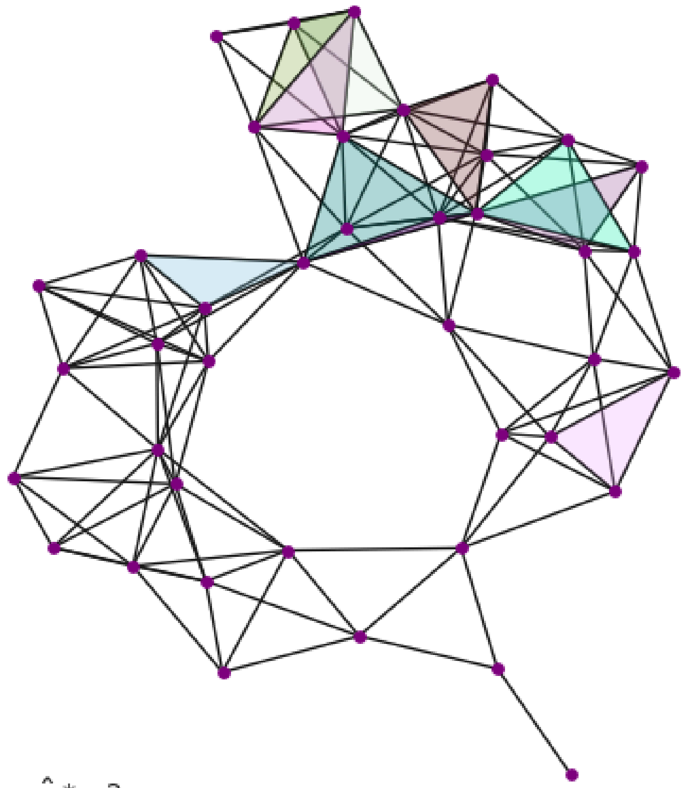
Inferred number of triangles: 62



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 2.3604 NMSE: 0.0077

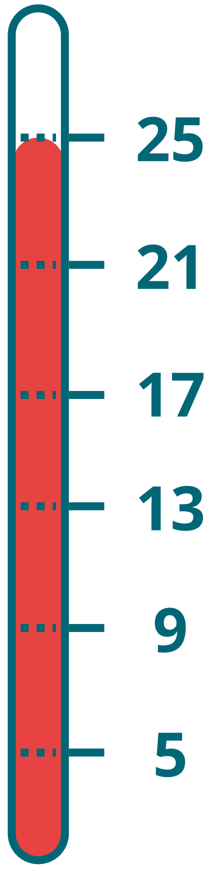
Pessimistic method

Inferred number of triangles: 11



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.9636 NMSE: 0.0163

Sparsity level

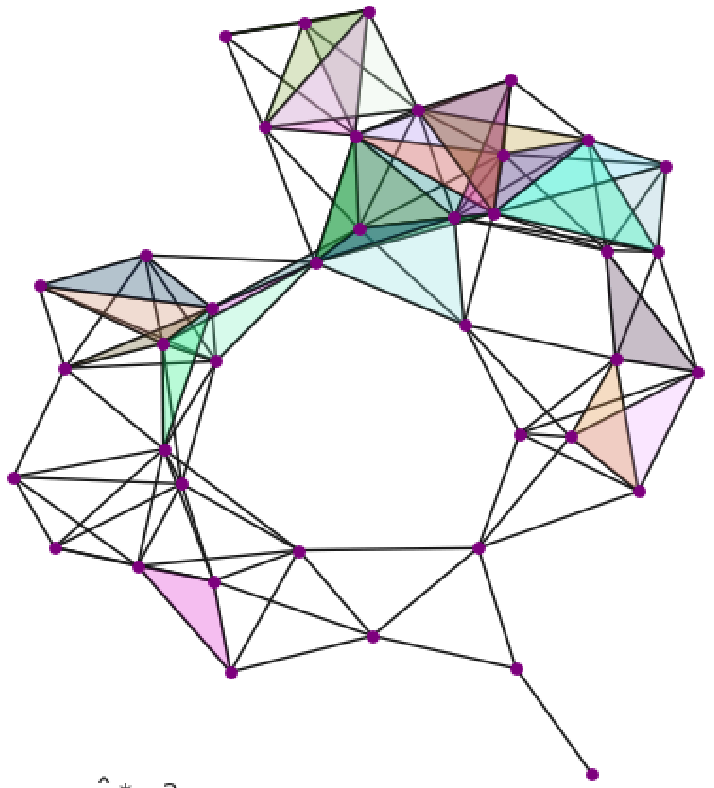


% of included triangles in the true topology

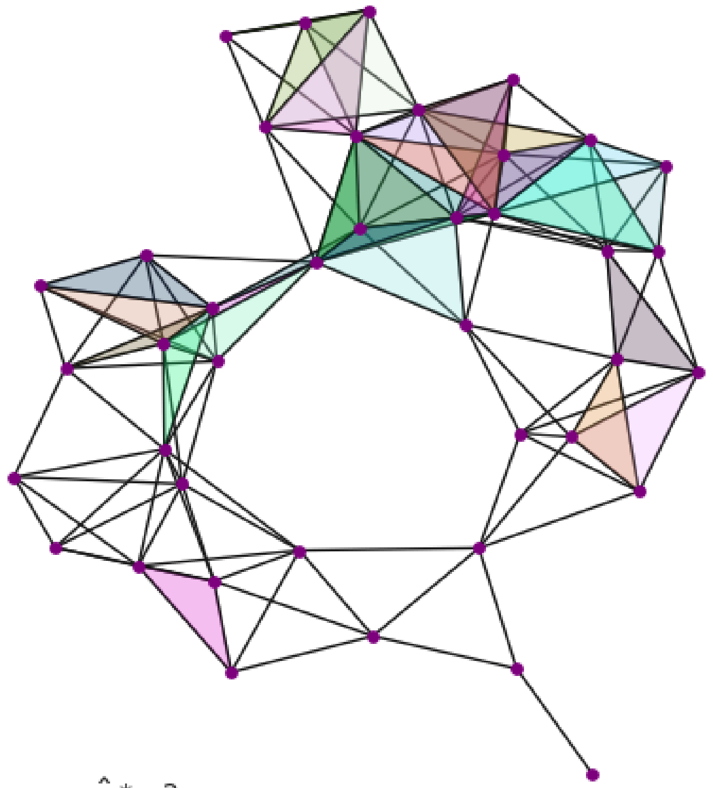
Joint Topology and Dictionary Learning

Results on synthetic data

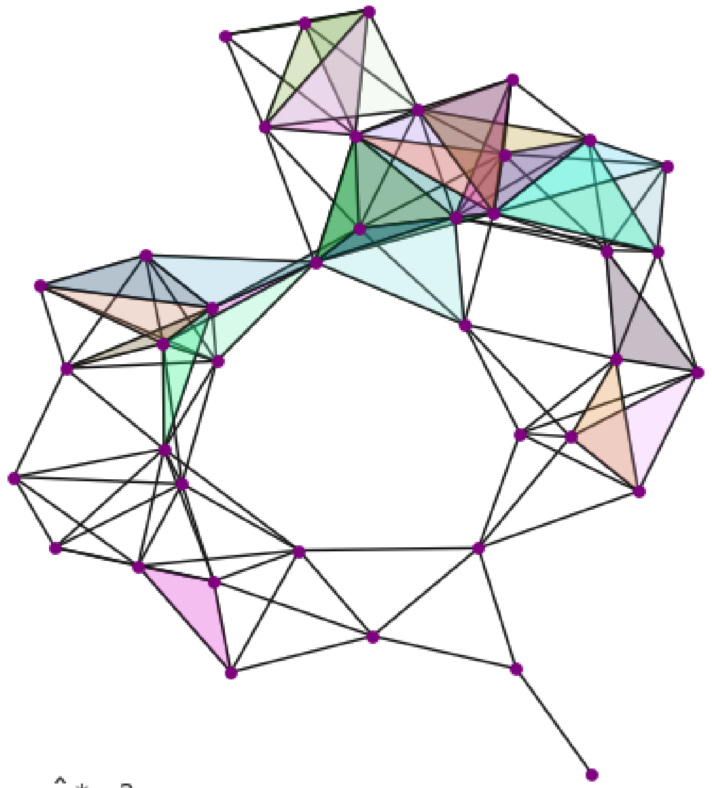
	Optimistic method	**Pessimistic method**
True number of triangles: 31	Inferred number of triangles: 31	Inferred number of triangles: 32



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0   NMSE: 0.0844



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0   NMSE: 0.0824



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.1558   NMSE: 0.0859

Sparsity level

25
21
17
13
9
5

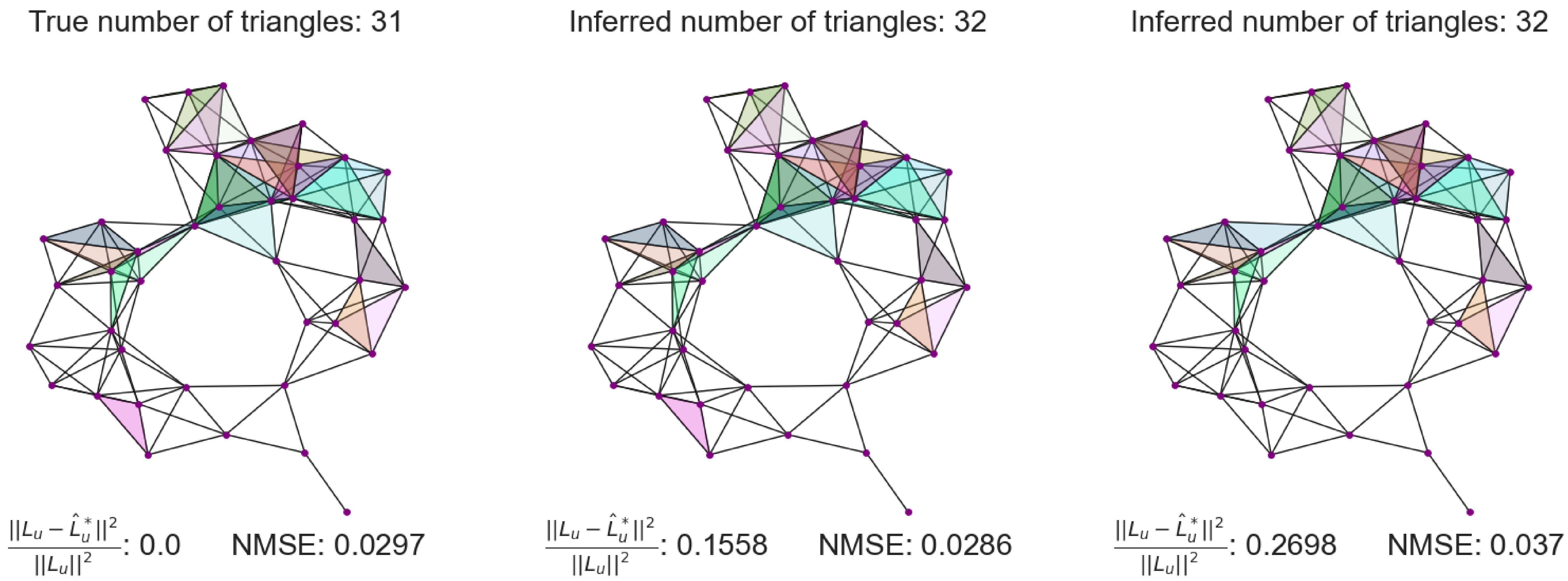
% of included triangles in the true topology

20% 50% 80% 90%

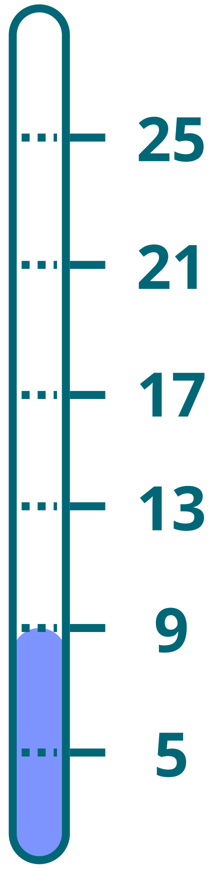
Joint Topology and Dictionary Learning

Results on synthetic data

Optimistic method **Pessimistic method**



Sparsity level

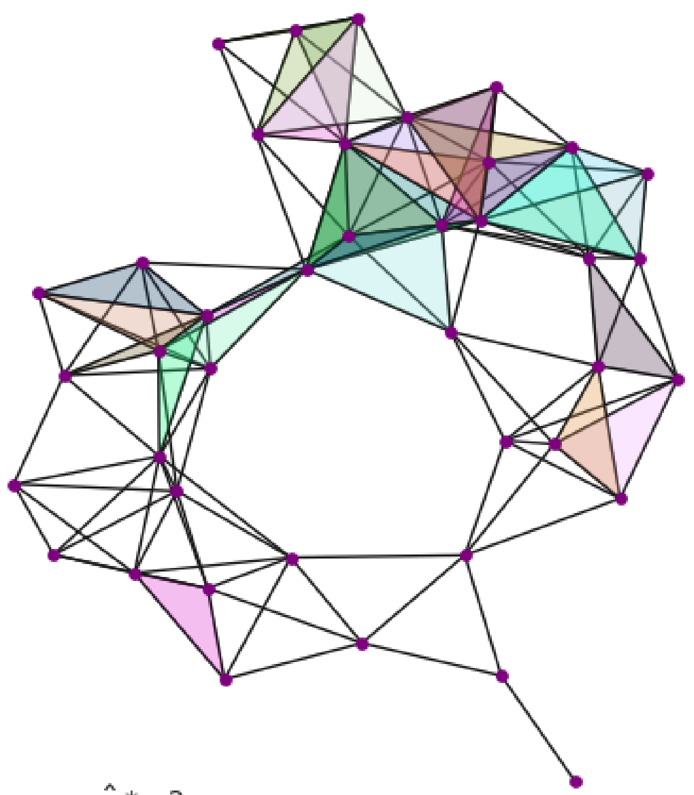


% of included triangles in the true topology

Joint Topology and Dictionary Learning

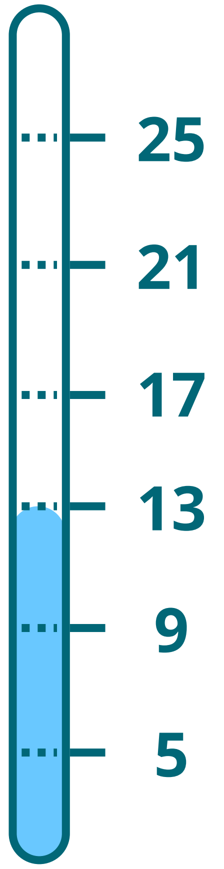
Results on synthetic data

	Optimistic method	Pessimistic method
True number of triangles: 31	Inferred number of triangles: 32	Inferred number of triangles: 25



$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.0 NMSE: 0.016	$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.1558 NMSE: 0.0163	$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.5295 NMSE: 0.0274

Sparsity level

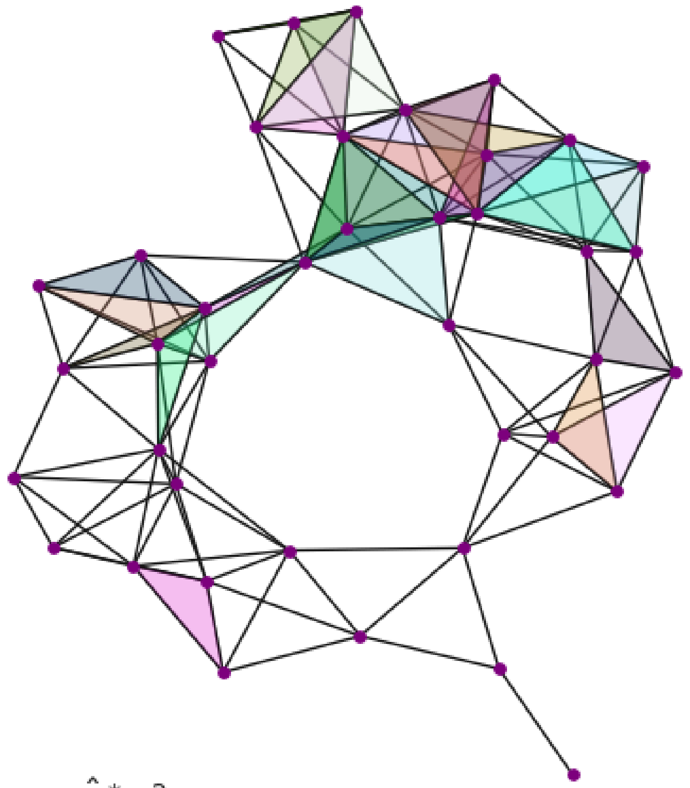
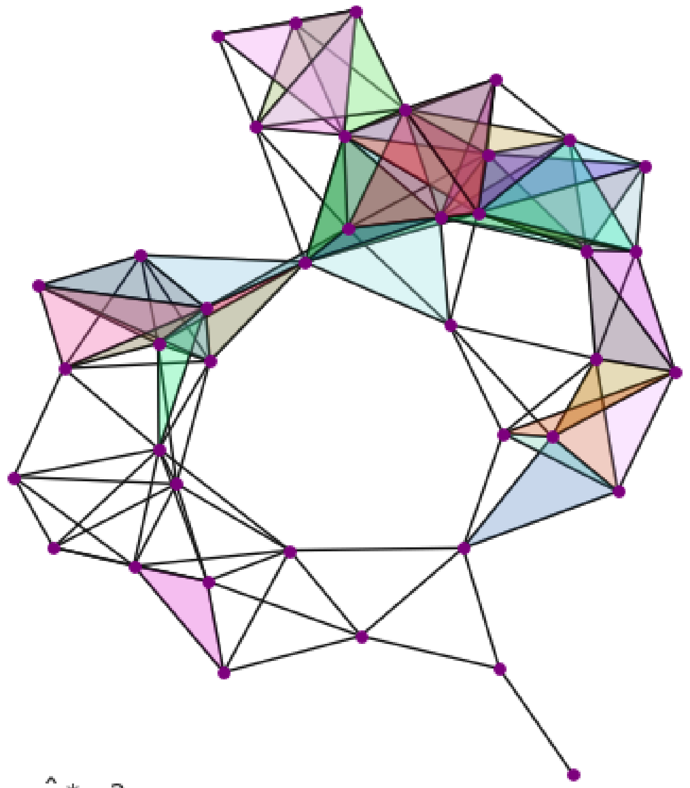
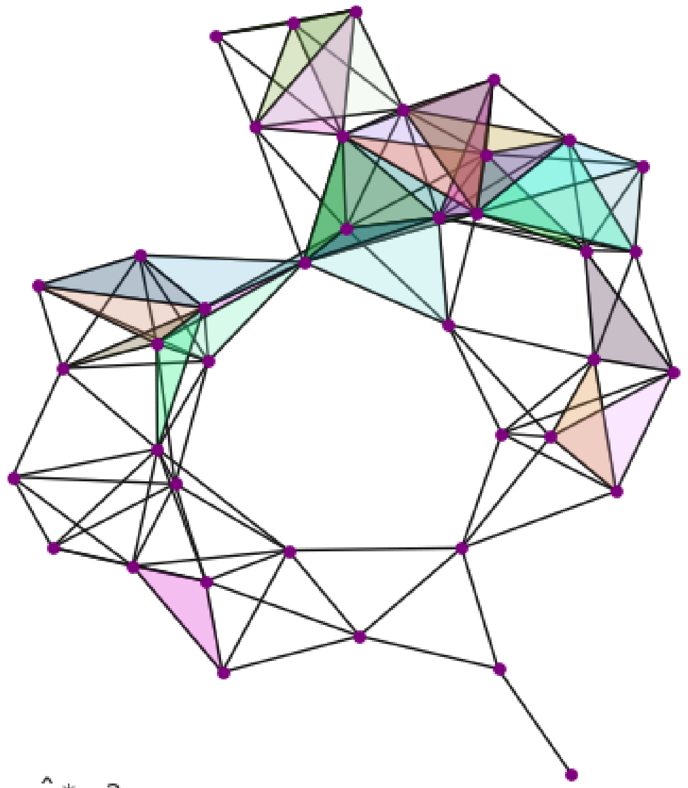


% of included triangles in the true topology

Joint Topology and Dictionary Learning

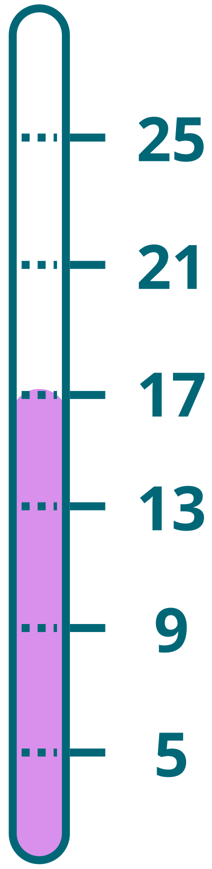
Results on synthetic data

	Optimistic method	Pessimistic method
True number of triangles: 31	Inferred number of triangles: 55	Inferred number of triangles: 34

$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.0 NMSE: 0.0176	$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.9052 NMSE: 0.0181	$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.2698 NMSE: 0.0095

Sparsity level



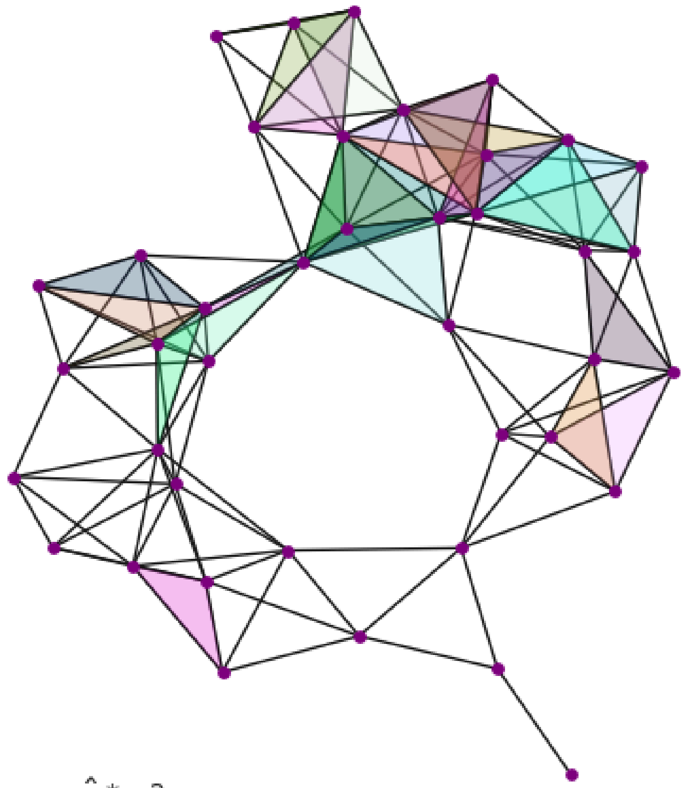
% of included triangles in the true topology

Joint Topology and Dictionary Learning

Results on synthetic data

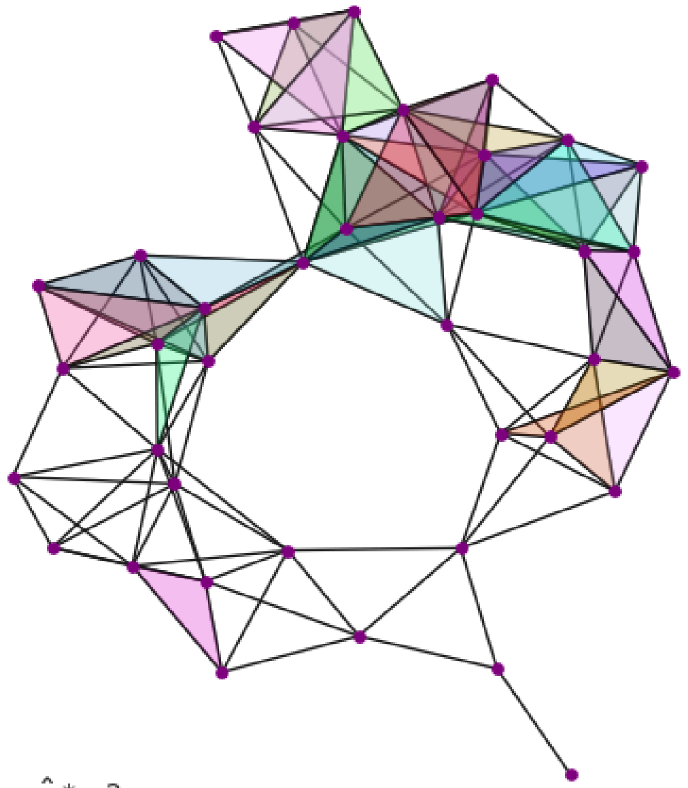
Optimistic method | **Pessimistic method**

True number of triangles: 31



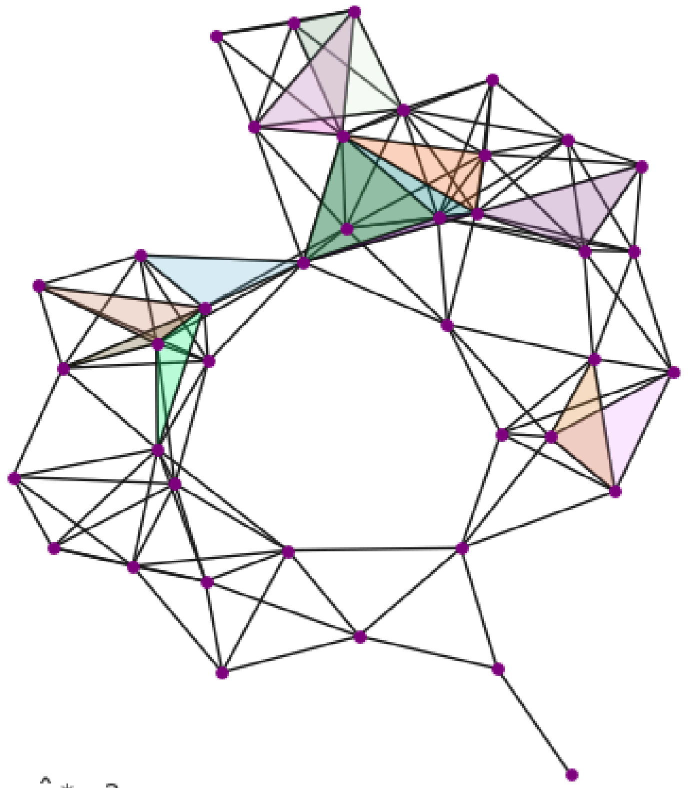
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0083

Inferred number of triangles: 50



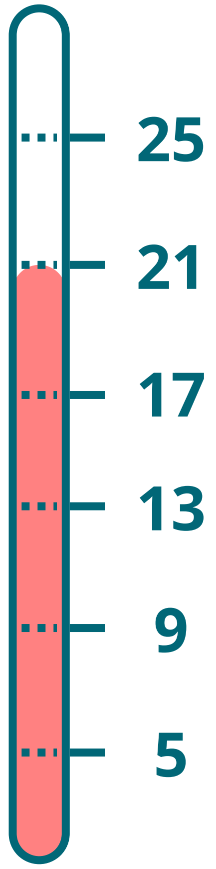
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.7857 NMSE: 0.0106

Inferred number of triangles: 14



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.7925 NMSE: 0.0217

Sparsity level

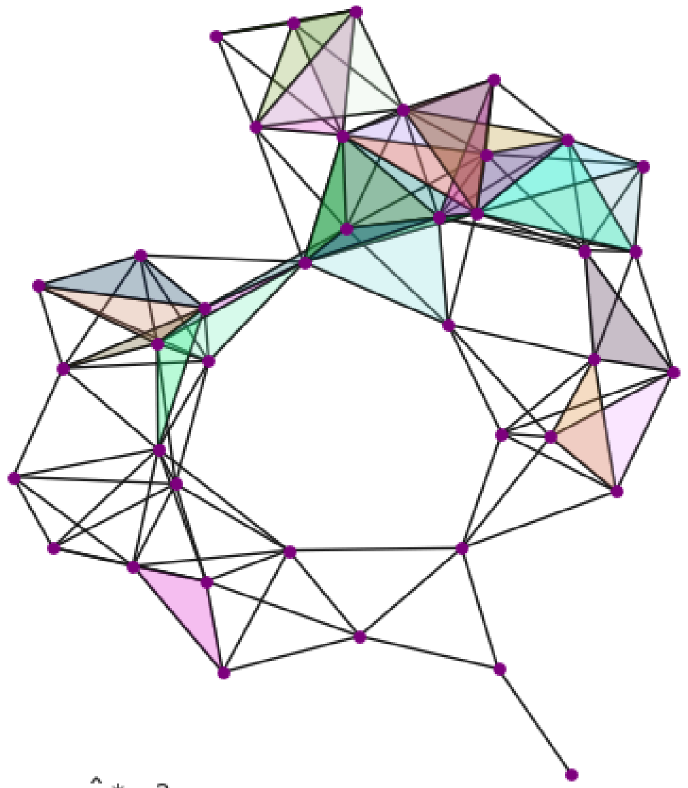


% of included triangles in the true topology

Joint Topology and Dictionary Learning

Results on synthetic data

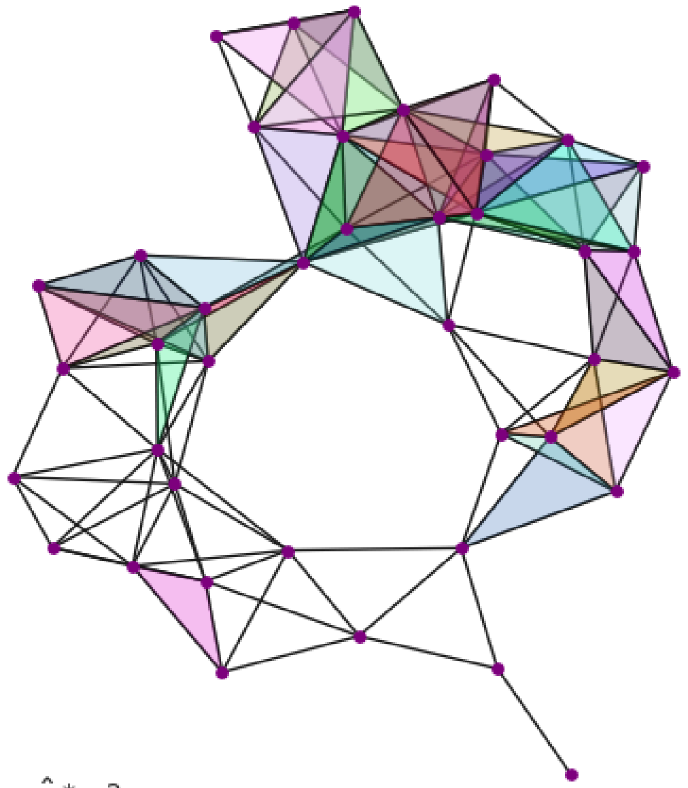
True number of triangles: 31



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0081

Optimistic method

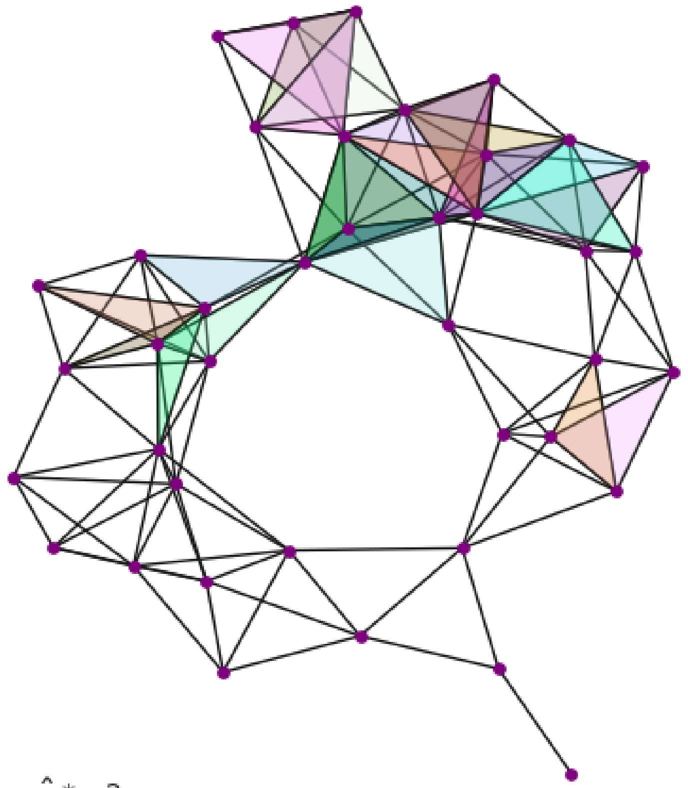
Inferred number of triangles: 57



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.9345 NMSE: 0.0076

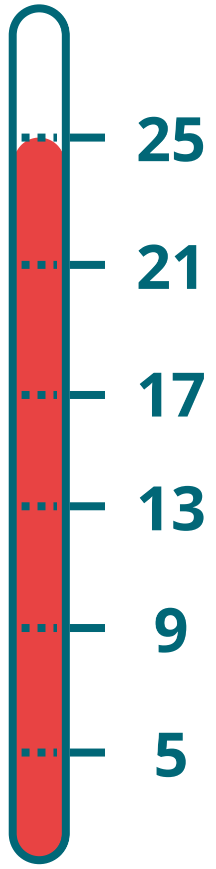
Pessimistic method

Inferred number of triangles: 29



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.4758 NMSE: 0.0096

Sparsity level

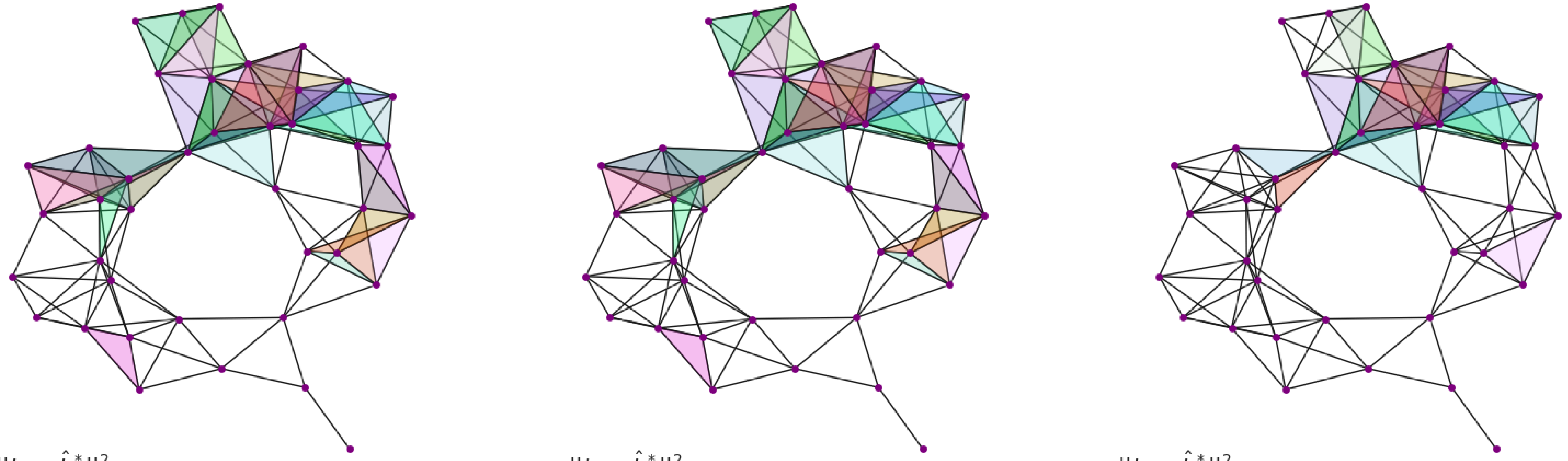


% of included triangles in the true topology

Joint Topology and Dictionary Learning

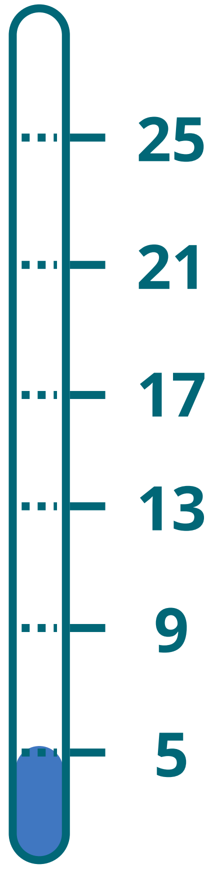
Results on synthetic data

	Optimistic method	**Pessimistic method**
True number of triangles: 49	Inferred number of triangles: 49	Inferred number of triangles: 28

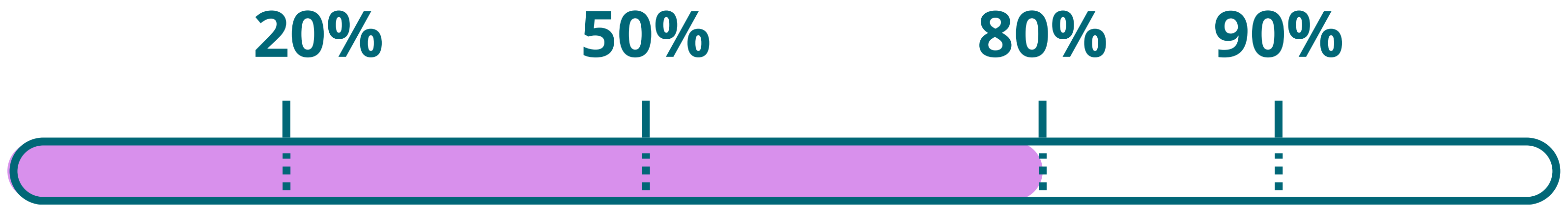


$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.0 NMSE: 0.0649	$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.0 NMSE: 0.0598	$\frac{\|\|L_u - \hat{L}_u^*\|\|^2}{\|\|L_u\|\|^2}$: 0.6091 NMSE: 0.1281

Sparsity level



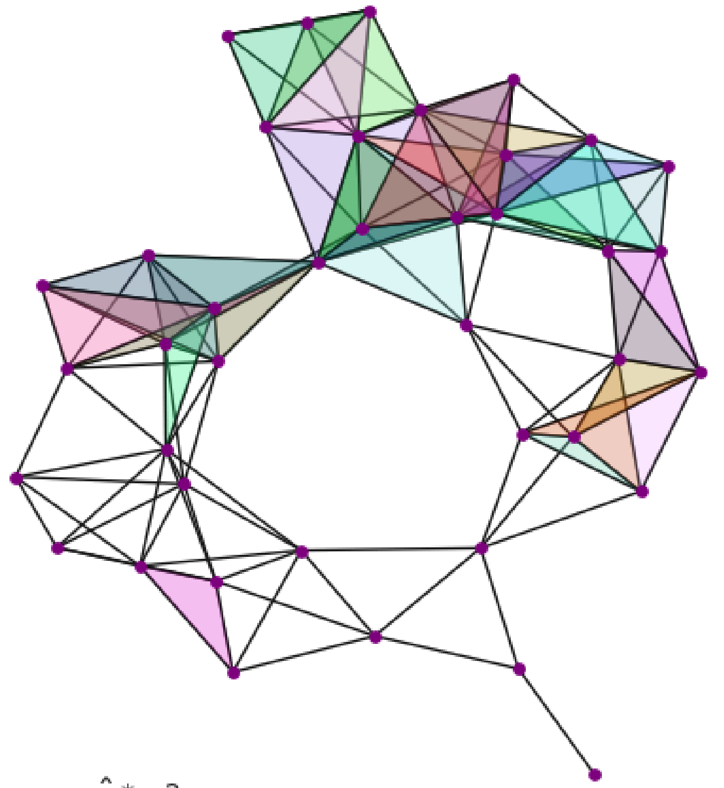
% of included triangles in the true topology



Joint Topology and Dictionary Learning

Results on synthetic data

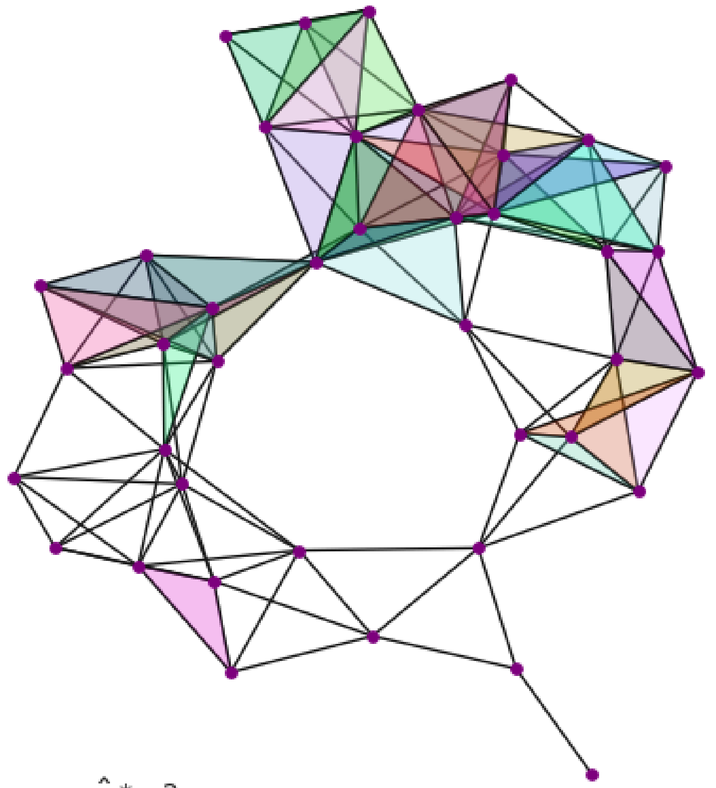
True number of triangles: 49



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0238

Optimistic method

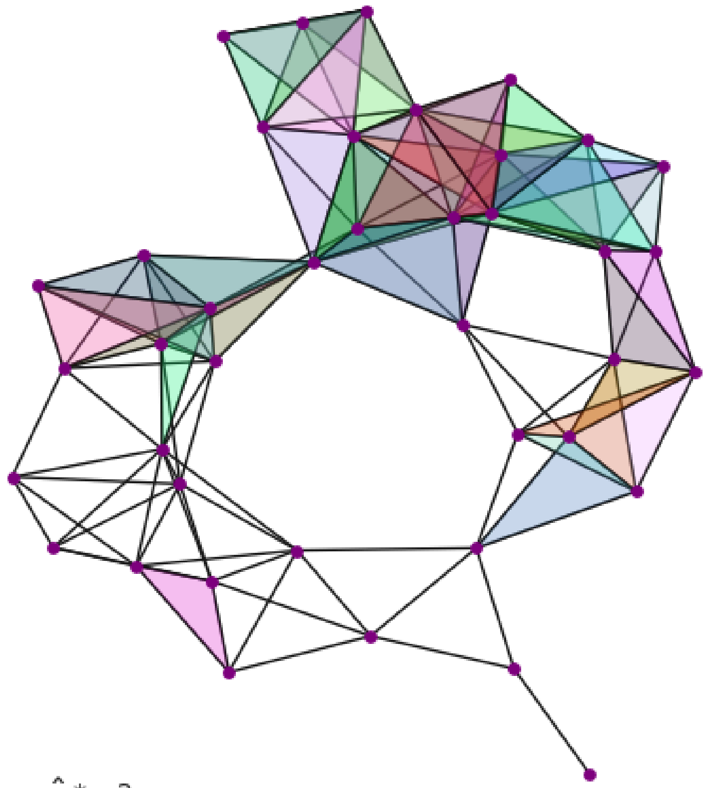
Inferred number of triangles: 49



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0205

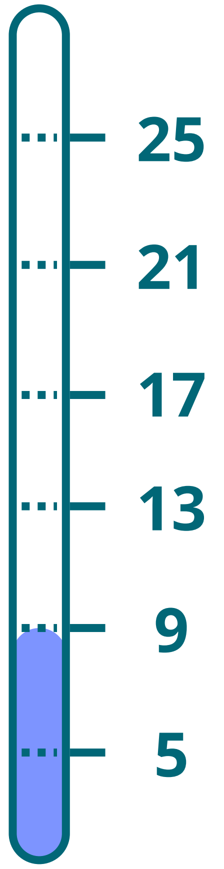
Pessimistic method

Inferred number of triangles: 62

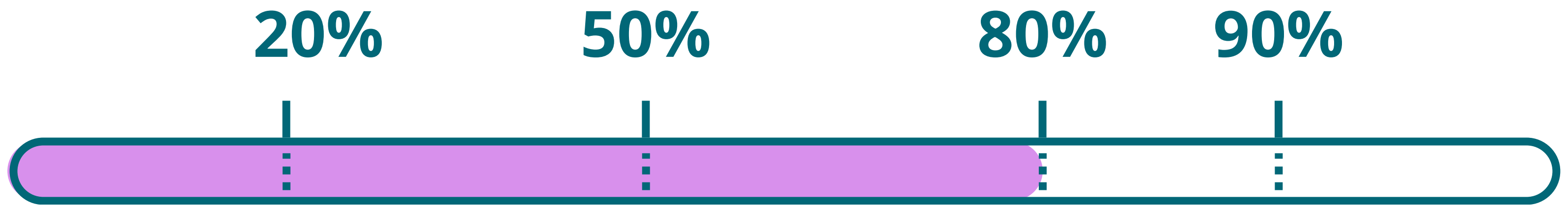


$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.9378 NMSE: 0.6433

Sparsity level



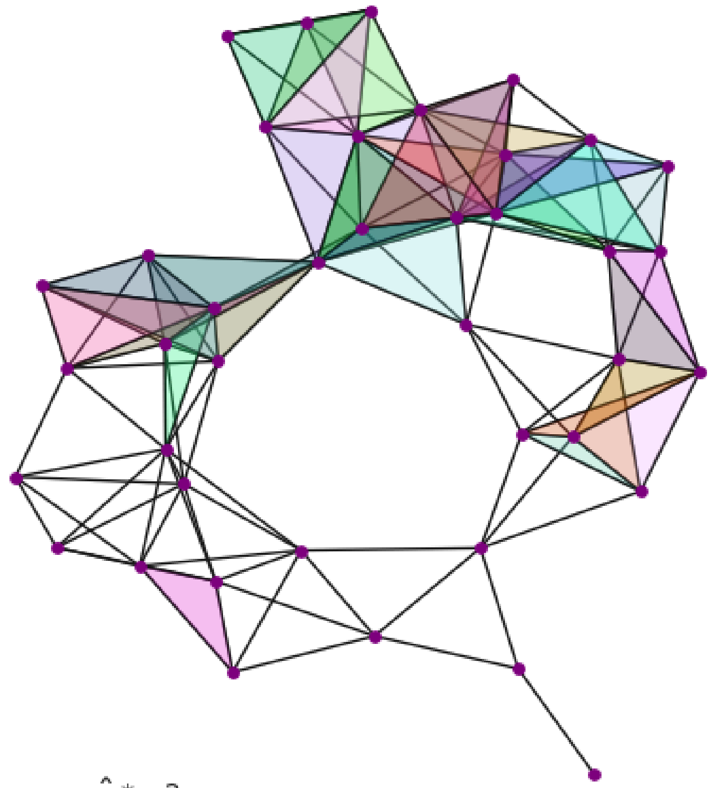
% of included triangles in the true topology



Joint Topology and Dictionary Learning

Results on synthetic data

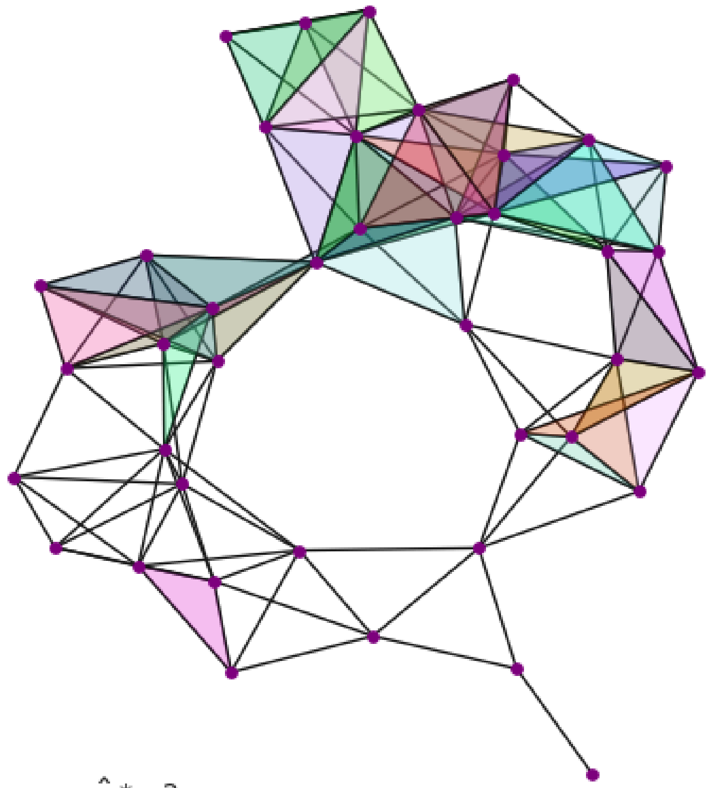
True number of triangles: 49



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0222

Optimistic method

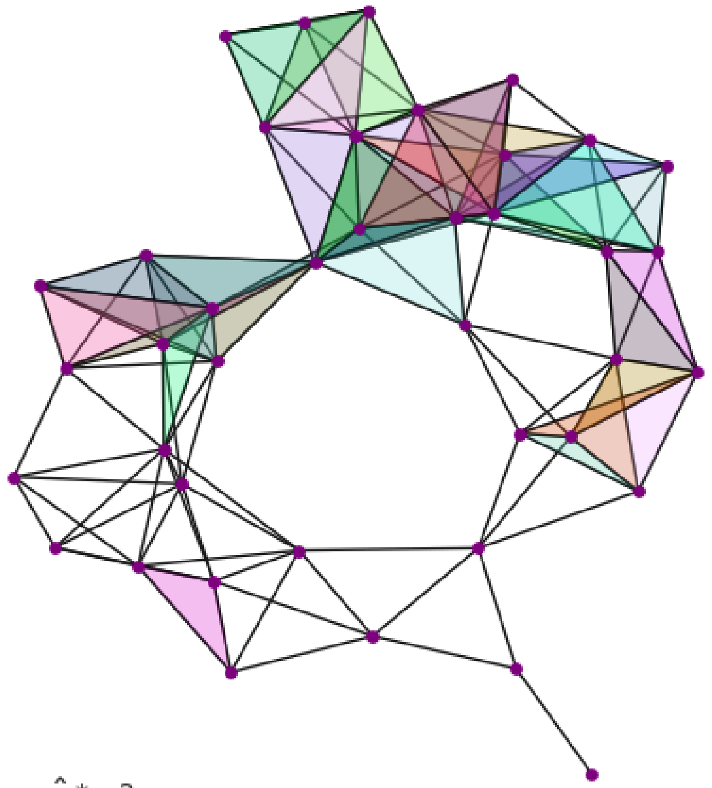
Inferred number of triangles: 49



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0124

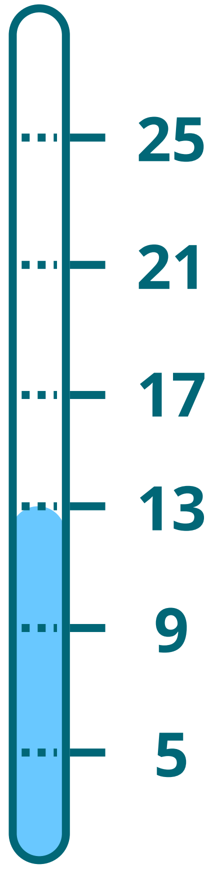
Pessimistic method

Inferred number of triangles: 50

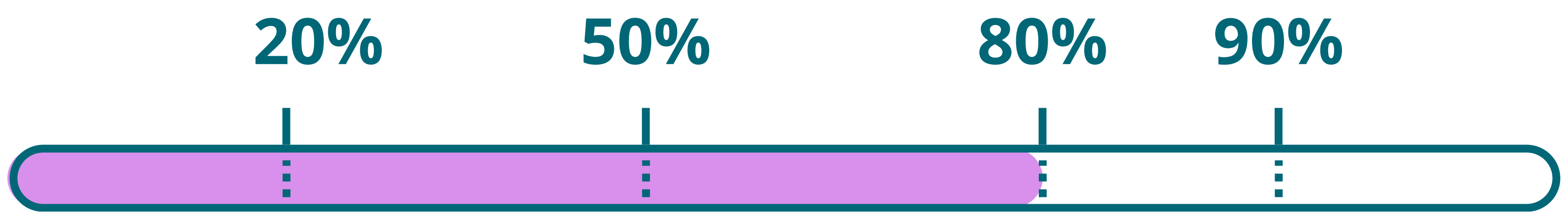


$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.1172 NMSE: 0.0137

Sparsity level



% of included triangles in the true topology

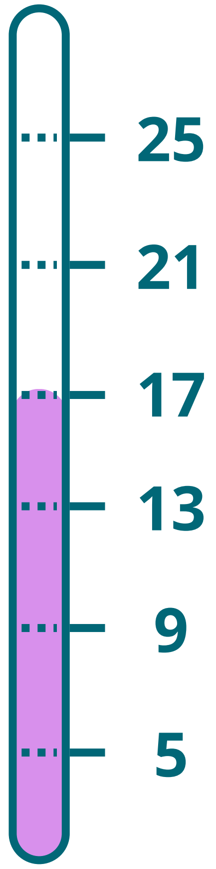


Joint Topology and Dictionary Learning

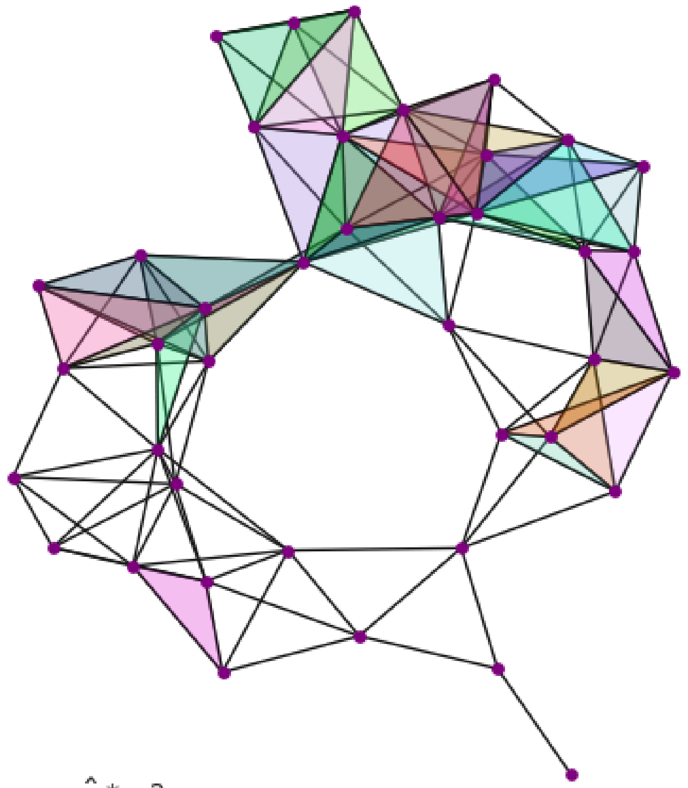
Results on synthetic data

Optimistic method | **Pessimistic method**

Sparsity level

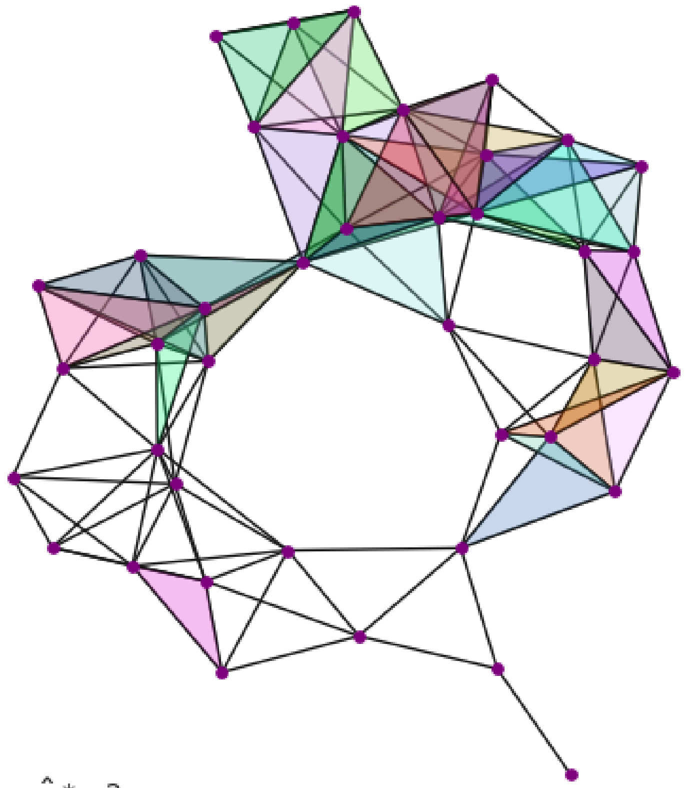


True number of triangles: 49



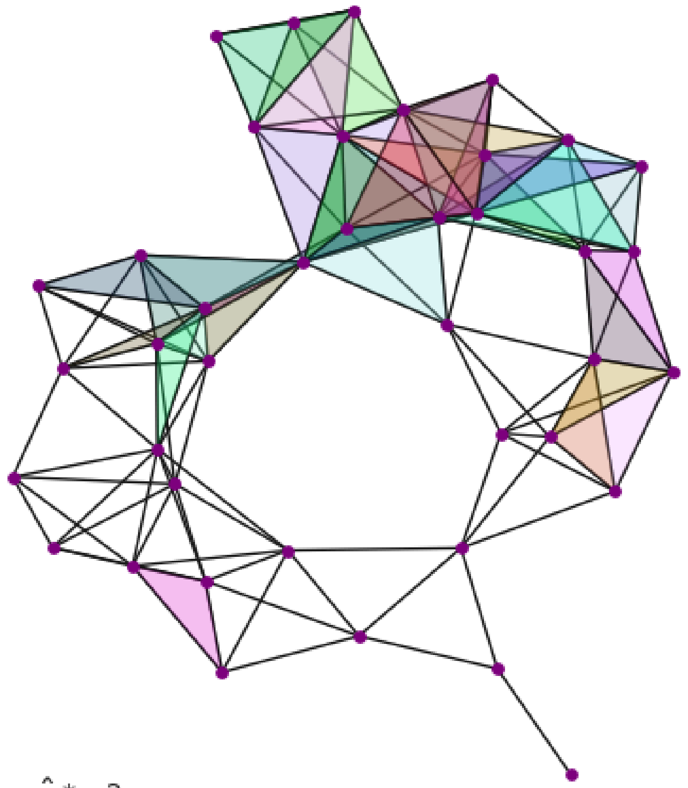
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0123

Inferred number of triangles: 50



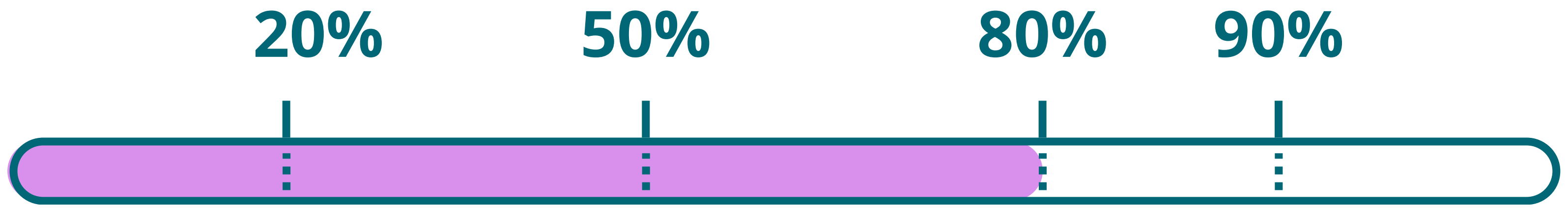
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.1172 NMSE: 0.0113

Inferred number of triangles: 45



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.3077 NMSE: 0.0114

% of included triangles in the true topology

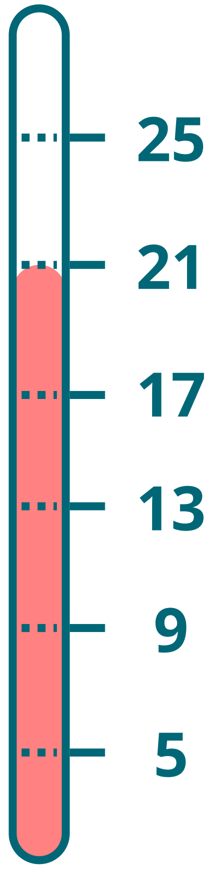


Joint Topology and Dictionary Learning

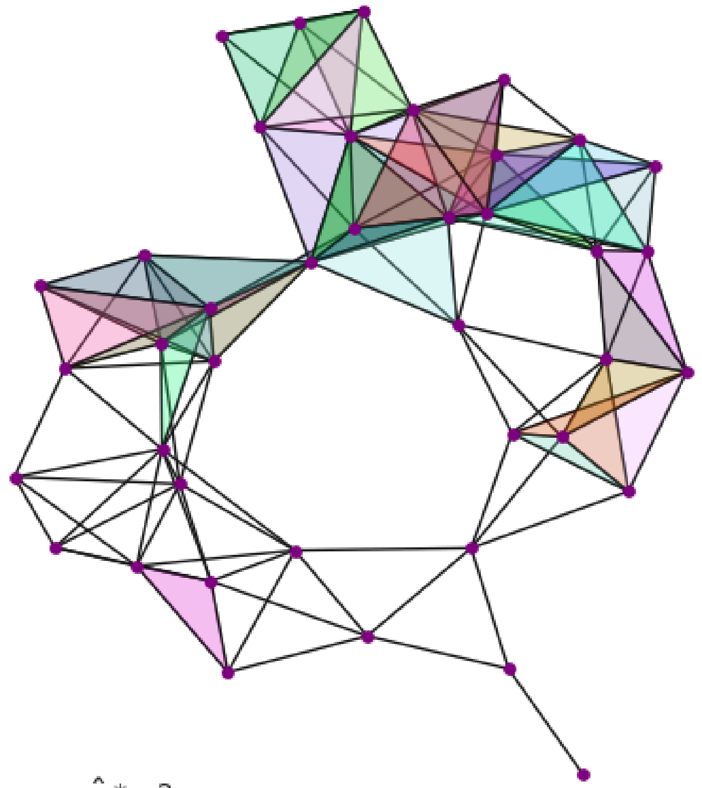
Results on synthetic data

Optimistic method | **Pessimistic method**

Sparsity level

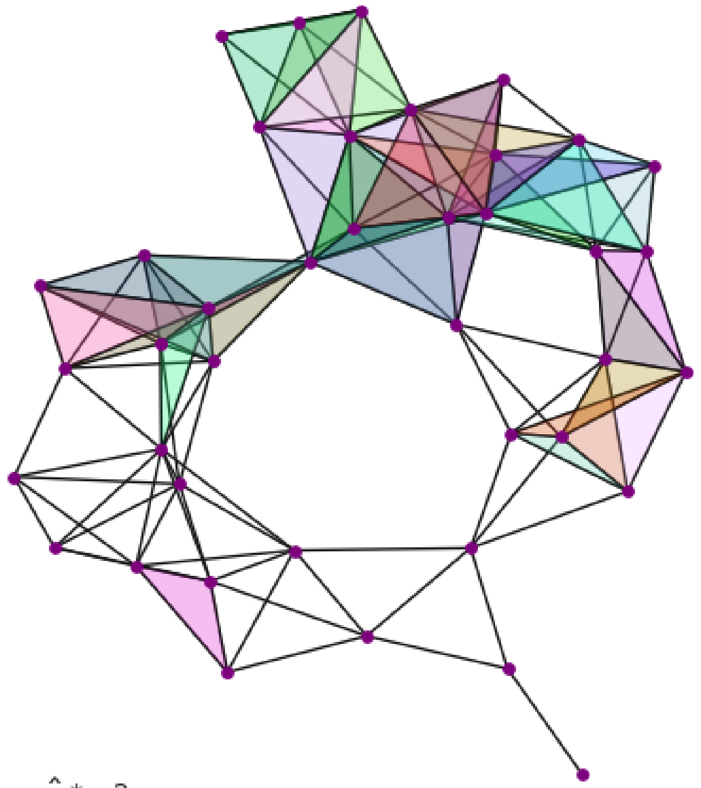


True number of triangles: 49



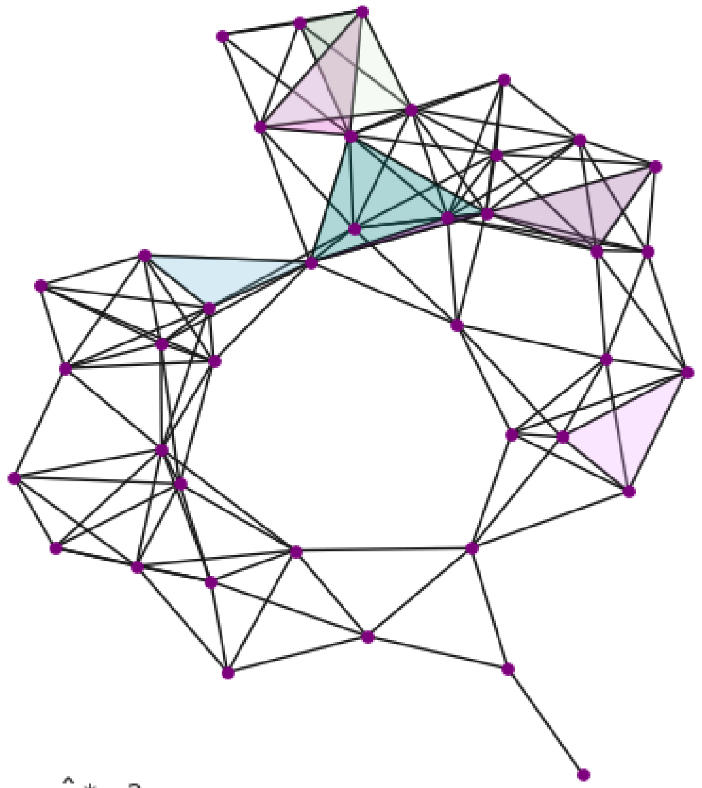
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0066

Inferred number of triangles: 50



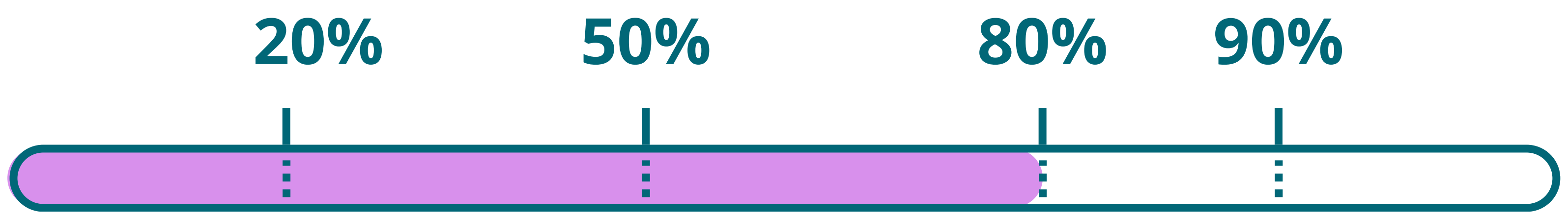
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.1172 NMSE: 0.0119

Inferred number of triangles: 8



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.9071 NMSE: 0.023

% of included triangles in the true topology

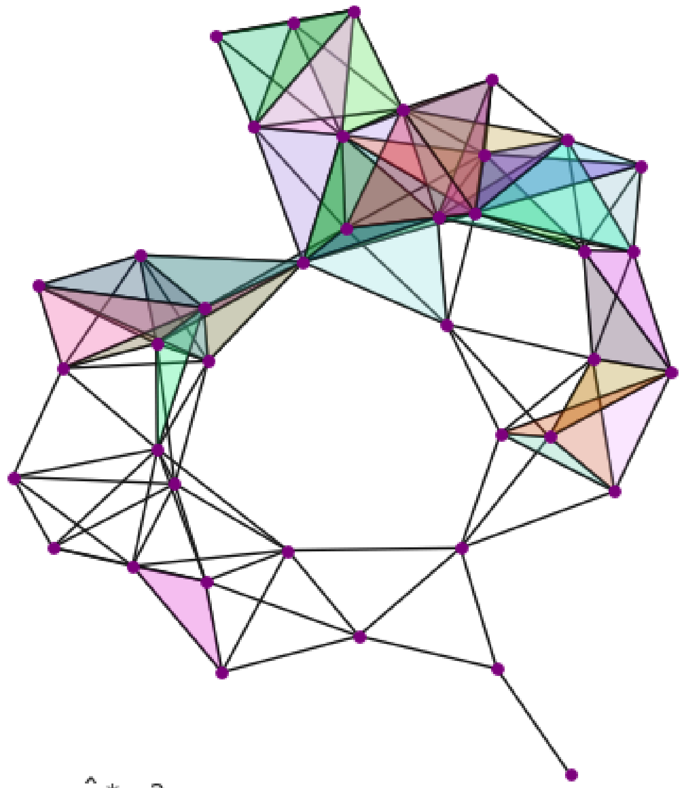


Joint Topology and Dictionary Learning

Results on synthetic data

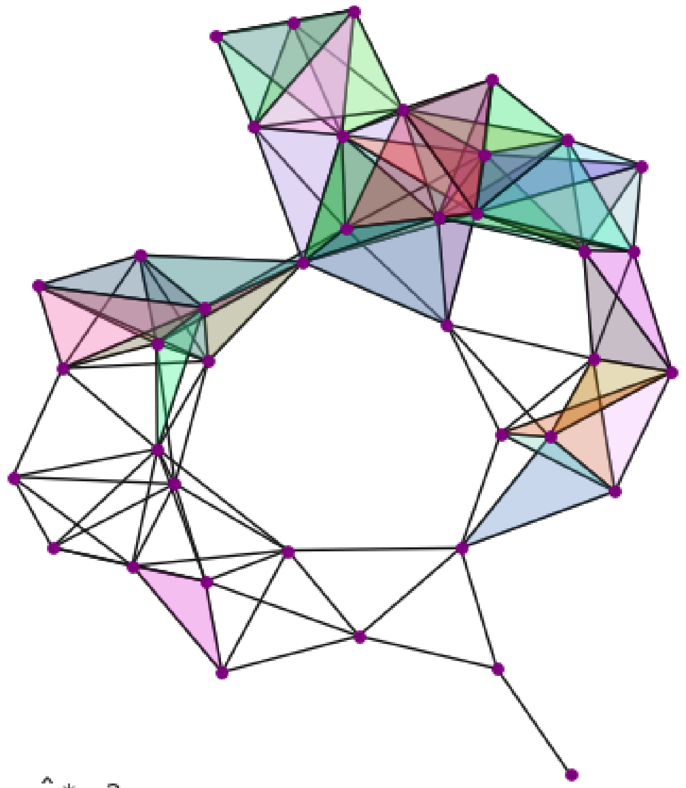
Optimistic method | **Pessimistic method**

True number of triangles: 49



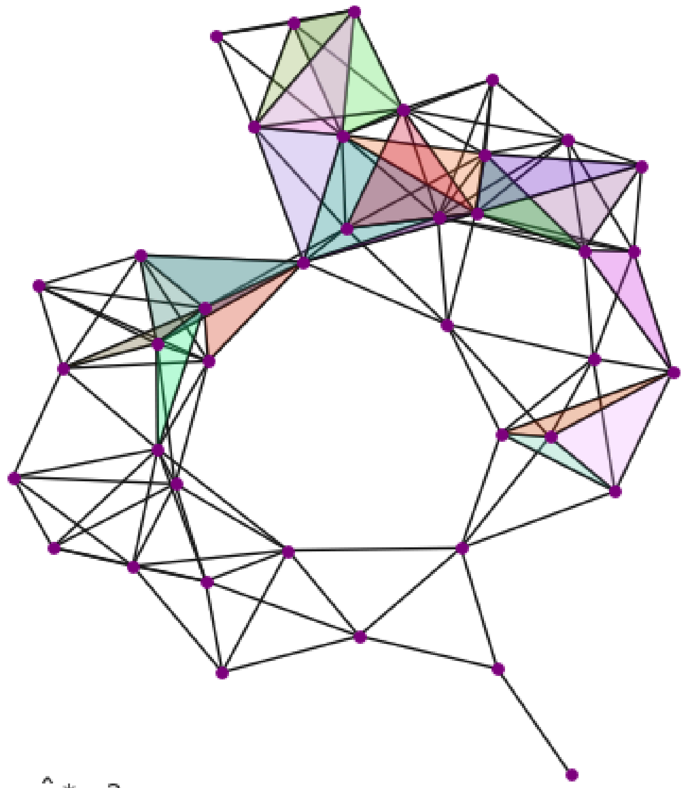
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0112

Inferred number of triangles: 60



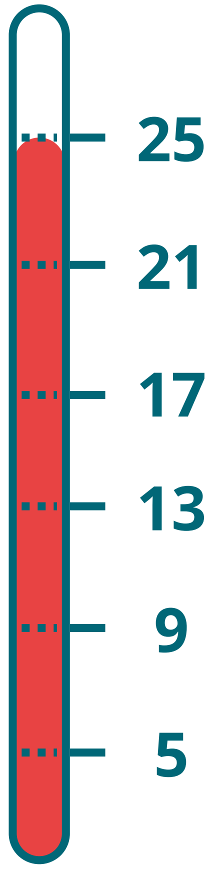
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.4262 NMSE: 0.0093

Inferred number of triangles: 23

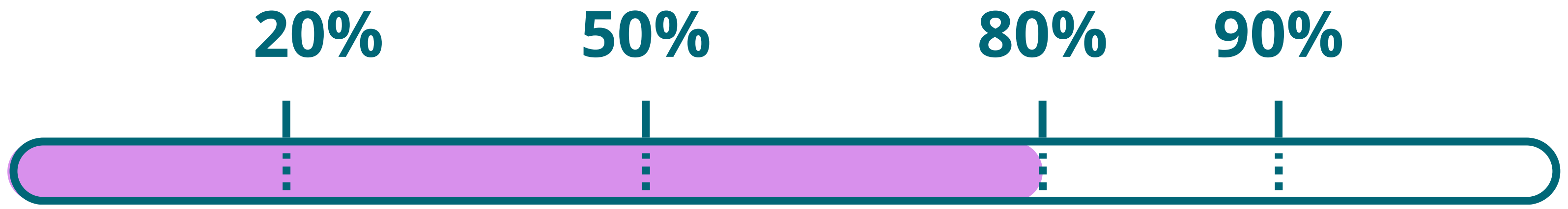


$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.7141 NMSE: 0.0158

Sparsity level



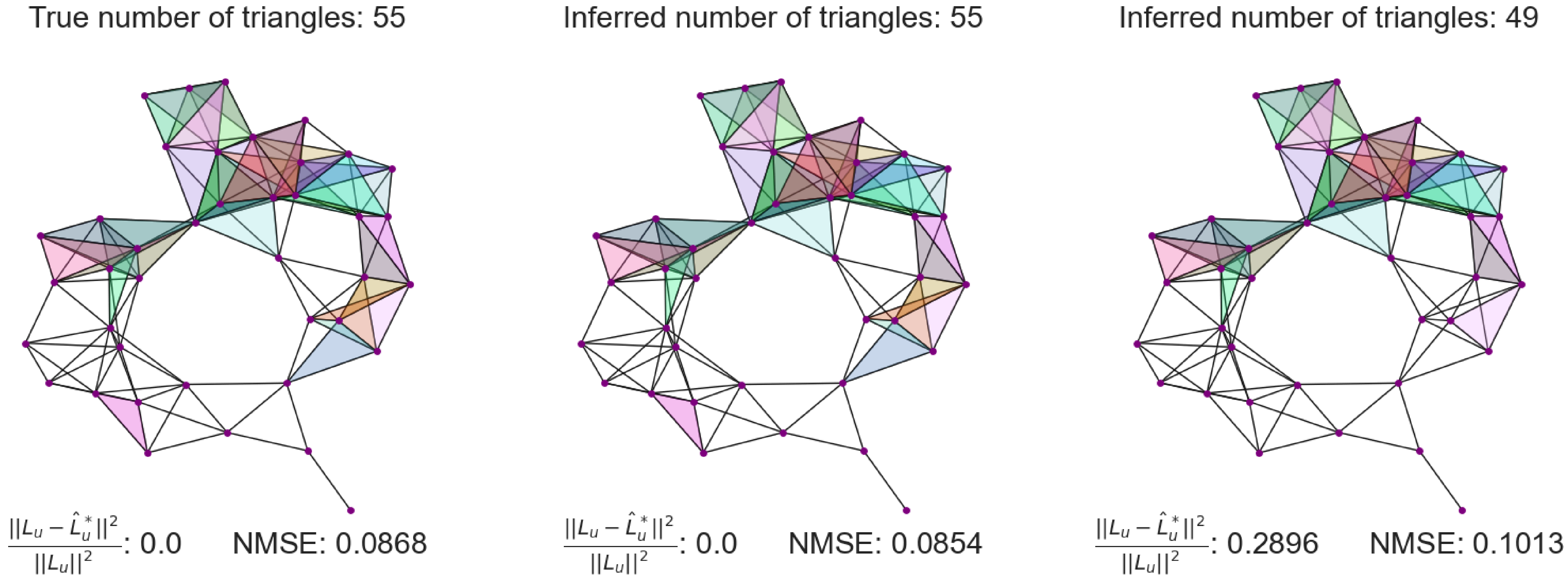
% of included triangles in the true topology



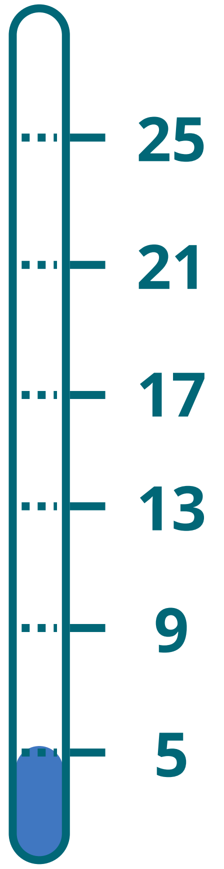
Joint Topology and Dictionary Learning

Results on synthetic data

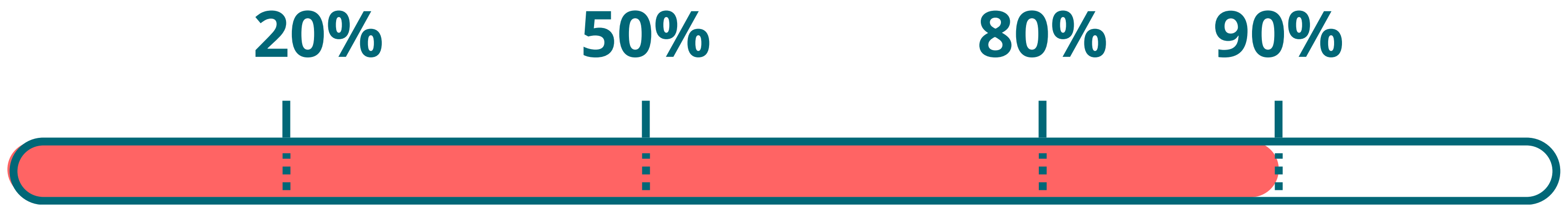
Optimistic method **Pessimistic method**



Sparsity level



% of included triangles in the true topology

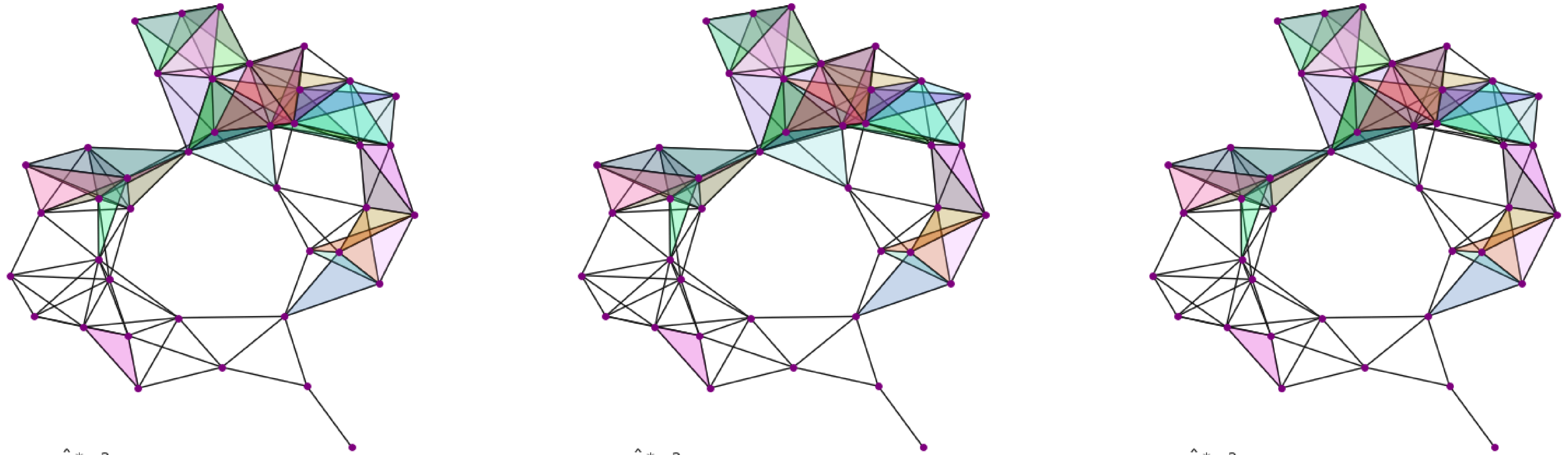


Joint Topology and Dictionary Learning

Results on synthetic data

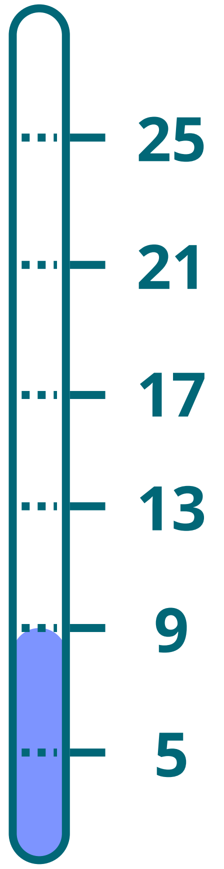
Optimistic method | **Pessimistic method**

True number of triangles: 55 | Inferred number of triangles: 55 | Inferred number of triangles: 49

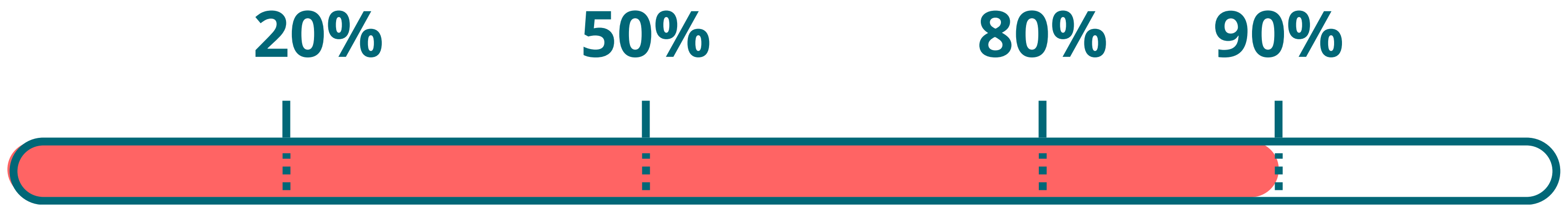


$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0315 | $\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0297 | $\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.3267 NMSE: 0.051

Sparsity level



% of included triangles in the true topology



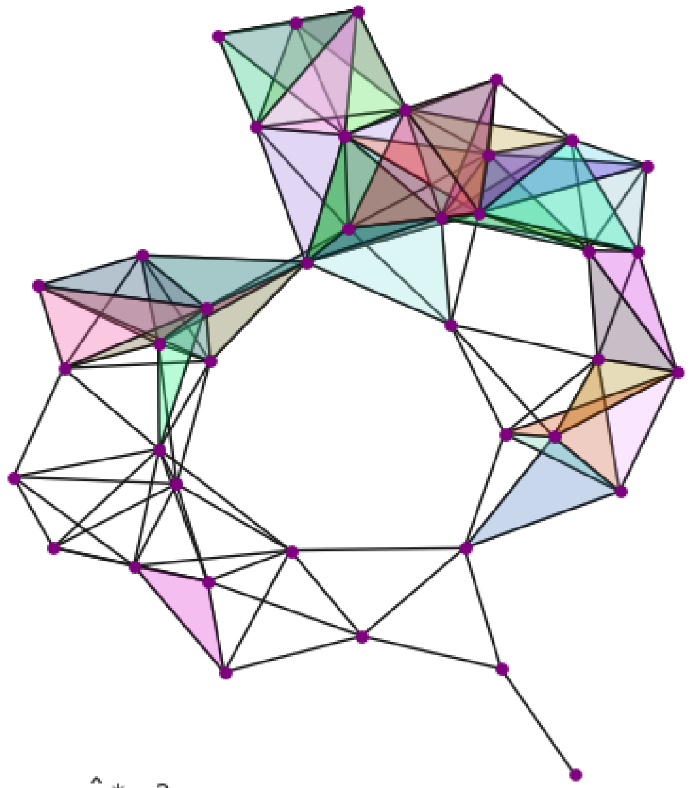
Joint Topology and Dictionary Learning

Results on synthetic data

Optimistic method | **Pessimistic method**

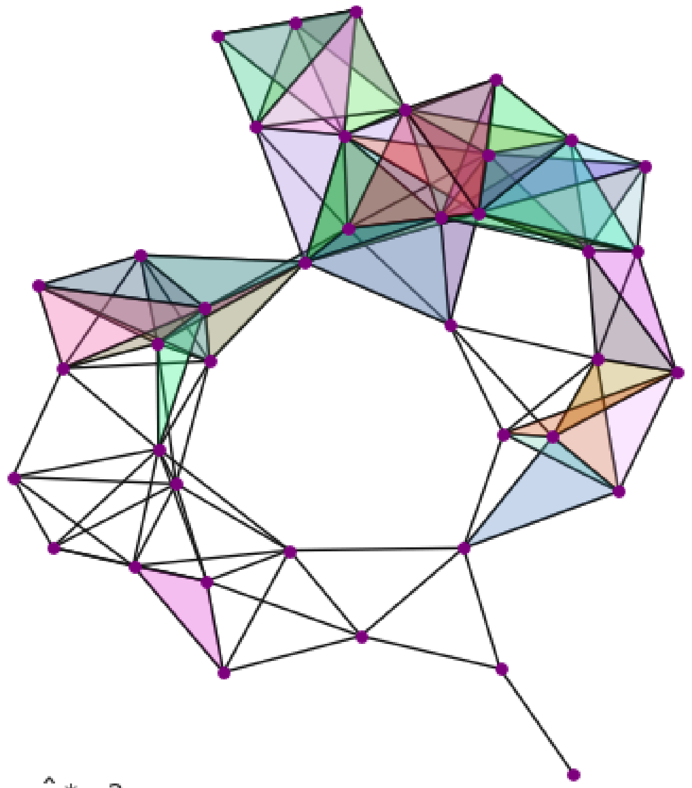
Sparsity level

True number of triangles: 55



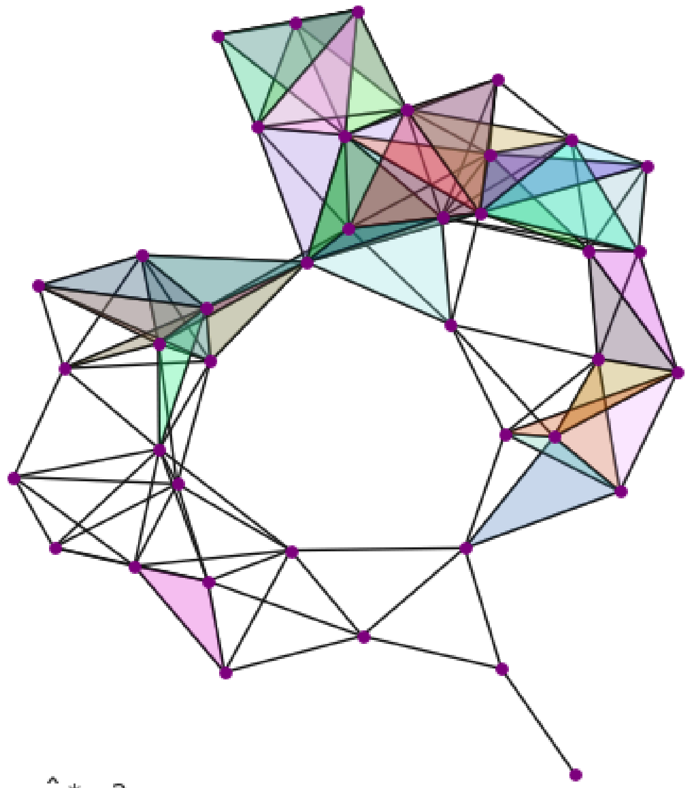
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0151

Inferred number of triangles: 61

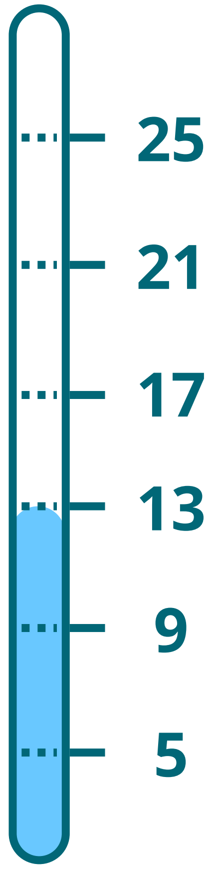


$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.2807 NMSE: 0.0284

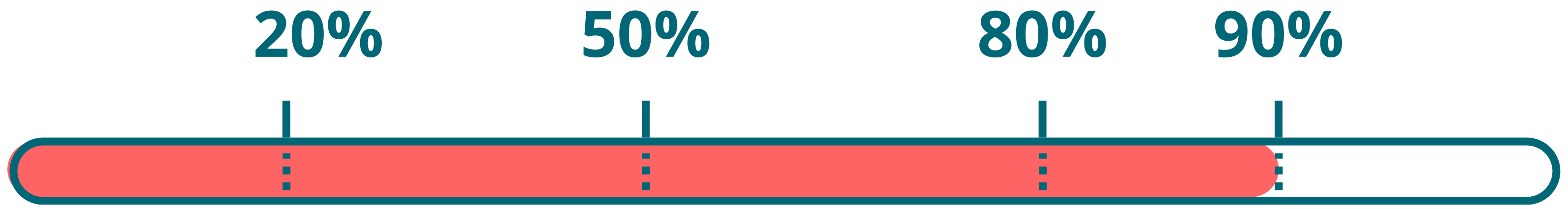
Inferred number of triangles: 48



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.3438 NMSE: 0.0291



% of included triangles in the true topology



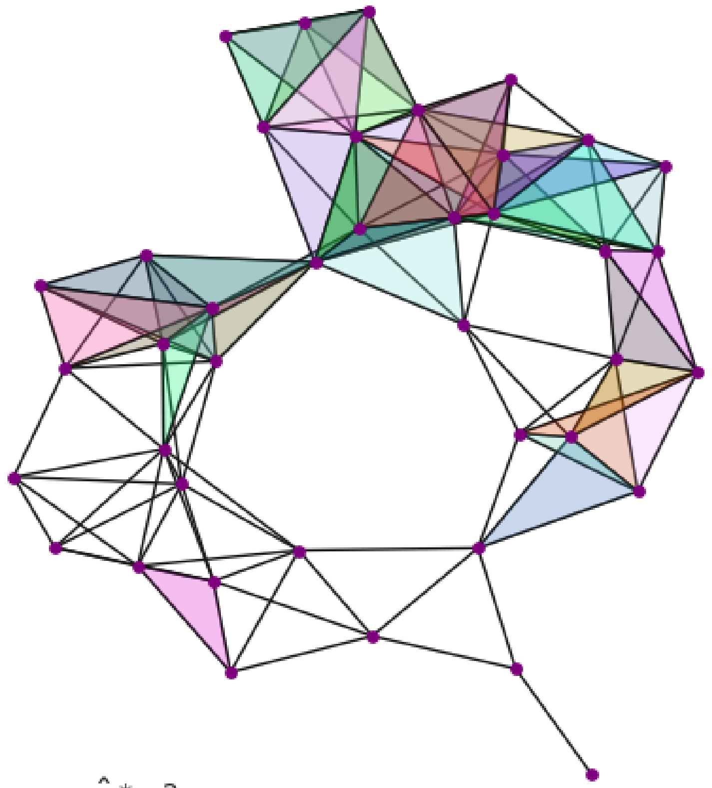
Joint Topology and Dictionary Learning

Results on synthetic data

Optimistic method | **Pessimistic method**

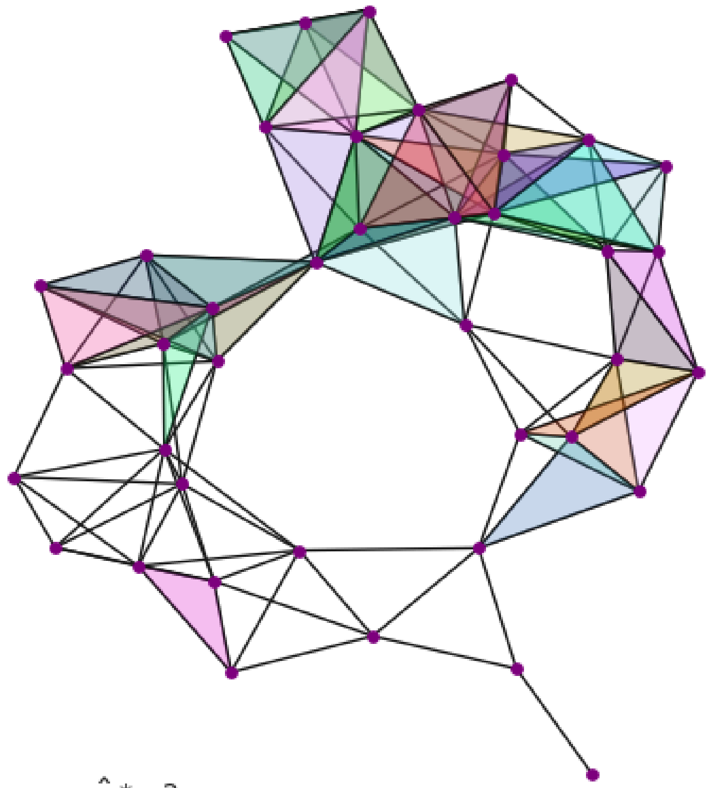
Sparsity level

True number of triangles: 55



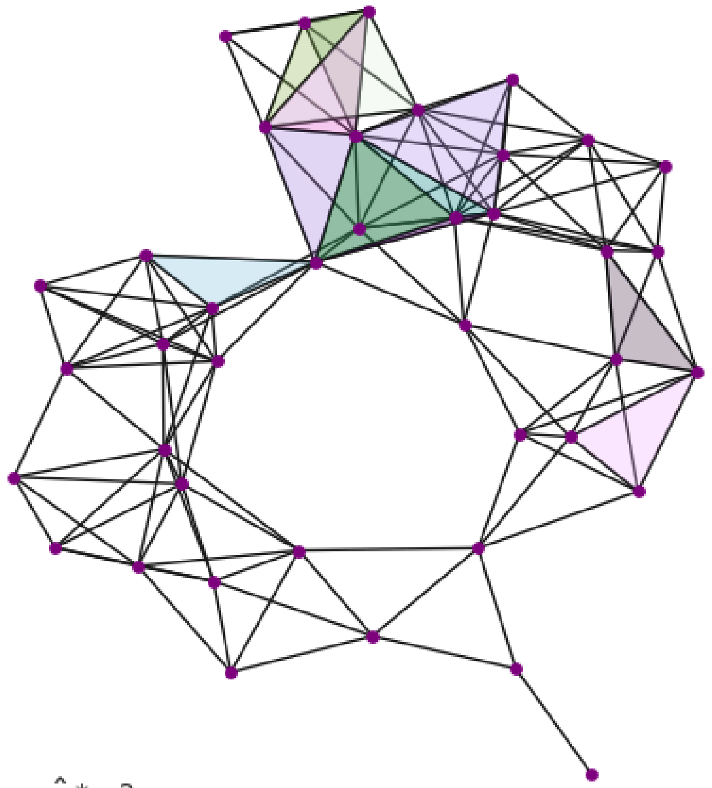
$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0069

Inferred number of triangles: 55

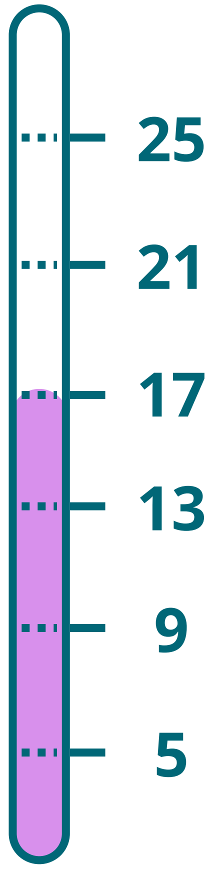


$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.0 NMSE: 0.0104

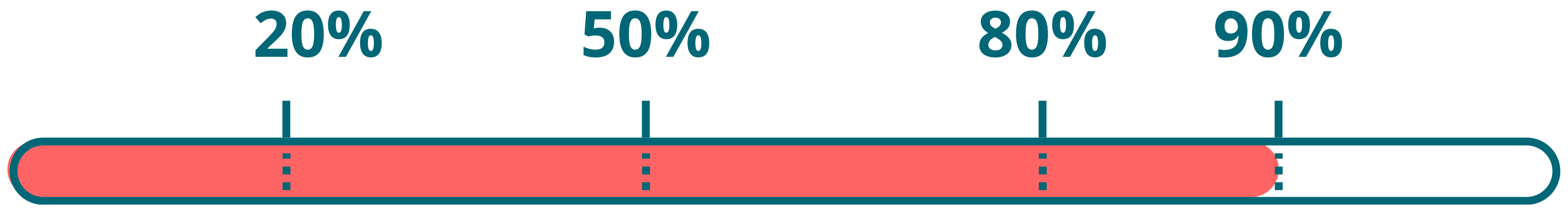
Inferred number of triangles: 12



$\frac{||L_u - \hat{L}_u^*||^2}{||L_u||^2}$: 0.8488 NMSE: 0.0367



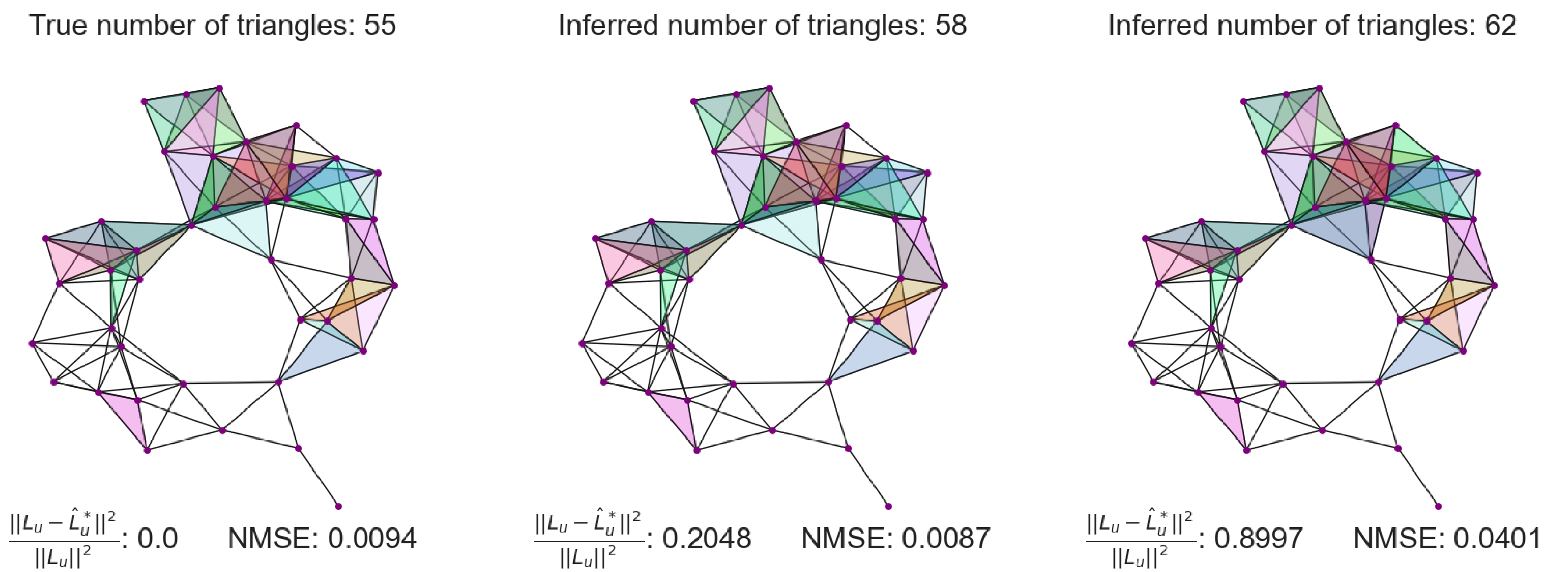
% of included triangles in the true topology

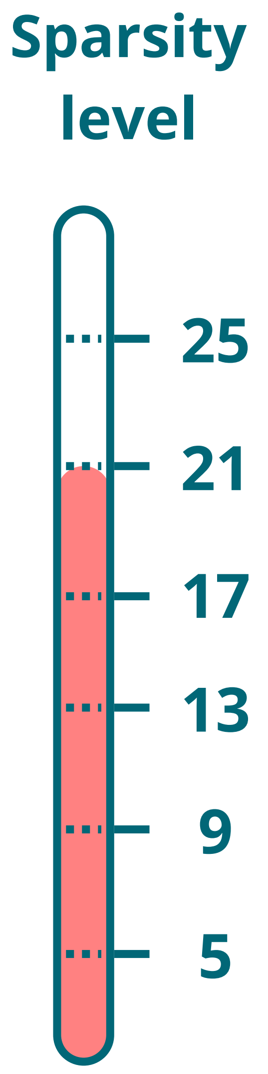


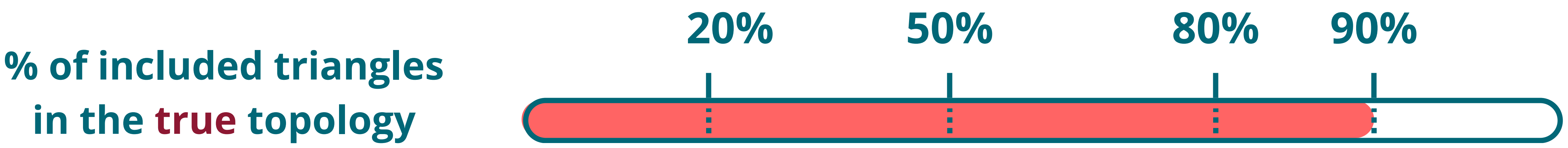
Joint Topology and Dictionary Learning

Results on synthetic data

Optimistic method **Pessimistic method**



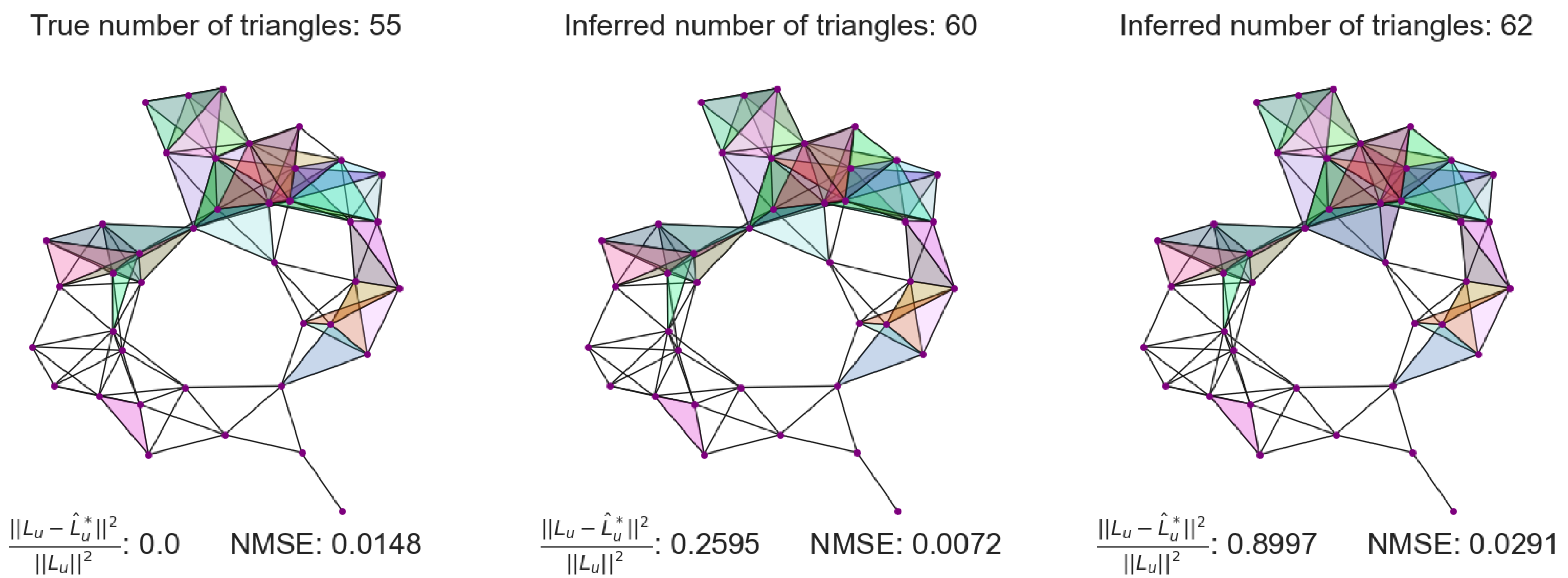




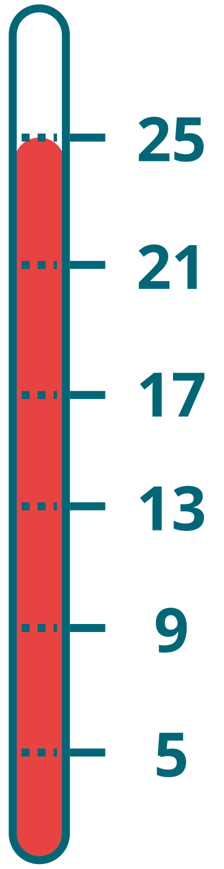
Joint Topology and Dictionary Learning

Results on synthetic data

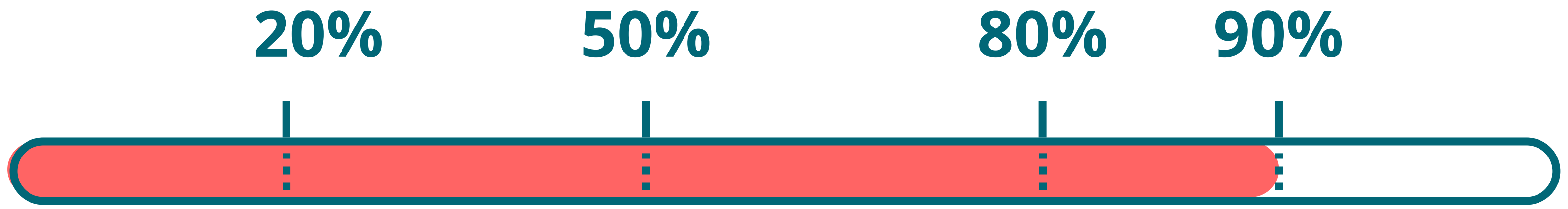
Optimistic method **Pessimistic method**



Sparsity level

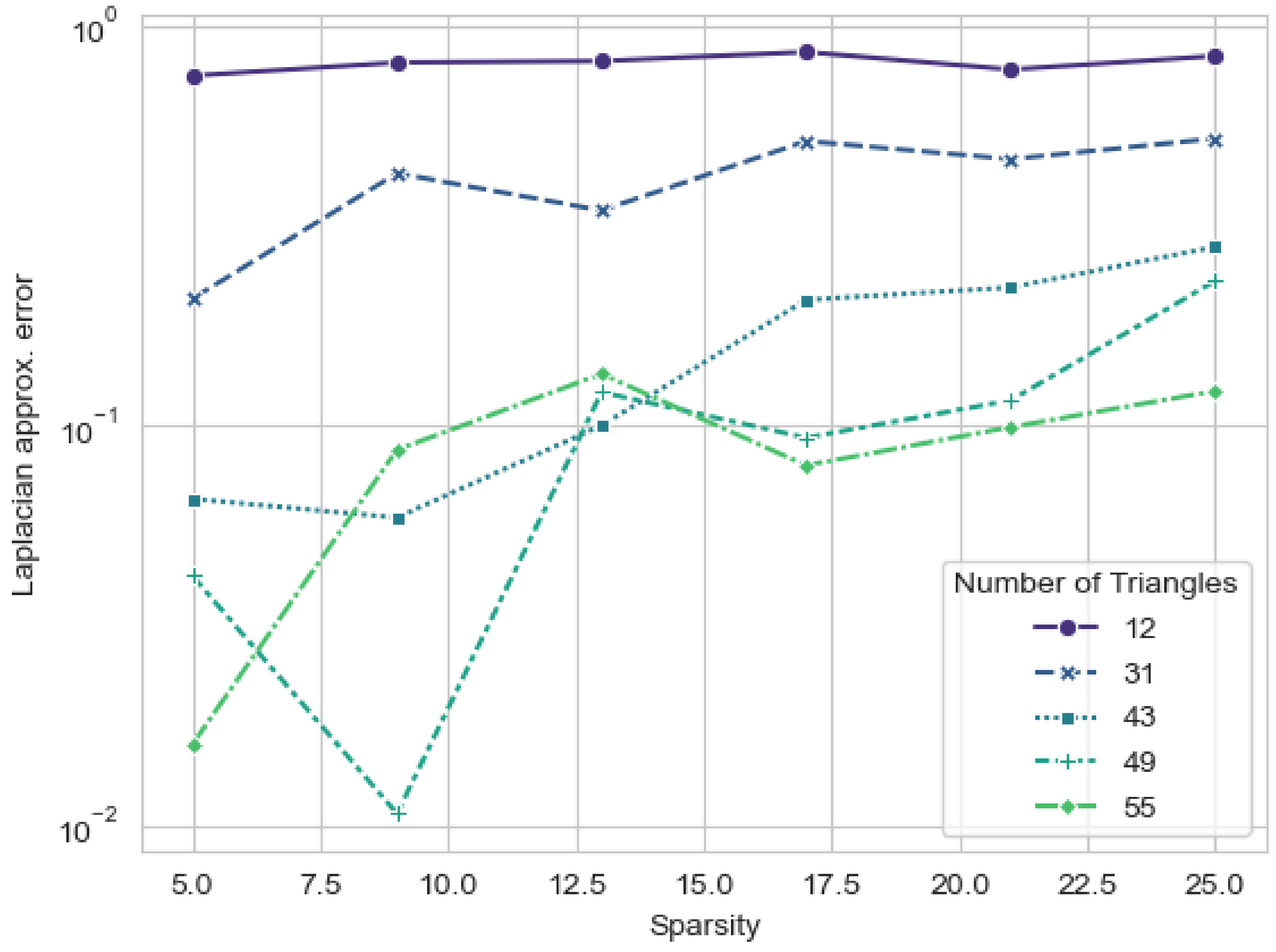


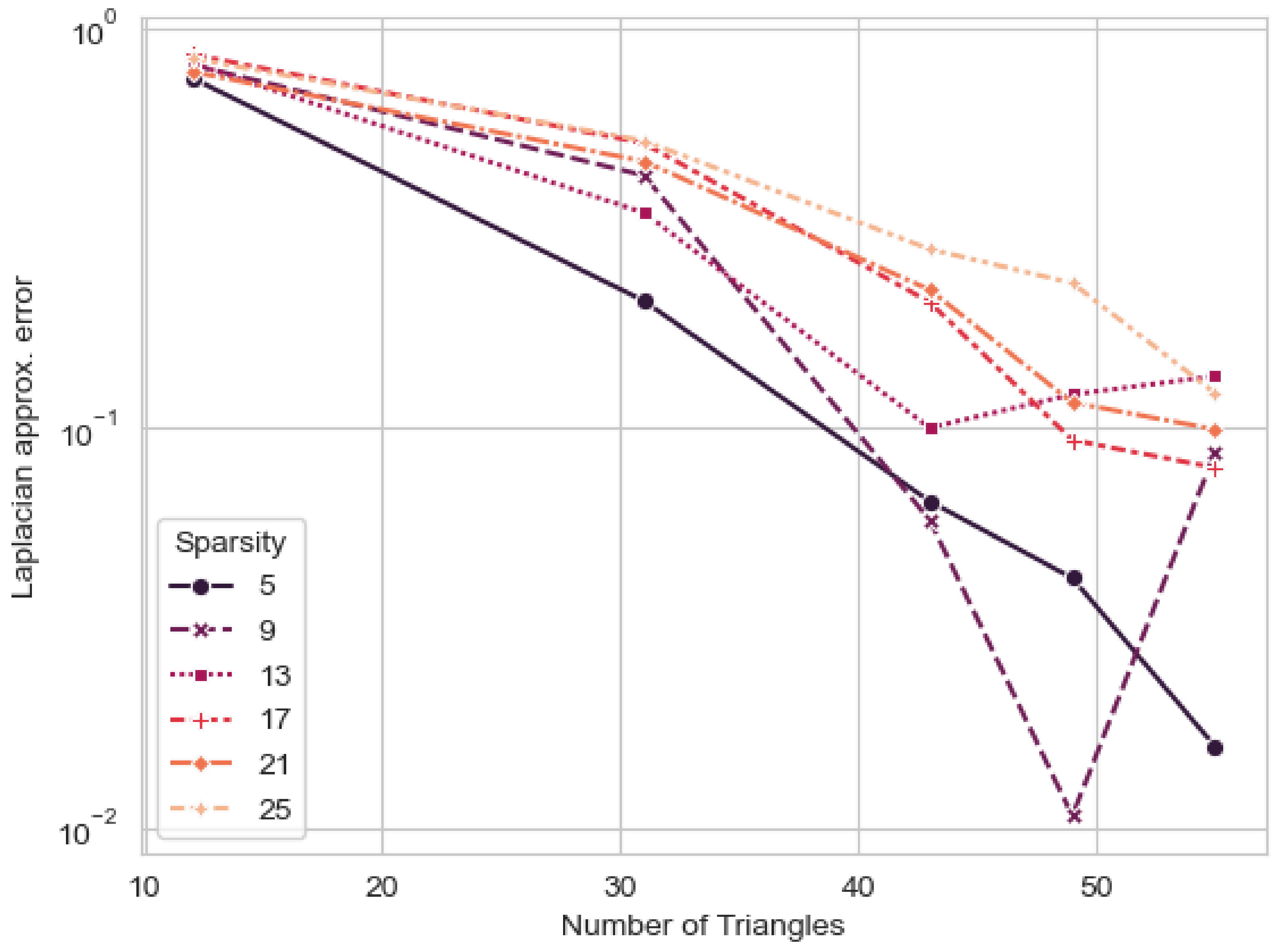
% of included triangles in the true topology



Joint Topology and Dictionary Learning

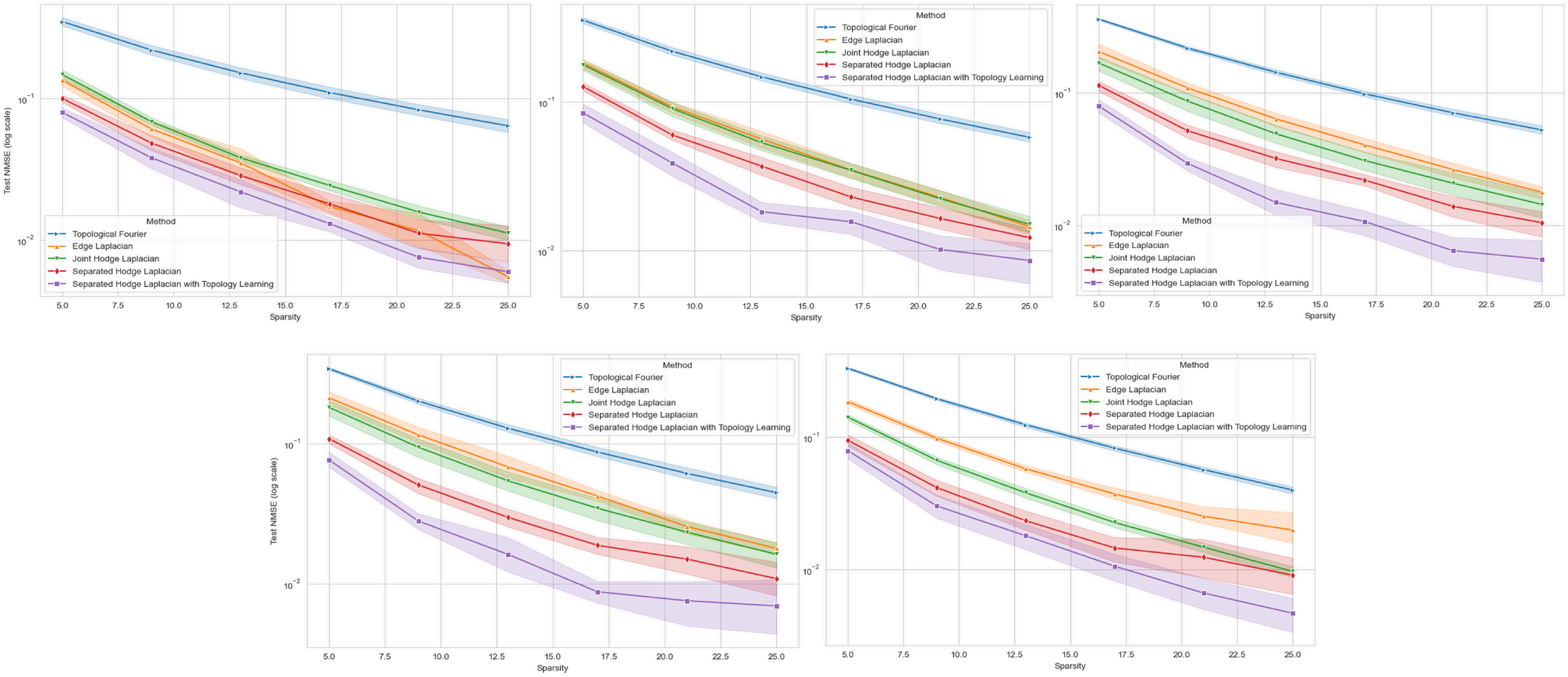
Results on synthetic data





Joint Topology and Dictionary Learning

Results on synthetic data



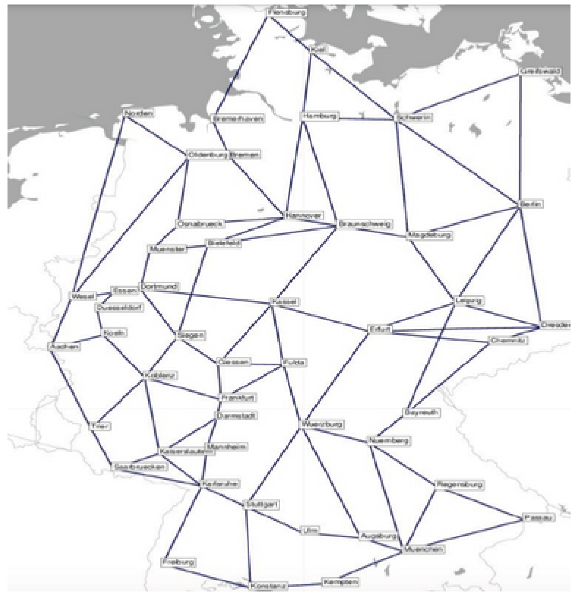
Real Data: DFN Network

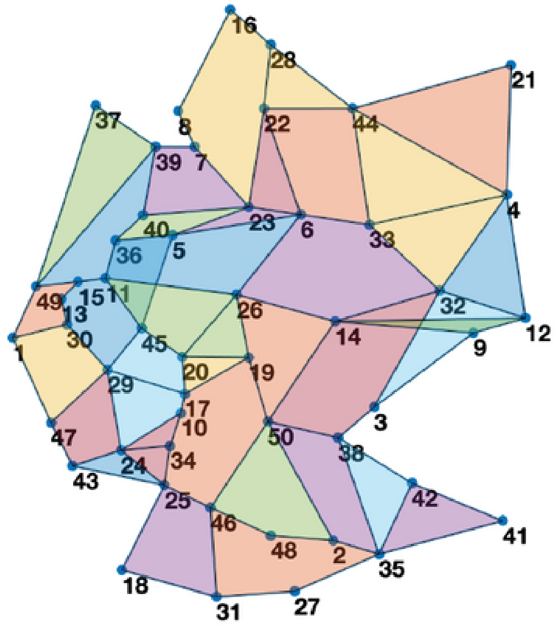
Facoltà di Ingegneria dell'Informazione,
Informatica e Statistica

- Traffic data over the German National Research and Education Network (DFN)

- Model the network as a **order 2 cell complex** with:
  - **50** nodes
  - **89** edges
  - **39** polygons

- The data traffic is aggregated daily over February 2005, and the data measurements are expressed in Mbit/sec and collected on each link for a total of **28** **edge signals**





Joint Topology and Dictionary Learning

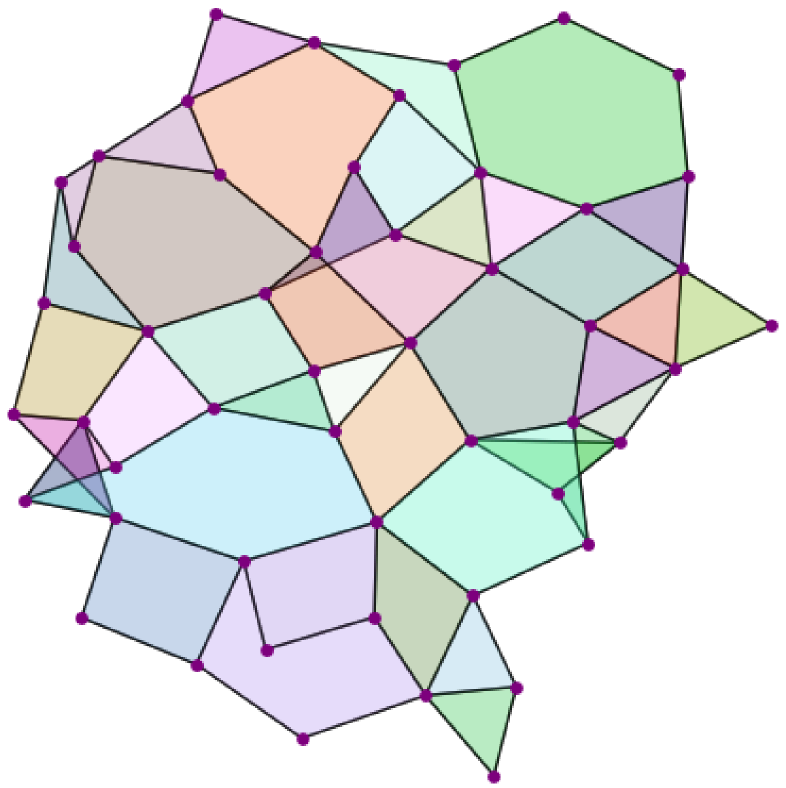
Results on real data

Optimistic method

Pessimistic method

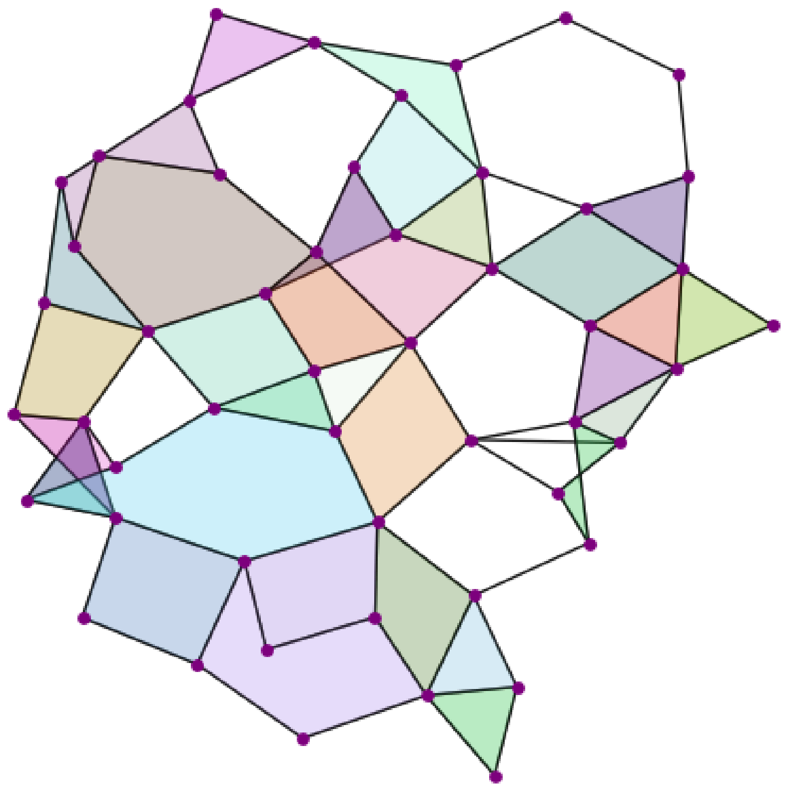
Sparsity level

Assumed number of polygons: 39



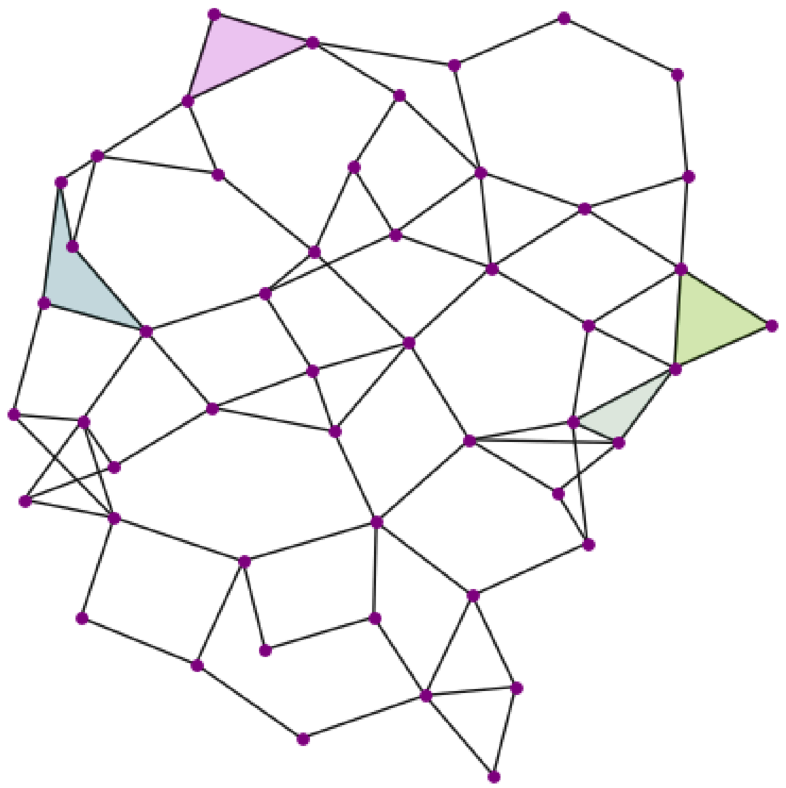
NMSE: 0.0692

Inferred number of polygons : 32

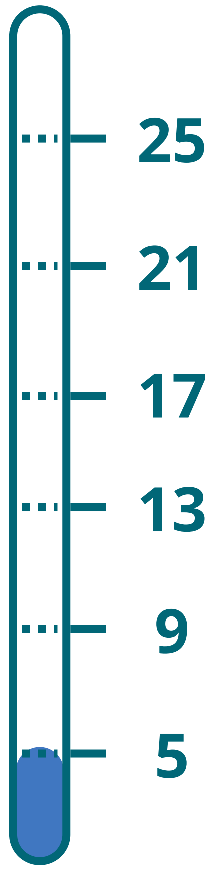


NMSE: 0.0556

Inferred number of polygons: 4



NMSE: 0.0707



Joint Topology and Dictionary Learning

Results on real data

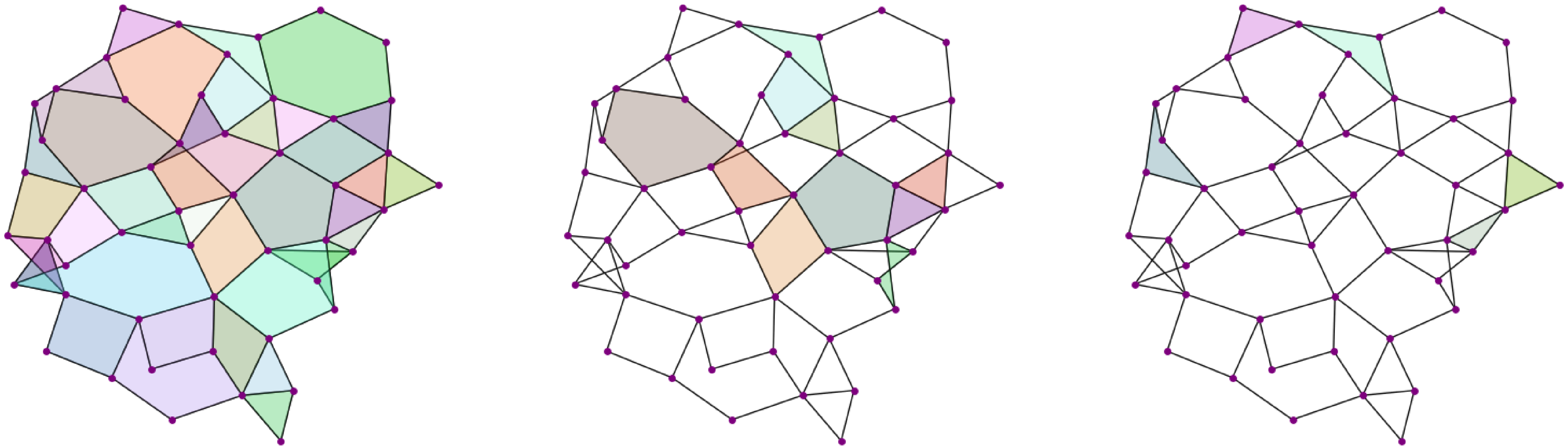
Optimistic method

Pessimistic method

Assumed number of polygons: 39

Inferred number of polygons : 10

Inferred number of polygons: 5

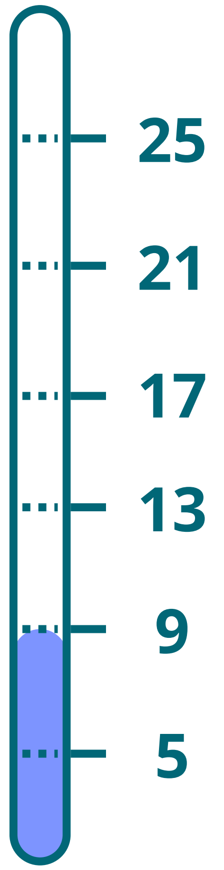


NMSE: 0.0147

NMSE: 0.0096

NMSE: 0.0102

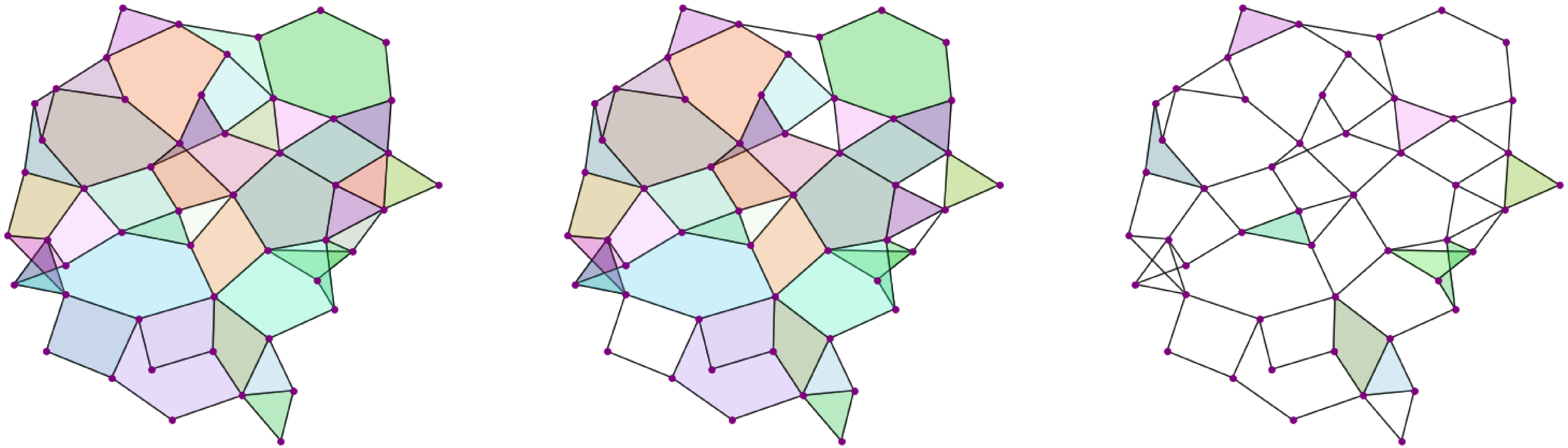
Sparsity level



Joint Topology and Dictionary Learning

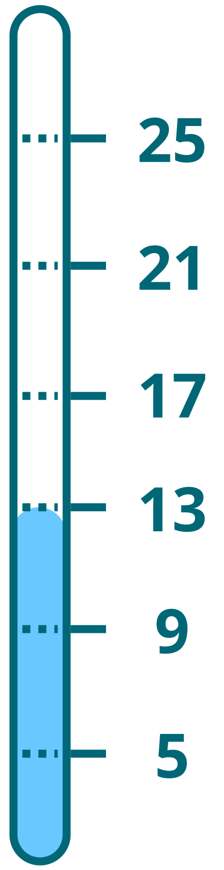
Results on real data

	Optimistic method	**Pessimistic method**
Assumed number of polygons: 39	Inferred number of polygons : 34	Inferred number of polygons: 9



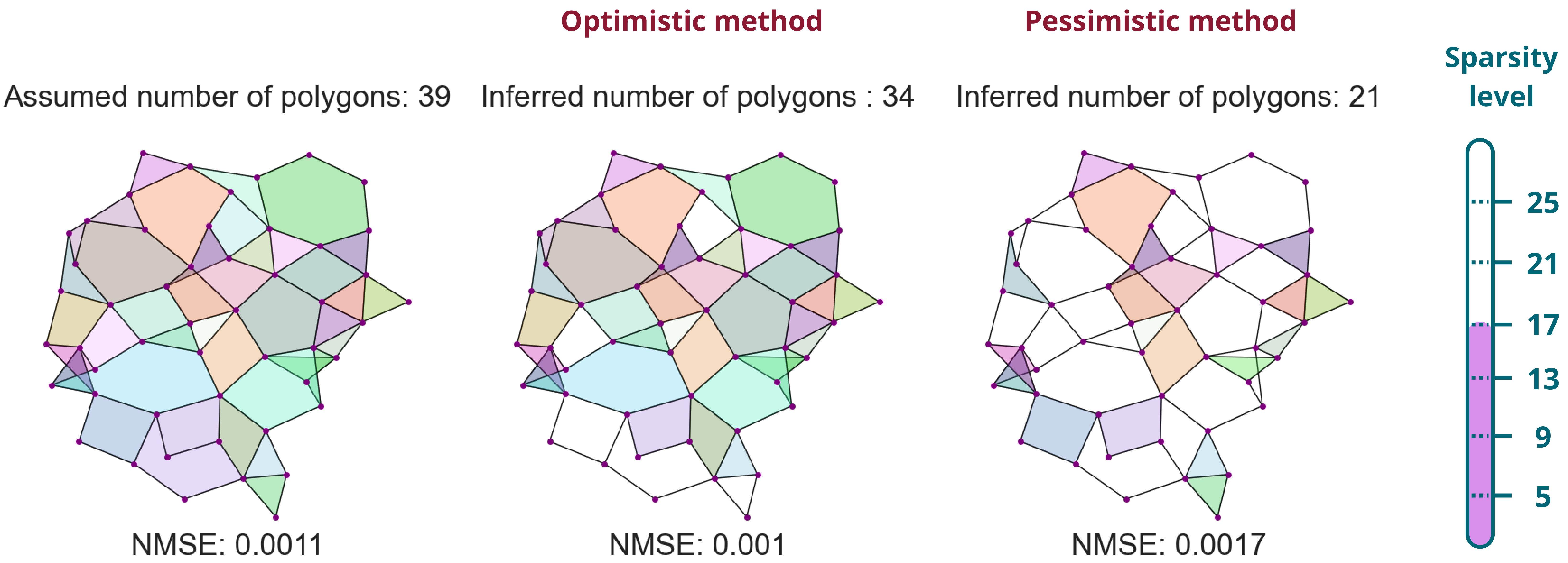
NMSE: 0.0028	NMSE: 0.0026	NMSE: 0.0017

Sparsity level



Joint Topology and Dictionary Learning

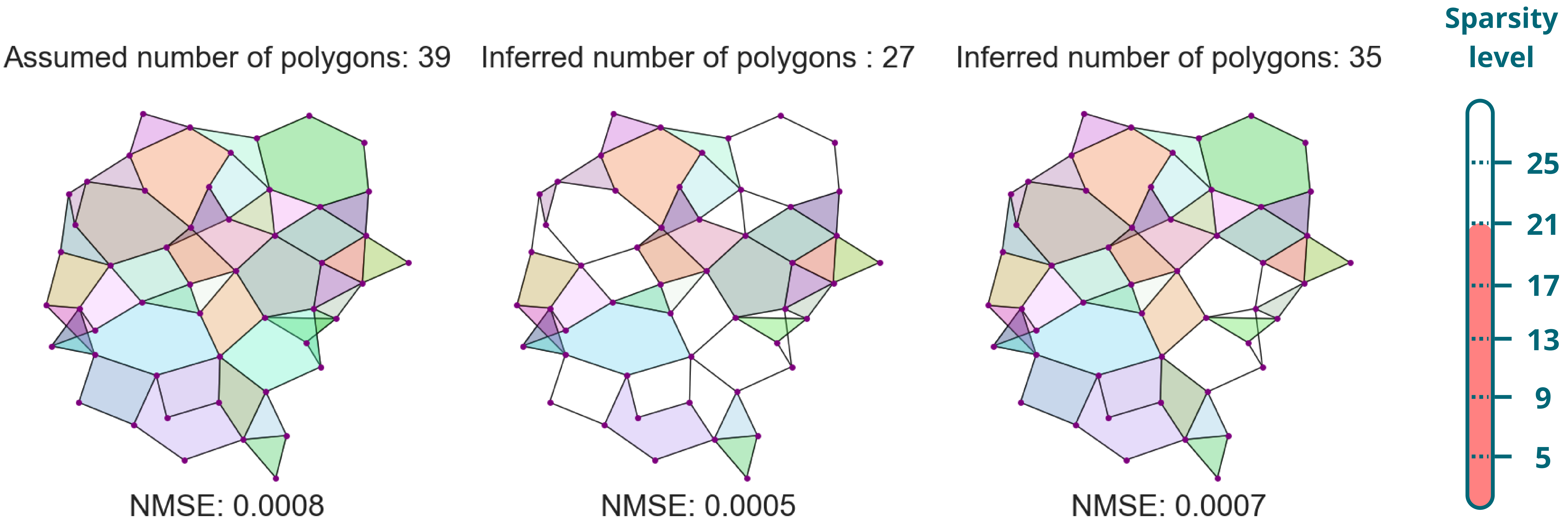
Results on real data



Joint Topology and Dictionary Learning

Results on real data

Optimistic method **Pessimistic method**

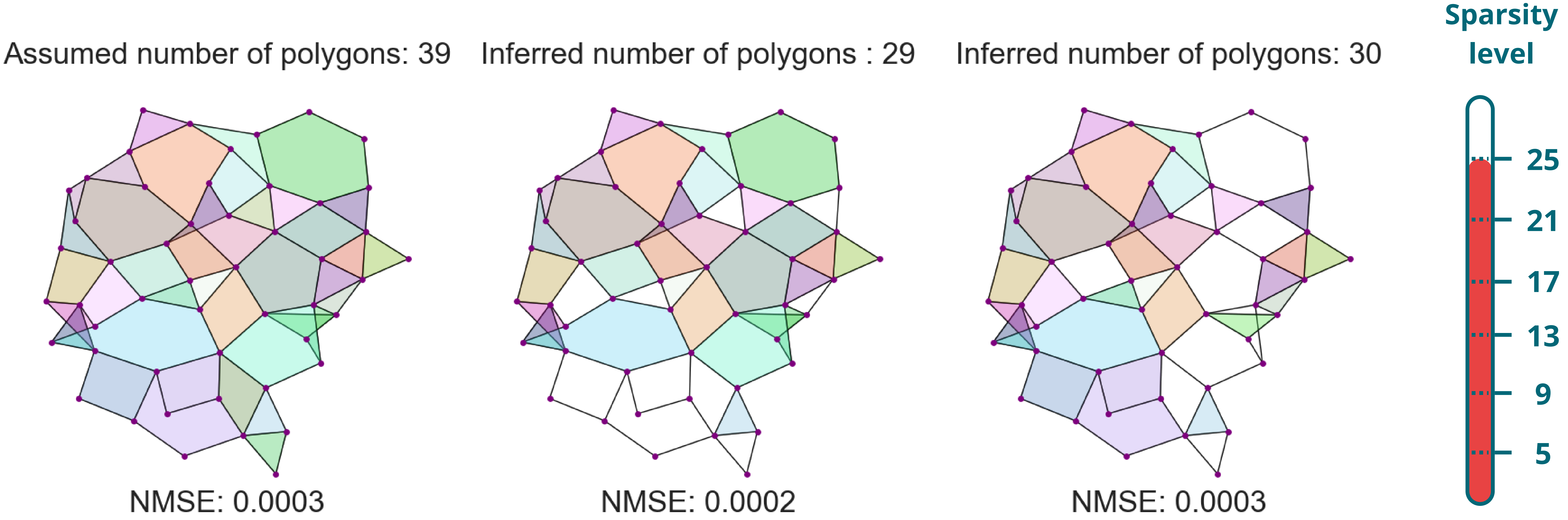


Joint Topology and Dictionary Learning

Results on real data

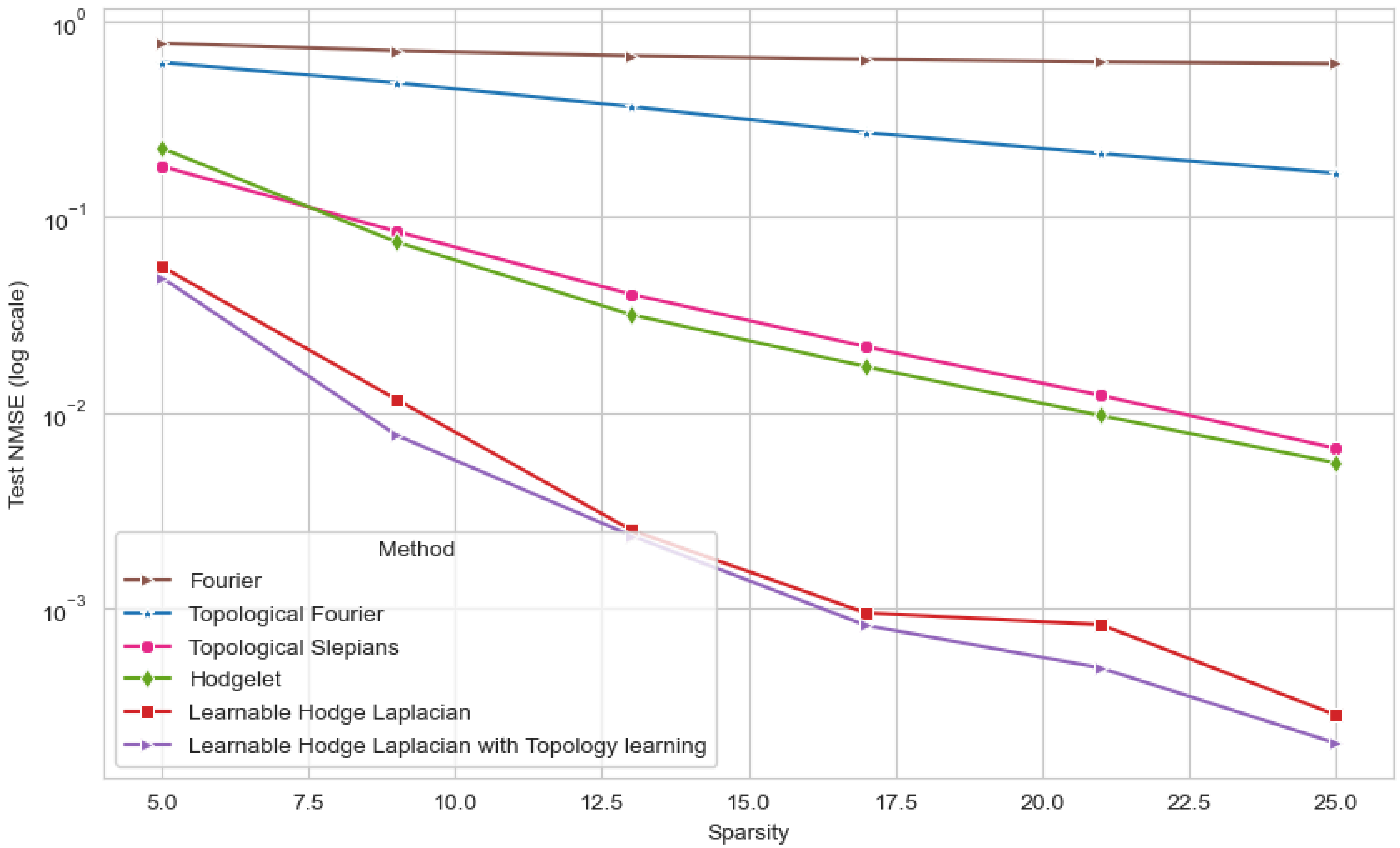
Optimistic method

Pessimistic method



Joint Topology and Dictionary Learning

Results on real data



Fourier-based dictionary

Topology-informed analytical dictionary

Proposed method with learnable dictionary

Conclusions

- The proposed algorithm empirically works better than compression techniques based on non-learnable dictionaries

- The Separated Hodge Laplacian parameterization is able to generalize better w.r.t. the other parameterization techniques

- The Topology Inference step is slow but the overall algorithm benefits the introduction of this further learning step.

- These entity of the benefits deriving from the introduction of topology learning step depend on the assumed sparsity level and the density in terms of k-th order cells in the true cell complex

Further improvements

- Improve the Topology learning step with faster non-greedy methods
- Setup a faster algorithm by using Model-based deep learning architectures

Master’s thesis in Data Science
a.y. 2023/24

Thank you for the attention!

Enrico Grimaldi