

# 1 映射

## 1.1

- (1) 不能构成映射, 如  $x_1 = 0$ , 则  $x_2 = 0, 1, \dots, 9$  均满足.
- (2) 能构成映射,  $\forall y_1 \in \mathbb{R}, \exists! y_2 = y_1^2$ .
- (3) 不能构成映射, 如  $y_1 = 1$ , 则  $y_2 = \pm 1$  均满足.

## 1.2

- (1)

$$R_f = \{2, -2, 0\}$$

- (2)

$$\begin{cases} |A| = 9 \\ |R_f| = 3 \end{cases} \Rightarrow n = 3^9.$$

## 1.3

- (1) 满射

$$\forall y \in \mathbb{Z}^+, \exists x = \pm(y-1) \in \mathbb{Z}, f(x) = y.$$

- (2) 既不是单射, 又不是满射. 值域为  $\{0, 1, 2\} \subseteq \mathbb{Z} \cup \{0\}$

$$\forall y \in \mathbb{Z} \cup \{0\} \begin{cases} y \in \{0, 1, 2\}, \exists x = 3k + y (k \in \mathbb{Z}), f(x) = y. \\ y \notin \{0, 1, 2\}, \forall x \in \mathbb{Z}, f(x) \neq y. \end{cases}$$

- (3)  $f, g$  均是双射.

$$\begin{cases} \forall y \in \mathbb{Z}, \exists! x = y - 1 \in \mathbb{Z}, f(x) = y. \\ \forall y \in \mathbb{Z}, \exists! x = y + 1 \in \mathbb{Z}, g(x) = y. \end{cases}$$

- (4) 满射.

$$\forall y \in \{0, 1\}, \exists x = 2k + y + 1 (k \in \mathbb{Z}), f(x) = y.$$

- (5) 既不是单射, 又不是满射.

$$\begin{cases} \forall y \in \mathbb{Z}, y < -16, \forall x \in \mathbb{Z}, f(x) > y. \\ \exists y = -15, f(0) = f(-2) = y. \end{cases}$$

## 1.4

证明.

$$\begin{cases} f: A \times B \rightarrow B \times A, (a, b) \mapsto (b, a) \\ g: B \times A \rightarrow A \times B, (b, a) \mapsto (a, b) \end{cases}$$

则

$$f \circ g = I_{A \times B}, \quad g \circ f = I_{B \times A}, \quad g = f^{-1}.$$

即  $f$  为双射, 即证  $|A \times B| = |B \times A|$ . □

## 1.5

证明.

(1)

$$\forall f(x) = \sum_{i=0}^n a_i \cdot x^i, \exists! g(x) = \sum_{i=1}^n i \cdot a_i \cdot x^{i-1}, \frac{d}{dx} f(x) = g(x). (a_i \in \mathbb{R})$$

值域为  $R[x]$ , 是满射不是双射.

$$\forall g(x) = \sum_{i=0}^n a_i \cdot x^i, \exists f(x) = a + \sum_{i=0}^n \frac{a_i}{i+1} \cdot x^{i+1}, \frac{d}{dx} f(x) = g(x). (a, a_i \in \mathbb{R})$$

(2)

$$\forall f(x) = \sum_{i=0}^n a_i \cdot x^i, \exists! g(x) = \sum_{i=0}^n \frac{a_i}{i+1} \cdot x^{i+1}, I(f(x)) = g(x). (a_i \in \mathbb{R})$$

值域为常数项为 0 的实系数多项式, 既不是满射, 也不是双射.

$$\forall g(x) = a (0 \neq a \in \mathbb{R}), \text{ 若 } \exists f(x) \in R[x], \text{ 满足 } I(f(x)) = g(x) = a \quad I(f(x)) = g(x).$$

则

$$a = g(x) = \int_0^x f(t) dt \xrightarrow{x=0} g(0) = \int_0^0 f(t) dt = 0 \neq a.$$

矛盾. □

## 1.6

证明.  $\forall (b_{i1}, b_{i2}, \dots, b_{in}) \in S(B),$

$$\exists! f \in F, f: A \rightarrow B, a_j \mapsto b_{ij}. \text{ 满足 } g(f) = (b_{i1}, b_{i2}, \dots, b_{in}).$$

即证  $g$  即是单射, 又是满射, 即  $g$  是从  $F$  到  $S(B)$  的双射.

$$|F| = |S(B)| = |B|^n = m^n.$$

□

## 1.7

证明.

$$(1) f(A \cup B) = f(A) \cup f(B)$$

$$(a) \forall y \in f(A \cup B), \exists x \in A \cup B, f(x) = y.$$

$$\begin{cases} x \in A, y = f(x) \in f(A) \\ x \in B, y = f(x) \in f(B) \end{cases} \Rightarrow y \in f(A) \cup f(B) \Rightarrow f(A \cup B) \subseteq f(A) \cup f(B).$$

$$(b) \forall y \in f(A) \cup f(B), y \in f(A) \text{ 或 } y \in f(B).$$

$$\begin{cases} y \in f(A), \exists x \in A, f(x) = y \\ y \in f(B), \exists x \in B, f(x) = y \end{cases} \Rightarrow y \in f(A \cup B) \Rightarrow f(A) \cup f(B) \subseteq f(A \cup B).$$

综上, 即证

$$f(A \cup B) = f(A) \cup f(B).$$

$$(2) f(A \cap B) \subseteq f(A) \cap f(B)$$

$$\forall y \in f(A \cap B), \exists x \in A \cap B, f(x) = y.$$

由  $x \in A \cap B$  有  $x \in A, x \in B$ , 即

$$\begin{cases} x \in A, y = f(x) \in f(A) \\ x \in B, y = f(x) \in f(B) \end{cases} \Rightarrow y \in f(A) \cap f(B) \Rightarrow f(A \cap B) \subseteq f(A) \cap f(B).$$

(3)

$$S = \mathbb{Z}, T = \{1\}, A = \{2k+1 | k \in \mathbb{Z}\}, B = \{2k | k \in \mathbb{Z}\}$$

取  $f: S \rightarrow T, n \mapsto 1$ , 则

$$\begin{cases} f(A \cap B) = f(\phi) = \phi. \\ f(A) \cap f(B) = \{1\} \end{cases} \Rightarrow f(A \cap B) \neq f(A) \cap f(B).$$

□

## 1.8

(1)  $f$  为单射:  $f(\tilde{A}) \subseteq \widetilde{f(A)}$ .

$$\forall y \in f(\tilde{A}) \subseteq S, \exists! x \in S \text{ 且 } x \in \tilde{A}, f(x) = y.$$

有

$$\forall x \in A, f(x) \neq y, y \notin f(A) \Rightarrow y \in \widetilde{f(A)}.$$

即

$$\forall y \in f(\tilde{A}) \subseteq S, y \in \widetilde{f(A)} \Rightarrow f(\tilde{A}) \subseteq \widetilde{f(A)}.$$

(2)  $f$  为满射:  $\widetilde{f(A)} \subseteq f(\tilde{A})$ .

$$\forall y \in \widetilde{f(A)}, \exists x \in S, f(x) = y.$$

若  $x \in A, y = f(x) \in f(A)$ , 矛盾. 故  $x \in \tilde{A}$ .

$$y = f(x) \in f(\tilde{A}) \Rightarrow \widetilde{f(A)} \subseteq f(\tilde{A}).$$

## 1.9

(1)

$$f \circ g = 3(3x+1) = 9x+3.$$

(2)

$$f \circ g = 3$$

(3)

$$g \circ f = 3(3x) + 1 = 9x + 1.$$

(4)

$$g \circ h = 3(3x + 2) + 1 = 9x + 7.$$

(5)

$$f \circ g \circ h = 3(3(3x + 2) + 1) = 27x + 21.$$

**1.10**

证明.

先证  $g \circ f$  是从  $A$  到  $C$  的映射.

$$\forall x \in A, \exists! y = f(x) \in B, \exists! z = g(y) \in C.$$

假设  $g \circ f$  不是从  $A$  到  $C$  的单射, 则等价于

$$\exists c \in C, \exists a_1, a_2 \in A, a_1 \neq a_2, g \circ f(a_1) = g \circ f(a_2) = c.$$

即

$$g(f(a_1)) = g(f(a_2)) = c \in C \xrightarrow{g} \exists! b \in B, g(b) = c, f(a_1) = f(a_2) = b \in B.$$

又

$$f \text{ 为单射} \Rightarrow \exists a \in A, f(a) = b.$$

即  $a_1 = a_2 = a$ , 矛盾, 即证  $g \circ f$  是从  $A$  到  $C$  的单射.

□

**1.11**

会发生矛盾.

(1) 先证  $g$  为满射若  $\exists y \in S, \forall x \in S, g(x) \neq y$ , 则  $f(y) \in S$ , 且  $f(y) \notin f \circ g(S)$ , 矛盾.(2) 再证  $g$  为单射若  $\exists a, b \in S, a \neq b$ , 且  $g(a) = g(b)$ , 则  $f(g(a)) = f(g(b))$ .又  $f \circ g(a) = a, f \circ g(b) = b, a \neq b$ , 矛盾.

综上, 即证  $g$  为双射. 故

$$\forall x \in S, f(g(x)) = f(f^{-1}(x)) = x, \Rightarrow \forall x \in S, g(x) = f^{-1}(x), g \circ f = I_S.$$

故矛盾.

### 1.12

(1)

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix} = (1)(2)(3)(4\ 6)(5).$$

(2)

$$\tau^2\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 5 & 6 & 3 \end{pmatrix} = (124563).$$

(3)

$$\sigma^2\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 4 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

(4)

$$\sigma^{-1}\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}.$$

### 1.13

(1)

$$(257)(78)(145) = \begin{pmatrix} 1 & 2 & 4 & 5 & 7 & 8 \\ 4 & 5 & 7 & 1 & 8 & 2 \end{pmatrix} = (147825).$$

(2)

$$(72815)(21)(476)(12) = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 2 & 7 & 4 & 6 & 1 \end{pmatrix} = (1576428).$$

### 1.14

(1)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} = (18)(2)(364)(57).$$

(2)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix} = (134)(26)(587).$$

(3)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix} = (13478652).$$

**1.15**

(1)

$$(47)(261)(567)(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 7 & 2 & 1 & 4 & 5 \end{pmatrix} = (1642375).$$

故阶为 7.

(2)

$$(163)(1357)(67)(12345) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 4 & 7 & 1 & 6 & 3 \end{pmatrix} = (125)(347)(6).$$

 $[3, 3, 1] = 3$ , 故阶为 3.**1.16**

证明.

(1) 由定理 3.9, 任何  $n$  元置换都可以表示成若干不相交的轮换之乘积.

(2) 任意轮换可以表示成对换之积.

(3) 任意对换  $\sigma = (ij)$  ( $j > i$ ), 有

$$(ij) = (i(i+1))((i+1)(i+2)) \dots ((j-1)j).$$

综上, 即证任意置换都可以表示成对换  $(12), \dots, ((n-1)n)$  的乘积.

□

**1.17**

证明.

(1)

$$\begin{aligned}
x_1x_2x_3 + x_1\bar{x}_2x_3 + x_1 + x_1\bar{x}_2\bar{x}_3 &= x_1(x_2x_3 + \bar{x}_2x_3) + x_1(x_2\bar{x}_3 + \bar{x}_2\bar{x}_3) \\
&= x_1x_3(x_2 + \bar{x}_2) + x_1\bar{x}_3(x_2 + \bar{x}_2) \\
&= x_1x_3 + x_1\bar{x}_3 \\
&= x_1(x_3 + \bar{x}_3) \\
&= x_1.
\end{aligned}$$

(2)

$$\begin{aligned}
x_1x_2 + x_2x_3 + \bar{x}_1x_3 &= x_1x_2 + (x_1 + \bar{x}_1)x_2x_3 + \bar{x}_1x_3 \\
&= x_1x_2 + x_1x_2x_3 + \bar{x}_1x_2x_3 + \bar{x}_1x_3 \\
&= x_1x_2(1 + x_3) + \bar{x}_1x_3(1 + x_2) \\
&= x_1x_2 + \bar{x}_1x_3.
\end{aligned}$$

□

**1.18**

证明. (1)

$$\begin{aligned}
f \cdot g + \bar{f} &= f \cdot (f + g) + \bar{f} \\
&= f \cdot f + f \cdot g + \bar{f} \\
&= f + \bar{f} + f \cdot g \\
&= 1 + f \cdot g \\
&= 1
\end{aligned}$$

(2)

$$\begin{aligned}
\bar{f} + g &= \bar{f} + f + g \\
&= 1 + g \\
&= 1
\end{aligned}$$



(3)

$$\begin{aligned} p \cdot \bar{q} &= \overline{\bar{p} + q} \\ &= \bar{1} \\ &= 0 \end{aligned}$$

□

**1.19**

(1)

$$f = x_1x_2 + x_1\bar{x}_2 + \bar{x}_1x_2 + \bar{x}_1\bar{x}_2.$$

(2)

$$g = x_1x_2 + \bar{x}_1\bar{x}_2.$$