

1

1.1

- (1) 不能构成映射, 如 $x_1 = 0$, 则 $x_2 = 0, 1, \dots, 9$ 均满足.
- (2) 能构成映射, $\forall y_1 \in \mathbb{R}, \exists! y_2 = y_1^2$.
- (3) 不能构成映射, 如 $y_1 = 1$, 则 $y_2 = \pm 1$ 均满足.

1.2

(1)

$$R_f = \{2, -2, 0\}$$

(2)

$$\begin{cases} |A| = 9 \\ |R_f| = 3 \end{cases} \Rightarrow n = 3^9.$$

1.3

(1) 满射

$$\forall y \in \mathbb{Z}^+, \exists x = \pm(y-1) \in \mathbb{Z}, f(x) = y.$$

(2) 既不是单射, 又不是满射. 值域为 $\{0, 1, 2\} \subseteq \mathbb{Z} \cup \{0\}$

$$\forall y \in \mathbb{Z} \cup \{0\} \begin{cases} y \in \{0, 1, 2\}, \exists x = 3k + y (k \in \mathbb{Z}), f(x) = y. \\ y \notin \{0, 1, 2\}, \forall x \in \mathbb{Z}, f(x) \neq y. \end{cases}$$

(3) f, g 均是双射.

$$\begin{cases} \forall y \in \mathbb{Z}, \exists! x = y - 1 \in \mathbb{Z}, f(x) = y. \\ \forall y \in \mathbb{Z}, \exists! x = y + 1 \in \mathbb{Z}, g(x) = y. \end{cases}$$

(4) 满射.

$$\forall y \in \{0, 1\}, \exists x = 2k + y + 1 (k \in \mathbb{Z}), f(x) = y.$$

(5) 既不是单射, 又不是满射.

$$\begin{cases} \forall y \in \mathbb{Z}, y < -16, \forall x \in \mathbb{Z}, f(x) > y. \\ \exists y = -15, f(0) = f(-2) = y. \end{cases}$$

1.4

证明.

$$\begin{cases} f: A \times B \rightarrow B \times A, (a, b) \mapsto (b, a) \\ g: B \times A \rightarrow A \times B, (b, a) \mapsto (a, b) \end{cases}$$

则

$$f \circ g = Id_{A \times B}, \quad g \circ f = Id_{B \times A}, \quad g = f^{-1}.$$

即 f 为双射, 即证 $|A \times B| = |B \times A|$. □

1.5

证明.

(1)

$$\forall f(x) = \sum_{i=0}^n a_i \cdot x^i, \exists! g(x) = \sum_{i=1}^n i \cdot a_i \cdot x^{i-1}, \frac{d}{dx} f(x) = g(x). (a_i \in \mathbb{R})$$

值域为 $R[x]$, 是满射不是双射.

$$\forall g(x) = \sum_{i=0}^n a_i \cdot x^i, \exists f(x) = a + \sum_{i=0}^n \frac{a_i}{i+1} \cdot x^{i+1}, \frac{d}{dx} f(x) = g(x). (a, a_i \in \mathbb{R})$$

(2)

$$\forall f(x) = \sum_{i=0}^n a_i \cdot x^i, \exists! g(x) = \sum_{i=0}^n \frac{a_i}{i+1} \cdot x^{i+1}, I(f(x)) = g(x). (a_i \in \mathbb{R})$$

值域为常数项为 0 的实系数多项式, 既不是满射, 也不是双射.

$$\forall g(x) = a (0 \neq a \in \mathbb{R}), \text{ 若 } \exists f(x) \in R[x], \text{ 满足 } I(f(x)) = g(x) = a \quad I(f(x)) = g(x).$$

则

$$a = g(x) = \int_0^x f(t) dt \xrightarrow{x=0} g(0) = \int_0^0 f(t) dt = 0 \neq a.$$

矛盾. □

1.6

证明. $\forall (b_{i1}, b_{i2}, \dots, b_{in}) \in S(B),$

$$\exists! f \in F, f: A \rightarrow B, a_j \mapsto b_{ij}. \text{ 满足 } g(f) = (b_{i1}, b_{i2}, \dots, b_{in}).$$

即证 g 即是单射, 又是满射, 即 g 是从 F 到 $S(B)$ 的双射.

$$|F| = |S(B)| = |B|^n = m^n.$$

□

1.7

证明.

$$(1) f(A \cup B) = f(A) \cup f(B)$$

$$(a) \forall y \in f(A \cup B), \exists x \in A \cup B, f(x) = y.$$

$$\begin{cases} x \in A, y = f(x) \in f(A) \\ x \in B, y = f(x) \in f(B) \end{cases} \Rightarrow y \in f(A) \cup f(B) \Rightarrow f(A \cup B) \subseteq f(A) \cup f(B).$$

$$(b) \forall y \in f(A) \cup f(B), y \in f(A) \text{ 或 } y \in f(B).$$

$$\begin{cases} y \in f(A), \exists x \in A, f(x) = y \\ y \in f(B), \exists x \in B, f(x) = y \end{cases} \Rightarrow y \in f(A \cup B) \Rightarrow f(A) \cup f(B) \subseteq f(A \cup B).$$

综上, 即证

$$f(A \cup B) = f(A) \cup f(B).$$

$$(2) f(A \cap B) \subseteq f(A) \cap f(B)$$

$$\forall y \in f(A \cap B), \exists x \in A \cap B, f(x) = y.$$

由 $x \in A \cap B$ 有 $x \in A, x \in B$, 即

$$\begin{cases} x \in A, y = f(x) \in f(A) \\ x \in B, y = f(x) \in f(B) \end{cases} \Rightarrow y \in f(A) \cap f(B) \Rightarrow f(A \cap B) \subseteq f(A) \cap f(B).$$

(3)

$$S = \mathbb{Z}, T = \{1\}, A = \{2k + 1 | k \in \mathbb{Z}\}, B = \{2k | k \in \mathbb{Z}\}$$

取 $f : S \rightarrow T, n \mapsto 1$, 则

$$\begin{cases} f(A \cap B) = f(\phi) = \phi. \\ f(A) \cap f(B) = \{1\} \end{cases} \Rightarrow f(A \cap B) \neq f(A) \cap f(B).$$

□

1.8(1) f 为单射:(2) f 为满射:**1.9**

(1)

$$f \circ g = 3(3x + 1) = 9x + 3.$$

(2)

$$f \circ g = 3$$

(3)

$$g \circ f = 3(3x) + 1 = 9x + 1.$$

(4)

$$g \circ h = 3(3x + 2) + 1 = 9x + 7.$$

(5)

$$f \circ g \circ h = 3(3(3x + 2) + 1) = 27x + 21.$$

1.10

证明.

□

1.11

(1)

(2)

1.12

(1)

(2)

(3)

(4)

1.13

(1)

(2)

1.14

(1)

(2)

(3)

1.15

(1)

(2)

1.16

证明.

□

1.17

(1)

(2)

1.18

(1)

(2)

(3)

1.19

(1)

(2)