

1 数论初步

1.1

证明.

(1)

$$\forall x|a, x|b \begin{cases} x > 0 & \xrightarrow{a>0, x|a} x \leq a \\ x < 0 & \xrightarrow{a>0} x < a \end{cases} \Rightarrow x < a \xrightarrow{a|a, a|b} (a, b) = a.$$

(2)

$$\left\{ \begin{array}{l} (a, b)|(a, b), (a, b)|b \\ \forall x|(a, b), x|b, \text{ 有 } x \leq (a, b). \text{ (证明同(1))} \end{array} \right. \Rightarrow ((a, b), b) = (a, b).$$

□

1.2

证明.

(1) 不妨假设 $\exists n > 0, (n, n+1) = d > 1$

$$\begin{aligned} (n, n+1) = d &\Rightarrow \exists x, y \in \mathbb{Z}, n = xd, n+1 = yd \\ &\Rightarrow 1 = (n+1) - n = (y-x)d > 0 \\ &\Rightarrow y > x, (y-x)d \geq d > 1 \\ &\Rightarrow \text{矛盾, 假设不成立.} \end{aligned}$$

(2) 可取 (n, k) , 证明如下

由推论 2.3, 取 $x = 1, a = n, b = k$, 有 $(n, k) = (n, n+k)$.

□

1.3

(1) $(314, 159) = 1$, 有解。由辗转相除法

$$314 = 159 \cdot 1 + 155$$

$$159 = 155 \cdot 1 + 4$$

$$155 = 4 \cdot 38 + 3$$

$$4 = 3 \cdot 1 + 1$$

即

$$1 = 4 - 3 \cdot 1$$

$$= 4 - (155 - 4 \cdot 38) \cdot 1$$

$$= (159 - 155 \cdot 1) \cdot 39 - 155$$

$$= 159 \cdot 39 - 155 \cdot 40$$

$$= 159 \cdot 39 - (314 - 159 \cdot 1) \cdot 40$$

$$= 159 \cdot 79 - 314 \cdot 40.$$

即 $x = -40, y = 79$.

(2) $(3141, 1592) = 1$, 有解。由辗转相除法

$$3141 = 1592 \cdot 1 + 1549$$

$$1592 = 1549 \cdot 1 + 43$$

$$1549 = 43 \cdot 36 + 1$$

即

$$1 = 1549 - 43 \cdot 36$$

$$= 1549 - (1592 - 1549 \cdot 1) \cdot 36$$

$$= 1549 \cdot 37 - 1592 \cdot 36$$

$$= (3141 - 1592 \cdot 1) \cdot 37 - 1592 \cdot 36$$

$$= 3141 \cdot 37 - 1592 \cdot 73.$$

即 $x = 37, y = -73$.

1.4

证明.

$$(0) \quad n = 1, n^3 - n = 0, \text{ 有 } 0 = 6 \cdot 0, 6|(n^3 - n).$$

$$(1) \quad n = 2, n^3 - n = 0, \text{ 有 } 6 = 6 \cdot 1, 6|(n^3 - n).$$

$$(2) \quad \text{假设 } n = k, k \in \mathbb{N} \text{ 时, 有 } 6|(k^3 - k), \text{ 则 } n = k + 1 \text{ 时有}$$

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k(k+1) \end{aligned}$$

显然有 $6|(k^3 - k)$, 下证 $6|3k(k+1)$

$$1^\circ \quad k = 1, 3k(k+1) = 6, \text{ 有 } 6 = 6 \cdot 1, 6|3k(k+1)$$

$$2^\circ \quad \text{若 } 6|3k(k+1), \text{ 则}$$

$$3(k+1)(k+2) = 3k(k+1) + 6(k+1) \Rightarrow 6|3(k+1)(k+2)$$

即证

$$\forall k \in \mathbb{N} \quad 6|3k(k+1) \Rightarrow 6|(k+1)^3 - (k+1)$$

综上, 即证

$$\forall n > 0, \quad 6|(n^3 - n).$$

□

1.5

证明.

$$\left\{ \begin{array}{ll} 3^4 \equiv 1(\text{mod } 10) & \Rightarrow 3^{4n} \equiv 1(\text{mod } 10) \\ & \Rightarrow 3^{m+4n} \equiv (-1)(\text{mod } 10). \\ 10|(3^m + 1) & \Rightarrow 3^m \equiv (-1)(\text{mod } 10) \end{array} \right.$$

即证

$$10|(3^{m+4n} + 1)$$

□

1.6

(1)

$$2345 = 5 \cdot 7 \cdot 67$$

(2)

$$3456 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

1.7

证明. 不妨假设 $\exists n > 0$, 使得 $n(n+1) = d^2$ 为平方数, 则有

$$n^2 < n(n+1) = d^2 < (n+1)^2 \Rightarrow n < d < n+1$$

不存在相邻整数间的整数, d 不存在, 假设不成立, 即证. □

1.8

证明. $n = 5! + 1 = 2 \cdot 3 \cdot 4 \cdot 5 + 1$.

(1)

$$n+1 = 2 \cdot 3 \cdot 4 \cdot 5 + 2 = 2 \cdot (3 \cdot 4 \cdot 5 + 1)$$

(2)

$$n+2 = 2 \cdot 3 \cdot 4 \cdot 5 + 3 = 3 \cdot (2 \cdot 4 \cdot 5 + 1)$$

(3)

$$n+3 = 2 \cdot 3 \cdot 4 \cdot 5 + 4 = 4 \cdot (2 \cdot 3 \cdot 5 + 1)$$

(4)

$$n+4 = 2 \cdot 3 \cdot 4 \cdot 5 + 5 = 5 \cdot (2 \cdot 3 \cdot 4 + 1)$$

□

1.9

(1) $(1, 1) = 1|2$, 方程有解, $x_0 = 0, y_0 = 2$ 为一组特解, 故通解为

$$\begin{cases} x = t & (t \in \mathbb{Z}) \\ y = 2 - t \end{cases}$$

(2) $(2, 1) = 1|2$, 方程有解, $x_0 = 0, y_0 = 2$ 为一组特解, 故通解为

$$\begin{cases} x = t & (t \in \mathbb{Z}) \\ y = 2 - 2t \end{cases}$$

(3) $(15, 16) = 1|17$, 方程有解, $x_0 = -17, y_0 = 17$ 为一组特解, 故通解为

$$\begin{cases} x = 16t - 17 & (t \in \mathbb{Z}) \\ y = 17 - 15t \end{cases}$$

1.10

(1) $(6, -15) = 3|51$, 方程有解, $x_0 = 11, y_0 = 1$ 为一组特解, 故通解为

$$\begin{cases} x = 11 - 5t & (t \in \mathbb{Z}) \\ y = 1 - 2t \end{cases}$$

又要求负整数解, 故 $x, y < 0, t \geq 3$, 即所以负整数解为

$$\begin{cases} x = 11 - 5t & (t \in \mathbb{Z}, t \geq 3) \\ y = 1 - 2t \end{cases}$$

(2) $(6, 15) = 3|51$, 方程有解, $x_0 = 6, y_0 = 1$ 为一组特解, 故通解为

$$\begin{cases} x = 6 + 5t & (t \in \mathbb{Z}) \\ y = 1 - 2t \end{cases}$$

又要求负整数解, 故 $x, y < 0, t$ 无解, 即无负整数解.

1.11

1.11.1 必须要用 30 张

设需要 x 张 5 分, y 张 1 角, $z = (30 - x - y)$ 张 2 角五分. 有

$$\begin{aligned} 0.05x + 0.1y + 0.25(30 - x - y) &= 5 \Leftrightarrow x + 2y + 5(30 - x - y) = 100 \\ &\Leftrightarrow 4x + 3y = 50 \end{aligned}$$

$(4, 3) = 1|50$, 方程有解, $x_0 = 2, y_0 = 14$ 为一组特解, 故通解为

$$\begin{cases} x = 2 + 3t & (t \in \mathbb{Z}) \\ y = 14 - 4t \end{cases}$$

又 $x, y, z \in \mathbb{N}$, 即

$$\begin{cases} 2 + 3t \geq 0 \\ 14 - 4t \geq 0 \\ 14 + t \geq 0 \end{cases} \xrightarrow{t \in \mathbb{Z}} t = 0, 1, 2, 3.$$

即有 4 种方案, 记 x 张 5 分, y 张 1 角, z 张 2 角五分, 则方案为

$$\begin{cases} x = 2 \\ y = 14 \\ z = 14 \end{cases} \quad \begin{cases} x = 5 \\ y = 10 \\ z = 15 \end{cases} \quad \begin{cases} x = 8 \\ y = 6 \\ z = 16 \end{cases} \quad \begin{cases} x = 11 \\ y = 2 \\ z = 17 \end{cases}$$

1.11.2 不多于 30 张

即求 $a, b, c \in \mathbb{N}$, $0.05a + 0.1b + 0.25c = 5$, 且 $a + b + c \leq 30$. 即解

$$\begin{cases} a + 2b + 5c = 100 \\ (a + b + c) \leq 30 \\ a, b, c \in \mathbb{N} \end{cases}$$

(1) $c \leq 13$

$$a + 2b + 5c < 2 \cdot (30 - c) + 5c = 3c + 60 \leq 99 < 100, \text{无解.}$$

(2) $c > 20$

$$a + 2b + 5c > 5c > 100, \text{无解.}$$

(3) 由 (1)(2) 可知 $14 \leq c \leq 20$

$$a + 2b = 100 - 5c, (1, 2) = 1|100 - 5c \Rightarrow \text{方程存在解.}$$

又 $a = 100 - 5c, b = 0$ 为一组特解, 故通解为

$$\begin{cases} a = 100 - 5c - 2t; \\ b = t. \end{cases} (t \in \mathbb{N})$$

$$\begin{cases} a+b+c=100-4k-t \leq 30 \\ a=100-5c-2t \geq 0 \end{cases} \Rightarrow \begin{cases} t \geq 70-4k \\ t \leq \left\lfloor \frac{100-5k}{2} \right\rfloor \end{cases}$$

即解的组数为

$$\begin{cases} \left\lfloor \frac{100-5k}{2} \right\rfloor - (70-4k) + 1; & (70-4k \geq 0) \\ \left\lfloor \frac{100-5k}{2} \right\rfloor + 1. & (70-4k < 0) \end{cases}$$

$$(a) \ (c=14) \text{ 解的组数为 } \left\lfloor \frac{100-5 \times 14}{2} \right\rfloor - (70-4 \times 14) + 1 = 2$$

$$(b) \ (c=15) \text{ 解的组数为 } \left\lfloor \frac{100-5 \times 15}{2} \right\rfloor - (70-4 \times 15) + 1 = 3$$

$$(c) \ (c=16) \text{ 解的组数为 } \left\lfloor \frac{100-5 \times 16}{2} \right\rfloor - (70-4 \times 16) + 1 = 5$$

$$(d) \ (c=17) \text{ 解的组数为 } \left\lfloor \frac{100-5 \times 17}{2} \right\rfloor - (70-4 \times 17) + 1 = 6$$

$$(e) \ (c=18) \text{ 解的组数为 } \left\lfloor \frac{100-5 \times 18}{2} \right\rfloor + 1 = 6$$

$$(f) \ (c=19) \text{ 解的组数为 } \left\lfloor \frac{100-5 \times 19}{2} \right\rfloor + 1 = 3$$

$$(g) \ (c=20) \text{ 解的组数为 } \left\lfloor \frac{100-5 \times 20}{2} \right\rfloor + 1 = 1$$

即共有 $2+3+5+6+6+3+1=26$ 种兑换方法.

1.12

设买了 x 个苹果, $12-x$ 个橘子, 每个苹果 y 分钱, 每个橘子 $y-3$ 分钱, 则有

$$\begin{cases} 0 \leq 12-x < x \\ xy + (12-x)(y-3) = 99 \end{cases} \Leftrightarrow \begin{cases} 6 < x \leq 12 \\ x+4y = 45 \end{cases}$$

$(1, 4) = 1|45$, 方程有解, $x_0 = 9, y_0 = 9$ 为一组特解, 故通解为

$$\begin{cases} x = 9 + 4t \\ y = 9 - t \end{cases} \quad (t \in \mathbb{Z})$$

又 $6 < x \leq 12$, 即 $t = 0, x = 9, 12-x = 3$, 买了 9 个苹果和 3 个橘子.

1.13

$$6k + 5 \equiv 6k + 1 \pmod{4}$$

又 $6k \equiv 6 \pmod{4}$, 有

$$\begin{aligned} 6k + 5 &\equiv 7 \pmod{4} \\ &\equiv 3 \pmod{4} \end{aligned}$$

1.14

证明.

(1) 分情况讨论 $6k, 6k + 2, 6k + 3, 6k + 4$ ($k \geq 1$) 即可, 不再赘述.

(2) 记素数为 $p, p > 3$.

(a) $p < 6$, 则 $p = 5$, 成立.

(b) $p > 6$, 有 $(6, p) = 1$, 故 p 属于 6 的缩系, 故 p 模 6 或与 1 或 5 同余.

□

1.15

证明.

不妨设这两个连续的立方数为 k^3 与 $(k+1)^3$.

$$\begin{aligned} (k+1)^3 - k^3 &\equiv 3k^2 + 3k + 1 \pmod{3} \\ &\equiv 1 \pmod{3} \end{aligned}$$

□

1.16

证明.

设该数为 $A = \overline{a_n a_{n-1} \dots a_1 a_0}$, 则

$$A = \sum_{i=0}^{i=n} a_i \cdot 10^i, \quad \sum_{i=0}^{i=n} a_i \equiv 0 \pmod{3}$$

又 $\forall k \in \mathbb{N}, 10^i \equiv 1(\text{mod}3)$, 故

$$\begin{aligned} A &\equiv \sum_{i=0}^{i=n} a_i \cdot 10^i (\text{mod}3) \\ &\equiv \sum_{i=0}^{i=n} a_i (\text{mod}3) \\ &\equiv 0 (\text{mod}3) \end{aligned}$$

□

1.17

证明.

(1)

$$10 \equiv -1(\text{mod}11) \Rightarrow 10^k \equiv (-1)^k(\text{mod}11)$$

(2) 设数为 $A = \overline{a_n a_{n-1} \dots a_1 a_0}$, 则

$$A \equiv 0(\text{mod}11) \Leftrightarrow \sum_{i=0}^n (-1)^i \cdot a_i \equiv 0(\text{mod}11)$$

即偶数位之和与奇数位之和的差能被 11 整除等价于该数也能被 11 整除.

□

1.18

(1)

$$\begin{aligned} 2x &\equiv 1(\text{mod}17) \\ &\equiv 18(\text{mod}17) \end{aligned} \xrightarrow{(2,17)=1} x \equiv 9(\text{mod}17)$$

(2) $(3, 18) = 3|6$, 故有 3 组解由 $x \equiv 2(\text{mod}6)$ 得原方程解为

$$x \equiv 2 + 6t(\text{mod}18) \quad (0 \leq t \leq 2).$$

即

$$x \equiv 2, 8, 14(\text{mod}18).$$

(3) $(4, 18) = 2|6$, 故有 2 组解解 $2x \equiv 3(\text{mod } 9)$

$$\begin{aligned} 2x &\equiv 3(\text{mod } 9) \\ &\equiv 12(\text{mod } 9) \end{aligned} \xrightarrow{(2,9)=1} x \equiv 6(\text{mod } 9).$$

即原方程解为

$$x \equiv 6 + 9t(\text{mod } 18) \quad (t = 0, 1) \Rightarrow x \equiv 6, 15(\text{mod } 18).$$

(4)

$$\begin{aligned} 3x &\equiv 1(\text{mod } 17) \\ &\equiv 18(\text{mod } 17) \end{aligned} \xrightarrow{(3,17)=1} x \equiv 6(\text{mod } 17).$$

1.19

(1) $(2, 3) = 1$, 有解。本题中

$$M = 2 \cdot 3 = 6, M_1 = 3, M_2 = 2.$$

由

$$\begin{cases} 3b_1 &\equiv 1(\text{mod } 2) \\ 2b_2 &\equiv 1(\text{mod } 3) \end{cases} \Rightarrow \begin{cases} b_1 &= 1 \\ b_2 &= 2 \end{cases}$$

从而

$$\begin{aligned} y &= 3 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 2 \\ &= 7 \end{aligned} \Rightarrow y \equiv 1(\text{mod } 6).$$

(2) $(41, 26) = 1$, 有解。原式等价于

$$\begin{cases} x &\equiv 31(\text{mod } 41) \\ x &\equiv 7(\text{mod } 26) \end{cases}$$

本题中

$$M = 41 \cdot 26, M_1 = 26, M_2 = 41.$$

由

$$\begin{cases} 26b_1 &\equiv 1(\text{mod } 41) \\ 41b_2 &\equiv 1(\text{mod } 26) \end{cases} \Rightarrow \begin{cases} b_1 &= 30 \\ b_2 &= 7 \end{cases}$$

从而

$$\begin{aligned} y &= 26 \cdot 31 \cdot 30 + 41 \cdot 7 \cdot 7 \\ &= 26819 \end{aligned} \Rightarrow y \equiv 605 \pmod{1066}.$$

(3) $(2, 3) = (2, 7) = (3, 7) = 1$, 有解。本题中

$$M = 2 \cdot 3 \cdot 7 = 42, M_1 = 21, M_2 = 14, M_3 = 6.$$

由

$$\begin{cases} 21b_1 \equiv 1 \pmod{2} \\ 14b_2 \equiv 1 \pmod{3} \\ 6b_3 \equiv 1 \pmod{7} \end{cases} \Rightarrow \begin{cases} b_1 = 1 \\ b_2 = 2 \\ b_3 = 6 \end{cases}$$

从而

$$\begin{aligned} y &= 21 \cdot 1 \cdot 1 + 14 \cdot 1 \cdot 2 + 6 \cdot 6 \cdot 6 \\ &= 265. \end{aligned} \Rightarrow y \equiv 13 \pmod{42}.$$

(4) 原式等价于

$$\begin{cases} x \equiv 3 \pmod{5} \\ x \equiv 3 \pmod{7} \\ x \equiv 3 \pmod{11} \end{cases}$$

$(5, 7) = (5, 11) = (7, 11) = 1$, 有解。本题中

$$M = 5 \cdot 7 \cdot 11 = 385, M_1 = 77, M_2 = 55, M_3 = 35$$

由

$$\begin{cases} 77b_1 \equiv 1 \pmod{5} \\ 55b_2 \equiv 1 \pmod{7} \\ 35b_3 \equiv 1 \pmod{11} \end{cases} \Rightarrow \begin{cases} b_1 = 3 \\ b_2 = 6 \\ b_3 = 6 \end{cases}$$

从而

$$\begin{aligned} y &= 77 \cdot 3 \cdot 3 + 55 \cdot 3 \cdot 6 + 35 \cdot 3 \cdot 6 \\ &= 2313. \end{aligned} \Rightarrow y \equiv 3 \pmod{385}.$$

1.20

设

$$\begin{cases} 3x \equiv m-1 & (\text{mod } 20) \\ 5y \equiv m & (\text{mod } 20) \\ 7z \equiv m+1 & (\text{mod } 20). \end{cases} \quad (1 \leq m \leq 18)$$

则

$$\begin{cases} 3x = 20(m-1) + (m-1) \\ 5y = 20m + m \\ 7z = 20(m+1) + (m+1). \end{cases} \Rightarrow \begin{cases} x = 7m-7 \\ y = \frac{21m}{5} \\ z = 3m+3. \end{cases} \Rightarrow 5|m, m \in \{5, 10, 15\}$$

即

$$\begin{cases} x = 28 \\ y = 21 \\ z = 18; \end{cases} \quad \begin{cases} x = 63 \\ y = 42 \\ z = 33; \end{cases} \quad \begin{cases} x = 98 \\ y = 63 \\ z = 48. \end{cases}$$

1.21

由题意有

$$\begin{cases} n \equiv 0 & (\text{mod } 2) \\ n+1 \equiv 0 & (\text{mod } 3) \\ n+2 \equiv 0 & (\text{mod } 4) \\ n+3 \equiv 0 & (\text{mod } 5) \\ n+4 \equiv 0 & (\text{mod } 6) \end{cases} \Leftrightarrow \begin{cases} n \equiv 0 & (\text{mod } 2) \\ n \equiv 2 & (\text{mod } 3) \\ n \equiv 2 & (\text{mod } 4) \\ n \equiv 2 & (\text{mod } 5) \\ n \equiv 2 & (\text{mod } 6) \end{cases}$$

由 $n=2$ 为一个特解, 有模 $[2, 3, 4, 5, 6] = 60$ 唯一解

$$n \equiv 2 \pmod{60}$$

故所求最小整数 $n(n > 2)$ 为

$$n = 62.$$

1.22

(1)

$$\phi(42) = \phi(2 \cdot 3 \cdot 7) = \phi(2) \cdot \phi(3) \cdot \phi(7) = 1 \cdot 2 \cdot 6 = 12.$$

(2)

$$\phi(420) = \phi(2^2 \cdot 3 \cdot 5 \cdot 7) = \phi(2^2) \cdot \phi(3) \cdot \phi(5) \cdot \phi(7) = 2 \cdot 2 \cdot 4 \cdot 6 = 96.$$

(3)

$$\phi(4200) = \phi(2^3 \cdot 3 \cdot 5^2 \cdot 7) = \phi(2^3) \cdot \phi(3) \cdot \phi(5^2) \cdot \phi(7) = 4 \cdot 2 \cdot 20 \cdot 6 = 960.$$

1.23

(1) 小于 18 且与 18 互素的正整数有

$$1, 5, 7, 11, 13, 17$$

(2)

$$1 \cdot 5 \equiv 5(\text{mod}18) \qquad 5 \cdot 5 \equiv 25(\text{mod}18)$$

$$\equiv 7(\text{mod}18)$$

$$7 \cdot 5 \equiv 35(\text{mod}18) \qquad 11 \cdot 5 \equiv 55(\text{mod}18)$$

$$\equiv 17(\text{mod}18) \qquad \equiv 1(\text{mod}18)$$

$$13 \cdot 5 \equiv 65(\text{mod}18) \qquad 17 \cdot 5 \equiv 85(\text{mod}18)$$

$$\equiv 11(\text{mod}18) \qquad \equiv 13(\text{mod}18)$$

仍为缩系，引理 2.1 成立.

1.24

设 m, n 有素数分解

$$m = m_1^{k_1} m_2^{k_2} \cdots m_x^{k_x} \cdot p^M, \quad n = n_1^{l_1} n_2^{l_2} \cdots n_y^{l_y} \cdot p^N$$

且

$$\forall 1 \leq i \leq x, 1 \leq j \leq y, \text{有 } m_i \neq n_j. \quad (m_i, n_j \text{ 均为素数})$$

$$\begin{aligned}
\phi(mn) &= \phi\left(p^{M+N} \cdot \prod_{i=1}^x m_i^{k_i} \cdot \prod_{j=1}^y n_j^{l_j}\right) \\
&= \phi(p^{M+N}) \cdot \prod_{i=1}^x \phi(m_i^{k_i}) \cdot \prod_{j=1}^y \phi(n_j^{l_j}) \\
&= p^{M+N} \cdot \left(1 - \frac{1}{p}\right) \cdot \prod_{i=1}^x m_i^{k_i} \left(1 - \frac{1}{m_i}\right) \cdot \prod_{j=1}^y n_j^{l_j} \left(1 - \frac{1}{n_j}\right) \\
&= mn \cdot \left(1 - \frac{1}{p}\right) \cdot \prod_{i=1}^x \left(1 - \frac{1}{m_i}\right) \cdot \prod_{j=1}^y \left(1 - \frac{1}{n_j}\right)
\end{aligned}$$

$$\begin{aligned}
\phi(m)\phi(n) &= \phi\left(p^M \cdot \prod_{i=1}^x m_i^{k_i}\right) \cdot \phi\left(p^N \cdot \prod_{j=1}^y n_j^{l_j}\right) \\
&= \phi(p^M) \cdot \phi(p^N) \cdot \prod_{i=1}^x \phi(m_i^{k_i}) \cdot \prod_{j=1}^y \phi(n_j^{l_j}) \\
&= p^M \cdot \left(1 - \frac{1}{p}\right) \cdot p^N \cdot \left(1 - \frac{1}{p}\right) \cdot \prod_{i=1}^x m_i^{k_i} \left(1 - \frac{1}{m_i}\right) \cdot \prod_{j=1}^y n_j^{l_j} \left(1 - \frac{1}{n_j}\right) \\
&= mn \cdot \left(1 - \frac{1}{p}\right)^2 \cdot \prod_{i=1}^x \left(1 - \frac{1}{m_i}\right) \cdot \prod_{j=1}^y \left(1 - \frac{1}{n_j}\right)
\end{aligned}$$

即

$$\phi(m)\phi(n) = \left(1 - \frac{1}{p}\right) \cdot \phi(mn)$$

1.25

证明.

显然有 $n \geq 0$, 否则 $\phi(n) \geq 0 > n$, 问题无意义.

(1) $6|n$ 即 $2|n, 3|n$, 不妨记 n 有素数分解

$$n = 2^p \cdot 3^q \cdot n_1^{k_1} n_2^{k_2} \cdots n_N^{k_N}. \quad (p, q \geq 1)$$

则

$$\begin{aligned}
 \phi(n) &= \phi(2^p \cdot 3^q \cdot n_1^{k_1} n_2^{k_2} \cdots n_N^{k_N}) \\
 &= \phi(2^p) \cdot \phi(3^q) \cdot \prod_{i=1}^N \phi(n_i^{k_i}) \\
 &= n \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \prod_{i=1}^N \left(1 - \frac{1}{n_i}\right) \\
 &= \frac{n}{3} \cdot \prod_{i=1}^N \left(1 - \frac{1}{n_i}\right) \\
 &\leq \frac{n}{3}
 \end{aligned}$$

即证, 且当且仅当 $N = 0$ 时等号成立.

(2) 由 T14 可知

$$n - 1 \equiv 5 \pmod{6}; \quad n + 1 \equiv 1 \pmod{6}; \quad n \equiv 0 \pmod{6}$$

即 $6|n$, 由 (1) 即证.

□

1.26

(1)

$$\begin{aligned}
 \frac{3}{2}\phi(3) &= \frac{3}{2} \cdot 2 = 3 = 1 + 2. & \frac{4}{2}\phi(4) &= \frac{4}{2} \cdot 2 = 4 = 1 + 3. \\
 \frac{5}{2}\phi(5) &= \frac{5}{2} \cdot 4 = 10 = 1 + 2 + 3 + 4. & \frac{6}{2}\phi(6) &= \frac{6}{2} \cdot 2 = 6 = 1 + 5. \\
 \frac{7}{2}\phi(7) &= \frac{7}{2} \cdot 6 = 21 = 1 + 2 + 3 + 4 + 5 + 6. & \frac{8}{2}\phi(8) &= \frac{8}{2} \cdot 4 = 16 = 1 + 3 + 5 + 7.
 \end{aligned}$$

(2) 对于整数 $n \geq 3$, 缩系中所有数的和等于 $\frac{1}{2}n \cdot \phi(n)$. 即

$$\sum_{\substack{(d,n)=1 \\ 1 \leq d \leq n-1}} d = \frac{1}{2}\phi(n) \cdot n.$$

(3) 证明.

$\forall 1 \leq d \leq n-1, (d, n) = 1$, 有 $(n-d, n) = 1$, 且 $d \neq n/2$, 故

$$\sum_{\substack{(d,n)=1 \\ 1 \leq d \leq n-1}} d = \sum_{\substack{(d,n)=1 \\ 1 \leq d < n/2}} d + (n-d) = \frac{1}{2}n \cdot \phi(n)$$

□

1.27

$$\begin{aligned} 314^{159} &\equiv (7 \cdot 45 - 1)^{159} \pmod{7} \\ &\equiv (-1)^{159} \pmod{7} \\ &\equiv (-1) \pmod{7} \\ &\equiv 6 \pmod{7} \end{aligned}$$

1.28

(1) 求末位即求模 10 余数

$$\begin{aligned} 7^{355} &\equiv (7^4)^{88} \cdot 7^3 \pmod{10} \\ &\equiv (2400 + 1)^{88} \cdot (340 + 3) \pmod{10} \\ &\equiv 3 \pmod{10} \end{aligned}$$

即末位为 3. 用欧拉定理 $7^{\phi(10)} \equiv 1 \pmod{10}$ 亦可.

(2) 求末两位即求模 100 余数

$$\begin{aligned} 7^{355} &\equiv (7^4)^{88} \cdot 7^3 \pmod{100} \\ &\equiv (2400 + 1)^{88} \cdot (300 + 43) \pmod{100} \\ &\equiv 43 \pmod{100} \end{aligned}$$

即末两位为 43. 用欧拉定理 $7^{\phi(100)} \equiv 1 \pmod{100}$ 亦可.

1.29

证明.

(1)

$$\begin{aligned}
 (k+1)^p - k^p &\equiv 1 \pmod{p} \Leftrightarrow p \mid (k+1)^p - k^p - 1 \\
 &\Leftrightarrow p \mid \sum_{i=1}^{p-1} C_p^i \cdot k^{p-i}
 \end{aligned}$$

有

$$C_p^i = \frac{p(p-1)\dots(p-i+1)}{i!} \in \mathbb{N}, \Rightarrow i! \mid p(p-1)\dots(p-i+1). \quad (1 \leq i \leq p-1)$$

又 $\forall 1 \leq i \leq p-1, (p, i) = 1$, 故 $(p, i!) = 1$, 即

$$i! \mid (p-1)\dots(p-i+1) \Rightarrow \frac{(p-1)\dots(p-i+1)}{i!} \in \mathbb{N}, \quad p \mid C_p^i.$$

故有

$$p \mid C_p^i \Rightarrow p \mid \sum_{i=1}^{p-1} C_p^i \cdot k^{p-i} \Rightarrow (k+1)^p - k^p \equiv 1 \pmod{p}.$$

(2) 对于任意素数 p 有 $p \nmid a$, 则

$$\begin{aligned}
 a^p &\equiv \sum_{k=0}^{a-1} ((k+1)^p - k^p) \pmod{p} \\
 &\equiv \sum_{k=0}^{a-1} 1 \pmod{p} \\
 &\equiv a \pmod{p}
 \end{aligned}$$

又 $(a^p, a) = a, (p, a) = 1$, 故有

$$a^{p-1} \equiv 1 \pmod{p}.$$

□

1.30

证明.

(1)

$$\forall 1 \leq k \leq p-1, (k, p) = 1 \text{ 且 } p \nmid k \Rightarrow k^{p-1} \equiv 1 \pmod{p}$$

故

$$\begin{aligned}\sum_{i=1}^{p-1} i^{p-1} &\equiv \sum_{i=1}^{p-1} 1 \pmod{p} \\ &\equiv p-1 \pmod{p} \\ &\equiv -1 \pmod{p}.\end{aligned}$$

(2)

$$\begin{aligned}\forall 1 \leq k \leq p-1, (k, p) = 1 \text{ 且 } p \mid k &\Rightarrow k^{p-1} \equiv 1 \pmod{p} \\ &\Rightarrow k^p \equiv k \pmod{p}\end{aligned}$$

故

$$\begin{aligned}\sum_{i=1}^{p-1} i^{p-1} &\equiv \sum_{i=1}^{p-1} i \pmod{p} \\ &\equiv \frac{p(p-1)}{2} \pmod{p} \\ &\equiv 0 \pmod{p}. \quad (2 \mid p-1)\end{aligned}$$

□

1.31

$$\begin{array}{lll}d(42) = d(2 \cdot 3 \cdot 7) & d(420) = d(2^2 \cdot 3 \cdot 5 \cdot 7) & d(4200) = d(2^3 \cdot 3 \cdot 5^2 \cdot 7) \\ = 2^3 & = 3 \cdot 2^3 & = 4 \cdot 3 \cdot 2^2 \\ = 8. & = 24. & = 48.\end{array}$$

$$\begin{aligned}
 \sigma(42) &= \sigma(2 \cdot 3 \cdot 7) \\
 &= \frac{2^2-1}{2-1} \cdot \frac{3^2-1}{3-1} \cdot \frac{7^2-1}{7-1} \\
 &= 96.
 \end{aligned}$$

$$\begin{aligned}
 \sigma(420) &= \sigma(2^2 \cdot 3 \cdot 5 \cdot 7) \\
 &= \frac{2^3-1}{2-1} \cdot \frac{3^2-1}{3-1} \cdot \frac{5^2-1}{5-1} \cdot \frac{7^2-1}{7-1} \\
 &= 1344.
 \end{aligned}$$

$$\begin{aligned}
 \sigma(4200) &= \sigma(2^3 \cdot 3 \cdot 5^2 \cdot 7) \\
 &= \frac{2^4-1}{2-1} \cdot \frac{3^2-1}{3-1} \cdot \frac{5^3-1}{5-1} \cdot \frac{7^2-1}{7-1} \\
 &= 14880.
 \end{aligned}$$

1.32

不妨设 n 有素数分解

$$n = n_1^{k_1} n_2^{k_2} \cdots n_x^{k_x} \Rightarrow \sigma(n) = (k_1+1)(k_2+1) \cdots (k_x+1) = 60$$

又

$$\lceil \log_2(10^4) \rceil = 13 \Rightarrow k_1 + k_2 + \cdots + k_x \leq 13$$

$$n = 2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5040 \quad \text{或} \quad n = 2^4 \cdot 5^2 \cdot 3 \cdot 7 = 8400$$

1.33

证明.

设 d_1, d_2, \dots, d_k 为 n 的全部因子 (相同因子算两遍), 则

$$\forall 1 \leq i \leq k, \exists! 1 \leq j \leq k, d_j = n/d_i$$

不妨取一个排列使得 $i+j = k+1$.

$$\begin{aligned}
 \sum_{d|n} \frac{1}{d} &= \frac{1}{n} \sum_{i=1}^k \frac{n}{d_i} \\
 &= \frac{1}{n} \sum_{i=1}^k d_{k+1-i} \\
 &= \frac{1}{n} \sum_{i=1}^k d_i \\
 &= \frac{1}{n} \sigma(n).
 \end{aligned}$$

□

1.34

证明.

不妨记偶完全数为

$$n = 2^{p-1} \cdot (2^p - 1) \quad (p, 2^p - 1 \text{ 均为素数})$$

由题意可得

$$2^{p-1} \cdot (2^p - 1) \equiv 6 \pmod{10} \quad \text{或} \quad 2^{p-1} \cdot (2^p - 1) \equiv 8 \pmod{10}$$

等价于

$$2^{p-2} \cdot (2^p - 1) \equiv 3 \pmod{5} \quad \text{或} \quad 2^{p-2} \cdot (2^p - 1) \equiv 4 \pmod{5}$$

$$\begin{cases} (2^{p-2}, 5) = 1 \\ (2^p - 1, 5) = 1 \end{cases} \Rightarrow \begin{cases} 2^{p-2} \equiv a \pmod{5} & (a \in \{1, 2, 3, 4\}) \\ 2^p - 1 \equiv b \pmod{5} & (b \in \{1, 2, 3, 4\}) \end{cases}$$

$$(1) \quad 2^{p-2} \equiv 1 \pmod{5}$$

$$\begin{aligned} 2^p - 1 &\equiv 2^2 \cdot 1 - 1 \pmod{5} \\ &\equiv 3 \pmod{5} \end{aligned} \Rightarrow \begin{aligned} 2^{p-2} \cdot (2^p - 1) &\equiv 3 \cdot 1 \pmod{5} \\ &\equiv 3 \pmod{5} \end{aligned}$$

$$(2) \quad 2^{p-2} \equiv 2 \pmod{5}$$

$$\begin{aligned} 2^p - 1 &\equiv 2^2 \cdot 2 - 1 \pmod{5} \\ &\equiv 2 \pmod{5} \end{aligned} \Rightarrow \begin{aligned} 2^{p-2} \cdot (2^p - 1) &\equiv 2 \cdot 2 \pmod{5} \\ &\equiv 4 \pmod{5} \end{aligned}$$

$$(3) \quad 2^{p-2} \equiv 3 \pmod{5}$$

$$\begin{aligned} 2^p - 1 &\equiv 2^2 \cdot 3 - 1 \pmod{5} \\ &\equiv 1 \pmod{5} \end{aligned} \Rightarrow \begin{aligned} 2^{p-2} \cdot (2^p - 1) &\equiv 3 \cdot 1 \pmod{5} \\ &\equiv 3 \pmod{5} \end{aligned}$$

$$(4) 2^{p-2} \equiv 4 \pmod{5}$$

$$\begin{aligned} 2^p - 1 &\equiv 2^2 \cdot 4 - 1 \pmod{5} \\ &\equiv 0 \pmod{5} \end{aligned} \Rightarrow 0 \notin \{1, 2, 3, 4\}, \text{该情况不存在.}$$

综上, 即证

$$2^{p-1} \cdot (2^p - 1) \equiv 6 \pmod{10} \quad \text{或} \quad 2^{p-1} \cdot (2^p - 1) \equiv 8 \pmod{10}$$

□

1.35

证明.

由题意可得

$$n = 2^{p-1} \cdot (2^p - 1) \quad (p, 2^p - 1 \text{ 均为素数})$$

又 $n > 6$, 故 $p > 2$, $2|p-1$, 不妨记 $p-1 = 2k$ ($k \in \mathbb{Z}^*$), 有

$$n = 2^{2k} \cdot (2^{2k+1} - 1) = 4^k \cdot (2 \cdot 4^k - 1)$$

又

$$\forall k \geq 1, 4^k \equiv m \pmod{9}, \text{ 有 } m \in \{4, 7, 1\}$$

$$(1) 4^k \equiv 4 \pmod{9}$$

$$\begin{aligned} 2 \cdot 4^k - 1 &\equiv 2 \cdot 4 - 1 \pmod{9} \\ &\equiv 7 \pmod{9} \end{aligned} \Rightarrow \begin{aligned} 4^k \cdot (2 \cdot 4^k - 1) &\equiv 4 \cdot 7 \pmod{9} \\ &\equiv 1 \pmod{9} \end{aligned}$$

$$(2) 4^k \equiv 7 \pmod{9}$$

$$\begin{aligned} 2 \cdot 4^k - 1 &\equiv 2 \cdot 7 - 1 \pmod{9} \\ &\equiv 4 \pmod{9} \end{aligned} \Rightarrow \begin{aligned} 4^k \cdot (2 \cdot 4^k - 1) &\equiv 7 \cdot 4 \pmod{9} \\ &\equiv 1 \pmod{9} \end{aligned}$$

$$(3) 4^k \equiv 1 \pmod{9}$$

$$\begin{aligned} 2 \cdot 4^k - 1 &\equiv 2 \cdot 1 - 1 \pmod{9} \\ &\equiv 1 \pmod{9} \end{aligned} \Rightarrow \begin{aligned} 4^k \cdot (2 \cdot 4^k - 1) &\equiv 1 \cdot 1 \pmod{9} \\ &\equiv 1 \pmod{9} \end{aligned}$$

综上, 即证

$$n \equiv 1 \pmod{9}.$$

□

1.36

证明. $\forall p, \sigma(p) = p + 1, \phi(p) = p - 1, d(p) = 2$, 有

$$\sigma(p) = \phi(p) + d(p) \Rightarrow \sum_{p \leq x} \sigma(p) = \sum_{p \leq x} \phi(p) + \sum_{p \leq x} d(p).$$

□

1.37

$$2^1 \equiv 2 \pmod{15}$$

$$2^2 \equiv 4 \pmod{15}$$

$$2^3 \equiv 8 \pmod{15}$$

$$2^4 \equiv 1 \pmod{15}$$

因此, 2模15的阶为4.

$$7^1 \equiv 7 \pmod{15}$$

$$7^2 \equiv 4 \pmod{15}$$

$$7^3 \equiv 13 \pmod{15}$$

$$7^4 \equiv 1 \pmod{15}$$

因此, 7模15的阶为4.

$$8^1 \equiv 8 \pmod{15}$$

$$8^2 \equiv 4 \pmod{15}$$

$$8^3 \equiv 2 \pmod{15}$$

$$8^4 \equiv 1 \pmod{15}$$

因此, 8模15的阶为4.

$$4^1 \equiv 4 \pmod{15}$$

$$4^2 \equiv 1 \pmod{15}$$

因此, 4模15的阶为2.

$$11^1 \equiv 11 \pmod{15}$$

$$11^2 \equiv 1 \pmod{15}$$

因此, 11模15的阶为2.

$$14^1 \equiv 14 \pmod{15}$$

$$14^2 \equiv 1 \pmod{15}$$

因此, 14模15的阶为2.

$$13^1 \equiv 13 \pmod{15}$$

$$13^2 \equiv 4 \pmod{15}$$

$$13^3 \equiv 7 \pmod{15}$$

$$13^4 \equiv 1 \pmod{15}$$

因此, 13模15的阶为4.

1.38

(1) 2 为 29 的原根.

$$n = \text{ind}_2 k \Leftrightarrow 2^n \equiv k \pmod{29}$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
n	0	1	5	2	22	6	12	3	10	23	25	7	18	13
k	15	16	17	18	19	20	21	22	23	24	25	26	27	28
n	27	4	21	11	9	24	17	26	20	8	16	19	15	14

(2) 由 2 为 29 的原根可知

$$\text{ind}_2 9 + \text{ind}_2 x = \text{ind}_2 2 \pmod{28}$$

又由 (1) 得 $\text{ind}_2 9 = 10$, $\text{ind}_2 2 = 1$, 有

$$\text{ind}_2 x = -9 \equiv 19 \pmod{28} \Rightarrow x \equiv 2^{19} \equiv 26 \pmod{29}.$$

(3) 由 2 为 29 的原根可知

$$9 \cdot \text{ind}_2 x = \text{ind}_2 2 \pmod{28}$$

又由 (1) 得 $\text{ind}_2 2 = 1$, 有

$$\begin{aligned} 9 \cdot \text{ind}_2 x &\equiv 1 \pmod{28} \Rightarrow \text{ind}_2 x \equiv 25 \pmod{28} \\ &\Rightarrow x \equiv 11 \pmod{29}. \end{aligned}$$

1.39

证明.

不对, 证明如下:

$$457^{911} \equiv 1 \pmod{10021} \Leftrightarrow \begin{cases} 457^{911} \equiv 1 \pmod{11} \\ 457^{911} \equiv 1 \pmod{911} \end{cases}$$

又

$$\begin{cases} 457^{10} \equiv 1 \pmod{11} \\ 457^{910} \equiv 1 \pmod{911} \end{cases} \Rightarrow \begin{cases} 457^{911} \equiv 6 \pmod{11} \\ 457^{911} \equiv 1457 \pmod{911} \end{cases}$$

故不对.

□

1.40

$\phi(\phi(37)) = 12$, 即 37 有 12 个原根.

又 $2^{36} \equiv 1 \pmod{37}$, 且

$$\begin{cases} 2^1 \equiv \quad \pmod{37}; & 2^2 \equiv \quad \pmod{37}; \\ 2^3 \equiv \quad \pmod{37}; & 2^4 \equiv \quad \pmod{37}; \\ 2^6 \equiv \quad \pmod{37}; & 2^9 \equiv \quad \pmod{37}; \\ 2^{12} \equiv \quad \pmod{37}; & 2^{18} \equiv \quad \pmod{37}; \end{cases} \Rightarrow 2 \text{ 为 } 37 \text{ 最小原根.}$$

由推论 2.7, 对于 $a = 2^i$ ($(i, 36) = 1$), a 模 37 的阶为 36. 即原根集合为

$$\{x \mid 1 \leq x \leq 36, x \equiv 2^i \pmod{37}\}, \text{ 且 } i \in \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35\}$$

经计算

$$\begin{aligned} 2^1 &\equiv 2 \pmod{37}, 2^5 \equiv 32 \pmod{37}, 2^7 \equiv 17 \pmod{37}, 2^{11} \equiv 13 \pmod{37}, \\ 2^{13} &\equiv 15 \pmod{37}, 2^{17} \equiv 18 \pmod{37}, 2^{19} \equiv 35 \pmod{37}, 2^{23} \equiv 5 \pmod{37}, \\ 2^{25} &\equiv 20 \pmod{37}, 2^{29} \equiv 24 \pmod{37}, 2^{31} \equiv 22 \pmod{37}, 2^{35} \equiv 19 \pmod{37}, \end{aligned}$$

即 37 的原根集合为

$$\{2, 5, 13, 15, 17, 18, 19, 20, 22, 24, 32, 35\}$$

1.41

证明.

设 $(-a)$ 模 q 的阶为 d .

$$\begin{aligned} q \mid (a^p + 1) &\Rightarrow a^p \equiv -1 \pmod{q} \\ &\Rightarrow -a^p \equiv 1 \pmod{q} \quad \Rightarrow d \mid p, d = 1 \text{ 或 } p. \\ &\Rightarrow (-a)^p \equiv 1 \pmod{q} \end{aligned}$$

(1) $d = 1$

$$\begin{aligned} -a \equiv 1 \pmod{q} &\Rightarrow a + 1 \equiv 0 \pmod{q} \\ &\Rightarrow q \mid (a + 1). \end{aligned}$$

(2) $d = p$

$$\begin{aligned} (-a)^p \equiv 1 \pmod{q} &\Rightarrow p \mid \phi(q), p \mid (q - 1) \\ &\Rightarrow 2p \mid (q - 1) \\ &\Rightarrow \exists k \in \mathbb{Z}, 2kp = q - 1, q \mid (2kp + 1) \end{aligned}$$

即证

$$q \mid (a + 1) \text{ 或 } q \mid (2kp + 1) \quad (k \text{ 为某个整数}).$$

□

1.42

证明.

6 的正因子为 1, 2, 3, 6, 则 $(a + 1)$ 模 p 的阶为 6 等价于

$$\begin{cases} (a + 1) \not\equiv 1 \pmod{p} & (a + 1)^2 \not\equiv 1 \pmod{p} \\ (a + 1)^3 \not\equiv 1 \pmod{p} & (a + 1)^6 \equiv 1 \pmod{p} \end{cases}$$

(1)

$$(a, p) = 1 \Rightarrow (a + 1) \not\equiv 1 \pmod{p} \Rightarrow p \nmid (a - 1).$$

(2) 由 $a^3 \equiv 1 \pmod{p}, a \not\equiv 1 \pmod{p}, a^2 \not\equiv 1 \pmod{p}$, 得

$$a^3 - 1 = (a - 1)(a^2 + a + 1) \equiv 0 \pmod{p} \Rightarrow p \mid (a^2 + a + 1)$$

$$(a + 1)^2 = a^2 + 2a + 1 \equiv a \not\equiv 1 \pmod{p}.$$

(3)

$$(a + 1)^3 \equiv a(a + 1) \equiv -a^3 \equiv -1 \pmod{p}.$$

(4)

$$(a + 1)^6 \equiv 1 \pmod{p}.$$

即证 $(a+1)$ 模 p 的阶为 6.

□