

妮可代数结构答案

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2023 年 3 月 9 日

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1 集合

1.1

- (1) 不相等.
- (2) 相等.
- (3) 相等.

1.2

证明.

$$\left\{ \begin{array}{l} A \subseteq B \Rightarrow \forall x \in A, x \in B. \\ B \subset C \Rightarrow \left\{ \begin{array}{l} \forall x \in B, x \in C \\ \exists x \in C, x \notin B \end{array} \right. \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \forall x \in A, x \in C \\ \exists x \in C, x \notin A \end{array} \right. \Rightarrow A \subset C.$$

□

1.3

- (1) 不成立.
- (2) 不成立.
- (3) 不成立.
- (4) 成立.
- (5) 成立.
- (6) 不成立.

1.4

- (1) 不成立.
- (2) 成立.
- (3) 成立.

1.5

证明.

(1)

$$A \cap (\overline{A} \cup B) = (A \cap \overline{A}) \cup (A \cap B) = \emptyset \cup (A \cap B) = A \cap B.$$

(2)

$$A \cup (A \cap B) = (A \cup A) \cap (A \cup B) = A \cap (A \cup B).$$

$$\begin{cases} A \subseteq A \cup (A \cap B) \\ A \supseteq A \cap (A \cup B) \end{cases} \Rightarrow A \cup (A \cap B) = A.$$

(3) (a)

$$\begin{aligned} \forall x \in \overline{\bigcap_i A_i} &\Rightarrow x \notin \bigcap_i A_i & \forall x \in \bigcup_i \overline{A_i} &\Rightarrow \exists 1 \leq k \leq n, x \in \overline{A_k} \\ &\Rightarrow \exists 1 \leq k \leq n, x \notin A_k & &\Rightarrow \exists 1 \leq k \leq n, x \notin A_k \\ &\Rightarrow \exists 1 \leq k \leq n, x \in \overline{A_k} & &\Rightarrow x \notin \bigcap_i A_i \\ &\Rightarrow x \in \bigcup_i \overline{A_i} & &\Rightarrow x \in \overline{\bigcap_i A_i} \\ &\Rightarrow \overline{\bigcap_i A_i} \subseteq \bigcup_i \overline{A_i} & &\Rightarrow \bigcup_i \overline{A_i} \subseteq \overline{\bigcap_i A_i} \end{aligned}$$

即证 $\overline{\bigcap_i A_i} = \bigcup_i \overline{A_i}$.

(b)

$$\begin{aligned} \forall x \in \overline{\bigcup_i A_i} &\Rightarrow x \notin \bigcup_i A_i & \forall x \in \bigcap_i \overline{A_i} &\Rightarrow \forall 1 \leq k \leq n, x \in \overline{A_k} \\ &\Rightarrow \forall 1 \leq k \leq n, x \notin A_k & &\Rightarrow \forall 1 \leq k \leq n, x \notin A_k \\ &\Rightarrow \forall 1 \leq k \leq n, x \in \overline{A_k} & &\Rightarrow x \notin \bigcup_i A_i \\ &\Rightarrow x \in \bigcap_i \overline{A_i} & &\Rightarrow x \in \overline{\bigcup_i A_i} \\ &\Rightarrow \overline{\bigcup_i A_i} \subseteq \bigcap_i \overline{A_i} & &\Rightarrow \bigcap_i \overline{A_i} \subseteq \overline{\bigcup_i A_i} \end{aligned}$$

即证 $\overline{\bigcup_i A_i} = \bigcap_i \overline{A_i}$.

□

1.6

证明.

$$(1) B \subseteq C \Rightarrow \forall x \in B, x \in C.$$

$$\forall x \in (A \cap B), x \in A \text{ 且 } x \in B \Rightarrow x \in A \text{ 且 } x \in C \Rightarrow x \in (A \cap C)$$

$$(2)$$

$$\begin{aligned} A \subseteq C, B \subseteq C &\Leftrightarrow A \cup C = C, B \cup C = C \\ &\Leftrightarrow (A \cup B) \cup C = A \cup (B \cup C) = A \cup C = C \\ &\Leftrightarrow (A \cup B) \subseteq C. \end{aligned}$$

$$(3) \text{ 若 } |A \cup B| > |A| + |B|, \text{ 则 } \exists x \in (A \cup B), \text{ 且 } x \notin A, x \notin B, \text{ 矛盾.}$$

$$|A \cup B| = |A| + |B| - |A \cap B|, |A \cup B| = |A| + |B| \text{ 当且仅当 } A \cap B = \phi \text{ 时.}$$

□

1.7

$$(1) \text{ 设所求集合为 } E.$$

1. (基础语句) 令 $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, 若 $x \in D$, 则 $x \in E$.
2. (归纳语句) 若 $x, y \in E$, 则 x 与 y 的连接 $\overline{xy} \in E$.
3. (终结语句) $x \in E$, 当且仅当 x 是由有限次 1, 2 得到的.

$$(2) \text{ 设所求集合为 } E.$$

1. (基础语句) 令 $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, 若 $x \in D$, 则 $x \in E, .x \in E$.
2. (归纳语句) 若 $x = a.b, y = c.d \in E$, 则 $\overline{ac}. \overline{bd} \in E$.
3. (终结语句) $x \in E$, 当且仅当 x 是由有限次 1, 2 得到的.

$$(3) \text{ 设所求集合为 } E.$$

1. (基础语句) $0, 10 \in E$.
2. (归纳语句) 若 $x = \overline{A0} \in E (x \neq 0)$, 则 $\overline{A00}, \overline{A10} \in E$.
3. (终结语句) $x \in E$, 当且仅当 x 是由有限次 1, 2 得到的.

2 数论初步

2.1

证明.

(1)

$$\forall x|a, x|b \begin{cases} x > 0 & \xrightarrow{a>0, x|a} x \leq a \\ x < 0 & \xrightarrow{a>0} x < a \end{cases} \Rightarrow x < a \xrightarrow{a|a, a|b} (a, b) = a.$$

(2)

$$\left\{ \begin{array}{l} (a, b)|(a, b), (a, b)|b \\ \forall x|(a, b), x|b, \text{ 有 } x \leq (a, b). \text{ (证明同(1))} \end{array} \right. \Rightarrow ((a, b), b) = (a, b).$$

□

2.2

证明.

(1) 不妨假设 $\exists n > 0, (n, n+1) = d > 1$

$$\begin{aligned} (n, n+1) = d &\Rightarrow \exists x, y \in \mathbb{Z}, n = xd, n+1 = yd \\ &\Rightarrow 1 = (n+1) - n = (y-x)d > 0 \\ &\Rightarrow y > x, (y-x)d \geq d > 1 \\ &\Rightarrow \text{矛盾, 假设不成立.} \end{aligned}$$

(2) 可取 (n, k) , 证明如下

由推论 2.3, 取 $x = 1, a = n, b = k$, 有 $(n, k) = (n, n+k)$.

□

2.3

(1) $(314, 159) = 1$, 有解。由辗转相除法

$$314 = 159 \cdot 1 + 155$$

$$159 = 155 \cdot 1 + 4$$

$$155 = 4 \cdot 38 + 3$$

$$4 = 3 \cdot 1 + 1$$

即

$$1 = 4 - 3 \cdot 1$$

$$= 4 - (155 - 4 \cdot 38) \cdot 1$$

$$= (159 - 155 \cdot 1) \cdot 39 - 155$$

$$= 159 \cdot 39 - 155 \cdot 40$$

$$= 159 \cdot 39 - (314 - 159 \cdot 1) \cdot 40$$

$$= 159 \cdot 79 - 314 \cdot 40.$$

即 $x = -40, y = 79$.

(2) $(3141, 1592) = 1$, 有解。由辗转相除法

$$3141 = 1592 \cdot 1 + 1549$$

$$1592 = 1549 \cdot 1 + 43$$

$$1549 = 43 \cdot 36 + 1$$

即

$$1 = 1549 - 43 \cdot 36$$

$$= 1549 - (1592 - 1549 \cdot 1) \cdot 36$$

$$= 1549 \cdot 37 - 1592 \cdot 36$$

$$= (3141 - 1592 \cdot 1) \cdot 37 - 1592 \cdot 36$$

$$= 3141 \cdot 37 - 1592 \cdot 73.$$

即 $x = 37, y = -73$.

2.4

证明.

$$(0) \quad n = 1, n^3 - n = 0, \text{ 有 } 0 = 6 \cdot 0, 6|(n^3 - n).$$

$$(1) \quad n = 2, n^3 - n = 0, \text{ 有 } 6 = 6 \cdot 1, 6|(n^3 - n).$$

$$(2) \quad \text{假设 } n = k, k \in \mathbb{N} \text{ 时, 有 } 6|(k^3 - k), \text{ 则 } n = k + 1 \text{ 时有}$$

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k(k+1) \end{aligned}$$

显然有 $6|(k^3 - k)$, 下证 $6|3k(k+1)$

$$1^\circ \quad k = 1, 3k(k+1) = 6, \text{ 有 } 6 = 6 \cdot 1, 6|3k(k+1)$$

$$2^\circ \quad \text{若 } 6|3k(k+1), \text{ 则}$$

$$3(k+1)(k+2) = 3k(k+1) + 6(k+1) \Rightarrow 6|3(k+1)(k+2)$$

即证

$$\forall k \in \mathbb{N} \quad 6|3k(k+1) \Rightarrow 6|(k+1)^3 - (k+1)$$

综上, 即证

$$\forall n > 0, \quad 6|(n^3 - n).$$

□

2.5

证明.

$$\left\{ \begin{array}{ll} 3^4 \equiv 1(\text{mod } 10) & \Rightarrow 3^{4n} \equiv 1(\text{mod } 10) \\ & \Rightarrow 3^{m+4n} \equiv (-1)(\text{mod } 10). \\ 10|(3^m + 1) & \Rightarrow 3^m \equiv (-1)(\text{mod } 10) \end{array} \right.$$

即证

$$10|(3^{m+4n} + 1)$$

□

2.6

(1)

$$2345 = 5 \cdot 7 \cdot 67$$

(2)

$$3456 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

2.7

证明. 不妨假设 $\exists n > 0$, 使得 $n(n+1) = d^2$ 为平方数, 则有

$$n^2 < n(n+1) = d^2 < (n+1)^2 \Rightarrow n < d < n+1$$

不存在相邻整数间的整数, d 不存在, 假设不成立, 即证. □

2.8

证明. $n = 5! + 1 = 2 \cdot 3 \cdot 4 \cdot 5 + 1$.

(1)

$$n+1 = 2 \cdot 3 \cdot 4 \cdot 5 + 2 = 2 \cdot (3 \cdot 4 \cdot 5 + 1)$$

(2)

$$n+2 = 2 \cdot 3 \cdot 4 \cdot 5 + 3 = 3 \cdot (2 \cdot 4 \cdot 5 + 1)$$

(3)

$$n+3 = 2 \cdot 3 \cdot 4 \cdot 5 + 4 = 4 \cdot (2 \cdot 3 \cdot 5 + 1)$$

(4)

$$n+4 = 2 \cdot 3 \cdot 4 \cdot 5 + 5 = 5 \cdot (2 \cdot 3 \cdot 4 + 1)$$

□**2.9**

(1) $(1, 1) = 1|2$, 方程有解, $x_0 = 0, y_0 = 2$ 为一组特解, 故通解为

$$\begin{cases} x = t \\ y = 2 - t \end{cases}, (t \in \mathbb{Z})$$

(2)

(3)

2.10

2.11

2.12

2.13

2.14

2.15

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2.35

2.36

2.37

2.38

3 映射

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