1 映射

1.1

(1) 不能构成映射，如 $x_1 = 0$，则 $x_2 = 0, 1, \ldots, 9$ 均满足.

(2) 能构成映射，$\forall\ y_1 \in \mathbb{R},\ \exists!\ y_2 = y_1^2$.

(3) 不能构成映射，如 $y_1 = 1$，则 $y_2 = \pm 1$ 均满足.

1.2

(1)

$$R_f = \{2, -2, 0\}$$

(2)

$$\begin{cases} |A| = 9 \\ |R_f| = 3 \end{cases} \quad \Rightarrow \ n = 3^9.$$

1.3

(1) 满射

$$\forall\ y \in \mathbb{Z}^+,\ \exists\ x = \pm(y-1) \in \mathbb{Z},\ f(x) = y.$$

(2) 既不是单射，又不是满射. 值域为 $\{0, 1, 2\} \subseteq \mathbb{Z} \cup \{0\}$

$$\forall\ y \in \mathbb{Z} \cup \{0\} \begin{cases} y \in \{0, 1, 2\},\ \exists\ x = 3k + y (k \in \mathbb{Z}),\ f(x) = y. \\ y \notin \{0, 1, 2\},\ \forall\ x \in \mathbb{Z},\ f(x) \neq y. \end{cases}$$

(3) f, g 均是双射.

$$\begin{cases} \forall\ y \in \mathbb{Z},\ \exists!\ x = y - 1 \in \mathbb{Z}, f(x) = y. \\ \forall\ y \in \mathbb{Z},\ \exists!\ x = y + 1 \in \mathbb{Z}, g(x) = y. \end{cases}$$

(4) 满射.

$$\forall\ y \in \{0, 1\},\ \exists\ x = 2k + y + 1 (k \in \mathbb{Z}),\ f(x) = y.$$

(5) 既不是单射，又不是满射.

$$\begin{cases} \forall\ y \in \mathbb{Z},\ y < -16,\ \forall\ x \in \mathbb{Z},\ f(x) > y. \\ \exists\ y = -15,\ f(0) = f(-2) = y. \end{cases}$$

1.4

证明.

$$\begin{cases} f: A\times B \to B\times A,\ (a,b)\mapsto (b,a) \\ g: B\times A \to A\times B,\ (b,a)\mapsto (a,b) \end{cases}$$

则

$$f\circ g = I_{A\times B},\quad g\circ f = I_{B\times A},\quad g = f^{-1}.$$

即 f 为双射，即证 $|A\times B| = |B\times A|$. □

1.5

证明.

(1)

$$\forall\ f(x)=\sum_{i=0}^{n} a_i\cdot x^i,\ \exists!\ g(x)=\sum_{i=1}^{n} i\cdot a_i\cdot x^{i-1},\ \frac{d}{dx}f(x)=g(x).\ (a_i\in\mathbb{R})$$

值域为 $R[x]$，是满射不是双射.

$$\forall\ g(x)=\sum_{i=0}^{n} a_i\cdot x^i,\ \exists\ f(x)=a+\sum_{i=0}^{n}\frac{a_i}{i+1}\cdot x^{i+1},\ \frac{d}{dx}f(x)=g(x).\ (a,a_i\in\mathbb{R})$$

(2)

$$\forall\ f(x)=\sum_{i=0}^{n} a_i\cdot x^i,\ \exists!\ g(x)=\sum_{i=0}^{n}\frac{a_i}{i+1}\cdot x^{i+1},\ I(f(x))=g(x).\ (a_i\in\mathbb{R})$$

值域为常数项为 0 的实系数多项式，既不是满射，也不是双射.

$$\forall\ g(x)=a\ (0\neq a\in\mathbb{R}),\ 若\exists\ f(x)\in R[x],满足I(f(x))=g(x)=a\ I(f(x))=g(x).$$

则

$$a=g(x)=\int_0^x f(t)\,dt\ \xrightarrow{x=0}\ g(0)=\int_0^0 f(t)\,dt=0\neq a.$$

矛盾.

□

1.6

证明. $\forall\ (b_{i1}, b_{i2}, \ldots, b_{in}) \in S(B)$，

$$\exists!\ f \in F,\ f : A \to B,\ a_j \mapsto b_{ij}.\text{满足}g(f) = (b_{i1}, b_{i2}, \ldots, b_{in}).$$

即证 g 即是单射，又是满射，即 g 是从 F 到 $S(B)$ 的双射.

$$|F| = |S(B)| = |B|^n = m^n.$$

□

1.7

证明.

(1) $f(A \cup B) = f(A) \cup f(B)$

(a) $\forall\ y \in f(A \cup B),\ \exists\ x \in A \cup B,\ f(x) = y.$

$$\begin{cases} x \in A, y = f(x) \in f(A) \\ x \in B, y = f(x) \in f(B) \end{cases} \Rightarrow\ y \in f(A) \cup f(B)\ \Rightarrow\ f(A \cup B) \subseteq f(A) \cup f(B).$$

(b) $\forall\ y \in f(A) \cup f(B),\ y \in f(A)\text{或}y \in f(B).$

$$\begin{cases} y \in f(A), \exists\ x \in A, f(x) = y \\ y \in f(B), \exists\ x \in B, f(x) = y \end{cases} \Rightarrow\ y \in f(A \cup B)\ \Rightarrow\ f(A) \cup f(B) \subseteq f(A \cup B).$$

综上，即证

$$f(A \cup B) = f(A) \cup f(B).$$

(2) $f(A \cap B) \subseteq f(A) \cap f(B)$

$$\forall\ y \in f(A \cap B),\ \exists\ x \in A \cap B,\ f(x) = y.$$

由 $x \in A \cap B$ 有 $x \in A, x \in B$，即

$$\begin{cases} x \in A,\ y = f(x) \in f(A) \\ x \in B,\ y = f(x) \in f(B) \end{cases} \Rightarrow\ y \in f(A) \cap f(B)\ \Rightarrow\ f(A \cap B) \subseteq f(A) \cap f(B).$$

(3)

$$S=\mathbb{Z}, T=\{1\}, A=\{2k+1|k\in\mathbb{Z}\}, B=\{2k|k\in\mathbb{Z}\}$$

取 $f:S\to T, n\mapsto 1$，则

$$\begin{cases} f(A\cap B)=f(\phi)=\phi. \\ f(A)\cap f(B)=\{1\} \end{cases} \quad\Rightarrow\quad f(A\cap B)\neq f(A)\cap f(B).$$

$\square$

1.8

(1) f 为单射：$f(\widetilde{A})\subseteq\widetilde{f(A)}$.

$$\forall\ y\in f(\widetilde{A})\subseteq S,\ \exists!\ x\in S\text{且}x\in\widetilde{A},\ f(x)=y.$$

有

$$\forall\ x\in A,\ f(x)\neq y,\ y\notin f(A)\ \Rightarrow\ y\in\widetilde{f(A)}.$$

即

$$\forall\ y\in f(\widetilde{A})\subseteq S,\ y\in\widetilde{f(A)}\ \Rightarrow\ f(\widetilde{A})\subseteq\widetilde{f(A)}.$$

(2) f 为满射：$\widetilde{f(A)}\subseteq f(\widetilde{A})$.

$$\forall\ y\in\widetilde{f(A)},\ \exists\ x\in S, f(x)=y.$$

若 $x\in A,\ y=f(x)\in f(A)$，矛盾. 故 $x\in\widetilde{A}$.

$$y=f(x)\in f(\widetilde{A})\ \Rightarrow\ \widetilde{f(A)}\subseteq f(\widetilde{A}).$$

1.9

(1)

$$f\circ g=3(3x+1)=9x+3.$$

(2)

$$f\circ g=3$$

(3)

$$g\circ f=3(3x)+1=9x+1.$$

(4)

$$g \circ h = 3(3x+2)+1 = 9x+7.$$

(5)

$$f \circ g \circ h = 3(3(3x+2)+1) = 27x+21.$$

1.10

证明.

先证 $g \circ f$ 是从 A 到 C 的映射.

$$\forall\ x \in A,\ \exists!\ y = f(x) \in B,\ \exists!\ z = g(y) \in C.$$

假设 $g \circ f$ 不是从 A 到 C 的单射，则等价于

$$\exists\ c \in C,\ \exists\ a_1, a_2 \in A,\ a_1 \neq a_2,\ g \circ f(a_1) = g \circ f(a_2) = c.$$

即

$$g(f(a_1)) = g(f(a_2)) = c \in C \xrightarrow{g} \exists!\ b \in B,\ g(b) = c,\ f(a_1) = f(a_2) = b \in B.$$

又

$$f\text{为单射} \Rightarrow \exists\ a \in A,\ f(a) = b.$$

即 $a_1 = a_2 = a$，矛盾，即证 $g \circ f$ 是从 A 到 C 的单射.

□

1.11

会发生矛盾.

(1) 先证 g 为满射

$$\text{若}\exists\ y \in S,\ \forall\ x \in S,\ g(x) \neq y,\ \text{则}f(y) \in S,\ \text{且}f(y) \notin f \circ g(S),\ \text{矛盾}.$$

(2) 再证 g 为单射

$$\text{若}\exists\ a, b \in S,\ a \neq b,\ \text{且}g(a) = g(b),\ \text{则}f(g(a)) = f(g(b)).$$

又 $f \circ\ g(a) = a,\ f \circ g(b) = b,\ a \neq b$，矛盾.

综上，即证 g 为双射. 故

$$\forall\ x \in\ S,\ f(g(x)) = f(f^{-1}(x)) = x,\ \ \Rightarrow\ \forall\ x \in\ S,\ g(x) = f^{-1}(x),\ g \circ f = I_S.$$

故矛盾.

1.12

(1)

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix} = (1)(2)(3)(4\ 6)(5).$$

(2)

$$\tau^2\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 5 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 & 3 \end{pmatrix}.$$

(3)

$$\sigma^2\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 4 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

(4)

$$\sigma^{-1}\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}.$$

1.13

(1)

$$(257)(78)(145) = \begin{pmatrix} 1 & 2 & 4 & 5 & 7 & 8 \\ 4 & 5 & 7 & 1 & 8 & 2 \end{pmatrix} = (147825).$$

(2)

$$(72815)(21)(476)(12) = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 2 & 7 & 4 & 6 & 1 \end{pmatrix} = (1576428).$$

1.14

(1)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} = (18)(2)(364)(57).$$

(2)

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\3&6&4&1&8&2&5&7\end{pmatrix}=(134)(26)(587).$$

(3)

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\3&1&4&7&2&5&8&6\end{pmatrix}=(13478652).$$

1.15

(1)

$$(47)(261)(567)(1234)=\begin{pmatrix}1&2&3&4&5&6&7\\6&3&7&2&1&4&5\end{pmatrix}=(1642375).$$

故阶为 7.

(2)

$$(163)(1357)(67)(12345)=\begin{pmatrix}1&2&3&4&5&6&7\\2&5&4&7&1&6&3\end{pmatrix}=(125)(347)(6).$$

$[3,3,1]=3$，故阶为 3.

1.16

证明. □

1.17

(1)

(2)

1.18

(1)

(2)

(3)

1.19

(1)

(2)