图论作业 (Week 13)

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Ch9

1.

Proof. $\forall\ e=(v,u)\in E(D)$，$e\in\alpha(v)$，$e\in\beta(u)$，有

$$\sum_{v\in V(D)}\sum_{e\in\alpha(v)}f(e)=\sum_{u\in V(D)}\sum_{e\in\beta(u)}f(e)$$

又 $\forall\ v\in V(D)-\{s,t\}$，$\sum\limits_{e\in\alpha(v)}f(e)=\sum\limits_{e\in\beta(v)}f(e)$, 可联立解得

$$\sum_{e\in\alpha(s)}f(e)+\sum_{e\in\alpha(t)}f(e)=\sum_{e\in\beta(s)}f(e)+\sum_{e\in\beta(t)}f(e)$$

即

$$\sum_{e\in\alpha(t)}f(e)-\sum_{e\in\beta(t)}f(e)=\sum_{e\in\beta(s)}f(e)-\sum_{e\in\alpha(s)}f(e)$$

□

2.

Proof.

(1) 首先证 $\forall\ e\in E(D)$，有 $c(e)\geq\overline{f}(e)\geq 0$.

(a) $e\notin P(s,t)$.

$$c(e)\geq f(e)=\overline{f}(e)\geq 0$$

(b) e 为正向边

$$c(e)=f(e)+c(e)-f(e)\geq f(e)+l(P)=\overline{f}(e)\geq 0$$

(c) e 为反向边

$$c(e)\geq f(e)>f(e)-l(P)\geq 0$$

(2) 再证 $\forall\ v\in V(D)-\{s,t\}$，都有 $\sum\limits_{e\in\alpha(v)}\overline{f}(e)-\sum\limits_{e\in\beta(v)}\overline{f}(e)=0$.

(a) $v\notin P(s,t)$. 此时

$$\forall\ e\in\alpha(v)\text{或}e\in\beta(v),\ \overline{f}(e)=f(e)\ \Rightarrow\ \sum_{e\in\alpha(v)}\overline{f}(e)-\sum_{e\in\beta(v)}\overline{f}(e)=\sum_{e\in\alpha(v)}f(e)-\sum_{e\in\beta(v)}f(e)=0$$

(b) $v \in P(s,t)$. 不妨设 $P(s,t) = s \cdots e_1 v e_2 \cdots t$. 取 e_1, e_2 均为正向边的情况.

$$
\begin{aligned}
&\sum_{e\in\alpha(v)} \overline{f}(e) - \sum_{e\in\beta(v)} \overline{f}(e) \\
=&\left[\overline{f}(e_1) + \sum_{e\in\alpha(v)-\{e_1\}} \overline{f}(e)\right] - \left[\overline{f}(e_2) + \sum_{e\in\beta(v)-\{e_2\}} \overline{f}(e)\right] \\
=&\left[l(P) + f(e_1) + \sum_{e\in\alpha(v)-\{e_1\}} f(e)\right] - \left[l(P) + f(e_2) + \sum_{e\in\beta(v)-\{e_2\}} f(e)\right] \\
=&\sum_{e\in\alpha(v)} f(e) - \sum_{e\in\beta(v)} f(e) \\
=&0.
\end{aligned}
$$

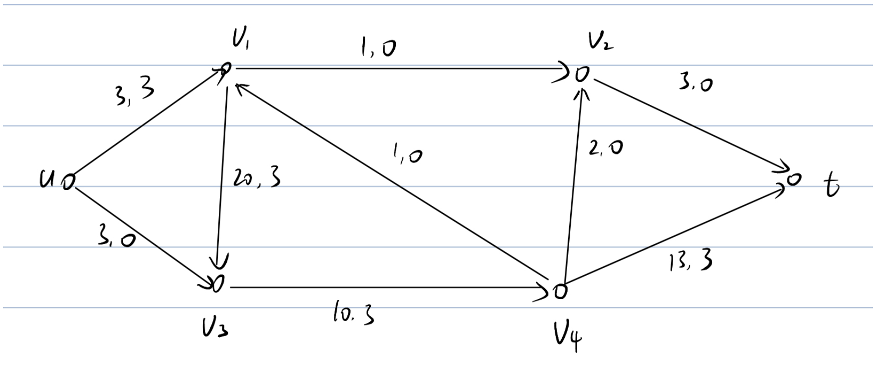
其余情况同理可得.

□

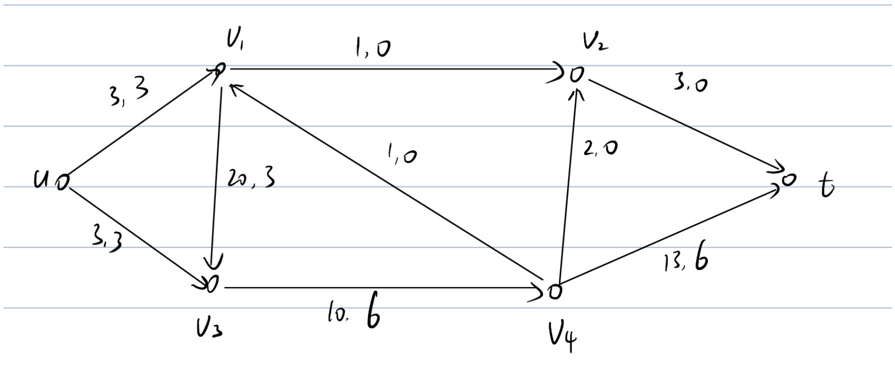
4.

(1) 取初始流 $f \equiv 0$.

(2) 找到可增载轨道 $uv_1v_3v_4t$，此时 $l(P) = 3$，有修正后流函数



(3) 找到可增载轨道 uv_3v_4t，此时 $l(P) = 3$，有修正后流函数



(4) 无可增载轨道，最大流为 6.

5.

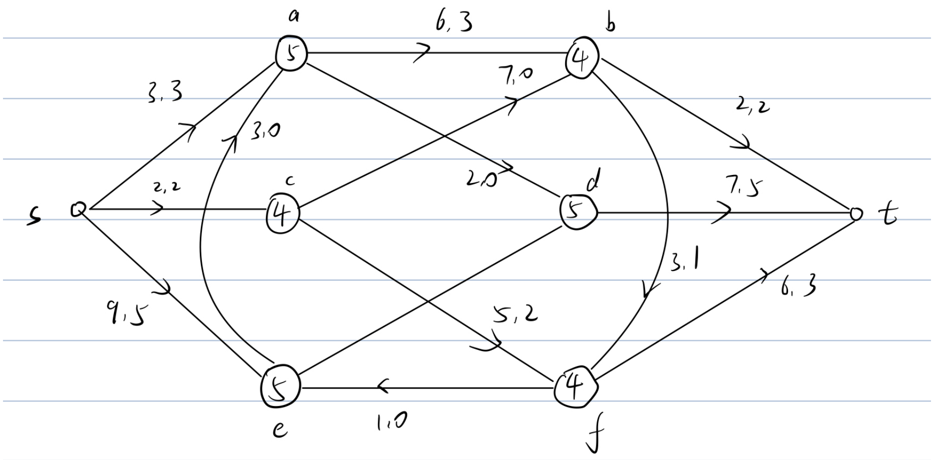
Proof. 对于任意的截，

$$C(S,\bar{S}) = \sum_{e\in(S,\bar{S})c} c(e) \ \Rightarrow\ C(S,\bar{S})\text{为整数, 因此最小截也为整数.}$$

由因为最大流最小截定理，最大流等于最小截，故最大流也一定是整数。 □

7.

如图



10.

Proof.

$$D\text{有可行流} \Leftrightarrow \text{伴随网络}N\text{的最大流使}\forall\ e \in V\text{都满载}$$

□

13.

不存在可行流，$\{c\}$ 需漏掉流，$\{d\}$ 需冒出流。