线性代数 homework (第五周)

PB20000113 孔浩宇

March 27, 2022

1 周二

1.1 习题四

24. 设 A 是奇数阶反对称复方阵，证明：$\det(A) = 0$.

Proof. 记 A 的阶数为 n (n 为奇数), 则有

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A) \Rightarrow \det(A) = 0$$

□

25. 设 A 是 $m \times n$ 矩阵,B 是 $n \times m$ 矩阵，证明：

$$\det(I_n - BA) = \det \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} = \det(I_m - AB).$$

Proof.

$$\begin{aligned} &\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \begin{pmatrix} I_m & -A \\ O & I_n \end{pmatrix} = \begin{pmatrix} I_m & O \\ B & I_n - BA \end{pmatrix} \\ \Rightarrow \det &\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \begin{pmatrix} I_m & -A \\ O & I_n \end{pmatrix} = \det \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} = \det \begin{pmatrix} I_m & O \\ B & I_n - BA \end{pmatrix} = \det(I_n - BA) \end{aligned}$$

$$\begin{aligned} &\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \begin{pmatrix} I_m & O \\ -B & I_n \end{pmatrix} = \begin{pmatrix} I_m - AB & A \\ O & I_n \end{pmatrix} \\ \Rightarrow \det &\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \begin{pmatrix} I_m & O \\ -B & I_n \end{pmatrix} = \det \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} = \det \begin{pmatrix} I_m - AB & O \\ B & I_n \end{pmatrix} = \det(I_m - AB) \end{aligned}$$

□

26. 设 A, B 是 n 阶方阵,λ 是数，证明:

(1) $(\lambda A)^* = \lambda^{n-1} A^*$; (2) $(AB)^* = B^* A^*$; (3) $\det(A^*) = (\det(A))^{n-1}$.

Proof. (1)

$$\begin{aligned} (\lambda A)^* &= \det(\lambda A) \cdot (\lambda A)^{-1} \\ &= \lambda^n \det(A) \cdot \lambda^{-1} \frac{A^*}{\det(A)} \\ &= \lambda^{n-1} A^*. \end{aligned}$$

(2)

$$\begin{aligned}(AB)^* &= \det(AB)\cdot(AB)^{-1}\\ &= \det(A)\cdot\det(B)\cdot B^{-1}A^{-1}\\ &= \det(A)\cdot\det(B)\cdot\frac{B^*}{\det(B)}\cdot\frac{A^*}{\det(A)}\\ &= B^*A^*.\end{aligned}$$

(3)

$$\begin{aligned}\det(A^*) &= \det\left(\det(A)\cdot A^{-1}\right)\\ &= (\det(A))^n\cdot(\det(A))^{-1}\\ &= (\det(A))^{n-1}.\end{aligned}$$

□

27. 设方阵 A 的逆矩阵 $A^{-1}=\begin{pmatrix}1&1&1\\1&2&1\\1&1&3\end{pmatrix}$, 求 A^*.

$$\begin{aligned}A^* &= \det(A)\cdot A^{-1} = \frac{1}{\det(A^{-1})}\cdot A^{-1}\\ &= \frac{1}{2}\cdot A^{-1} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&1&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&\frac{3}{2}\end{pmatrix}.\end{aligned}$$

28. 设方阵 A 的伴随矩阵 $A^*=\begin{pmatrix}0&0&0&1\\0&0&2&0\\0&-1&0&0\\4&0&0&0\end{pmatrix}$，求 A.

$$\begin{aligned}&(\det(A))^3 = \det(A^*) = -8 \quad \Rightarrow \det(A) = -2.\\ \Rightarrow &A^{-1} = \frac{A^*}{\det(A)} = \begin{pmatrix}0&0&0&-\frac{1}{2}\\0&0&-1&0\\0&\frac{1}{2}&0&0\\-2&0&0&0\end{pmatrix}.\\ \Rightarrow &A = \frac{(A^{-1})^*}{\det(A^{-1})} = -2\cdot\begin{pmatrix}0&0&0&\frac{1}{4}\\0&0&-1&0\\0&\frac{1}{2}&0&0\\1&0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0&-\frac{1}{2}\\0&0&2&0\\0&-1&0&0\\-2&0&0&0\end{pmatrix}.\end{aligned}$$

29. 设 n 阶方阵 A 的每行、每列元素之和都是 0，证明:A^* 的所有元素都相等.

Proof. $\forall 1 \le i,j,x,y \le n$, 有 $a_{xl} = -\sum\limits_{k\ne x} a_{kl}, a_{ky} = -\sum\limits_{l\ne y} a_{kl}$.

$$
\begin{aligned}
A_{ij} &= (-1)^{i+j}\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} \\
&= (-1)^{i+j}\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -\sum\limits_{k\ne x} a_{k1} & \cdots & -\sum\limits_{k\ne x} a_{k,j-1} & -\sum\limits_{k\ne x} a_{k,j+1} & \cdots & -\sum\limits_{k\ne x} a_{kn} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} \\
&= (-1)^{i+j}\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -a_{i1} & \cdots & -a_{i,j-1} & -a_{i,j+1} & \cdots & -a_{in} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} \\
&= (-1)^{i+j+(i-1-x)+1}\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{x-1,1} & \cdots & a_{x-1,j-1} & a_{x-1,j+1} & \cdots & a_{x-1,n} \\ a_{x+1,1} & \cdots & a_{x+1,j-1} & a_{x+1,j+1} & \cdots & a_{x+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} \\
&= (-1)^{x+j}\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{x-1,1} & \cdots & a_{x-1,j-1} & a_{x-1,j+1} & \cdots & a_{x-1,n} \\ a_{x+1,1} & \cdots & a_{x+1,j-1} & a_{x+1,j+1} & \cdots & a_{x+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} \\
&= A_{xj}. \quad (\forall 1 \le i,j,x \le n)
\end{aligned}
$$

A^T 也满足每行、每列元素之和都是 0，则可得

$$A_{ij} = (A^T)_{ji} = (A^T)_{yi} = A_{iy}. \quad (\forall 1 \le i,j,y \le n)$$

综合以上可得

$$A_{ij} = A_{xy}. \quad (\forall 1 \le i,j,x,y \le n)$$

□

30. 设 A 是方阵，证明：线性方程组 $Ax = 0$ 有非零解当且仅当 $\det(A) = 0$.

Proof. 记 $A=(a_{ij})_{n\times n} \quad (\exists a_{ij}\neq 0)$.

$$
\begin{aligned}
Ax=0\text{有非零解} &\Leftrightarrow \exists x=(x_1,x_2,\dots,x_n)^T\neq 0, Ax=0;\\
&\Leftrightarrow \exists x=(x_1,x_2,\dots,x_n)^T\neq 0, \sum_{j=1}^{n} x_j(a_{1j},\dots,a_{nj})^T=(0,\dots,0)^T;\\
&\Leftrightarrow \exists x=(x_1,x_2,\dots,x_n)^T\neq 0,\\
&\qquad \det(A)=\frac{1}{x_k}\det\begin{pmatrix} a_{11} & a_{12} & \cdots & \sum\limits_{j=1}^{n} x_j a_{1j} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & \sum\limits_{j=1}^{n} x_j a_{2j} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & \sum\limits_{j=1}^{n} x_j a_{nj} & \cdots & a_{nn}\end{pmatrix}=0. \quad (x_k\neq 0)
\end{aligned}
$$

□

2 周四

2.1 习题四

31. 用 Cramer 法则求解下列线性方程组:

$$
(1)\begin{cases} x_1-x_2+x_3=3,\\ x_1+2x_2+4x_3=5,\\ x_1+3x_2+9x_3=7;\end{cases}\qquad
(2)\begin{cases} 2x_1+x_2-5x_3+x_4=8,\\ x_1-3x_2-6x_4=9,\\ 2x_2-x_3+2x_4=-5,\\ x_1+4x_2-7x_3+6x_4=0.\end{cases}
$$

(1)

$$
\Delta=\begin{vmatrix}1&-1&1\\1&2&4\\1&3&9\end{vmatrix}=12,\ \Delta_1=\begin{vmatrix}3&-1&1\\5&2&4\\7&3&9\end{vmatrix}=36,\ \Delta_2=\begin{vmatrix}1&3&1\\1&5&4\\1&7&9\end{vmatrix}=4,\ \Delta_3=\begin{vmatrix}1&-1&3\\1&2&5\\1&3&7\end{vmatrix}=4.
$$

故方程组的解为

$$
x_1=3,\ x_2=\frac{1}{3},\ x_3=\frac{1}{3}.
$$

(2)

$$
\Delta=\begin{vmatrix}2&1&-5&1\\1&-3&0&-6\\0&2&-1&2\\1&4&-7&6\end{vmatrix}=27,\qquad
\Delta_1=\begin{vmatrix}8&1&-5&1\\9&-3&0&-6\\-5&2&-1&2\\0&4&-7&6\end{vmatrix}=81,
$$

$$
\Delta_2=\begin{vmatrix}2&8&-5&1\\1&9&0&-6\\0&-5&-1&2\\1&0&-7&6\end{vmatrix}=-108,\qquad
\Delta_3=\begin{vmatrix}2&1&8&1\\1&-3&9&-6\\0&2&-5&2\\1&4&0&6\end{vmatrix}=-27,
$$

$$
\Delta_4=\begin{vmatrix}2&1&-5&8\\1&-3&0&9\\0&2&-1&-5\\1&4&-7&0\end{vmatrix}=27.
$$

故方程组的解为

$$
x_1=3,\ x_2=-4,\ x_3=-1,\ x_4=1.
$$

32. 设是 $x_0,x_1,\dots,x_n$ 及 $y_0,y_1,\dots,y_n$ 是任给实数, 其中 $x_i(0\leqslant i\leqslant n)$ 两两不等. 证明: 存在唯一的次数不超过 n 的多项式 $p(x)$, 满足 $p(x_i)=y_i, i=0,1,\dots,n$.

Proof. 设 $p(x)=a_0+a_1x+\cdots+a_nx^n$, 则求 $p(x)$ 满足 $p(x_i)=y_i\ (i=0,1,\ldots,n)$ 等价于解

$$\begin{pmatrix}1 & x_0 & \cdots & {x_0}^n\\ 1 & x_1 & \cdots & {x_1}^n\\ \vdots & \vdots & & \vdots\\ 1 & x_n & \cdots & {x_n}^n\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ \vdots\\ a_n\end{pmatrix}=\begin{pmatrix}y_0\\ y_1\\ \vdots\\ y_n\end{pmatrix}$$

又 $\det\begin{pmatrix}1 & x_0 & \cdots & {x_0}^n\\ 1 & x_1 & \cdots & {x_1}^n\\ \vdots & \vdots & & \vdots\\ 1 & x_n & \cdots & {x_n}^n\end{pmatrix}=\prod\limits_{1\leqslant i<j\leqslant n}(a_j-a_i)\neq 0\xrightarrow{\text{有唯一解}}\begin{pmatrix}a_0\\ a_1\\ \vdots\\ a_n\end{pmatrix}=\begin{pmatrix}1 & x_0 & \cdots & {x_0}^n\\ 1 & x_1 & \cdots & {x_1}^n\\ \vdots & \vdots & & \vdots\\ 1 & x_n & \cdots & {x_n}^n\end{pmatrix}^{-1}\begin{pmatrix}y_0\\ y_1\\ \vdots\\ y_n\end{pmatrix}.$

即证存在唯一的 $a_0,a_1,\ldots,a_n$ 使得 $p(x)=a_0+a_1x+\cdots+a_nx^n$ 满足 $p(x_i)=y_i\ (i=0,1,\ldots,n)$. □

34. 证明: 初等方阵具有以下性质:

(1) $T_{ij}(\lambda)T_{ij}(\mu)=T_{ij}(\lambda+\mu)$;

(2) 当 $i\neq q$ 且 $j\neq p$ 时,$T_{ij}(\lambda)T_{pq}(\mu)=T_{pq}(\mu)T_{ij}(\lambda)$;

(3) $D_i(-1)S_{ij}=S_{ij}D_j(-1)=T_{ji}(1)T_{ij}(-1)T_{ji}(1)$.

Proof. (1)

$$T_{ij}(\lambda)T_{ij}(\mu)=(I+\lambda E_{ij})(I+\mu E_{ij})=I+(\lambda+\mu)E_{ij}=T_{ij}(\lambda+\mu).$$

(2)

$$T_{ij}(\lambda)T_{pq}(\mu)=(I+\lambda E_{ij})(I+\mu E_{pq})=I+\lambda E_{ij}+\mu E_{pq}+\lambda\mu E_{ij}E_{pq}.$$
$$T_{pq}(\mu)T_{ij}(\lambda)=(I+\mu E_{pq})(I+\lambda E_{ij})=I+\mu E_{pq}+\lambda E_{ij}+\lambda\mu E_{pq}E_{ij}.$$

又 $i\neq q$ 且 $j\neq p$, 可得

$$E_{ij}E_{pq}=E_{pq}E_{ij}=O\Rightarrow T_{ij}(\lambda)T_{pq}(\mu)=T_{pq}(\mu)T_{ij}(\lambda).$$

(3)

$$\begin{aligned}D_i(-1)S_{ij}&=(I-2E_{ii})(I-E_{ii}-E_{jj}+E_{ij}+E_{ji})\\&=I-E_{ii}-E_{jj}-E_{ij}+E_{ji};\end{aligned}$$

$$\begin{aligned}S_{ij}D_j(-1)&=(I-E_{ii}-E_{jj}+E_{ij}+E_{ji})(I-2E_{jj})\\&=I-E_{ii}-E_{jj}-E_{ij}+E_{ji};\end{aligned}$$

$$\begin{aligned}T_{ji}(1)T_{ij}(-1)T_{ji}(1)&=(I+E_{ji})(I-E_{ij})(I+E_{ji})\\&=(I+E_{ji}-E_{ij}-E_{jj})(I+E_{ji})\\&=I-E_{ii}-E_{jj}-E_{ij}+E_{ji}.\end{aligned}$$

即证:

$$D_i(-1)S_{ij}=S_{ij}D_j(-1)=T_{ji}(1)T_{ij}(-1)T_{ji}(1).$$

□

35. 求下列矩阵的逆矩阵:

(1)

$$\begin{pmatrix}1&0&1&-4&1&0&0&0\\-1&-3&-4&-2&0&1&0&0\\2&-1&4&4&0&0&1&0\\2&3&-3&2&0&0&0&1\end{pmatrix}\xrightarrow[-2r_1\to r_4]{r_1\to r_2,-2r_1\to r_3}\begin{pmatrix}1&0&1&-4&1&0&0&0\\0&-3&-3&-6&1&1&0&0\\0&-1&2&12&-2&0&1&0\\0&3&-5&10&-2&0&0&1\end{pmatrix}$$

$$\xrightarrow[3r_3\to r_4]{-3r_3\to r_2}\begin{pmatrix}1&0&1&-4&1&0&0&0\\0&0&-9&-42&7&1&-3&0\\0&-1&2&12&-2&0&1&0\\0&0&1&46&-8&0&3&1\end{pmatrix}\xrightarrow[r_3\leftrightarrow r_4]{r_2\leftrightarrow r_3}\begin{pmatrix}1&0&1&-4&1&0&0&0\\0&-1&2&12&-2&0&1&0\\0&0&1&46&-8&0&3&1\\0&0&-9&-42&7&1&-3&0\end{pmatrix}$$

$$\xrightarrow[9r_3\to r_4]{-2r_3\to r_2}\begin{pmatrix}1&0&1&-4&1&0&0&0\\0&-1&0&-80&14&0&-5&0\\0&0&1&46&-8&0&3&1\\0&0&0&372&-65&1&-24&9\end{pmatrix}\xrightarrow[\frac{1}{372}r_4]{-r_2}\begin{pmatrix}1&0&1&-4&1&0&0&0\\0&1&0&80&-14&0&5&0\\0&0&1&46&-8&0&3&1\\0&0&0&1&-\frac{65}{372}&\frac{1}{372}&\frac{2}{31}&\frac{3}{124}\end{pmatrix}$$

$$\xrightarrow[-80r_4\to r_2,4r_4\to r_1]{-46r_4\to r_3}\begin{pmatrix}1&0&1&0&\dfrac{28}{93}&\dfrac{1}{93}&\dfrac{8}{31}&\dfrac{3}{31}\\0&1&0&0&-\dfrac{2}{93}&-\dfrac{20}{93}&-\dfrac{5}{31}&\dfrac{2}{31}\\0&0&1&0&\dfrac{7}{186}&-\dfrac{23}{186}&\dfrac{1}{31}&-\dfrac{7}{62}\\0&0&0&1&-\dfrac{65}{372}&\dfrac{1}{372}&\dfrac{2}{31}&\dfrac{3}{124}\end{pmatrix}$$

$$\xrightarrow{-r_3\to r_1}\begin{pmatrix}1&0&0&0&\dfrac{49}{186}&\dfrac{25}{186}&\dfrac{7}{31}&\dfrac{13}{62}\\0&1&0&0&-\dfrac{2}{93}&-\dfrac{20}{93}&-\dfrac{5}{31}&\dfrac{2}{31}\\0&0&1&0&\dfrac{7}{186}&-\dfrac{23}{186}&\dfrac{1}{31}&-\dfrac{7}{62}\\0&0&0&1&-\dfrac{65}{372}&\dfrac{1}{372}&\dfrac{2}{31}&\dfrac{3}{124}\end{pmatrix}$$

即逆矩阵为

$$\begin{pmatrix}\dfrac{49}{186}&\dfrac{25}{186}&\dfrac{7}{31}&\dfrac{13}{62}\\-\dfrac{2}{93}&-\dfrac{20}{93}&-\dfrac{5}{31}&\dfrac{2}{31}\\\dfrac{7}{186}&-\dfrac{23}{186}&\dfrac{1}{31}&-\dfrac{7}{62}\\-\dfrac{65}{372}&\dfrac{1}{372}&\dfrac{2}{31}&\dfrac{3}{124}\end{pmatrix}.$$

(2)

$$
\begin{pmatrix}
1 & 4 & -1 & -1 & 1 & 0 & 0 & 0 \\
1 & -2 & -1 & 1 & 0 & 1 & 0 & 0 \\
-3 & 3 & -4 & -2 & 0 & 0 & 1 & 0 \\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}
\xrightarrow[3r_1\to r_3]{-r_1\to r_2}
\begin{pmatrix}
1 & 4 & -1 & -1 & 1 & 0 & 0 & 0 \\
0 & -6 & 0 & 2 & -1 & 1 & 0 & 0 \\
0 & 15 & -7 & -5 & 3 & 0 & 1 & 0 \\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\xrightarrow[-15r_4\to r_3]{-4r_4\to r_1,6r_4\to r_2}
\begin{pmatrix}
1 & 0 & 3 & 3 & 1 & 0 & 0 & -4 \\
0 & 0 & -6 & -4 & -1 & 1 & 0 & 6 \\
0 & 0 & 8 & 10 & 3 & 0 & 1 & -15 \\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}
\xrightarrow[r_2\leftrightarrow r_4]{-\frac{1}{8}r_3}
\begin{pmatrix}
1 & 0 & 3 & 3 & 1 & 0 & 0 & -4 \\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & \frac{5}{4} & \frac{3}{8} & 0 & \frac{1}{8} & -\frac{15}{8} \\
0 & 0 & -6 & -4 & -1 & 1 & 0 & 6
\end{pmatrix}
$$

$$
\xrightarrow[r_3\to r_2,6r_3\to r_4]{-3r_3\to r_1}
\begin{pmatrix}
1 & 0 & 0 & -\frac{3}{4} & -\frac{1}{8} & 0 & -\frac{3}{8} & \frac{13}{8} \\
0 & 1 & 0 & \frac{1}{4} & \frac{3}{8} & 0 & \frac{1}{8} & \frac{7}{8} \\
0 & 0 & 1 & \frac{5}{4} & \frac{3}{8} & 0 & \frac{1}{8} & -\frac{15}{8} \\
0 & 0 & 0 & \frac{7}{2} & \frac{5}{4} & 1 & \frac{3}{4} & -\frac{21}{4}
\end{pmatrix}
\xrightarrow{\frac{2}{7}r_4}
\begin{pmatrix}
1 & 0 & 0 & -\frac{3}{4} & -\frac{1}{8} & 0 & -\frac{3}{8} & \frac{13}{8} \\
0 & 1 & 0 & \frac{1}{4} & \frac{3}{8} & 0 & \frac{1}{8} & \frac{7}{8} \\
0 & 0 & 1 & \frac{5}{4} & \frac{3}{8} & 0 & \frac{1}{8} & -\frac{15}{8} \\
0 & 0 & 0 & 1 & \frac{5}{14} & \frac{2}{7} & \frac{3}{14} & -\frac{3}{2}
\end{pmatrix}
$$

$$
\xrightarrow[\frac{3}{4}r_1,-\frac{1}{4}r_4\to r_2]{-\frac{5}{4}r_4\to r_3}
\begin{pmatrix}
1 & 0 & 0 & 0 & \frac{1}{7} & \frac{3}{14} & -\frac{3}{14} & \frac{1}{2} \\
0 & 1 & 0 & 0 & \frac{2}{7} & -\frac{1}{14} & \frac{1}{14} & -\frac{1}{2} \\
0 & 0 & 1 & 0 & -\frac{1}{14} & -\frac{5}{14} & -\frac{1}{7} & 0 \\
0 & 0 & 0 & 1 & \frac{5}{14} & \frac{2}{7} & \frac{3}{14} & -\frac{3}{2}
\end{pmatrix}
$$

即逆矩阵为

$$
\begin{pmatrix}
\frac{1}{7} & \frac{3}{14} & -\frac{3}{14} & \frac{1}{2} \\
\frac{2}{7} & -\frac{1}{14} & \frac{1}{14} & -\frac{1}{2} \\
-\frac{1}{14} & -\frac{5}{14} & -\frac{1}{7} & 0 \\
\frac{5}{14} & \frac{2}{7} & \frac{3}{14} & -\frac{3}{2}
\end{pmatrix}.
$$

(3)

$$\begin{pmatrix} & & & 1 & 1 & & \\ & & 1 & 1 & & 1 & \\ & \iddots & \iddots & \vdots & & & \ddots \\ 1 & 1 & \cdots & 1 & & & 1 \end{pmatrix} \xrightarrow[1\leqslant i\leqslant n/2]{r_i\leftrightarrow r_{n-i}} \begin{pmatrix} 1 & 1 & \cdots & 1 & & & 1 \\ & \ddots & \ddots & \vdots & & 1 & \\ & & 1 & 1 & \iddots & & \\ & & & 1 & 1 & & \end{pmatrix}$$

$$\xrightarrow{-r_2\to r_1}\xrightarrow{-r_3\to r_2}\cdots\xrightarrow{-r_n\to r_{n-1}} \begin{pmatrix} 1 & & & & & -1 & 1 \\ & 1 & & & \iddots & 1 & \\ & & \ddots & -1 & \iddots & & \\ & & 1 & 1 & & & \end{pmatrix}$$

即逆矩阵为

$$\begin{pmatrix} & & & -1 & 1 \\ & & \iddots & 1 & \\ & -1 & \iddots & & \\ 1 & 1 & & & \end{pmatrix}.$$

(4) 记 A_i 的阶数为 n_i.

$$\begin{pmatrix} & & A_1 & I_{n_1} & & \\ & A_2 & & & I_{n_2} & \\ \iddots & & & & & \ddots \\ A_k & & & & & I_{n_k} \end{pmatrix} \xrightarrow[\text{多次相邻对换}]{\text{类似 }23(4)} \begin{pmatrix} A_k & & & & I_{n_k} \\ & \ddots & & \iddots & \\ & & A_1 \quad I_{n_1} & & \end{pmatrix}$$

左乘 $diag(A_k^{-1},\dots,A_1^{-1})$, 得

$$\begin{pmatrix} I_{n_k} & & & & A_k^{-1} \\ & \ddots & & \iddots & \\ & & I_{n_1} \quad A_1^{-1} & & \end{pmatrix}$$

即逆矩阵为

$$\begin{pmatrix} & & & A_k^{-1} \\ & & A_{k-1}^{-1} & \\ & \iddots & & \\ A_1^{-1} & & & \end{pmatrix}.$$

(5) 记原矩阵为 A, 逆矩阵 (若存在) 记为 $A^{-1}=\frac{1}{\det(A)}A^*=B=(b_{ij})$.

$$A_{ij}=\begin{cases} -\prod\limits_{\substack{k=1\\k\neq i,j}}^{n} a_k; & (i\neq j) \\ \prod\limits_{\substack{k=1\\k\neq i}}^{n} a_k+\sum\limits_{\substack{l=1\\l\neq i}}^{n}\left(\prod\limits_{\substack{k=1\\k\neq i,l}}^{n} a_k\right). & (i=j) \end{cases}$$

(1) $\exists\, i\neq j,\ a_i=a_j=0$. 则 $\det(A)=0$, 无逆矩阵.

(2) $\exists!\, m,\ a_m=0$. 则 $\det(A)=\prod\limits_{\substack{i=1\\i\neq m}}^{n} a_i\neq 0$.

(a) $i=j$

$$b_{ii}=\frac{A_{ij}}{c\cdot\prod\limits_{\substack{i=1\\i\neq m}} a_i}=\begin{cases} \dfrac{1}{a_i}; & (i\neq m) \\ 1+\sum\limits_{\substack{j=1\\j\neq m}}^{n}\dfrac{1}{a_j}. & (i=m) \end{cases}$$

(b) $i \neq j$

$$b_{ii} = \frac{A_{ij}}{c \cdot \prod\limits_{\substack{i=1 \\ i \neq m}} a_i} = \begin{cases} 0\ ; & (i,j \neq m) \\ -\dfrac{1}{a_j}\ ; & (i = m) \\ -\dfrac{1}{a_i}\ . & (j = m) \end{cases}$$

则

$$A^{-1} = \begin{pmatrix} \frac{1}{a_1} & & & -\frac{1}{a_1} & & & \\ & \ddots & & \vdots & & & \\ & & \frac{1}{a_{m-1}} & -\frac{1}{a_{m-1}} & & & \\ -\frac{1}{a_1} & \cdots & -\frac{1}{a_{m-1}} & 1 + \sum\limits_{\substack{j=1 \\ j \neq m}}^{n} \frac{1}{a_j} & -\frac{1}{a_m} & \cdots & -\frac{1}{a_n} \\ & & & -\frac{1}{a_{m+1}} & -\frac{1}{a_{m+1}} & & \\ & & & \vdots & & \ddots & \\ & & & -\frac{1}{a_n} & & & \frac{1}{a_n} \end{pmatrix}$$

(3) $\forall\ i,\ a_i \neq 0.$ 则 $\det(A) = \left(1 + \sum\limits_{i=1}^{n} \frac{1}{a_i}\right) \prod\limits_{i=1}^{n} a_i.$

$$\text{记} c = \left(1 + \sum_{i=1}^{n} \frac{1}{a_i}\right), \quad b_{ji} = \frac{A_{ij}}{c \cdot \prod\limits_{i=1} a_i} = \begin{cases} -\dfrac{1}{a_i a_j c}; & (i \neq j) \\ \dfrac{1}{a_i} - \dfrac{1}{{a_i}^2 c}. & (i = j) \end{cases}$$

则

$$A^{-1} = \begin{pmatrix} \frac{1}{a_1} - \frac{1}{{a_1}^2 c} & -\frac{1}{a_1 a_2 c} & \cdots & -\frac{1}{a_1 a_n c} \\ -\frac{1}{a_1 a_2 c} & \frac{1}{a_2} - \frac{1}{{a_2}^2 c} & & \vdots \\ \vdots & & \ddots & \vdots \\ -\frac{1}{a_1 a_n c} & \cdots & \cdots & \frac{1}{a_n} - \frac{1}{{a_n}^2 c} \end{pmatrix}$$