线性代数 homework (第三周)

PB20000113 孔浩宇

April 17, 2022

1 周二

1.1 习题五

17. 设向量组 $\alpha_1, \ldots, \alpha_r$ 线性无关, 且 $\alpha_1, \ldots, \alpha_r$ 可以由向量组 $\beta_1, \ldots, \beta_r$ 线性表示, 则 $\beta_1, \ldots, \beta_r$ 也线性无关.

Proof.
$$r = \operatorname{rank}(\alpha_1, \ldots, \alpha_r) \leqslant \operatorname{rank}(\beta_1, \ldots, \beta_r) \leqslant r \Rightarrow \operatorname{rank}(\beta_1, \ldots, \beta_r) = r.$$
□

19. 求下列向量组的极大无关组与秩:

(3) $\overrightarrow{a_1} = (0,1,2,3), \quad \overrightarrow{a_2} = (1,2,3,4), \quad \overrightarrow{a_3} = (3,4,5,6), \quad \overrightarrow{a_4} = (4,3,2,1), \quad \overrightarrow{a_5} = (6,5,4,3).$

解：记极大无关组为 S.

(a) $S_1 = \{\alpha_1\}$.

(b) 假设存在 x 使得 $x \cdot \alpha_1 = \alpha_2$, 则 $0 \cdot x = 1$, 显然无解, 故取 $S_2 = S_1 \cup \{\alpha_2\} = \{\alpha_1, \alpha_2\}$.

(c) 假设存在 x, y 使得 $x \cdot \alpha_1 + y \cdot \alpha_2 = \alpha_3$, 则有

$$\begin{cases} y &= 3, \\ x + 2y &= 4, \\ 2x + 3y &= 5, \\ 3x + 4y &= 6. \end{cases} \Rightarrow \begin{cases} x &= -2 \\ y &= 3. \end{cases} \Rightarrow S_3 = S_2 = \{\alpha_1, \alpha_2\}.$$

(d) 假设存在 x, y 使得 $x \cdot \alpha_1 + y \cdot \alpha_2 = \alpha_4$, 则有

$$\begin{cases} y &= 4, \\ x + 2y &= 3, \\ 2x + 3y &= 2, \\ 3x + 4y &= 1. \end{cases} \Rightarrow \begin{cases} x &= -5 \\ y &= 4. \end{cases} \Rightarrow S_4 = S_3 = \{\alpha_1, \alpha_2\}.$$

(e) 假设存在 x, y 使得 $x \cdot \alpha_1 + y \cdot \alpha_2 = \alpha_5$, 则有

$$\begin{cases} y &= 6, \\ x + 2y &= 5, \\ 2x + 3y &= 4, \\ 3x + 4y &= 3. \end{cases} \Rightarrow \begin{cases} x &= -7 \\ y &= 6. \end{cases} \Rightarrow S_5 = S_4 = \{\alpha_1, \alpha_2\}.$$

综上, 极大无关组 $S = \{\alpha_1, \alpha_2\}, \operatorname{rank}(\alpha_1, \ldots, \alpha_6) = 2.$

22. 设向量组 $\alpha_1, \ldots, \alpha_m$ 的秩为 r, 则其中任何 r 个线性无关的向量构成 $\alpha_1, \ldots, \alpha_m$ 的极大无关组.

Proof. 设有任何 r 个线性无关的向量 $\alpha_{i1},\ldots,\alpha_{ir}$ 构成一个向量组 S_1,$\forall\ \alpha_j\notin S_1, S_2=S_1\cup\{\alpha_j\}$.

$$r+1>r=\operatorname{rank}(\alpha_1,\ldots,\alpha_m)\Rightarrow S_2\text{线性相关, 又}S_1\text{线性无关}\Rightarrow S_1\text{构成}\alpha_1,\ldots,\alpha_m\text{的极大无关组.}$$

□

23. 设向量组 $\alpha_1,\ldots,\alpha_m$ 的秩为 r, 如果 $\alpha_1,\ldots,\alpha_m$ 可以由它的 r 个向量线性表示, 则这 r 个向量构成 $\alpha_1,\ldots,\alpha_m$ 的极大无关组.

Proof. 将这 r 个向量记为 $\alpha_{i1},\ldots,\alpha_{ir}$.

$$r=\operatorname{rank}(\alpha_1,\ldots,\alpha_m)\leqslant\operatorname{rank}(\alpha_{i1},\ldots,\alpha_{ir})\leqslant r\Rightarrow\operatorname{rank}(\alpha_{i1},\ldots,\alpha_{ir})=r$$

$$\xrightarrow{\text{T22 结论}}\operatorname{rank}(\alpha_{i1},\ldots,\alpha_{ir})\text{构成}\alpha_1,\ldots,\alpha_m\text{的极大无关组.}$$

□

2 周四

2.1 习题五

20. 求下列矩阵的秩, 并求出它的行向量空间的一组基.

(2) 进行初等变换:

$$\begin{pmatrix}3&6&1&5\\1&4&-1&3\\-1&-10&5&-7\\4&-2&8&0\end{pmatrix}\to\begin{pmatrix}0&-6&4&-6\\1&4&-1&3\\0&-6&4&-4\\0&-18&12&-12\end{pmatrix}\to\begin{pmatrix}0&0&0&-2\\1&0&0&0\\0&-6&4&-4\\0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&4&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix}$$

即

$$\operatorname{rank}\begin{pmatrix}3&6&1&5\\1&4&-1&3\\-1&-10&5&-7\\4&-2&8&0\end{pmatrix}=\operatorname{rank}\begin{pmatrix}1&0&0&0\\0&4&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix}=3.$$

记原矩阵为 $(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)^T$, 由初等变换可得 $\alpha_4-4\cdot\alpha_2=3\cdot(\alpha_2+\alpha_3)$

24. 证明:$\operatorname{rank}(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s)\leqslant\operatorname{rank}(\alpha_1,\ldots,\alpha_r)+\operatorname{rank}(\beta_1,\ldots,\beta_s)$.

Proof. 记 $A=(\alpha_1,\ldots,\alpha_r), B=(\beta_1,\ldots,\beta_s)$.

$$\operatorname{rank}(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s)=\operatorname{rank}\begin{pmatrix}A&B\end{pmatrix}\leqslant\operatorname{rank}(A)+\operatorname{rank}(B)=\operatorname{rank}(\alpha_1,\ldots,\alpha_r)+\operatorname{rank}(\beta_1,\ldots,\beta_s)$$

□

27. 设 A,B 是同阶矩阵. 证明:$\operatorname{rank}(A+B)\leqslant\operatorname{rank}(A)+\operatorname{rank}(B)$.

Proof.

$$\operatorname{rank}(A+B)\leqslant\max\{\operatorname{rank}(A),\operatorname{rank}(B),\operatorname{rank}(A+B)\}\leqslant\operatorname{rank}\begin{pmatrix}A&B\end{pmatrix}\leqslant\operatorname{rank}(A)+\operatorname{rank}(B)$$

□

34. 以向量组 $\alpha_1=(3,1,0),\alpha_2=(6,3,2),\alpha_3=(1,3,5)$ 为基, 求 $\beta=(2,-1,2)$ 的坐标.

设 $\beta=x\cdot\alpha_1+y\cdot\alpha_2+z\cdot\alpha_3$, 有

$$\begin{cases}3x+6y+\ \ z&=2\\ \ \ x+3y+3z&=-1\\ \qquad 2y+5z&=2\end{cases}\Rightarrow\begin{cases}x&=-76\\y&=\ \ 41\\z&=-16\end{cases}\text{, 即}\beta=-76\alpha_1+41\alpha_2-16\alpha_3.$$

35. 设 $\alpha_1 = (3, 2, -1, 4), \alpha_2 = (2, 3, 0, -1)$.

(1) 将 α_1, α_2 扩充为 $\mathbb{R}^4$ 的一组基;

(2) 给出标准基在该组基下的表示;

(3) 求 $\beta = (1, 3, 4, -2)$ 在该组基下的坐标.

解:

(1) 显然 α_1, α_2 线性无关, 取 $\alpha_3 = (0, 0, 1, 0), \alpha_4 = (0, 0, 0, 1)$, 有

$$\operatorname{rank}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \operatorname{rank}\begin{pmatrix} 3 & 2 & -1 & 4 \\ 2 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3$$

即 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 即为 $\mathbb{R}^4$ 的一组基.

(2)

$$\begin{cases} e_1 & = \frac{3}{5}\alpha_1 - \frac{2}{5}\alpha_2 + \frac{3}{5}\alpha_3 - \frac{14}{5}\alpha_4 \\ e_2 & = -\frac{2}{5}\alpha_1 + \frac{3}{5}\alpha_2 - \frac{2}{5}\alpha_3 + \frac{11}{5}\alpha_4 \\ e_3 & = \alpha_3 \\ e_4 & = \alpha_4 \end{cases}$$

(3)

$$\beta = (1, 3, 4, -2)\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = (1, 3, 4, -2)\begin{pmatrix} \frac{3}{5} & -\frac{2}{5} & \frac{3}{5} & -\frac{14}{5} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} & \frac{11}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \left(-\frac{3}{5}, \frac{7}{5}, \frac{17}{5}, \frac{9}{5}\right)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

即 $\beta = -\frac{3}{5}\alpha_1 + \frac{7}{5}\alpha_2 + \frac{17}{5}\alpha_3 + \frac{9}{5}\alpha_4$.