

线性代数 homework (第三周)

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1 周二

1.1 习题四 23

23. 计算下列行列式:

(1)

$$\begin{vmatrix} 1 & 0 & 1 & -4 \\ -1 & -3 & -4 & -2 \\ 2 & -1 & 4 & 4 \\ 2 & 3 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & -3 & -3 & -6 \\ 2 & -1 & 2 & 12 \\ 2 & 3 & -5 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ 2 & -1 & 3 & 14 \\ 2 & 3 & -8 & 4 \end{vmatrix} = -372.$$

(2)

$$\begin{vmatrix} 1 & 4 & -1 & -1 \\ 1 & -2 & -1 & 1 \\ -3 & 3 & -4 & -2 \\ 0 & 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -6 & 0 & 2 \\ -3 & 15 & -7 & -5 \\ 0 & 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ -3 & -2 & -7 & -5 \\ 0 & -2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ -2 & -7 & -5 \\ -2 & -1 & -1 \end{vmatrix} = 28.$$

(3)

$$\begin{vmatrix} x+a & x+b & x+c \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix} = \begin{vmatrix} x+a & x+b & x+c \\ y-x & y-x & y-x \\ z-x & z-x & z-x \end{vmatrix} = \begin{vmatrix} x+a & b-a & c-a \\ y-x & 0 & 0 \\ z-x & 0 & 0 \end{vmatrix} = 0.$$

(7)

$$\text{记 } D_n = \begin{vmatrix} a_1 & & & & b_1 \\ & \ddots & & & \\ & & a_n & b_n & \\ & & c_n & d_n & \\ & \ddots & & & \\ c_1 & & & & d_1 \end{vmatrix} \Rightarrow \begin{cases} D_1 &= a_1 d_1 - b_1 c_1, \\ D_2 &= (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \end{cases}.$$

假设 $D_n = \prod_{i=1}^n (a_i d_i - b_i c_i)$.

(a) $n = 1$ 成立.

(b) 假设 $D_n = \prod_{i=1}^n (a_i d_i - b_i c_i)$ 对 $n = k$ 成立, 则

$$D_{k+1} = (a_1 d_1 - b_1 c_1) \begin{vmatrix} a_2 & & & & b_2 \\ & \ddots & & & \\ & & a_{k+1} & b_{k+1} & \\ & & c_{k+1} & d_{k+1} & \\ & \ddots & & & \\ c_2 & & & & d_2 \end{vmatrix} = \prod_{i=1}^{k+1} (a_i d_i - b_i c_i)$$

$n = k + 1$ 时原式仍成立.

综合 (a)(b) 即证 $D_n = \prod_{i=1}^n (a_i d_i - b_i c_i)$ 对任意 $n \in \mathbb{N}^*$ 成立.

即 $D_n = \prod_{i=1}^n (a_i d_i - b_i c_i)$.

1.2 补充题

1. 对于 a, b, c, d 为不全为 0 的实数

$$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$$

求 M 的行列式.

做多项式 $f(x) = a + bx + cx^2 + dx^3$, 令 $\omega_k = \frac{2k\pi i}{4} = \frac{k\pi i}{2}$.

$$\begin{aligned} \begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} &= \det \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_4^2 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 & \omega_4^3 \end{pmatrix} \\ &= \det \begin{pmatrix} f(\omega_1) & f(\omega_2) & f(\omega_3) & f(\omega_4) \\ \omega_1 f(\omega_1) & \omega_2 f(\omega_2) & \omega_3 f(\omega_3) & \omega_4 f(\omega_4) \\ \omega_1^2 f(\omega_1) & \omega_2^2 f(\omega_2) & \omega_3^2 f(\omega_3) & \omega_4^2 f(\omega_4) \\ \omega_1^3 f(\omega_1) & \omega_2^3 f(\omega_2) & \omega_3^3 f(\omega_3) & \omega_4^3 f(\omega_4) \end{pmatrix} \\ &= f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4) \begin{vmatrix} 1 & 1 & 1 & 1 \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_4^2 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 & \omega_4^3 \end{vmatrix} \end{aligned}$$

$$\text{又 } \omega_k \text{ 各不相同, } \begin{vmatrix} 1 & 1 & 1 & 1 \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_4^2 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 & \omega_4^3 \end{vmatrix} \neq 0, \Rightarrow \begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4).$$

即:

$$\begin{aligned} \begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} &= f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4) \\ &= (a+b+c+d)(a-b+c-d)(a+bi-c-di)(a-bi-c+di) \\ &= [(a+c)^2 - (b+d)^2][(a-c)^2 + (b-d)^2]. \end{aligned}$$

2. 写出行列式 M 中含有 x^4 和 x^3 的项.

$$M = \begin{vmatrix} x & 1 & 2 & 3 \\ x & x & 1 & 2 \\ 2 & 3 & x & 1 \\ x & 2 & 3 & x \end{vmatrix}.$$

$$M = x \cdot \begin{vmatrix} x & 1 & 2 \\ 3 & x & 1 \\ 2 & 3 & x \end{vmatrix} - x \cdot \begin{vmatrix} 1 & 2 & 3 \\ 3 & x & 1 \\ 2 & 3 & x \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ x & 1 & 2 \\ 2 & 3 & x \end{vmatrix} - x \cdot \begin{vmatrix} 1 & 2 & 3 \\ x & 1 & 2 \\ 3 & x & 1 \end{vmatrix}$$

(1) x^4 : 仅第一部分有 x^4 , 易得 M 中含有 x^4 的项为 x^4 .

(2) x^3 : 仅第二、四部分有 x^3 , 易得 M 中含有 x^3 的项为 $-x^3 - 3x^3 = -4x^3$.

3. 计算行列式 (参考第四题)

$$M = \begin{vmatrix} 1 & 0 & 1 & 0 & 0 \\ -3 & 1 & 3 & 1 & 0 \\ 2 & -3 & 2 & 3 & 1 \\ 0 & -2 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

记 $f(x) = x^3 - 3x^2 + 2x, g(x) = x^2 + 3x + 2$.

$$\text{解 } f(x) = 0 \Rightarrow \begin{cases} x_1 = 0, \\ x_2 = 1, \\ x_3 = 2. \end{cases} \Rightarrow M = 1^2 \cdot g(x_1) \cdot g(x_2) \cdot g(x_3) = 144.$$

4. 对于多项式 $A, B \in \mathbb{C}[x]$ 有如下的展开

$$A = a_0x^d + a_1x^{d-1} + \cdots + a_d$$

$$B = b_0x^e + b_1x^{e-1} + \cdots + b_e$$

并且它们的根分别为 $\lambda_1, \dots, \lambda_d$ 与 μ_1, \dots, μ_e . 我们定义它们的 Sylvester 行列式为

$$M = \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_d & a_{d-1} & \cdots & \vdots & b_e & b_{e-1} & \cdots & \vdots \\ 0 & a_d & \ddots & \vdots & 0 & b_e & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{d-1} & \vdots & \vdots & \ddots & b_{e-1} \\ 0 & 0 & \cdots & a_d & 0 & 0 & \cdots & b_e \end{pmatrix}$$

这对多项式的结式 (Resultant) 定义为他们的 Sylvester 多项式的行列式, 即

$$\text{res}(A, B) = \det M.$$

请验证有如下的关系:

$$\text{res}(A, B) = a_0^e b_0^d \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq e}} (\lambda_i - \mu_j)$$

Remarque. 根据结果我们知道, 结式可以用来判断两个多项式是否有公共的根, 进而可以知道两个多项式是否互素。除此以外, 结式在数论、代数几何、交换代数中都有广泛的应用。

Remarque. 事实: 当 A 和 B 互素时, M 可逆。

Remarque. 当 $d = 3, e = 2$ 时, M 如下

$$\begin{pmatrix} a_0 & 0 & b_0 & 0 & 0 \\ a_1 & a_0 & b_1 & b_0 & 0 \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & 0 & b_2 & b_1 \\ 0 & a_3 & 0 & 0 & b_2 \end{pmatrix}.$$

Proof. 1.

$$M^T = N = \begin{pmatrix} a_0 & a_1 & \cdots & a_d & & & & \\ & a_0 & a_1 & \cdots & a_d & & & \\ & & \ddots & \ddots & & \ddots & & \\ & & & a_0 & a_1 & \cdots & a_d & \\ b_0 & b_1 & \cdots & b_e & & & & \\ & b_0 & b_1 & \cdots & b_e & & & \\ & & \ddots & \ddots & & \ddots & & \\ & & & b_0 & a_1 & \cdots & b_e \end{pmatrix}$$

$$\text{记 } N(B) = \begin{pmatrix} a_0 & a_1 & \cdots & a_d & & & \\ & a_0 & a_1 & \cdots & a_d & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \cdots & a_d \\ b_0 & b_1 & \cdots & b_e - B & & & \\ & b_0 & b_1 & \cdots & b_e - B & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & a_1 & \cdots & b_e - B \end{pmatrix}.$$

记 $x^i = x_i, l = d + e - 1$, 考虑关于 x_l, x_{l-1}, \dots, x_0 的线性方程组:

$$(1) \begin{cases} a_0 x_l + a_1 x_{l-1} + \cdots + a_d x_{e-1} = 0 \\ a_0 x_{l-1} + a_1 x_{l-2} + \cdots + a_d x_{e-2} = 0 \\ \vdots \\ a_0 x_d + a_1 x_{d-1} + \cdots + a_d x_0 = 0 \end{cases}$$

$$(2) \begin{cases} b_0 x_l + b_1 x_{l-1} + \cdots + (b_e - B) x_{d-1} = 0 \\ b_0 x_{l-1} + b_1 x_{l-2} + \cdots + (b_e - B) x_{d-2} = 0 \\ \vdots \\ b_0 x_e + b_1 x_{e-1} + \cdots + (b_e - B) x_0 = 0 \end{cases}$$

(1) 对于 A 的解 $\lambda_1, \dots, \lambda_d$ 成立; (2) 对于 B 为恒等式.

又 $x_0 = 1, A$ 的解 $\lambda_1, \dots, \lambda_d$ 对应 $N(B)X = 0$ 的 d 组非平凡解, 即 $\text{rank } N(B) < d + e, \det N(B) = 0$:

$$\det N(B) = c_0 B^d + c_1 B^{d-1} + \cdots + c_d = 0$$

$\det N(B)$ 作为 B 的 d 次多项式, 它的 d 个根为

$$B(\lambda_1), B(\lambda_2), \dots, B(\lambda_d)$$

在上述 $\det N(B)$ 的展开式中, 显然有

$$c_0 = (-1)^d a_0^e, \text{res}(A, B) = \det N(0) = c_d$$

由根和系数的关系, 得:

$$\prod_{i=1}^d B(\lambda_i) = (-1)^d \frac{c_d}{c_0}$$

即:

$$\text{res}(A, B) = a_0^e \prod_{i=1}^d B(\lambda_i)$$

又 $B = b_0 \prod_{j=1}^e (x - \mu_j)$:

$$\text{res}(A, B) = a_0^e b_0^d \prod_{i=1}^d \prod_{j=1}^e (\lambda_i - \mu_j) = a_0^e b_0^d \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq e}} (\lambda_i - \mu_j)$$

□

Proof. 2. 记 $f(x) = m_0x^e + m_1x^{e-1} + m_e, g(x) = n_0x^d + n_1x^{d-1} + n_d$. 解:

$$f(x)A + g(x)B = 1 \Leftrightarrow \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_d & a_{d-1} & \cdots & \vdots & b_e & b_{e-1} & \cdots & \vdots \\ 0 & a_d & \ddots & \vdots & 0 & b_e & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{d-1} & \vdots & \vdots & \ddots & b_{e-1} \\ 0 & 0 & \cdots & a_d & 0 & 0 & \cdots & b_e \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_e \\ n_0 \\ n_1 \\ \vdots \\ n_d \end{pmatrix} = MX = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} f(x)A + g(x)B = 1 \text{ 有解} &\Leftrightarrow f(x), g(x) \text{ 互素} \\ MX = (0, 0, \dots, 0, 1)^T \text{ 有解} &\Leftrightarrow \det(M) \neq 0 \\ f(x), g(x) \text{ 互素} &\Leftrightarrow \det(M) \neq 0 \end{aligned}$$

即:

$$\det(M) = 0 \Leftrightarrow \exists 1 \leq i \leq d, 1 \leq j \leq e, \lambda_i = \mu_j$$

可得:

$$\det(M) = \xi \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq e}} (\lambda_i - \mu_j) (\xi \text{ 为系数})$$

取 $\lambda_1 = \dots = \lambda_d = 0$, 则 $a_0 \neq 0, a_1 = \dots = a_d = 0$

$$\det(M) = \begin{vmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ 0 & 0 & \cdots & \vdots & b_e & b_{e-1} & \cdots & \vdots \\ 0 & 0 & \ddots & \vdots & 0 & b_e & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & b_{e-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & b_e \end{vmatrix} = a_0^e b_e^d = a_0^e ((-1)^e b_0 \cdot \mu_1 \cdot \mu_2 \cdots \mu_e)^d$$

又

$$\begin{aligned} \det(M) &= a_0^e ((-1)^e b_0 \cdot \mu_1 \cdot \mu_2 \cdots \mu_e)^d \\ &= a_0^e b_0^d (-1)^{de} (\mu_1 \cdot \mu_2 \cdots \mu_e)^d \\ &= a_0^e b_0^d \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq e}} \lambda_i - \mu_j \end{aligned}$$

可推出 $\xi = a_0^e b_0^d$, 即证 $\text{res}(A, B) = a_0^e b_0^d \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq e}} (\lambda_i - \mu_j)$. □

2 周四

2.1 习题四

21. 求以下排列的逆序数, 并指出其奇偶性:

(1) (6, 8, 1, 4, 7, 5, 3, 2, 9) 逆序数 19, 奇

- (2) (6,4,2,1,9,7,3,5,8) 逆序数 15, 奇
 (3) (7,5,2,3,9,8,1,6,4) 逆序数 20, 偶

23. (4)

$$\begin{vmatrix} & & & A_1 \\ & & A_2 & \\ & \ddots & & \\ A_k & & & \end{vmatrix} \xrightarrow{n_1 \sum_{i=2}^k n_i \text{次相邻对调}} (-1)^{\sum_{i=2}^k n_1 n_i} \begin{vmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{vmatrix}$$

即:

$$\begin{vmatrix} & & & A_1 \\ & & A_2 & \\ & \ddots & & \\ A_k & & & \end{vmatrix} = (-1)^{\sum_{i < j} n_i n_j} \begin{vmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{vmatrix} = (-1)^{\sum_{i < j} n_i n_j} \det(A_1) \cdots \det(A_k)$$

(6)

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+a_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1+a_1 & 1 & \cdots & 1 \\ 0 & 1 & 1+a_2 & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & 1 \\ 0 & 1 & \cdots & 1 & 1+a_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \ddots & \vdots \\ -1 & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & a_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 + \sum_{i=1}^n \frac{1}{a_i} & 0 & 0 & \cdots & 0 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \ddots & \vdots \\ -1 & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & a_n \end{vmatrix} = \left(1 + \sum_{i=1}^n \frac{1}{a_i}\right) \prod_{j=1}^n a_j.$$

(8) (a) $n = 1$

$$|a_1 - b_1| = a_1 - b_1.$$

(b) $n = 2$

$$\begin{vmatrix} a_1 - b_1 & a_1 - b_2 \\ a_2 - b_1 & a_2 - b_2 \end{vmatrix} = (a_1 - b_1)(a_2 - b_2) - (a_1 - b_2)(a_2 - b_1).$$

(c) $n \geq 3$

$$\begin{vmatrix} a_1 - b_1 & a_1 - b_2 & \cdots & a_1 - b_n \\ a_2 - b_1 & a_2 - b_2 & \cdots & a_2 - b_n \\ \vdots & \vdots & & \vdots \\ a_n - b_1 & a_n - b_2 & \cdots & a_n - b_n \end{vmatrix} = \begin{vmatrix} a_1 - b_1 & b_1 - b_2 & \cdots & b_1 - b_n \\ a_2 - b_1 & b_1 - b_2 & \cdots & b_1 - b_n \\ \vdots & \vdots & & \vdots \\ a_n - b_1 & b_1 - b_2 & \cdots & b_1 - b_n \end{vmatrix} = 0.$$

2.2 补充题

1. 确定 i, j , 使得 (1245*i*6*j*97) 分别为奇排列、偶排列。

- (1) $i = 3, j = 8$, 逆序数为 3, 奇排列;
 (2) $i = 8, j = 3$, 逆序数为 6, 偶排列.

2. 将 λ 作为变量, $a_{ij} (1 \leq i, j \leq n)$ 作为常数, 则

$$f_n(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

为 λ 的多项式, 求这个多项式的 n 次项和 $n-1$ 次项。

$$\text{记 } f_n(\lambda) = (\lambda - a_{nn})f_{n-1}(\lambda) + a_{12} \begin{vmatrix} -a_{21} & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & \lambda - a_{33} & \cdots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \cdots & \lambda - a_{nn} \end{vmatrix} + \cdots.$$

显然, 后 $n-1$ 个项中 λ 的次数均不超过 $n-2$, 仅第一项中含 n 次项和 $n-1$ 次项, 即在 $f_{n-1}(\lambda)$ 中找 $n-1$ 次项和 $n-2$ 次项.

依此类推, 可得 $f(x)$ 的 n 次项和 $n-1$ 次项均在 $\prod_{i=1}^n (\lambda - a_{ii})$ 里.

$$\begin{aligned} n\text{次项} &: \lambda^n; \\ n-1\text{次项} &: -\sum_{i=1}^n a_{ii}\lambda^{n-1}. \end{aligned}$$