# 线性代数 homework (第七周)

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## 1 周四

### 1.1 习题五

9. 判别下列线性方程组是否线性相关:

$$(1)\begin{cases} -x_1+2x_2+3\ x_3=3, \\ 5x_1 \qquad\quad - \quad x_3=5, \\ 8x_1-6x_2-10x_3=7; \end{cases} \qquad (2)\begin{cases} 2x_1+3x_2+\ \ x_3+5x_4=2, \\ 3x_1+2x_2+4x_3+2x_4=3, \\ \ x_1+\ \ x_2+2x_3+4x_4=1. \end{cases}$$

(1) 令 $\begin{cases} l_1 &= -x_1+2x_2+3x_3-4, \\ l_2 &= 5x_1-x_3+1, \\ l_3 &= 8x_1-6x_2-10x_3+13. \end{cases}$, 假设 $l_3=\lambda_1 l_1+\lambda_2 l_2$, 则

$$\begin{cases} -\lambda_1+5\lambda_2 &= 8, \\ 2\lambda_1 &= -6, \\ 3\lambda_1-\ \lambda_2 &= -10, \\ -4\lambda_1+\ \lambda_2 &= 13. \end{cases} \Rightarrow \begin{cases} \lambda_1 &= -3, \\ \lambda_2 &= 1. \end{cases} \Rightarrow l_3=-3l_1+l_2,\text{原方程组线性相关.}$$

(2) 令 $\begin{cases} l_1 &= 2x_1+3x_2+x_3+5x_4-2, \\ l_2 &= 3x_1+2x_2+4x_3+2x_4-3, \\ l_3 &= x_1+x_2+2x_3+4x_4-1. \end{cases}$, 假设 $l_3=\lambda_1 l_1+\lambda_2 l_2$, 则

$$\begin{cases} 2\lambda_1+3\lambda_2 &= 1, \\ 3\lambda_1+2\lambda_2 &= 1, \\ \lambda_1+4\lambda_2 &= 2, \\ 5\lambda_1+2\lambda_2 &= 4, \\ -2\lambda_1-3\lambda_2 &= -1, \end{cases} \Rightarrow \text{无解, 即原方程组不线性相关.}$$

10. 判断下列向量组是否线性相关:

(2) $a_1=(2,1,2,-4), a_2=(1,0,5,2), a_3=(-1,2,0,3)$.
假设有 $a_3=m\cdot a_1+n\cdot a_2$, 则

$$\begin{cases} 2m+n &= -1, \\ m &= 2, \\ 2m+5n &= 0, \\ -4m+2n &= 3. \end{cases} \Rightarrow \text{无解, 向量组线性无关.}$$

(4) $a_1=(1,-1,0,0), a_2=(0,1,-1,0), a_3=(0,0,1,-1), a_4=(-1,0,0,1)$.

假设有 $a_4 = m\cdot a_1 + n\cdot a_2 + p\cdot a_3$, 则

$$\begin{cases} m &= -1,\\ -m+n &= 0,\\ -n+p &= 0,\\ -p &= 1.\end{cases} \Rightarrow \begin{cases} m &= -1,\\ n &= -1,\\ p &= -1.\end{cases} \Rightarrow a_4 = -a_1 - a_2 - a_3, \text{向量组线性相关.}$$

12. 下列说法是否正确？为什么？
    (1) 错误. 例如取 $\alpha_1 = (1,1), \alpha_2 = (0,0)$, 有 $\alpha_2 = 0\cdot\alpha_1$, 线性相关. 但不存在 $\lambda$, 使得 $\alpha_1 = \lambda\cdot\alpha_2$.
    (2) 错误. 例如取 $\alpha_1 = (1,0), \alpha_2 = (0,1), \alpha_3 = (1,1)$, 有 $\{\alpha_1,\alpha_2\}, \{\alpha_1,\alpha_3\}\{\alpha_2,\alpha_3\}$ 线性无关, 但有 $\alpha_3 = \alpha_1 + \alpha_2$.
    (5) 错误. 例如取 $s = 2$, 则 $\alpha_1+\alpha_2, \alpha_2+\alpha_1$ 线性相关.
    (6) 正确.
    若 $s$ 为偶数, 则 $(\alpha_1+\alpha_2)+(\alpha_3+\alpha_4)+\cdots+(\alpha_{s-1}+\alpha_s) = (\alpha_2+\alpha_3)+(\alpha_4+\alpha_5)+\cdots+(\alpha_s+\alpha_1)$, 线性相关;
    若 $s$ 为奇数, 取 $\alpha_{i+s} = \alpha_i$ $(1\leqslant i\leqslant s)$. 则 $\alpha_i = \sum\limits_{k=0}^{s-1}(-1)^k(\alpha_{i+k}+\alpha_{i+k+1})$, 两向量组等价.
    即 $\operatorname{rank}\{\alpha_1+\alpha_2,\ldots,\alpha_s+\alpha_1\} = \operatorname{rank}\{\alpha_1,\ldots,\alpha_s\} < s$, 即证线性相关.

15. 证明: 非零向量组 $\alpha_1,\ldots,\alpha_s$ 线性无关的充要条件是，每个 $\alpha_i (1 < i \leqslant s)$ 都不能用它面前的向量线性表示.

*Proof.*
(1) 必要性: 显然.
(2) 充分性: 假设有非零向量组 $\alpha_1,\ldots,\alpha_s$ 每个 $\alpha_i(1<i\leqslant s)$ 都不能用它面前的向量线性表示, 且线性相关.
由 $\alpha_1,\ldots,\alpha_s$ 线性相关, 得 $\exists\, \alpha_i = \sum\limits_{j\neq i}\lambda_j\alpha_j$, 且 $\lambda_j$ 不全为 0.
取 $k = \max\{\, j \mid \lambda_j \neq 0\}$,
(a) $k < i$. 矛盾, 不存在这样的非零向量组.
(b) $k > i$. 则有 $\alpha_k = \alpha_i - \sum\limits_{j\neq k,i}\alpha_j$, 矛盾, 不存在这样的非零向量组.
即假设不成立, 充分性即证.

综上, 即证非零向量组 $\alpha_1,\ldots,\alpha_s$ 线性无关的充要条件是，每个 $\alpha_i(1<i\leqslant s)$ 都不能用它面前的向量线性表示. □

16. 设向量组 $\alpha_1,\ldots,\alpha_s$ 线性无关,$\beta = \lambda_1\alpha_1+\cdots+\lambda_s\alpha_s$. 如果 $\lambda_i \neq 0$, 则用 $\beta$ 代替 $\alpha_i$ 后, 向量组 $\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_{i+1},\ldots,\alpha_s$ 线性无关.

*Proof.*

$$\alpha_j = \begin{cases} \alpha_i, \\ \\ \dfrac{1}{\lambda_i}\left(\beta - (\lambda_1\alpha_1+\cdots+\lambda_{i-1}\alpha_{i-1}+\lambda_{i+1}\alpha_{i+1}+\cdots+\lambda_s\alpha_s)\right).\end{cases}$$

$$\Rightarrow \{\alpha_1,\ldots,\alpha_s\} \text{ 与 } \{\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_{i+1},\ldots,\alpha_s\} \text{ 等价;}$$

$$\Rightarrow \operatorname{rank}\{\alpha_1,\ldots,\alpha_s\} = \operatorname{rank}\{\alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_{i+1},\ldots,\alpha_s\} = s;$$

$$\Rightarrow \alpha_1,\ldots,\alpha_{i-1},\beta,\alpha_{i+1},\ldots,\alpha_s \text{线性无关.}$$

□