线性代数 homework (第五周)

PB20000113 孔浩宇

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1 周二

1.1 习题四

36. (1) 做初等变换

$$\begin{pmatrix} 3 & 2 & -1 & 9 \\ -2 & 1 & -4 & 2 \\ -1 & -2 & 3 & -2 \\ 3 & 2 & -1 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -4 & -1 & 9 \\ 0 & 5 & -10 & 6 \\ -1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -1 & 9 \\ 0 & 5 & -10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 35 & 0 & -84 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -84 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

即

$$\operatorname{rank} \begin{pmatrix} 3 & 2 & -1 & 9 \\ -2 & 1 & -4 & 2 \\ -1 & -2 & 3 & -2 \\ 3 & 2 & -1 & 9 \end{pmatrix} = 3.$$

37. 对于 a, b 的各种取值, 讨论实矩阵 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix}$ 的秩.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} \xrightarrow[\substack{-3r_1 \to r_3}]{-2r_1 \to r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & a-6 \\ 0 & b-6 & 0 \end{pmatrix} \xrightarrow[\substack{-3c_1 \to c_3}]{-2c_1 \to c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & a-6 \\ 0 & b-6 & 0 \end{pmatrix}$$

即

$$\operatorname{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank} \begin{pmatrix} 0 & a-6 \\ b-6 & 0 \end{pmatrix}.$$

(1) $a - 6 = 0, b - 6 = 0$

$$\operatorname{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 1. \quad (a = 6, b = 6)$$

(2) $a - 6 = 0, b - 0 \neq 0$

$$\operatorname{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank} \begin{pmatrix} 0 & 0 \\ b-6 & 0 \end{pmatrix} = 2. \quad (a = 6, b \neq 6)$$

(3) $a - 6 \neq 0, b - 6 = 0$

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \text{rank} \begin{pmatrix} 0 & a - 6 \\ 0 & 0 \end{pmatrix} = 2. \quad (a \neq 6, b = 6)$$

(4) $a - 6 \neq 0, b - 6 \neq 0$

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \text{rank} \begin{pmatrix} 0 & a - 6 \\ b - 6 & 0 \end{pmatrix} = 3. \quad (a \neq 6, b \neq 6)$$

39. 设 A 是 n 阶矩阵, 证明:$\text{rank}(A^*) = \begin{cases} n, & \text{rank}(A) = n, \\ 1, & \text{rank}(A) = n - 1, \\ 0, & \text{rank}(A) \leqslant n - 2. \end{cases}$

Proof.

(0) 引理 1: $\text{rank} \begin{pmatrix} A & O \\ C & B \end{pmatrix} \geqslant \text{rank}(A) + \text{rank}(B).$

Proof. 记 $a = \text{rank}(A), b = \text{rank}(B)$, 则存在可逆方阵 P_1, Q_1, P_2, Q_2 使

$$P_1 A Q_1 = \begin{pmatrix} I_a & O \\ O & O \end{pmatrix}, \quad P_2 B Q_2 = \begin{pmatrix} I_b & O \\ O & O \end{pmatrix}$$

取可逆方阵

$$P = \begin{pmatrix} P_1 & O \\ O & P_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix}$$

则

$$S = P \begin{pmatrix} A & O \\ C & B \end{pmatrix} Q = \begin{pmatrix} P_1 A Q_1 & O \\ P_2 C Q_1 & P_2 B Q_2 \end{pmatrix} = \begin{pmatrix} diag(I_a, O) & O \\ P_2 C Q_2 & diag(I_b, O) \end{pmatrix}$$

存在 $a + b$ 阶子式 $\begin{vmatrix} I_a & O \\ * & I_b \end{vmatrix} = 1 \neq 0$, 因此

$$\text{rank} \begin{pmatrix} A & O \\ C & B \end{pmatrix} = \text{rank}(S) \geqslant a + b = \text{rank}(A) + \text{rank}(B).$$

$\square$

引理 2: $\text{rank}(A) + \text{rank}B - n \leqslant \text{rank}(AB).$ (Sylvecter 秩不等式)

Proof. 即

$$\text{rank}(A) + \text{rank}(B) \leqslant \text{rank}(I_n) + \text{rank}(AB) = \text{rank} \begin{pmatrix} AB & O \\ O & I_n \end{pmatrix}$$

进行初等变换

$$\begin{pmatrix} I_n & A \\ O & I_n \end{pmatrix} \begin{pmatrix} AB & O \\ O & I_n \end{pmatrix} \begin{pmatrix} I_n & O \\ -B & I_n \end{pmatrix} = \begin{pmatrix} O & A \\ -B & I_n \end{pmatrix}, \quad \begin{pmatrix} O & A \\ -B & I_n \end{pmatrix} \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix} = \begin{pmatrix} A & O \\ I_n & B \end{pmatrix}$$

即有

$$\text{rank} \begin{pmatrix} AB & O \\ O & I_n \end{pmatrix} = \text{rank} \begin{pmatrix} A & O \\ I_n & B \end{pmatrix} \geqslant \text{rank}(A) + \text{rank}(B).$$

$\square$

(1) $\text{rank}(A) = n$

$$\text{rank}(A^*) = \text{rank}(A \cdot A^*) = \text{rank}(\det(A)I_n) = n.$$

(2) $\mathrm{rank}(A) = n - 1$

$$\begin{cases} A \cdot A^* = O & \Rightarrow \mathrm{rank}(A^*) + \mathrm{rank}(A) \leqslant n \\[2mm] \mathrm{rank}(A) = n - 1 & \Rightarrow \exists\, A_{ij} \neq 0, \mathrm{rank}(A^*) > 0 \end{cases} \Rightarrow 0 < \mathrm{rank}(A^*) \leqslant n - \mathrm{rank}(A) = 1.$$

显然有

$$\mathrm{rank}(A^*) = 1.$$

(3) $\mathrm{rank}(A) \leqslant n - 2$

$$\mathrm{rank}(A) \leqslant n - 2 \Rightarrow \forall\, i, j, A_{ij} = (-1)^{i+j} M_{ij} = 0 \Rightarrow \mathrm{rank}(A^*) = \mathrm{rank}(O) = 0.$$

$\square$

41. 证明下列关于秩的等式和不等式: (其中 A, B, C 是使运算有意义的矩阵)

(1) $\max\left(\mathrm{rank}(A), \mathrm{rank}(B), \mathrm{rank}(A + B)\right) \leqslant \mathrm{rank}\begin{pmatrix} A & B \end{pmatrix}$;

(2) $\mathrm{rank}\begin{pmatrix} A & B \end{pmatrix} \leqslant \mathrm{rank}(A) + \mathrm{rank}(B)$;

(3) $\mathrm{rank}\begin{pmatrix} A & C \\ O & B \end{pmatrix} \geqslant \mathrm{rank}(A) + \mathrm{rank}(B)$.

Proof.

(1) 任意 A, B 的非零子式均为 $\begin{pmatrix} A & B \end{pmatrix}$ 的非零子式, 即

$$\mathrm{rank}\begin{pmatrix} A & B \end{pmatrix} \geqslant \mathrm{rank}(A), \quad \mathrm{rank}\begin{pmatrix} A & B \end{pmatrix} \geqslant \mathrm{rank}(B)$$

记 $A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{pmatrix}, B = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix}$. 则

$$\mathrm{rank}\begin{pmatrix} A & B \end{pmatrix} = \mathrm{rank}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\} \geqslant \mathrm{rank}\{\boldsymbol{a}_1 + \boldsymbol{b}_1, \ldots, \boldsymbol{a}_n + \boldsymbol{b}_n\} = \mathrm{rank}(A + B).$$

综上

$$\max\left(\mathrm{rank}(A), \mathrm{rank}(B), \mathrm{rank}(A + B)\right) \leqslant \mathrm{rank}\begin{pmatrix} A & B \end{pmatrix}.$$

(2) 记 $a = \mathrm{rank}(A), b = \mathrm{rank}(B), A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m \end{pmatrix}, B = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix}$.

$$\forall\, \{\boldsymbol{a}_{i1}, \boldsymbol{a}_{i2}, \ldots, \boldsymbol{a}_{ia}\} \subseteq \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\}, \ \exists\, x_1, \ldots, x_a, x_1 \cdot \boldsymbol{a}_{i1} + \cdots + x_a \cdot \boldsymbol{a}_{ia} = 0.$$

$$\forall\, \{\boldsymbol{b}_{j1}, \boldsymbol{b}_{j2}, \ldots, \boldsymbol{b}_{jb}\} \subseteq \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}, \ \exists\, y_1, \ldots, y_b, y_1 \cdot \boldsymbol{b}_{j1} + \cdots + y_b \cdot \boldsymbol{b}_{jb} = 0.$$

$$\forall\, \{\boldsymbol{a}_{i1}, \ldots, \boldsymbol{a}_{ia}, \boldsymbol{b}_{j1}, \ldots, \boldsymbol{b}_{jb}\} \subseteq \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$$

$$\exists\, x_1, \ldots, x_a, y_1, \ldots, y_b, x_1 \cdot \boldsymbol{a}_{i1} + \cdots + x_a \cdot \boldsymbol{a}_{ia} + y_1 \cdot \boldsymbol{b}_{j1} + \cdots + y_b \cdot \boldsymbol{b}_{jb} = 0.$$

即

$$\mathrm{rank}\begin{pmatrix} A & B \end{pmatrix} = \mathrm{rank}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\} \leqslant a + b = \mathrm{rank}(A) + \mathrm{rank}(B).$$

(3) 记 $a = \mathrm{rank}(A), b = \mathrm{rank}(B)$, 则存在可逆方阵 P_1, Q_1, P_2, Q_2 使

$$P_1 A Q_1 = \begin{pmatrix} I_a & O \\ O & O \end{pmatrix}, \quad P_2 B Q_2 = \begin{pmatrix} I_b & O \\ O & O \end{pmatrix}$$

取可逆方阵

$$P = \begin{pmatrix} P_1 & O \\ O & P_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix}$$

则

$$S = P \begin{pmatrix} A & C \\ O & B \end{pmatrix} Q = \begin{pmatrix} P_1 A Q_1 & P_1 C Q_2 \\ O & P_2 B Q_2 \end{pmatrix} = \begin{pmatrix} diag(I_a, O) & P_1 C Q_2 \\ O & diag(I_b, O) \end{pmatrix}$$

存在 $a+b$ 阶子式 $\begin{vmatrix} I_a & * \\ O & I_b \end{vmatrix} = 1 \neq 0$, 因此

$$\mathrm{rank}\begin{pmatrix} A & C \\ O & B \end{pmatrix} = \mathrm{rank}(S) \geqslant a+b = \mathrm{rank}(A) + \mathrm{rank}(B).$$

$\square$

43. 设 n 阶方阵 A 满足 $A^2 = I$, 证明:$\mathrm{rank}(I+A) + \mathrm{rank}(I-A) = n$.

Proof. 进行初等变换

$$\begin{pmatrix} I+A & O \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} I+A & I-A \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} I+A & 2I \\ O & I-A \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} O & 2I \\ -\frac{1}{2}(I-A)(I+A) & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} O & 2I \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} O & 2I \\ O & O \end{pmatrix}$$

即

$$\mathrm{rank}(I+A) + \mathrm{rank}(I-A) = \mathrm{rank}\begin{pmatrix} I+A & O \\ O & I-A \end{pmatrix} = \mathrm{rank}\begin{pmatrix} O & 2I \\ O & O \end{pmatrix} = n.$$

$\square$

2 周四

2.1 习题五

3. 在 $\mathbb{F}^4$ 中, 判断向量 $\boldsymbol{b}$ 能否写成 $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ 的线性组合. 若能, 请写出一种表示方式.

   (1) $\boldsymbol{a}_1 = (-1, 3, 0, -5)$, $\boldsymbol{a}_2 = (2, 0, 7, -3)$,
       $\boldsymbol{a}_3 = (-4, 1, -2, 6)$, $\boldsymbol{b} = (8, 3, -1, -25)$;

   (2) $\boldsymbol{a}_1 = (3, -5, 2, -4)^T$, $\boldsymbol{a}_2 = (-1, 7, -3, 6)^T$,
       $\boldsymbol{a}_3 = (3, 11, -5, 10)^T$, $\boldsymbol{b} = (2, -30, 13, -26)^T$;

假设存在 $x, y, z \in \mathbb{R}, \boldsymbol{b} = x \cdot \boldsymbol{a}_1 + y \cdot \boldsymbol{a}_2 + z \cdot \boldsymbol{a}_3$.

(1)

$$\boldsymbol{b} = x \cdot \boldsymbol{a}_1 + y \cdot \boldsymbol{a}_2 + z \cdot \boldsymbol{a}_3 \Leftrightarrow \begin{cases} -x + 2y - 4z &= 8, \\ 3x \quad\quad\quad + z &= 3, \\ \quad\quad 7y - 2z &= -1, \\ -5x - 3y + 6z &= -25. \end{cases} \Leftrightarrow \begin{cases} x &= 2, \\ y &= -1, \\ z &= -3. \end{cases}$$

   能, $\boldsymbol{b} = 2\boldsymbol{a}_1 - \boldsymbol{a}_2 - 3\boldsymbol{a}_3$.

(2)

$$\boldsymbol{b} = x \cdot \boldsymbol{a}_1 + y \cdot \boldsymbol{a}_2 + z \cdot \boldsymbol{a}_3 \Leftrightarrow \begin{cases} 3x - y + 3z &= 2, \\ -5x + 7y + 11z &= -30, \\ 2x - 3y - 5z &= 13, \\ -4x + 6y + 10z &= -26. \end{cases} \Leftrightarrow \begin{cases} x &= -1, \\ y &= -5, \\ z &= 0. \end{cases}$$

   能, $\boldsymbol{b} = -\boldsymbol{a}_1 - 5\boldsymbol{a}_2$.

4. 设 $\boldsymbol{a}_1 = (1, 0, 0, 0), \boldsymbol{a}_2 = (1, 1, 0, 0), \boldsymbol{a}_3 = (1, 1, 1, 0), \boldsymbol{a}_4 = (1, 1, 1, 1)$. 证明:$\mathbb{F}^4$ 中任何向量都可以写成的线性组合, 且表示唯一.

Proof.

$$\forall \vec{\nu} \in \mathbb{F}^4, \ \exists!(x_1, x_2, x_3, x_4), \vec{\nu} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + x_3 \boldsymbol{e}_3 + x_4 \boldsymbol{e}_4.$$

$$又 \begin{cases} \boldsymbol{e}_1 &= \boldsymbol{a}_1, \\ \boldsymbol{e}_2 &= \boldsymbol{a}_2 - \boldsymbol{a}_1, \\ \boldsymbol{e}_3 &= \boldsymbol{a}_3 - \boldsymbol{a}_2, \\ \boldsymbol{e}_4 &= \boldsymbol{a}_4 - \boldsymbol{a}_3. \end{cases} \Rightarrow \exists!(x_1, x_2, x_3, x_4), \vec{\nu} = x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + x_3 \boldsymbol{a}_3 + x_4 \boldsymbol{a}_4.$$

□

5. 设 $P_i = (x_i, y_i, z_i), i = 1, 2, 3, 4$ 是三维几何空间中的点. 证明:$P_i, i = 1, 2, 3, 4$ 共面的条件是

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Proof. 记 $\alpha_1 = \overrightarrow{P_1 P_4}, \alpha_2 = \overrightarrow{P_2 P_4}, \alpha_3 = \overrightarrow{P_3 P_4}.$

四点共面 $\Leftrightarrow \alpha_1, \alpha_2, \alpha_3$ 共面 $\Leftrightarrow \exists\, m, n \in \mathbb{R}, m\alpha_1 + n\alpha_2 = \alpha_3, \begin{pmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - y_4 \end{pmatrix} = m \begin{pmatrix} x_1 - x_4 \\ y_1 - y_4 \\ z_1 - y_4 \end{pmatrix} + n \begin{pmatrix} x_2 - x_4 \\ y_2 - y_4 \\ z_2 - y_4 \end{pmatrix}.$

即

$$P_i, i = 1, 2, 3, 4 共面 \Leftrightarrow \begin{pmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - y_4 \end{pmatrix}, \begin{pmatrix} x_1 - x_4 \\ y_1 - y_4 \\ z_1 - y_4 \end{pmatrix}, \begin{pmatrix} x_2 - x_4 \\ y_2 - y_4 \\ z_2 - y_4 \end{pmatrix} 线性相关 \Leftrightarrow \begin{vmatrix} x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\ y_1 - y_4 & y_2 - y_4 & y_3 - y_4 \\ z_1 - z_4 & z_2 - z_4 & z_3 - z_4 \end{vmatrix} = 0.$$

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 & 0 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 & 0 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 & 0 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \begin{vmatrix} x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\ y_1 - y_4 & y_2 - y_4 & y_3 - y_4 \\ z_1 - z_4 & z_2 - z_4 & z_3 - z_4 \end{vmatrix} = 0. \Leftrightarrow P_i, i = 1, 2, 3, 4 共面.$$

□

6. 设 $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ 是三维几何空间中的四个向量. 证明它们必线性相关.