线性代数 homework (第三周)

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1 周二

1.1 习题四

7. 计算下列方阵的 k 次幂，$k \geqslant 1$.

$$(1)\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix};\qquad (2)\begin{pmatrix}a & b\\ -b & a\end{pmatrix};\qquad (3)\begin{pmatrix}1 & a & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & a\\ 0 & 0 & 0 & 1\end{pmatrix};$$

$$(4)\begin{pmatrix}1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1\\ & & & 1\end{pmatrix}_{n\times n};\qquad (5)\begin{pmatrix}a_1b_1 & a_1b_2 & \cdots & a_1b_n\\ a_2b_1 & a_2b_2 & \cdots & a_2b_n\\ \vdots & \vdots & \ddots & \vdots\\ a_nb_1 & a_nb_2 & \cdots & a_nb_n\end{pmatrix}.$$

解:

(1) 记 $A_k = \begin{pmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{pmatrix}$.

$$A_1 = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}, {A_1}^2 = \begin{pmatrix}\cos^2\theta-\sin^2\theta & 2\sin\cos\theta\\ -2\sin\cos\theta & \cos^2\theta-\sin^2\theta\end{pmatrix} = \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{pmatrix} = A_2.$$

不妨设 ${A_1}^n = A_n$.

(a) $n = 1$, 成立.

(b) 若 $n = k(k \geqslant 1)$ 时成立, 则 $n = k + 1$ 时:

$$\begin{aligned}{A_1}^{k+1} = A_k\cdot A_1 &= \begin{pmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\\ &= \begin{pmatrix}\cos k\theta\cos\theta - \sin k\theta\sin\theta & \sin\theta\cos k\theta + \sin k\theta\cos\theta\\ -\sin\theta\cos k\theta - \sin k\theta\cos\theta & \cos k\theta\cos\theta - \sin k\theta\sin\theta\end{pmatrix}\\ &= \begin{pmatrix}\cos(k+1)\theta & \sin(k+1)\theta\\ -\sin(k+1)\theta & \cos(k+1)\theta\end{pmatrix}\\ &= A_{k+1}.\end{aligned}$$

综合 $(a)(b)$, 即证 ${A_1}^k = A_k(k \geqslant 1)$, 即

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^k = \begin{pmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{pmatrix}.$$

(2) 1° $a = b = 0$

$$\begin{pmatrix}a & b\\ -b & a\end{pmatrix}^k = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}^k = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

2° $ab \neq 0$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^k = \left(\sqrt{a^2+b^2}\begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ \frac{-b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}\right)^k \xrightarrow[\cos\theta=\frac{a}{\sqrt{a^2+b^2}}]{\sin\theta=\frac{b}{\sqrt{a^2+b^2}}} \left(\sqrt{a^2+b^2}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right)^k.$$

$$\Rightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^k = (a^2+b^2)^{k/2}\begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}.$$

(3)

$$\begin{pmatrix} 1 & a & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}^k = \left(I + \begin{pmatrix} 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)^k \xrightarrow{A=\begin{pmatrix} 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}} (I+A)^k = \sum_{i=0}^k \binom{k}{i} A^{k-i}$$

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 2a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^3 = O \Rightarrow (I+A)^k = I + kA + \frac{k(k-1)}{2}A^2 = \begin{pmatrix} 1 & ka & k & k(k-1)a \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & ka \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(4)

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}_{n\times n}^k = \left(I + \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}_{n\times n}\right)^k \xrightarrow{A_m=\begin{cases} a_{ij} = 1 & j = i+m \\ a_{ij} = 0 & j \neq i+m \end{cases}} = (I+A_1)^k$$

$${A_1}^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & 0 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} = A_2 \Rightarrow {A_1}^k = \begin{cases} A_k & k \leqslant n-1 \\ 0 & k \geqslant n \end{cases}$$

$$\Rightarrow (I+A_1)^k = \begin{cases} \sum\limits_{i=0}^{k} C_k^i A_i & k \leqslant n-1; \\ \sum\limits_{i=0}^{n-1} C_k^i A_i & k \geqslant n. \end{cases}$$

(a) $k \leqslant n-1$

$$(I+A_1)^k = \begin{pmatrix} 1 & C_k^1 & \cdots & C_k^k & 0 & \cdots & 0 \\ & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & & \ddots & 0 \\ & & & \ddots & \ddots & & C_k^k \\ & & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & C_k^1 \\ & & & & & & 1 \end{pmatrix}.$$

(b) $k \geqslant n$

$$(I+A_1)^k = \begin{pmatrix} 1 & C_k^1 & \cdots & C_k^{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & C_k^1 \\ & & & 1 \end{pmatrix}.$$

(5)

$$
\begin{aligned}
\begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix}^k &= \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} \right)^k \\
&= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \left(\begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \right)^{k-1} \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} \\
&= (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^{k-1} \begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix}
\end{aligned}
$$

10. 证明: 与任意 n 阶方阵都乘法可交换的方阵一定是数量阵.

Proof. 设 A 是与任意 n 阶方阵都乘法可交换的方阵.

$$\forall 1 \leqslant i \leqslant n, E_{ij}A = AE_{ij}.$$

$$
\begin{aligned}
E_{ij}A = (b_{mn}) &= \begin{cases} b_{mn} = a_{jn} & m = i; \\ b_{mn} = 0 & m \neq i. \end{cases}, \\
AE_{ij} = (c_{mn}) &= \begin{cases} c_{mn} = a_{mi} & n = j; \\ c_{mn} = 0 & n \neq i. \end{cases} \\
\Rightarrow a_{ii} = a_{jj}, a_{jn} = 0, a_{mi} = 0 (m \neq i, n \neq j) &\Rightarrow \begin{cases} a_{ii} = a_{jj}, & \forall i, j. \\ a_{ij} = 0, & \forall i \neq j. \end{cases}
\end{aligned}
$$

即证 A 为数量阵. □

12. 设 $A_1, A_2, \ldots, A_k$ 都是 n 阶可逆方阵. 证明:

$$(A_1A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}.$$

Proof. 假设 $(A_1A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$ 对于 $1 \leqslant n \leqslant k$ 成立.

(1) $n = 1, (A_1)^{-1} = {A_1}^{-1}$, 成立.

(2) 假设 m 时成立 $(1 \leqslant m \leqslant k-1)$, 则

$$(A_1A_2 \cdots A_{m+1})^{-1} = {A_{m+1}}^{-1}(A_1A_2 \cdots A_m)^{-1} = {A_{m+1}}^{-1}{A_m}^{-1} \cdots {A_2}^{-1}{A_1}^{-1}.$$

综合 (1)(2), $(A_1A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$ 对于 $1 \leqslant n \leqslant k$ 成立.

即证

$$(A_1A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}.$$

□

2 周四

2.1 习题四

13. 设方阵 A 满足 $A^k = O$,k 为正整数. 证明:$I+A$ 可逆. 并求 $(I+A)^{-1}$.

Proof.

$$
\begin{aligned}
&0 = x^k = (x+1)\sum_{i=1}^{k}(-1)^{i+1}\cdot x^{k-i} + (-1)^k \\
\Rightarrow &O = A^k = (I+A)\cdot\sum_{i=1}^{k}(-1)^{i+1}\cdot A^{k-i} + (-1)^k I \\
\Rightarrow &(I+A)\cdot\sum_{i=1}^{k}(-1)^{i+1}\cdot A^{k-i} = (-1)^k I \\
\Rightarrow &(I+A)\cdot\sum_{i=1}^{k}(-1)^{i+1+k}\cdot A^{k-i} = I \\
\Rightarrow &(I+A)^{-1} = \sum_{i=1}^{k}(-1)^{i+1+k}\cdot A^{k-i}.
\end{aligned}
$$

即证 $I+A$ 可逆, 且

$$(I+A)^{-1} = \sum_{i=1}^{k}(-1)^{i+1+k}\cdot A^{k-i}.$$

□

14. 设方阵 A 满足 $I-2A-3A^2+4A^3+5A^4-6A^5=O$. 证明:$I-A$ 可逆. 并求 $(I-A)^{-1}$.

Proof.

$$
\begin{aligned}
&0 = 1-2x-3x^2+4x^3+5x^4-6x^5 = (1-x)\cdot(6x^4+x^3-3x^2+2)-1 \\
\Rightarrow &O = I-2A-3A^2+4A^3+5A^4-6A^5 = (I-A)\cdot(6A^4+A^3-3A^2+2I)-I \\
\Rightarrow &(I-A)\cdot(6A^4+A^3-3A^2+2I) = I \\
\Rightarrow &(I-A)^{-1} = 6A^4+A^3-3A^2+2I.
\end{aligned}
$$

即证 $I-A$ 可逆, 且

$$(I-A)^{-1} = 6A^4+A^3-3A^2+2I.$$

□

17. 证明:$(A_1A_2\cdots A_k)^T = {A_k}^T\cdots{A_2}^T{A_1}^T$ (假设其中的矩阵乘法有意义).

Proof. 假设 $(A_1A_2\cdots A_n)^T = {A_n}^T\cdots{A_2}^T{A_1}^T, (\forall\ 1\le n\le k)$

(1) $n=1$

$$(A_1)^T = {A_1}^T, \text{成立}.$$

(2) 假设 $n-1$ 时成立, 则:

$$
\begin{aligned}
(A_1A_2\cdots A_{n-1}A_n)^T &= ((A_1A_2\cdots A_{n-1})A_n)^T \\
&= {A_n}^T(A_1A_2\cdots A_{n-1})^T \\
&= {A_n}^T{A_{n-1}}^T\cdots{A_2}^T{A_1}^T, \text{成立}.
\end{aligned}
$$

综合 (1)(2), $(A_1A_2\cdots A_n)^T = {A_n}^T\cdots{A_2}^T{A_1}^T, (\forall\ 1\le n\le k)$ 成立. 即证.

$\square$

18. 求所有满足 $A^2=O, B^2=I, \overline{C}^T C=I$ 的 2 阶复方阵 A, B, C.

解: 分别设三个矩阵为 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} (a,b,c,d\in\mathbb{C})$

(1)

$$A^2=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}=O\Rightarrow\begin{cases} a^2+bc & =d^2+bc=0 \\ c(a+d) & =b(a+d)=0 \end{cases}$$

1° $a+d\neq 0$

$$\begin{cases} a+d & \neq 0 \\ c(a+d) & =0 \\ b(a+d) & =0 \end{cases}\Rightarrow\begin{cases} b & =0 \\ c & =0. \end{cases}\quad\begin{cases} a^2+bc & =0 \\ d^2+bc & =0 \\ bc & =0 \end{cases}\Rightarrow\begin{cases} a & =0 \\ d & =0. \end{cases}\Rightarrow a+d\neq 0, \text{矛盾}.$$

2° $a+d=0$

$$\begin{cases} a^2+bc & =d^2+bc=0 \\ c(a+d) & =b(a+d)=0 \\ a+d & =0 \end{cases}\Leftrightarrow\begin{cases} a^2+bc & =0 \\ a+d & =0 \end{cases}\Rightarrow A=\begin{pmatrix} \pm\sqrt{-bc} & b \\ c & \mp\sqrt{-bc} \end{pmatrix}.$$

即:

$$A=\begin{pmatrix} \pm\sqrt{-bc} & b \\ c & \mp\sqrt{-bc} \end{pmatrix}.$$

(2)

$$B^2=\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2=\begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}=I\Rightarrow\begin{cases} a^2+bc & =d^2+bc=1 \\ c(a+d) & =b(a+d)=0 \end{cases}$$

1° $a+d\neq 0$

$$\begin{cases} (a+d)c & =0 \\ (a+d)d & =0 \\ a+d & \neq 0 \end{cases}\Rightarrow\begin{cases} c & =0 \\ d & =0. \end{cases}\Rightarrow\begin{cases} a^2 & =1 \\ d^2 & =1 \\ a+d & \neq 0 \end{cases}\Rightarrow\begin{cases} a & =1 \\ d & =1; \end{cases}\begin{cases} a & =-1 \\ d & =-1. \end{cases}\Rightarrow B=\pm I.$$

2° $a+d=0$

$$\begin{cases} a^2+bc & =d^2+bc=1 \\ c(a+d) & =b(a+d)=0 \\ a+d & =0 \end{cases}\Leftrightarrow\begin{cases} a^2+bc & =1 \\ a+d & =0 \end{cases}\Rightarrow B=\begin{pmatrix} \pm\sqrt{1-bc} & b \\ c & \mp\sqrt{1-bc} \end{pmatrix}.$$

即:

$$B=\begin{pmatrix} \pm\sqrt{1-bc} & b \\ c & \mp\sqrt{1-bc} \end{pmatrix},\quad \pm I.$$

(3)

$$\overline{C}^T C=\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} \overline{a}a+\overline{c}c & \overline{a}b+\overline{c}d \\ a\overline{b}+c\overline{d} & \overline{b}b+\overline{d}d \end{pmatrix}=\begin{pmatrix} |a|^2+|c|^2 & \overline{a}b+\overline{c}d \\ \overline{\overline{a}b+\overline{c}d} & |b|^2+|d|^2 \end{pmatrix}=I$$

$$\begin{cases} |a|^2+|c|^2 & =1 \\ |b|^2+|d|^2 & =1 \end{cases}\xrightarrow[\lambda,\mu\in[0,2\pi)]{\alpha,\beta,\gamma,\delta\in[0,2\pi)}\begin{cases} a=\cos\lambda e^{i\alpha}, & b=\cos\mu e^{i\beta}, \\ c=\sin\lambda e^{i\gamma}, & d=\sin\mu e^{i\delta}. \end{cases}$$

$$\Rightarrow \overline{a}b+\overline{c}d=\cos\lambda\cdot\cos\mu\cdot e^{i(\beta-\alpha)}+\sin\lambda\cdot\sin\mu\cdot e^{i(\delta-\gamma)}=0$$

$$\Rightarrow\begin{cases} \beta-\alpha & =\delta-\gamma \\ \cos\lambda\cdot\cos\mu+\sin\lambda\cdot\sin\mu & =0. \end{cases}\Longleftrightarrow\begin{cases} \beta+\gamma & =\delta+\alpha; \\ \cos(\lambda-\mu) & =0 \end{cases}$$

即

$$C=\begin{pmatrix} \cos(\mu+\pi/2)\cdot e^{i\alpha} & \cos\mu\cdot e^{i\beta} \\ \sin(\mu+\pi/2)\cdot e^{i\gamma} & \sin\mu\cdot e^{\beta+\gamma-\alpha} \end{pmatrix} \text{或} \begin{pmatrix} \cos(\mu+3\pi/2)\cdot e^{i\alpha} & \cos\mu\cdot e^{i\beta} \\ \sin(\mu+3\pi/2)\cdot e^{i\gamma} & \sin\mu\cdot e^{\beta+\gamma-\alpha} \end{pmatrix}.$$

19. 证明: 不存在 n 阶复方阵 A, B 满足 $AB - BA = I$.

Proof. 设 $A = (a_{ij}), B = (b_{ij}), AB = C = (c_{ij}), BA = D = (d_{ij})$, 则:

$$
\begin{aligned}
&tr(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki},\\
&tr(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = tr(AB),\\
&tr(AB - BA) = te(AB) - tr(BA) = 0,\\
&tr(I_n) = n \neq 0 \quad \Rightarrow AB - BA \neq I_n, \forall A, B \in \mathbb{C}^{n\times n}.
\end{aligned}
$$

即证不存在 n 阶复方阵 A, B 满足 $AB - BA = I$. □

20. 证明: 可逆上 (下) 三角、准对角、对称、反对称方阵的逆矩阵仍然分别是上 (下) 三角、准对角、对称、反对称方阵.

Proof. (1) 可逆上 (下) 三角方阵

$$
\begin{aligned}
&\forall A = (a_{ij})_{n\times n}, 有a_{ij} = 0(i < j), a_{ii} \neq 0; A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1}\\ A_{12} & A_{22} & \cdots & A_{n2}\\ \vdots & \vdots & & \vdots\\ A_{1n} & A_{2n} & \cdots & A_{nn}\end{pmatrix}\\
&\Rightarrow \forall i < j, 设A_{ij} = \det(c_{pq}), 则c_{pq} = 0, for\ p < q\ or\ p = q = i\\
&\Rightarrow \forall i < j, A_{ij} = \prod_{k=1}^{n-1} c_{kk} = \frac{c_{ii}}{a_{ii}\cdot a_{jj}}\prod_{k=1}^{n} a_{kk} = 0\\
&\Rightarrow A^{-1}仍为上三角方阵.
\end{aligned}
$$

(2) 可逆下三角方阵

$$
\begin{aligned}
&\forall A = (a_{ij})_{n\times n}, 有a_{ij} = 0(i > j), a_{ii} \neq 0; A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1}\\ A_{12} & A_{22} & \cdots & A_{n2}\\ \vdots & \vdots & & \vdots\\ A_{1n} & A_{2n} & \cdots & A_{nn}\end{pmatrix}\\
&\Rightarrow \forall i > j, 设A_{ij} = \det(c_{pq}), 则c_{pq} = 0, for\ p > q\ or\ p = q = i\\
&\Rightarrow \forall i > j, A_{ij} = \prod_{k=1}^{n-1} c_{kk} = \frac{c_{ii}}{a_{ii}\cdot a_{jj}}\prod_{k=1}^{n} a_{kk} = 0\\
&\Rightarrow A^{-1}仍为下三角方阵.
\end{aligned}
$$

(3) 准对角方阵

$$
\forall 可逆A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n\end{pmatrix}, A^{-1} = \begin{pmatrix} {A_1}^{-1} & & \\ & \ddots & \\ & & {A_n}^{-1}\end{pmatrix} 仍为准对角方阵.
$$

(4) 对称方阵

$$
\forall 可逆A = A^T, \left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}, 即\ A^{-1}\ 仍为对称方阵.
$$

(5) 反对称方阵

$$
\forall 可逆A^T = -A, \left(A^{-1}\right)^T = \left(A^T\right)^{-1} = (-A)^{-1} = -A^{-1}, 即\ A^{-1}\ 仍为反对称方阵.
$$

(6) 第一问其他做法

1. $n=2$, 可验证.
2. 假设 $n-1$ 时成立, 则:

$$A_n{A_n}^{-1}=\begin{pmatrix}A_{n-1} & \gamma_n\\ 0 & a_n\end{pmatrix}\begin{pmatrix}B_{n-1} & \alpha_n\\ \beta_n & b_n\end{pmatrix}=\begin{pmatrix}A_{n-1}B_{n-1}+\gamma_n\beta_n & A_{n-1}\alpha_n+b_n\gamma_n\\ a_n\beta_n & a_nb_n\end{pmatrix}=I.$$

$\Rightarrow\begin{cases}a_n\beta_n & =0\\ a_nb_n & =1\end{cases}\Rightarrow\begin{cases}\beta_n & =0\\ b_n & =\frac{1}{a_n}\end{cases}\Rightarrow A_{n-1}B_{n-1}=I_{n-1}\Rightarrow B_{n-1}={A_{n-1}}^{-1}$,为上三角阵.

$\Rightarrow {A_n}^{-1}$为上三角阵.

$\square$