线性代数 homework (第五周)

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April 1, 2022

1 周二

1.1 习题四

36. (1) 做初等变换

$$\begin{pmatrix} 3 & 2 & -1 & 9 \\ -2 & 1 & -4 & 2 \\ -1 & -2 & 3 & -2 \\ 3 & 2 & -1 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -4 & -1 & 9 \\ 0 & 5 & -10 & 6 \\ -1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -1 & 9 \\ 0 & 5 & -10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 35 & 0 & -84 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -84 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

即

$$\operatorname{rank} \begin{pmatrix} 3 & 2 & -1 & 9 \\ -2 & 1 & -4 & 2 \\ -1 & -2 & 3 & -2 \\ 3 & 2 & -1 & 9 \end{pmatrix} = 3.$$

37. 对于 a, b 的各种取值, 讨论实矩阵 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix}$ 的秩.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} \xrightarrow[-3r_1 \to r_3]{-2r_1 \to r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & a-6 \\ 0 & b-6 & 0 \end{pmatrix} \xrightarrow[-3c_1 \to c_3]{-2c_1 \to c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & a-6 \\ 0 & b-6 & 0 \end{pmatrix}$$

即

$$\operatorname{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank} \begin{pmatrix} 0 & a-6 \\ b-6 & 0 \end{pmatrix}.$$

(1) $a-6=0, b-6=0$

$$\operatorname{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 1. \quad (a=6, b=6)$$

(2) $a-6=0, b-0 \neq 0$

$$\operatorname{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank} \begin{pmatrix} 0 & 0 \\ b-6 & 0 \end{pmatrix} = 2. \quad (a=6, b \neq 6)$$

(3) $a-6\neq 0, b-6=0$

$$\operatorname{rank}\begin{pmatrix}1&2&3\\2&4&a\\3&b&9\end{pmatrix}=1+\operatorname{rank}\begin{pmatrix}0&a-6\\0&0\end{pmatrix}=2.\quad (a\neq 6, b=6)$$

(4) $a-6\neq 0, b-6\neq 0$

$$\operatorname{rank}\begin{pmatrix}1&2&3\\2&4&a\\3&b&9\end{pmatrix}=1+\operatorname{rank}\begin{pmatrix}0&a-6\\b-6&0\end{pmatrix}=3.\quad (a\neq 6, b\neq 6)$$

39. 设 A 是 n 阶矩阵, 证明:$\operatorname{rank}(A^*)=\begin{cases}n, & \operatorname{rank}(A)=n,\\ 1, & \operatorname{rank}(A)=n-1,\\ 0, & \operatorname{rank}(A)\leqslant n-2.\end{cases}$

Proof.

(0) 引理 1: $\operatorname{rank}\begin{pmatrix}A&O\\C&B\end{pmatrix}\geqslant \operatorname{rank}(A)+\operatorname{rank}(B)$.

Proof. 记 $a=\operatorname{rank}(A), b=\operatorname{rank}(B)$, 则存在可逆方阵 P_1,Q_1,P_2,Q_2 使

$$P_1AQ_1=\begin{pmatrix}I_a&O\\O&O\end{pmatrix},\quad P_2BQ_2=\begin{pmatrix}I_b&O\\O&O\end{pmatrix}$$

取可逆方阵

$$P=\begin{pmatrix}P_1&O\\O&P_2\end{pmatrix},\quad Q=\begin{pmatrix}Q_1&O\\O&Q_2\end{pmatrix}$$

则

$$S=P\begin{pmatrix}A&O\\C&B\end{pmatrix}Q=\begin{pmatrix}P_1AQ_1&O\\P_2CQ_1&P_2BQ_2\end{pmatrix}=\begin{pmatrix}diag(I_a,O)&O\\P_2CQ_2&diag(I_b,O)\end{pmatrix}$$

存在 $a+b$ 阶子式 $\begin{vmatrix}I_a&O\\ *&I_b\end{vmatrix}=1\neq 0$, 因此

$$\operatorname{rank}\begin{pmatrix}A&O\\C&B\end{pmatrix}=\operatorname{rank}(S)\geqslant a+b=\operatorname{rank}(A)+\operatorname{rank}(B).$$

□

引理 2: $\operatorname{rank}(A)+\operatorname{rank}B-n\leqslant\operatorname{rank}(AB)$. (Sylvecter 秩不等式)

Proof. 即

$$\operatorname{rank}(A)+\operatorname{rank}(B)\leqslant\operatorname{rank}(I_n)+\operatorname{rank}(AB)=\operatorname{rank}\begin{pmatrix}AB&O\\O&I_n\end{pmatrix}$$

进行初等变换

$$\begin{pmatrix}I_n&A\\O&I_n\end{pmatrix}\begin{pmatrix}AB&O\\O&I_n\end{pmatrix}\begin{pmatrix}I_n&O\\-B&I_n\end{pmatrix}=\begin{pmatrix}O&A\\-B&I_n\end{pmatrix},\quad \begin{pmatrix}O&A\\-B&I_n\end{pmatrix}\begin{pmatrix}O&-I_n\\I_n&O\end{pmatrix}=\begin{pmatrix}A&O\\I_n&B\end{pmatrix}$$

即有

$$\operatorname{rank}\begin{pmatrix}AB&O\\O&I_n\end{pmatrix}=\operatorname{rank}\begin{pmatrix}A&O\\I_n&B\end{pmatrix}\geqslant\operatorname{rank}(A)+\operatorname{rank}(B).$$

□

(1) $\operatorname{rank}(A)=n$

$$\operatorname{rank}(A^*)=\operatorname{rank}(A\cdot A^*)=\operatorname{rank}(\det(A)I_n)=n.$$

(2) $\operatorname{rank}(A)=n-1$

$$\begin{cases} A\cdot A^*=O & \Rightarrow \operatorname{rank}(A^*)+\operatorname{rank}(A)\leqslant n \\ \operatorname{rank}(A)=n-1 & \Rightarrow \exists\, A_{ij}\neq 0, \operatorname{rank}(A^*)>0 \end{cases} \Rightarrow 0<\operatorname{rank}(A^*)\leqslant n-\operatorname{rank}(A)=1.$$

显然有

$$\operatorname{rank}(A^*)=1.$$

(3) $\operatorname{rank}(A)\leqslant n-2$

$$\operatorname{rank}(A)\leqslant n-2 \Rightarrow \forall\, i,j, A_{ij}=(-1)^{i+j}M_{ij}=0 \Rightarrow \operatorname{rank}(A^*)=\operatorname{rank}(O)=0.$$

□

41. 证明下列关于秩的等式和不等式: (其中 A,B,C 是使运算有意义的矩阵)

(1) $\max\left(\operatorname{rank}(A),\operatorname{rank}(B),\operatorname{rank}(A+B)\right)\leqslant \operatorname{rank}\begin{pmatrix}A & B\end{pmatrix}$;

(2) $\operatorname{rank}\begin{pmatrix}A & B\end{pmatrix}\leqslant \operatorname{rank}(A)+\operatorname{rank}(B)$;

(3) $\operatorname{rank}\begin{pmatrix}A & C\\ O & B\end{pmatrix}\geqslant \operatorname{rank}(A)+\operatorname{rank}(B)$.

Proof.

(1) 任意 A,B 的非零子式均为 $\begin{pmatrix}A & B\end{pmatrix}$ 的非零子式, 即

$$\operatorname{rank}\begin{pmatrix}A & B\end{pmatrix}\geqslant \operatorname{rank}(A),\quad \operatorname{rank}\begin{pmatrix}A & B\end{pmatrix}\geqslant \operatorname{rank}(B)$$

记 $A=\begin{pmatrix}\boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n\end{pmatrix}, B=\begin{pmatrix}\boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n\end{pmatrix}$. 则

$$\operatorname{rank}\begin{pmatrix}A & B\end{pmatrix}=\operatorname{rank}\left\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n,\boldsymbol{b}_1,\ldots,\boldsymbol{b}_n\right\}\geqslant \operatorname{rank}\left\{\boldsymbol{a}_1+\boldsymbol{b}_1,\ldots,\boldsymbol{a}_n+\boldsymbol{b}_n\right\}=\operatorname{rank}(A+B).$$

综上

$$\max\left(\operatorname{rank}(A),\operatorname{rank}(B),\operatorname{rank}(A+B)\right)\leqslant \operatorname{rank}\begin{pmatrix}A & B\end{pmatrix}.$$

(2) 记 $a=\operatorname{rank}(A), b=\operatorname{rank}(B), A=\begin{pmatrix}\boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m\end{pmatrix}, B=\begin{pmatrix}\boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n\end{pmatrix}$.

$$\forall\left\{\boldsymbol{a}_{i1},\boldsymbol{a}_{i2},\ldots,\boldsymbol{a}_{ia}\right\}\subseteq\left\{\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_m\right\},\ \exists\, x_1,\ldots,x_a, x_1\cdot\boldsymbol{a}_{i1}+\cdots+x_a\cdot\boldsymbol{a}_{ia}=0.$$

$$\forall\left\{\boldsymbol{b}_{j1},\boldsymbol{b}_{j2},\ldots,\boldsymbol{b}_{jb}\right\}\subseteq\left\{\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_n\right\},\ \exists\, y_1,\ldots,y_b, y_1\cdot\boldsymbol{b}_{j1}+\cdots+y_b\cdot\boldsymbol{b}_{jb}=0.$$

$\forall\left\{\boldsymbol{a}_{i1},\ldots,\boldsymbol{a}_{ia},\boldsymbol{b}_{j1},\ldots,\boldsymbol{b}_{jb}\right\}\subseteq\left\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,\boldsymbol{b}_1,\ldots,\boldsymbol{b}_n\right\}$

$$\exists\, x_1,\ldots,x_a,y_1,\ldots,y_b, x_1\cdot\boldsymbol{a}_{i1}+\cdots+x_a\cdot\boldsymbol{a}_{ia}+y_1\cdot\boldsymbol{b}_{j1}+\cdots+y_b\cdot\boldsymbol{b}_{jb}=0.$$

即

$$\operatorname{rank}\begin{pmatrix}A & B\end{pmatrix}=\operatorname{rank}\left\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,\boldsymbol{b}_1,\ldots,\boldsymbol{b}_n\right\}\leqslant a+b=\operatorname{rank}(A)+\operatorname{rank}(B).$$

(3) 记 $a=\operatorname{rank}(A), b=\operatorname{rank}(B)$, 则存在可逆方阵 P_1,Q_1,P_2,Q_2 使

$$P_1AQ_1=\begin{pmatrix}I_a & O\\ O & O\end{pmatrix},\quad P_2BQ_2=\begin{pmatrix}I_b & O\\ O & O\end{pmatrix}$$

取可逆方阵

$$P=\begin{pmatrix}P_1 & O\\ O & P_2\end{pmatrix},\quad Q=\begin{pmatrix}Q_1 & O\\ O & Q_2\end{pmatrix}$$

则

$$S=P\begin{pmatrix}A & C\\ O & B\end{pmatrix}Q=\begin{pmatrix}P_1AQ_1 & P_1CQ_2\\ O & P_2BQ_2\end{pmatrix}=\begin{pmatrix}diag(I_a,O) & P_1CQ_2\\ O & diag(I_b,O)\end{pmatrix}$$

存在 $a+b$ 阶子式 $\begin{vmatrix} I_a & * \\ O & I_b \end{vmatrix} = 1 \neq 0$, 因此

$$\operatorname{rank}\begin{pmatrix} A & C \\ O & B \end{pmatrix} = \operatorname{rank}(S) \geqslant a+b = \operatorname{rank}(A) + \operatorname{rank}(B).$$

□

43. 设 n 阶方阵 A 满足 $A^2 = I$, 证明:$\operatorname{rank}(I+A) + \operatorname{rank}(I-A) = n$.

Proof. 进行初等变换

$$\begin{pmatrix} I+A & O \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} I+A & I-A \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} I+A & 2I \\ O & I-A \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} O & 2I \\ -\frac{1}{2}(I-A)(I+A) & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} O & 2I \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} O & 2I \\ O & O \end{pmatrix}$$

即

$$\operatorname{rank}(I+A) + \operatorname{rank}(I-A) = \operatorname{rank}\begin{pmatrix} I+A & O \\ O & I-A \end{pmatrix} = \operatorname{rank}\begin{pmatrix} O & 2I \\ O & O \end{pmatrix} = n.$$

□

2 周四

2.1 习题五

3. 在 $\mathbb{F}^4$ 中, 判断向量 $\boldsymbol{b}$ 能否写成 $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ 的线性组合. 若能, 请写出一种表示方式.

(1) $\boldsymbol{a}_1 = (-1, 3, 0, -5)$, $\boldsymbol{a}_2 = (2, 0, 7, -3)$,
$\boldsymbol{a}_3 = (-4, 1, -2, 6)$, $\boldsymbol{b} = (8, 3, -1, -25)$;

(2) $\boldsymbol{a}_1 = (3, -5, 2, -4)^T$, $\boldsymbol{a}_2 = (-1, 7, -3, 6)^T$,
$\boldsymbol{a}_3 = (3, 11, -5, 10)^T$, $\boldsymbol{b} = (2, -30, 13, -26)^T$;

假设存在 $x, y, z \in \mathbb{R}, \boldsymbol{b} = x \cdot \boldsymbol{a}_1 + y \cdot \boldsymbol{a}_2 + z \cdot \boldsymbol{a}_3$.

(1)

$$\boldsymbol{b} = x \cdot \boldsymbol{a}_1 + y \cdot \boldsymbol{a}_2 + z \cdot \boldsymbol{a}_3 \Leftrightarrow \begin{cases} -x + 2y - 4z &= 8, \\ 3x \qquad + \ z &= 3, \\ \qquad 7y - 2z &= -1, \\ -5x - 3y + 6z &= -25. \end{cases} \Leftrightarrow \begin{cases} x &= 2, \\ y &= -1, \\ z &= -3. \end{cases}$$

能, $\boldsymbol{b} = 2\boldsymbol{a}_1 - \boldsymbol{a}_2 - 3\boldsymbol{a}_3$.

(2)

$$\boldsymbol{b} = x \cdot \boldsymbol{a}_1 + y \cdot \boldsymbol{a}_2 + z \cdot \boldsymbol{a}_3 \Leftrightarrow \begin{cases} 3x - \ y + \ 3z &= 2, \\ -5x + 7y + 11z &= -30, \\ 2x - 3y - \ 5z &= 13, \\ -4x + 6y + 10z &= -26. \end{cases} \Leftrightarrow \begin{cases} x &= -1, \\ y &= -5, \\ z &= 0. \end{cases}$$

能, $\boldsymbol{b} = -\boldsymbol{a}_1 - 5\boldsymbol{a}_2$.

4. 设 $\boldsymbol{a}_1 = (1, 0, 0, 0), \boldsymbol{a}_2 = (1, 1, 0, 0), \boldsymbol{a}_3 = (1, 1, 1, 0), \boldsymbol{a}_4 = (1, 1, 1, 1)$. 证明:$\mathbb{F}^4$ 中任何向量都可以写成的线性组合, 且表示唯一.

Proof.

$$\forall\ \vec{\nu} \in \mathbb{F}^4,\ \exists!(x_1,x_2,x_3,x_4), \vec{\nu} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4.$$

$$又\begin{cases}\boldsymbol{e}_1 &= \boldsymbol{a}_1,\\ \boldsymbol{e}_2 &= \boldsymbol{a}_2 - \boldsymbol{a}_1,\\ \boldsymbol{e}_3 &= \boldsymbol{a}_3 - \boldsymbol{a}_2,\\ \boldsymbol{e}_4 &= \boldsymbol{a}_4 - \boldsymbol{a}_3.\end{cases} \Rightarrow\ \exists!(x_1,x_2,x_3,x_4), \vec{\nu} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 + x_4\boldsymbol{a}_4.$$

□

5. 设 $P_i = (x_i, y_i, z_i), i = 1,2,3,4$ 是三维几何空间中的点. 证明:$P_i, i = 1,2,3,4$ 共面的条件是

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1\end{vmatrix} = 0.$$

Proof. 记 $\alpha_1 = \overrightarrow{P_1P_4}, \alpha_2 = \overrightarrow{P_2P_4}, \alpha_3 = \overrightarrow{P_3P_4}$.

四点共面 $\Leftrightarrow \alpha_1,\alpha_2,\alpha_3$共面 $\Leftrightarrow \exists\, m,n \in \mathbb{R}, m\alpha_1 + n\alpha_2 = \alpha_3, \begin{pmatrix} x_3 - x_4\\ y_3 - y_4\\ z_3 - y_4\end{pmatrix} = m\begin{pmatrix} x_1 - x_4\\ y_1 - y_4\\ z_1 - y_4\end{pmatrix} + n\begin{pmatrix} x_2 - x_4\\ y_2 - y_4\\ z_2 - y_4\end{pmatrix}$.

即

$$P_i, i = 1,2,3,4\text{共面} \Leftrightarrow \alpha_1,\alpha_2,\alpha_3\text{线性相关} \Leftrightarrow \begin{vmatrix} x_1 - x_4 & x_2 - x_4 & x_3 - x_4\\ y_1 - y_4 & y_2 - y_4 & y_3 - y_4\\ z_1 - z_4 & z_2 - z_4 & z_3 - z_4\end{vmatrix} = 0.$$

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1\end{vmatrix} = \begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 & 0\\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 & 0\\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 & 0\\ x_4 & y_4 & z_4 & 1\end{vmatrix} = 0. \Leftrightarrow P_i, i = 1,2,3,4\text{共面}.$$

□

6. 设 $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ 是三维几何空间中的四个向量. 证明它们必线性相关.

Proof. 设 $\boldsymbol{a}_i = (x_i, y_i, z_i)^T\ (i = 1,2,3,4)$, 记 $A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{pmatrix}$.

证 $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ 线性相关, 即证

$$A\lambda = \begin{pmatrix} x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} = 0$$

有非零解. 又 $\mathrm{rank}A < 3, A\lambda = 0$ 有非零解, 即证 $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ 线性相关. □