线性代数 homework (第五周)

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1 周二

1.1 习题四

36. (1) 做初等变换

$$\begin{pmatrix} 3 & 2 & -1 & 9 \\ -2 & 1 & -4 & 2 \\ -1 & -2 & 3 & -2 \\ 3 & 2 & -1 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -4 & -1 & 9 \\ 0 & 5 & -10 & 6 \\ -1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -1 & 9 \\ 0 & 5 & -10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 35 & 0 & -84 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -84 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

即

$$\operatorname{rank}\begin{pmatrix} 3 & 2 & -1 & 9 \\ -2 & 1 & -4 & 2 \\ -1 & -2 & 3 & -2 \\ 3 & 2 & -1 & 9 \end{pmatrix} = 3.$$

37. 对于 a, b 的各种取值, 讨论实矩阵 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix}$ 的秩.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} \xrightarrow[-3r_1\to r_3]{-2r_1\to r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & a-6 \\ 0 & b-6 & 0 \end{pmatrix} \xrightarrow[-3c_1\to c_3]{-2c_1\to c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & a-6 \\ 0 & b-6 & 0 \end{pmatrix}$$

即

$$\operatorname{rank}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank}\begin{pmatrix} 0 & a-6 \\ b-6 & 0 \end{pmatrix}.$$

(1) $a-6=0, b-6=0$

$$\operatorname{rank}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 1. \quad (a=6, b=6)$$

(2) $a-6=0, b-0\neq 0$

$$\operatorname{rank}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \operatorname{rank}\begin{pmatrix} 0 & 0 \\ b-6 & 0 \end{pmatrix} = 2. \quad (a=6, b\neq 6)$$

(3) $a-6\neq 0, b-6=0$

$$\operatorname{rank}\begin{pmatrix}1&2&3\\2&4&a\\3&b&9\end{pmatrix}=1+\operatorname{rank}\begin{pmatrix}0&a-6\\0&0\end{pmatrix}=2.\quad (a\neq 6, b=6)$$

(4) $a-6\neq 0, b-6\neq 0$

$$\operatorname{rank}\begin{pmatrix}1&2&3\\2&4&a\\3&b&9\end{pmatrix}=1+\operatorname{rank}\begin{pmatrix}0&a-6\\b-6&0\end{pmatrix}=3.\quad (a\neq 6, b\neq 6)$$

39. 设 A 是 n 阶矩阵, 证明:$\operatorname{rank}(A^*)=\begin{cases}n, & \operatorname{rank}(A)=n,\\ 1, & \operatorname{rank}(A)=n-1,\\ 0, & \operatorname{rank}(A)\leqslant n-2.\end{cases}$

Proof.

(0) 引理 1: $\operatorname{rank}\begin{pmatrix}A&O\\C&B\end{pmatrix}\geqslant \operatorname{rank}(A)+\operatorname{rank}(B)$.

Proof. 记 $a=\operatorname{rank}(A), b=\operatorname{rank}(B)$, 则存在可逆方阵 P_1,Q_1,P_2,Q_2 使

$$P_1AQ_1=\begin{pmatrix}I_a&O\\O&O\end{pmatrix},\quad P_2BQ_2=\begin{pmatrix}I_b&O\\O&O\end{pmatrix}$$

取可逆方阵

$$P=\begin{pmatrix}P_1&O\\O&P_2\end{pmatrix},\quad Q=\begin{pmatrix}Q_1&O\\O&Q_2\end{pmatrix}$$

则

$$S=P\begin{pmatrix}A&O\\C&B\end{pmatrix}Q=\begin{pmatrix}P_1AQ_1&O\\P_2CQ_1&P_2BQ_2\end{pmatrix}=\begin{pmatrix}diag(I_a,O)&O\\P_2CQ_2&diag(I_b,O)\end{pmatrix}$$

存在 $a+b$ 阶子式 $\begin{vmatrix}I_a&O\\ *&I_b\end{vmatrix}=1\neq 0$, 因此

$$\operatorname{rank}\begin{pmatrix}A&O\\C&B\end{pmatrix}=\operatorname{rank}(S)\geqslant a+b=\operatorname{rank}(A)+\operatorname{rank}(B).$$

□

引理 2: $\operatorname{rank}(A)+\operatorname{rank}B-n\leqslant \operatorname{rank}(AB)$. (Sylvecter 秩不等式)

Proof. 即

$$\operatorname{rank}(A)+\operatorname{rank}(B)\leqslant \operatorname{rank}(I_n)+\operatorname{rank}(AB)=\operatorname{rank}\begin{pmatrix}AB&O\\O&I_n\end{pmatrix}$$

进行初等变换

$$\begin{pmatrix}I_n&A\\O&I_n\end{pmatrix}\begin{pmatrix}AB&O\\O&I_n\end{pmatrix}\begin{pmatrix}I_n&O\\-B&I_n\end{pmatrix}=\begin{pmatrix}O&A\\-B&I_n\end{pmatrix},\quad \begin{pmatrix}O&A\\-B&I_n\end{pmatrix}\begin{pmatrix}O&-I_n\\I_n&O\end{pmatrix}=\begin{pmatrix}A&O\\I_n&B\end{pmatrix}$$

即有

$$\operatorname{rank}\begin{pmatrix}AB&O\\O&I_n\end{pmatrix}=\operatorname{rank}\begin{pmatrix}A&O\\I_n&B\end{pmatrix}\geqslant \operatorname{rank}(A)+\operatorname{rank}(B).$$

□

(1) $\operatorname{rank}(A)=n$

$$\operatorname{rank}(A^*)=\operatorname{rank}(A\cdot A^*)=\operatorname{rank}(\det(A)I_n)=n.$$

(2) $\text{rank}(A) = n - 1$

$$\left\{\begin{array}{ll} A \cdot A^* = O & \Rightarrow \text{rank}(A^*) + \text{rank}(A) \leqslant n \\ & \\ \text{rank}(A) = n-1 & \Rightarrow \exists\ A_{ij} \neq 0, \text{rank}(A^*) > 0 \end{array}\right. \Rightarrow 0 < \text{rank}(A^*) \leqslant n - \text{rank}(A) = 1.$$

显然有

$$\text{rank}(A^*) = 1.$$

(3) $\text{rank}(A) \leqslant n - 2$

$$\text{rank}(A) \leqslant n-2 \Rightarrow \forall\ i, j, A_{ij} = (-1)^{i+j} M_{ij} = 0 \Rightarrow \text{rank}(A^*) = \text{rank}(O) = 0.$$

□

41. 证明下列关于秩的等式和不等式: (其中 A, B, C 是使运算有意义的矩阵)

(1) $\max\left(\text{rank}(A), \text{rank}(B), \text{rank}(A+B)\right) \leqslant \text{rank}\begin{pmatrix} A & B \end{pmatrix}$;

(2) $\text{rank}\begin{pmatrix} A & B \end{pmatrix} \leqslant \text{rank}(A) + \text{rank}(B)$;

(3) $\text{rank}\begin{pmatrix} A & C \\ O & B \end{pmatrix} \geqslant \text{rank}(A) + \text{rank}(B)$.

Proof.

(1) 任意 A, B 的非零子式均为 $\begin{pmatrix} A & B \end{pmatrix}$ 的非零子式, 即

$$\text{rank}\begin{pmatrix} A & B \end{pmatrix} \geqslant \text{rank}(A), \quad \text{rank}\begin{pmatrix} A & B \end{pmatrix} \geqslant \text{rank}(B)$$

记 $A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{pmatrix}, B = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix}$. 则

$$\text{rank}\begin{pmatrix} A & B \end{pmatrix} = \text{rank}\left\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\right\} \geqslant \text{rank}\left\{\boldsymbol{a}_1 + \boldsymbol{b}_1, \ldots, \boldsymbol{a}_n + \boldsymbol{b}_n\right\} = \text{rank}(A+B).$$

综上

$$\max\left(\text{rank}(A), \text{rank}(B), \text{rank}(A+B)\right) \leqslant \text{rank}\begin{pmatrix} A & B \end{pmatrix}.$$

(2) 记 $a = \text{rank}(A), b = \text{rank}(B), A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m \end{pmatrix}, B = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix}$.

$$\forall\left\{\boldsymbol{a}_{i1}, \boldsymbol{a}_{i2}, \ldots, \boldsymbol{a}_{ia}\right\} \subseteq \left\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\right\},\ \exists\ x_1, \ldots, x_a, x_1 \cdot \boldsymbol{a}_{i1} + \cdots + x_a \cdot \boldsymbol{a}_{ia} = 0.$$

$$\forall\left\{\boldsymbol{b}_{j1}, \boldsymbol{b}_{j2}, \ldots, \boldsymbol{b}_{jb}\right\} \subseteq \left\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\right\},\ \exists\ y_1, \ldots, y_b, y_1 \cdot \boldsymbol{b}_{j1} + \cdots + y_b \cdot \boldsymbol{b}_{jb} = 0.$$

$\forall\left\{\boldsymbol{a}_{i1}, \ldots, \boldsymbol{a}_{ia}, \boldsymbol{b}_{j1}, \ldots, \boldsymbol{b}_{jb}\right\} \subseteq \left\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\right\}$

$$\exists\ x_1, \ldots, x_a, y_1, \ldots, y_b, x_1 \cdot \boldsymbol{a}_{i1} + \cdots + x_a \cdot \boldsymbol{a}_{ia} + y_1 \cdot \boldsymbol{b}_{j1} + \cdots + y_b \cdot \boldsymbol{b}_{jb} = 0.$$

即

$$\text{rank}\begin{pmatrix} A & B \end{pmatrix} = \text{rank}\left\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\right\} \leqslant a + b = \text{rank}(A) + \text{rank}(B).$$

(3) 记 $a = \text{rank}(A), b = \text{rank}(B)$, 则存在可逆方阵 P_1, Q_1, P_2, Q_2 使

$$P_1 A Q_1 = \begin{pmatrix} I_a & O \\ O & O \end{pmatrix}, \quad P_2 B Q_2 = \begin{pmatrix} I_b & O \\ O & O \end{pmatrix}$$

取可逆方阵

$$P = \begin{pmatrix} P_1 & O \\ O & P_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix}$$

则

$$S = P\begin{pmatrix} A & C \\ O & B \end{pmatrix}Q = \begin{pmatrix} P_1 A Q_1 & P_1 C Q_2 \\ O & P_2 B Q_2 \end{pmatrix} = \begin{pmatrix} diag(I_a, O) & P_1 C Q_2 \\ O & diag(I_b, O) \end{pmatrix}$$

存在 $a+b$ 阶子式 $\begin{vmatrix} I_a & * \\ O & I_b \end{vmatrix} = 1 \neq 0$, 因此

$$\operatorname{rank}\begin{pmatrix} A & C \\ O & B \end{pmatrix} = \operatorname{rank}(S) \geqslant a+b = \operatorname{rank}(A) + \operatorname{rank}(B).$$

□

43. 设 n 阶方阵 A 满足 $A^2 = I$, 证明:$\operatorname{rank}(I+A) + \operatorname{rank}(I-A) = n$.

Proof. 进行初等变换

$$\begin{pmatrix} I+A & O \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} I+A & I-A \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} I+A & 2I \\ O & I-A \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} O & 2I \\ -\frac{1}{2}(I-A)(I+A) & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} O & 2I \\ O & I-A \end{pmatrix} \longrightarrow \begin{pmatrix} O & 2I \\ O & O \end{pmatrix}$$

即

$$\operatorname{rank}(I+A) + \operatorname{rank}(I-A) = \operatorname{rank}\begin{pmatrix} I+A & O \\ O & I-A \end{pmatrix} = \operatorname{rank}\begin{pmatrix} O & 2I \\ O & O \end{pmatrix} = n.$$

□