线性代数 homework (第三周)

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1 周二

1.1 习题四 23

23. 计算下列行列式:

(1)

$$\begin{vmatrix} 1 & 0 & 1 & -4 \\ -1 & -3 & -4 & -2 \\ 2 & -1 & 4 & 4 \\ 2 & 3 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & -3 & -3 & -6 \\ 2 & -1 & 2 & 12 \\ 2 & 3 & -5 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ 2 & -1 & 3 & 14 \\ 2 & 3 & -8 & 4 \end{vmatrix} = -372.$$

(2)

$$\begin{vmatrix} 1 & 4 & -1 & -1 \\ 1 & -2 & -1 & 1 \\ -3 & 3 & -4 & -2 \\ 0 & 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -6 & 0 & 2 \\ -3 & 15 & -7 & -5 \\ 0 & 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ -3 & 0 & -7 & -5 \\ 0 & -2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ 0 & -7 & -5 \\ -2 & -1 & -1 \end{vmatrix} = 14.$$

(3)

$$\begin{vmatrix} x+a & x+b & x+c \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix} = \begin{vmatrix} x+a & x+b & x+c \\ y-x & y-x & y-x \\ z-x & z-x & z-x \end{vmatrix} = \begin{vmatrix} x+a & b-a & c-a \\ y-x & 0 & 0 \\ z-x & 0 & 0 \end{vmatrix} = 0.$$

(7)

$$\text{记} D_n = \begin{vmatrix} a_1 & & & & & b_1 \\ & \ddots & & & \iddots & \\ & & a_n & b_n & & \\ & & c_n & d_n & & \\ & \iddots & & & \ddots & \\ c_1 & & & & & d_1 \end{vmatrix} \Rightarrow \begin{cases} D_1 & = a_1d_1 - b_1c_1, \\ D_2 & = (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \end{cases}.$$

假设 $D_n = \prod\limits_{i=1}^{n}(a_id_i - b_ic_i)$.

(a) $n = 1$ 成立.

(b) 假设 $D_n = \prod\limits_{i=1}^{n}(a_id_i - b_ic_i)$ 对 $n = k$ 成立, 则

$$D_{k+1} = (a_1d_1 - b_1c_1) \begin{vmatrix} a_2 & & & & & b_2 \\ & \ddots & & & \iddots & \\ & & a_{k+1} & b_{k+1} & & \\ & & c_{k+1} & d_{k+1} & & \\ & \iddots & & & \ddots & \\ c_2 & & & & & d_2 \end{vmatrix} = \prod_{i=1}^{k+1}(a_id_i - b_ic_i)$$

$n = k + 1$ 时原式仍成立.

综合 $(a)(b)$ 即证 $D_n=\prod_{i=1}^{n}(a_id_i-b_ic_i)$ 对任意 $n\in\mathbb{N}^*$ 成立.

即 $D_n=\prod_{i=1}^{n}(a_id_i-b_ic_i)$.

1.2 补充题

1. 对于 a,b,c,d 为不全为 0 的实数

$$\begin{pmatrix} a & b & c & d\\ d & a & b & c\\ c & d & a & b\\ b & c & d & a \end{pmatrix}$$

求 M 的行列式.

做多项式 $f(x)=a+bx+cx^2+dx^3$, 令 $\omega_k=\frac{2k\pi i}{4}=\frac{k\pi i}{2}$.

$$\begin{aligned}\begin{vmatrix} a & b & c & d\\ d & a & b & c\\ c & d & a & b\\ b & c & d & a \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 & 1\\ \omega_1 & \omega_1^2 & \omega_1^3 & \omega_1^4\\ \omega_2 & \omega_2^2 & \omega_2^3 & \omega_2^4\\ \omega_3 & \omega_3^2 & \omega_3^3 & \omega_3^4 \end{vmatrix} &= \det\begin{pmatrix} a & b & c & d\\ d & a & b & c\\ c & d & a & b\\ b & c & d & a \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1\\ \omega_1 & \omega_2 & \omega_3 & \omega_4\\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_4^2\\ \omega_1^3 & \omega_2^3 & \omega_3^3 & \omega_4^3 \end{pmatrix}\\ &= \det\begin{pmatrix} f(\omega_1) & f(\omega_2) & f(\omega_3) & f(\omega_4)\\ \omega_1 f(\omega_1) & \omega_2 f(\omega_2) & \omega_3 f(\omega_3) & \omega_4 f(\omega_4)\\ \omega_1^2 f(\omega_1) & \omega_2^2 f(\omega_2) & \omega_3^2 f(\omega_3) & \omega_4^2 f(\omega_4)\\ \omega_1^3 f(\omega_1) & \omega_2^3 f(\omega_2) & \omega_3^3 f(\omega_3) & \omega_4^3 f(\omega_4) \end{pmatrix}\\ &= f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)\begin{vmatrix} 1 & 1 & 1 & 1\\ \omega_1 & \omega_1^2 & \omega_1^3 & \omega_1^4\\ \omega_2 & \omega_2^2 & \omega_2^3 & \omega_2^4\\ \omega_3 & \omega_3^2 & \omega_3^3 & \omega_3^4 \end{vmatrix}\end{aligned}$$

又 ω_k 各不相同, $\begin{vmatrix} 1 & 1 & 1 & 1\\ \omega_1 & \omega_2 & \omega_3 & \omega_4\\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_4^2\\ \omega_1^3 & \omega_2^3 & \omega_3^3 & \omega_4^3 \end{vmatrix}\neq 0,\Rightarrow \begin{vmatrix} a & b & c & d\\ d & a & b & c\\ c & d & a & b\\ b & c & d & a \end{vmatrix}=f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)$.

即:

$$\begin{aligned}\begin{vmatrix} a & b & c & d\\ d & a & b & c\\ c & d & a & b\\ b & c & d & a \end{vmatrix} &= f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)\\ &= (a+b+c+d)(a-b+c-d)(a+bi-c-di)(a-bi-c+di)\\ &= [(a+c)^2-(b+d)^2][(a-c)^2+(b-d)^2].\end{aligned}$$

2. 写出行列式 M 中含有 x^4 和 x^3 的项.

$$M=\begin{vmatrix} x & 1 & 2 & 3\\ x & x & 1 & 2\\ 2 & 3 & x & 1\\ x & 2 & 3 & x \end{vmatrix}.$$

$$M=x\cdot\begin{vmatrix} x & 1 & 2\\ 3 & x & 1\\ 2 & 3 & x \end{vmatrix}-x\cdot\begin{vmatrix} 1 & 2 & 3\\ 3 & x & 1\\ 2 & 3 & x \end{vmatrix}+2\cdot\begin{vmatrix} 1 & 2 & 3\\ x & 1 & 2\\ 2 & 3 & x \end{vmatrix}-x\cdot\begin{vmatrix} 1 & 2 & 3\\ x & 1 & 2\\ 3 & x & 1 \end{vmatrix}$$

(1) x^4: 仅第一部分有 x^4, 易得 M 中含有 x^4 的项为 x^4.

(2) x^3: 仅第二、四部分有 x^3, 易得 M 中含有 x^3 的项为 $-x^3-3x^3=-4x^3$.

3. 计算行列式 (参考第四题)

$$M=\begin{vmatrix}1&0&1&0&0\\-3&1&3&1&0\\2&-3&2&3&1\\0&-2&0&2&3\\0&0&0&0&2\end{vmatrix}$$

记 $f(x)=x^3-3x^2+2x, g(x)=x^2+3x+2$.

$$\text{解}f(x)=0\Rightarrow\begin{cases}x_1&=0,\\x_2&=1,\\x_3&=2.\end{cases}\Rightarrow M=1^2\cdot g(x_1)\cdot g(x_2)\cdot g(x_3)=144.$$

4. 对于多项式 $A,B\in\mathbb{C}[x]$ 有如下的展开

$$A=a_0x^d+a_1x^{d-1}+\cdots+a_d$$
$$B=b_0x^e+b_1x^{e-1}+\cdots+b_e$$

并且它们的根分别为 $\lambda_1,\ldots,\lambda_d$ 与 $\mu_1,\ldots,\mu_e$. 我们定义它们的 Sylvester 行列式为

$$M=\begin{pmatrix}a_0&0&\cdots&0&b_0&0&\cdots&0\\a_1&a_0&\cdots&0&b_1&b_0&\cdots&0\\a_2&a_1&\ddots&0&b_2&b_1&\ddots&0\\\vdots&\vdots&\ddots&a_0&\vdots&\vdots&\ddots&b_0\\a_d&a_{d-1}&\cdots&\vdots&b_e&b_{e-1}&\cdots&\vdots\\0&a_d&\ddots&\vdots&0&b_e&\ddots&\vdots\\\vdots&\vdots&\ddots&a_{d-1}&\vdots&\vdots&\ddots&b_{e-1}\\0&0&\cdots&a_d&0&0&\cdots&b_e\end{pmatrix}$$

这对多项式的结式 (Resultant) 定义为他们的 Sylvester 多项式的行列式，即

$$res(A,B)=\det M.$$

请验证有如下的关系:

$$res(A,B)={a_0}^e{b_0}^d\prod_{\substack{1\le i\le d\\1\le j\le e}}(\lambda_i-\mu_j)$$

Remarque. 根据结果我们知道，结式可以用来判断两个多项式是否有公共的根，进而可以知道两个多项式是否互素。除此以外，结式在数论、代数几何、交换代数中都有广泛的应用。

Remarque. 事实: 当 A 和 B 互素时，M 可逆。

Remarque. 当 $d=3,e=2$ 时，M 如下

$$\begin{pmatrix}a_0&0&b_0&0&0\\a_1&a_0&b_1&b_0&0\\a_2&a_1&b_2&b_1&b_0\\a_3&a_2&0&b_2&b_1\\0&a_3&0&0&b_2\end{pmatrix}.$$

Proof. 1.

$$M^T=N=\begin{pmatrix}a_0&a_1&\cdots&a_d&&&&\\&a_0&a_1&\cdots&a_d&&&\\&&\ddots&\ddots&&&\ddots&\\&&&a_0&a_1&\cdots&&a_d\\b_0&b_1&\cdots&b_e&&&&\\&b_0&b_1&\cdots&b_e&&&\\&&\ddots&\ddots&&&\ddots&\\&&&b_0&a_1&\cdots&&b_e\end{pmatrix}$$

$$
记N(B)=\begin{pmatrix}
a_0 & a_1 & \cdots & a_d & & & \\
 & a_0 & a_1 & \cdots & a_d & & \\
 & & \ddots & \ddots & & \ddots & \\
 & & & a_0 & a_1 & \cdots & a_d \\
b_0 & b_1 & \cdots & b_e-B & & & \\
 & b_0 & b_1 & \cdots & b_e-B & & \\
 & & \ddots & \ddots & & \ddots & \\
 & & & b_0 & a_1 & \cdots & b_e-B
\end{pmatrix}.
$$

记 $x^i=x_i, l=d+e-1$, 考虑关于 $x_l, x_{l-1}, \ldots, x_0$ 的线性方程组:

$$
(1)\begin{cases}
a_0x_l \quad + a_1x_{l-1} + \cdots + a_dx_{e-1} &= 0 \\
a_0x_{l-1} + a_1x_{l-2} + \cdots + a_dx_{e-2} &= 0 \\
\qquad\qquad \vdots & \\
a_0x_d \quad + a_1x_{d-1} + \cdots + a_dx_0 &= 0
\end{cases}
$$

$$
(2)\begin{cases}
b_0x_l \quad + b_1x_{l-1} + \cdots + (b_e-B)x_{d-1} &= 0 \\
b_0x_{l-1} + b_1x_{l-2} + \cdots + (b_e-B)x_{d-2} &= 0 \\
\qquad\qquad \vdots & \\
b_0x_e \quad + b_1x_{e-1} + \cdots + (b_e-B)x_0 &= 0
\end{cases}
$$

(1) 对于 A 的解 $\lambda_1, \ldots, \lambda_d$ 成立;(2) 对于 B 为恒等式.
又 $x_0=1$,A 的解 $\lambda_1, \ldots, \lambda_d$ 对应 $N(B)X=0$ 的 d 组非平凡解, 即 $rank\ N(B)<d+e, \det N(B)=0$:

$$
\det N(B)=c_0B^d+c_1B^{d-1}+\cdots+c_d=0
$$

$\det N(B)$ 作为 B 的 d 次多项式, 它的 d 个根为

$$
B(\lambda_1), B(\lambda_2), \ldots, B(\lambda_d)
$$

在上述 $\det N(B)$ 的展开式中, 显然有

$$
c_0=(-1)^d{a_0}^e, res(A,B)=\det N(0)=c_d
$$

由根和系数的关系, 得:

$$
\prod_{i=1}^{d} B(\lambda_i)=(-1)^d\frac{c_d}{c_0}
$$

即:

$$
res(A,B)={a_0}^e\prod_{i=1}^{d} B(\lambda_i)
$$

又 $B=b_0\prod_{j=1}^{e}(x-\mu_j)$:

$$
res(A,B)={a_0}^e{b_0}^d\prod_{i=1}^{d}\prod_{j=1}^{e}(\lambda_i-\mu_j)={a_0}^e{b_0}^d\prod_{\substack{1\le i\le d\\ 1\le j\le e}}(\lambda_i-\mu_j)
$$

$\square$

Proof. 2. 记 $f(x) = m_0x^e + m_1x^{e-1} + m_e, g(x) = n_0x^d + n_1x^{d-1} + n_d$. 解:

$$f(x)A + g(x)B = 1 \Leftrightarrow \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_d & a_{d-1} & \cdots & \vdots & b_e & b_{e-1} & \cdots & \vdots \\ 0 & a_d & \ddots & \vdots & 0 & b_e & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{d-1} & \vdots & \vdots & \ddots & b_{e-1} \\ 0 & 0 & \cdots & a_d & 0 & 0 & \cdots & b_e \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_e \\ n_0 \\ n_1 \\ \vdots \\ n_d \end{pmatrix} = MX = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} f(x)A + g(x)B = 1\text{有解} &\Leftrightarrow f(x), g(x)\text{互素} \\ MX = (0, 0, \ldots, 0, 1)^T\text{有解} &\Leftrightarrow \det(M) \neq 0 \\ f(x), g(x)\text{互素} &\Leftrightarrow \det(M) \neq 0 \end{aligned}$$

即:

$$\det(M) = 0 \Leftrightarrow \exists 1 \leq i \leq d, 1 \leq j \leq e, \lambda_i = \mu_j$$

可得:

$$\det(M) = \xi \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq e}} (\lambda_i - \mu_j)(\xi\text{为系数})$$

取 $\lambda_1 = \cdots = \lambda_d = 0$, 则 $a_0 \neq 0, a_1 = \cdots = a_d = 0$

$$\det(M) = \begin{vmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ 0 & 0 & \cdots & \vdots & b_e & b_{e-1} & \cdots & \vdots \\ 0 & 0 & \ddots & \vdots & 0 & b_e & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & b_{e-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & b_e \end{vmatrix} = {a_0}^e {b_e}^d = {a_0}^e((-1)^e b_0 \cdot \mu_1 \cdot \mu_2 \cdots \mu_e)^d$$

又

$$\begin{aligned} \det(M) &= {a_0}^e((-1)^e b_0 \cdot \mu_1 \cdot \mu_2 \cdots \mu_e)^d \\ &= {a_0}^e {b_0}^d (-1)^{de} (\mu_1 \cdot \mu_2 \cdots \mu_e)^d \\ &= {a_0}^e {b_0}^d \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq e}} \lambda_i - \mu_j \end{aligned}$$

可推出 $\xi = {a_0}^e {b_0}^d$, 即证 $res(A, B) = {a_0}^e {b_0}^d \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq e}} (\lambda_i - \mu_j)$. □

2 周四

2.1 习题四

21. 求以下排列的逆序数，并指出其奇偶性:

(1) (6,8,1,4,7,5,3,2,9)　逆序数 19, 奇

(2) (6,4,2,1,9,7,3,5,8)　逆序数 15, 奇

(3) (7,5,2,3,9,8,1,6,4)　逆序数 20, 偶

23. (4)

$$\begin{vmatrix} & & & A_1 \\ & & A_2 & \\ & \iddots & & \\ A_k & & & \end{vmatrix} \xrightarrow{n_1 \sum\limits_{i=2}^{k} n_i \text{次相邻对调}} (-1)^{\sum\limits_{i=2}^{k} n_1 n_i} \begin{vmatrix} A_1 & & & \\ & & & A_2 \\ & & \iddots & \\ & A_k & & \end{vmatrix}$$

即:

$$\begin{vmatrix} & & & A_1 \\ & & A_2 & \\ & \iddots & & \\ A_k & & & \end{vmatrix} = (-1)^{\sum\limits_{i<j} n_i n_j} \begin{vmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{vmatrix} = (-1)^{\sum\limits_{i<j} n_i n_j} \det(A_1)\cdots\det(A_k)$$

(6)

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+a_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1+a_1 & 1 & \cdots & 1 \\ 0 & 1 & 1+a_2 & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & 1 \\ 0 & 1 & \cdots & 1 & 1+a_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \ddots & \vdots \\ -1 & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & a_n \end{vmatrix}$$

$$= \begin{vmatrix} 1+\sum\limits_{i=1}^{n} \frac{1}{a_i} & 0 & 0 & \cdots & 0 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \ddots & \vdots \\ -1 & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & a_n \end{vmatrix} = \left(1+\sum_{i=1}^{n} \frac{1}{a_i}\right) \prod_{j=1}^{n} a_j.$$

(8) (a) $n=1$

$$\begin{vmatrix} a_1 - b_1 \end{vmatrix} = a_1 - b_1.$$

(b) $n=2$

$$\begin{vmatrix} a_1-b_1 & a_1-b_2 \\ a_2-b_1 & a_2-b_2 \end{vmatrix} = (a_1-b_1)(a_2-b_2)-(a_1-b_2)(a_2-b_1).$$

(c) $n \geq 3$

$$\begin{vmatrix} a_1-b_1 & a_1-b_2 & \cdots & a_1-b_n \\ a_2-b_1 & a_2-b_2 & \cdots & a_2-b_n \\ \vdots & \vdots & & \vdots \\ a_n-b_1 & a_n-b_2 & \cdots & a_n-b_n \end{vmatrix} = \begin{vmatrix} a_1-b_1 & b_1-b_2 & \cdots & b_1-b_n \\ a_2-b_1 & b_1-b_2 & \cdots & b_1-b_n \\ \vdots & \vdots & & \vdots \\ a_n-b_1 & b_1-b_2 & \cdots & b_1-b_n \end{vmatrix} = 0.$$

2.2 补充题

1. 确定 i, j, 使得 $(1245i6j97)$ 分别为奇排列、偶排列。

(1) $i=3, j=8$, 逆序数为 4, 偶排列;

(2) $i=8, j=3$, 逆序数为 7, 奇排列.

2. 将 λ 作为变量，$a_{ij}(1 \leq i,j \leq n)$ 作为常数，则

$$f_n(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

为 λ 的多项式，求这个多项式的 n 次项和 $n-1$ 次项。

$$
\text{记} f_n(\lambda) = (\lambda - a_{nn}) f_{n-1}(\lambda) + a_{12} \begin{vmatrix} -a_{21} & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & \lambda - a_{33} & \cdots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \cdots & \lambda - a_{nn} \end{vmatrix} + \cdots .
$$

显然, 后 $n-1$ 个项中 λ 的次数均不超过 $n-2$, 仅第一项中含 n 次项和 $n-1$ 次项, 即在 $f_{n-1}(\lambda)$ 中找 $n-1$ 次项和 $n-2$ 次项.

依此类推, 可得 $f(x)$ 的 n 次项和 $n-1$ 次项均在 $\prod_{i=1}^{n}(\lambda - a_{ii})$ 里.

$$
n\text{次项} : \lambda^n;
$$

$$
n-1\text{次项} : -\sum_{i=1}^{n} a_{ii} \lambda^{n-1}.
$$