线性代数 homework (第三周)

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1 周二

1.1 习题五

17. 设向量组 $\alpha_1, \ldots, \alpha_r$ 线性无关, 且 $\alpha_1, \ldots, \alpha_r$ 可以由向量组 $\beta_1, \ldots, \beta_r$ 线性表示, 则 $\beta_1, \ldots, \beta_r$ 也线性无关.

    Proof.
    $$r = \operatorname{rank}(\alpha_1, \ldots, \alpha_r) \leqslant \operatorname{rank}(\beta_1, \ldots, \beta_r) \leqslant r \Rightarrow \operatorname{rank}(\beta_1, \ldots, \beta_r) = r.$$
    □

19. 求下列向量组的极大无关组与秩:

    (3) $\overrightarrow{a_1} = (0,1,2,3)$, $\overrightarrow{a_2} = (1,2,3,4)$, $\overrightarrow{a_3} = (3,4,5,6)$, $\overrightarrow{a_4} = (4,3,2,1)$, $\overrightarrow{a_5} = (6,5,4,3)$.

解：记极大无关组为 S.

(a) $S_1 = \{\alpha_1\}$.

(b) 假设存在 x 使得 $x \cdot \alpha_1 = \alpha_2$, 则 $0 \cdot x = 1$, 显然无解, 故取 $S_2 = S_1 \cup \{\alpha_2\} = \{\alpha_1, \alpha_2\}$.

(c) 假设存在 x, y 使得 $x \cdot \alpha_1 + y \cdot \alpha_2 = \alpha_3$, 则有

$$\begin{cases} y &= 3, \\ x + 2y &= 4, \\ 2x + 3y &= 5, \\ 3x + 4y &= 6. \end{cases} \Rightarrow \begin{cases} x &= -2 \\ y &= 3. \end{cases} \Rightarrow S_3 = S_2 = \{\alpha_1, \alpha_2\}.$$

(d) 假设存在 x, y 使得 $x \cdot \alpha_1 + y \cdot \alpha_2 = \alpha_4$, 则有

$$\begin{cases} y &= 4, \\ x + 2y &= 3, \\ 2x + 3y &= 2, \\ 3x + 4y &= 1. \end{cases} \Rightarrow \begin{cases} x &= -5 \\ y &= 4. \end{cases} \Rightarrow S_4 = S_3 = \{\alpha_1, \alpha_2\}.$$

(e) 假设存在 x, y 使得 $x \cdot \alpha_1 + y \cdot \alpha_2 = \alpha_5$, 则有

$$\begin{cases} y &= 6, \\ x + 2y &= 5, \\ 2x + 3y &= 4, \\ 3x + 4y &= 3. \end{cases} \Rightarrow \begin{cases} x &= -7 \\ y &= 6. \end{cases} \Rightarrow S_5 = S_4 = \{\alpha_1, \alpha_2\}.$$

综上, 极大无关组 $S = \{\alpha_1, \alpha_2\}, \operatorname{rank}(\alpha_1, \ldots, \alpha_6) = 2$.

22. 设向量组 $\alpha_1, \ldots, \alpha_m$ 的秩为 r, 则其中任何 r 个线性无关的向量构成 $\alpha_1, \ldots, \alpha_m$ 的极大无关组.

Proof. 设有任何 r 个线性无关的向量 $\alpha_{i1},\dots,\alpha_{ir}$ 构成一个向量组 S_1,$\forall\ \alpha_j \notin S_1, S_2 = S_1 \cup \{\alpha_j\}$.

$$r+1 > r = \operatorname{rank}(\alpha_1,\dots,\alpha_m) \Rightarrow S_2\text{线性相关, 又}S_1\text{线性无关} \Rightarrow S_1\text{构成}\alpha_1,\dots,\alpha_m\text{的极大无关组.}$$

□

23. 设向量组 $\alpha_1,\dots,\alpha_m$ 的秩为 r, 如果 $\alpha_1,\dots,\alpha_m$ 可以由它的 r 个向量线性表示, 则这 r 个向量构成 $\alpha_1,\dots,\alpha_m$ 的极大无关组.

Proof. 将这 r 个向量记为 $\alpha_{i1},\dots,\alpha_{ir}$.

$$r = \operatorname{rank}(\alpha_1,\dots,\alpha_m) \leqslant \operatorname{rank}(\alpha_{i1},\dots,\alpha_{ir}) \leqslant r \Rightarrow \operatorname{rank}(\alpha_{i1},\dots,\alpha_{ir}) = r$$
$$\xrightarrow{\text{T22 结论}} \operatorname{rank}(\alpha_{i1},\dots,\alpha_{ir})\text{构成}\alpha_1,\dots,\alpha_m\text{的极大无关组.}$$

□

2 周四

2.1 习题五