

线性代数 homework (第五周)

PB20000113 孔浩宇

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1 周二

1.1 习题四

36. (1) 做初等变换

$$\begin{pmatrix} 3 & 2 & -1 & 9 \\ -2 & 1 & -4 & 2 \\ -1 & -2 & 3 & -2 \\ 3 & 2 & -1 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -4 & -1 & 9 \\ 0 & 5 & -10 & 6 \\ -1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -1 & 9 \\ 0 & 5 & -10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 35 & 0 & -84 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -84 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

即

$$\text{rank} \begin{pmatrix} 3 & 2 & -1 & 9 \\ -2 & 1 & -4 & 2 \\ -1 & -2 & 3 & -2 \\ 3 & 2 & -1 & 9 \end{pmatrix} = 3.$$

37. 对于 a, b 的各种取值, 讨论实矩阵 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix}$ 的秩.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} \xrightarrow[-3r_1 \rightarrow r_3]{-2r_1 \rightarrow r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & a-6 \\ 0 & b-6 & 0 \end{pmatrix} \xrightarrow[-3c_1 \rightarrow c_3]{-2c_1 \rightarrow c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & a-6 \\ 0 & b-6 & 0 \end{pmatrix}$$

即

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \text{rank} \begin{pmatrix} 0 & a-6 \\ b-6 & 0 \end{pmatrix}.$$

(1) $a-6=0, b-6=0$

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \text{rank} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 1. \quad (a=6, b=6)$$

(2) $a-6=0, b-0 \neq 0$

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \text{rank} \begin{pmatrix} 0 & 0 \\ b-6 & 0 \end{pmatrix} = 2. \quad (a=6, b \neq 6)$$

(3) $a - 6 \neq 0, b - 6 = 0$

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \text{rank} \begin{pmatrix} 0 & a-6 \\ 0 & 0 \end{pmatrix} = 2. \quad (a \neq 6, b = 6)$$

(4) $a - 6 \neq 0, b - 6 \neq 0$

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ 3 & b & 9 \end{pmatrix} = 1 + \text{rank} \begin{pmatrix} 0 & a-6 \\ b-6 & 0 \end{pmatrix} = 3. \quad (a \neq 6, b \neq 6)$$

39. 设 A 是 n 阶矩阵, 证明: $\text{rank}(A^*) = \begin{cases} n, & \text{rank}(A) = n, \\ 1, & \text{rank}(A) = n - 1, \\ 0, & \text{rank}(A) \leq n - 2. \end{cases}$

Proof.

(0) 引理 1: $\text{rank} \begin{pmatrix} A & O \\ C & B \end{pmatrix} \geq \text{rank}(A) + \text{rank}(B).$

Proof. 记 $a = \text{rank}(A), b = \text{rank}(B)$, 则存在可逆方阵 P_1, Q_1, P_2, Q_2 使

$$P_1 A Q_1 = \begin{pmatrix} I_a & O \\ O & O \end{pmatrix}, \quad P_2 B Q_2 = \begin{pmatrix} I_b & O \\ O & O \end{pmatrix}$$

取可逆方阵

$$P = \begin{pmatrix} P_1 & O \\ O & P_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix}$$

则

$$S = P \begin{pmatrix} A & O \\ C & B \end{pmatrix} Q = \begin{pmatrix} P_1 A Q_1 & O \\ P_2 C Q_1 & P_2 B Q_2 \end{pmatrix} = \begin{pmatrix} \text{diag}(I_a, O) & O \\ P_2 C Q_2 & \text{diag}(I_b, O) \end{pmatrix}$$

存在 $a + b$ 阶子式 $\begin{vmatrix} I_a & O \\ * & I_b \end{vmatrix} = 1 \neq 0$, 因此

$$\text{rank} \begin{pmatrix} A & O \\ C & B \end{pmatrix} = \text{rank}(S) \geq a + b = \text{rank}(A) + \text{rank}(B).$$

□

引理 2: $\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB)$. (Sylvester 秩不等式)

Proof. 即

$$\text{rank}(A) + \text{rank}(B) \leq \text{rank}(I_n) + \text{rank}(AB) = \text{rank} \begin{pmatrix} AB & O \\ O & I_n \end{pmatrix}$$

进行初等变换

$$\begin{pmatrix} I_n & A \\ O & I_n \end{pmatrix} \begin{pmatrix} AB & O \\ O & I_n \end{pmatrix} \begin{pmatrix} I_n & O \\ -B & I_n \end{pmatrix} = \begin{pmatrix} O & A \\ -B & I_n \end{pmatrix}, \quad \begin{pmatrix} O & A \\ -B & I_n \end{pmatrix} \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix} = \begin{pmatrix} A & O \\ I_n & B \end{pmatrix}$$

即有

$$\text{rank} \begin{pmatrix} AB & O \\ O & I_n \end{pmatrix} = \text{rank} \begin{pmatrix} A & O \\ I_n & B \end{pmatrix} \geq \text{rank}(A) + \text{rank}(B).$$

□

(1) $\text{rank}(A) = n$

$$\text{rank}(A^*) = \text{rank}(A \cdot A^*) = \text{rank}(\det(A)I_n) = n.$$

$$(2) \text{rank}(A) = n - 1$$

$$\begin{cases} A \cdot A^* = O & \Rightarrow \text{rank}(A^*) + \text{rank}(A) \leq n \\ \text{rank}(A) = n - 1 & \Rightarrow \exists A_{ij} \neq 0, \text{rank}(A^*) > 0 \end{cases} \Rightarrow 0 < \text{rank}(A^*) \leq n - \text{rank}(A) = 1.$$

显然有

$$\text{rank}(A^*) = 1.$$

$$(3) \text{rank}(A) \leq n - 2$$

$$\text{rank}(A) \leq n - 2 \Rightarrow \forall i, j, A_{ij} = (-1)^{i+j} M_{ij} = 0 \Rightarrow \text{rank}(A^*) = \text{rank}(O) = 0.$$

□

41. 证明下列关于秩的等式和不等式: (其中 A, B, C 是使运算有意义的矩阵)

$$(1) \max(\text{rank}(A), \text{rank}(B), \text{rank}(A + B)) \leq \text{rank} \begin{pmatrix} A & B \end{pmatrix};$$

$$(2) \text{rank} \begin{pmatrix} A & B \end{pmatrix} \leq \text{rank}(A) + \text{rank}(B);$$

$$(3) \text{rank} \begin{pmatrix} A & C \\ O & B \end{pmatrix} \geq \text{rank}(A) + \text{rank}(B).$$

Proof.

(1) 任意 A, B 的非零子式均为 $\begin{pmatrix} A & B \end{pmatrix}$ 的非零子式, 即

$$\text{rank} \begin{pmatrix} A & B \end{pmatrix} \geq \text{rank}(A), \quad \text{rank} \begin{pmatrix} A & B \end{pmatrix} \geq \text{rank}(B)$$

记 $A = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n), B = (\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n)$. 则

$$\text{rank} \begin{pmatrix} A & B \end{pmatrix} = \text{rank} \{\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b}_1, \dots, \mathbf{b}_n\} \geq \text{rank} \{\mathbf{a}_1 + \mathbf{b}_1, \dots, \mathbf{a}_n + \mathbf{b}_n\} = \text{rank}(A + B).$$

综上

$$\max(\text{rank}(A), \text{rank}(B), \text{rank}(A + B)) \leq \text{rank} \begin{pmatrix} A & B \end{pmatrix}.$$

(2) 记 $a = \text{rank}(A), b = \text{rank}(B), A = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_m), B = (\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n)$.

$$\forall \{\mathbf{a}_{i1}, \mathbf{a}_{i2}, \dots, \mathbf{a}_{ia}\} \subseteq \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}, \exists x_1, \dots, x_a, x_1 \cdot \mathbf{a}_{i1} + \dots + x_a \cdot \mathbf{a}_{ia} = 0.$$

$$\forall \{\mathbf{b}_{j1}, \mathbf{b}_{j2}, \dots, \mathbf{b}_{jb}\} \subseteq \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}, \exists y_1, \dots, y_b, y_1 \cdot \mathbf{b}_{j1} + \dots + y_b \cdot \mathbf{b}_{jb} = 0.$$

$$\forall \{\mathbf{a}_{i1}, \dots, \mathbf{a}_{ia}, \mathbf{b}_{j1}, \dots, \mathbf{b}_{jb}\} \subseteq \{\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{b}_1, \dots, \mathbf{b}_n\}$$

$$\exists x_1, \dots, x_a, y_1, \dots, y_b, x_1 \cdot \mathbf{a}_{i1} + \dots + x_a \cdot \mathbf{a}_{ia} + y_1 \cdot \mathbf{b}_{j1} + \dots + y_b \cdot \mathbf{b}_{jb} = 0.$$

即

$$\text{rank} \begin{pmatrix} A & B \end{pmatrix} = \text{rank} \{\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{b}_1, \dots, \mathbf{b}_n\} \leq a + b = \text{rank}(A) + \text{rank}(B).$$

(3) 记 $a = \text{rank}(A), b = \text{rank}(B)$, 则存在可逆方阵 P_1, Q_1, P_2, Q_2 使

$$P_1 A Q_1 = \begin{pmatrix} I_a & O \\ O & O \end{pmatrix}, \quad P_2 B Q_2 = \begin{pmatrix} I_b & O \\ O & O \end{pmatrix}$$

取可逆方阵

$$P = \begin{pmatrix} P_1 & O \\ O & P_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix}$$

则

$$S = P \begin{pmatrix} A & C \\ O & B \end{pmatrix} Q = \begin{pmatrix} P_1 A Q_1 & P_1 C Q_2 \\ O & P_2 B Q_2 \end{pmatrix} = \begin{pmatrix} \text{diag}(I_a, O) & P_1 C Q_2 \\ O & \text{diag}(I_b, O) \end{pmatrix}$$

存在 $a+b$ 阶子式 $\begin{vmatrix} I_a & * \\ O & I_b \end{vmatrix} = 1 \neq 0$, 因此

$$\text{rank} \begin{pmatrix} A & C \\ O & B \end{pmatrix} = \text{rank}(S) \geq a+b = \text{rank}(A) + \text{rank}(B).$$

□

43. 设 n 阶方阵 A 满足 $A^2 = I$, 证明: $\text{rank}(I+A) + \text{rank}(I-A) = n$.

Proof. 进行初等变换

$$\begin{aligned} \begin{pmatrix} I+A & O \\ O & I-A \end{pmatrix} &\rightarrow \begin{pmatrix} I+A & I-A \\ O & I-A \end{pmatrix} \rightarrow \begin{pmatrix} I+A & 2I \\ O & I-A \end{pmatrix} \\ &\rightarrow \begin{pmatrix} O & 2I \\ -\frac{1}{2}(I-A)(I+A) & I-A \end{pmatrix} \rightarrow \begin{pmatrix} O & 2I \\ O & I-A \end{pmatrix} \rightarrow \begin{pmatrix} O & 2I \\ O & O \end{pmatrix} \end{aligned}$$

即

$$\text{rank}(I+A) + \text{rank}(I-A) = \text{rank} \begin{pmatrix} I+A & O \\ O & I-A \end{pmatrix} = \text{rank} \begin{pmatrix} O & 2I \\ O & O \end{pmatrix} = n.$$

□

2 周四

2.1 习题五

3. 在 \mathbb{F}^4 中, 判断向量 \mathbf{b} 能否写成 $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ 的线性组合. 若能, 请写出一种表示方式.

- (1) $\mathbf{a}_1 = (-1, 3, 0, -5), \mathbf{a}_2 = (2, 0, 7, -3),$
 $\mathbf{a}_3 = (-4, 1, -2, 6), \mathbf{b} = (8, 3, -1, -25);$
 (2) $\mathbf{a}_1 = (3, -5, 2, -4)^T, \mathbf{a}_2 = (-1, 7, -3, 6)^T,$
 $\mathbf{a}_3 = (3, 11, -5, 10)^T, \mathbf{b} = (2, -30, 13, -26)^T;$

假设存在 $x, y, z \in \mathbb{R}, \mathbf{b} = x \cdot \mathbf{a}_1 + y \cdot \mathbf{a}_2 + z \cdot \mathbf{a}_3$.

(1)

$$\mathbf{b} = x \cdot \mathbf{a}_1 + y \cdot \mathbf{a}_2 + z \cdot \mathbf{a}_3 \Leftrightarrow \begin{cases} -x + 2y - 4z = 8, \\ 3x + y = 3 \\ 7y - 2z = -1 \\ -5x - 3y + 6z = -25 \end{cases}$$

(2)

4. 设 $\mathbf{a}_1 = (1, 0, 0, 0), \mathbf{a}_2 = (1, 1, 0, 0), \mathbf{a}_3 = (1, 1, 1, 0), \mathbf{a}_4 = (1, 1, 1, 1)$. 证明: \mathbb{F}^4 中任何向量都可以写成的线性组合, 且表示唯一.
5. 设 $P_i = (x_i, y_i, z_i), i = 1, 2, 3, 4$ 是三维几何空间中的点. 证明: $P_i, i = 1, 2, 3, 4$ 共面的条件是

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

6. 设 $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ 是三维几何空间中的四个向量. 证明它们必线性相关.