# 概统作业 (Week 13)

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## 1 (P234 T23)

(1)

$$E(\hat{\theta}_1) = E(\overline{X} + a_n) = E(\overline{X}) + E(a_n) = E(X) + a_n$$

又

$$E(X) = \int_{-\infty}^{+\infty} x f(x;\theta) dx = \int_{\theta}^{+\infty} x \cdot e^{-(x-\theta)} = (-x-1)e^{-(x-\theta)}\Big|_{\theta}^{+\infty} = \theta + 1.$$

可以得到

$$E(\hat{\theta}_1) = \theta + 1 + a_n = \theta \ \Rightarrow \ a_n = -1.$$

有 $x$ 的分布函数

$$F(x) = \int_{-\infty}^{x} f(x,\theta) dx = \begin{cases} 0, & (x < \theta) \\ \int_{\theta}^{x} e^{-(x-\theta)} dx & (x \geq \theta) \end{cases} = \begin{cases} 0, & (x < \theta) \\ 1 - e^{-(x-\theta)} & (x \geq \theta) \end{cases}$$

记 $X_{(1)} = \min\{X_1, X_2, \dots, X_n\}$，则有 $x < \theta$ 时，$X_{(1)}$ 概率密度 $f_1(x) = 0$，$x \geq \theta$ 时，有

$$f_1(x) = n[1 - F(X)]^{n-1} f(x) = n e^{-n(x-\theta)}$$

即

$$f_1(x) = \begin{cases} 0, & (x < \theta) \\ n e^{-n(x-\theta)}. & (x \geq \theta) \end{cases}$$

有

$$E(\hat{\theta}_2) = E(X_{(1)}) + E(b_n) = \int_{-\infty}^{+\infty} x f_1(x) dx + b_n = \left(-x - \frac{1}{n}\right) e^{-n(x-\theta)}\Big|_{\theta}^{+\infty} + b_n = \theta + \frac{1}{n} + b_n$$

可以得到

$$E(\hat{\theta}_2) = \theta + \frac{1}{n} + b_n = \theta \ \Rightarrow \ b_n = -\frac{1}{n}$$

即

$$a_n = -1, \quad b_n = -\frac{1}{n}.$$

(2)

$$Var(\hat{\theta}_1) = Var\left(\overline{X} - 1\right) = \frac{1}{n} Var(X) = \frac{1}{n}\left[E(X^2) - E(X)^2\right]$$

又

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x;\theta) dx = \int_{\theta}^{+\infty} x^2 \cdot e^{-(x-\theta)} dx = \theta^2 + 2\theta + 2$$

故

$$Var(\hat{\theta}_1)=\frac{1}{n}\left[(\theta^2+2\theta+2)-(\theta+1)^2\right]=\frac{1}{n}.$$

由方差的性质有

$$Var(\hat{\theta}_2)=Var\left(X_{(1)}-\frac{1}{n}\right)=Var\left(X_{(1)}\right)=E\left({X_{(1)}}^2\right)-\left[E\left(X_{(1)}\right)\right]^2$$

又

$$E\left({X_{(1)}}^2\right)=\int_{-\infty}^{+\infty}x^2f_1(x)dx=\int_{\theta}^{+\infty}x^2\cdot ne^{-n(x-\theta)}dx=\theta^2+\frac{2}{n}\theta+\frac{2}{n^2}$$

故

$$Var(\hat{\theta}_2)=\left(\theta^2+\frac{2}{n}\theta+\frac{2}{n^2}\right)-\left(\theta+\frac{1}{n}\right)^2=\frac{1}{n^2}$$

有 $Var(\hat{\theta}_1)\geq Var(\tilde{\theta}_2)$，当且仅当 $n=1$ 时取等，故 $\hat{\theta}_2$ 更有效.

# 2 (P236 T39)

(1) 有 $x$ 的密度函数

$$f(x;\theta)=\begin{cases}\dfrac{2x}{\theta}e^{-x^2/\theta}, & (x\geq 0)\\ 0. & (\text{其他})\end{cases}$$

可得

$$E(X)=\int_{\infty}^{+\infty}xf(x;\theta)dx=\int_{0}^{+\infty}\frac{2x^2}{\theta}e^{-\frac{x^2}{\theta}}dx=-\int_{0}^{+\infty}xde^{-\frac{x^2}{\theta}}=\int_{0}^{+\infty}e^{-\frac{x^2}{\theta}}dx-\int_{0}^{+\infty}d\left(xe^{-\frac{x^2}{\theta}}\right)$$

又

$$\int_{0}^{+\infty}e^{-\frac{x^2}{\theta}}dx=\sqrt{\theta}\int_{0}^{+\infty}e^{-\left(\frac{x}{\sqrt{\theta}}\right)^2}d\left(\frac{x}{\sqrt{\theta}}\right)\xrightarrow{x=\sqrt{\theta}t}\sqrt{\theta}\int_{0}^{+\infty}e^{-t^2}dt=\frac{\pi\theta}{2}.$$

$$\int_{0}^{+\infty}d\left(xe^{-\frac{x^2}{\theta}}\right)\xrightarrow{t=xe^{-\frac{x^2}{\theta}}}\int_{0}^{0}dt=0.$$

故

$$E(X)=\frac{\pi\theta}{2}.$$

另

$$E(X^2)=\int_{\infty}^{+\infty}x^2f(x;\theta)dx=\int_{0}^{+\infty}\frac{2x^3}{\theta}e^{-\frac{x^2}{\theta}}dx\xrightarrow{t=\frac{x^2}{\theta}}\theta\int_{0}^{+\infty}te^{-t}dt=\theta.$$

(2) 设 $x_1,x_2,...,x_n$ 为样本观测值，似然函数为

$$L(\theta)=\prod_{i=1}^{n}f(x_i)=\begin{cases}\dfrac{2^n\prod\limits_{i=1}^{n}x_i}{\theta^n}\cdot e^{-\frac{1}{\theta}\sum\limits_{i=1}^{n}{x_i}^2}, & (x_1,x_2,...,x_n>0)\\ 0. & (\text{其他})\end{cases}$$

当 $x_1,x_2,...,x_n>0$ 时，有

$$\ln L(\theta)=n\ln 2+\sum_{i=1}^{n}x_i-n\ln\theta-\frac{1}{\theta}\sum_{i=1}^{n}{x_i}^2$$

令

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} {x_i}^2 = 0.$$

得

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2.$$

又有 $\ln L(\theta)$ 二阶导

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{1}{\theta^2}\left(n - \frac{2}{\theta}\sum_{i=1}^{n} {x_i}^2\right) \xrightarrow{\theta=\hat{\theta}} \frac{\partial^2 \ln L(\theta)}{\partial \theta^2}\Big|_{\theta=\hat{\theta}} = -\frac{n}{\theta^2} < 0.$$

故 $\hat{\theta} = \frac{1}{n}\sum\limits_{i=1}^{n} {x_i}^2$ 为 $\theta$ 的最大似然估计量.

(3) 存在，$a = \theta$. 已知 $\{{X_i}^2\}$ 是独立同分布的随机变量序列，记

$$\mu = E(X^2) = \theta, \quad S_n = \sum_{i=1}^{n} X_i$$

由大数定律有，对 $\forall\ \varepsilon > 0$，

$$\lim_{n\to\infty} \mathbb{P}\left(|S_n/n - \mu| \geq \varepsilon\right) = 0 \xrightarrow{\hat{\theta}=S_n/n} \lim_{n\to\infty} \mathbb{P}\left(\left|\hat{\theta} - \theta\right| \geq \varepsilon\right) = 0.$$

即证对于 $a = \theta$ 有

$$\hat{\theta} \xrightarrow{P} a.$$

# 3 (P238 T57)

*Proof.*

对于 $X$ 有密度函数及分布函数

$$f(x) = \begin{cases} \dfrac{1}{\theta}, & (x \in (0,\theta)) \\ 0. & (\text{其他}) \end{cases} \qquad F(x) = \begin{cases} 0, & (x \leq 0) \\ \dfrac{x}{\theta}, & (x \in (0,\theta)) \\ 1. & (x \geq \theta) \end{cases}$$

有 $\max\{X_1, X_2, \ldots, X_n\}$ 的分布函数及密度函数

$$F_{\max}(x) = \begin{cases} 0, & (x \leq 0) \\ \dfrac{x^n}{\theta^n}, & (x \in (0,\theta)) \\ 1. & (x \geq \theta) \end{cases} \qquad f_{\max}(x) = \begin{cases} \dfrac{n \cdot x^{n-1}}{\theta^n}, & (x \in (0,\theta)) \\ 0. & (\text{其他}) \end{cases}$$

有

$$E(\hat{\theta}) = \int_{-\infty}^{+\infty} x \cdot f_{\max}(x)dx = \int_0^{\theta} \frac{n \cdot x^n}{\theta^n}dx = \frac{n \cdot \theta}{n+1}$$

$$\lim_{n\to+\infty} E(\hat{\theta}) = \lim_{n\to+\infty} \frac{n \cdot \theta}{n+1} = \theta.$$

故 $\hat{\theta}$ 为 $\theta$ 的相合估计量. 又

$$E(\hat{\theta}) \neq \theta$$

故 $\hat{\theta}$ 不是 $\theta$ 的无偏估计量.

□

## 4 (P263 T16)

(1) 记样本数据分别为 $x_1, x_2, \dots, x_{10}$，有

$$\sum_{i=1}^{10} (x_i - \mu)^2 = 2.9$$

$1 - \alpha = 0.95$，$\alpha = 0.05$，有

$$\chi^2_{0.025}(10) = 3.2470, \quad \chi^2_{0.975}(10) = 20.4832$$

有置信区间

$$\left[ \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\chi^2_{0.975}(10)}, \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\chi^2_{0.025}(10)} \right] = \left[ \frac{2.9}{20.4832}, \frac{2.9}{3.2470} \right] = [0.1416, 0.8931]$$

(2)

$$\overline{x} = \frac{49.5 + 50.4 + 49.7 + 51.1 + 49.4 + 49.7 + 50.8 + 49.9 + 50.3 + 50.0}{10} = 50.08$$

$$(n-1)s^2 = \sum_{i=1}^{10} (x_i - \overline{x})^2 = 2.8360, \quad \chi^2_{0.025}(9) = 2.7004, \quad \chi^2_{0.975}(9) = 19.0228$$

有置信区间

$$\left[ \frac{(n-1)s^2}{\chi^2_{0.975}(9)}, \frac{(n-1)s^2}{\chi^2_{0.975}(9)} \right] = \left[ \frac{2.8360}{19.0228}, \frac{2.8360}{2.7004} \right] = [0.1491, 1.0502]$$

## 5 (P263 T19)

记 $Y = \dfrac{X}{\theta}$，$Y_i = \dfrac{X_i}{\theta}$，$X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$，$Y_n = \max\{Y_1, Y_2, \dots, Y_n\}$，则有

$$Y \sim U(0,1), \quad X_{(n)} = \theta \cdot Y_{(n)}, \quad P\left(X_{(n)} \leq \theta \leq c_n X_{(n)}\right) = P\left({c_n}^{-1} \leq Y_{(n)} \leq 1\right)$$

有 $Y_{(n)}$ 的分布函数及密度函数

$$F_{\max}(y) = \begin{cases} 0, & (y \leq 0) \\ y^n, & (y \in (0,1)) \\ 1. & (y \geq 1) \end{cases} \qquad f_{\max}(y) = \begin{cases} n \cdot y^{n-1}, & (y \in (0,1)) \\ 0. & (\text{其他}) \end{cases}$$

若有 $c_n$ 使得 $\left[X_{(n)}, c_n X_{(n)}\right]$ 为 $\theta$ 的 $1-\alpha$ 置信系数，则有 $\forall\ \theta \in \Theta$

$$P\left(X_{(n)} \leq \theta \leq c_n X_{(n)}\right) = 1 - \alpha \ \Leftrightarrow\ P\left({c_n}^{-1} \leq Y_{(n)} \leq 1\right) = 1 - F_{\max}({c_n}^{-1}) = 1 - \alpha$$

又显然 $0 \leq c_n \leq 1$，有

$$1 - F_{\max}({c_n}^{-1}) = 1 - c_{n-n} = 1 - \alpha \ \Rightarrow\ c_n = \alpha^{-\frac{1}{n}}$$

即存在 $c_n = \alpha^{-\frac{1}{n}}$ 满足要求.