概统作业 (Week 4)

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1 (P82 第 12 题)

(1) 记一次产卵数量为 X.

$$P(Y=k)=\sum_{n=k}^{+\infty}P(X=n,Y=k)=\sum_{n=k}^{+\infty}P(X=n)P(Y=k\mid X=n)$$

又

$$P(X=n)P(Y=k\mid X=n)=\frac{\lambda^n}{n!}e^{-\lambda}\cdot\binom{n}{k}\cdot p^k(1-p)^{n-k}\quad(n\geq k)$$

故

$$\begin{aligned}P(Y=k)&=\sum_{n=k}^{+\infty}\frac{\lambda^n}{n!}e^{-\lambda}\cdot\frac{n!\cdot p^k}{k!\cdot(n-k)!}\cdot(1-p)^{n-k}\\&=\frac{\lambda^k\cdot p^k}{k!}e^{-\lambda}\cdot\sum_{n=k}^{+\infty}\frac{(\lambda-\lambda p)^{n-k}}{(n-k)!}\\&=\frac{\lambda^k\cdot p^k}{k!}e^{-\lambda}\cdot e^{\lambda-\lambda p}\\&=\frac{\lambda^k\cdot p^k}{k!}e^{-\lambda p}.\end{aligned}$$

同理有

$$P(Z=k)=\frac{\lambda^k\cdot(1-p)^k}{k!}e^{-\lambda(1-p)}.$$

即 Y 服从参数为 λp 的泊松分布，Z 服从参数为 $\lambda(1-p)$ 的泊松分布.

(2)

$$P(Y=m,Z=n)=\frac{\lambda^{m+n}}{(m+n)!}e^{-\lambda}\cdot\frac{(m+n)!}{m!\cdot n!}p^m(1-p)^n=\frac{\lambda^{m+n}\cdot p^m(1-p)^n}{m!\cdot n!}\cdot e^{-\lambda}.$$

$$P(Y=m)P(Z=n)=\frac{\lambda^m\cdot p^m}{m!}e^{-\lambda p}\cdot\frac{\lambda^n\cdot(1-p)^n}{n!}e^{-\lambda(1-p)}=\frac{\lambda^{m+n}\cdot p^m(1-p)^n}{m!\cdot n!}\cdot e^{-\lambda}.$$

$P(Y=m,Z=n)=P(Y=m)P(Z=n)$，故 Y,Z 相互独立.

2 (P83 第 19 题)

由 $F(x)$ 的右连续性有

$$\lim_{x\to-1+}F(x)=F(-1)\ \Rightarrow\ -a+b=\frac{1}{8}.$$

由于

$$F(1) = P(X \le 1) = P(X < 1) + P(X = 1)$$

且 $P(X < 1) = \lim\limits_{x \to 1-} F(x) = a + b$，可得

$$a + b + \frac{1}{4} = 1$$

联立以上方程解得

$$a = \frac{5}{16},\ b = \frac{7}{16}.$$

3 (P83 第 21 题)

(1)

$$\int_{-\infty}^{+\infty} \frac{a}{1+x^2}\,dx = 1 \ \Leftrightarrow\ a \cdot \arctan x \,|_{-\infty}^{+\infty} = 1 \ \Leftrightarrow\ a \cdot (\frac{\pi}{2} - (-\frac{\pi}{2})) = 1 \ \Leftrightarrow\ a = \frac{1}{\pi}.$$

即 $a = \frac{1}{\pi}$.

(2)

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,dt = \frac{1}{\pi}(\arctan(x) + \frac{\pi}{2}).$$

(3)

$$P(|X| < 1) = P(-1 < X < 1) = F(1) - F(-1) = \frac{1}{\pi}[\arctan(1) - \arctan(-1)] = \frac{1}{2}.$$

4 (P84 第 26 题)

单次观测到 $x > 2$ 的概率为

$$p = P(X > 2) = \frac{4-2}{4-1} = \frac{2}{3}.$$

设三次独立观测中观测值大于 2 的次数为 Y，则

$$P(Y \ge 2) = P(Y = 2) + P(Y = 3) = 3 \cdot p^2(1-p) + p^3 = \frac{20}{27}.$$

即三次独立观测中至少两次观测值大于 2 的概率为 $\frac{20}{27}$.

5 (P84 第 28 题)

(1) 设检修时间为 X，$f(x) = e^{-X}$ $(X > 0)$.

$$P(X > 2) = 1 - P(X \le 2) = 1 - F(2) = e^{-2}.$$

(2)

$$P(X > 4 | X > 2) = P(X > 2) = e^{-2}.$$

6 (P84 第 29 题)

单次等待时间超过 10min 的概率为

$$p = P(X > 10) = 1 - F(10) = e^{-2}.$$

设一个月内未接受服务而离开的次数为 Y，则

$$P(Y \ge 1) = 1 - P(Y = 0) = 1 - (1-p)^5 = 1 - (1-e^{-2})^5 \approx 0.5167.$$