1.(P176,37)

(1)证明 $Cov(X_1,X_2)=Cov(X_1,E(X_2|X_1))$

(2)假设存在常数c，$E(X_2|X_1)=1+cX_1$，证明

$$c=\frac{Cov(X_1,X_2)}{Var(X_1)}$$

证明:

(1)

$$\begin{aligned}Cov(X_1,X_2)&=E(X_1-EX_1)(X_2-EX_2)\\&=EX_1X_2-EX_1EX_2\\&=E(E(X_2|X_1)X_1)-EX_1E(E(X_2|X_1))\\&=Cov(X_1,E(X_2|X_1))\end{aligned}$$

(2)

$$\begin{aligned}\frac{Cov(X_1,X_2)}{Var(X_1)}&=\frac{Cov(X_1,E(X_2|X_1))}{Var(X_1)}\\&=\frac{Cov(X_1,1+cX_1)}{Var(X_1)}\\&=\frac{cCov(X_1,X_1)}{Var(X_1)}\\&=\frac{cVar(X_1)}{Var(X_1)}\\&=c\end{aligned}$$

2.(P174,27)试对下列常见的分布求矩母函数:

(1)二项分布$B(n,p)$

(2)参数为λ的泊松分布

(3)参数为λ的指数分布

(4)正态分布$N(\mu,\sigma^2)$

解:

(1)$X\sim B(n,p)$则

$$\begin{aligned}M_X(s)&=Ee^{sX}\\&=\sum_{k=0}^{n}e^{sk}C_n^kp^k(1-p)^{n-k}\\&=\sum_{k=0}^{n}C_n^k(pe^s)^k(1-p)^{n-k}\\&=(pe^s+1-p)^n\end{aligned}$$

(2)$X\sim P(\lambda)$则

$$\begin{aligned}M_X(s)&=Ee^{sX}\\&=\sum_{k=0}^{\infty}e^{sk}\frac{\lambda^k}{k!}e^{-\lambda}\\&=e^{\lambda(e^s-1)}\sum_{k=0}^{\infty}\frac{(\lambda e^s)^k}{k!}e^{-\lambda e^s}\\&=e^{\lambda(e^s-1)}\end{aligned}$$

(3)$X\sim Exp(\lambda)$则

$$\begin{aligned}M_X(s)&=Ee^{sX}\\&=\int_0^{\infty}e^{sx}\lambda e^{-\lambda x}\mathrm{d}x\\&=\frac{\lambda}{\lambda-s},s<\lambda\end{aligned}$$

(4)$X\sim N(\mu,\sigma^2)$则

$$\begin{aligned}M_X(s)&=Ee^{sX}\\&=\int_{-\infty}^{\infty}e^{sx}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}\mathrm{d}x\\&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu-\sigma^2 s)^2}{2\sigma^2}+\mu s+\frac{1}{2}\sigma^2s^2\right\}\mathrm{d}x\\&=e^{\mu s+\frac{1}{2}\sigma^2s^2}\end{aligned}$$

3.(P175,32)设随机变量(X,Y)服从$N(\mu,\mu,\sigma^2,\sigma^2,\rho)$，其中$\rho>0$，问是否存在两个常数$\alpha,\beta$使得$Cov(\alpha X+\beta Y,\alpha X-\beta Y)=0$.如果存在请求出，否则请说明原因

解:

因为$Cov(\alpha X+\beta Y,\alpha X-\beta Y)=\alpha^2Var(X)-\beta^2Var(Y)$

同时又因为X,Y的边缘分布均为$N(\mu,\sigma^2)$，有$Var(X)=Var(Y)$，则上式为0需要$\alpha^2=\beta^2$所以仅需$|\alpha|=|\beta|$即可

4.(P176,38)若$E(X_2|X_1)=1$，证明

$$Var(X_1X_2)\geqslant Var(X_1)$$

证明:

因为

$$E(X_2|X_1)=\int_{-\infty}^{\infty}x_2f(x_2|x_1)\mathrm{d}x_2=1$$

所以$EX_2=E(E(X_2|X_1))=1$

$$E(X_1X_2)^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1^2x_2^2f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1^2x_2^2f(x_2|x_1)f_1(x_1)\mathrm{d}x_1\mathrm{d}x_2$$

而

$$EX_1^2=\int_{-\infty}^{\infty}x_1^2f_1(x_1)\mathrm{d}x_1$$

因此

$$E(X_1X_2)^2-EX_1^2=\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}x_2^2f(x_1|x_1)\mathrm{d}x_2-1)x_1^2f_1(x_1)\mathrm{d}x_1$$

且

$$\int_{-\infty}^{\infty}x_2^2f(x_1|x_1)\mathrm{d}x_2-1=\int_{-\infty}^{\infty}(x_2^2-x_2)f(x_2|x_1)\mathrm{d}x_2$$

因为$EX_2=1$，所以上式$\leqslant 0$ 又$EX_1X_2=E(E(X_1X_2|X_1))=E(E(X_2|X_1)X_1)=E(X_1)$

$$Var(X_1X_2)-Var(X_1)=E(X_1X_2)^2-(EX_1X_2)^2-EX_1^2+(EX_1)^2=E(X_1X_2)^2-EX_1^2\geqslant 0$$

得证

5.若$\{X_n\}_{n\geqslant1},\{Y_n\}_{n\geqslant1}$是两个随机变量序列，$X_n$依概率收敛到$X$，$Y_n$依概率收敛到$Y$，证明$X_n+Y_n$依概率收敛到$X+Y$

证明:

由题，$\forall\varepsilon>0$，有$\lim_{n\to\infty}P(|X_n-X|\geqslant\varepsilon)=\lim_{n\to\infty}P(|Y_n-Y|\geqslant\varepsilon)=0$，因为

$$|X_n+Y_n-X-Y|\leqslant|X_n-X|+|Y_n-Y|$$

所以

$$\{|X_n+Y_n-X-Y|\geqslant\varepsilon\}\subset\{|X_n-X|\geqslant\frac{\varepsilon}{2}\}\cup\{|Y_n-Y|\geqslant\frac{\varepsilon}{2}\}$$

则

$$P(|X_n+Y_n-X-Y|\geqslant\varepsilon)\leqslant P(|X_n-X|\geqslant\frac{\varepsilon}{2})+P(|Y_n-Y|\geqslant\frac{\varepsilon}{2})\to0,n\to\infty$$

即

$$\lim_{n\to\infty} P(|X_n + Y_n - X - Y| \geqslant \varepsilon) = 0$$

有$X_n + Y_n$依概率收敛到$X + Y$