概统作业 (Week 9)

PB20000113 孔浩宇

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1 (P173 T16)

(1)

$$E(sgn(X)) = \int_{-2}^{1} sgn(x) \cdot f(x) dx = \frac{1}{3} \cdot \left(\int_{-2}^{0} sgn(x) dx + \int_{0}^{1} sgn(x) dx \right) = \frac{1}{3} \cdot (-1) = -\frac{1}{3}.$$

$$E(sgn^2(X)) = \int_{-2}^{1} sgn^2(x) \cdot f(x) dx = \frac{1}{3} \cdot \left(\int_{-2}^{0} sgn^2(x) dx + \int_{0}^{1} sgn^2(x) dx \right) = \frac{1}{3} \cdot 3 = 1.$$

因此

$$Var(sgn(X)) = E(sgn^2(X)) - (E(sgn(X)))^2 = 1 - \frac{1}{9} = \frac{8}{9}$$

(2)

$$E(sgn(X) \cdot X) = \int_{-\infty}^{+\infty} sgn(x) \cdot x \cdot f(x) dx = 2 \int_{0}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = -\sqrt{\frac{2}{\pi}} \cdot e^{-\frac{x^2}{2}} \Big|_0^{+\infty} = \sqrt{\frac{2}{\pi}}.$$

2 (P173 T18)

(1)

$$\begin{aligned} F(y) &= P(Y \le y) \\ &= P(Y \le y, X = 1) + P(Y \le y, X = 2) \\ &= P(Y \le y | X = 1) \cdot P(X = 1) + P(Y \le y | X = 2) \cdot P(X = 2) \\ &= \frac{1}{2} \left[P(Y \le y | X = 1) + P(Y \le y | X = 2) \right] \end{aligned}$$

(a) $y < 0$

$$F(y) = 0.$$

(b) $0 \le y < 1$

$$F(y) = \frac{1}{2} y + \frac{1}{2} \cdot \frac{y}{2} = \frac{3}{4} y.$$

(c) $1 \le y < 2$

$$F(y) = \frac{1}{2} + \frac{1}{2} \cdot \frac{y}{2} = \frac{1}{4} y + 2.$$

(d) $y \ge 2$

$$F(y) = 1.$$

故

$$F_Y(y) = \begin{cases} 0, & (y < 0) \\ \frac{3}{4} y, & (0 \le y < 1) \\ \frac{y}{4} + \frac{1}{2}, & (1 \le y < 2) \\ 1. & (y \ge 2) \end{cases}$$

(2)

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{3}{4} & (0 \le y < 1) \\ \frac{1}{4} & (1 \le y < 2) \\ 0. & \text{其他} \end{cases}$$

$$E(Y) = \int_{-\infty}^{+\infty} y \cdot f_Y(y) dy = \frac{3}{4}\int_0^1 y dy + \frac{1}{4}\int_1^2 y dy = \frac{3}{8}y^2\Big|_0^1 + \frac{1}{8}y^2\Big|_1^2 = \frac{3}{4}.$$

3 (P174 T24)

(1)

$$E(X+Y-3\mu) = E(X) + E(Y) - 3\mu = \mu + \mu - 3\mu = 0.$$

$$Var(X+Y-3\mu) = Var(X) + Var(Y) + 2Cov(X,Y) = \sigma^2 + 2\sigma^2 + 2\cdot\sigma\cdot\sqrt{2}\sigma\cdot\frac{\sqrt{2}}{4} = 4\sigma^2.$$

故

$$X+Y-3\mu \sim N(0, 4\sigma^2)$$

记 $Z = X+Y-3\mu$，有

$$f_Z(z) = \frac{1}{2\sqrt{2\pi}\sigma}e^{-\frac{z^2}{8\sigma^2}}$$

记 $(X+Y-3\mu)_+ = \max\{X+Y-3\mu, 0\} = \max\{z, 0\} = g(z)$，有

$$E(g(z)) = \int_{-\infty}^{+\infty} g(z)\cdot f_Z(z)dz = \int_0^{+\infty} z\cdot\frac{1}{2\sqrt{2\pi}\sigma}e^{-\frac{z^2}{8\sigma^2}}dz = \sqrt{\frac{2}{\pi}}\sigma.$$

(2)

$$E\left(g^2(z)\right) = \int_{-\infty}^{+\infty} g^2(z)\cdot f_Z(z)dz = \int_0^{+\infty} z^2\cdot\frac{1}{2\sqrt{2\pi}\sigma}e^{-\frac{z^2}{8\sigma^2}}dz = \frac{1}{2}E(z^2) = \frac{1}{2}\left[Var(z) + (E(z))^2\right] = 2\sigma^2.$$

$$Var(g(z)) = E(g^2(z)) - (E(z))^2 = (2 - \frac{2}{\pi})\sigma.$$

4 (P174 T26)

(1) 记 $\varphi(x_1, x_2, x_3) = x_1 + x_2 + x_3$，则

$$E(X_1+X_2+X_3) = \oiint\limits_{x_1^2+x_2^2+x_3^2=1} \varphi(x_1,x_2,x_3)dS = \oiint\limits_{x_1^2+x_2^2+x_3^2=1} x_1 dS + \oiint\limits_{x_1^2+x_2^2+x_3^2} x_2 dS + \oiint\limits_{x_1^2+x_2^2+x_3^2} x_3 dS = 0.$$

(2)

$$E\left[(X_1+X_2+X_3)^2\right] = \oiint\limits_{x_1^2+x_2^2+x_3^2=1} (x_1+x_2+x_3)^2 dS = \oiint\limits_{x_1^2+x_2^2+x_3^2=1} 1 + 2(x_1x_2 + x_1x_3 + x_2x_3)dS$$

又

$$\oiint\limits_{x_1^2+x_2^2+x_3^2=1} x_1x_2 dS = \oiint\limits_{x_1^2+x_2^2+x_3^2=1} x_1x_3 dS = \oiint\limits_{x_1^2+x_2^2+x_3^2=1} x_2x_3 dS = 0.$$

$$Var(X_1+X_2+X_3) = E\left[(X_1+X_2+X_3)^2\right] - \left[E(X_1+X_2+X_3)\right]^2 = \oiint\limits_{x_1^2+x_2^2+x_3^2=1} dS = 4\pi.$$

5 (P178 T55)

Proof.

(1) (a) 证明 1：

$$\forall\ X \in [0,1],\ X^2 \le X \ \Rightarrow\ Var(X) = E(X^2) - E(X)^2 \le E(X) - E(X)^2$$

记 $t = E(X)$，则 $0 \le t \le 1$，有

$$Var(X) = t(1-t) \le \left(\frac{t + (1-t)}{2}\right)^2 = \frac{1}{4}.$$

(b) 证明 2：

取 $Y = X - \frac{1}{2}$ ，有

$$Var(Y) = Var(X),\ -\frac{1}{2} \le Y \le \frac{1}{2} \ \Rightarrow\ 0 \le Y^2 \le \frac{1}{4} \ \Rightarrow\ E(Y^2) \le \frac{1}{4}$$

故

$$Var(X) = Var(Y) = E(Y^2) - (E(Y))^2 \le \frac{1}{4} - (E(Y))^2 \le \frac{1}{4}$$

当且仅当 $E(X) = \frac{1}{2}$ 且 $E(X) = E(X^2)$ 时等号成立，即 X 服从 $P(X=1) = P(X=0) = \frac{1}{2}$ 的分布时.

(2) 当 $0 < a \le X \le b$ 时，有 $Var(X) \le \frac{(b-a)^2}{4}$，当且仅当 X 服从 $P(X=a) = P(X=b) = \frac{1}{2}$ 的分布时取等.

取 $Y = X - \frac{a+b}{2}$ ，有

$$Var(Y) = Var(X),\ \frac{a-b}{2} \le Y \le \frac{b-a}{2} \ \Rightarrow\ 0 \le Y^2 \le \frac{(b-a)^2}{4} \ \Rightarrow\ E(Y^2) \le \frac{(b-a)^2}{4}$$

故

$$Var(X) = Var(Y) = E(Y^2) - (E(Y))^2 \le \frac{(b-a)^2}{4} - (E(Y))^2 \le \frac{(b-a)^2}{4}$$

□