HW-6

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1、解：

(1)、
$$f_{X,Y}(x,y) = P(X=x, Y=y)$$
$$= P(Y=y \mid X=x)\cdot P(X=x)$$
$$= (1-p)^{y-x-1}\cdot p \cdot (1-p)^{x-1}\cdot p$$
$$= p^2(1-p)^{y-2}$$

$\therefore$ (X,Y) 的联合分布律为 $f_{X,Y}(x,y) = p^2(1-p)^{y-2}$ $(x=1,2,\cdots,\ y=2,3,\cdots)$

(2)、关于 x 的边缘分布：
$$f_X(x) = \sum_{y=x+1}^{\infty} f_{X,Y}(x,y)$$
$$= \sum_{y=x+1}^{\infty} p^2(1-p)^{y-2}$$
$$= p^2(1-p)^{x-1}\sum_{t=0}^{\infty}(1-p)^t$$
$$= p^2(1-p)^{x-1}\lim_{k\to\infty}\frac{1-(1-p)^k}{p}$$
$$= p^2(1-p)^{x-1}\frac{1}{p}$$
$$= p(1-p)^{x-1} \qquad x=1,2,\cdots$$

关于 y 的边缘分布：
$$f_Y(y) = \sum_{x=1}^{y-1} f_{X,Y}(x,y)$$
$$= \sum_{x=1}^{y-1} p^2(1-p)^{y-2}$$
$$= (y-1)p^2(1-p)^{y-2} \qquad y=2,3,\cdots$$

2、解：

(1)、由题意可知
$$F(X,Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy$$
$$= \begin{cases} 0 & x\le 0,\ y\le 0 \\ \int_0^y\int_0^x f(x,y)\,dx\,dy & 0<x<\frac{\pi}{2},\ 0<y<\frac{\pi}{2} \\ \int_0^{\frac{\pi}{2}}\int_0^x f(x,y)\,dx\,dy & 0<x<\frac{\pi}{2},\ y\geq\frac{\pi}{2} \\ \int_0^y\int_0^{\frac{\pi}{2}} f(x,y)\,dx\,dy & x\geq\frac{\pi}{2},\ 0<y<\frac{\pi}{2} \\ \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} f(x,y)\,dx\,dy & x\geq\frac{\pi}{2},\ y\geq\frac{\pi}{2} \end{cases}$$

$$=\begin{cases} 0 & x\le 0,\ y\le 0 \\ \int_0^y\int_0^x \cos x\cos y\,dx\,dy & 0<x<\frac{\pi}{2},\ 0<y<\frac{\pi}{2} \\ \int_0^{\frac{\pi}{2}}\int_0^x \cos x\cos y\,dx\,dy & 0<x<\frac{\pi}{2},\ y\geqslant\frac{\pi}{2} \\ \int_0^y\int_0^{\frac{\pi}{2}} \cos x\cos y\,dx\,dy & x\geqslant\frac{\pi}{2},\ 0<y<\frac{\pi}{2} \\ \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos x\cos y\,dx\,dy & x\geqslant\frac{\pi}{2},\ y\geqslant\frac{\pi}{2} \end{cases}$$

$$=\begin{cases} 0 & x\le 0,\ y\le 0 \\ \sin x\sin y & 0<x<\frac{\pi}{2},\ 0<y<\frac{\pi}{2} \\ \sin x & 0<x<\frac{\pi}{2},\ y\geqslant\frac{\pi}{2} \\ \sin y & x\geqslant\frac{\pi}{2},\ 0<y<\frac{\pi}{2} \\ 1 & x\geqslant\frac{\pi}{2},\ y\geqslant\frac{\pi}{2} \end{cases}$$

(2). $P(0<X<\frac{\pi}{4},\ \frac{\pi}{4}<Y<\frac{\pi}{2})$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} f(x,y)\,dx\,dy$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} \cos x\cos y\,dx\,dy$

$= \frac{\sqrt{2}}{2}-\frac{1}{2}$

3、解：

(1). ∵ $F(x,y)$ 为 (X,Y) 的联合分布函数

∴ $F(+\infty,+\infty)=1$

即 $a(b+\frac{\pi}{2})(c+\frac{\pi}{2})=1$

$F(x,-\infty)$ 恒为 0

即 $a(b+\arctan x)(c-\frac{\pi}{2})=0 \Rightarrow c=\frac{\pi}{2}$

$F(-\infty,y)$ 恒为 0

即 $a(b-\frac{\pi}{2})(c+\arctan y)=0 \Rightarrow b=\frac{\pi}{2}$

∴ $a=\frac{1}{\pi^2}$

(2). 由(1)可得，$F(x,y)=\frac{1}{\pi^2}(\frac{\pi}{2}+\arctan x)(\frac{\pi}{2}+\arctan y)$，$x,y\in R$

∴ $f(x,y)=\frac{\partial^2 F}{\partial x\partial y}$

$=\frac{1}{\pi^2(1+x^2)(1+y^2)}$

$\therefore P(X>0, Y>0)$

$= \int_0^{+\infty}\int_0^{+\infty} f(x,y)\,dx\,dy$

$= \frac{1}{\pi^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(1+x^2)(1+y^2)}\,dx\,dy$

$= \frac{1}{\pi^2}\cdot\frac{\pi^2}{4}$

$= \frac{1}{4}$

4、解：由题意可知 $0.2+a+b+0.3=1$

$\because \{X=-1\}$ 和 $\{X+Y=0\}$ 相互独立

且 $P(X=-1)=0.2+a$，$P(X+Y=0)=a+b$

$P(X=-1, X+Y=0)=a$

$\therefore P(X=-1, X+Y=0)=P(X=-1)\,P(X+Y=0)$

$a=(a+b)(0.2+a)$

$\therefore a=0.2$，$b=0.3$

5、解：

(1)、由题意可知 $f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}$

$\therefore f(x,y)=f_{Y|X}(y|x)\cdot f_X(x)$

$=\frac{3y^2}{x^3}\,3x^2$

$=\frac{9y^2}{x}$

$$\therefore f(x,y)=\begin{cases}\frac{9y^2}{x} & 0<y<x<1\\ 0 & 其他\end{cases}$$

(2)、$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)\,dx$

$=\int_y^1\frac{9y^2}{x}\,dx$

$=-9y^2\ln y \quad 0<y<1$

$$\therefore f_Y(y)=\begin{cases}-9y^2\ln y & 0<y<1\\ 0 & 其他.\end{cases}$$