1.(P172, 4)设X为一个连续型随机变量, 试对下列各种情形, 计算$Var(X)$.

(1) 若X的密度函数为

$$f(x)=\frac{x}{\sigma^2}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}, x>0,$$

其中$\sigma>0$为常数, 则称X服从瑞利(Rayleigh)分布;

(2) 若X的密度函数为

$$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, 0<x<1,$$

其中$\alpha,\beta>0$为常数, $\Gamma(x)$为Γ函数, 则称X服从β分布;

(3) 若X的密度函数为

$$f(x)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\}, x>0,$$

其中$k,\lambda>0$为常数, 则称X服从韦布尔分布.

解:

(1)$EX=\frac{\sqrt{2\pi}\sigma}{2}$

$$\begin{aligned}EX^2&=\int_0^\infty x^2f(x)\mathrm{d}x\\&=\int_0^\infty\frac{x^3}{\sigma^2}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}\mathrm{d}x\\&=\int_0^\infty\frac{t}{2\sigma^2}\exp\left\{-\frac{t}{2\sigma^2}\right\}\mathrm{d}t\\&=t\exp\left\{-\frac{t}{2\sigma^2}\right\}\Bigg|_\infty^0+\int_0^\infty\exp\left\{-\frac{t}{2\sigma^2}\right\}\mathrm{d}t\\&=2\sigma^2\exp\left\{-\frac{t}{2\sigma^2}\right\}\Bigg|_\infty^0\\&=2\sigma^2\end{aligned}$$

$$Var(X)=EX^2-(EX)^2=2\sigma^2-\frac{\pi}{2}\sigma^2$$

(2)$EX=\frac{\alpha}{\alpha+\beta}$

$$\begin{aligned}EX^2&=\int_0^1x^2f(x)\mathrm{d}x\\&=\int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+1}(1-x)^{\beta-1}\mathrm{d}x\\&=\int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+2-1}(1-x)^{\beta-1}\mathrm{d}x\\&=\frac{B(\alpha+2,\beta)}{B(\alpha,\beta)}\\&=\frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}\end{aligned}$$

$$Var(X)=EX^2-(EX)^2=\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

(3)$EX=\lambda\Gamma\left(\frac{1}{k}+1\right)$ 令$t=\left(\frac{x}{\lambda}\right)^k$, 则$\mathrm{d}t=\frac{kx^{k-1}}{\lambda^k}\mathrm{d}x$, $x=\lambda t^{1/k}$

$$\begin{aligned}EX^2&=\int_0^\infty x^2f(x)\mathrm{d}x\\&=\int_0^\infty k\lambda\left(\frac{x}{\lambda}\right)^{k+1}\exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\}\mathrm{d}x\\&=\int_0^\infty\lambda^2t^{2/k+1-1}e^{-t}\mathrm{d}t\\&=\lambda^2\Gamma(\frac{2}{k}+1)\end{aligned}$$

$$Var(X)=EX^2-(EX)^2=\lambda^2\left[\Gamma\left(\frac{2}{k}+1\right)-\Gamma^2\left(\frac{1}{k}+1\right)\right]$$

2.(P173,16)设X为一随机变量, 它的符号函数定义为

$$sgn(x)=\begin{cases}0, & X=0,\\ 1, & X>0,\\ -1, & X<0.\end{cases}$$

(1)若X服从$U(-2,1)$, 试求$Var(sgn(X))$;

(2)若X服从标准正态分布, 试求$E[sgn(X)X]$.

解:

(1)$E(sgn(X))=\int_{-\infty}^{\infty} sgn(x)f(x)\mathrm{d}x=\int_{-2}^{0}-\frac{1}{3}\mathrm{d}x+\int_{0}^{1}\frac{1}{3}\mathrm{d}x=-\frac{1}{3}$, $E(sgn(X)^2)=\int_{-\infty}^{\infty} f(x)\mathrm{d}x=1$,
$Var(sgn(X))=E(sgn(X)^2)-(E(sgn(X)))^2=1-\frac{1}{9}=\frac{8}{9}$

(2)$E[sgn(X)X]=\int_{-\infty}^{\infty} sgn(x)xf(x)\mathrm{d}x=\int_{-\infty}^{\infty}|x|\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}\mathrm{d}x=\sqrt{\frac{2}{\pi}}$

3.(P173,10)设随机变量X只能取有限个正值$x_1,x_2,\ldots,x_k(k\geqslant 2)$, 证明:

$$\lim_{n\to\infty}\frac{E(X^{n+1})}{E(X^n)}=\max_{1\leqslant i\leqslant k}x_i.$$

证明: 设$P(X=x_i)=p_i,i=1,2,\ldots,k$, 其中$x_t=\max_{1\leqslant i\leqslant k}x_i$, 因此$\lim_{n\to\infty}(x_i/x_t)^n=0,i\neq t$.

则$E(X^n)=\sum_{i=1}^{k}p_ix_i^n$

$$\begin{aligned}\lim_{n\to\infty}\frac{E(X^{n+1})}{E(X^n)}&=\lim_{n\to\infty}\frac{\sum_{i=1}^{k}p_ix_i^{n+1}}{\sum_{i=1}^{k}p_ix_i^n}\\&=\lim_{n\to\infty}x_t\frac{\sum_{i=1}^{k}p_i(x_i/x_t)^{n+1}}{\sum_{i=1}^{k}p_i(x_i/x_t)^n}\\&=x_t\frac{p_t}{p_t}\\&=x_t=\max_{1\leqslant i\leqslant k}x_i.\end{aligned}$$

4.(P172,5)设随机变量X的密度函数为

$f(x)=ax^2+bx+c$,$0<x<1$,

且已知$E(X)=0.5,Var(X)=0.15$, 试求常数a,b,c.

解:

由题, $EX^2=Var(X)+(EX)^2=0.4$

$$\begin{cases}\int_0^1 f(x)\mathrm{d}x=1\\ \int_0^1 xf(x)\mathrm{d}x=0.5\\ \int_0^1 x^2f(x)\mathrm{d}x=0.4\end{cases}$$

解得$a=12,b=-12,c=3$.

5.证明下面两个广义切比雪夫不等式:

令$\sigma^2=Var(X)$. $\forall x,a>0$,

$$P(X-EX\geqslant x)\leqslant\frac{\sigma^2+a^2}{(x+a)^2},$$

$$P(X-EX\geqslant x)\leqslant\frac{\sigma^2}{x^2+\sigma^2}.$$

根据这两个不等式, 证明X的中位数$m(X)$满足: $|EX-m(X)|\leqslant\sigma$.

证明: 令$T=X-EX$, 则$ET=0,ET^2=\sigma^2$取$x+a\geqslant 0$, 则利用Markov不等式$P(Y\geqslant\varepsilon)\leqslant\frac{EY}{\varepsilon}$, 有

$$P(X-EX\geqslant x)=P(T+a\geqslant x+a\geqslant 0)=P\{(T+a)^2\geqslant(x+a)^2\}\leqslant\frac{E(T+a)^2}{(x+a)^2}=\frac{ET^2+2aET+a^2}{(x+a)^2}=\frac{\sigma^2+a^2}{(x+a)^2}$$

在上式中, 取$a=\frac{\sigma^2}{x}$, 得

$$P(X-EX\geqslant x)\leqslant\frac{\sigma^2+\sigma^4/x^2}{(x+\sigma^2/x)^2}=\frac{\sigma^2}{x^2+\sigma^2}$$

取$x=\sigma$, 则$P(X-EX\geqslant\sigma)\leqslant\frac{1}{2}$, 而$P\{X\geqslant m(X)\}=\frac{1}{2}$, 得$EX-m(X)\geqslant-\sigma$

同时，令$Y=-X$，有$P(-Y\leqslant -EY-\sigma)=P(X\leqslant EX-\sigma)\leqslant \frac{1}{2}$，得$EX-m(X)\leqslant \sigma$，故$|EX-m(X)|\leqslant \sigma$