概统作业 (Week 8)

PB20000113 孔浩宇

May 4, 2023

1 (P122 T48)

Proof.

(1) 有 X 和 Y 边缘分布

$$f(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}e^{\{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\}}\ \Rightarrow\ X\sim N(0,1),\ Y\sim N(0,1)$$

$$f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\ f_Y(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}.$$

有 Z 分布

$$f_Z(z)=P(Z\le z)=\iint_{\frac{y-\rho x}{\sqrt{1-\rho^2}}\le z}f(x,y)dxdy=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{\sqrt{1-\rho^2}z+\rho x}dyf(x,y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

有 X,Z 联合分布

$$f_{(X,Z)}(x,z)=f(x,\sqrt{1-\rho^2}z+\rho x)\cdot|J|=\sqrt{1-\rho^2}\cdot f(x,\sqrt{1-\rho^2}z+\rho x)=\frac{1}{2\pi}e^{\{-\frac{1}{2}(x^2+z^2)\}}=f_X(x)\cdot f_Z(z).$$

即证.

(2)

$$\begin{aligned}P(XY<0)&=1-P(XY\ge 0)\\&=1-P(XY>0)\\&=1-P(X>0,Y>0)-P(X<0,Y<0)\end{aligned}$$

由

$$f(-x,-y)=f(x,y)\ \Rightarrow\ P(X<0,Y<0)=P(-x<0,-y<0)=P(X>0,Y>0).$$

即

$$P(XY<0)=1-2P(X>0,Y>0).$$

有

$$\begin{aligned}P(X>0,Y>0)&=P(X>0,(\sqrt{1-\rho^2}Z+\rho X)>0)\\&=P(X>0,(\sqrt{1-\rho^2}Z+\rho X)>0)\\&=P(X>0,Z>\frac{-\rho X}{\sqrt{1-\rho^2}})\\&=\iint_{x>0,\sqrt{1-\rho^2}z+\rho x>0}f_{(X,Z)}(x,z)dxdz.\end{aligned}$$

做代换

$$\begin{cases} x = r\cos\alpha \\ z = r\sin\alpha \end{cases} \quad \Rightarrow \ |J| = \frac{\partial(x,z)}{\partial(r,\alpha)} = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -r\sin\alpha & -r\cos\alpha \end{vmatrix} = r. \ (0 \le \alpha < 2\pi,\ r > 0)$$

记 $\rho = \cos\beta$, $\sqrt{1-\rho^2} = \sin\beta$ $(0 < \beta < \frac{\pi}{2})$，有

$$\iint_{x>0,\sqrt{1-\rho^2}z+\rho x>0} f_{(X,Z)}(x,z)dxdz. = \iint_{\sin\alpha>0,\cos(\alpha-\beta)>0} r\cdot\frac{1}{2\pi}e^{-\frac{r^2}{2}}drd\theta.$$

$$\begin{aligned} P(X>0,Y>0) &= \iint_{\sin\alpha>0,\cos(\alpha-\beta)>0} r\cdot\frac{1}{2\pi\cdot\sin\beta}e^{-\frac{r^2}{2}}drd\theta \\ &= \int_{\sin\alpha>0,\cos(\alpha-\beta)>0}\frac{1}{2\pi\cdot\sin\beta}d\theta\int_0^{+\infty} -e^{-\frac{r^2}{2}}d(-\frac{r^2}{2}) \\ &= \int_0^{\frac{\pi}{2}}\frac{1}{2\pi}d\theta + \int_{\frac{3\pi}{2}+\beta}^{2\pi}\frac{1}{2\pi}d\theta \\ &= \frac{\pi-\beta}{2\pi} \\ &= \frac{1}{2} - \frac{\arccos\rho}{2\pi} \end{aligned}$$

故

$$P(XY<0) = 1 - P(X>0,Y>0) = \pi^{-1}\arccos\rho.$$

□

2 (P171 T1)

(0) 对 $0<p<1$，记 $q=1-p$，$\forall\ X\in\{4,\ 5,\ 6,\ 7\}$，有

$$P(X=n) = \binom{n-1}{n-4}\cdot(p^4\cdot q^{n-4} + q^4\cdot p^{n-4})$$

$$E(X) = 4\cdot P(X=4) + 5\cdot P(X=5) + 6\cdot P(X=6) + 7\cdot P(X=7)$$

(1) 对 $p=0.5$，有

$$E(X) = 4\cdot\frac{1}{8} + 5\cdot 4\cdot\frac{1}{16} + 6\cdot 10\cdot\frac{1}{32} + 7\cdot 20\cdot\frac{1}{64} = \frac{93}{16}$$

(1) 对 $p=0.6$，有

$$E(X) = 4\cdot\frac{3^4+2^4}{5^4} + 5\cdot 4\cdot\frac{3^4\cdot 2+2^4\cdot 3}{5^5} + 6\cdot 10\cdot\frac{3^4\cdot 2^2+2^4\cdot 3^2}{5^6} + 7\cdot 20\cdot\frac{3^4\cdot 2^3+2^4\cdot 3^3}{5^7} = \frac{17804}{3125}$$

3 (P171 T2)

Proof.

(1)

$$E(X) = \sum_{k=1}^{\infty} k\cdot P(X=k) = \sum_{k=1}^{\infty}\sum_{n=1}^{k} P(X=k) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} P(X=k) = \sum_{n=1}^{\infty} P(X\ge n)$$

(2)

$$E(X) = \int_0^{\infty} tf(t)dt = \int_0^{\infty} f(t)dt\int_0^t dx = \int_0^{\infty} dx\int_x^{\infty} f(t)dt = \int_0^{\infty}(1-F(x))dx.$$

(3) 以 $I(A)$ 表示事件 A 的示性函数，则有

$$E(X)=E\left(\int_0^X dx\right)=E\left(\int_0^{\infty} I(X>x)dx\right)=\int_0^{\infty} E\left[I(X>x)\right]=\int_0^{\infty} P(X>x)dx$$

即证

$$E(X)=\int_0^{\infty}(1-F(x))dx.$$

□

4 (P171 T8)

(1) 用 $E(T_i)$ 表示从第 $i-1$ 种到第 i 种需要买的卡片的期望，则单次买到第 i 种卡片的概率为 $p=\dfrac{n}{n-i+1}$. 记 $q=1-p$.

$$E(T_i)=\sum_{k=1}^{\infty} k\cdot q^{k-1}\cdot p=p\cdot\left(\sum_{k=1}^{\infty} q^k\right)'=\frac{1}{p}=\frac{n}{n-i+1}.$$

$$E(X_n)=\sum_{i=1}^{n}E(T_i)=\sum_{i=1}^{n}\frac{n}{n-i+1}=n\cdot\sum_{i=1}^{n}\frac{1}{i}=12\cdot\sum_{i=1}^{12}\frac{1}{i}=\frac{86021}{2310}\approx 37.24$$

(2)

$$\lim_{n\to\infty}E\left(\frac{X_n}{n\ln n}\right)=\lim_{n\to\infty}\frac{E(X_n)}{n\ln n}=\lim_{n\to\infty}\frac{n\cdot\sum\limits_{i=1}^{n}\frac{1}{i}}{n\ln n}=\lim_{n\to\infty}\frac{\sum\limits_{i=1}^{n}\frac{1}{i}}{\ln n}$$

记 $x_n=\sum\limits_{i=1}^{n}\frac{1}{i}-\ln n$，有

$$x_n\geq\sum_{i=1}^{n}\ln(1+\frac{1}{i})-\ln n=\ln(n+1)-\ln n\geq 0.$$

$$x_{n+1}-x_n=\frac{1}{n+1}-\ln\frac{n+1}{n}=\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)\leq\frac{1}{n+1}-\frac{1}{n+1}=0.$$

由 $\{x_n\}$ 单调有界可知 $\lim\limits_{n\to\infty}x_n$ 存在，记为 x. 则

$$\lim_{n\to\infty}\frac{\sum\limits_{i=1}^{n}\frac{1}{i}}{\ln n}=\lim_{n\to\infty}\frac{\ln n+x}{\ln n}=\lim_{n\to\infty}\left(1+\frac{x}{\ln n}\right)=1.$$

即

$$E\left(\frac{X_n}{n\ln n}\right)=1.$$

5 (P173 T11)

Proof. 不妨记 $P(X=x_i)=p_i$，其中 $p_i>0$ $(i=1,2,\ldots,k)$，$\sum\limits_{i=1}^{k}p_i=1$，不妨记 $\max\limits_{1\leq i\leq k}x_i=x_m$，有

$$E[X^n]=\sum_{i=1}^{k}x_i^n p_i\ \Rightarrow\ \lim_{n\to\infty}\frac{E[X^n]}{x_m^n}=p_m.$$

故

$$\lim_{n\to\infty}\frac{E[X^{n+1}]}{E[X^n]}=\lim_{n\to\infty}\frac{E[X^{n+1}]}{x_m^{n+1}}\frac{x_m^n\cdot x_m}{E[X^n]}=x_m=\max_{1\leq i\leq k}x_i.$$

□