概统作业 (Week 6)

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April 23, 2023

1 (P118 T20)

由题意可得 (x, y) 密度函数 $f(x,y) = \dfrac{1}{4}$，有

$$F_Z(z) = \iint_{|x-y|\le z} f(x,y)dxdy = \begin{cases} 0 & (z \le 0) \\ \dfrac{-z^2+4z}{4} & (0 \le z < 1) \\ 1 & (z \ge 1) \end{cases} \Rightarrow f_Z(z) = \begin{cases} 1 - \dfrac{z}{2} & (0 < z < 1) \\ 0 & \text{其他}. \end{cases}$$

2 (P118 T22)

(1)

$$\text{若}(x,y) \in D,\ f(x,y) = \frac{1}{S} = 2.\ \Rightarrow\ f(x,y) = \begin{cases} 2 & (0 \le x \le y \le 1) \\ 0 & \text{其他} \end{cases}$$

(2)

$$f_1(x) = \int_x^1 f(x,y)dy = \begin{cases} 2(1-x) & (0 \le x \le 1) \\ 0 & \text{其他} \end{cases}$$

$$f_2(y) = \int_0^y f(x,y)dx = \begin{cases} 2y & (0 \le y \le 1) \\ 0 & \text{其他} \end{cases}$$

(3)

$$\text{若}f_2(y) > 0,\ \text{即}0 \le y \le 1,\ f_{X|Y}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{2}{2y} = \frac{1}{y}.\ \Rightarrow\ f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{y} & (0 \le y \le 1) \\ \\ 0 & \text{其他}. \end{cases}$$

(4) 对于 $0 \le y \le 1$，有

$$P(X \le 0.5|Y=y) = \begin{cases} \displaystyle\int_0^{0.5} f_{X|Y}(x|y)dx & (0.5 \le y \le 1) \\ \\ \displaystyle\int_0^{y} f_{X|Y}(x|y)dx & (0 \le y < 0.5) \end{cases} = \begin{cases} \dfrac{1}{2y} & (0.5 \le y \le 1) \\ \\ 1. & (0 \le y < 0.5) \end{cases}$$

考虑 $(x,y) \notin D$，有

$$P(X \le 0.5|Y=y) = \begin{cases} \dfrac{1}{2y} & (0.5 \le y \le 1) \\ \\ 1 & (0 \le y < 0.5) \\ \\ 0 & \text{其他}. \end{cases}$$

3 (P119 T27)

由题意可得 $f_X(x)=e^{-x}\cdot I_{(0,+\infty)}(x)$，$f_Y(y)=\dfrac{1}{\sqrt{2\pi}}\cdot e^{-\frac{y^2}{2}}$，对 $x,y\geq 0$ 有

$$f_{|Y|}(y)=f_Y(y)+f_Y(-y)=\sqrt{\frac{2}{\pi}}\cdot e^{-\frac{y^2}{2}},\quad f_{(X,|Y|)}(x,y)=f_X(x)\cdot f_{|Y|}(y)=\sqrt{\frac{2}{\pi}}\cdot e^{-x-\frac{y^2}{2}}$$

即

$$f_{(X,|Y|)}(x,y)=\begin{cases}\sqrt{\dfrac{2}{\pi}}\cdot e^{-x-\frac{y^2}{2}} & (x,y\geq 0)\\ \\ 0. & \text{其他.}\end{cases}$$

4 (P120 T35)

(1)

$$\iint_{x>0,y>0} f(x,y)dxdy=1\ \Rightarrow\ \frac{A}{12}=1\ \Rightarrow\ A=12.$$

(2) X 与 Y 独立.

$$f_X(x)=\int_0^{+\infty}f(x,y)dy=\begin{cases}3\cdot e^{-3x} & (x>0)\\ \\ 0. & \text{其他.}\end{cases}$$

$$f_Y(y)=\int_0^{+\infty}f(x,y)dx=\begin{cases}4\cdot e^{-4y} & (y>0)\\ \\ 0. & \text{其他.}\end{cases}$$

有

$$f(x,y)=f_X(x)\cdot f_Y(y)\ \Rightarrow\ X\text{ 与 }Y\text{ 独立.}$$

(3) 对于 $z>0$，有

$$F_Z(z)=\iint_{x+y\leq z}f(x,y)dxdy=\int_0^z dx\int_0^{z-x}dy\cdot 12\cdot e^{-(3x+4y)}=-4\cdot e^{-3z}+3\cdot e^{-4z}+1.$$

故

$$F_Z(z)=\begin{cases}-4\cdot e^{-3z}+3\cdot e^{-4z}+1. & (z>0)\\ 0. & (z\leq 0)\end{cases}$$

可得密度函数

$$f_Z(z)=[F_Z(z)]'=\begin{cases}12(e^{-3z}-e^{-4z}). & (z>0)\\ 0. & (z\leq 0).\end{cases}$$

(4)

$$f_1(x|x+y=1)=\begin{cases}\dfrac{f(x,1-x)}{f_Z(1)} & (0<x<1)\\ \\ 0 & \text{其他.}\end{cases}=\begin{cases}\dfrac{e^x}{e-1} & (0<x<1)\\ \\ 0 & \text{其他.}\end{cases}$$

$$\Rightarrow\ P(X>0.5|X+Y=1)=\int_{0.5}^{+\infty}f_1(x|x+y=1)dx=\frac{e-\sqrt{e}}{e-1}.$$

5 (P121 T42)

Proof.

(1) 先求 X 与 Y，Y 与 Z，X 与 Z 这三组的联合分布密度函数

$$f_{(X,Y)}(x,y)=\int_0^{2\pi} f(x,y,z)dz=\begin{cases}\frac{1}{4\pi^2}, & (0\le x,y\le 2\pi)\\ 0, & \text{其他}.\end{cases}$$

$$f_{(Y,Z)}(y,z)=\int_0^{2\pi} f(x,y,z)dx=\begin{cases}\frac{1}{4\pi^2}, & (0\le y,z\le 2\pi)\\ 0, & \text{其他}.\end{cases}$$

$$f_{(X,Z)}(x,z)=\int_0^{2\pi} f(x,y,z)dy=\begin{cases}\frac{1}{4\pi^2}, & (0\le x,z\le 2\pi)\\ 0, & \text{其他}.\end{cases}$$

(2) 再求 X，Y，Z 三个的边缘分布密度函数

$$f_X(x)=\int_0^{2\pi} dy\int_0^{2\pi} dz\cdot f(x,y,z)=\frac{1}{2\pi}.$$

$$f_Y(y)=\int_0^{2\pi} dx\int_0^{2\pi} dz\cdot f(x,y,z)=\frac{1}{2\pi}.$$

$$f_Z(z)=\int_0^{2\pi} dx\int_0^{2\pi} dy\cdot f(x,y,z)=\frac{1}{2\pi}.$$

(3) 由 (1),(2) 有

$$f_{(X,Y)}(x,y)=f_X(x)\cdot f_Y(y).\quad f_{(Y,Z)}(y,z)=f_Y(y)\cdot f_Z(z).\quad f_{(X,Z)}(x,z)=f_X(x)\cdot f_Z(z).$$

故 X，Y，Z 两两独立，又

$$f_X(x)\cdot f_Y(y)\cdot f_Z(z)=\frac{1}{8\pi^3}\neq f(x,y,z).$$

故 X，Y，Z 不相互独立.

□