1.(P172, 4)设X为一个连续型随机变量, 试对下列各种情形, 计算EX.

(1) 若X的密度函数为

$$f(x)=\frac{x}{\sigma^2}\exp\left\{-\frac{x^2}{2\sigma^2}\right\},x>0,$$

其中$\sigma>0$为常数, 则称X服从瑞利(Rayleigh)分布;

(2) 若X的密度函数为

$$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1},0<x<1,$$

其中$\alpha,\beta>0$为常数, $\Gamma(x)$为Γ函数, 则称X服从β分布;

(3) 若X的密度函数为

$$f(x)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\},x>0,$$

其中$k,\lambda>0$为常数, 则称X服从韦布尔分布.

解:

(1)

$$\begin{aligned}EX&=\int_0^{\infty}xf(x)\mathrm{d}x\\&=\int_0^{\infty}\frac{x^2}{\sigma^2}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}\mathrm{d}x\\&=-x\exp\left\{-\frac{x^2}{2\sigma^2}\right\}\Big|_0^{\infty}+\int_0^{\infty}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}\mathrm{d}x\\&=\sqrt{2}\sigma\frac{\sqrt{\pi}}{2}\\&=\frac{\sqrt{2\pi}\sigma}{2}\end{aligned}$$

(2)

$$\begin{aligned}EX&=\int_0^1xf(x)\mathrm{d}x\\&=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1x^{(\alpha+1)-1}(1-x)^{\beta-1}\mathrm{d}x\\&=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\\&=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\\&=\frac{\alpha}{\alpha+\beta}\end{aligned}$$

(3)作换元$t=\left(\frac{x}{\lambda}\right)^k$, 则$\mathrm{d}t=k\left(\frac{x}{\lambda}\right)^{k-1}\mathrm{d}x,x=\lambda t^{1/k}$

$$\begin{aligned}EX&=\int_0^{\infty}k\left(\frac{x}{\lambda}\right)^k\exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\}\mathrm{d}x\\&=\int_0^{\infty}\lambda t^{(1/k+1)-1}e^{-t}\mathrm{d}t\\&=\lambda\Gamma(\frac{1}{k}+1)\end{aligned}$$

2.(P173, 13)设随机变量X的密度函数为$f(x)=2(x-1),1<x<2$, 试求随机变量$Y=e^X$和$Z=\frac{1}{X}$的数学期望.

解：

$EY = \int_1^2 e^x f(x)\mathrm{d}x = \int_1^2 2(x-1)e^x\mathrm{d}x = 2\int_1^2 x\mathrm{d}e^x - 2\int_1^2 e^x\mathrm{d}x = 2xe^x|_1^2 - 4\int_1^2 e^x\mathrm{d}x = 2e$

$EZ = \int_1^2 \frac{1}{x} f(x)\mathrm{d}x = 2\int_1^2 \frac{x-1}{x}\mathrm{d}x = 2 - 2\ln 2$

> **3.(P173, 17)**设随机变量X的密度函数为
>
> $$f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty.$$
>
> 试求$E(\min\{|X|, 1\})$.

解：设$Y = \min\{|X|, 1\}$，则Y的概率密度函数为

$$f(y) = \begin{cases} \frac{1}{\pi(1+y^2)}, & |y| < 1, \\ \frac{1}{2}, & |y| = 1. \end{cases}$$

则$EY = \int_{-\infty}^{\infty} yf(y) = \int_{-1}^{1} |x|f(x)\mathrm{d}x + \frac{1}{2} = \frac{1}{\pi}\ln 2 + \frac{1}{2}$

> **4.(P173, 18)**设随机变量X的分布律为$P(X=1) = P(X=2) = \frac{1}{2}$, 在给定$X = i$的条件下，随机变量$Y$服从均匀分布$U(0, i)(i = 1, 2)$.
>
> (1)求Y的分布函数;
>
> (2)求期望$E(Y)$.

解：

由题，$f(y|X=1) = I(0,1), f(y|X=2) = \frac{1}{2}I(0,2)$

则$f(y) = \begin{cases} \frac{3}{4}, & 0 < y \leqslant 1, \\ \frac{1}{4}, & 1 < y \leqslant 2, \\ 0, & \text{其他}. \end{cases}$，$F(y) = \begin{cases} 0, & y \leqslant 0, \\ \frac{3}{4}y, & 0 < y \leqslant 1, \\ \frac{y+2}{4}, & 1 < y \leqslant 2, \\ 1, & y > 2. \end{cases}$

$EY = \int_{-\infty}^{\infty} yf(y)\mathrm{d}y = \int_0^1 \frac{3}{4}y\mathrm{d}y + \int_1^2 \frac{1}{4}y\mathrm{d}y = \frac{3}{4}$

> **5.**设$X_1, X_2 \ldots$为一列独立同分布的随机变量, 随机变量N只取正整数值, 且N与$\{X_n\}$独立, 试证明: $E(\sum_{i=1}^{N} X_i) = E(X_1)E(N)$. (提示：利用条件期望的平滑公式/全期望公式)

证明：

根据全期望公式，有

$$E\left(\sum_{i=1}^{N} X_i\right) = E\left[E\left(\sum_{i=1}^{N} X_i | N = n\right)\right]$$

而由$X_1, X_2, \ldots$独立同分布，且与N独立，有

$$E\left(\sum_{i=1}^{N} X_i | N = n\right) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = nE(X_1)$$

所以

$$E\left(\sum_{i=1}^{N} X_i\right) = E[NE(X_1)] = E(X_1)E(N)$$