

概统作业 (Week 8)

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1 (P122 T48)

Proof.

(1) 有 X 和 Y 边缘分布

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\}} \Rightarrow X \sim N(0, 1), Y \sim N(0, 1)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.$$

有 Z 分布

$$f_Z(z) = P(Z \leq z) = \iint_{\frac{y-\rho x}{\sqrt{1-\rho^2}} \leq z} f(x, y) dx dy = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{\sqrt{1-\rho^2}z + \rho x} dy f(x, y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

有 X, Z 联合分布

$$f_{(X, Z)}(x, z) = f(x, \sqrt{1-\rho^2}z + \rho x) \cdot |J| = \sqrt{1-\rho^2} \cdot f(x, \sqrt{1-\rho^2}z + \rho x) = \frac{1}{2\pi} e^{\{-\frac{1}{2}(x^2 + z^2)\}} = f_X(x) \cdot f_Z(z).$$

即证.

(2)

$$\begin{aligned} P(XY < 0) &= 1 - P(XY \geq 0) \\ &= 1 - P(XY > 0) \\ &= 1 - P(X > 0, Y > 0) - P(X < 0, Y < 0) \end{aligned}$$

由

$$f(-x, -y) = f(x, y) \Rightarrow P(X < 0, Y < 0) = P(-x < 0, -y < 0) = P(X > 0, Y > 0).$$

即

$$P(XY < 0) = 1 - 2P(X > 0, Y > 0).$$

有

$$\begin{aligned} P(X > 0, Y > 0) &= P(X > 0, (\sqrt{1-\rho^2}Z + \rho X) > 0) \\ &= P(X > 0, (\sqrt{1-\rho^2}Z + \rho X) > 0) \\ &= P(X > 0, Z > \frac{-\rho X}{\sqrt{1-\rho^2}}) \\ &= \iint_{x>0, \sqrt{1-\rho^2}z + \rho x > 0} f_{(X, Z)}(x, z) dx dz. \end{aligned}$$

做代换

$$\begin{cases} x = r \cos \alpha \\ z = r \sin \alpha \end{cases} \Rightarrow |J| = \frac{\partial(x, z)}{\partial(r, \alpha)} = \begin{vmatrix} \cos \alpha & \sin \alpha \\ -r \sin \alpha & -r \cos \alpha \end{vmatrix} = r. \quad (0 \leq \alpha < 2\pi, r > 0)$$

记 $\rho = \cos \beta$, $\sqrt{1 - \rho^2} = \sin \beta$ ($0 < \beta < \frac{\pi}{2}$), 有

$$\iint_{x>0, \sqrt{1-\rho^2}z+\rho x>0} f_{(X,Z)}(x, z) dx dz = \iint_{\sin \alpha > 0, \cos(\alpha-\beta) > 0} r \cdot \frac{1}{2\pi} e^{-\frac{r^2}{2}} dr d\theta.$$

$$\begin{aligned} P(X > 0, Y > 0) &= \iint_{\sin \alpha > 0, \cos(\alpha-\beta) > 0} r \cdot \frac{1}{2\pi \cdot \sin \beta} e^{-\frac{r^2}{2}} dr d\theta \\ &= \int_{\sin \alpha > 0, \cos(\alpha-\beta) > 0} \frac{1}{2\pi \cdot \sin \beta} d\theta \int_0^{+\infty} -e^{-\frac{r^2}{2}} d\left(-\frac{r^2}{2}\right) \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2\pi} d\theta + \int_{\frac{3\pi}{2}+\beta}^{2\pi} \frac{1}{2\pi} d\theta \\ &= \frac{\pi - \beta}{2\pi} \\ &= \frac{1}{2} - \frac{\arccos \rho}{2\pi} \end{aligned}$$

故

$$P(XY < 0) = 1 - P(X > 0, Y > 0) = \pi^{-1} \arccos \rho.$$

□

2 (P171 T1)

(0) 对 $0 < p < 1$, 记 $q = 1 - p$, $\forall X \in \{4, 5, 6, 7\}$, 有

$$P(X = n) = \binom{n-1}{n-4} \cdot (p^4 \cdot q^{n-4} + q^4 \cdot p^{n-4})$$

$$E(X) = 4 \cdot P(X = 4) + 5 \cdot P(X = 5) + 6 \cdot P(X = 6) + 7 \cdot P(X = 7)$$

(1) 对 $p = 0.5$, 有

$$E(X) = 4 \cdot \frac{1}{8} + 5 \cdot 4 \cdot \frac{1}{16} + 6 \cdot 10 \cdot \frac{1}{32} + 7 \cdot 20 \cdot \frac{1}{64} = \frac{93}{16}$$

(1) 对 $p = 0.6$, 有

$$E(X) = 4 \cdot \frac{3^4 + 2^4}{5^4} + 5 \cdot 4 \cdot \frac{3^4 \cdot 2 + 2^4 \cdot 3}{5^5} + 6 \cdot 10 \cdot \frac{3^4 \cdot 2^2 + 2^4 \cdot 3^2}{5^6} + 7 \cdot 20 \cdot \frac{3^4 \cdot 2^3 + 2^4 \cdot 3^3}{5^7} = \frac{17804}{3125}$$

3 (P171 T2)

Proof.

(1)

$$E(X) = \sum_{k=1}^{\infty} k \cdot P(X = k) = \sum_{k=1}^{\infty} \sum_{n=1}^k P(X = k) = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P(X = k) = \sum_{n=1}^{\infty} P(X \geq n)$$

(2)

$$E(X) = \int_0^{\infty} t f(t) dt = \int_0^{\infty} f(t) dt \int_0^t dx = \int_0^{\infty} dx \int_x^{\infty} f(t) dt = \int_0^{\infty} (1 - F(x)) dx.$$

(3) 以 $I(A)$ 表示事件 A 的示性函数, 则有

$$E(X) = E\left(\int_0^X dx\right) = E\left(\int_0^\infty I(X > x) dx\right) = \int_0^\infty E[I(X > x)] dx = \int_0^\infty P(X > x) dx$$

又 $P(X > x) = 1 - F(x)$, 即证

$$E(X) = \int_0^\infty (1 - F(x)) dx.$$

□

4 (P171 T8)

(1) 用 $E(T_i)$ 表示从第 $i-1$ 种到第 i 种需要买的卡片的期望, 则单次买到第 i 种卡片的概率为 $p = \frac{n}{n-i+1}$. 记 $q = 1 - p$.

$$E(T_i) = \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot p = p \cdot \left(\sum_{k=1}^{\infty} q^k\right)' = \frac{1}{p} = \frac{n}{n-i+1}.$$

$$E(X_n) = \sum_{i=1}^n E(T_i) = \sum_{i=1}^n \frac{n}{n-i+1} = n \cdot \sum_{i=1}^n \frac{1}{i} = 12 \cdot \sum_{i=1}^{12} \frac{1}{i} = \frac{86021}{2310} \approx 37.24$$

(2)

$$\lim_{n \rightarrow \infty} E\left(\frac{X_n}{n \ln n}\right) = \lim_{n \rightarrow \infty} \frac{E(X_n)}{n \ln n} = \lim_{n \rightarrow \infty} \frac{n \cdot \sum_{i=1}^n \frac{1}{i}}{n \ln n} = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \frac{1}{i}}{\ln n}$$

记 $x_n = \sum_{i=1}^n \frac{1}{i} - \ln n$, 有

$$x_n \geq \sum_{i=1}^n \ln\left(1 + \frac{1}{i}\right) - \ln n = \ln(n+1) - \ln n \geq 0.$$

$$x_{n+1} - x_n = \frac{1}{n+1} - \ln \frac{n+1}{n} = \frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right) \leq \frac{1}{n+1} - \frac{1}{n+1} = 0.$$

由 $\{x_n\}$ 单调有界可知 $\lim_{n \rightarrow \infty} x_n$ 存在, 记为 x . 则

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \frac{1}{i}}{\ln n} = \lim_{n \rightarrow \infty} \frac{\ln n + x}{\ln n} = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{\ln n}\right) = 1.$$

即

$$E\left(\frac{X_n}{n \ln n}\right) = 1.$$

5 (P173 T11)

Proof. 不妨记 $P(X = x_i) = p_i$, 其中 $p_i > 0$ ($i = 1, 2, \dots, k$), $\sum_{i=1}^k p_i = 1$, 不妨记 $\max_{1 \leq i \leq k} x_i = x_m$, 有

$$E[X^n] = \sum_{i=1}^k x_i^n p_i \Rightarrow \lim_{n \rightarrow \infty} \frac{E[X^n]}{x_m^n} = p_m.$$

故

$$\lim_{n \rightarrow \infty} \frac{E[X^{n+1}]}{E[X^n]} = \lim_{n \rightarrow \infty} \frac{E[X^{n+1}]}{x_m^{n+1}} \frac{x_m^n \cdot x_m}{E[X^n]} = x_m = \max_{1 \leq i \leq k} x_i.$$

□