概统作业 (Week 10)

PB20000113 孔浩宇

May 13, 2023

1

(0) 选 D.

(1)

$$E(Y_1)=\int_{-\infty}^{+\infty} y\cdot\frac{f_1(y)+f_2(y)}{2}dy=\frac{1}{2}\cdot\left(\int_{-\infty}^{+\infty} y\cdot f_1(y)dy+\int_{-\infty}^{+\infty} y\cdot f_2(y)dy\right)=\frac{E(X_1)+E(X_2)}{2}.$$

$$E(Y_2)=E\left(\frac{X_1+X_2}{2}\right)=\frac{1}{2}\cdot(E(X_1+X_2))=\frac{E(X_1)+E(X_2)}{2}=E(Y_1).$$

(2)

$$E(Y_1^2)=\int_{-\infty}^{+\infty} y^2\cdot\frac{f_1(y)+f_2(y)}{2}dy=\frac{1}{2}\cdot\left(\int_{-\infty}^{+\infty} y^2\cdot f_1(y)dy+\int_{-\infty}^{+\infty} y^2\cdot f_2(y)dy\right)=\frac{E(X_1^2)+E(X_2^2)}{2}.$$

$$Var(Y_1)=E(Y_1^2)-E(Y_1)^2=\frac{E(X_1^2)+E(X_2^2)}{2}-\left(\frac{E(X_1)+E(X_2)}{2}\right)^2$$

又 $Var(X_1)=E(X_1^2)-E(X_1)^2$，$Var(X_2)=E(X_2^2)-E(X_2)^2$，有

$$Var(Y_1)=\frac{Var(X_1)+Var(X_2)}{4}+\frac{E(X_1^2)+E(X_2^2)-2E(X_1)E(X_2)}{4}$$

对于 Y_2，有

$$Var(Y_2)=Var\left(\frac{X_1+X_2}{2}\right)=\frac{Var(X_1)+Var(X_2)}{4}$$

又由方差 $Var(X)=E(X^2)-E(X)^2\geq 0$，有

$$\begin{cases} E(X_1^2) & \geq E(X_1)^2 \\ E(X_2^2) & \geq E(X_2)^2 \end{cases} \Rightarrow\ E(X_1^2)+E(X_2^2)-2E(X_1)E(X_2)\geq E(X_1)^2+E(X_2)^2-2E(X_1)E(X_2)\geq 0.$$

故

$$Var(Y_1)-Var(Y_2)\geq 0\ \Rightarrow\ Var(Y_1)\geq Var(Y_2).$$

2 (P174 T27)

(1)

$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}\ (k=0,1,\dots,n)$$

$$M_X(s)=E(e^{sX})=\sum_{k=0}^{n}e^{sk}\binom{n}{k}p^k(1-p)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}(pe^s)^k(1-p)^{n-k}=(pe^s+1-p)^n.$$

(2)

$$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}\ (k=0,1,2,...)$$

$$M_X(s)=\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}e^{sk}=e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda e^{s^k}}{k!}=e^{-\lambda}e^{\lambda e^s}=e^{\lambda(e^s-1)}.$$

3 (P175 T30)

(1)

$$E(X)=(0.05+0.05+0.1)-(0.1+0.2+0.2)=-0.3.$$

$$E(Y)=(0.2+0.15+0.1)-(0.1+0.05+0.05)=0.25.$$

$$E(XY)=(0.1+0.1)-(0.2+0.05)=-0.05.$$

$$Cov(X,Y)=E(XY)-E(X)E(Y)=-0.05+0.075=0.025.$$

(2)

X \ Y	0	1
0	0.1	0.2
1	0.25	0.45

$$E(X^2)=(0.05+0.05+0.1)+(0.1+0.2+0.2)=0.7.$$

$$E(Y^2)=(0.2+0.15+0.1)+(0.1+0.05+0.05)=0.65$$

$$E(X^2Y^2)=0.45.$$

$$Cov(X^2,Y^2)=E(X^2Y^2)-E(X^2)E(Y^2)=0.45-0.7\times 0.65=-0.005.$$

4 (P175 T32)

(1)

$$Cov(\alpha X+\beta Y,\alpha X-\beta Y)=\alpha^2Var(X)-\beta^2Var(Y)=(\alpha^2-\beta^2)\sigma^2.$$

(2) 又 $(\alpha X+\beta Y,\alpha X-\beta Y)$ 服从二维正态分布，故当 $\alpha=\pm\beta$ 时，$\alpha X+\beta Y,\alpha X-\beta Y$ 相互独立.

5 (P175 T34)

(1)

$$E(X)=\iint_{(x,y)\in G}xdxdy=0.\qquad E(Y)=\iint_{(x,y)\in G}ydxdy=0.$$

$$E(XY)=\iint_{(x,y)\in G}xydxdy=0.$$

故

$$Cov(X,Y)=E(XY)-E(X)E(Y)=0.$$

(2)

$$f(x,y) = \frac{1}{S} = \frac{1}{2}.$$

$$f_X(x) = \int_{-\infty}^{+\infty} dy \int_{|x|+|y|\le 1} f(x,y)dx = \begin{cases} 1+x, & (-1 \le x \le 0) \\ 1-x, & (0 < x \le 1) \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} dx \int_{|x|+|y|\le 1} f(x,y)dy = \begin{cases} 1+y, & (-1 \le y \le 0) \\ 1-y, & (0 < y \le 1) \end{cases}$$

$$f(x,y) \ne f_X(x)f_Y(y) \;\Rightarrow\; X,Y\text{不相互独立}.$$