HW7

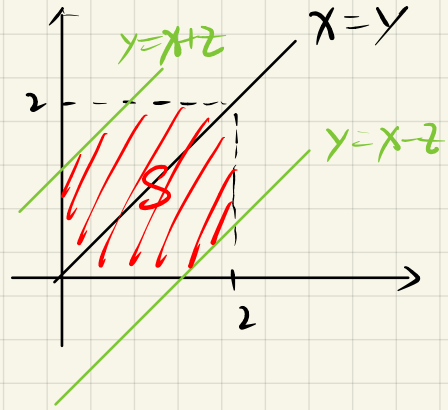
71. $f(x,y) = \frac{1}{4}$, 设 $Z = |X-Y|$

则 $F_Z(z) = P(Z \le z) = \iint_{|x-y| \le z} \frac{1}{4}\, dxdy$

$= \iint_{\substack{0<x<y<2 \\ y \le x+z}} \frac{1}{4}\, dxdy + \iint_{\substack{0<y<x<2 \\ y \ge x-z}} \frac{1}{4}\, dxdy$

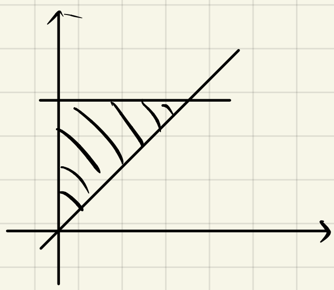
$= \frac{1}{4} \cdot S$

$= z - \frac{1}{4} z^2 \quad (z \in [0,2])$



$f_Z(z) = \dfrac{dF_Z(z)}{dz} = 1 - \frac{1}{2} z \quad (z \in [0,2])$

72. (1) $f(x,y) = 2 \quad (x,y \in D)$



(2) $f_X(x) = \int_{y \in D} f(x,y)\, dy$

$= \int_x^1 2\, dy$

$= 2 - 2x \quad (x \in [0,y])$

$\Rightarrow f_X(x) = \begin{cases} 2-2x & (x \in [0,y]) \\ 0 & \text{others} \end{cases}$

$f_Y(y) = \int_{x \in D} f(x,y)\, dx$

$= \int_0^y 2\, dx$

$= 2y \quad (y \in [x,1])$

$\Rightarrow f_Y(y) = \begin{cases} 2y & (y \in [x,1]) \\ 0, & \text{others} \end{cases}$

(3) $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{2}{2y} = \dfrac{1}{y} \quad (x,y \in D)$

(4) $P(X \le 0.5 \mid Y = y) = \int_0^{0.5} f_{X|Y}(x|y)\,dx$

$= \dfrac{1}{2y} \qquad y \in (\frac{1}{2}, 1]$

$\Rightarrow P = \begin{cases} \dfrac{1}{2y} & y \in (\frac{1}{2}, 1] \\ 1 & y \in [0, \frac{1}{2}] \end{cases}$

T3. $f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}$

$f_{|Y|}(y) = \begin{cases} \dfrac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} & y > 0 \\ 0 & y \le 0 \end{cases}$

由 X 与 Y 独立，故 X 与 $|Y|$ 独立

有 $f(x,y) = f_X(x) \cdot f_{|Y|}(y)$

$= \dfrac{2}{\sqrt{2\pi}} e^{-x-\frac{y^2}{2}} \quad (x, y > 0)$

T4. (1). $\int_{x,y\in D} f(x,y)\,dx\,dy = 1$

$\Leftrightarrow A\int_0^{+\infty} e^{-3x} \int_0^{+\infty} e^{-4y}\,dy\,dx = 1$

$\Leftrightarrow \frac{1}{12}A = 1$

$\Rightarrow A = 12$

(2) $f_X(x) = \int_0^{+\infty} f(x,y)\,dy$

$= \int_0^{+\infty} 12e^{-3x} \cdot e^{-4y}\,dy$

$$= 3e^{-3x}$$

$$f_Y(y) = \int_0^{+\infty} f(x,y)\,dx$$

$$= 4e^{-4y}$$

由 $f(x,y) = f_X(x)\cdot f_Y(y)$ 有，X,Y 独立

(3) $$f_Z(z) = \int_{-\infty}^{+\infty} f(x, z-x)\,dx$$

$$= \int_0^z 12\cdot e^{-(4z-x)}\,dx$$

$$= 12\cdot e^{-4z}(e^z-1) \quad (z\in[0,+\infty))$$

(4) 设 $Z_1 = X$，$Z_2 = X+Y$，求 Z_1, Z_2 联合分布、

有反函数：$\begin{cases} x = z_1 \\ y = z_2 - z_1 \end{cases}$

则 $$f(z_1,z_2) = 12e^{-(4z_2-z_1)}\cdot\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$$

$$= 12e^{-4z_2+z_1}$$

$$f_{Z_2}(z_2) = \int_0^{z_2} 12e^{-4z_2+z_1}\,dz_1$$

$$= 12\cdot e^{-4z_2}(e^{z_2}-1) \quad (z_2\in[0,+\infty))$$

则 $$f_{Z_1|Z_2}(z_1|z_2) = \frac{f(z_1,z_2)}{f_{Z_2}(z_2)}$$

$$= \frac{e^{-4z_2+z_1}}{e^{-4z_2}(e^{z_2}-1)}$$

$$= \frac{e^{z_1}}{e^{z_2}-1}$$

$$P(z_1>0.5 \mid z_2=1) = \int_{\substack{z_1>0.5 \\ z_2=1}} f_{z_1|z_2}\, dz_1$$

$$= \int_{0.5}^{1} \frac{e^{z_1}}{e-1}\, dz_1$$

$$= \frac{e-e^{\frac{1}{2}}}{e-1}$$

T5、 先证 X, Y 独立：

$$f(x,y) = \int_{-\infty}^{+\infty} f(x,y,z)\, dz$$

$$= \frac{1}{8\pi^3} \int_0^{2\pi} 1-\sin x \sin y \sin z \, dz$$

$$= \frac{1}{4\pi^2}$$

$$f_X(x) = \iint_{y,z\in D} f(x,y,z)\, dy\, dz = \frac{1}{2\pi}$$

$$f_Y(y) = \frac{1}{2\pi}$$

由 $f(x,y) = f_X(x) \cdot f_Y(y)$ 有 X, Y 独立

由 x, y, z 轮换对称，同理有 X, Z 独立与 Y, Z 独立、

即 X, Y, Z 两两独立

由 $f_X(x) = f_Y(y) = f_Z(z) = \frac{1}{2\pi}$

故 $f(x,y,z) \neq f_X \cdot f_Y \cdot f_Z$

故三者不相互独立