概统作业 (Week 5)

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1 (P84 第 27 题)

Proof.

(1) 先证 $\forall\, x = \frac{n}{m} \in (0,1)$，有 $F(x) = x$. 不妨取正整数 $n < m$.

$$F(1) - F(\frac{m-1}{m}) = F(\frac{m-1}{m}) - F(\frac{m-2}{m}) = \cdots = F(\frac{1}{m}) - F(0)$$

又

$$\sum_{i=1}^{m} F(\frac{i}{m}) - F(\frac{i-1}{m}) = F(1) - F(0) = 1. \;\Rightarrow\; F(\frac{1}{m}) = \frac{1}{m}, F(\frac{n}{m}) = \frac{n}{m}.$$

(2) 再证 $\forall\, x \in (0,1)$，有 $F(x) = x$.

(a) 若 $x \in \mathbb{Q}$，有 $m, n \in \mathbb{N}$，使 $x = \frac{n}{m}$，由 (1) 可知成立.

(b) 若 $x \notin \mathbb{Q}$，则存在有理数列 $\{Q_n\}$ 满足

$$\lim_{n\to+\infty} Q_n = x \;\Rightarrow\; F(x) = \lim_{n\to+\infty} F(Q_n) = \lim_{n\to+\infty} Q_n = x.$$

综上可得

$$F(x) = x.\ (0 < x < 1) \;\Rightarrow\; f(x) = I_{(0,1)}(x). \;\Rightarrow\; X \sim U(0,1).$$

□

2 (P85 第 32 题)

(1)

$$P(96 \le R \le 104) = \frac{104-96}{105-95} = 0.8.$$

即比例为 0.8.

(2) 取 $X = \frac{R-100}{2} \sim N(0,1)$，则

$$P(96 \le R \le 104) = P(-2 \le X \le 2) = 2\Phi(2) - 1 = 0.9544.$$

3 (P85 第 31 题)

取 $Y = \frac{X-1}{2} \sim N(0,1)$.

(1)

$$P(0 \le X \le 4) = P(-0.5 \le Y \le 1.5) = \Phi(1.5) + \Phi(-0.5) - 1 = 0.6247.$$

$$P(X > 2.4) = P(Y > 0.7) = 1 - \Phi(0.7) = 0.2420.$$

$$P(|X| > 2) = 2P(X > 2) = 2P(Y > 0.5) = 2(1 - \Phi(0.5)) = 0.6170.$$

(2)

$$
\begin{aligned}
P(X>c)=2P(X\le c) \Leftrightarrow\ & P(Y>\frac{c-1}{2})=2P(Y\le\frac{c-1}{2})\\
\Leftrightarrow\ & 1-\Phi(\frac{c-1}{2})=2\Phi(\frac{c-1}{2})\\
\Leftrightarrow\ & \Phi(\frac{c-1}{2})=\frac{1}{3},\ \Phi(\frac{1-c}{2})=\frac{2}{3}\\
\Leftrightarrow\ & c\approx 0.14.
\end{aligned}
$$

4 (P86 第 49 题)

(1)

$$
F_1(y)=P(Y\le y)=P(\frac{1-X}{X}\le y)=\begin{cases}0 & (y<0)\\ \dfrac{y}{1+y} & (y\ge 0)\end{cases}
$$

由 $f_1(y)=F_1'(y)$，有

$$
f_1(y)=\begin{cases}0 & (y<0)\\ \dfrac{1}{(1+y)^2} & (y\ge 0)\end{cases}
$$

(2)

$$
F_2(z)=P(Z\le z)=P(XI_{(a,1]}(X)\le z)=\begin{cases}0 & (z<0)\\ a & (0\le z<a)\\ z & (a\le z\le 1)\\ 1 & (z>1)\end{cases}
$$

Z 不是连续型随机变量，故不存在密度函数.

(3)

$$
F_3(w)=P(W\le w)=P(X^2+XI_{[0,b]}(X)\le w)=\begin{cases}0 & (w<0)\\ \dfrac{\sqrt{4w+1}-1}{2} & (0\le w<b^2)\\ \dfrac{\sqrt{4w+1}-1}{2}+\sqrt{w}-b & (b^2\le w<b^2+b)\\ \sqrt{w} & (b^2+b\le w\le 1)\\ 1 & (w>1)\end{cases}
$$

由 $f_3(w)=F_3'(w)$，有

$$
f_3(w)=\begin{cases}\dfrac{1}{\sqrt{4w+1}} & (0\le w<b^2)\\ \dfrac{1}{\sqrt{4w+1}}+\dfrac{1}{2\sqrt{w}} & (b^2\le w<b^2+b)\\ \dfrac{1}{2\sqrt{w}} & (b^2+b\le w\le 1)\\ 0 & (w<0\text{或}w>1)\end{cases}
$$

5 (P116 第 2 题)

(1)

$$P(X=1|Z=0)=\frac{P(X=1,Z=0)}{P(Z=0)}=\frac{2\times\frac{1}{6}\times\frac{2}{6}}{\frac{1}{2}\times\frac{1}{2}}=\frac{4}{9}.$$

(2) $X=\{0,1,2\}$, $Y=\{0,1,2\}$, $X+Y\le 2$.

$$P(X=0,Y=0)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4} \qquad P(X=1,Y=0)=2\times\frac{1}{6}\times\frac{1}{2}=\frac{1}{6}$$

$$P(X=0,Y=1)=2\times\frac{1}{3}\times\frac{1}{2}=\frac{1}{3} \qquad P(X=1,Y=1)=2\times\frac{1}{6}\times\frac{1}{3}=\frac{1}{9}$$

$$P(X=0,Y=2)=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9} \qquad P(X=2,Y=0)=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$$

又 $X+Y\le 2$，故

$$P(X=1,Y=2)=P(X=2,Y=1)=P(X=2,Y=2)=0.$$

分布列如图

X \ Y	0	1	2
0	1/4	1/3	1/9
1	1/6	1/9	0
2	1/36	0	0