# 概统作业 (Week 8)

PB20000113 孔浩宇

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## 1 (P122 T48)

*Proof.*

(1) 有 $X$ 和 $Y$ 边缘分布

$$f(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}e^{\{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\}}\ \Rightarrow\ X\sim N(0,1),\ Y\sim N(0,1)$$

$$f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\ f_Y(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}.$$

有 $Z$ 分布

$$f_Z(z)=P(Z\le z)=\iint_{\frac{y-\rho x}{\sqrt{1-\rho^2}}\le z}f(x,y)dxdy=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{\sqrt{1-\rho^2}z+\rho x}dyf(x,y)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{z^2}{2}}$$

有 $X,Z$ 联合分布

$$f_{(X,Z)}(x,z)=f(x,\sqrt{1-\rho^2}z+\rho x)=\frac{1}{2\pi\sqrt{1-\rho^2}}e^{\{-\frac{1}{2}(x^2+z^2)\}}=f_X(x)\cdot f_Z(z).$$

即证.

(2)

$$\begin{aligned}
P(XY<0)&=P(X\cdot(\sqrt{1-\rho^2}Z+\rho X)<0)\\
&=P(X>0,(\sqrt{1-\rho^2}Z+\rho X)<0)+P(X<0,(\sqrt{1-\rho^2}Z+\rho X)>0)\\
&=P(X>0,Z<\frac{-\rho X}{\sqrt{1-\rho^2}})+P(X<0,Z>\frac{-\rho X}{\sqrt{1-\rho^2}})\\
&=\int_0^{+\infty}dx\int_{-\infty}^{\frac{-\rho X}{\sqrt{1-\rho^2}}}f_{(X,Z)}(x,z)dz\\
&=\int\\
&=\pi^{-1}\arccos\rho.
\end{aligned}$$

□

## 2 (P171 T1)

(0) 对 $0<p<1$，记 $q=1-p$，$\forall\ X\in\{4,\ 5,\ 6,\ 7\}$，有

$$P(X=n)=\binom{n-1}{n-4}\cdot(p^4\cdot q^{n-4}+q^4\cdot p^{n-4})$$

$$E(X)=4\cdot P(X=4)+5\cdot P(X=5)+6\cdot P(X=6)+7\cdot P(X=7)$$

(1) 对 $p=0.5$，有

$$E(X)=4\cdot\frac{1}{8}+5\cdot 4\cdot\frac{1}{16}+6\cdot 10\cdot\frac{1}{32}+7\cdot 20\cdot\frac{1}{64}=\frac{93}{16}$$

(1) 对 $p=0.6$，有

$$E(X)=4\cdot\frac{3^4+2^4}{5^4}+5\cdot 4\cdot\frac{3^4\cdot 2+2^4\cdot 3}{5^5}+6\cdot 10\cdot\frac{3^4\cdot 2^2+2^4\cdot 3^2}{5^6}+7\cdot 20\cdot\frac{3^4\cdot 2^3+2^4\cdot 3^3}{5^7}=\frac{17804}{3125}$$

# 3 (P171 T2)

*Proof.*

(1)

$$E(X)=\sum_{k=1}^{\infty}k\cdot P(X=k)=\sum_{k=1}^{\infty}\sum_{n=1}^{k}P(X=k)=\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}P(X=k)=\sum_{n=1}^{\infty}P(X\geq n)$$

(2)

$$E(X)=\int_0^{\infty}tf(t)dt=\int_0^{\infty}f(t)dt\int_0^{t}dx=\int_0^{\infty}dx\int_x^{\infty}f(t)dt=\int_0^{\infty}(1-F(x))dx.$$

(3)

□

# 4 (P171 T8)

(1) 用 $E(T_i)$ 表示从第 $i-1$ 种到第 $i$ 种需要买的卡片的期望, 则单次买到第 $i$ 种卡片的概率为 $p=\dfrac{n}{n-i+1}$. 记 $q=1-p$.

$$E(T_i)=\sum_{k=1}^{\infty}k\cdot q^{k-1}\cdot p=p\cdot\left(\sum_{k=1}^{\infty}q^k\right)'=\frac{1}{p}=\frac{n}{n-i+1}.$$

$$E(X_n)=\sum_{i=1}^{n}E(T_i)=\sum_{i=1}^{n}\frac{n}{n-i+1}=n\cdot\sum_{i=1}^{n}\frac{1}{i}=12\cdot\sum_{i=1}^{12}\frac{1}{i}=\frac{86021}{2310}\approx 37.24$$

(2)

$$\lim_{n\to\infty}E\left(\frac{X_n}{n\ln n}\right)=\lim_{n\to\infty}\frac{E(X_n)}{n\ln n}=\lim_{n\to\infty}\frac{n\cdot\sum\limits_{i=1}^{n}\frac{1}{i}}{n\ln n}=\lim_{n\to\infty}\frac{\sum\limits_{i=1}^{n}\frac{1}{i}}{\ln n}$$

记 $x_n=\sum\limits_{i=1}^{n}\frac{1}{i}-\ln n$，有

$$x_n\geq\sum_{i=1}^{n}\ln(1+\frac{1}{i})-\ln n=\ln(n+1)-\ln n\geq 0.$$

$$x_{n+1}-x_n=\frac{1}{n+1}-\ln\frac{n+1}{n}=\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)\leq\frac{1}{n+1}-\frac{1}{n+1}=0.$$

由 $\{x_n\}$ 单调有界可知 $\lim\limits_{n\to\infty}x_n$ 存在，记为 $x$. 则

$$\lim_{n\to\infty}\frac{\sum\limits_{i=1}^{n}\frac{1}{i}}{\ln n}=\lim_{n\to\infty}\frac{\ln n+x}{\ln n}=\lim_{n\to\infty}\left(1+\frac{x}{\ln n}\right)=1.$$

即

$$E\left(\frac{X_n}{n\ln n}\right)=1.$$

## 5 (P173 T11)

*Proof.* 不妨记 $P(X = x_i) = p_i$，其中 $p_i > 0$ $(i = 1, 2, \dots, k)$，$\sum\limits_{i=1}^{k} p_i = 1$，不妨记 $\max\limits_{1 \le i \le k} x_i = x_m$，有

$$E[X^n] = \sum_{i=1}^{k} x_i^n p_i \ \Rightarrow\ \lim_{n\to\infty} \frac{E[X^n]}{x_m^n} = p_m.$$

故

$$\lim_{n\to\infty} \frac{E[X^{n+1}]}{E[X^n]} = \lim_{n\to\infty} \frac{E[X^{n+1}]}{x_m^{n+1}} \frac{x_m^n \cdot x_m}{E[X^n]} = x_m = \max_{1\le i\le k} x_i.$$

□