

概统作业 (Week 13)

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1 (P234 T23)

(1)

$$E(\hat{\theta}_1) = E(\bar{X} + a_n) = E(\bar{X}) + E(a_n) = E(X) + a_n$$

又

$$E(X) = \int_{-\infty}^{+\infty} xf(x; \theta)dx = \int_{\theta}^{+\infty} x \cdot e^{-(x-\theta)} = (-x-1)e^{-(x-\theta)} \Big|_{\theta}^{+\infty} = \theta + 1.$$

可以得到

$$E(\hat{\theta}_1) = \theta + 1 + a_n = \theta \Rightarrow a_n = -1.$$

有 x 的分布函数

$$F(x) = \int_{-\infty}^x f(x, \theta)dx = \begin{cases} 0, & (x < \theta) \\ \int_{\theta}^x e^{-(x-\theta)}dx & (x \geq \theta) \end{cases} = \begin{cases} 0, & (x < \theta) \\ 1 - e^{-(x-\theta)} & (x \geq \theta) \end{cases}$$

记 $X_{(1)} = \min\{X_1, X_2, \dots, X_n\}$, 则有 $x < \theta$ 时, $X_{(1)}$ 概率密度 $f_1(x) = 0$, $x \geq \theta$ 时, 有

$$f_1(x) = n[1 - F(X)]^{n-1}f(x) = ne^{-n(x-\theta)}$$

即

$$f_1(x) = \begin{cases} 0, & (x < \theta) \\ ne^{-n(x-\theta)}. & (x \geq \theta) \end{cases}$$

有

$$E(\hat{\theta}_2) = E(X_{(1)}) + E(b_n) = \int_{-\infty}^{+\infty} xf_1(x)dx + b_n = \left(-x - \frac{1}{n}\right)e^{-n(x-\theta)} \Big|_{\theta}^{+\infty} + b_n = \theta + \frac{1}{n} + b_n$$

可以得到

$$E(\hat{\theta}_2) = \theta + \frac{1}{n} + b_n = \theta \Rightarrow b_n = -\frac{1}{n}$$

即

$$a_n = -1, \quad b_n = -\frac{1}{n}.$$

(2)

$$Var(\hat{\theta}_1) = Var(\bar{X} - 1) = \frac{1}{n}Var(X) = \frac{1}{n}[E(X^2) - E(X)^2]$$

又

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x; \theta)dx = \int_{\theta}^{+\infty} x^2 \cdot e^{-(x-\theta)}dx = \theta^2 + 2\theta + 2$$

故

$$\text{Var}(\hat{\theta}_1) = \frac{1}{n} [(\theta^2 + 2\theta + 2) - (\theta + 1)^2] = \frac{1}{n}.$$

由方差的性质有

$$\text{Var}(\hat{\theta}_2) = \text{Var}\left(X_{(1)} - \frac{1}{n}\right) = \text{Var}(X_{(1)}) = E(X_{(1)}^2) - [E(X_{(1)})]^2$$

又

$$E(X_{(1)}^2) = \int_{-\infty}^{+\infty} x^2 f_1(x) dx = \int_{\theta}^{+\infty} x^2 \cdot n e^{-n(x-\theta)} dx = \theta^2 + \frac{2}{n}\theta + \frac{2}{n^2}$$

故

$$\text{Var}(\hat{\theta}_2) = \left(\theta^2 + \frac{2}{n}\theta + \frac{2}{n^2}\right) - \left(\theta + \frac{1}{n}\right)^2 = \frac{1}{n^2}$$

有 $\text{Var}(\hat{\theta}_1) \geq \text{Var}(\hat{\theta}_2)$, 当且仅当 $n = 1$ 时取等, 故 $\hat{\theta}_2$ 更有效.

2 (P236 T39)

(1) 有 x 的密度函数

$$f(x; \theta) = \begin{cases} \frac{2x}{\theta} e^{-x^2/\theta}, & (x \geq 0) \\ 0. & (\text{其他}) \end{cases}$$

可得

$$E(X) = \int_{-\infty}^{+\infty} x f(x; \theta) dx = \int_0^{+\infty} \frac{2x^2}{\theta} e^{-\frac{x^2}{\theta}} dx = - \int_0^{+\infty} x d e^{-\frac{x^2}{\theta}} = \int_0^{+\infty} e^{-\frac{x^2}{\theta}} dx - \int_0^{+\infty} d \left(x e^{-\frac{x^2}{\theta}} \right)$$

又

$$\int_0^{+\infty} e^{-\frac{x^2}{\theta}} dx = \sqrt{\theta} \int_0^{+\infty} e^{-\left(\frac{x}{\sqrt{\theta}}\right)^2} d \left(\frac{x}{\sqrt{\theta}} \right) \xrightarrow{x=\sqrt{\theta}t} \sqrt{\theta} \int_0^{+\infty} e^{-t^2} dt = \frac{\pi\theta}{2}.$$

$$\int_0^{+\infty} d \left(x e^{-\frac{x^2}{\theta}} \right) \xrightarrow{t=x e^{-\frac{x^2}{\theta}}} \int_0^0 dt = 0.$$

故

$$E(X) = \frac{\pi\theta}{2}.$$

另

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x; \theta) dx = \int_0^{+\infty} \frac{2x^3}{\theta} e^{-\frac{x^2}{\theta}} dx \xrightarrow{t=\frac{x^2}{\theta}} \theta \int_0^{+\infty} t e^{-t} dt = \theta.$$

(2) 设 x_1, x_2, \dots, x_n 为样本观测值, 似然函数为

$$L(\theta) = \prod_{i=1}^n f(x_i) = \begin{cases} \frac{2^n \prod_{i=1}^n x_i}{\theta^n} \cdot e^{-\frac{1}{\theta} \sum_{i=1}^n x_i^2}, & (x_1, x_2, \dots, x_n > 0) \\ 0. & (\text{其他}) \end{cases}$$

当 $x_1, x_2, \dots, x_n > 0$ 时, 有

$$\ln L(\theta) = n \ln 2 + \sum_{i=1}^n x_i - n \ln \theta - \frac{1}{\theta} \sum_{i=1}^n x_i^2$$

令

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i^2 = 0.$$

得

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i^2.$$

又有 $\ln L(\theta)$ 二阶导

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{1}{\theta^2} \left(n - \frac{2}{\theta} \sum_{i=1}^n x_i^2 \right) \xrightarrow{\theta=\hat{\theta}} \frac{\partial^2 \ln L(\theta)}{\partial \theta^2} \Big|_{\theta=\hat{\theta}} = -\frac{n}{\theta^2} < 0.$$

故 $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i^2$ 为 θ 的最大似然估计量.

(3) 存在, $a = \theta$. 已知 $\{X_i^2\}$ 是独立同分布的随机变量序列, 记

$$\mu = E(X^2) = \theta, \quad S_n = \sum_{i=1}^n X_i^2$$

由大数定律有, 对 $\forall \varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|S_n/n - \mu| \geq \varepsilon) = 0 \xrightarrow{\hat{\theta}=S_n/n} \lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\theta} - \theta| \geq \varepsilon) = 0.$$

即证对于 $a = \theta$ 有

$$\hat{\theta} \xrightarrow{P} a.$$

3 (P238 T57)

Proof.

对于 X 有密度函数及分布函数

$$f(x) = \begin{cases} \frac{1}{\theta}, & (x \in (0, \theta)) \\ 0, & (\text{其他}) \end{cases} \quad F(x) = \begin{cases} 0, & (x \leq 0) \\ \frac{x}{\theta}, & (x \in (0, \theta)) \\ 1, & (x \geq \theta) \end{cases}$$

有 $\max\{X_1, X_2, \dots, X_n\}$ 的分布函数及密度函数

$$F_{\max}(x) = \begin{cases} 0, & (x \leq 0) \\ \frac{x^n}{\theta^n}, & (x \in (0, \theta)) \\ 1, & (x \geq \theta) \end{cases} \quad f_{\max}(x) = \begin{cases} \frac{n \cdot x^{n-1}}{\theta^n}, & (x \in (0, \theta)) \\ 0, & (\text{其他}) \end{cases}$$

有

$$E(\hat{\theta}) = \int_{-\infty}^{+\infty} x \cdot f_{\max}(x) dx = \int_0^{\theta} \frac{n \cdot x^n}{\theta^n} dx = \frac{n \cdot \theta}{n+1}$$
$$\lim_{n \rightarrow +\infty} E(\hat{\theta}) = \lim_{n \rightarrow +\infty} \frac{n \cdot \theta}{n+1} = \theta.$$

故 $\hat{\theta}$ 为 θ 的相合估计量. 又

$$E(\hat{\theta}) \neq \theta$$

故 $\hat{\theta}$ 不是 θ 的无偏估计量.

□

4 (P263 T16)

(1) 记样本数据分别为 x_1, x_2, \dots, x_{10} , 有

$$\sum_{i=1}^{10} (x_i - \mu)^2 = 2.9$$

$1 - \alpha = 0.95$, $\alpha = 0.05$, 有

$$\chi_{0.025}^2(10) = 3.2470, \quad \chi_{0.975}^2(10) = 20.4832$$

取枢轴量

$$Q = \sum_{i=1}^{10} \left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi_{10}^2$$

由

$$P \left(\chi_{1-\alpha/2}^2(10) \leq \frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} \leq \chi_{\alpha/2}^2(10) \right) = 1 - \alpha$$

有置信区间

$$\left[\frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\chi_{0.975}^2(10)}, \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\chi_{0.025}^2(10)} \right] = \left[\frac{2.9}{20.4832}, \frac{2.9}{3.2470} \right] = [0.1416, 0.8931]$$

(2)

$$\bar{x} = \frac{49.5 + 50.4 + 49.7 + 51.1 + 49.4 + 49.7 + 50.8 + 49.9 + 50.3 + 50.0}{10} = 50.08$$

$$(n-1)s^2 = \sum_{i=1}^{10} (x_i - \bar{x})^2 = 2.8360, \quad \chi_{0.025}^2(9) = 2.7004, \quad \chi_{0.975}^2(9) = 19.0228$$

选取

$$K = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

由

$$P \left(\frac{(n-1)S^2}{\chi_{\alpha/2}^2(n-1)} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi_{1-\alpha/2}^2(n-1)} \right) = 1 - \alpha$$

有置信区间

$$\left[\frac{(n-1)s^2}{\chi_{0.975}^2(9)}, \frac{(n-1)s^2}{\chi_{0.025}^2(9)} \right] = \left[\frac{2.8360}{19.0228}, \frac{2.8360}{2.7004} \right] = [0.1491, 1.0502]$$

5 (P263 T19)

记 $Y = \frac{X}{\theta}$, $Y_i = \frac{X_i}{\theta}$, $X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$, $Y_n = \max\{Y_1, Y_2, \dots, Y_n\}$, 则有

$$Y \sim U(0, 1), \quad X_{(n)} = \theta \cdot Y_{(n)}, \quad P(X_{(n)} \leq \theta \leq c_n X_{(n)}) = P(c_n^{-1} \leq Y_{(n)} \leq 1)$$

有 $Y_{(n)}$ 的分布函数及密度函数

$$F_{\max}(y) = \begin{cases} 0, & (y \leq 0) \\ y^n, & (y \in (0, 1)) \\ 1, & (y \geq 1) \end{cases} \quad f_{\max}(y) = \begin{cases} n \cdot y^{n-1}, & (y \in (0, 1)) \\ 0, & (\text{其他}) \end{cases}$$

若有 c_n 使得 $[X_{(n)}, c_n X_{(n)}]$ 为 θ 的 $1 - \alpha$ 置信系数, 则有 $\forall \theta \in \Theta$

$$P(X_{(n)} \leq \theta \leq c_n X_{(n)}) = 1 - \alpha \Leftrightarrow P(c_n^{-1} \leq Y_{(n)} \leq 1) = 1 - F_{\max}(c_n^{-1}) = 1 - \alpha$$

又显然 $0 \leq c_n \leq 1$, 有

$$1 - F_{\max}(c_n^{-1}) = 1 - c_{n-n} = 1 - \alpha \Rightarrow c_n = \alpha^{-\frac{1}{n}}$$

即存在 $c_n = \alpha^{-\frac{1}{n}}$ 满足要求.