概统作业 (Week 7)

PB20000113 孔浩宇

May 2, 2023

1 (P118 T20)

由题意可得 (x, y) 密度函数 $f(x, y) = \dfrac{1}{4}$，有

$$F_Z(z) = \iint_{|x-y| \leq z} f(x, y) dx dy = \begin{cases} 0 & (z \leq 0) \\ \dfrac{-z^2 + 4z}{4} & (0 \leq z < 1) \\ 1 & (z \geq 1) \end{cases} \Rightarrow f_Z(z) = \begin{cases} 1 - \dfrac{z}{2} & (0 < z < 1) \\ 0 & \text{其他.} \end{cases}$$

2 (P118 T22)

(1)

$$\text{若} (x, y) \in D, \ f(x, y) = \frac{1}{S} = 2. \ \Rightarrow \ f(x, y) = \begin{cases} 2 & (0 \leq x \leq y \leq 1) \\ 0 & \text{其他} \end{cases}$$

(2)

$$f_1(x) = \int_x^1 f(x, y) dy = \begin{cases} 2(1 - x) & (0 \leq x \leq 1) \\ 0 & \text{其他} \end{cases}$$

$$f_2(y) = \int_0^y f(x, y) dx = \begin{cases} 2y & (0 \leq y \leq 1) \\ 0 & \text{其他} \end{cases}$$

(3)

$$\text{若} f_2(y) > 0, \ \text{即} 0 \leq y \leq 1, \ f_{X|Y}(x|y) = \frac{f(x, y)}{f_2(y)} = \frac{2}{2y} = \frac{1}{y}. \ \Rightarrow \ f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{y} & (0 \leq y \leq 1) \\ \\ 0 & \text{其他.} \end{cases}$$

(4) 对于 $0 \leq y \leq 1$，有

$$P(X \leq 0.5 | Y = y) = \begin{cases} \displaystyle\int_0^{0.5} f_{X|Y}(x|y) dx & (0.5 \leq y \leq 1) \\ \displaystyle\int_0^y f_{X|Y}(x|y) dx & (0 \leq y < 0.5) \end{cases} = \begin{cases} \dfrac{1}{2y} & (0.5 \leq y \leq 1) \\ 1. & (0 \leq y < 0.5) \end{cases}$$

考虑 $(x, y) \notin D$，有

$$P(X \leq 0.5 | Y = y) = \begin{cases} \dfrac{1}{2y} & (0.5 \leq y \leq 1) \\ \\ 1 & (0 \leq y < 0.5) \\ \\ 0 & \text{其他.} \end{cases}$$

3 (P119 T27)

由题意可得 $f_X(x) = e^{-x} \cdot I_{(0,+\infty)}(x),\ f_Y(y) = \dfrac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$, 对 $x, y \geq 0$ 有

$$f_{|Y|}(y) = f_Y(y) + f_Y(-y) = \sqrt{\frac{2}{\pi}} \cdot e^{-\frac{y^2}{2}}, \quad f_{(X,|Y|)}(x,y) = f_X(x) \cdot f_{|Y|}(y) = \sqrt{\frac{2}{\pi}} \cdot e^{-x-\frac{y^2}{2}}$$

即

$$f_{(X,|Y|)}(x,y) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \cdot e^{-x-\frac{y^2}{2}} & (x, y \geq 0) \\ \\ 0. & \text{其他.} \end{cases}$$

4 (P120 T35)

(1)
$$\iint_{x>0,y>0} f(x,y)dxdy = 1 \ \Rightarrow\ \frac{A}{12} = 1 \ \Rightarrow\ A = 12.$$

(2) X 与 Y 独立.

$$f_X(x) = \int_0^{+\infty} f(x,y)dy = \begin{cases} 3 \cdot e^{-3x} & (x>0) \\ \\ 0. & \text{其他.} \end{cases}$$

$$f_Y(y) = \int_0^{+\infty} f(x,y)dx = \begin{cases} 4 \cdot e^{-4y} & (y>0) \\ \\ 0. & \text{其他.} \end{cases}$$

有
$$f(x,y) = f_X(x) \cdot f_Y(y) \ \Rightarrow\ X \text{ 与 } Y \text{ 独立.}$$

(3) 对于 $z > 0$, 有

$$F_Z(z) = \iint_{x+y \leq z} f(x,y)dxdy = \int_0^z dx \int_0^{z-x} dy \cdot 12 \cdot e^{-(3x+4y)} = -4 \cdot e^{-3z} + 3 \cdot e^{-4z} + 1.$$

故
$$F_Z(z) = \begin{cases} -4 \cdot e^{-3z} + 3 \cdot e^{-4z} + 1. & (z>0) \\ 0. & (z \leq 0) \end{cases}$$

可得密度函数
$$f_Z(z) = [F_Z(z)]' = \begin{cases} 12(e^{-3z} - e^{-4z}). & (z>0) \\ 0. & (z \leq 0). \end{cases}$$

(4)
$$f_1(x|x+y=1) = \begin{cases} \dfrac{f(x,1-x)}{f_Z(1)} & (0<x<1) \\ \\ 0 & \text{其他.} \end{cases} = \begin{cases} \dfrac{e^x}{e-1} & (0<x<1) \\ \\ 0 & \text{其他.} \end{cases}$$

$$\Rightarrow\ P(X>0.5|X+Y=1) = \int_{0.5}^{+\infty} f_1(x|x+y=1)dx = \frac{e-\sqrt{e}}{e-1}.$$

5 (P121 T42)

Proof.

(1) 先求 X 与 Y，Y 与 Z，X 与 Z 这三组的联合分布密度函数

$$f_{(X,Y)}(x,y) = \int_0^{2\pi} f(x,y,z)dz = \begin{cases} \frac{1}{4\pi^2}, & (0 \leq x,y \leq 2\pi) \\ \\ 0, & \text{其他}. \end{cases}$$

$$f_{(Y,Z)}(y,z) = \int_0^{2\pi} f(x,y,z)dx = \begin{cases} \frac{1}{4\pi^2}, & (0 \leq y,z \leq 2\pi) \\ \\ 0, & \text{其他}. \end{cases}$$

$$f_{(X,Z)}(x,z) = \int_0^{2\pi} f(x,y,z)dy = \begin{cases} \frac{1}{4\pi^2}, & (0 \leq x,z \leq 2\pi) \\ \\ 0, & \text{其他}. \end{cases}$$

(2) 再求 X，Y，Z 三个的边缘分布密度函数

$$f_X(x) = \int_0^{2\pi} dy \int_0^{2\pi} dz \cdot f(x,y,z) = \frac{1}{2\pi}.$$

$$f_Y(y) = \int_0^{2\pi} dx \int_0^{2\pi} dz \cdot f(x,y,z) = \frac{1}{2\pi}.$$

$$f_Z(z) = \int_0^{2\pi} dx \int_0^{2\pi} dy \cdot f(x,y,z) = \frac{1}{2\pi}.$$

(3) 由 (1),(2) 有

$$f_{(X,Y)}(x,y) = f_X(x) \cdot f_Y(y). \quad f_{(Y,Z)}(y,z) = f_Y(y) \cdot f_Z(z). \quad f_{(X,Z)}(x,z) = f_X(x) \cdot f_Z(z).$$

故 X，Y，Z 两两独立，又

$$f_X(x) \cdot f_Y(y) \cdot f_Z(z) = \frac{1}{8\pi^3} \neq f(x,y,z).$$

故 X，Y，Z 不相互独立.

$\square$