Planar Graphs

Definition:

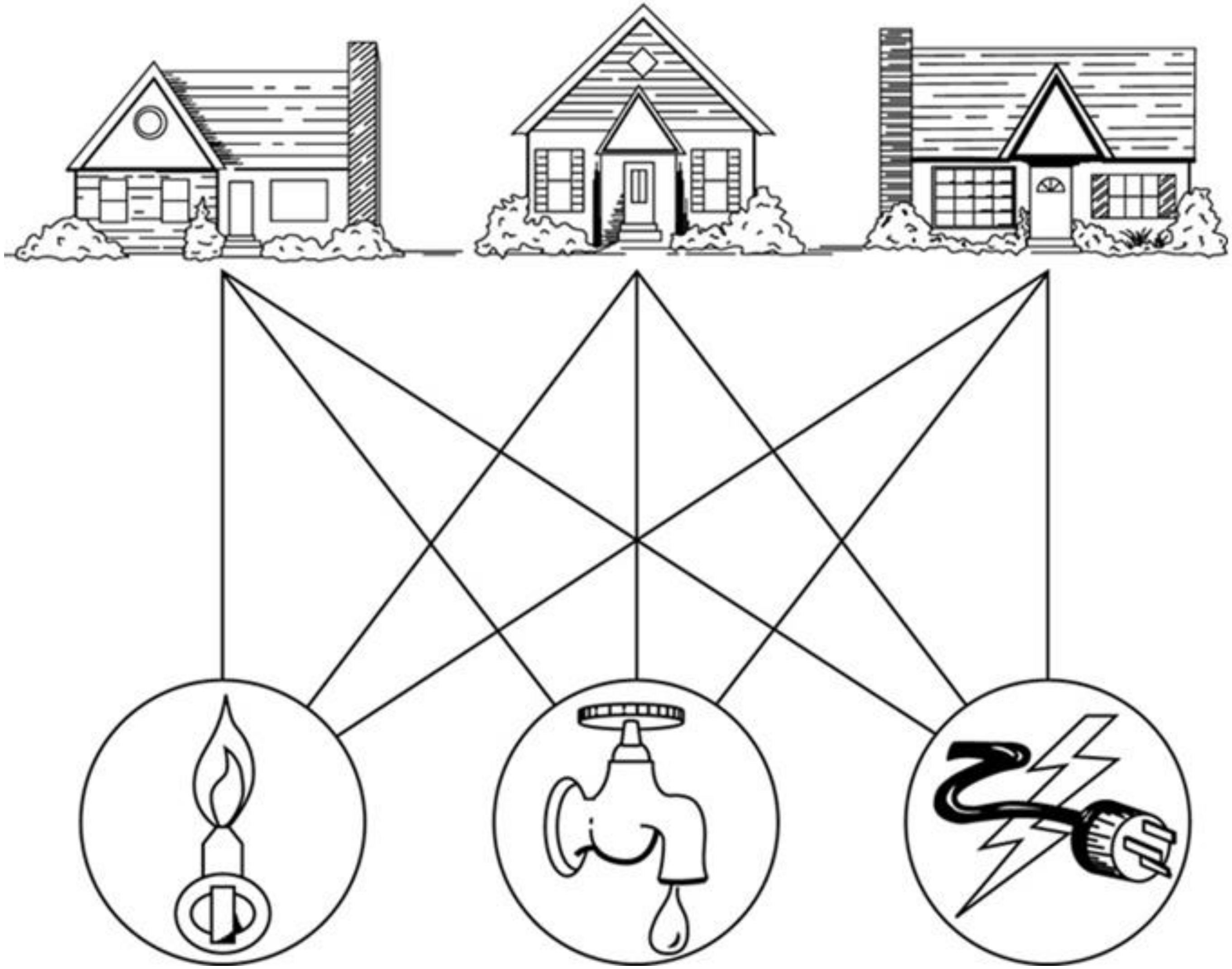
A graph that can be drawn in the plane without any of its edges intersecting is called a planar graph.. A graph that is so drawn in the plane is also said to be embedded (or imbedded) in the plane.

Applications:

(1) circuit layout problems.

(2) Three house and three utilities problem.

The House-and-Utilities Problem

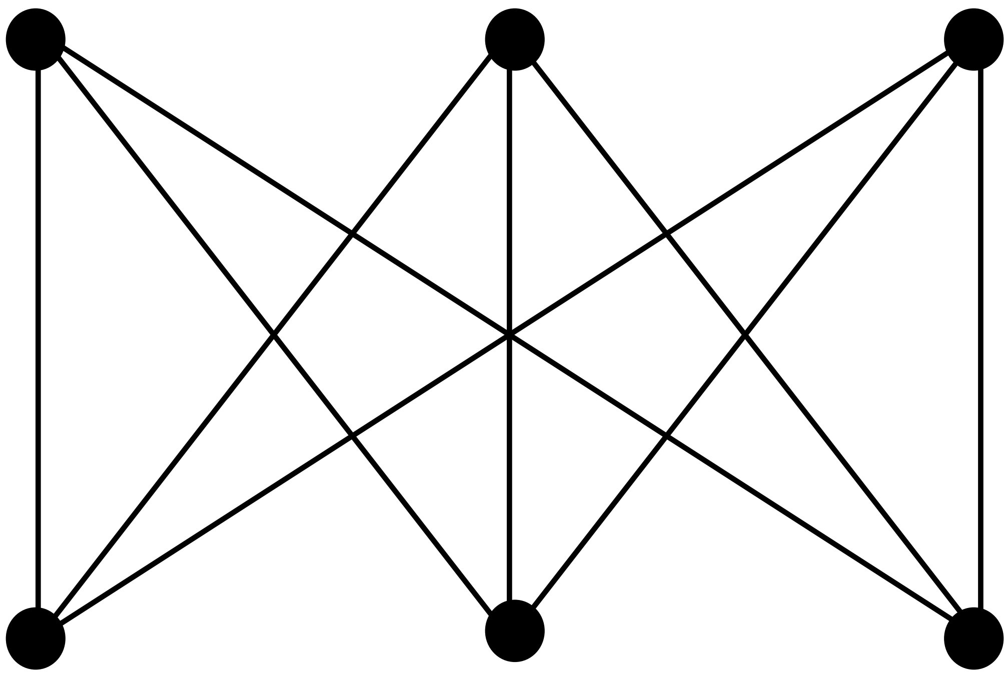


Planar Graphs

- Consider the previous slide. Is it possible to join the three houses to the three utilities in such a way that none of the connections cross?

Planar Graphs

- Phrased another way, this question is equivalent to: Given the complete bipartite graph $K_{3,3}$, can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?



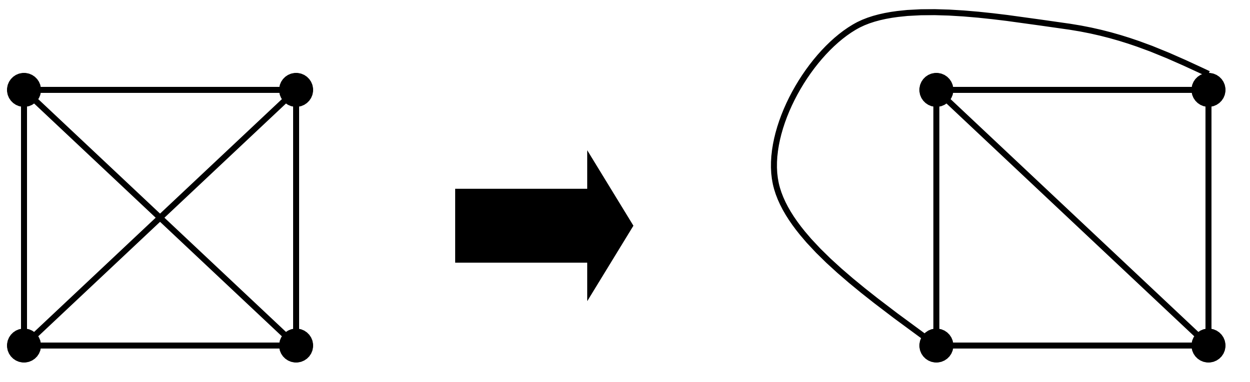
$K_{3,3}$

Planar Graphs

- A graph is called *planar* if it can be drawn in the plane without any edges crossing.
- A crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint.
- Such a drawing is called a *planar representation* of the graph.

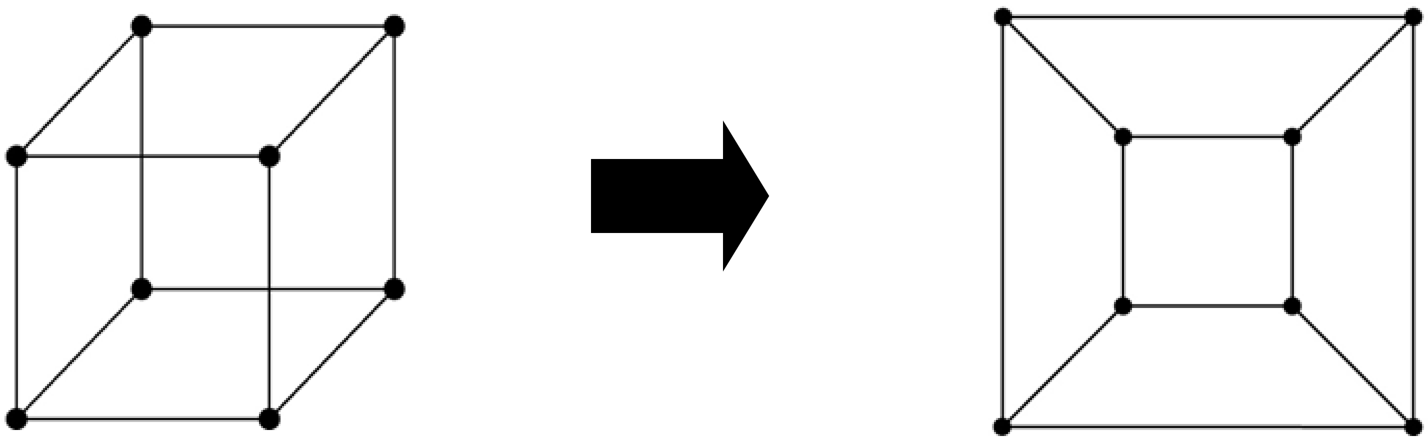
Example

A graph may be planar even if it is usually drawn with crossings, since it may be possible to draw it in another way without crossings.



Example

A graph may be planar even if it represents a 3-dimensional object.

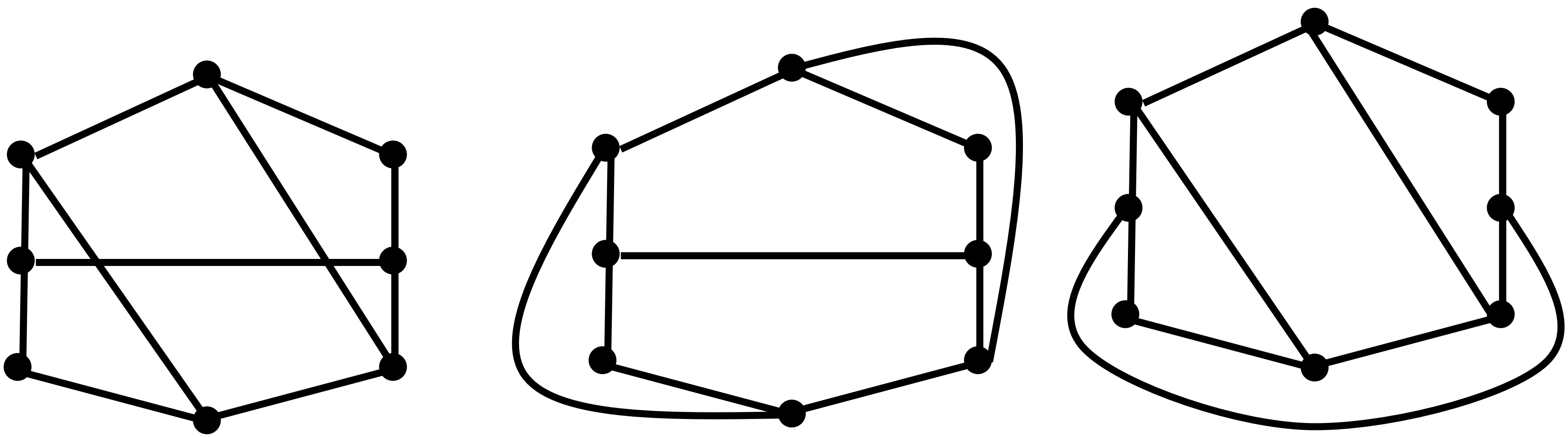


Planar Graphs

- We can prove that a particular graph is planar by showing how it can be drawn without any crossings.
- However, not all graphs are planar.
- It may be difficult to show that a graph is nonplanar. We would have to show that there is *no way* to draw the graph without any edges crossing.

Definition:

A planar graph G that is drawn in the plane so that no two edges intersect (that is, G is embedded in the plane) is called a plane graph.



(a) planar, not a plane graph

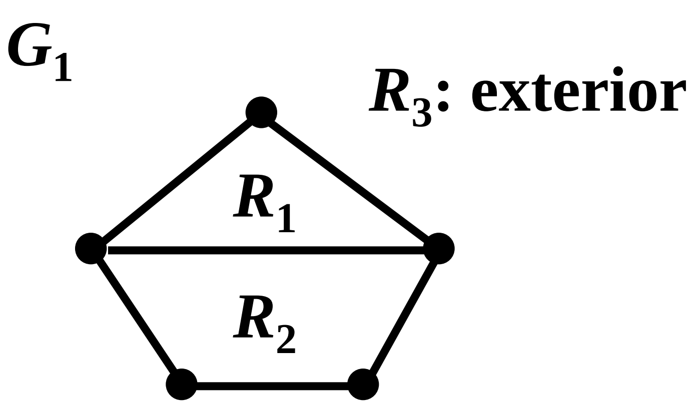
(b) a plane graph

(c) another plane graph

Note: A given planar graph can give rise to several different plane graph.

Definition:

Let G be a plane graph. The connected pieces of the plane that remain when the vertices and edges of G are removed are called the regions of G.

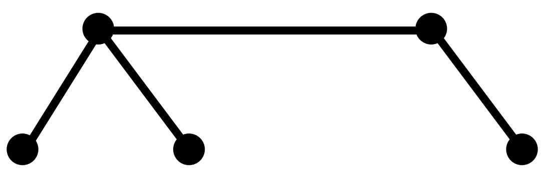


G_1 has 3 regions.

Definition:

Every plane graph has exactly one unbounded region, called the exterior region. The vertices and edges of G that are incident with a region R form a subgraph of G called the boundary of R.

G_2

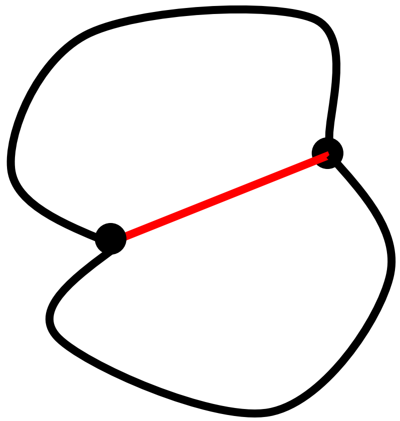


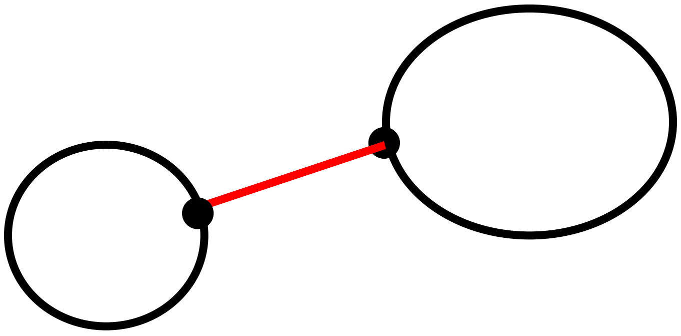
G_2 has only 1 region.

Observe:

(1) Each <u>cycle edge</u> belongs to the boundary of two regions.

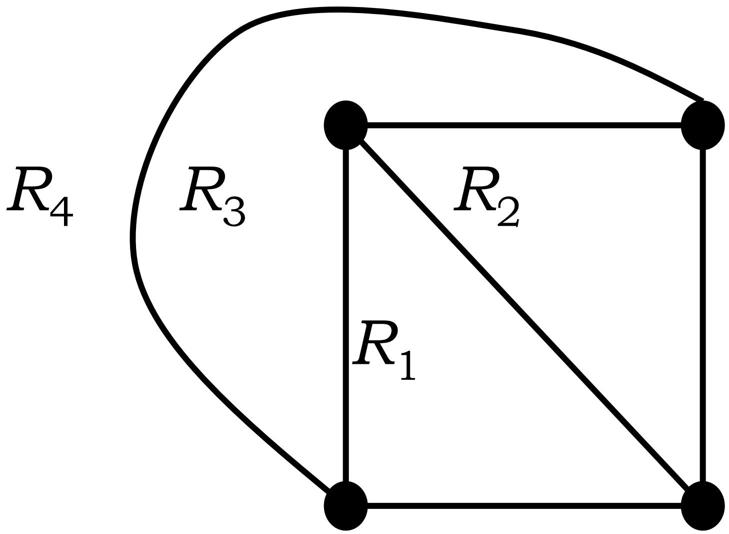
(2) Each <u>bridge</u> is on the boundary of only one region. (exterior)

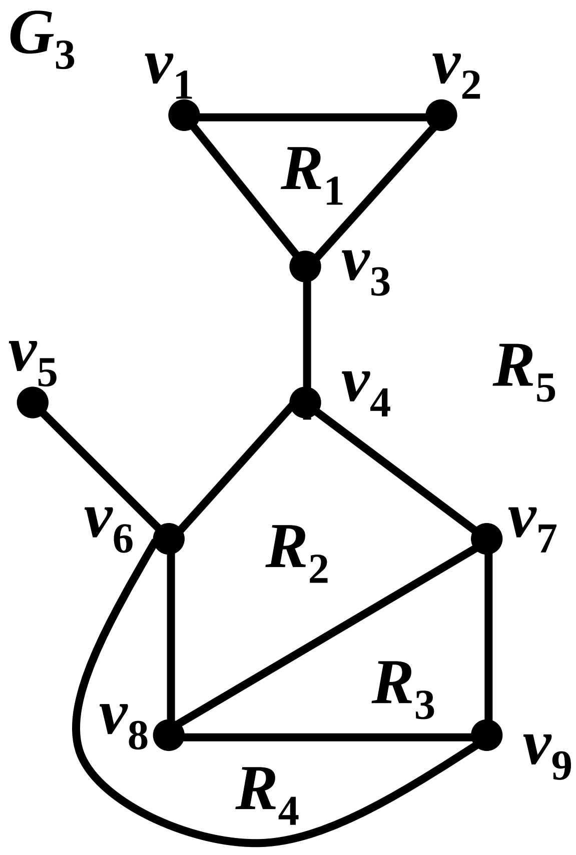




Regions

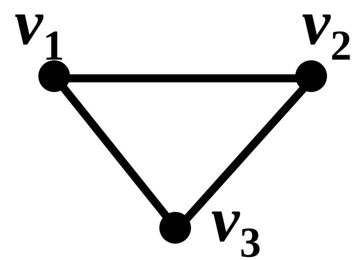
- Euler showed that all planar representations of a graph split the plane into the same number of *regions*, including an unbounded region.



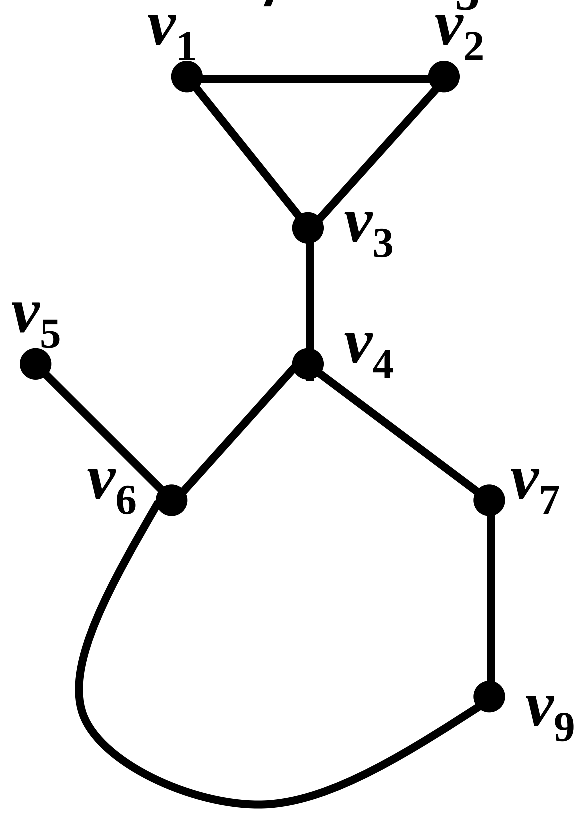


G_3 has 5 regions.

Boundary of R_1:

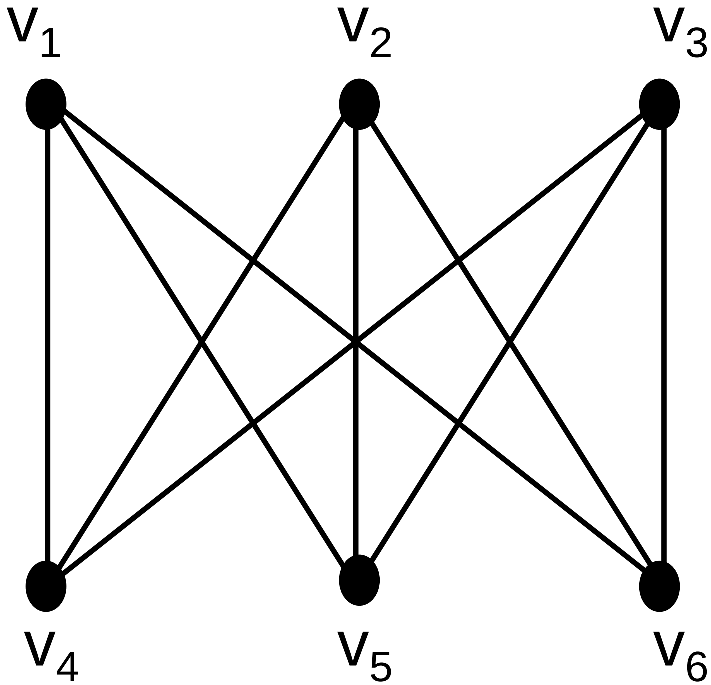


Boundary of R_5:



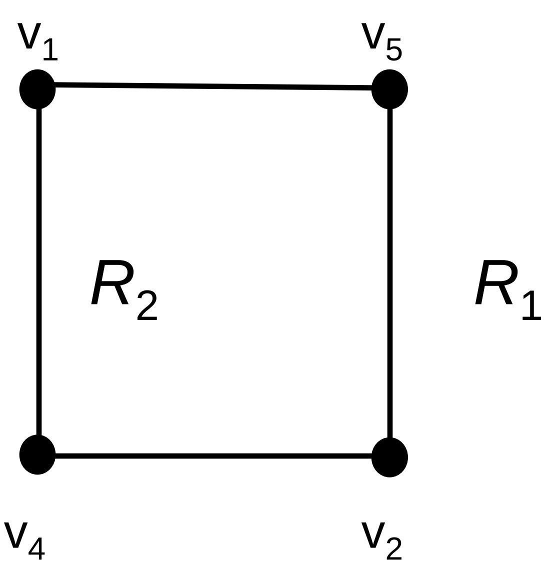
Regions

- In any planar representation of $K_{3,3}$, vertex v_1 must be connected to both v_4 and v_5, and v_2 also must be connected to both v_4 and v_5.



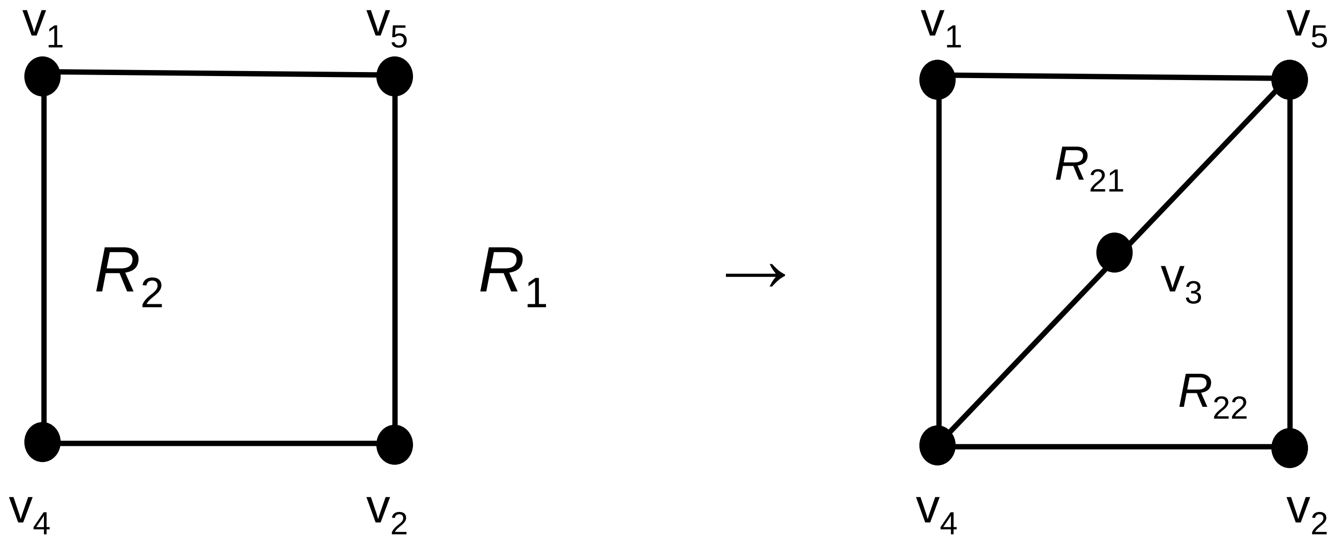
Regions

- The four edges $\{v_1, v_4\}$, $\{v_4, v_2\}$, $\{v_2, v_5\}$, $\{v_5, v_1\}$ form a closed curve that splits the plane into two regions, R_1 and R_2.



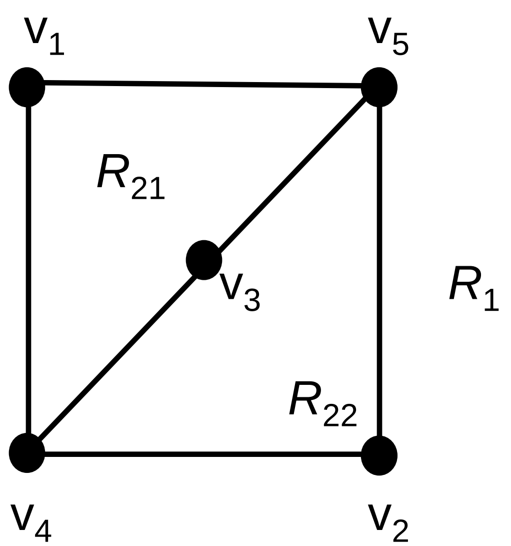
Regions

- Next, we note that v_3 must be in either R_1 or R_2.
- Assume v_3 is in R_2. Then the edges $\{v_3, v_4\}$ and $\{v_3, v_5\}$ separate R_2 into two subregions, R_{21} and R_{22}.



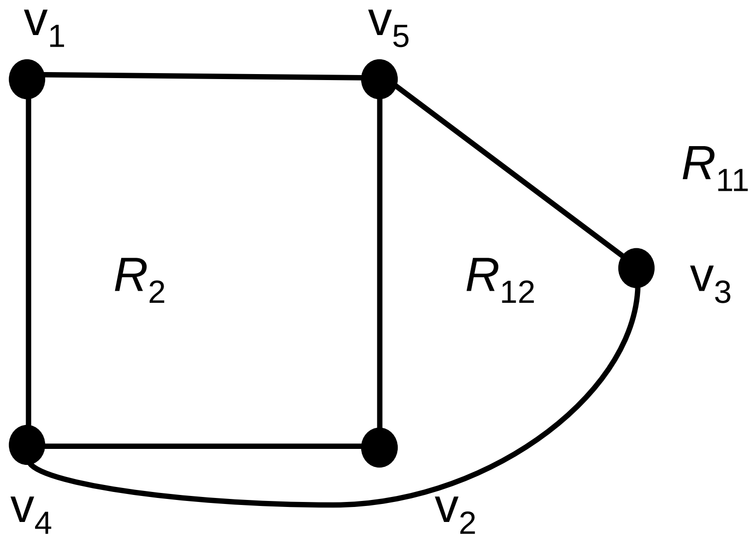
Regions

- Now there is no way to place vertex v_6 without forcing a crossing:
  - If v_6 is in R_1 then $\{v_6, v_3\}$ must cross an edge
  - If v_6 is in R_{21} then $\{v_6, v_2\}$ must cross an edge
  - If v_6 is in R_{22} then $\{v_6, v_1\}$ must cross an edge



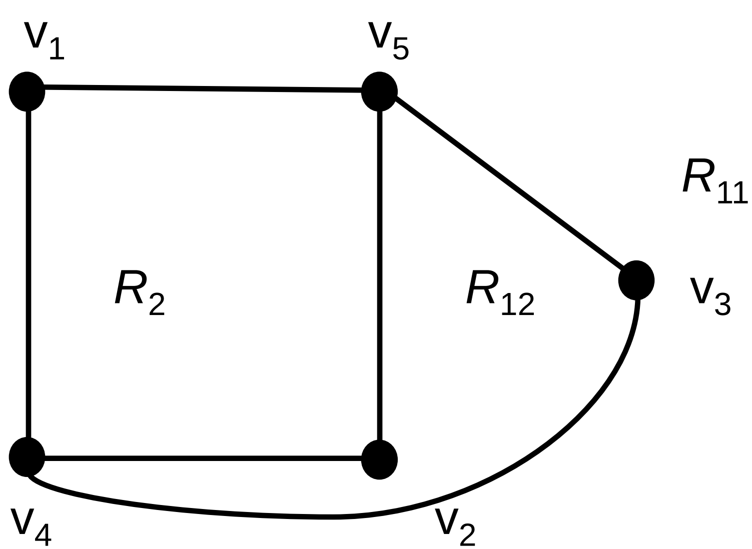
Regions

- Alternatively, assume v_3 is in R_1. Then the edges $\{v_3, v_4\}$ and $\{v_3, v_5\}$ separate R_1 into two subregions, R_{11} and R_{12}.



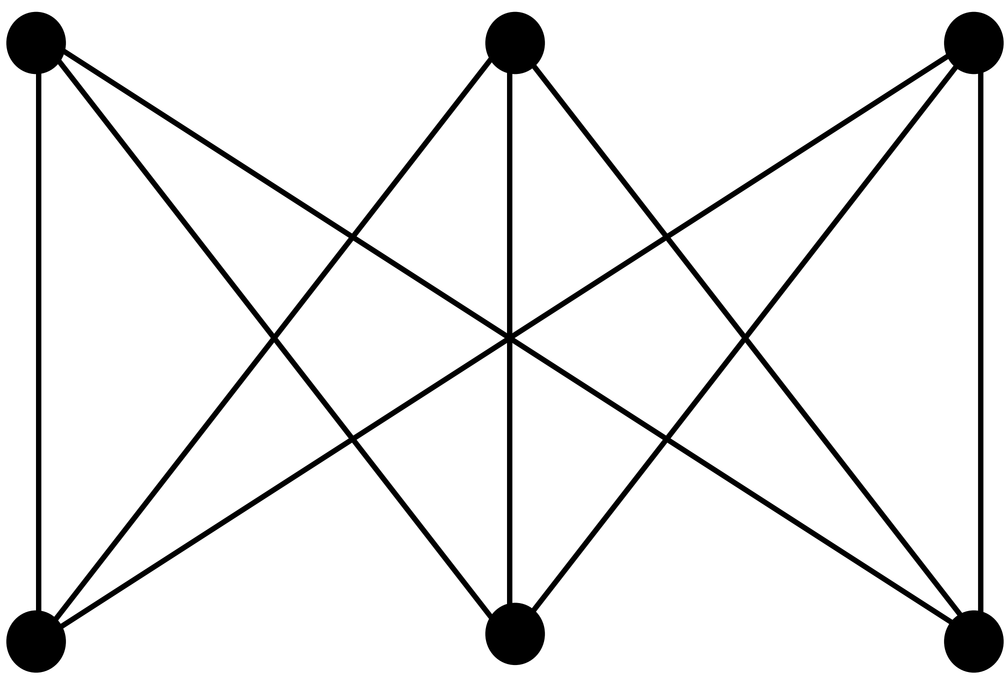
Regions

- Now there is no way to place vertex v_6 without forcing a crossing:
  - If v_6 is in R_2 then $\{v_6, v_3\}$ must cross an edge
  - If v_6 is in R_{11} then $\{v_6, v_2\}$ must cross an edge
  - If v_6 is in R_{12} then $\{v_6, v_1\}$ must cross an edge



Planar Graphs

- Consequently, the graph $K_{3,3}$ must be nonplanar.



$K_{3,3}$

Regions

- Euler devised a formula for expressing the relationship between the number of vertices, edges, and regions of a planar graph.
- These *may* help us determine if a graph can be planar or not.

Theorem 1: (Euler's Formula)
If G is a connected plane graph with p vertices, q edges, and r regions, then

$$p - q + r = 2.$$

pf: (by induction on q)

(basis) If $q = 0$, then $G \cong K_1$; so $p = 1$, $r = 1$, and $p - q + r = 2$.

(inductive) Assume the result is true for any graph with $q = k - 1$ edges, where $k \geq 1$.

Let G be a graph with k edges. Suppose G has p vertices and r regions.

If G is a tree, then G has p vertices, $p-1$ edges and 1 region.

$\Rightarrow p - q + r = p - (p-1) + 1 = 2$.

If G is not a tree, then some edge e of G is on a cycle.

Hence $G-e$ is a connected plane graph having order p and size $k-1$, and $r-1$ regions.

$\Rightarrow p - (k-1) + (r-1) = 2$ (by assumption)

$\Rightarrow p - k + r = 2$

#

Theorem : Euler's Formula

- If *G* is a connected planar graph, then any plane graph depiction of *G* has $\boldsymbol{r = e - v} + 2$ regions.
  - Recall: *Connected* planar graphs have paths between each pair of vertices.
  - ***v*** = number of vertices
  - ***e*** = number of edges
  - ***r*** = number of regions
- This is important because there are many different plane graph depictions that can be drawn for a planar graph, however, the number of regions will not change.

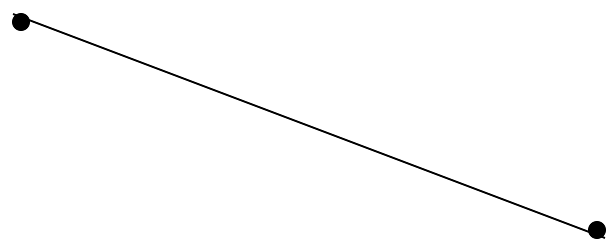
Proof of Euler's Formula

- Let's draw a plane graph depiction of *G*, edge by edge.

- Let G_n denote the connected plane graph after *n* edges have been added.
- Let v_n denote the number of vertices in G_n
- Let e_n denote the number of edges in G_n
- Let r_n denote the number of regions in G_n

Proof of Euler's Formula (cont'd)

- Let's start by drawing G_1



$$v_1 = 2$$
$$e_1 = 1$$
$$r_1 = 1$$

Euler's formula is valid for G_1, since
$r = e - v + 2$
1=1-2+2

We obtain G_2 from G_1 by adding an edge at one of the vertices of G_1.

Proof of Euler's Formula (cont'd)

In general, we can obtain G_n from G_{n-1} by adding an n^{th} edge to one of the vertices of G_{n-1}.

- The new edge might link two vertices already in G_{n-1}.
- Or, the new edge might add another vertex to G_{n-1}.

We will use the method of induction to complete the proof:

We have shown that the theorem is true for G_1.

Next, let's assume that it is true for G_{n-1} for any n>1, and prove it is true for G_n.

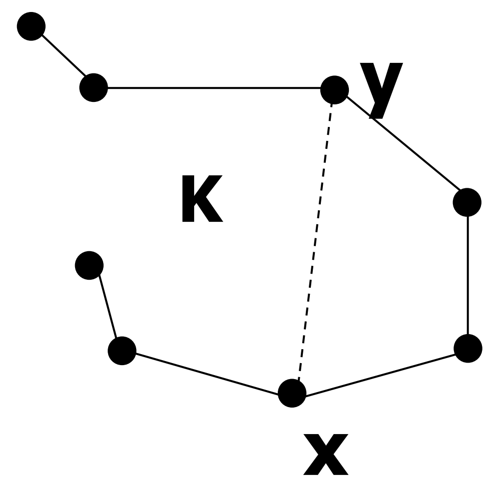
Let (x,y) be the n^{th} edge that is added to G_{n-1} to get G_n.

There are two cases to consider.

Proof of Euler's Formula (cont'd)

In the first case, *x* and *y* are both in G_{n-1} .

Then they are on the boundary of a common region K , possibly an unbounded region.



Edge (*x*, *y*) splits K into two regions.

Then, $r_n = r_{n-1} + 1$

$$e_n = e_{n-1} + 1$$

$$v_n = v_{n-1}$$

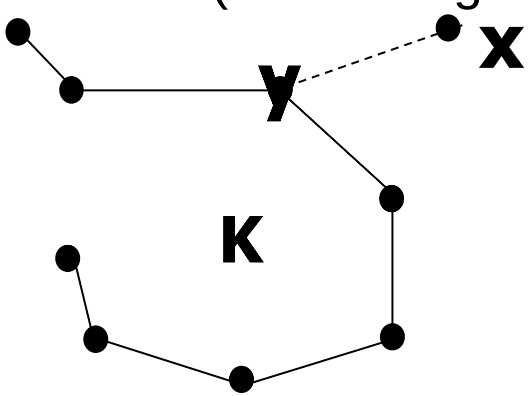
Each side of Euler's formula grows by one.

So, if the formula was true for G_{n-1} , it will also be true for G_n .

Proof of Euler's Formula (cont'd)

In the second case, one of the vertices *x*,*y* is not in G_{n-1} . Let's say that it is *x*.

Then, adding (*x*,*y*) implies that *x* is also added, but that no new regions are formed (no existing regions are split).



$$r_n = r_{n-1}$$

$$e_n = e_{n-1} + 1$$

$$v_n = v_{n-1} + 1$$

So, the value on each side of Euler's equation is unchanged.
The validity of Euler's formula for G_{n-1} implies its validity for G_n .

By induction, Euler's formula is true for all G_n 's and the full graph *G*.

Example of Euler’s Formula

- **How many regions would there be in a plane graph with 10 vertices each of degree 3?**

By Theorem 1, Sec. 1.3: $\sum(\text{degrees of vertices}) = 2\mathbf{e}$

$$=> (10 * 3) = 30 = 2\mathbf{e}$$

$$15 = \mathbf{e}$$

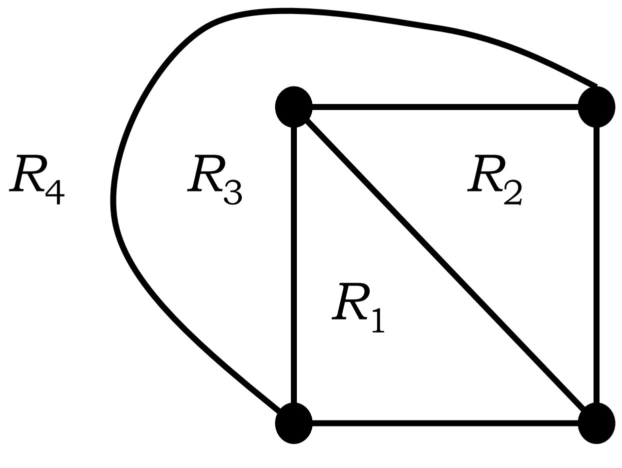
By Euler's Formula, $\mathbf{r} = \mathbf{e} - \mathbf{v} + 2$

$$\mathbf{r} = 15\text{-}10+2 = 7$$

Answer: There are 7 regions.

Example of Euler's Formula

- Let *G* be a connected planar simple graph with *e* edges and *v* vertices. Let *r* be the number of regions in a planar representation of *G*. Then $r = e - v + 2$.

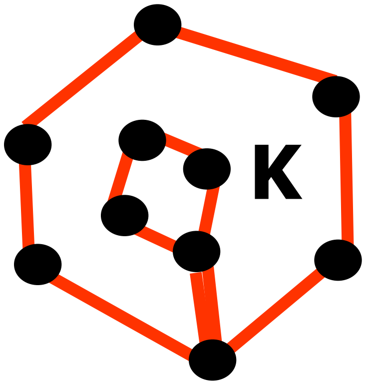


\# of edges, $e = 6$
\# of vertices, $v = 4$
\# of regions, $r = e - v + 2 = 4$

Corollary

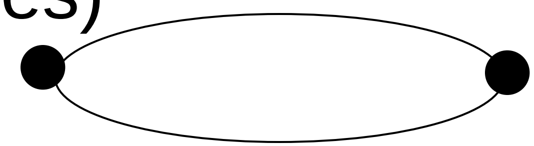
- If G is a connected planar graph with **e** > 1, then **e** ≤ 3**v** – 6

- Proof:
  - Define the degree of a region as the number of edges on its boundary. If an edge occurs twice along the boundary, then count it twice. The region K has degree 12.

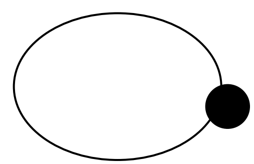


Proof of Corollary continued

- Note that no region can be less than degree 3.
  - A region of degree 2 would be bounded by two edges joining the same pair of vertices (parallel edges)



  - A region of degree 1 would be bounded by a loop edge.



- Neither of these is allowed, and so a region must have at least degree 3.

Proof of Corollary continued

Since $2\mathbf{e} = \sum$(degrees of r), we know that $2\mathbf{e} \geq 3\mathbf{r} \Rightarrow \frac{2}{3}\mathbf{e} \geq \mathbf{r}$

Also, we know $\mathbf{r} = \mathbf{e} - \mathbf{v} + 2$ from Euler's Formula.

Substitute Euler's Formula in to get:

$$\frac{2}{3}\mathbf{e} \geq \mathbf{e} - \mathbf{v} + 2$$

$$\Rightarrow \quad 0 \geq (\frac{1}{3}\mathbf{e} - \mathbf{v} + 2) * 3$$

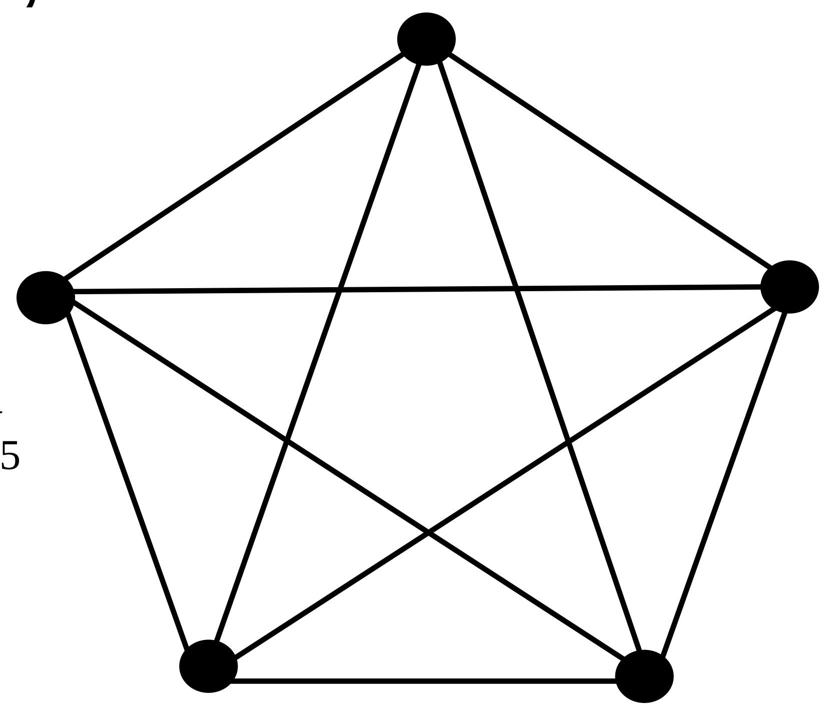
$$\Rightarrow \quad 0 \geq \mathbf{e} - 3\mathbf{v} + 6$$

$$\Rightarrow \quad \mathbf{e} \leq 3\mathbf{v} - 6$$

Corollary

- Corollary 1: If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$.
- Is K_5 planar?

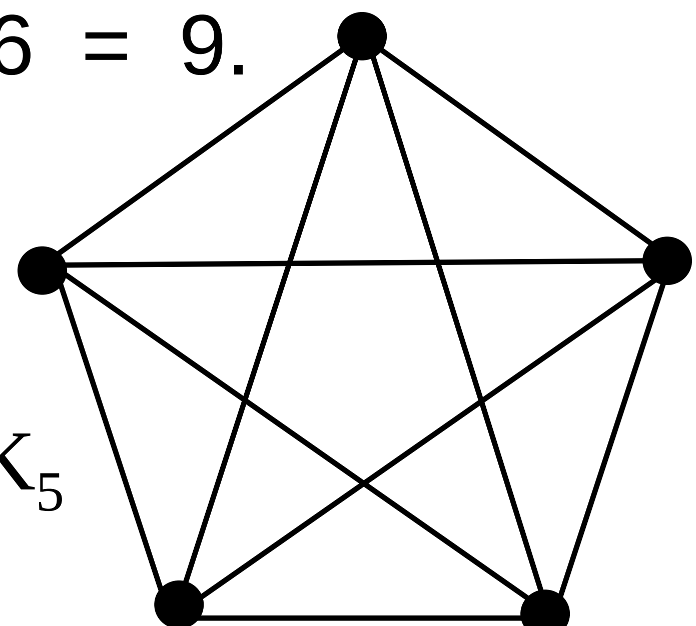
K_5



Example of Corollary

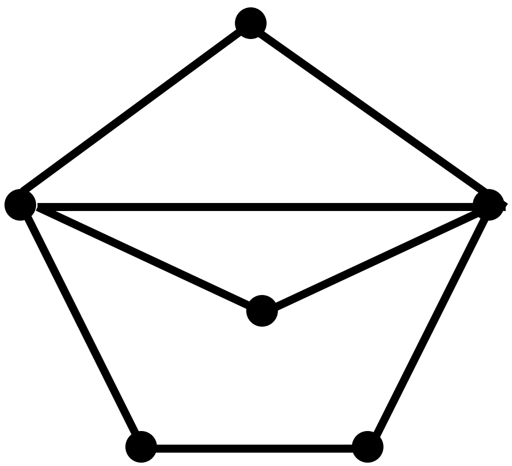
- K_5 has 5 vertices and 10 edges.
- We see that $v \geq 3$.
- So, if K_5 is planar, it must be true that $e \leq 3v - 6$.
- $3v - 6 = 3*5 - 6 = 15 - 6 = 9$.
- So e must be ≤ 9.
- But $e = 10$.
- So, K_5 is nonplanar.

K_5

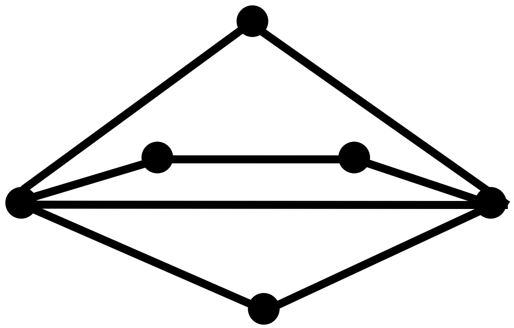


Two embeddings of a planar graph

(a)



(b)



Definition:

A plane graph G is called maximal planar if, for every pair u, v of nonadjacent vertices of G, the graph $G+uv$ is nonplanar.

Thus, in any embedding of a maximal planar graph G of order at least 3, the boundary of every region of G is a triangle.

Theorem 2: If G is a maximal planar graph with $p \geq 3$ vertices and q edges, then

$$q = 3p - 6.$$

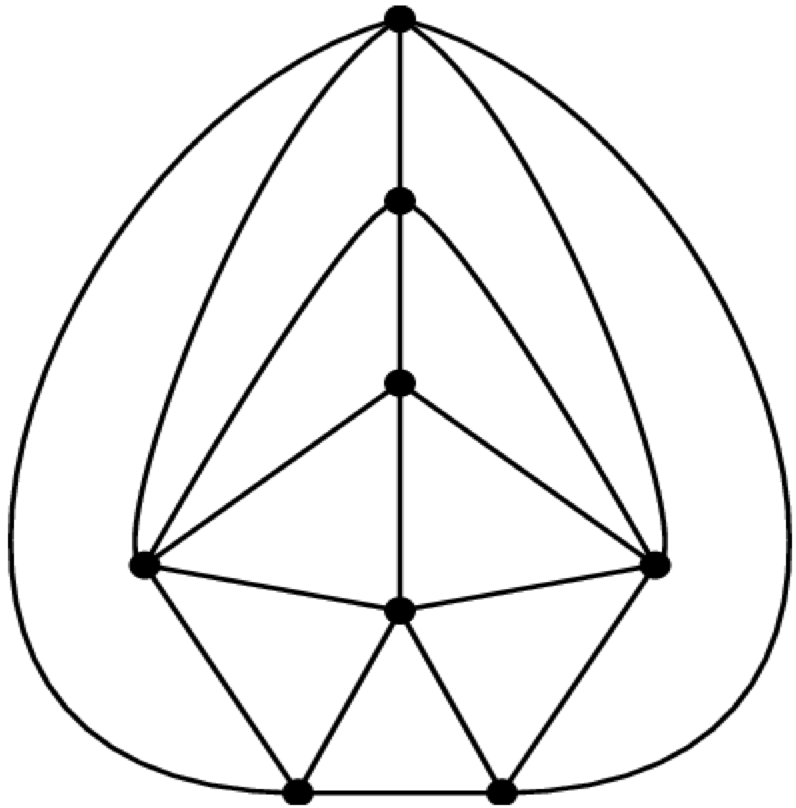
pf: Embed the graph G in the plane, resulting in r regions. $\Rightarrow p - q + r = 2.$

Since the boundary of every region of G is a triangle, every edge lies on the boundary of two regions.

$$\Rightarrow \sum_{\forall\, Region\, R} |\{The\ edges\ of\ the\ boundry\ of\ R| = 3\mathrm{r} = 2\mathrm{q}.$$

$$\Rightarrow p - q + 2q/3 = 2.$$

$$\Rightarrow q = 3p - 6$$



Maximal Planar Graph

Cor. 2(a): If G is a maximal planar bipartite graph with $p \geq 3$ vertices and q edges, then $q = 2p - 4$.

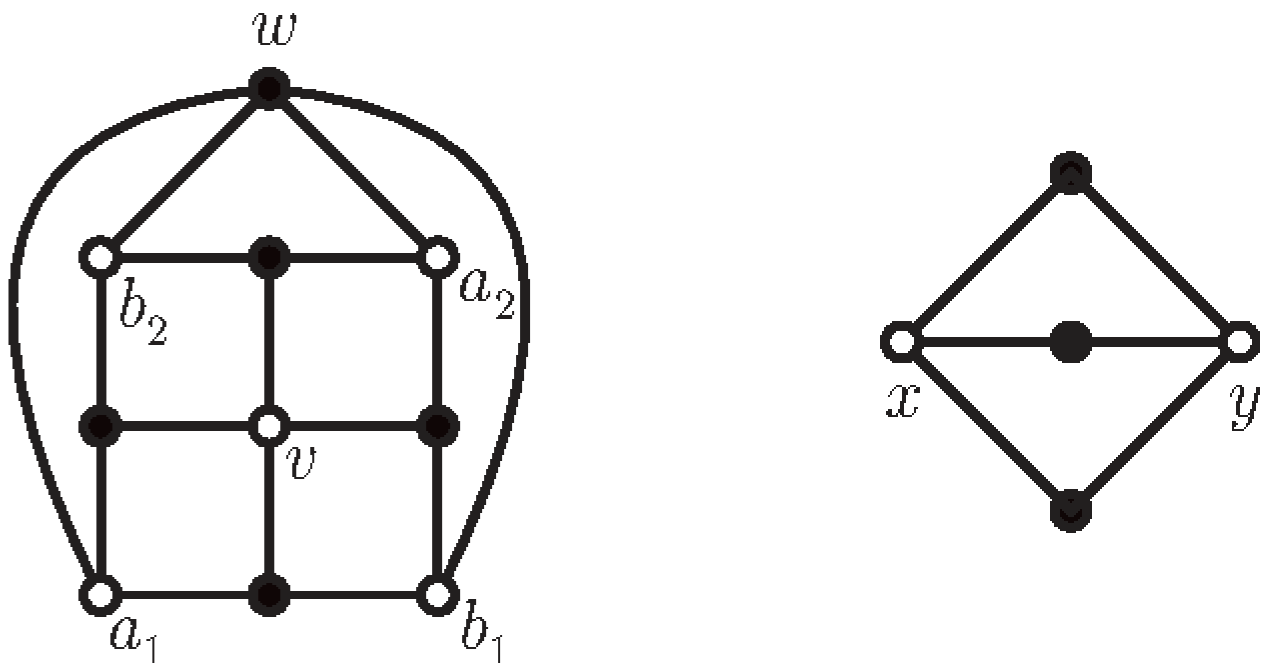
pf: The boundary of every region is a 4-cycle.

$$4r = 2q \Rightarrow p - q + q/2 = 2 \Rightarrow q = 2p - 4.$$

Cor. 2(b): If G is a planar graph with $p \geq 3$ vertices and q edges, then

$$q \leq 3p - 6.$$

pf: If G is not maximal planar, we can add edges to G to produce a maximal planar graph.



Maximal Planar Bipartite Graph

Theorem 3: Every planar graph contains a vertex of degree 5 or less.

pf: Let G be a planar graph of p **vertices and** q **edges**.

If $\deg(v) \geq 6$ for every $v \in V(G)$

$$\Rightarrow \quad \sum_{v \in V(G)} \deg(v) \geq 6p$$

$$\Rightarrow \quad 2q \geq 6p \qquad \rightarrow\leftarrow$$

Corollary

- Corollary 2: If *G* is a connected planar simple graph, then *G* must have a vertex of degree not exceeding 5.

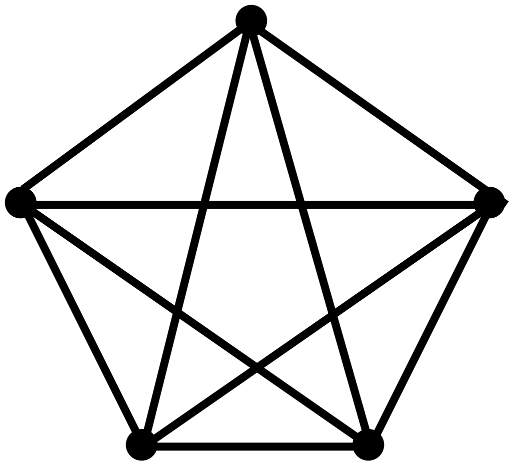
If G has one or two vertices, it is true;
thus, we assume that G has at least three vertices.

If the degree of each vertex were at least 6, then by Handshaking Theorem, $2e \geq 6v$, i.e., $e \geq 3v$,

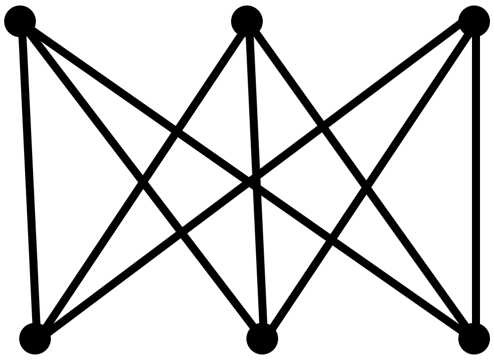
$$2e = \sum_{v \in V} \deg(v)$$

but this contradicts the inequality from
Corollary 1: 1 $e \leq 3v - 6$.

Two important nonplanar graph



K_5



$K_{3,3}$

Theorem 4: The graphs K_5 and $K_{3,3}$ are nonplanar.

pf: **(1)** K_5 has $p = 5$ vertices and $q = 10$ **edges**.

$q > 3p - 6 \quad \Rightarrow$ K_5 **is nonplanar.**

(2) Suppose $K_{3,3}$ is planar, and consider any embedding of $K_{3,3}$ in the plane.

Suppose **the** embedding has r regions.

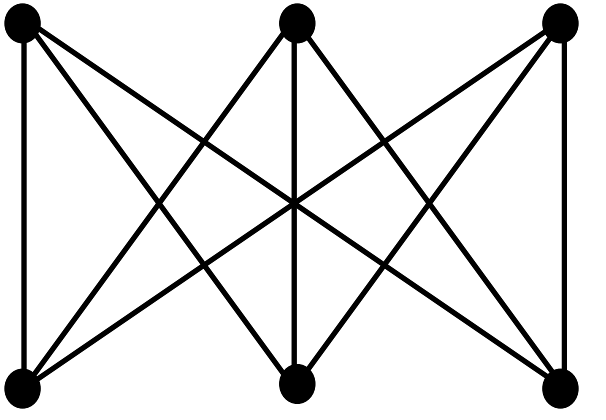
$p - q + r = 2 \quad \Rightarrow \quad r = 5$

$K_{3,3}$ is bipartite
$\Rightarrow$ The boundary of every region has $\geq$4 edges.

$$\Rightarrow 4r \leq \sum_{\forall\, Region\, R} |The\ edges\ of\ the\ boundry\ of\ R| = 2q = 18. \qquad \rightarrow\leftarrow$$

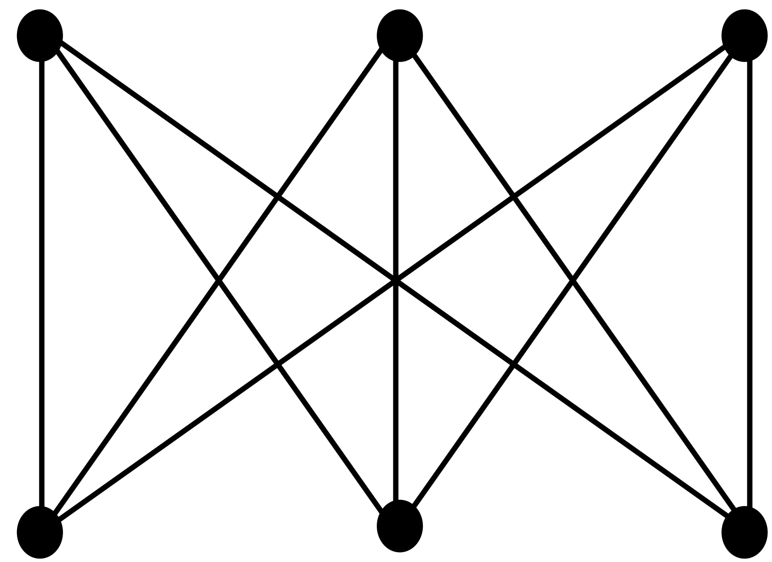
Corollary

- Corollary 3: If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v - 4$.
- Is $K_{3,3}$ planar?



Example of Corollary

- $K_{3,3}$ has 6 vertices and 9 edges.
- Obviously, v $\geq$ 3 and there are no circuits of length 3.
- If $K_{3,3}$ were planar, then $e \leq 2v - 4$ would have to be true.
- $2v - 4 = 2*6 - 4 = 8$
- So e must be ≤ 8.
- But $e = 9$.
- So $K_{3,3}$ is nonplanar.



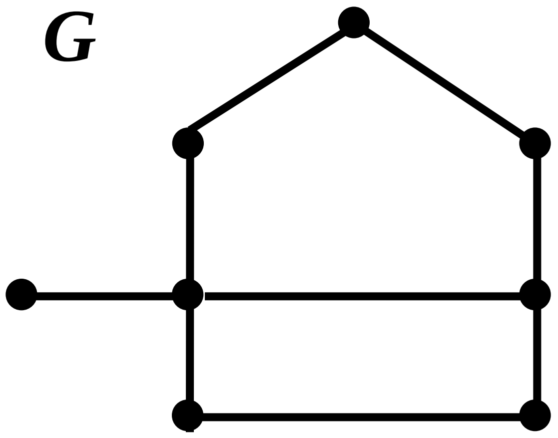
$K_{3,3}$

Definition:

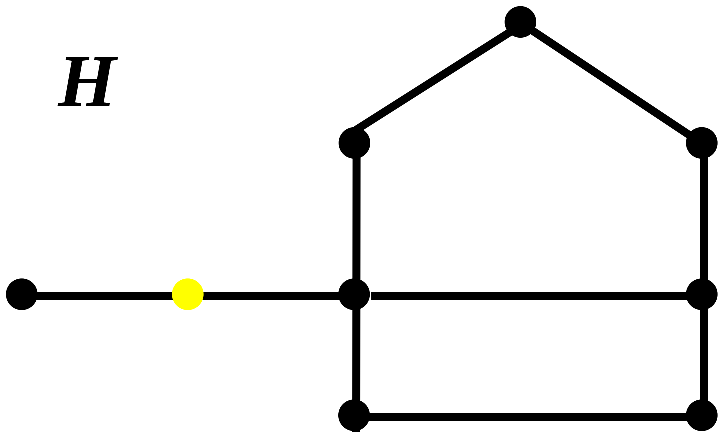
A subdivision of a graph G is a graph obtained by inserting vertices (of degree 2) into the edges of G.

Subdivisions of graphs.

G

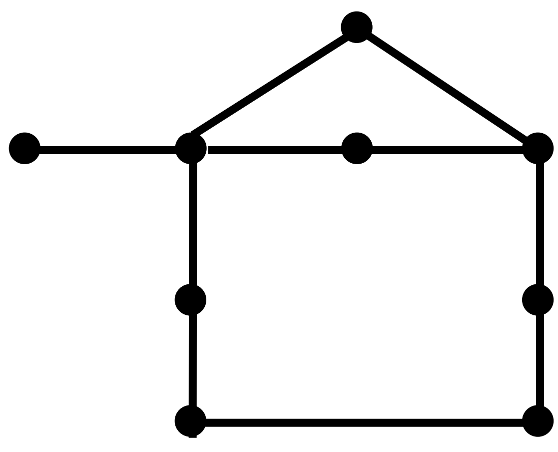


H



H is a subdivision of ***G***.

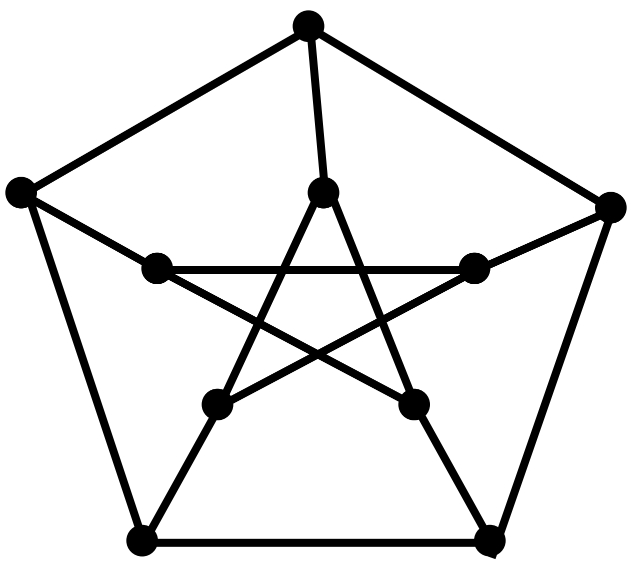
F



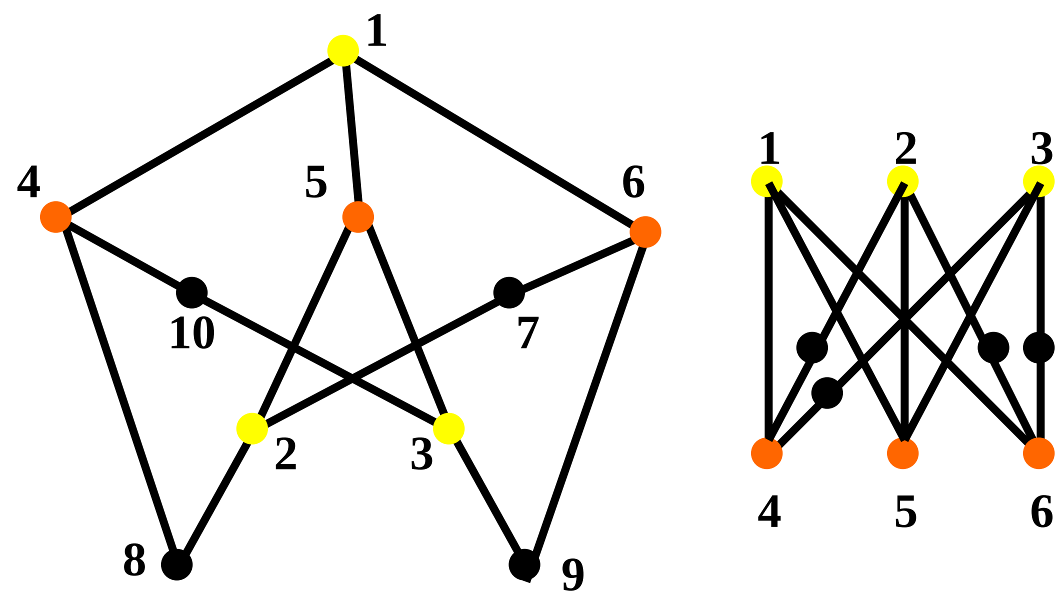
F is not a subdivision of ***G***.

Theorem 5: (Kuratowski's Theorem)
A graph is planar if and only if it contains no subgraph that is isomorphic to or is a subdivision of K_5 or $K_{3,3}$.

The Petersen graph is nonplanar.



(a) Petersen



(b) Subdivision of $K_{3,3}$