MATH1851

Notes for HKU · Spring 2024

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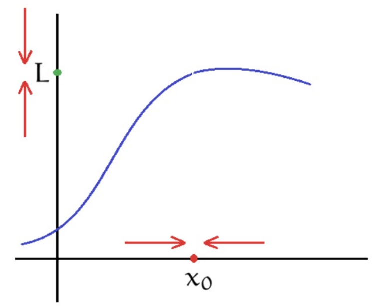
1 Limits and Continuity

1.1 Introduction to the concept of limit

We can conceptualize that the limit of a function $f(x)$ is L as x approaches c, given that we can make $f(x)$ *as close to L as we want* for all x sufficiently close to a, from both sides, *without actually letting x be a*. We can write this as:

$$\lim_{x \to a} f(x) = L$$



1.2 One-sided limits

There are two sides that x can tend to a number. We can write it as $x \to n^-$ and $x \to n^+$, which represents from the negative (left) / positive (right) side.

1.3 Existence of limits

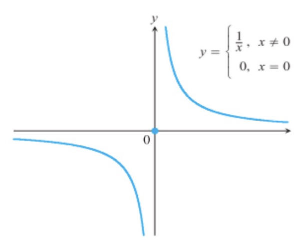
Condition for limit to exist

The limit for a function $f(x)$ only exists if and only if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

WARNING: If the **limit is** ∞ it **doesn't exist**.

For this example, when $x \to 0^-, y \to -\infty$.
Similarly, as $x \to 0^+, y \to +\infty$.
Hence, we can conclude that the limit for this function as $x \to 0$ doesn't exist.



1.4 Continuity

Continuity

A function $f(x)$ is *continuous* at $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

Continuity properties

For f, g continuous at c, the following are also continuous at c:

1. $f \pm g$
2. kf
3. fg
4. $\frac{f}{g}$, given that $g(c) \neq 0$

Intermediate value theorem (IVT)

If a function f is continuous on $[a, b]$, there is a number c in $[a, b]$ where $f(c)$ in $[f(a), f(b)]$.

To prove that there is a root, we can use the IVT by showing there is a **change of sign** between the interval given.
To show that there's *only one solution*, check if $f'(x) > 0$ or < 0 (strict $>$) in the interval.

1.5 Computing limits

Indeterminate forms

Indeterminate forms are forms that cannot be solved by simply substituting the value of x into the function. They are:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 1^\infty, \quad \infty^0$$

Note that $\infty + \infty = \infty$. Related: L'Hopital's rule

Using the limit laws

For functions f, g and using $\lim\limits_{x \to a} = \bigcirc$ for simpler notation:

1. $lim_{x \to a} c = c$
2. $\bigcirc(f \pm g) = \bigcirc f \pm \bigcirc g$
3. $\bigcirc(k \cdot f) = \bigcirc k \cdot \bigcirc f$
4. $\bigcirc(f^n) = (\bigcirc f)^n$
5. $\bigcirc(fg) = \bigcirc f \bigcirc g$
6. $\bigcirc(\frac{f}{g}) = \frac{\bigcirc f}{\bigcirc g}$, given that $\bigcirc g \neq 0$. *This strict condition prevents indeterminate forms.*

We can use these laws to break a limit into separate limits, and compute that way. Also note that:

7. $\bigcirc f(g) = f(\bigcirc g)$, given that f is **continuous** at $\bigcirc g$

Limit of a polynomial

For the limit of a polynomial $p(x)$:

$$lim_{x \to a} p(x) = p(a)$$

This can be proven easily with the limit laws above.

Techniques to compute limits

To solve for limits, we have to get the expression to the right form - a polynomial, for us to substitute our limit value into the function.
To do this, often we have to **factorize** or **rationalize**.

Example 1.1 *Indeterminate forms by substitution*

This applies limit law #5. As substituting into the function directly gives $0/0$*, we have to change it into a form such that we could apply the limit laws directly.*

$$\begin{aligned} lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} &= \frac{(x-2)(x+6)}{(x-2)x} \\ &= \frac{x+6}{x} \\ \textit{Substituting 2 gives} &= 4 \end{aligned}$$

The squeeze / sandwich theorem

Suppose $f(x) \leq g(x) \leq h(x)$ in the range $[a, b]$, for c in $[a, b]$:

$$\lim_{x\to c} f \leq \lim_{x\to c} g \leq \lim_{x\to c} h$$

We will make use of the fact that the limits can be equal to solve for the limit of $g(x)$.

Example 1.2 *Squeezing a function*

When we can't seem to factorize a function, we can try squeezing it between two other functions.

$$\lim_{x\to 0} x^2 \cos\frac{1}{x}$$

We know the limits of the function $\cos\frac{1}{x}$*, so we can start from there.*

$$\textit{Given that } x \neq 0, -1 \leq \cos\frac{1}{x} \leq 1$$

$$\textit{Multiplying } x^2 \textit{ on both sides}, -x^2 \leq \cos x^2\frac{1}{x} \leq x^2$$

$$\textit{As } \lim_{x\to 0} \pm x^2 = 0, \quad \textit{we can conclude that } \lim_{x\to 0} x^2 \cos\frac{1}{x} = 0$$

1.6 Infinite limits

Determining infinite limits

If $f(x)$ gets (negatively) arbitrarily large when x approaches a, we can say:

$$\lim_{x\to a} f(x) = (-)\infty$$

After we know that the limit *may* be infinity, we then have to make sure that *the limit is the same from both sides*, so that the limit is actually ∞. We can do so by plugging numbers which are approaching the limit from both sides.

Example 1.3 *Infinite limit exists*

$$\lim_{x \to 0} \frac{6}{x^2}$$

Consider both $\lim_{x \to 0^-} \frac{6}{x^2}, \lim_{x \to 0^+} \frac{6}{x^2}$:

$$\lim_{x \to 0} \frac{6}{x^2} = \infty$$

Example 1.4 *Infinite limit doesn't exist*

$$\lim_{x \to 4} \frac{3}{(4-x)^3}$$

Checking both sides, we can conclude that the limit doesn't exist, as:

$$\lim_{x \to 4^+} \frac{3}{(4-x)^3} = -\infty, \quad \lim_{x \to 4^-} \frac{3}{(4-x)^3} = \infty$$

1.7 Limits at infinity

Infinity operations

Note the following operations:

1. $\infty + k = \infty$
2. For $k < 0, \; k\infty = -\infty$

Determining limits of infinity

It is not hard to see that, for rational numbers n:

$$\lim_{x \to \pm\infty} \frac{k}{x^n} = 0$$

The easiest way to determine the limit would be to *factorize* the function so that we can use the fact above.

Determining limits of infinity of polynomials

Using the above fact, we can see that for a polynomial $p(x)$ with degree n and largest coefficient a_n:

$$\lim_{x \to \pm\infty} p(x) = a_n x^n$$

Which means we can *only consider the largest term in a polynomial* for limits of infinity.

Example 1.5 *Indeterminate forms by substitution of infinity*

Substituting ∞ *into the function gives* $\infty - \infty - \infty$*, which is indeterminate. Hence, we must factorize it.*

$$\begin{aligned} \lim_{x \to \infty} 2x^4 - x^2 - 8x &= \lim_{x \to \infty} [x^4(2 - \frac{1}{x^2} - \frac{8}{x^3})] \\ &= \infty \times 2 \\ &= \infty \end{aligned}$$

Or we can just simply use the theorem above and consider $\lim_{x \to \infty} 2x^4$ *only to give* ∞.

Example 1.6 *Factor polynomials limit to infinity*

We can simply consider the largest terms on each side and give the final answer easily.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x} &= \lim_{x\to-\infty}\frac{\sqrt{3x^2}}{-2x} \\
&= \lim_{x\to-\infty}\frac{\sqrt{3}|x|}{-2x} \leftarrow \sqrt{x^2}=|x| \\
&= \lim_{x\to-\infty}\frac{-\sqrt{3}}{-2} \leftarrow |c|, c<0=-c \\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

Note that, as we are considering the negative limit of infinity, we need to add - to the abs sign on line 3.

1.8 Asymptotes

Vertical asymptotes

f will have v-asymptotes at $x = a$ if any $\pm$ is true:

$$\lim_{x\to a^\pm} f(x) = \pm\infty$$

Horizontal asymptotes

f will have h-asymptotes at $y = L$ if any $\pm$ is true:

$$\lim_{x\to\pm\infty} f(x) = L$$

Related: Graphing functions

2 Trigonometry review

Trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan x = \frac{\sin x}{\cos x}$
3. $\csc x = \frac{1}{\sin x}$
4. $\sec x = \frac{1}{\cos x}$
5. $\cot x = \frac{1}{\tan x}$
6. $\sin 2x = 2 \sin x \cos x$
7. $\cos 2x = \cos^2 x - \sin^2 x$
8. $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$
9. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
10. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
11. $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
12. $1 + tan^2 x = sec^2 x$
13. $1 + cot^2 x = csc^2 x$

Related: Techniques of integration

3 Derivatives

First principle

The first principle is the definition of the derivative of a function $f(x)$ at $x = a$:

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiability

The function is differentiable at $x = a$ if:

$$\lim_{x \to a^-} f' = \lim_{x \to a^+} f'$$

3.1 Differentiation formulas and rules

Basic formulas

- We can differentiate individual items: $(f \pm g)' = f' \pm g'$
- We can factor out a multiplicative constant: $(cf)' = cf'$
- Derivative of a constant is 0: $\frac{d}{dx}k = 0$
- Power rule: $\frac{d}{dx}x^n = nx^{n-1}$

Chain rule

Shorthand: **d**1x2 + **d**2x1

$$(u(v))' = u'(v)v' \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product rule

Shorthand: **d** from outside to inside

$$(uv)' = uv' + vu' \qquad \text{or} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule

Shorthand: move lower **d** upper **-** **d** lower x upper, lower square

$$(\frac{u}{v})' = \frac{vu' - uv'}{v^2} \qquad \text{or} \qquad \frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

	$f(x)$	$f'(x)$
1.	a^x	$\ln a \cdot a^x$
2.	e^{kx}	ke^{kx}
3.	$\ln kx$	x^{-1}
4.	$\lvert x \rvert$	$\frac{\lvert x \rvert}{x}$
5.	$\sin kx$	$k \cos kx$
6.	$\cos kx$	$-k \sin kx$
7.	$\tan kx$	$k \sec^2 kx$
8.	$\csc x$	$-\csc x \cot x$
9.	$\sec x$	$\sec x \tan x$
10.	$\cot x$	$-\csc^2 x$
11.	$\sin^{-1} x$	$(\sqrt{1-x^2})^{-1}$
12.	$\cos^{-1} x$	$-(\sqrt{1-x^2})^{-1}$
13.	$\tan^{-1} x$	$(1+x^2)^{-1}$

Note that the trigo derivatives can be extended to hyperbolic trigo functions, with the except of $\frac{d}{dx}\text{sech}u = -\text{sech}u \tanh u$.

3.2 Techniques of differentiation

Parametric differentiation

For a parametric equation $y = f(t)$ and $x = g(t)$:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Derivative of Inverse functions

At c, we first find $f'(x)$ and the value of $f^{-1}(c)$, then we apply the formula to find the value of ${f^{-1}}'(c)$:

$${f^{-1}}'(c) = \frac{1}{f'(f^{-1}(c))}$$

$f^{-1}(c)$ can be found by solving $f(x) = c$.

Implicit differentiation

Differentiate all xy, add y' behind all differentiations of y.

To find y' for $y^2 = x^2 + \sin(xy)$:

$$\begin{aligned} y^2 &= x^2 + \sin(xy) \\ 2yy' &= \frac{d}{dx}(\sin(xy)) \\ 2yy' &= 2x + \cos(xy)(xy' + y) \\ \text{Then we simply collect terms of } y' \\ y' &= \frac{2x + y\cos(xy)}{2y - x\cos(xy)} \end{aligned}$$

To find the second derivative y'', differentate the expression and substitute y' back in.

Logarithmic differentiation

For $y = f(x), \quad y' = f(x) \times (\ln f(x))'$. (Takes natural log for both sides)

To find $\frac{dy}{dx}$ for $y = x^x$:

$$\begin{aligned} y &= x^x \\ \ln y &= x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= \ln x + 1 \\ \frac{dy}{dx} &= x^x(\ln x + 1) = x^x(x \ln x)' \end{aligned}$$

L'Hopital's rule

For any $a \in [\mathbb{R}, \pm\infty]$, if $\lim_{x\to a}(\frac{f}{g})$ is in indeterminate form after substitution, we can conclude:

$$\lim_{x\to a}(\frac{f}{g}) = \lim_{x\to a}(\frac{f'}{g'})$$

To find $\lim_{x\to -\infty} xe^x$, we first check if the limit is indeterminate, then we can apply the rule:

$$\begin{aligned}
\lim_{x\to -\infty} xe^x &\implies \infty \times 0 \\
\lim_{x\to -\infty} xe^x &= \lim_{x\to -\infty} \frac{x}{e^{-x}} \\
&= \lim_{x\to -\infty} \frac{1}{-e^{-x}} \text{ (rule applied)} \\
&= 0
\end{aligned}$$

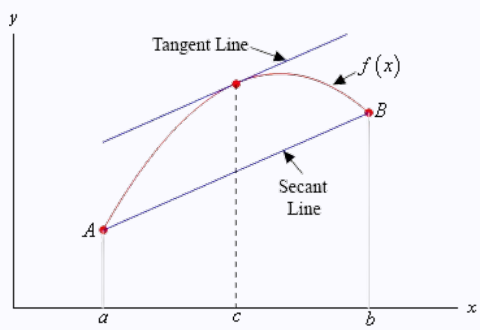
3.3 Important theorems

Mean value theorem (MVT)

For $f(x)$ that is *continuous* in $[a, b]$ and *differentiable* in (a, b):

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad c \in (a, b)$$

The theorem tells us that, in described conditions, there must be a point c where the slope of the *tangent line* is equal to the slope of the line from $a \to b$ (secant line).

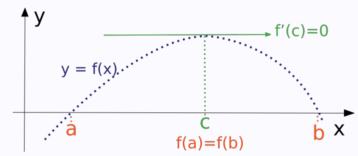


Source: https://tutorial.math.lamar.edu/Classes/CalcI/MeanValueTheorem.aspx

Rolle's theorem

For $f(x)$ that is *continuous* in $[a, b]$ and *differentiable* in (a, b):

$$\forall\ f(a) = f(b) \text{ there exists } f'(c) = 0, \quad c \in (a, b)$$



For $F' = f$, if F has 4 roots, then f has 3 roots.

3.4 Extremum points

Critical and inflection points

A **critical point** is a point where $f'(x) = 0$ or undefined, or the end-points of the domain if inclusive.
An **inflection point** is a point where $f''(x) = 0$ or undefined, that the concavity of the function changes.

Max/minimum points

The **absolute maximum/minimum points** are the points where the function has the largest/smallest value in the entire domain.
The **local maximum/minimum points** are the points where the function has the largest/smallest value in a small interval around the point.

3.5 Determining shape of graph

Concavity

Concavity is the direction of the curve, and can be described by the values of f' and f'':

f''	$-$	$+$	$+$	$-$
f'	$-$	$+$	$-$	$+$
f	$\searrow$	$\swarrow$	$\nwarrow$	$\nearrow$

Note: Arrows goes clock-wise.

	Step	Expression
1.	Determine domain of function	
2.	Special points without continuity?	
3.	Axis intercepts	$(f(x) = 0, 0)$ $(0, f(0))$
4.	Critical points	$f'(x) = 0$ or undefined
5.	Point maxima?	$+f''(x)$ or $-f''(x)$
6.	Inflection points	$f''(x) = 0$ or undefined
7.	Horizontal asymptotes	$\lim_{x \to \pm\infty} f(x) = n$?
8.	Vertical asymptotes	$\lim_{x \to a^{\pm}} f(x) = \pm\infty$?
9.	Area strictly increasing/decreasing?	$f'(x) > 0$ or $f'(x) < 0$
10.	Area concavity?	$- + + - / - + - + /$ $\searrow\swarrow\nwarrow\nearrow$

Related: Definition of asymptotes

Higher derivatives

Derivatives of higher order (e.g. $f''(x), f'''(x)$) can be expressed as $f^{(n)}(x)$

4 Integrals

Definition of natural logs

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$$

4.1 Definite integrals

Signed areas

Signed area is the area between the curve and the x-axis, where the area above the x-axis is positive and below is negative.

Properties of definite intergrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
4. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
5. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

4.2 Fundamental theorem of calculus

Fundamental theorem of calculus (FTC)

If $f(x)$ is continuous in $[a, b]$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where $F(x)$ is the definite integral of $f(x)$.

Second fundamental theorem of calculus

If $f(x)$ is continuous in $[a, b]$, then:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Find $g'(x)$ for $g(x) = \int_1^{x^2} \cos t \; dt$:

We first define $G(u) = \int_1^u \cos t \; dt$, then we can apply the chain rule:

$$\begin{aligned} g'(x) &= (G(x^2))' \\ &= G'(x^2) \cdot 2x \\ &= \cos x^2 \cdot 2x \end{aligned}$$

Cases of area 0

For $\int_a^b f(x)dx$:

- If $b = a$, then the area is 0 (no width)
- If $f(x) = f(-x)$ and $a = -b$, then the area is 0 (symmetry)

4.3 Techniques of integration

Overview

- By **substitution**: "sub $g(x) \to u$, find du and replace all functions of $x(dx)$ with $u(du)$"
- By **part**: "$\int ab = a \int b - \int (a' \times \int b)$"
- By **joining recurring parts**: "$\int ab = f(ab) + \int ab \to 2 \int ab = f(ab)$"
- By **partial fractions**: "$\frac{k}{f(x)g(x)} = \frac{A}{f(x)} + \frac{B}{g(x)}$"
- By **trigonometric identities**
- By **multiplying fractions by one**: "Multiplying $\frac{\sec^2 x}{\sec^2 x}$ to fit trigo form"

Integration by substitution

For $\int f(g)g'\,dx$:

1. Let $u = g$
2. Find $du = g'dx$
3. Change limits in terms of u if definite
4. Replace all $g \to u$ and $g'dx \to du$
5. Integrate, then subsitute $u \to g$

Integration by parts

$$\int ab = a \int b - \int (a' \times \int b)$$

Tips: Let a (differentiating term) to the first term you see on the list:

1. **L**ogarithmic
2. **I**nverse trigo
3. **A**lebratic (polynomial)
4. **T**rigo
5. **E**xponential

Deriving the IBP formula

The formula is derived from the product rule of differentiation:

$$\begin{aligned}(ab)' &= a'b + b'a\\ b'a &= (ab)' - a'b\\ \int b'a &= ab - \int a'b\\ \int ab &= a\int b - \int (a' \times \int b)\end{aligned}$$

Trigo-identities substitution tips

Use the following trigonometric identities to simplify the integral:

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1-\cos 2x}{2}$
- $\cos^2 x = \frac{1+\cos 2x}{2}$
- $2\sin x \cos x = \sin 2x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\sec^2 x = 1 + \tan^2 x$
- $\csc^2 x = 1 + \cot^2 x$

Use the following substitutions for similar expressions:

$$\text{Expressions of the form} \begin{cases} a^2 - f(x)^2 & \to f(x) = a\sin\theta \\ a^2 + f(x)^2 & \to f(x) = a\tan\theta \\ f(x)^2 - a^2 & \to f(x) = a\sec\theta \end{cases}$$

Remember that this uses *integration by substitution*, so we need to find dx in terms of θ.

4.4 Solids of revolution

Solids of revolution

A 3D shape formed by rotating a *region* around an axis.

A region is defined by 2 curves and an interval:

1. The vertical region bounded by an outer cruve $R(x)$ and an inner curve $r(x)$.
2. The inner curve is $y = 0$ for the region bounded by the x-axis.
3. The horizontal region is bounded by an interval $x:\ [a, b]$

Note that for the methods below, the terms x and y can be *switched* to fit the problem.
$\{f(x) \to \circlearrowleft x\}$ denotes using a function of x to find the volume of a solid of revolution around the x-axis.
If the volume is generated by the rotation about an axis other than the main axes, we can simply *shift* the function to fit the main axes.

Volume by Washers (Disk)

$$\{f(x) \to \circlearrowleft x\}: \quad V = \pi \int_a^b R^2(x) - r^2(x) dx$$

For rotating about $y = n$: $\forall f(x) \to f(x) - n$

Volume by Cylindrical shells

$$\{f(x) \to \circlearrowleft y\}: \quad V = 2\pi \int_a^b [x][R^2(x) - r^2(x)] dx$$

For rotating about $x = n$: $[x] \to [x - n]$

4.5 Arcs and surfaces

Arc length

The *arc length* of a curve $y = f(x)$ from $x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + f'^2(x)} dx$$

And for parametized equations $x = f(t)$, $y = g(t)$:

$$L = \int_a^b \sqrt{f'^2(t) + g'^2(t)} dt$$

Area surfaces of revolution

Used to find the *surface area* generated by rotating a curve along an axis.

$$\{f(x) \to \circlearrowleft x\}: \quad A = 2\pi \int_a^b f(x)\sqrt{1 + f'^2(x)} dx$$

And for parametized equations $x = f(t)$, $y = g(t)$:

$$A = 2\pi \int_a^b g(t)\sqrt{f'^2(t) + g'^2(t)} dt$$

5 First Order Differential Equations

Differential equations are equations that involve a function and its derivatives.

Order of differential equations

A n ordered differential equation is an equation of the form:

$$F(x, y, y' \dots y'_n) = 0$$

Where y'_n is the nth derivative of y with respect to x. The highest degree of the derivative is n for a n-ordered differential equation. Note that y is really just $y(x)$ (A function of x)

5.1 Solving linear 1st-ODEs

Linear differential equations

Linear differential equations does not contain non-linear functions. (e.g. $\sin y$) Otherwise, it's a non-linear ODE.

Solving by integrating factors

We can solve a linear 1st-ODE as followed, given a **particular solution** of $y(x)$:

$$y' + p(x)y = q(x) \quad : \quad \times\ e^{\int p(x)}$$

The multiplied integration factor $e^{p(x)}$ will give us a product of the *product rule*, then we simply integrate both sides to solve for y. Make sure that the **coefficient of y' is 1**.

Separable equations

We can solve a separable equation as followed, given a **particular solution** of $y(x)$:

$$\frac{dy}{dx} = f(x)g(y)$$
$$g(y)^{-1}dy = f(x)dx$$
$$\int g(y)^{-1}\, dy = \int f(x)\, dx \Leftarrow [(x, y) \to c]$$

5.2 Solving non-linear 1st-ODEs

Bernoulli's equation

A non-linear 1st-ODE of the form can be solved by:

$$y' + p(x)y = q(x)y^n \quad n \in \mathbb{R} \quad : \quad \text{sub } u = y^{1-n} \to y' = \frac{1}{1-n}u^{\frac{1}{1-n}-1}u'$$

The subsitution $u = y^{1-n}$ will turn the equation into a linear ODE, then simply solve using integrating factors.

Riccati's equation

A non-linear 1st-ODE of the form can be solved by the following, given a **particular solution** of $y(x)$:

$$y' = p(x)y^2 + q(x)y + r(x) \quad : \quad \text{sub } y = y(x) + u^{-1}$$

Homogeneous equations

A homogeneous equation has it's x and y terms **in the same degree**. (e.g. $x^2 + xy + y^2 = 0$)
A *homogeneous* 1st-ODE of the form can be solved by the following, given a **particular solution** of $y(x)$:

$$y' = f(\frac{y}{x}) \quad : \quad \text{sub } u = \frac{y}{x} \rightarrow y' = u + xu'$$

We can **divide the formula** by x^n or y^n to get the equation in the desired form (*every term is the ratio* $\frac{y}{x}$). Otherwise, we can shift the origin using $X = x - n$ and $Y = y - m$.
After substitution, we will get a **separable equation** after the substitution, and the particular solution is used.

5.3 Exact equations

Partial derivatives

A partial derivative is a derivative of a function with respect to one of its variables, with the others held constant. The following notation expresses the partial derivative of f with respect to x:

$$\frac{\partial f}{\partial x}$$

$$F = 2x + y$$
$$\frac{\partial F}{\partial x} = 2$$

Exact equations

An exact equation is simply a 1st-ODE where $dF = 0$.
The expressed equation dF is exact if:

$$dF = Mdx + Ndy \quad : \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solving exact equations

To find the solution of an exact equation:

$$M = \frac{\partial F}{\partial x}, \quad N = \frac{\partial F}{\partial y}$$

$$\partial F = M \partial x$$

$$F = \int M dx + g(y)$$

$$\frac{\partial F}{\partial y} = N$$

$$\frac{\partial \int M dx + g(y)}{\partial y} = N \rightarrow g'(y)$$

$$\int g'(y)\, dy = g(y) \rightarrow F$$

$g(y)$ is present as we are integrating partially with respect to x, and $g(y)$ is the constant of integration. Hence, the solution would be:

$$\int M dx + g(y) = c$$

6 Second Order Differential Equations

6.1 Solving homogeneous linear 2nd-ODEs

Constant coefficient Homogeneous 2nd-ODEs

The term homogeneous is used differently from the previous section.

A homogeneous 2nd-ODE is of the form:

$$ay'' + by' + cy = \mathbf{0}$$

Where a, b, c are constants. (Coefficients are *constants*)

We first use the following substitution:

$$ay'' + by' + cy = 0: \quad y = e^{\lambda x} \implies a\lambda^2 + b\lambda + c = 0 \to \lambda$$

To find the general solution, we put the λ roots into the **quadratic characteristic equation**:

1. $\lambda_1 \neq \lambda_2: \quad y = c_1\mathbf{e}^{\lambda_1 x} + c_2\mathbf{e}^{\lambda_2 x}$
2. $\lambda_1 = \lambda_2: \quad y = c_1\mathbf{e}^{\lambda x} + c_2 x\mathbf{e}^{\lambda x}$
3. $\{\lambda_{1,2} = \alpha \pm \beta i\} \in \mathbb{C}: \quad y = c_1\mathbf{e}^{\alpha x}\cos(\beta x) + c_2\mathbf{e}^{\alpha x}\sin(\beta x)$

If given **particular solutions** of y and y', we can solve for c_1 and c_2 by finding y' with our general solution and substituting.

Cauchy-Euler equations

A Cauchy-Euler equation is a slight variation of homogenous 2nd-ODEs, which is of the following form and can be solved by:

$$ax^2y'' + bxy' + cy = 0 \quad : \quad y = x^{\lambda} \implies a(\lambda^2 - \lambda) + b\lambda + c = 0$$

The general solutions is similar to that of the homogeneous 2nd-ODEs, but with **all terms of** $x \to \ln x$:

1. $e^{\lambda x} \to x^{\lambda}$
2. $e^{\lambda x} \to x^{\lambda}, \quad x \to \ln x$
3. $e^{\alpha x} \to x^{\alpha}, \quad \beta x \to \beta \ln x$

6.2 Solving non-homogeneous linear 2nd-ODEs

Constant coefficient Non-homogeneous 2nd-ODEs

A non-homogeneous 2nd-ODE is of the form:

$$ay'' + by' + cy = g(x)$$

Where a, b, c are constants. (Coefficients are *constants*)

We first solve for $\lambda_{1,2}$ for the **complementary homogenous function** F_c:

$$F_c: \; ay'' + by' + cy = 0$$

The general solution for the non-homogeneous 2nd-ODE F is:

$$F = F_c + F_p$$

Where F_p is a **particular solution** of F. To solve for F_p, we can use the following methods:

1. **Method of undetermined coefficients**
2. **Variation of parameters**

Method of undetermined coefficients

To solve for F_p for a non-homogeneous 2nd-ODE, let F_p as the following if $g(x)$ consists of:

- $e^{ax} \to F_p = Ae^{ax}$
- $\sin x$ and / or $\cos x \to F_p = A\sin x + B\cos x$
- $x^n \to F_p = Ax^n = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0$ (polynomial of degree n)

Then substitute $F_p \to y$ into the non-homogeneous 2nd-ODE and solve for A and B. Finally, the general solution is $F = F_c + F_p$.