ENGG1300

Notes for HKU · Spring 2024

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Contents

1 Vectors

Basics

Vector can be represented by $\vec{F}$ or $\boldsymbol{F}$. The magnitude can be represented by $|F|$ or F.

Unit vectors

They are vectors with magnitude 1. $\hat{A} = \frac{\vec{A}}{|\vec{A}|}$

1.1 Coplanar vectors

Cartesian vector notation

In two dimensions, the Cartesian unit vectors $\boldsymbol{i}, \boldsymbol{j}$ are used to designate the directions of the x and y axes respectively.

$$F = F_x \boldsymbol{i} + F_y \boldsymbol{j}$$

Where $F_{x/y}$ is the x/y component of F. And to find the x/y components we can use trigonometry:

$$F_x = F\cos\theta, \quad F_y = F\sin\theta$$

Where the angle θ is the angle between F and the x-axis.

Resultant force

The resultant force F_R can be found by the sum of the components of F:

$$F_R = \sum F$$

In Cartesian form, it's the same as adding all the terms together: $F_R = (F_{x1} + F_{x2})\boldsymbol{i} + (F_{y1} + F_{y2})\boldsymbol{j}$

Orientation of vector

We always consider the angle between F & F_x. It can be found by $\theta = \tan^{-1}\frac{F_y}{F_x}$.

Magnitude of forces

The magnitude will simply be the square root of the sum of squared components of the force:

$$|F| = \sqrt{F_x^2 + F_y^2 + \dots}$$

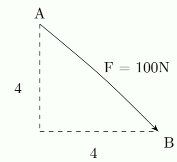
Moving vectors by their position vectors

We can move a force F to a point using its position vector r (pointing to their tail), using the following definition:

$$\vec{F} = |F| \times \hat{F} = |F| \times \frac{r}{|r|}$$

The position vectors are in Cartesian form, so the moved vectors will also be in Cartesian form.

Consider the following graph:



First, we consider r_{AB}. From the graph:

$$r_{\vec{AB}} = 4\boldsymbol{i} + 4\boldsymbol{j}$$

Then, we find the magnitude of r_{AB}:

$$|r_{AB}| = \sqrt{4^2 + 4^2} = 5.65\ldots$$

Finally, we apply our formula for F:

$$\begin{aligned}\vec{F} = |F| \times \frac{r_{AB}}{|r_{AB}|} &= 100 \times (\frac{r_x}{5.65}\boldsymbol{i} + \frac{r_y}{5.65}\boldsymbol{j}) \\ &= 100 \times (\frac{4}{5.65}\boldsymbol{i} + \frac{4}{5.65}\boldsymbol{j}) \\ &= 70.7\boldsymbol{i} + 70.7\boldsymbol{j}N\end{aligned}$$

1.2 Vectors in 3D

The concepts above can be extended to 3D simply by adding another variable to the system.

2 Moment of forces

Definition of moment

The moment of a force is a measure of its *tendency* to cause a body to rotate about a specific point. The moment about a point O, when F is applied a distance d from the point is:

$$M_O = F \times d$$

Keep in mind that *positive* moment is *anti-clockwise*.

2.1 Coplanar / 2D moment

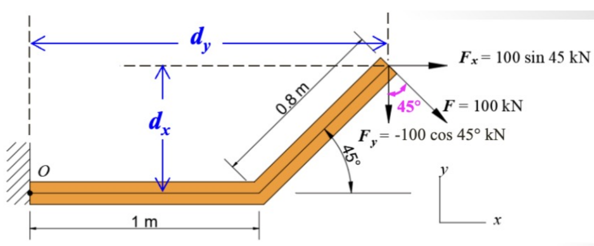
Resultant moments

The resultant moment is the **sum** of all moments present on the point, given by:

$$M_R = \sum M_O$$

2.1.1 The moment of a non-linearly attached force

One simple way is to find the *components* of the force, and sum their individual moments together. The following is a simple example:



After finding the component forces of F, we can deduce the resultant moment to be:

$$\begin{aligned} M_O &= F_x \times 0.8 \sin 45 \deg + F_y \times (1 + 0.8 \cos 45 \deg) \\ &= -150.7kNm \end{aligned}$$

2.2 Non-coplanar / 3D moment

Moments in a 3D system

Consider position vector $\vec{r}$ drawn from O to any point on the *line of action* of F. The moment can hence be given by:

$$M_O = r \times F$$

Finding the moment via cross products Cartesian vectors

The cross product C given by A and B is:

$$A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = C$$

The cross-product for vectors going in the *same direction* is 0. (i.e. $n\boldsymbol{k} \times m\boldsymbol{k} = 0$)

Resultant moments

The resultant moment is simply the **sum** of *couple moments* and moments of forces:

$$(M_R)_O = \sum M_O + \sum M$$

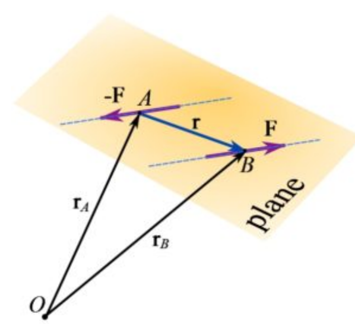
You can interpret $(M_R)_O$ as the resultant moment about point O.

2.3 Couple moments

Couples are *two parallel forces* that have the same magnitude but have *opposite directions*, separated by a *perpendicular distance* d. The magnitude of the moment is given by:

$$M = Fd$$

Notice that there's no point mentioned so far. For couple moment, it is **always the same about any point**. Let's assume for any point O (refer to graph), the moment is:



$$\begin{aligned} M_O &= r_B \times F + r_A \times -F \\ &= (r_B - r_A) \times F \\ &= r \times F \quad \text{which is independent of } O \end{aligned}$$

Hence, we can say that couple moments are **free vectors**.

3 Axially loaded members

Axial loading refers to the application of a force along the axis of the member.

3.1 Stress and strain

Axial stress

Axial stress is the stress that is *parallel* to the cross-sectional area of the member. It is given by:

$$\sigma = \frac{F}{A} \quad (Nm^{-2})$$

Where F is the force applied, and A is the cross-sectional area of the member. Note that $1Pa = 1Nm^{-2}$.

Eccentric loading and stress

When a force is applied *off-centre* to the member, the stress at each end is given by:

$$\sigma = \frac{\sum F}{wd} \pm \frac{6F \times e}{d \times w^2}$$

Where w is the width of the member, d is the depth of the member, and e is the eccentric distance from the centroid of the member to the point of application of the force.
The $\pm$ sign is used to denote the *maximum* and *minimum* stress on opposite sides.

Axial strain

Axial strain is the ratio of the change in length to the original length of the member. It is given by:

$$\epsilon = \frac{\Delta x}{x} \quad \text{(Ratio)}$$

Where Δx is the change in length, and L is the original length of the member.

3.2 Materials

Strength

Material strength is defined as the **maximum stress** that can be resisted by the material.

Young's Modulus

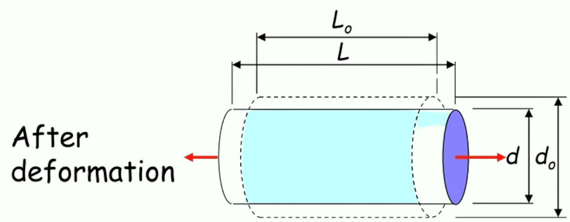
Young's modulus is the ratio of stress to strain, given by:

$$E = \frac{\sigma}{\epsilon} = \frac{Fx}{A\Delta x} \quad (Pa, Nm^{-2})$$

Poisson's ratio

Poisson's ratio is the ratio of lateral strain ϵ_l to axial strain ϵ, given by:

$$\nu = \frac{\epsilon_l}{\epsilon} \quad \text{(Ratio)}$$



The lateral strain ϵ_l is $\frac{\Delta d}{d_0}$.

3.3 Hydrostatic pressure

Hydrostatic/water pressure

The water pressure acting on any surface is *always perpendicular* to the surface, and the pressure is given by:

$$p = \rho g h \quad (Pa, Nm^{-2})$$

Where ρ is the density of water, and h is the depth of the water.
Hence, we can see that the water pressure **increases linearly with depth**.

Water pressure load on slanted surface

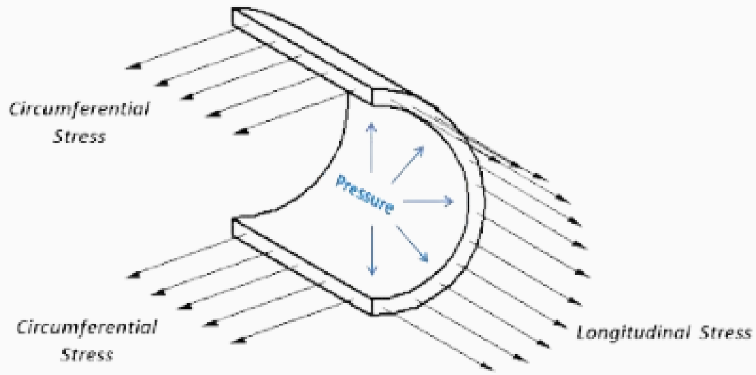
The load exerted on a slanted surface by water pressure F is given by and located at:

$$F = \frac{\rho g d w L}{2} \quad @\frac{1}{3}L \;/\; \frac{2}{3}d$$

Where d is the depth of water, w is the width of the volume, and L is the length of the surface.

Water pipe

1. **Internal pressure** refers to the pressure inside the pipe.
2. **Internal force** refers to the pressure's effect onto a side of the pipe.
3. **Internal stress** refers to the stress caused by the internal force, acting along the *circumferential direction* of the pipe. This is also called the **hoop stress**.



4 Statics

4.1 Equilibrium of rigid bodies

The 3 equations of equilibrium

A rigid body is in equilibrium if:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_O = 0$$

Where O is any point.

Static equilibrium

A system is in **static equilibrium** if it experiences no acceleration when loads are applied to it.

Free body diagrams (FBD)

Free body diagrams are diagrams that show all the forces acting on a body. They are useful in determining the forces that cause the body to be in equilibrium.

4.2 Members

A member is a *straight* and *long* structural element that is subjected to axial forces. The following are the types of members:

4.3 Static systems

Supports and reaction forces

Each support will restrict the movement of the rigid body in a certain way. If a certain degree of freedom is restricted, a reaction force will be present to counteract the force that would have caused the movement.

Support type	**Restricted movement and reaction forces**	**Shape**
Fixed	x, y, r (moment)	Flat
Pinned	x, y	Triangle
Roller	y	Circle

Stable structures

A structure is said *stable* if all members remain in place under any loading conditions.

Static conditions of a system

A system's state of equilibrium can be determined by the number of restraints present:

1. Insufficient restraints → **non-static system** (contains mechanisms)
2. Sufficient restraints → **static system**

A static system is said to be **statically indeterminate** if the number of unknowns is larger than the number of equations of equilibrium.

Degree of indeterminacy (Trusses)

$$I = m + r - 2j$$

Where m is the number of members, r is the number of reaction forces, j is the number of joints.

Degree of indeterminacy (Frames)

$$I = r - 3 + 3n$$

Where n is the number of members that we can cut through for the frame to be statically determinate.

4.4 Loading

Types of loading

1. **Concentrated load** is a force applied at a single point. (N)
2. **Distributed load** is a force applied over a length. (N/m)

Distributed load

A *distributed load* (w) is a force that is distributed over a length (l), that has the unit N/m.
The **resultant force** is the supposed area of the load. To consider the **moment** by a distributed load, we can treat the load as a *single force* acting at the **centroid** of the load.

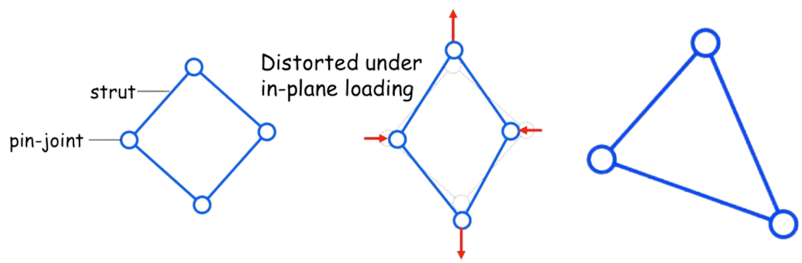
- Unifrom → $F = wl \quad @\frac{1}{2}l$
- Triangular → $F = \frac{1}{2}wl \quad @\frac{1}{3}l$ (from the base of the load)

5 Structural analysis

5.1 Pin-joined trusses

A **truss** is a structure that consists of *straight members* connected at their ends by *pin joints*.

A triangular truss is a statically determinate structure, but a quadrangular truss is not.

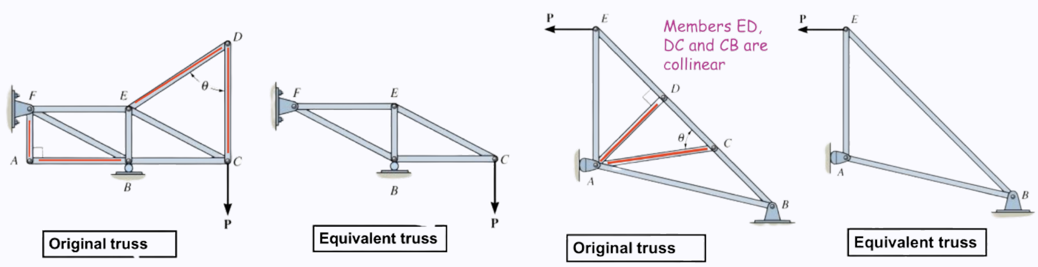


Zero-force members

The following are zero-force members given that **no external load or support reaction is applied to the joint**.

- **2 non-collinear members** of a two-member-joint
- **3rd member** of a three-member-joint, where the other two are collinear.

Additionally, members that provide **no structural support against the applied load** are also considered to be zero-force members.
Removing zero-force members allows us to simplify a truss structure.



5.2 Statically determinate trusses

To construct an *internally* statically determinate truss, we begin with a triangular pin-jointed truss and then successively adding two new members with a new joint.

We say the truss is only *internally* statically determinate because the external reactions are not yet known.

5.3 Truss analysis

When we analyze a truss, we assume that forces are only applied at joints.

Internal vs. External forces

- Forces acting on the members $\rightarrow$ **Internal forces**
- Forces acting on joints $\rightarrow$ **External forces**

Sign convention of analysis

As we focus on the analysis of **internal forces acting on joints**, we always draw:

- Tension forces $\rightarrow$ **pointing away** from the joint $\{\leftarrow \circ \boxed{\rightarrow\ T\ \leftarrow} \circ \rightarrow\}$
- Compression forces $\rightarrow$ **pointing towards** the joint $\{\rightarrow \circ \boxed{\leftarrow\ C\ \rightarrow} \circ \leftarrow\}$

During analysis, we always draw forces **pointing towards** the joint $\{\rightarrow \circ \leftarrow\}$, which means we treat **compression as positive**.

We also treat **downward forces as positive** $\{+\downarrow\}$.

5.4 Equilibrium sections

If we cut out any section of an equilibrium system, the cut will be in equilibrium. That is, the **external forces** are balanced by **internal forces**.

Therefore, to solve for the forces in a system, we simply cut out a section that we deem solvable (by practice!) and draw it's FBD. The forces includes *all external forces in the cut* as well as the *interal forces of the cut members*.

5.5 Cable anaylsis

Cable forces

Cables are **always in tension**. The tension force is always in the direction of the cable.

Some characteristics of cable structures:

- Supports are always 2 inverted pinned supports
- Applied forces all point downwards due to gravity
- Horizontal reaction forces at supports are equal in magnitude

Hence, we can use the following steps to solve for cable forces:

1. Find reaction forces by considering joint equilibrium
2. Tension in cables attached to a support can be found by $\sqrt{(R_x)^2 + (R_y)^2}$
3. Tension in cables not attached to a support can be found by $\sqrt{(R_x)^2 + (R_y - \sum y)^2}$

6 Centroids and moment of inertia

6.1 Centroids

Expression of centroids

We use $\bar{x}$ and $\bar{y}$ to denote the *centroid* of a shape. They are the horizontal and vertical distance about a given **reference axis**.

The centroid is the **centre of mass** of the shape. For an axis that the shape is *symmetric about*, the centroid will be **on the axis**.

The centroid is also known as the *first moment of area*.

Centroid of composite shapes

The centroid of a shape can be found by:

$$\bar{x} = \frac{\sum(A_i x_i)}{\sum A_i}, \quad \bar{y} = \frac{\sum(A_i y_i)}{\sum A_i}$$

Where A_i is the area of the composition shape, and x_i, y_i are the distances of the composition shape's centroid from the reference axes.

6.2 Moment of inertia

The reference axis for moment of inertia

We measure the moment of inertia about a **reference axis**. This is because at different points, the moment of inertia will be different. (*Unlike the use of reference axes in centroids, where the reference axes is just relative*).

For finding the moment of inertia at the centroid, we simply set the reference axis at the centroid.

Moment of inertia / Second moment of area

The moment of inertia is a measure of an object's resistance to changes in its rotation, about a reference axis. It is given by:

$$I_x = \int y^2 \, dA, \quad I_y = \int x^2 \, dA$$

Where x and y are the distances from the axis. This units are m^4.

Moment of inertia for rectangles

The moment of inertia for a rectangle is given by:

$$I_x = \frac{bh^3}{12}, \quad I_y = \frac{hb^3}{12}$$

Where b is the width of the rectangle, and h is the height of the rectangle. This can be derived by the formulas above.

It is important that the **direction** where b extends is **parallel to the reference axis** (so width $\neq$ longest length)

Moment of inertia for circles

The moment of inertia for a circle is given by:

$$I_x = I_y = \frac{\pi r^4}{4}$$

Where r is the radius of the circle.

Parallel axis theorem

The moment of inertia about an axis parallel to a reference axis for a shape is given by:

$$I_{x'} = I_x + Ad^2$$

Where A is the area of the shape, and d is the distance between the **reference axis** and the **parallel axis**.

Moment of inertia for composite shapes

The moment of inertia for a composite shape is simply the sum of the moments of inertia of the individual shapes **about the same axis**. This can be expressed as:

$$I_x = I_{x1\to\bar{y}} + I_{x2\to\bar{y}}, \quad I_y = I_{y1\to\bar{x}} + I_{y2\to\bar{x}}$$

Related: Bending stress

7 Bending forces and stress

7.1 Internal force in beams

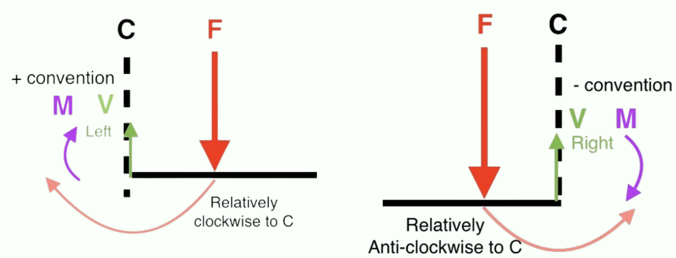
When we consider the internal forces of a beam, we take a *cross section* and consider the forces acting on it. The internal forces are:

- **Shear force** V is the force acting parallel to the cross-section.
- **Bending moment** M is the moment generated by the variation of forces acting perpendicularly to the cross-section.

Sign convention of internal forces

$\{+\circlearrowright\}$ The idea is that if the acting force is relatively **clockwise** to the cross-section C, it is **positive**. The following is the signs of internal forces for a *positive* moment:

- **V:** $\circlearrowright$ (L $\uparrow$ R $\downarrow$)
- **M:** $\curvearrowright\curvearrowleft$ (sagging moment is *positive*)



Couple moments can be treated as a rotational moment at clock-direction the arrows are pointing to.

Finding internal forces by side

To find the internal forces at cross-section C, we consider the *left* and *right* side of the section:

- **V:** $\sum F$ on either side
- **M:** $\sum M_C$ on either side about C

Note that the sums are all **signed by convention**. (e.g. A force F is positive if it points upwards on the left side.)
This also gives us the fact that $\sum F_L = \sum F_R$ and $\sum M_L = \sum M_R$, signed by convention on both sides.

7.2 Diagramming internal forces

Diagramming internal forces

The two internal force diagrams are $V(x) - x$ and $M(x) - x$ diagrams. $V(x)$ gives the shear force V at distance x from the *left side* of the beam.

Relation between internal forces

The following is the relation between the internal forces:

$$\frac{dM}{dx} = V \quad \frac{dV}{dx} = -w$$

Where w is the distributed load acting on the beam.

- This tells us that the slope of the $M(x)$ diagram at an interval is the value of V at that interval.
- This tells us that the slope of the $V(x)$ diagram at an interval is the value of $-w$ at that interval.

7.2.1 Steps to follow

1. Solve for reaction forces
2. Draw the FBD of a cut from the leftmost side to a location with distance x from the leftmost force *not analysed*
3. Analyse the FBD of the cut (solve for V and M)
4. Draw the graphs in the cuts and repeat for all cuts

Figurative steps:

1. $\rightarrow R_A, R_B ...$
2. $\{\overset{R}{\uparrow}\ \underbrace{=====}_{x}\ \overset{V}{\downarrow}\overset{M}{\circlearrowleft}\ \dots\}$
3. $\rightarrow V, M$
4. $\rightarrow V(x), M(x)$

To analyse a distributed load w, consider the FBD of the whole system on the left, then let x be the distance from the leftmost force *analysed*:

$$\{\underset{\underset{R}{\uparrow}}{=}\underset{2}{==}\overset{\overset{F}{\downarrow}}{=}\ \underbrace{==\overset{\overset{w\cdot x}{\downarrow}}{=}==}_{x}\ \overset{V}{\downarrow}\overset{M}{\circlearrowleft}\ \dots\}$$

7.2.2 Signing conventions

By convention, we consider the **positive convention** with respect to the cross-section. That means, if the force is **clockwise** to the cross-section, it is **positive**..

7.3 Bending stress

Pure bending

A **section** of the beam is said to be under *pure bending* if:

$$V = 0, \quad M \text{ constant}$$

Netural axis

The neutral axis always **passes through the centroid** of the beam. It is *netural* as the **length remains unchanged** for the surface that the axis passes through.
The distance from the netural axis is denoted y (**positive downwards**).

Bending section modulus

The bending section modulus W is given by:

$$W = \frac{I}{y_{max}} \; m^3$$

Bending stress

The bending stress σ is given by:

$$\sigma = \frac{My}{I}$$

Where M is the bending moment, y is the distance from the **neutral axis** (centroid), and I is the **moment of inertia** of the section.
The formula also tells us that the bending stress is *proportional* to the distance from the neutral axis, and the neutral axis experiences *no stress*.
Related: Moment of inertia

Maximum tensile and compressive stress

$$\sigma_{max} = \frac{M_{\pm}}{W_{max}}$$

We have to find the max/min bending moment in order to calculate the respective max/min tensile and compressive stresses. Note that **tensile stress is positive** (unlike tension which is negative).

$$\text{max/min } M \rightarrow \sigma^{\pm}_{\text{max}}$$

The max/min bending moment is found by the **extreme** of the $M(x) - x$ diagram.