Lab 3: Signal Processing Report

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1 At Home

1.1 Theoretical Study of the Sine Wave

Give that x is a real-valued sine wave,

$$x(t) = acos(2\pi f_0 t + \phi)$$

1. Write Fourier transform of x. Plot magnitude spectrum (be precise concerning pulse weights); scale: 1cm for f_0, abscissa from -10cm to +10cm.

$$\begin{aligned} x(t) &= \frac{a}{2}[e^{j2\pi f_0 t+\phi} + e^{j2\pi f_0 t+\phi}] \\ |F_{cc}(f)| &= \left|\frac{a}{2}\right| |\delta(f - f_0) + \delta(f + f_0)| \end{aligned}$$

2. This sine wave is sampled with sampling frequency $f_s > 2f_0$. Write expression of sampled signal x_s. Write s0s its Fourier transform $F_{dc}x_s(\lambda)$. Plot magnitude spectrum for $\frac{1}{f_s}F_{dc}x_s\left(\frac{f}{f_s}\right)$, with $f_s = 10f_0$.

   Let $\lambda_0 = f_0 T_s$.

$$\begin{aligned} x(nT_s) &= acos(2\pi f_0 T_s n + \phi) \\ &= \frac{a}{2}[e^{j\phi}e^{2j\pi\lambda_0 n} + e^{-j\phi}e^{-2j\pi\lambda_0 n}] \end{aligned}$$

   The discreet Fourier Transform of x is given by the formula,

$$\begin{aligned} |F_{dc}x\lambda| &= \left|\frac{a}{2}\right| |\sqcup(\lambda - \lambda_0) + \sqcup(\lambda + \lambda_0)| \\ \left|\frac{1}{f_s}F_{dc}x_s\left(\frac{f}{f_s}\right)\right| &= \left|\frac{a}{20f_0}\right| |\sqcup(\lambda - \lambda_0) + \sqcup(\lambda + \lambda_0)| \end{aligned}$$

3. Write Fourier transform of the windowed sampled signal $y = x_s rect_{N_t}$ by means of Dirichlet kernel D_{N_t}.

$$\begin{aligned} y &= x_s rect_{N_t} \\ F_{dc}y &= Fdcx_s * D_{N_t} \\ &= \left|\frac{a}{2}\right| |e^{j\phi} D_{N_t}(\lambda - \lambda_0) + e^{-j\phi} D_{N_t}(\lambda + \lambda_0)| \end{aligned}$$

4. A frequency sampling is performed: $F_{dc}y(\lambda)$ is computed only for frequencies multiple of $\frac{1}{N_f}$, that is to say $\lambda = \frac{k}{N_f}$, for all integers k. Write Y_{N_f} by means of discrete comb $1_{N_f,0}$in the case where $N_f = N_t$ and there exists an integer k_0 such that $\lambda_0 = k_0/N_f$.

   Substituting $\lambda = \frac{k}{N_f}$ in the previous question,

$$F_{dc}y(k/N_f) = \left|\frac{a}{2}\right| |e^{j\phi} D_{N_t}(\frac{k}{N_f} - \frac{k_0}{N_f}) + e^{-j\phi} D_{N_t}(\frac{k}{N_f} + \frac{k_0}{N_f})|$$

1.2 Numerical Implementation

1. If $N_f = N_t$, we recognize the discreet Fourier transform.

2. If $N_f = N_t$, the signal is zero-padded, i.e. the number of points remains the same in both, the frequency and the time domain.

3. If $N_f < N_t$, we compute the Fast Fourier Transform (FFT) instead of the Discreet Fourier Transform (DFT). In this case, some information is lost in the frequency domain.

B.8 Space Shuttle Altitude Simulation

B.8.1 Transfer Function and State Space Representation

a) Write the transfer function $\mathcal{L}h(s)$ of the system

Input torque u and attack angle ff :

$$\frac{\ddot{\alpha}}{w_0^2} \quad \alpha - \alpha_{\text{nom}} = Gu$$

By defining $\tilde{\alpha} = \alpha - \alpha_{\text{nom}}$, the function becomes:

$$\frac{\ddot{\tilde{\alpha}}}{w_0^2} - \tilde{\alpha} = Gu$$

Taking two-sided Laplace transform of the function:

$$\begin{aligned}
s^2 \frac{\mathcal{L}\tilde{\alpha}(s)}{w_0^2} \quad \mathcal{L}\tilde{\alpha}(s) &= G\mathcal{L}u(s) \\
\Leftrightarrow \quad \mathcal{L}\tilde{\alpha}(s)\left(\frac{s^2}{w_0^2} \quad 1\right) &= G\mathcal{L}u(s) \\
\Rightarrow \mathcal{L}h(s) = \frac{\mathcal{L}\tilde{\alpha}(s)}{\mathcal{L}u(s)} &= \frac{Gw_0^2}{s^2 \quad w_0^2}
\end{aligned}$$

Substituting the values of G and ω_0, we have:

$$\mathcal{L}h(s) = \frac{4}{s^2 \quad 4}$$

b) Calculate the transfer function $\mathcal{Z}\tilde{h}(z)$ with sampling period $T_s = 0.1\text{s}$ by step invariance method

We have:

$$\begin{aligned}
\mathcal{L}(h * \text{step})(s) &= \mathcal{L}h(s)\mathcal{L}\text{step}(s) \\
&= \frac{Gw_0^2}{s(s^2 \quad w_0^2)} \\
&= \frac{G}{s} - \frac{Gs}{s^2 \quad w_0^2}
\end{aligned}$$

Express the function above in time domain by taking inverse Laplace transform, then sample it with sampling period T_s we have:

$$\begin{aligned}
(h * \text{step})(t) &= G(1 - \cos w_0 t)\text{step}(t) \\
(h * \text{step})(nT_s) &= G[1 - \cos(w_0 n T_s)]\text{step}(nT_s) \\
&= G[1 - \cos(w_0 n T_s)]\text{step}[n]
\end{aligned}$$

Then, applying z-transform on the equation:

$$\mathcal{Z}(h * \text{step})(z) = G\left[\frac{1}{1 - z^{-1}} - \frac{1 - z^{-1}\cos(w_0 T_s)}{1 - 2z^{-1}\cos(w_0 T_s) \quad z^{-2}}\right]$$

Therefore, the transfer function $\mathcal{Z}\tilde{h}(z)$ is:

$$\begin{aligned}
\mathcal{Z}\tilde{h}(z) = \frac{\mathcal{Z}(h * \text{step})(z)}{\mathcal{Z}\text{step}(z)} &= G\left[1 - \frac{1 - z^{-1}\cos(w_0 T_s)}{1 - 2z^{-1}\cos(w_0 T_s) \quad z^{-2}}(1 - z^{-1})\right] \\
&= G\frac{[1 - \cos(w_0 T_s)](z^{-1} \quad z^{-2})}{1 - 2z^{-1}\cos(w_0 T_s) \quad z^{-2}}
\end{aligned}$$

$$\text{Given} \begin{cases} w_0 & = 2\text{rad.s}^{-1} \\ T_s & = 0.1\text{s} \\ G & = 1\text{rad.N}^{-1}.\text{m}^{-1} \end{cases}$$

$$\mathcal{Z}\tilde{h}(z) = \frac{0.02(z^{-1} \quad z^{-2})}{1 - 1.96z^{-1} \quad z^{-2}}$$

c) State space representation of the system

Let $\omega = \dot{\alpha}$. Since $\tilde{\alpha} = \alpha - \alpha_{\text{nom}}$, it can be deduced that $\dot{\alpha} = \dot{\tilde{\alpha}}$, thus $\omega = \dot{\tilde{\alpha}}$. In order to express the system in state space representation, let:

$$\begin{cases} x_1 & = \tilde{\alpha} \\ x_2 & = \omega \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = \dot{\tilde{\alpha}} & = \omega \\ \dot{x}_2 = \dot{\omega} & = \ddot{\tilde{\alpha}} = -\omega_0^2\tilde{\alpha} \quad \omega_0^2 G u \end{cases}$$

Rewrite the equations above in matrix form, we have:

$$\begin{bmatrix} \dot{\tilde{\alpha}} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \omega \end{bmatrix} \quad \begin{bmatrix} 0 \\ G\omega_0^2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \omega \end{bmatrix}$$

They correspond to the state space representation:

$$\begin{cases} \dot{x}(t) & = \mathbf{A}x(t) \quad \mathbf{B}u(t) \\ y(t) & = \mathbf{C}x(t) \quad \mathbf{D}u(t) \end{cases}$$

Where:

$$x = \begin{bmatrix} \tilde{\alpha} \\ \omega \end{bmatrix}; \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ G\omega_0^2 \end{bmatrix}; \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}; \qquad \mathbf{D} = 0$$

The dimension of state vector x is 2. Again, by substituting the value of G and ω_0 we have:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}; \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}; \qquad \mathbf{D} = 0$$

d) Write the discretized state space representation using step invariance method

General formulation of discrete state space representation:

$$\begin{cases} \tilde{x}[n \quad 1] & = \tilde{\mathbf{A}}\tilde{x}[n] \quad \tilde{\mathbf{B}}u_s[n] \\ \tilde{y}[n] & = \tilde{\mathbf{C}}\tilde{x}[n] \quad \tilde{\mathbf{D}}u_s[n] \end{cases}$$

Firstly, by using the equation $\mathbf{A}v = \lambda v$, we can find the eigenvalues of matrix $\mathbf{A}$:

$$\begin{cases} \lambda_1 & = i\omega_0 \\ \lambda_2 & = -i\omega_0 \end{cases}$$

Thus:

$$\mathbf{\Lambda} = \begin{bmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{bmatrix}$$

$$\Rightarrow \quad \mathrm{e}^{\mathbf{\Lambda} T_s} = \begin{bmatrix} \mathrm{e}^{i\omega_0 T_s} & 0 \\ 0 & \mathrm{e}^{-i\omega_0 T_s} \end{bmatrix}$$

Based on the eigenvalues, the eigenvectors can be calculated:

$$\lambda_1 = i\omega_0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ i\omega_0 \end{bmatrix}$$
$$\lambda_2 = -i\omega_0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ -i\omega_0 \end{bmatrix}$$

Thus, $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix}$. The matrix $\tilde{\mathbf{A}}$ is computed using the formula:

$$\begin{aligned}
\tilde{\mathbf{A}} &= \mathbf{P}\mathrm{e}^{\mathbf{\Lambda}T_s}\mathbf{P}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{i\omega_0 T_s} & 0 \\ 0 & \mathrm{e}^{-i\omega_0 T_s} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \mathrm{e}^{i\omega_0 T_s} & \mathrm{e}^{-i\omega_0 T_s} \\ i\omega_0\mathrm{e}^{i\omega_0 T_s} & -i\omega_0\mathrm{e}^{-i\omega_0 T_s} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2i\omega_0} \\ \frac{1}{2} & \frac{-1}{2i\omega_0} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{e^{i\omega_0 T_s} \quad e^{-i\omega_0 T_s}}{2} & \dfrac{e^{i\omega_0 T_s} - e^{-i\omega_0 T_s}}{2i\omega_0} \\ i\omega_0\dfrac{e^{i\omega_0 T_s} - e^{-i\omega_0 T_s}}{2} & \dfrac{e^{i\omega_0 T_s} \quad e^{-i\omega_0 T_s}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \cos(\omega_0 T_s) & \dfrac{\sin(w_0 T_s)}{\omega_0} \\ \omega_0 \sin(\omega_0 T_s) & \cos(\omega_0 T_s) \end{bmatrix}
\end{aligned}$$

Using the formula:

$$\tilde{\mathbf{B}} = \int_0^{T_s} \mathrm{e}^{\mathbf{A}\tau}\mathbf{B}d\tau = \int_0^{T_s} \tilde{\mathbf{A}}\mathbf{B}d\tau$$

Substituting the value of $\tilde{\mathbb{A}}$ and $\mathbf{B}$, we can compute the matrix $\tilde{\mathbf{B}}$:

$$\begin{aligned}
\tilde{\mathbf{B}} &= \int_0^{T_s} \begin{bmatrix} \cos(\omega_0\tau) & \sin(\omega_0\tau)/\omega_0 \\ -\omega_0\sin(\omega_0\tau) & \cos(\omega_0\tau) \end{bmatrix} \begin{bmatrix} 0 \\ G\omega_0^2 \end{bmatrix} d\tau \\
&= \int_0^{T_s} \begin{bmatrix} G\omega_0\sin(\omega_0\tau) \\ G\omega_0^2\cos(\omega_0\tau) \end{bmatrix} d\tau \\
&= \begin{bmatrix} -G\cos(\omega_0\tau) \\ G\omega_0\sin(\omega_0\tau) \end{bmatrix}_0^{T_s} \\
&= \begin{bmatrix} -G\cos(\omega_0 T_s) \\ G\omega_0\sin(\omega_0 T_s) \end{bmatrix} - \begin{bmatrix} -G \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} G[1-\cos(\omega_0 T_s)] \\ G\omega_0\sin(\omega_0 T_s) \end{bmatrix}
\end{aligned}$$

Additionally, we have $\tilde{\mathbf{C}} = \mathbf{C}$ and $\tilde{\mathbf{D}} = \mathbf{D}$. Hence, the matrices of discrete state space representation of the system are:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \cos(\omega_0 T_s) & \dfrac{\sin(w_0 T_s)}{\omega_0} \\ \omega_0\sin(\omega_0 T_s) & \cos(\omega_0 T_s) \end{bmatrix}; \qquad \tilde{\mathbf{B}} = \begin{bmatrix} G[1-\cos(\omega_0 T_s)] \\ G\omega_0\sin(\omega_0 T_s) \end{bmatrix}; \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}; \qquad \mathbf{D} = 0$$

Substituting the values, we have:

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0.98 & 0.10 \\ -0.40 & 0.98 \end{bmatrix}; \qquad \tilde{\mathbf{B}} = \begin{bmatrix} 0.02 \\ 0.40 \end{bmatrix}; \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}; \qquad \mathbf{D} = 0$$

B.8.2 Pulse Width Modulator

Prove that if u is constant between $-E$ and E, then $\bar{u}_m = u$

Since the actual torque input of the shuttle is $u_m = E\text{sign}(u - p)$ where p is a triangular waveform with period T_P, we can plot u and u_m as shown in Figure B.8.1:

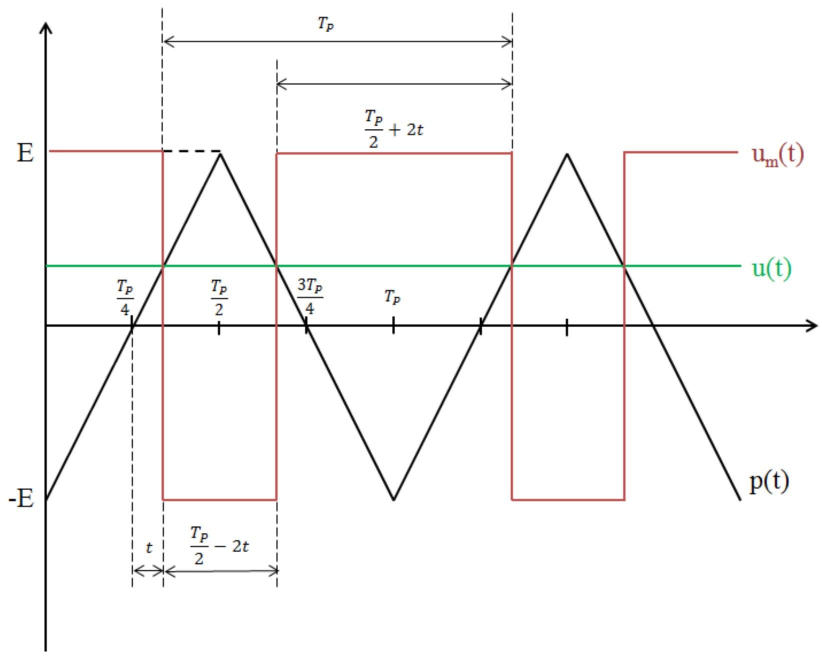


Figure B.8.1: Plot of controller signal u, actual torque input u_m and triangular waveform p

From Figure B.8.1, it is shown that signal u_m is periodic with the period equal to that of the triangular waveform p: T_P. In each period, the value of u_m is:

$$\begin{cases} -E & \text{for a duration of } t_1 = \frac{T_P}{2} - 2t \\ E & \text{for a duration of } t_2 = \frac{T_P}{2} \quad 2t \end{cases}$$

Where t can be found from the relationship:

$$\frac{E}{T_P/4} = \frac{u}{t}$$
$$\Rightarrow \quad t = \frac{uT_P}{4E}$$

The mean value $\bar{u}_m$ of the actual torque input:

$$\begin{aligned} \bar{u}_m &= \frac{-Et_1 \quad Et_2}{t_1 \quad t_2} \\ &= \frac{E[(\frac{T_P}{2} \quad 2t) - (\frac{T_P}{2} - 2t)]}{(\frac{T_P}{2} \quad 2t) \ (\quad \frac{T_P}{2} - 2t)} \\ &= \frac{4E}{T_P} t \\ &= \frac{4E}{T_P} \frac{uT_P}{4E} \\ &= u \end{aligned}$$

Hence the mean value of actual torque input u_m is equal to constant u.

1.3 B.8.3 Open Loop Simulation

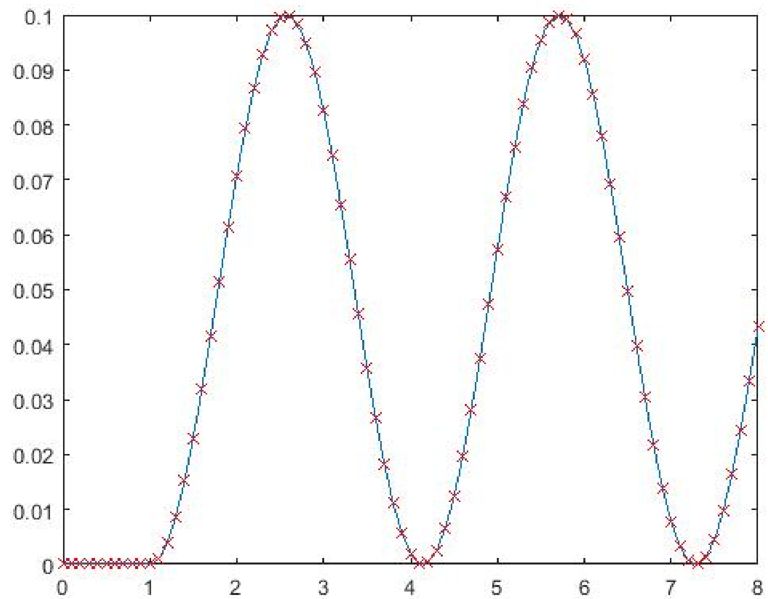


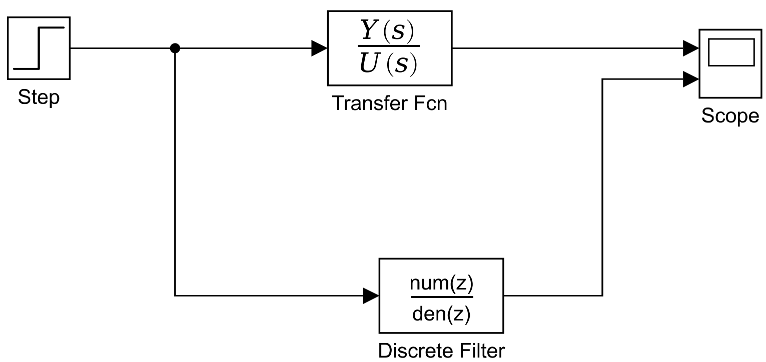
Figure 1: Plot for Open Loop Simulation

The Matlab function which describes the differential equation of the state vector of the continuous time system is as follows:

```
function xpoint = navettecontinue(t,x,A,B)

u=0.05*(t >= 1);
xpoint=A*x+B*u;

end
```

A Matlab function which describe the recursion of the state vector of the discrete time system is:

```
function xeplus = navettediscrete(n,xe,Atilde,Btilde,ts)

u=0.05*(ts*n >= 1);
xeplus=Atilde*xe+Btilde*u;

end
```


Step
$\frac{Y(s)}{U(s)}$
Transfer Fcn
Scope
$\frac{\text{num}(z)}{\text{den}(z)}$
Discrete Filter

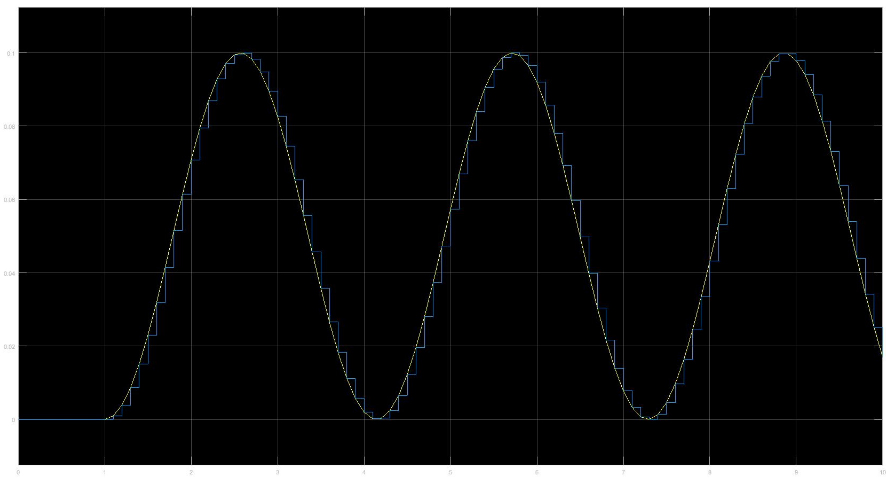


Figure 2: Open Loop Simulation using Simulink

1.4 B.8.5 Closed Loop Simulation

The simulink models for a closedloop system and a closedloop PID controller are as follows:

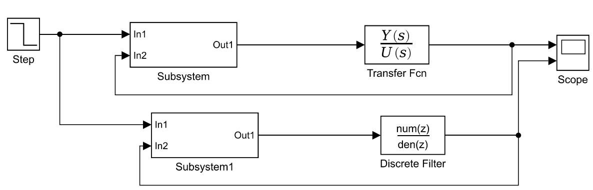


Figure 3: Closedloop Simulation

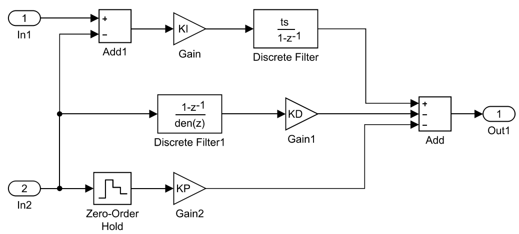


Figure 4: Closedloop Simulation for a PID Controller

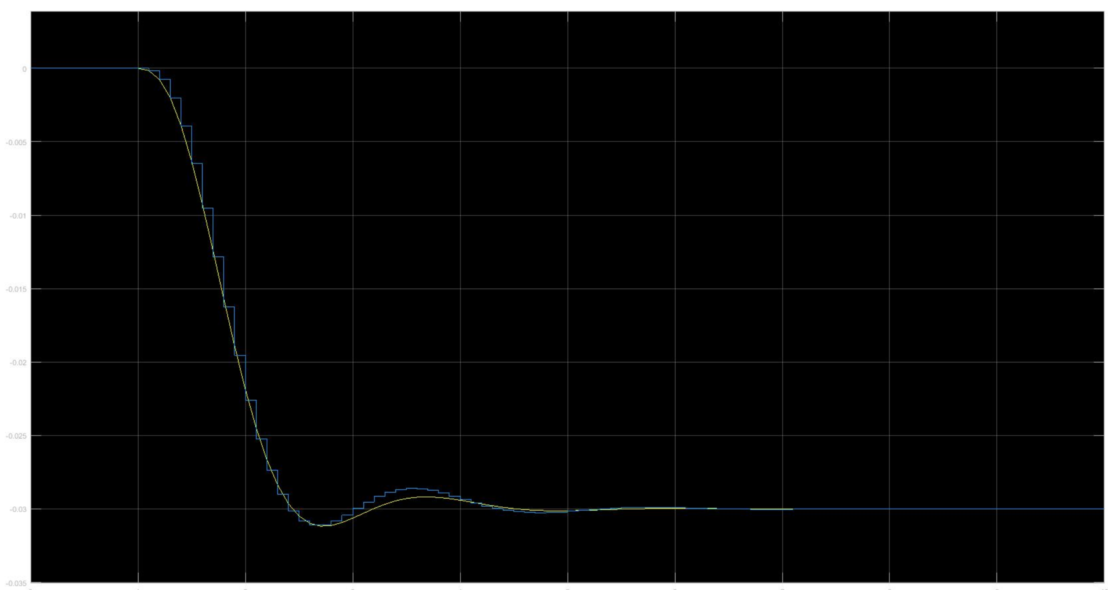


Figure 5: Closedloop Simulation Plot

1.5 Closedloop Simulation with PWD

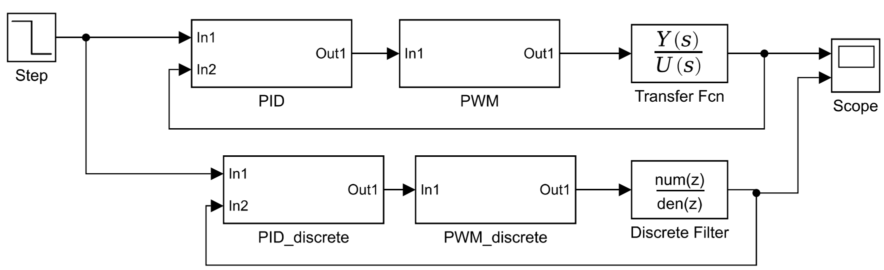
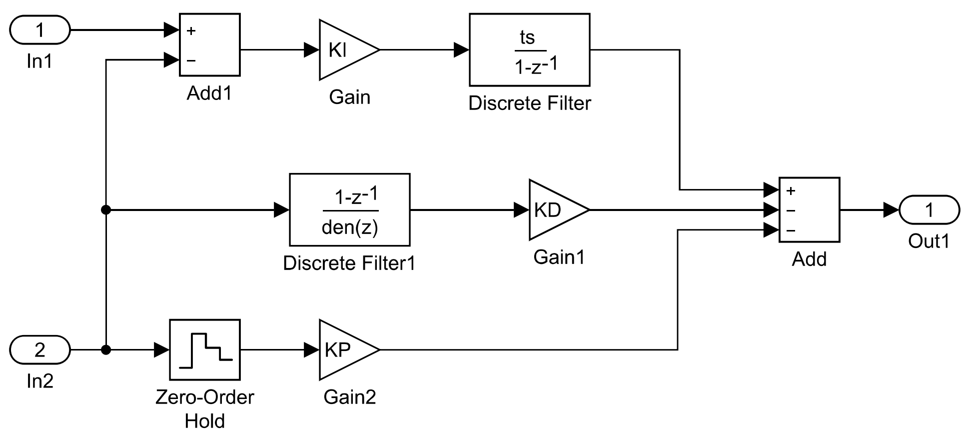


Figure 6: PWD Simulation


1
In1
+
–
Add1
KI
Gain
ts
1-z-1
Discrete Filter
1-z-1
den(z)
Discrete Filter1
KD
Gain1
+
–
–
Add
1
Out1
2
In2
Zero-Order
Hold
KP
Gain2

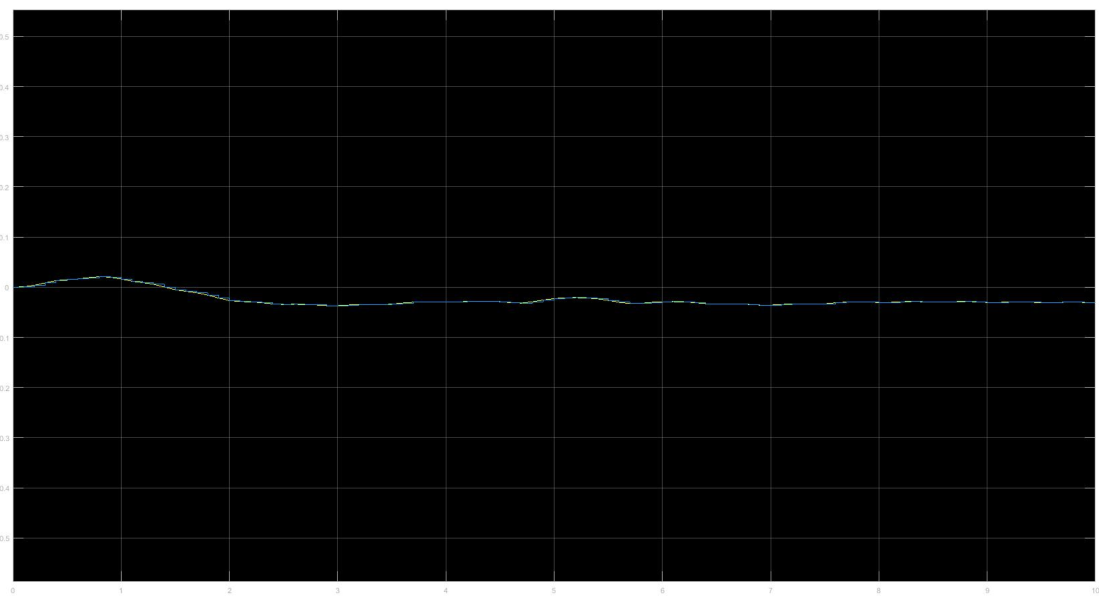


Figure 7: Plot for Closedloop Simulation for PWM